

Unraveling pentaquarks with the Born-Oppenheimer effective theory

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The hidden-charm pentaquark states $P_{c\bar{c}}(4312)^+$, $P_{c\bar{c}}(4380)^+$, $P_{c\bar{c}}(4440)^+$, and $P_{c\bar{c}}(4457)^+$, all with isospin $I = 1/2$, were discovered by the LHCb Collaboration in the decay process $\Lambda_b^0 \rightarrow J/\psi p K^-$. Although their quantum numbers remain undetermined, these states have generated significant theoretical interest. We analyze their spectrum and decay patterns—including those of their spin partners—within the Born-Oppenheimer (BO) effective field theory (BOEFT), a framework grounded in QCD. At leading order in BOEFT, we identify these pentaquark states as bound states in BO potentials that exhibit at short distance a repulsive octet behavior and a nonperturbative shift due to the adjoint baryons masses, while asymptotically approaching the $\Sigma_c \bar{D}$ threshold. We further incorporate $\mathcal{O}(1/m_Q)$ spin-dependent corrections to compute pentaquark multiplet spin splittings. Based on the spectrum, semi-inclusive decay widths to J/ψ and η_c , and the decay width ratios to $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$, we provide the first theoretical predictions for the adjoint baryon masses, which can be confirmed by future lattice QCD studies. Moreover, our analysis supports the quantum number assignments: $J^P = (1/2)^-$ for $P_{c\bar{c}}(4312)^+$, $(3/2)^-$ for $P_{c\bar{c}}(4380)^+$, $(1/2)^-$ for $P_{c\bar{c}}(4457)^+$, and $(3/2)^-$ for $P_{c\bar{c}}(4440)^+$. We also present results for the lowest bottom pentaquarks.

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I. INTRODUCTION

QCD, the established theory of the strong interaction, describes the dynamics of quarks and gluons. The property of color confinement confines quarks and gluons into color-singlet hadrons and allows for a richer spectrum of bound states than conventional mesons (quark-antiquark pairs) or baryons (three-quark states) [1–3] including exotic hadrons such as tetraquarks (four-quark states), pentaquarks (five-quark states), hybrids (bound states of quarks with gluonic excitations), and glueballs (bound states composed purely of gluons) and so on; see Ref. [4] for a review. The past two decades have seen a surge of discoveries of XYZ states that are exotic hadrons with at least two heavy quarks and are often located near or above open-flavor thresholds. The Belle experiment in 2003 discovered the first XYZ state,

$\chi_{c1}(3872)$, containing a charm quark-antiquark ($c\bar{c}$) pair, in the $B \rightarrow K\pi^+\pi^- J/\psi$ decay channel [5]. This breakthrough paved the way for the discovery of dozens of new XYZ hadrons by various experiments at the B-factories (BABAR, Belle, Belle2), τ -charm facilities (BES, BESIII), and also proton-(anti)proton colliders (CDF, D0, LHCb, ATLAS, CMS) (see Refs. [4,6–11] for reviews). To date, approximately 55 XYZ states with at least two heavy quarks have been observed [12]. Among them are six pentaquark states discovered by LHCb in the $c\bar{c}$ sector: four isospin half-integer ($I = 1/2$) states $P_{c\bar{c}}(4312)^+$, $P_{c\bar{c}}(4380)^+$, $P_{c\bar{c}}(4440)^+$, and $P_{c\bar{c}}(4457)^+$ with quark content $c\bar{c}qqq$ and two isospin integer ($I = 0$) states $P_{c\bar{c}}(4338)^0$ and $P_{c\bar{c}}(4459)^0$ with quark content $c\bar{c}sqq$, where $q \equiv (u, d)$ [13–16]. Understanding the nature of these pentaquark states remains an elusive problem.

The LHCb Collaboration first reported the discovery of two hidden-charm pentaquark states in the $J/\psi p$ invariant mass distribution of the $\Lambda_b^0 \rightarrow J/\psi p K^-$ decay [13]. These included a broad state $P_{c\bar{c}}(4380)^+$ with mass 4380 ± 30 MeV and width 210 ± 90 MeV, and a narrower state $P_{c\bar{c}}(4450)^+$ with mass 4449.8 ± 3 MeV and width 39 ± 20 MeV. Four years later, in 2019, a subsequent updated analysis with larger data sample (Run 1 and Run 2) by the LHCb Collaboration found a new narrow state $P_{c\bar{c}}(4312)^+$ and further resolved the

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originally reported $P_{c\bar{c}}(4450)^+$ into two distinct narrow resonances: $P_{c\bar{c}}(4440)^+$ and $P_{c\bar{c}}(4457)^+$ [14]. The broad state $P_{c\bar{c}}(4380)^+$ was neither confirmed nor refuted in the updated analysis, which was only sensitive to narrow peaks. A clear signal for this state was not seen in the $J/\psi p$ mass spectrum in Ref. [14], and therefore the existence of $P_{c\bar{c}}(4380)^+$ needs further experimental confirmation. In our work, we refer to a “ $P_{c\bar{c}}(4380)^+$ ” state, whose mass aligns with the value reported in Ref. [13], although our predicted decay width is significantly smaller and inconsistent with that analysis, which is now considered obsolete. Table I shows the four hidden-charm pentaquark states reported in the particle data group (PDG) [17]. The quantum number J^P assignment of these states is not yet known.

The hidden-charm pentaquark states $P_{c\bar{c}}$ were theoretically anticipated well before their experimental observation [18–25]. After the LHCb Collaboration discovery, a wide range of theoretical frameworks has emerged to explain the nature of the pentaquark states such as hadronic molecules [26–58], compact pentaquark states [51,59–68], baryo-charmonia or hadro-charmonia [69–74], cusp effects [51], triangle singularities [75], and virtual states [76]. Given that the pentaquark states $P_{c\bar{c}}$ in Table I have masses within close proximity of the $\Sigma_c^{(*)}\bar{D}^{(*)}$ thresholds, the states have been widely interpreted as hadronic molecules with S-wave $\Sigma_c^{(*)}\bar{D}^{(*)}$ constituents. In this interpretation, $P_{c\bar{c}}(4312)^+$ is a $(1/2)^- \Sigma_c \bar{D}$ state and $P_{c\bar{c}}(4380)^+$ is a $(3/2)^- \Sigma_c^* \bar{D}$ state, albeit with a smaller width compared to the state observed in the original LHCb analysis [28,29,55]. $P_{c\bar{c}}(4440)^+$ and $P_{c\bar{c}}(4457)^+$ are both $\Sigma_c \bar{D}^*$ states, and can be assigned either $(1/2)^-$ and $(3/2)^-$, or $(3/2)^-$ and $(1/2)^-$, respectively. In contrast, the compact pentaquark model attributes positive parity to some of these states. Within the molecular framework, the heavy-quark spin symmetry implies that there is a multiplet of seven pentaquark states (including three additional states near the $\Sigma_c^* \bar{D}^*$ threshold with respect to the four states listed in Table I). As remarked above, in the molecular framework, the mass spectrum alone is insufficient to determine the J^P numbers of $P_c(4440)$ and $P_c(4457)$. Many studies have addressed this J^P issue [31,43,44,55,56,77–80]. We discuss the quantum number assignment in Sec. III.

The pentaquark states $P_{c\bar{c}}(4312)^+$, $P_{c\bar{c}}(4440)^+$, and $P_{c\bar{c}}(4457)^+$ have been investigated using lattice QCD by calculating the S-wave scattering of $\Sigma_c \bar{D}$ and $\Sigma_c \bar{D}^*$ via Lüscher’s method only in the $I(J^P) = \frac{1}{2}(1/2)^-$ channel at a pion mass of 294 MeV and with a lattice spacing of 0.08 fm. Two bound-state poles were found with one corresponding to $P_{c\bar{c}}(4312)^+$ and the other to either $P_{c\bar{c}}(4440)^+$ or $P_{c\bar{c}}(4457)^+$ [81].

Beyond spectroscopy, decay properties are also key to understanding the structure of the pentaquark states. Several approaches have been used to study the decay of charm pentaquark states $P_{c\bar{c}}$ into $J/\psi p$, $\eta_c p$, $\Lambda_c \bar{D}^{(*)}$ and $\Sigma_c \bar{D}^{(*)}$ thresholds [29,41,45,55,57,58,79,82,83]. Most recently, the LHCb Collaboration performed a search for $P_{c\bar{c}}(4312)^+$, $P_{c\bar{c}}(4440)^+$, and $P_{c\bar{c}}(4457)^+$ states in the prompt $\Lambda_c \bar{D}^{(*)}$, $\Lambda_c \bar{D}^{(*)}$, and $\Lambda_c \pi \bar{D}^{(*)}$ mass spectra. No significant signal was observed [84]. Thus far, the only observed decay channel for the $P_{c\bar{c}}$ pentaquark states remain $J/\psi p$.

In this paper, we use the QCD effective field theory called BOEFT to address both the spectra and decays of quarkonium pentaquark states $Q\bar{Q}qqq$, without making any *a priori* assumption on their quark configurations. The BOEFT is derived from QCD on the basis of symmetries and scale separations and results in coupled channel Schrödinger equations governing the dynamics of the states [85]. A crucial input to the Schrödinger equations are the static energies or potentials between the heavy quarks due to light quarks or gluons. These are nonperturbative functions of the heavy quark separation to be computed in lattice QCD. Currently, there are no lattice QCD results for the pentaquark static energies. However, BOEFT constrains the behavior of the static energies based on the symmetries both at short and long distances. While BOEFT requires lattice input for the static energies computed from some generalized Wilson loops, the factorization inherent the framework ensures that only a small set of universal, flavor-independent nonperturbative correlators are needed, greatly simplifying the problem.

The Born-Oppenheimer picture has been used for a long time to study QCD bound states such as heavy quarkonium hybrids [86–94]. It has been recast into an EFT framework known as BOEFT [85,95–99]. For hybrids, the BOEFT has

TABLE I. List of nonstrange pentaquark states with quark content $c\bar{c}qqq$, where $q = (u, d)$ that are listed on PDG [17]. All the states have $I = 1/2$ but J^P quantum numbers are unknown.

State (PDG)	M (MeV)	Γ (MeV)	$I(J^P)$	Decay modes
$P_{c\bar{c}}(4312)^+$	$4311.9^{+7.0}_{-0.9}$	10 ± 5	$1/2(?)$	$J/\psi p$
$P_{c\bar{c}}(4380)^+$	4380 ± 30	210 ± 90^a	$1/2(?)$	$J/\psi p$
$P_{c\bar{c}}(4440)^+$	$4440^{+4.0}_{-5.0}$	21^{+10}_{-11}	$1/2(?)$	$J/\psi p$
$P_{c\bar{c}}(4457)^{+b}$	$4457.3^{+4.0}_{-1.8}$	$6.4^{+6.0}_{-2.8}$	$1/2(?)$	$J/\psi p$

^aThe reported width is based on the original analysis [13], which is nowadays considered obsolete.

^bEarlier, this state was $P_{c\bar{c}}(4450)$.

been used to study the spectrum, incorporating spin corrections in the hybrid multiplets through spin-dependent potentials [95,100–104], and computing semi-inclusive decays to low-lying quarkonium states [100,105,106]. In addition, decays of hybrids into open-flavor threshold states have been investigated within this framework [107–109]. The BOEFT formalism has been also applied to study spectra and decays of doubly heavy baryons [110–113] and recently the lowest-lying tetraquark multiplets, including their spin splittings [114,115].

The paper is organized as follows: in Sec. II, we give a brief description of the BOEFT, information on the quantum numbers, expressions of the pentaquark static energies relevant for lattice QCD computation, and the expected short and long distance behavior of the static energies. In Sec. III, we first write the coupled Schrödinger equations that follow from BOEFT for pentaquark states ignoring spin corrections and then we include spin corrections using first-order perturbation theory to identify the experimental states. We discuss the different scenarios for the quantum number J^P assignments to the pentaquark states in the charm sector. In Sec. IV, we examine the decays of the pentaquark states into J/ψ , η_c , $\Lambda_c \bar{D}$, and $\Lambda_c \bar{D}^*$ based on the scenarios in Sec. III. In Sec. V, we give the results for the bottom pentaquark states $P_{b\bar{b}}$ based on the preferred scenario fixed by the spectrum and decays of the charm pentaquark $P_{c\bar{c}}$ states. Finally, Sec. VI contains discussion and conclusion.

II. BORN-OPPENHEIMER EFFECTIVE THEORY

Heavy quarkonium systems and hidden-heavy systems are hadrons made of a heavy quark-antiquark pair ($c\bar{c}$ or $b\bar{b}$) bound to light d.o.f. (LDF) such as gluons g or light quarks q or antiquarks $\bar{q}$. The large heavy quark mass scale, m_Q , introduces significant simplifications: the scale is perturbative ($m_Q \gg \Lambda_{\text{QCD}}$, with Λ_{QCD} the nonperturbative hadronic scale), and the quarks move nonrelativistically, i.e., $v \ll 1$, where v is the relative velocity of the heavy quark-antiquark pair in the bound system.

The simplest hidden-heavy systems are quarkonium states ($Q\bar{Q}$), which are color singlet heavy quark-antiquark bound states with no valence LDF in the bound state. In contrast, pentaquark states $Q\bar{Q}qqq$ are examples of hidden-heavy systems with three light quarks as LDF bound to the $Q\bar{Q}$ pair to form a color-singlet state. Other examples include hybrids ($Q\bar{Q}g$), where a valence gluon contributes to the dynamics and tetraquarks ($Q\bar{Q}q\bar{q}$), which contain light quark-antiquark pair as valence LDF. The relevant energy scales to describe such systems made of two nonrelativistic heavy quarks ($Q\bar{Q}$ or even QQ) are the mass scale m_Q , the relative momentum scale $m_Q v \sim 1/r$, where r is the relative distance between the heavy quarks, and the heavy-quark-antiquark binding energy scale $m_Q v^2$. Such scales are hierarchically ordered as $m_Q \gg m_Q v \gg m_Q v^2$, where $v \ll 1$ is the

relative velocity of the heavy (anti)quark in the bound system. In addition, in QCD, there is the nonperturbative energy scale of the LDF, Λ_{QCD} . Apart from the heavy quark mass, the treatment of the other energy scales depends on their proximity to Λ_{QCD} , as nonperturbative methods have to be used if the scales are close to it. The challenge of dealing with multiple intertwined energy scales in nonrelativistic bound states of QCD has been systematically addressed by substituting QCD with simpler yet equivalent nonrelativistic effective field theories [88,116–121].

The suitable EFT for describing systems with two heavy quarks after integrating out the heavy quark mass scale m_Q is nonrelativistic QCD (NRQCD) [116,117]. NRQCD is formulated as a systematic expansion in $1/m_Q$. For exotic states such as pentaquarks, which are typically extended objects, we assume that the LDF responsible for their binding satisfy the hierarchy condition $m_Q v \gtrsim \Lambda_{\text{QCD}} \gg m_Q v^2$. This hierarchy implies that the nonrelativistic motion of the $Q\bar{Q}$ pair evolves with a much larger time scale compared to the LDF time scale, $1/\Lambda_{\text{QCD}}$. It also implies that mixing between states separated by energy gaps of order Λ_{QCD} is parametrically suppressed.¹ This separation of scales naturally motivates the use of the BO approximation, well known from molecular physics [122,123]. In the context of QCD bound states, focusing on the lower-energy dynamics at the scale $m_Q v^2$ by integrating out all the higher-energy scales above $m_Q v^2$ leads to the BOEFT [85,95–99]. In BOEFT, the LDF dynamics is encoded in the static potentials between the heavy quarks. In this work, we apply the formalism developed in Refs. [85,105,109] to the study of spectrum and decays of $Q\bar{Q}qqq$ pentaquark states. We consider both semi-inclusive decays to quarkonium states like J/ψ , η_c , and decays to baryon-antimeson $\Lambda_c \bar{D}$, and $\Lambda_c \bar{D}^*$ thresholds.

A. BO quantum numbers

In the *static limit*, eigenstates and eigenvalues of the QCD Hamiltonian in the $Q\bar{Q}$ sector are referred to as *static states* and *static energies*, respectively. In this limit, the heavy quarks act as static color sources, while the different LDF configurations identify different static energies that are labeled by the *BO quantum numbers* Λ_η^σ , representations of the cylindrical symmetry group D_{cyl} . The quantity Λ is defined as $\Lambda \equiv |\lambda| = |\mathbf{K} \cdot \hat{\mathbf{r}}|$, where λ is the eigenvalue of the projection of the LDF total angular momentum (spin) $\mathbf{K}$ along the heavy quark pair axis $\hat{\mathbf{r}}$. For integer values of Λ , we use the standard notation of capital Greek letters: $\Sigma, \Pi, \Delta, \dots$ for $\Lambda = 0, 1, 2, \dots$. The index η is the *CP* eigenvalue, if *CP* is a good quantum number, elsewhere it is the parity P , and is denoted by $g = +1$ and $u = -1$. The index $\sigma = \pm 1$ is the eigenvalue of the reflection operator with respect to a plane passing through the

¹Note that states like quarkonium and pentaquarks cannot mix due to differences in quantum numbers such as isospin.

$\hat{\mathbf{r}}$ axis. The index σ is explicitly written only for Σ states, which are not degenerate under reflection. The static energies are also characterized by the flavor quantum numbers of the LDFs, such as isospin I and projection m_I , baryon number b , and so on.

Assuming that the angular momentum operator $\mathbf{K}^2$ has eigenvalues $k(k+1)$, which restricts the projection to $\Lambda \leq k$, we introduce the shorthand notation

$$\kappa \equiv \{k^{PC}, f\}, \quad (2.1)$$

where f denotes the LDF flavor indices.² The quarkonium states are isoscalar states with $\kappa = \{0^{++}, I = 0\}$, while pentaquark states have nonzero isospin and nonzero LDF spin k . It is important to note that k is a good quantum number only at short distances ($r \rightarrow 0$), where the symmetry group of the system becomes the spherical symmetry $O(3)$, but not at large distances, where the symmetry of the system is $D_{\infty h}$.

For $Q\bar{Q}qqq$ pentaquarks, the total spin $\mathbf{K}$ of the three light quarks can only assume half-integer values. As a result, the standard notation for the BO-quantum numbers Λ_η^σ is not applicable: there are no established labels for Λ in the half-integer case, the reflection quantum number σ does not appear without a $\lambda = 0$ state, and, because CP transforms a light qqq state into a $\bar{q}\bar{q}\bar{q}$ state, the CP eigenstates are even and odd linear combinations of the two.³ For notation, we choose to label the LDF states by k^P and represent the BO quantum numbers as $(\Lambda)_\eta$, where η refers only to parity P , and $\Lambda = |\mathbf{K} \cdot \hat{\mathbf{r}}|$. The ground state with given BO quantum number is labeled $(\Lambda)_\eta$, whereas excited states with the same quantum numbers are labeled $(\Lambda)_\eta', (\Lambda)_\eta'', \dots$

B. Pentaquark static energies: Operator and behavior at short distance

While quarkonium $Q\bar{Q}$ states are bound states of a heavy quark-antiquark pair with no LDF, pentaquark $Q\bar{Q}qqq$ states are bound states with three light quarks as LDF. In the $r \rightarrow 0$ (short-distance) limit, where r is the separation between the heavy quark and antiquark pair, the color quantum number combination of the $Q\bar{Q}$ pair results in both color singlet and color octet configurations. In the color octet case, the $(Q\bar{Q})_8$ pair binds with the LDF that are in the adjoint representation of $SU(3)$ to form color singlet hadrons. We refer to those LDF states with quantum numbers k^P in the adjoint representation as *adjoint hadrons*. In the context of the current work, the color octet case describes pentaquarks $[(Q\bar{Q})_8 + (qqq)_8]$, while

the color singlet case corresponds to quarkonia $(Q\bar{Q})_1$ or quarkonia with light hadrons $[(Q\bar{Q})_1 + (qqq)_1]$ [85,99].⁴

The static energy—also known as the *BO potential*—for the pentaquark state with BO-quantum number $(\Lambda)_\eta$ can be computed from the large time behavior of the logarithm of an appropriate gauge-invariant correlator, as follows:

$$\begin{aligned} E_{(\Lambda)_\eta}(r) &\equiv E_{\kappa,|\lambda|}(r) \\ &= \lim_{T \rightarrow \infty} \frac{i}{T} \log[\langle \text{vac} | \mathcal{O}_{\kappa,\lambda}(T/2, \mathbf{r}, \mathbf{R}) \\ &\quad \times \mathcal{O}_{\kappa,\lambda}^\dagger(-T/2, \mathbf{r}, \mathbf{R}) | \text{vac} \rangle], \end{aligned} \quad (2.2)$$

where κ is given by Eq. (2.1) and $|\text{vac}\rangle$ denotes the NRQCD vacuum. The gauge-invariant interpolating operator $\mathcal{O}_{\kappa,\lambda}$ is a function of the relative coordinate $\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2$ and the c.m. coordinate $\mathbf{R} \equiv (\mathbf{x}_1 + \mathbf{x}_2)/2$ of the $Q\bar{Q}$ pair, $\mathbf{x}_1$ and $\mathbf{x}_2$ being the space locations of the quark and antiquark. The LDF isospin or flavor quantum numbers are not explicitly written. For a heavy quark-antiquark octet, $\mathcal{O}_{\kappa,\lambda}$ can be given in terms of NRQCD fields by

$$\begin{aligned} \mathcal{O}_{\kappa,\lambda}(t, \mathbf{r}, \mathbf{R}) &= \chi^\dagger(t, \mathbf{R} + \mathbf{r}/2) \phi(t; \mathbf{R} + \mathbf{r}/2, \mathbf{R}) P_{\kappa,\lambda}^{\alpha\dagger} H_{8,\kappa}^{\alpha,a}(t, \mathbf{R}) \\ &\quad \times T^a \phi(t; \mathbf{R}, \mathbf{R} - \mathbf{r}/2) \psi(t, \mathbf{R} - \mathbf{r}/2), \end{aligned} \quad (2.3)$$

where the field ψ is the Pauli spinor that annihilates the heavy quark and χ is the Pauli spinor that creates the heavy antiquark; they satisfy canonical equal time anticommutation relations. The matrices T^a ($a = 1, \dots, 8$) are the $SU(3)$ generators and $\phi(t; \mathbf{x}, \mathbf{y})$ is the Wilson line

$$\phi(t; \mathbf{x}, \mathbf{y}) = \mathcal{P} \exp \left[-ig \int_{\mathbf{y}}^{\mathbf{x}} d\mathbf{z} \cdot \mathbf{A}(t, \mathbf{z}) \right], \quad (2.4)$$

with $\mathcal{P}$ the path ordering operator. The operators $H_{8,\kappa}^{\alpha,a}(\mathbf{R})$ are related to adjoint baryons when the LDF are three light quarks. The projection vectors $P_{\kappa,\lambda}^\alpha$ with α the vector or spin index project onto eigenstates of $\mathbf{K} \cdot \hat{\mathbf{r}}$ with eigenvalue λ , thereby fixing the cylindrical symmetry group $D_{\infty h}$ quantum numbers. They are given in terms of Wigner D-matrices and read [85]

$$P_{\kappa\lambda}^\alpha(\theta, \varphi) = \sqrt{\frac{8\pi^2}{2k+1}} D_{\kappa\lambda}^{\alpha*}(0, \theta, \varphi), \quad (2.5)$$

where $\alpha = k, \dots, 0, \dots, -k$, as we are using a spherical basis. Specific expressions of different projection vectors are in Appendix F of Ref. [85]. For quarkonium, the quantum numbers of the LDF are $\kappa = 0^{++}$, which implies a trivial form of the projection vector: $P_{00} = 1$. Indeed,

²For notational ease, we explicitly mention whenever we suppress the flavor index f .

³See Sec. IV C in Ref. [85] for more details.

⁴Lattice QCD calculations show that these states are not sufficiently bound to form a multiquark state [124,125].

setting $P_{\kappa,\lambda}^\alpha H_{8,\kappa}^{\alpha,a}(t, \mathbf{R}) T^a = \mathbb{1}$ in Eq. (2.3) identifies the color singlet $(Q\bar{Q})_1$ state corresponding to quarkonium.

In the current work, we consider the lowest pentaquark states $Q\bar{Q}qqq$, where $q = (u, d)$, listed in Table I, all of which have isospin $I = 1/2$. Without orbital excitations, the three light quarks have positive parity and can have total spin either as a spin doublet $[k^P = (1/2)^+]$ or a spin quartet $[k^P = (3/2)^+]$. Their isospin can similarly be either a doublet ($I = 1/2$) or quartet ($I = 3/2$). However, the light quarks being indistinguishable, the Pauli exclusion principle imposes constraints on the whole color-spin-isospin

combinations, which must be totally antisymmetric under quark exchange. As a result, in the color-octet sector, all combinations of spin and isospin are allowed except for the case where both are quartets ($I = 3/2, k = 3/2$).⁵ For light quark spin $k^P = (1/2)^+$, we denote the BO potential $E_{(1/2)_g}(r)$ and for light quark spin $k^P = (3/2)^+$, we denote the two BO potentials corresponding to the two projections along the $Q\bar{Q}$ axis as $E_{(1/2)'_g}(r)$ and $E_{(3/2)_g}(r)$.

The light quark interpolating operator $H_{8,\kappa}^{\alpha,a}(t, \mathbf{x})$ for $I = 1/2, I_3 = \pm 1/2$, and $k = 1/2$ is given by [85]

$$\begin{aligned} H_{8,I_3=\pm 1/2,(1/2)^+}^{\alpha,a}(t, \mathbf{x}) &= [(\delta_{\alpha\beta_1}\sigma_{\beta_2\beta_3}^2 + \delta_{\alpha\beta_2}\sigma_{\beta_1\beta_3}^2 + \delta_{\alpha\beta_3}\sigma_{\beta_1\beta_2}^2)(\delta_{I_3f_1}\tau_{f_2f_3}^2 + \delta_{I_3f_2}\tau_{f_1f_3}^2 + \delta_{I_3f_3}\tau_{f_1f_2}^2)(T_2)_{l_1,l_2,l_3}^a \\ &+ (\delta_{\alpha\beta_1}\sigma_{\beta_2\beta_3}^2 + \delta_{\alpha\beta_2}\sigma_{\beta_3\beta_1}^2 + \delta_{\alpha\beta_3}\sigma_{\beta_2\beta_1}^2)(\delta_{I_3f_1}\tau_{f_2f_3}^2 + \delta_{I_3f_2}\tau_{f_3f_1}^2 + \delta_{I_3f_3}\tau_{f_2f_1}^2)(T_3)_{l_1,l_2,l_3}^a \\ &+ (\delta_{\alpha\beta_1}\sigma_{\beta_3\beta_2}^2 + \delta_{\alpha\beta_2}\sigma_{\beta_3\beta_1}^2 + \delta_{\alpha\beta_3}\sigma_{\beta_1\beta_2}^2)(\delta_{I_3f_1}\tau_{f_3f_2}^2 + \delta_{I_3f_2}\tau_{f_3f_1}^2 + \delta_{I_3f_3}\tau_{f_1f_2}^2)(T_1)_{l_1,l_2,l_3}^a] \\ &\times (P_+ q_{l_1f_1}(t, \mathbf{x}))^{\beta_1} (P_+ q_{l_2f_2}(t, \mathbf{x}))^{\beta_2} (P_+ q_{l_3f_3}(t, \mathbf{x}))^{\beta_3}, \end{aligned} \quad (2.6)$$

and for $I = 1/2, I_3 = \pm 1/2$, and $k = 3/2$, it is given by

$$\begin{aligned} H_{8,I_3=\pm 1/2,(3/2)^+}^{\alpha,a}(t, \mathbf{x}) &= [(C_{1m1/2\beta_1}^{3/2\alpha}(\mathbf{e}_m \cdot \boldsymbol{\sigma} i \sigma^2)_{\beta_2\beta_3} + C_{1m1/2\beta_2}^{3/2\alpha}(\mathbf{e}_m \cdot \boldsymbol{\sigma} i \sigma^2)_{\beta_1\beta_3} + C_{1m1/2\beta_3}^{3/2\alpha}(\mathbf{e}_m \cdot \boldsymbol{\sigma} i \sigma^2)_{\beta_1\beta_2}) \\ &\times (C_{1m1/2f_1}^{1/2I_3}(\mathbf{e}_m \cdot \boldsymbol{\tau} i \tau^2)_{f_2f_3} + C_{1m1/2f_2}^{1/2I_3}(\mathbf{e}_m \cdot \boldsymbol{\tau} i \tau^2)_{f_1f_3} + C_{1m1/2f_3}^{1/2I_3}(\mathbf{e}_m \cdot \boldsymbol{\tau} i \tau^2)_{f_1f_2})(T_2)_{l_1,l_2,l_3}^a \\ &+ (C_{1m1/2\beta_1}^{3/2\alpha}(\mathbf{e}_m \cdot \boldsymbol{\sigma} i \sigma^2)_{\beta_2\beta_3} + C_{1m1/2\beta_2}^{3/2\alpha}(\mathbf{e}_m \cdot \boldsymbol{\sigma} i \sigma^2)_{\beta_3\beta_1} + C_{1m1/2\beta_3}^{3/2\alpha}(\mathbf{e}_m \cdot \boldsymbol{\sigma} i \sigma^2)_{\beta_2\beta_1}) \\ &\times (C_{1m1/2f_1}^{1/2I_3}(\mathbf{e}_m \cdot \boldsymbol{\tau} i \tau^2)_{f_2f_3} + C_{1m1/2f_2}^{1/2I_3}(\mathbf{e}_m \cdot \boldsymbol{\tau} i \tau^2)_{f_3f_1} + C_{1m1/2f_3}^{1/2I_3}(\mathbf{e}_m \cdot \boldsymbol{\tau} i \tau^2)_{f_2f_1})(T_3)_{l_1,l_2,l_3}^a \\ &+ (C_{1m1/2\beta_1}^{3/2\alpha}(\mathbf{e}_m \cdot \boldsymbol{\sigma} i \sigma^2)_{\beta_3\beta_2} + C_{1m1/2\beta_2}^{3/2\alpha}(\mathbf{e}_m \cdot \boldsymbol{\sigma} i \sigma^2)_{\beta_3\beta_1} + C_{1m1/2\beta_3}^{3/2\alpha}(\mathbf{e}_m \cdot \boldsymbol{\sigma} i \sigma^2)_{\beta_1\beta_2}) \\ &\times (C_{1m1/2f_1}^{1/2I_3}(\mathbf{e}_m \cdot \boldsymbol{\tau} i \tau^2)_{f_3f_2} + C_{1m1/2f_2}^{1/2I_3}(\mathbf{e}_m \cdot \boldsymbol{\tau} i \tau^2)_{f_3f_1} + C_{1m1/2f_3}^{1/2I_3}(\mathbf{e}_m \cdot \boldsymbol{\tau} i \tau^2)_{f_1f_2})(T_1)_{l_1,l_2,l_3}^a] \\ &\times (P_+ q_{l_1f_1}(t, \mathbf{x}))^{\beta_1} (P_+ q_{l_2f_2}(t, \mathbf{x}))^{\beta_2} (P_+ q_{l_3f_3}(t, \mathbf{x}))^{\beta_3}, \end{aligned} \quad (2.7)$$

where repeated indices are summed over, α, β_i ($i = 1, 2, 3$) are the spin or vector indices, f_i ($i = 1, 2, 3$) are the isospin or flavor index, l_i ($i = 1, 2, 3$) are the color index, $C_{1m1/2\beta_i}^{3/2\alpha}$ and $C_{1m1/2f_i}^{1/2I_3}$ are Clebsch-Gordon coefficients and the color matrices $(T_i)^a$ ($i = 1, 2, 3$) are defined as

$$\begin{aligned} (T_1)_{ijk}^a &= \epsilon_{ijk} T_{li}^a, & (T_2)_{ijk}^a &= \epsilon_{ilk} T_{lj}^a, \\ (T_3)_{ijk}^a &= \epsilon_{ijl} T_{lk}^a. \end{aligned} \quad (2.8)$$

The projector $P_+ = (1 + \gamma^0)/2$ in Eqs. (2.6) and (2.7) is required due to positive parity [98,126]; it selects a two-component Pauli spinor, σ^2 and τ^2 are the antisymmetric spin and isospin Pauli matrices, σ^i and τ^i are the spin and isospin Pauli matrices, and $\mathbf{e}_m$ are the spherical basis vectors

$$\begin{aligned} \mathbf{e}_0 &= (0, 0, 1), & \mathbf{e}_{-1} &= -(1, i, 0)/\sqrt{2}, \\ \mathbf{e}_{+1} &= (1, -i, 0)/\sqrt{2}. \end{aligned} \quad (2.9)$$

Replacing the Clebsch-Gordon coefficient $C_{1,m,1/2,\beta_i}^{3/2,\alpha}$ in Eq. (2.7) with $C_{1,m,1/2,\beta_i}^{1/2,\alpha}$ yields another light-quark interpolating operator, $H_{8,I_3=\pm 1/2,(3/2)^+}^{\alpha,a}(t, \mathbf{x})$, for $I = 1/2, I_3 = \pm 1/2$, and $k = 1/2$. As discussed in Secs. II C and II D 1, and summarized in Tables II and III, there are in fact two distinct $k^P = (1/2)^+$ configurations. The interpolating operator in Eq. (2.6) corresponds to the light quark pair-quark combination $k_{qq}^P \otimes k_q^P = 0^+ \otimes (1/2)^+$, while the

⁵There is no fully antisymmetric combination in color for three identical quarks in a color octet configuration. For details, see Appendix H in Ref. [85].

TABLE II. J^P multiplets for the lowest pentaquark states $Q\bar{Q}qqq$ where $q = (u, d)$ and P denotes the parity of the state. We represent the BO quantum numbers ($D_{\infty h}$ representation) in the third column as $(\Lambda)_{\eta}$, where $\eta = g$ denotes positive parity and $\Lambda = |\mathbf{K} \cdot \mathbf{r}|$. There are two distinct multiplets corresponding to two distinct adjoint baryons with light spin parity $(1/2)^+$: one multiplet with $k^P = (1/2)^+$ corresponds to states near the $\Lambda_c/\Lambda_b - \bar{D}/\bar{B}$ thresholds, while the other multiplet with $k^P = (1/2)^+$ together with the multiplet with $k^P = (3/2)^+$ corresponds to states near the $\Sigma_c/\Sigma_b - \bar{D}/\bar{B}$ thresholds. In the table, we only show the J^P multiplets associated with the lowest pentaquark states near the $\Sigma_c/\Sigma_b - \bar{D}/\bar{B}$ thresholds, since no experimental states have been seen observed near the $\Lambda_c/\Lambda_b - \bar{D}/\bar{B}$ thresholds.

$Q\bar{Q}$ color state	Light spin k^P	BO quantum # $D_{\infty h}$	l	$J^P \{s=0, s=1\}$
Octet 8	$(1/2)^+$	$(1/2)_g$	1/2	$\{1/2^-, (1/2, 3/2)^-\}$
	$(3/2)^+$	$\{(1/2)'_g, (3/2)_g\}$	3/2	$\{3/2^-, (1/2, 3/2, 5/2)^-\}$

TABLE III. Total LDF spin-parity k^P quantum numbers of light quark pair-quark ($qq \otimes q$) combinations for the lowest light-quark states forming $\bar{B}\bar{M}$ pair thresholds. The light-quark states have quantum numbers k_{qq}^P and k_q^P . The BO quantum numbers corresponding to k^P are listed in the third column. The prime in the third row indicates the excited state of the same BO quantum number. The first row, which accounts for the spin singlet qq pair, is relevant for the $\Lambda_c\bar{D}$ or $\Lambda_b\bar{B}$ thresholds with isospin $I = 1/2$, while the second and third rows, which account for the spin triplet qq pair, are relevant for the $\Sigma_c\bar{D}$ or $\Sigma_b\bar{B}$ thresholds, and include both isospin $I = 1/2$ and $I = 3/2$.

$k_{qq}^P \otimes k_q^P$	k^P	BO quantum $D_{\infty h}$
$0^+ \otimes (1/2)^+$	$(1/2)^+$	$(1/2)_g$
$1^+ \otimes (1/2)^+$	$(1/2)^+$	$(1/2)_g$
	$(3/2)^+$	$\{(1/2)'_g, (3/2)_g\}$

operator obtained via the above substitution corresponds to $k_{qq}^P \otimes k_q^P = 1^+ \otimes (1/2)^+$.

Following Ref. [85], the short-distance behavior of the pentaquark BO potential $E_{(\Lambda)_{\eta}}^{(0)}(r)$ can be written as

$$E_{(\Lambda)_{\eta}}(r) = V_o(r) + \Lambda_{\kappa} + \mathcal{O}(r^2), \quad (2.10)$$

where $V_o(r) = \alpha_s/(6r)$ is the color octet potential and Λ_{κ} is the adjoint baryon mass with $\kappa = \{(1/2)^+, I = 1/2\}$ and $\{(3/2)^+, I = 1/2\}$. Equation (2.10) incorporates the information that, at leading order in the multipole expansion, several BO potentials become degenerate as they depend solely on the adjoint baryon quantum number $\kappa \equiv k^P$ (see the second column of Table II). The degeneracy is broken by the $\mathcal{O}(r^2)$ terms that arise from the multipole expansion [88].

C. Pentaquark multiplets

The lowest quarkonium pentaquarks are bound states in the BO potentials (static energies) with BO-quantum number $(1/2)_g$ corresponding to the adjoint baryon $k^P = (1/2)^+$, or BO potentials with BO-quantum numbers $(1/2)'_g$ and $(3/2)_g$ corresponding to the adjoint baryon $k^P = (3/2)^+$ (see Table II). The ordering of the adjoint baryon masses for $(1/2)^+$ and $(3/2)^+$ is not known from lattice QCD. We consider both adjoint baryons for the ground state. For $k^P = (3/2)^+$, we need to consider both static energies

$(1/2)'_g$ and $(3/2)_g$ because they are degenerate in the short distance limit.⁶ The states associated with the static energies $(1/2)'_g$ and $(3/2)_g$ mix, and their dynamics are governed by a set of coupled Schrödinger equations [85]. Higher static energies are assumed to be separated from the static energies $(1/2)_g$, $(1/2)'_g$, and $(3/2)_g$ by a gap of order Λ_{QCD} (at least true for the hybrid static energies), and their modes are integrated out when integrating out LDF of energy or momentum of order Λ_{QCD} .

We define the total angular momentum of the state as $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where the angular momentum $\mathbf{L} = \mathbf{L}_{Q\bar{Q}} + \mathbf{K}$ is the sum of the orbital angular momentum $\mathbf{L}_{Q\bar{Q}}$ of the $Q\bar{Q}$ pair and the spin $\mathbf{K}$ of the LDF, and $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ is the total spin of the $Q\bar{Q}$ pair. The associated quantum numbers are as follows: $J(J+1)$ and m_J are the eigenvalue of $\mathbf{J}^2$ and J_3 respectively, $l(l+1)$ is the eigenvalue of $\mathbf{L}^2$, and $s(s+1)$ is the eigenvalue of $\mathbf{S}^2$. In addition, we denote the quantum number of the angular momentum $\mathbf{L}_{Q\bar{Q}}$ by L_Q . In Table II, we show the results for the J^P multiplets of the lowest $Q\bar{Q}qqq$ pentaquark states, where P denotes the parity of the state. Notably, there are two distinct multiplets corresponding to two different adjoint baryon configurations with light

⁶An analogous degeneracy at short distances occurs in the case of the hybrid static energies Σ_u^- and Π_u [88,89,127] and the tetraquark static energies Σ_g^+ and Π_g [85,114].

spin-parity $k^P = (1/2)^+$: one multiplet is associated with states near the $\Lambda_c/\Lambda_b - \bar{D}/\bar{B}$ thresholds, while the other—along with the $k^P = (3/2)^+$ multiplet—is associated with states near the $\Sigma_c/\Sigma_b - \bar{D}/\bar{B}$ thresholds. Hence, in total, there are 10 possible J^P states. However, three of these states are excluded as no experimental states have been seen observed near the $\Lambda_c/\Lambda_b - \bar{D}/\bar{B}$ thresholds. As we will see in Sec. II D, this has implications on the behavior of the BO potentials.

As seen in Table II, all the ground-state pentaquark states have negative parity given by $P = (-1)^{l+k}$ [85]. The $Q\bar{Q}$ pair has negative intrinsic parity, while the three light quarks have positive intrinsic parity. For $k^P = (1/2)^+$ with $l = 1/2$, the negative parity P implies $L_Q = 0$ and for $k^P = (3/2)^+$ with $l = 3/2$, the negative parity implies $L_Q = 0, 2$.

D. Behavior of pentaquark BO potentials and parametrization

1. Mixing with baryon-antimeson threshold

The BOEFT constrains the behavior of the BO (static) potentials at short and large distances based on the BO quantum number [85]. At short distances ($r \rightarrow 0$), the form of the potential is fixed by the potential NRQCD multipole expansion: for color octet ($Q\bar{Q}$)₈, the potential is the sum of the repulsive color octet potential, adjoint hadron mass and higher-order corrections of $\mathcal{O}(\Lambda_{\text{QCD}}^3 r^2)$ [85,88]. At large distances ($r \rightarrow \infty$), the form depends on the LDF. In the quenched approximation, where the LDF consist solely of gluons (quarkonium and hybrid configurations), lattice QCD calculations show that the static potentials grow linearly with the interquark distance r (see recent lattice results in Refs. [128–130]). In contrast, in the unquenched case, the LDF include light quarks, and the relevant long-distance behavior is governed by the heavy-light hadron pair thresholds, such as the heavy baryon–heavy antimeson threshold (denoted by $\bar{\text{B}}\bar{\text{M}}$ for notational ease) in the $Q\bar{Q}qqq$ pentaquark system.

The LDF quantum numbers of the $\bar{\text{B}}\bar{\text{M}}$ thresholds are inferred from the light quark spin, k_q^P , in the heavy antimeson and the light quark pair spin, k_{qq}^P , in the heavy baryon: $k_q^P = (1/2)^+$ for the ground-state heavy antimeson and $k_{qq}^P = 0^+$ and 1^+ for the ground-state heavy baryon, where we assume that the $k^P = 0^+$ state is lower in energy.⁷ The total LDF spin-parity k^P and the corresponding BO quantum numbers in the Λ_η^σ representation for the lowest $\bar{\text{B}}\bar{\text{M}}$ thresholds are listed in Table III. The BO quantum number conservation between adjoint baryons

(Table II) and $\bar{\text{B}}\bar{\text{M}}$ thresholds (Table III) implies that the $Q\bar{Q}$ pentaquark BO static potentials with repulsive color octet behavior at $r \rightarrow 0$ decrease as r increases and eventually evolve smoothly into $\bar{\text{B}}\bar{\text{M}}$ thresholds at $r \rightarrow \infty$ [85,115]. This constraint severely restricts the potentials capable of supporting bound states. These are only the pentaquark potentials that cross the $\bar{\text{B}}\bar{\text{M}}$ threshold and then approaches it from below. If the pentaquark potential decreases monotonically as r increases and connects to the $\bar{\text{B}}\bar{\text{M}}$ threshold from above, it cannot support bound states [85,115].

The LHCb experiment has observed four half-integer isospin ($I = 1/2$) pentaquark states: $P_{c\bar{c}}(4312)^+$, $P_{c\bar{c}}(4380)^+$, $P_{c\bar{c}}(4440)^+$, and $P_{c\bar{c}}(4457)^+$, all with quark content $c\bar{c}qqq$, where $q \equiv (u, d)$ [13,14]. The J^P quantum numbers are unknown. All of these states are below the spin-isospin averaged $\Sigma_c\bar{D}$ threshold (which includes $\Sigma_c\bar{D}$, $\Sigma_c^*\bar{D}$, $\Sigma_c\bar{D}^*$, and $\Sigma_c^*\bar{D}^*$) and are above the spin-isospin averaged $\Lambda_c\bar{D}$ threshold (which includes $\Lambda_c\bar{D}$ and $\Lambda_c\bar{D}^*$). Notably, no states have been observed near the spin-isospin averaged $\Lambda_c\bar{D}$ or $\Lambda_b\bar{B}$ thresholds. This suggests that the static potential with BO quantum number $(1/2)_g$ corresponding to the adjoint baryon $(1/2)^+$ for $k_{qq}^P = 0^+$ (first row in Table III) decreases monotonically with r and joins the $\Lambda_c/\Lambda_b - \bar{D}/\bar{B}$ thresholds from above—thus failing to support bound states, as sketched in Fig. 1. In contrast, for the BO potentials with quantum numbers $(1/2)_g$ and $(1/2)'_g, (3/2)_g$ corresponding to the two adjoint baryons $(1/2)^+$ and $(3/2)^+$ with $k_{qq}^P = 1^+$ (second and third rows of Table III) and asymptotically connecting to the $\Sigma_c/\Sigma_b - \bar{D}/\bar{B}$ thresholds, two scenarios appear possible:

- (1) Only the potentials with BO quantum numbers $\{(1/2)'_g, (3/2)_g\}$ cross the $\Sigma_c/\Sigma_b - \bar{D}/\bar{B}$ thresholds and then approach them from below, supporting bound states (see Fig. 1), while the potential with BO quantum number $(1/2)_g$ joins the $\Sigma_c/\Sigma_b - \bar{D}/\bar{B}$ thresholds from above, thus not supporting any bound state. This implies that the lowest pentaquark states are only four, those associated with the multiplet of the adjoint baryon $(3/2)^+$ (see Table II). This number matches the number of observed pentaquark states reported in Table I.
- (2) All the static potentials with BO quantum numbers $(1/2)_g$ and $\{(1/2)'_g, (3/2)_g\}$ cross the $\Sigma_c/\Sigma_b - \bar{D}/\bar{B}$ thresholds and then approach them from below, supporting bound states near the potential minima (see Fig. 1). This implies that the lowest pentaquark states are the seven pentaquark states associated with the multiplet of the adjoint baryons $(1/2)^+$ and $(3/2)^+$ (see Table II). Of these seven states, four can be associated with the observed pentaquark states reported in Table I; the remaining three must be understood as

⁷This is supported by the observation that Λ -baryons have lower mass than Σ -baryons.

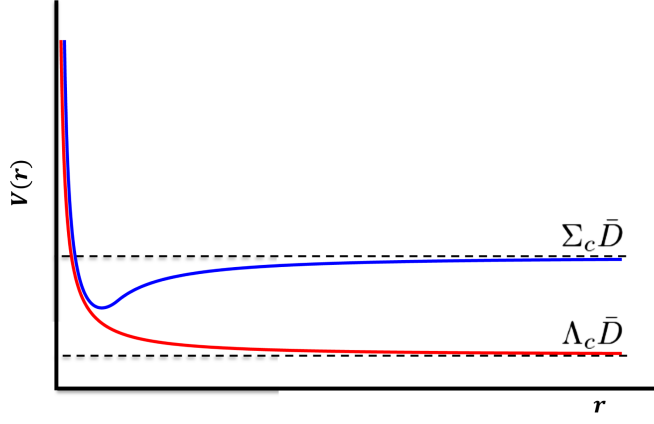


FIG. 1. Illustrative figure showing BO potentials that decrease monotonically with r and connect to the baryon-antimeson thresholds at large distances. The blue curve represents a BO potential that approaches the $\Sigma_c \bar{D}$ threshold from below, potentially supporting a bound state. In contrast, the red curve corresponds to a BO potential that approaches the $\Lambda_c \bar{D}$ threshold from above, in which case, making it unlikely to support any bound state. Furthermore, though not shown here, a BO potential may dip slightly below a threshold, such as the $\Lambda_c \bar{D}$ one, cross it again, and then approach it from above, in which case it could support a resonant state [115].

yet undetected. The first scenario has been analyzed in Ref. [131]. In the current work, we analyze the second one. We only consider $I = 1/2$ adjoint baryons and exclusively focus on the static potentials that asymptotically connect to the $\Sigma_c/\Sigma_b - \bar{D}/\bar{B}$ thresholds.

2. Quark masses and parametrization

The charm and bottom quark masses that we use in the current work are the renormalon-subtracted (RS) charm and bottom masses [132,133] defined at the renormalon subtraction scale $\nu_f = 1$ GeV and computed in [95]: $m_c^{\text{RS}} = 1.477$ GeV and $m_b^{\text{RS}} = 4.863$ GeV. The physical spin-isospin averaged $\bar{B}\bar{M}$ thresholds are $E_{\Sigma_c \bar{D}} = 4.470$ GeV, and $E_{\Sigma_b \bar{B}} = 11.140$ GeV [17]. We choose the zero of energy to be the spin-isospin averaged $\Sigma_c \bar{D}$ threshold for charm and $\Sigma_b \bar{B}$ threshold for bottom. For completeness, the spin-isospin averaged $\Lambda_c \bar{D}$ and $\Lambda_b \bar{B}$ thresholds are $E_{\Lambda_c \bar{D}} = 4.260$ GeV and $E_{\Lambda_b \bar{B}} = 10.933$ GeV [17].

There are no lattice QCD computations of the pentaquark static potentials. Therefore, we model the potentials based on the short-distance and long-distance behavior dictated by BOEFT [85], as follows:

$$E_{(\Lambda)_\eta}(r) = \begin{cases} V_o^{\text{RS}}(r, \nu_f) + \Lambda_\kappa + A_{\Lambda_\eta} r^2 & r < R_{\Lambda_\eta} \\ F_{\Lambda_\eta} e^{-r/d}/r & r > R_{\Lambda_\eta} \end{cases}, \quad (2.11)$$

where $\Lambda_\eta \in [(1/2)_g, \{(1/2)_g^+, (3/2)_g\}]$. For the short-distance part ($r < R_{\Lambda_\eta}$), we use the RS octet potential $V_o^{\text{RS}}(r, \nu_f)$ up to order α_s^3 in perturbation theory⁸ and $\nu_f = 1$ GeV is the renormalon subtraction scale. The constant Λ_κ is the adjoint baryon mass with $\kappa = \{(1/2)^+, I = 1/2\}$ and $\{(3/2)^+, I = 1/2\}$. Due to the lack of lattice data, we choose the parameters A_{Λ_η} in Eq. (2.11) to be the same as the parameters for the lowest hybrid potentials in Ref. [94], as follows:

$$\begin{aligned} A_{(1/2)_g} &= 0.042 \text{ GeV}^3, & A_{(1/2)_g'} &= 0.0065 \text{ GeV}^3, \\ A_{(3/2)_g} &= 0.0726 \text{ GeV}^3. \end{aligned} \quad (2.12)$$

For the long-distance part ($r > R_{\Lambda_\eta}$) in Eq. (2.11), we use the one-pion exchange potential motivated from molecular models [28,29,44,52,80,134]. The parameters F_{Λ_η} and R_{Λ_η} are determined by imposing continuity up to the first derivatives. We treat the adjoint baryon masses $\Lambda_{(1/2)^+}$ and $\Lambda_{(3/2)^+}$ as adjustable parameters to reproduce the $P_{c\bar{c}}(4312)^+$, $P_{c\bar{c}}(4380)^+$, $P_{c\bar{c}}(4440)^+$, and $P_{c\bar{c}}(4457)^+$ pentaquark masses within the experimental uncertainties.

III. PENTAQUARK SPECTRUM AND IDENTIFICATION WITH EXPERIMENTAL STATES

A. Schrödinger equations without spin splittings

Pentaquarks ($Q\bar{Q}qqq$) are exotic hadrons that are color-singlet bound states of a color octet $Q\bar{Q}$ pair coupled to three light quarks qqq . We focus here on the lowest-lying pentaquark states that can be built from the $(1/2)_g$ and $\{(1/2)_g', (3/2)_g\}$ static energies corresponding to adjoint baryons with quantum numbers $\kappa = \{(1/2)^+, I = 1/2\}$ and $\{(3/2)^+, I = 1/2\}$ (see multiplets in Table II). For $k^P = (1/2)^+$, two values of λ ($\pm 1/2$) are possible and for $k^P = (3/2)^+$, four values of λ ($\pm 1/2$ and $\pm 3/2$) are possible. The pentaquark state $|P_N\rangle$ with quantum numbers $N \equiv \{n, J, m_J, l, s\}$, where n is the principal quantum number, can be written in the rest frame as [85]

⁸An expression of the RS octet potential can be found in Appendix B of Ref. [95].

⁹Varying the short-distance coefficients A_{Λ_η} by 20% requires adjusting the adjoint baryon mass by only a few MeV to reproduce the pentaquark masses within experimental uncertainties. Furthermore, replacing the one-pion exchange potential in Eq. (2.11) with a two-pion exchange potential in the long-distance part, the adjoint baryon masses $\Lambda_{(1/2)^+}$ and $\Lambda_{(3/2)^+}$ have to be changed by around 100 MeV and 40 MeV respectively, to reproduce the pentaquark masses within experimental uncertainties.

$$|P_N\rangle = \sum_{\lambda} \int d^3 r |\mathbf{r}\rangle |\kappa, \lambda\rangle \Psi_{\kappa\lambda}^{(N)}(\mathbf{r}). \quad (3.1)$$

The state $|\mathbf{r}\rangle$ stands for the static heavy quark-antiquark pair at the c.m. ($\mathbf{R} = 0$) with relative separation $\mathbf{r}$, $|\kappa, \lambda\rangle$ for the LDF identified by the quantum numbers κ and λ , and the integration over the coordinates $\mathbf{r}$ implies that we are beyond the static limit. In the short-distance limit $r \rightarrow 0$,

the static states are approximate eigenstates of $\mathbf{K}^2$, whose eigenvalues are labeled by k .

The wave functions $\Psi_{\kappa\lambda}^{(N)}$ in Eq. (3.1) are eigenfunctions of $\mathbf{K} \cdot \hat{\mathbf{r}}$ but not of parity. Under parity, the index λ goes into $-\lambda$, so, the parity eigenfunctions are constructed as a linear combinations of $\Psi_{\kappa\lambda}^{(N)}(\mathbf{r})$ and $\Psi_{\kappa-\lambda}^{(N)}(\mathbf{r})$.¹⁰ Following Ref. [85], the parity eigenfunctions for the pentaquark multiplet in Table II corresponding to $k^P = (1/2)^+$ can be written as

$$\Psi_{(1/2)^+, P=-}^{(N)}(\mathbf{r}) = \frac{1}{\sqrt{2}} \sum_{m_l, m_s} \mathcal{C}_{J m_l s}^{m_l m_s} \left(\mathbf{P}_{\frac{11}{22}} v_{l, m_l}^{1/2}(\theta, \phi) + \mathbf{P}_{\frac{1}{2}-\frac{1}{2}} v_{l, m_l}^{-1/2}(\theta, \phi) \right) \psi_{(1/2)^+}^{(N)}(r) \chi_{sm_s}. \quad (3.2)$$

Similarly, the parity eigenfunctions for the pentaquark multiplet in Table II corresponding to $k^P = (3/2)^+$ are

$$\begin{aligned} \Psi_{(3/2)^+, P=-}^{(N)}(\mathbf{r}) = \frac{1}{\sqrt{2}} \sum_{m_l, m_s} \mathcal{C}_{J m_l s}^{m_l m_s} \left[\left(\mathbf{P}_{\frac{33}{22}} v_{l, m_l}^{3/2}(\theta, \phi) + \mathbf{P}_{\frac{3}{2}-\frac{3}{2}} v_{l, m_l}^{-3/2}(\theta, \phi) \right) \psi_{(3/2)^+}^{(N)}(r) \right. \\ \left. + \left(\mathbf{P}_{\frac{31}{22}} v_{l, m_l}^{1/2}(\theta, \phi) + \mathbf{P}_{\frac{3}{2}-\frac{1}{2}} v_{l, m_l}^{-1/2}(\theta, \phi) \right) \psi_{(1/2)^+}^{(N)}(r) \right] \chi_{sm_s}, \end{aligned} \quad (3.3)$$

where $\psi_{(1/2)^+}^{(N)}(r)$, $\psi_{(3/2)^+}^{(N)}(r)$, and $\psi_{(1/2)^+}^{(N)}(r)$ are the radial wave functions, χ_{sm_s} denotes the heavy quark-antiquark $Q\bar{Q}$ pair spin wave function, $\mathcal{C}_{J m_l s}^{m_l m_s}$ are suitable Clebsch-Gordon coefficients, and the index P in the subscript on the lhs of Eqs. (3.2) and (3.3) denotes the parity of the states in Table II. The angular eigenfunctions $v_{l, m_l}^{\lambda}(\theta, \phi)$ are generalizations of the spherical harmonics for systems with cylindrical symmetry [123] and are given by

$$v_{lm}^{\lambda}(\theta, \phi) = \frac{(-1)^{l+m}}{2^l} \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!(l-\lambda)!(l+\lambda)!}} \mathbf{P}_{lm}^{\lambda}(\cos \theta) e^{im\phi}, \quad (3.4)$$

$$\mathbf{P}_{lm}^{\lambda}(x) = (1-x)^{(m-\lambda)/2} (1+x)^{(m+\lambda)/2} \partial_x^{l+m} (1-x)^{l+\lambda} (1+x)^{l-\lambda}. \quad (3.5)$$

The pentaquark wave functions in Eqs. (3.2) and (3.3) are tensor wave functions. Their Schrödinger equations can be derived from the equations of motion of the BOEFT Lagrangian [85]. The radial Schrödinger equation for the pentaquark multiplet in Table II with parity $P = -$ and associated with the adjoint baryon $k^P = (1/2)^+$ is

$$\left[-\frac{1}{m_Q r^2} \partial_r r^2 \partial_r + \frac{(l-1/2)(l+1/2)}{m_Q r^2} + E_{(1/2)_g} \right] \psi_{(1/2)^+}^{(N)} = \mathcal{E}_{1/2} \psi_{(1/2)^+}^{(N)}. \quad (3.6)$$

For the pentaquark multiplet in Table II with parity $P = -$ and associated with the adjoint baryon $k^P = (3/2)^+$, the two static energies $E_{(1/2)_g}$ and $E_{(3/2)_g}$ mix at short distances, so, the radial Schrödinger equations are a set of coupled equations:

$$\left[-\frac{1}{m_Q r^2} \partial_r r^2 \partial_r + \frac{1}{m_Q r^2} \begin{pmatrix} l(l-1) + \frac{9}{4} & -\sqrt{3l(l+1) - \frac{9}{4}} \\ -\sqrt{3l(l+1) - \frac{9}{4}} & l(l+1) - \frac{3}{4} \end{pmatrix} + \begin{pmatrix} E_{(1/2)_g} & 0 \\ 0 & E_{(3/2)_g} \end{pmatrix} \right] \begin{pmatrix} \psi_{(1/2)^+}^{(N)} \\ \psi_{(3/2)^+}^{(N)} \end{pmatrix} = \mathcal{E}_{3/2} \begin{pmatrix} \psi_{(1/2)^+}^{(N)} \\ \psi_{(3/2)^+}^{(N)} \end{pmatrix}, \quad (3.7)$$

where $\mathcal{E}_{1/2}$ and $\mathcal{E}_{3/2}$ are the eigenenergies for which we have suppressed the label (N) for simplicity. These eigenenergies provide the binding energies with respect to the spin-isospin averaged $B\bar{M}$ threshold—specifically, the $\Sigma_c \bar{D}$ and $\Sigma_b \bar{B}$

¹⁰The projection vectors $P_{\kappa\lambda}^{\alpha}$ in Eq. (2.5) project on states with definite $\mathbf{K} \cdot \hat{\mathbf{r}}$. States labeled by the index $\lambda = 0$ are eigenstates of $\mathbf{K} \cdot \hat{\mathbf{r}}$ and parity as well [85,95].

thresholds—that are set to zero. The static potentials $E_{(1/2)_g}$, $E_{(1/2)'_g}$, and $E_{(3/2)_g}$ are given by Eqs. (2.11) and (2.12) and are independent of the heavy-quark spin. Since we treat Λ_κ in Eq. (2.11) as a free parameter, the eigenenergies $\mathcal{E}_{1/2}$ and $\mathcal{E}_{3/2}$ are functions of the respective adjoint baryon masses.

Without spin-dependent corrections, we obtain the spin-averaged pentaquark spectrum with masses given by $E_{\Sigma_c \bar{D}} + \mathcal{E}_{1/2}$ and $E_{\Sigma_c \bar{D}} + \mathcal{E}_{3/2}$ in the charm sector and $E_{\Sigma_b \bar{B}} + \mathcal{E}_{1/2}$ and $E_{\Sigma_b \bar{B}} + \mathcal{E}_{3/2}$ in the bottom sector. For $\mathcal{E}_{1/2} = \mathcal{E}_{3/2} = 0$, the seven pentaquark states in Table II lie exactly at the spin-isospin averaged $\Sigma_c \bar{D}$ or $\Sigma_b \bar{B}$ thresholds.

B. Spin-splittings

Quantitative results of the pentaquark spectrum that reproduce the experimental masses of the states listed in Table I require incorporating spin-dependent corrections, which depend on the heavy-quark spins S_1, S_2 , or their total spin S . The BOEFT potential at $\mathcal{O}(1/m_Q^0)$ includes only static contributions, and solving the Schrödinger equation in this potential yields degenerate spin multiplets. However, BOEFT allows for the systematic inclusion of the spin-dependent corrections to the BO potential, which for hybrids and tetraquarks appear at order $1/m_Q$ [98,100–104,114,115]. The general form of the spin corrections to the BO potentials in terms of generalized static Wilson loops is known [98], but preliminary lattice calculations of them exist at the moment only for hybrids [104].

In the absence of lattice data, we evaluate the spin splittings within the ground-state $Q\bar{Q}$ pentaquark multiplets in Table II using a method analogous to the one developed for the $Q\bar{Q}$ tetraquarks in Refs. [114,115]. The spin splittings in the multiplet are estimated from the spin splittings in the heavy-light baryon-antimeson pair states $B\bar{M}$. As discussed, the BO static potentials for the $Q\bar{Q}$ pentaquark asymptotically approach (smoothly connect to) the $B\bar{M}$ energy at large separations ($r \rightarrow \infty$), owing to the conservation of BO quantum numbers [85]. Since the spin-dependent interactions responsible for the $B\bar{M}$ threshold

splittings arise at $\mathcal{O}(1/m_Q)$, the pentaquark spin-dependent BO potential at $\mathcal{O}(1/m_Q)$ reduces at large r to a constant potential V_{SS} that depends on the heavy-quark and anti-quark spins S_1 and S_2 , as follows:

$$V_{SS} = \frac{2\Delta_1^Q}{3} \mathbf{S}_1 \cdot \mathbf{K}_1 + \Delta_2^Q \mathbf{S}_2 \cdot \mathbf{K}_2, \quad (3.8)$$

where $\mathbf{K}_1$ and $\mathbf{K}_2$ denote the total light-quark spin in the heavy baryon B and heavy antimeson $\bar{M}$, respectively. The prefactor $2/3$ in the first term on the rhs of Eq. (3.8) is such that the parameter Δ_1^Q , which is proportional to $1/m_Q$, is equal to the spin splitting between the heavy baryons B^* and B : $\Delta_1^c = 64.6$ MeV for Σ_c baryons and $\Delta_1^b = 19.4$ MeV for Σ_b baryons, and $\Delta_1^{c/b} = 0$ MeV for Λ_c or Λ_b baryons [17]. The parameter Δ_2^Q is proportional to $1/m_Q$ and is equal to the spin splitting between the heavy mesons M^* and M : $\Delta_2^c = 141.3$ MeV for D mesons and $\Delta_2^b = 45.2$ MeV for B mesons [17]. The energies of the $\Sigma_{c/b}^*$ and $\Sigma_{c/b}$ baryons relative to spin-isospin average are $\frac{\Delta_1^Q}{3}$ and $-\frac{2\Delta_1^Q}{3}$. The energies of the D^*/B^* and D/B mesons relative to spin-isospin average are $\frac{\Delta_2^Q}{4}$ and $-\frac{3\Delta_2^Q}{4}$. The different heavy-light baryon antimeson thresholds relative to spin-isospin average threshold are $\left(-\frac{2\Delta_1^Q}{3} - \frac{3\Delta_2^Q}{4}\right)$ for $B\bar{M}$, $\left(\frac{\Delta_1^Q}{3} - \frac{3\Delta_2^Q}{4}\right)$ for $B^*\bar{M}$, $\left(-\frac{2\Delta_1^Q}{3} + \frac{\Delta_2^Q}{4}\right)$ for $B\bar{M}^*$, $\left(\frac{\Delta_1^Q}{3} + \frac{\Delta_2^Q}{4}\right)$ for $B^*\bar{M}^*$, where $B = \{\Sigma_c, \Sigma_b\}$ and $\bar{M} = \{\bar{D}, \bar{B}\}$.

Because of the form of the assumed spin interaction in Eq. (3.8), it is convenient to move from a basis of the total heavy-quark spin, labeled by s , and light-quark spin, labeled by k , to a basis of the $\bar{Q}q$ and Qqq total angular momenta labeled by $j_{\bar{Q}q}$ and j_{Qqq} , respectively. These quantum numbers match the ones of the heavy baryon-heavy antimeson pair states $B\bar{M}$, $B^*\bar{M}$, $B\bar{M}^*$, and $B^*\bar{M}^*$. The change of basis may be expressed in terms of Wigner $9j$ symbols and found, e.g., in Refs. [29,82,109]. It reads for the pentaquark states listed in Table II with $J^P = (1/2)^-$ as follows:

$$\begin{aligned} |s=0, k=1/2\rangle_{J^P=(1/2)^-} &= \frac{1}{2} |j_{Qqq}=1/2, j_{\bar{Q}q}=0\rangle + \frac{1}{2\sqrt{3}} |j_{Qqq}=1/2, j_{\bar{Q}q}=1\rangle \\ &\quad + \sqrt{\frac{2}{3}} |j_{Qqq}=3/2, j_{\bar{Q}q}=1\rangle, \\ |s=1, k=1/2\rangle_{J^P=(1/2)^-} &= \frac{1}{2\sqrt{3}} |j_{Qqq}=1/2, j_{\bar{Q}q}=0\rangle + \frac{5}{6} |j_{Qqq}=1/2, j_{\bar{Q}q}=1\rangle \\ &\quad - \frac{\sqrt{2}}{3} |j_{Qqq}=3/2, j_{\bar{Q}q}=1\rangle, \\ |s=1, k=3/2\rangle_{J^P=(1/2)^-} &= \sqrt{\frac{2}{3}} |j_{Qqq}=1/2, j_{\bar{Q}q}=0\rangle - \frac{\sqrt{2}}{3} |j_{Qqq}=1/2, j_{\bar{Q}q}=1\rangle \\ &\quad - \frac{1}{3} |j_{Qqq}=3/2, j_{\bar{Q}q}=1\rangle. \end{aligned} \quad (3.9)$$

Similarly, for $J^P = (3/2)^-$ states, we have

$$\begin{aligned}
 |s = 1, k = 1/2\rangle_{J^P=(3/2)^-} &= \frac{1}{\sqrt{3}}|j_{Qqq} = 3/2, j_{\bar{Q}q} = 0\rangle - \frac{1}{3}|j_{Qqq} = 1/2, j_{\bar{Q}q} = 1\rangle \\
 &\quad + \frac{\sqrt{5}}{3}|j_{Qqq} = 3/2, j_{\bar{Q}q} = 1\rangle, \\
 |s = 0, k = 3/2\rangle_{J^P=(3/2)^-} &= -\frac{1}{2}|j_{Qqq} = 3/2, j_{\bar{Q}q} = 0\rangle + \frac{1}{\sqrt{3}}|j_{Qqq} = 1/2, j_{\bar{Q}q} = 1\rangle \\
 &\quad + \sqrt{\frac{5}{12}}|j_{Qqq} = 3/2, j_{\bar{Q}q} = 1\rangle, \\
 |s = 1, k = 3/2\rangle_{J^P=(3/2)^-} &= \sqrt{\frac{5}{12}}|j_{Qqq} = 3/2, j_{\bar{Q}q} = 0\rangle + \frac{\sqrt{5}}{3}|j_{Qqq} = 1/2, j_{\bar{Q}q} = 1\rangle \\
 &\quad - \frac{1}{6}|j_{Qqq} = 3/2, j_{\bar{Q}q} = 1\rangle,
 \end{aligned} \tag{3.10}$$

and for $J^P = (5/2)^-$, we have

$$|s = 1, k = 3/2\rangle_{J^P=(5/2)^-} = |j_{Qqq} = 3/2, j_{\bar{Q}q} = 1\rangle. \tag{3.11}$$

To compute the effects of the spin-dependent potential V_{SS} , we use first-order perturbation theory. We first consider the pentaquark multiplet in Table II associated with the adjoint baryon $k^P = (1/2)^+$. To first order in Δ_1^Q and Δ_2^Q , the energies E_{kj}^s of the three states in the spin multiplet are

$$\begin{aligned}
 E_{\frac{1}{2}\frac{1}{2}}^{s=0} &= \mathcal{E}_{1/2}, & E_{\frac{1}{2}\frac{3}{2}}^{s=1} &= \mathcal{E}_{1/2} - \frac{4\Delta_1^Q}{9} + \frac{\Delta_2^Q}{6}, \\
 E_{\frac{1}{2}\frac{3}{2}}^{s=1} &= \mathcal{E}_{1/2} + \frac{2\Delta_1^Q}{9} - \frac{\Delta_2^Q}{12},
 \end{aligned} \tag{3.12}$$

where $\mathcal{E}_{1/2}$ is the eigenenergy obtained from the Schrödinger equation (3.6), the first subscript corresponds to k , and the second subscript corresponds to J . We next consider the pentaquark multiplet in Table II associated with the adjoint baryon $k^P = (3/2)^+$. To first order in Δ_1^Q and Δ_2^Q , the energies of the four states in the spin multiplet are

$$\begin{aligned}
 E_{\frac{3}{2}\frac{3}{2}}^{s=0} &= \mathcal{E}_{3/2}, & E_{\frac{3}{2}\frac{1}{2}}^{s=1} &= \mathcal{E}_{3/2} - \frac{5\Delta_1^Q}{9} - \frac{5\Delta_2^Q}{12}, \\
 E_{\frac{3}{2}\frac{3}{2}}^{s=1} &= \mathcal{E}_{3/2} - \frac{2\Delta_1^Q}{9} - \frac{\Delta_2^Q}{6}, & E_{\frac{3}{2}\frac{5}{2}}^{s=1} &= \mathcal{E}_{3/2} + \frac{\Delta_1^Q}{3} + \frac{\Delta_2^Q}{4},
 \end{aligned} \tag{3.13}$$

where $\mathcal{E}_{3/2}$ is the eigenenergy obtained from the Schrödinger equations (3.7), the first subscript corresponds to k , and the second subscript corresponds to J .

In Eqs. (3.12) and (3.13), there are three pentaquark states with $J^P = (1/2)^-$ and $J^P = (3/2)^-$. If the eigenenergies $\mathcal{E}_{1/2}$ and $\mathcal{E}_{3/2}$ are nearly degenerate, then the mixing

of states with the same J^P quantum numbers cannot be treated as a perturbation. To correctly determine the physical spectrum or eigenstates, it is necessary to diagonalize the 3×3 submatrix of V_{SS} in Eq. (3.8) for $J^P = (1/2)^-$ and $J^P = (3/2)^-$. As we show in Eqs. (3.18), (3.19), (3.22), and (3.23), the resulting physical eigenstates (pentaquark states) turn out to be superpositions of states with heavy quark-antiquark pair spin $s = 0$ and $s = 1$.

The 3×3 submatrix for $J^P = (1/2)^-$ is given by

$$M_{(1/2)^-} = \begin{pmatrix} E_{\frac{1}{2}\frac{1}{2}}^{s=0} & -\frac{2\Delta_1^Q}{3\sqrt{3}} - \frac{\Delta_2^Q}{4\sqrt{3}} & -\frac{2\Delta_1^Q}{3\sqrt{6}} - \frac{\Delta_2^Q}{\sqrt{6}} \\ -\frac{2\Delta_1^Q}{3\sqrt{3}} - \frac{\Delta_2^Q}{4\sqrt{3}} & E_{\frac{1}{2}\frac{3}{2}}^{s=1} & \frac{\sqrt{2}\Delta_1^Q}{9} - \frac{\Delta_2^Q}{3\sqrt{2}} \\ -\frac{2\Delta_1^Q}{3\sqrt{6}} - \frac{\Delta_2^Q}{\sqrt{6}} & \frac{\sqrt{2}\Delta_1^Q}{9} - \frac{\Delta_2^Q}{3\sqrt{2}} & E_{\frac{3}{2}\frac{1}{2}}^{s=1} \end{pmatrix}, \tag{3.14}$$

where the rows and columns correspond to $|s = 0, k = 1/2\rangle_{J^P=(1/2)^-}$, $|s = 1, k = 1/2\rangle_{J^P=(1/2)^-}$, and $|s = 1, k = 3/2\rangle_{J^P=(1/2)^-}$. Similarly, for $J^P = (3/2)^-$, the resulting matrix is given by

$$M_{(3/2)^-} = \begin{pmatrix} E_{\frac{1}{2}\frac{3}{2}}^{s=1} & \frac{\Delta_1^Q}{3\sqrt{3}} + \frac{\Delta_2^Q}{2\sqrt{3}} & \frac{\sqrt{5}\Delta_1^Q}{9} - \frac{\sqrt{5}\Delta_2^Q}{6} \\ \frac{\Delta_1^Q}{3\sqrt{3}} + \frac{\Delta_2^Q}{2\sqrt{3}} & E_{\frac{3}{2}\frac{3}{2}}^{s=0} & -\frac{\sqrt{5}\Delta_1^Q}{3\sqrt{3}} + \frac{\sqrt{5}\Delta_2^Q}{4\sqrt{3}} \\ \frac{\sqrt{5}\Delta_1^Q}{9} - \frac{\sqrt{5}\Delta_2^Q}{6} & -\frac{\sqrt{5}\Delta_1^Q}{3\sqrt{3}} + \frac{\sqrt{5}\Delta_2^Q}{4\sqrt{3}} & E_{\frac{3}{2}\frac{5}{2}}^{s=1} \end{pmatrix}, \tag{3.15}$$

where the rows and columns correspond to $|s = 1, k = 1/2\rangle_{J^P=(3/2)^-}$, $|s = 0, k = 3/2\rangle_{J^P=(3/2)^-}$, and $|s = 1, k = 3/2\rangle_{J^P=(3/2)^-}$. We denote the eigenvalues of the matrices $M_{(1/2)^-}$ and $M_{(3/2)^-}$ by $E_{M_{1/2}}^i$ and $E_{M_{3/2}}^i$ ($i = 1, 2, 3$), respectively. The eigenvectors of the matrices

$M_{(1/2)^-}$ and $M_{(3/2)^-}$ are identified with the physical pentaquark states, which are superpositions of the multiplet states (Table II) with well-defined heavy quark-antiquark pair spin s and LDF spin k . The state with $J^P = (5/2)^-$ corresponds to only $s = 1$, $k = 3/2$ and $l = 3/2$. Note that $E_{M_{1/2}}^i$ and $E_{M_{3/2}}^i$ depend on the adjoint baryon masses $\Lambda_{(1/2)^+}$ and $\Lambda_{(3/2)^+}$ through the eigenenergies $\mathcal{E}_{1/2}$ and $\mathcal{E}_{3/2}$ obtained from the Schrödinger equations (3.6) and (3.7).

After incorporating spin-dependent corrections, the masses of the pentaquark states are given as follows: for $J^P = (1/2)^-$ the masses are $E_{\Sigma_c \bar{D}} + E_{M_{1/2}}^i$ in the charm sector and $E_{\Sigma_b \bar{B}} + E_{M_{1/2}}^i$ in the bottom sector, while for $J^P = (3/2)^-$ they are $E_{\Sigma_c \bar{D}} + E_{M_{3/2}}^i$ in the charm sector and $E_{\Sigma_b \bar{B}} + E_{M_{3/2}}^i$ in the bottom sector. Finally, for $J^P = (5/2)^-$, the masses are given by $E_{\Sigma_c \bar{D}} + E_{\frac{35}{22}}^{s=1}$ in the charm sector and $E_{\Sigma_b \bar{B}} + E_{\frac{35}{22}}^{s=1}$ in the bottom sector.

In the next subsection, we focus only on the charm pentaquarks. We discuss different plausible scenarios for

identifying states with the experimentally observed charm pentaquarks $P_{c\bar{c}}$ listed in Table I.

C. Identification with experimental $P_{c\bar{c}}$ states

1. Scenario 0: $\mathcal{E}_{1/2} = \mathcal{E}_{3/2} = 0$

For illustration, we first discuss the simplest scenario where the adjoint baryon masses $\Lambda_{(1/2)^+}$ and $\Lambda_{(3/2)^+}$ are chosen in such a way to set the eigenenergies of the Schrödinger equations (3.6) and (3.7) to zero: $\mathcal{E}_{1/2} = \mathcal{E}_{3/2} = 0$. This scenario was considered by Voloshin to study the decay properties of $P_{c\bar{c}}$ states in the molecular picture in Ref. [41]. Without including the spin-dependent corrections from V_{SS} given in Eq. (3.8), the states are exactly at the spin-isospin averaged $\Sigma_c \bar{D}$ threshold. The critical values of the adjoint baryon masses (in the RS scheme¹¹) that yield $\mathcal{E}_{1/2} = \mathcal{E}_{3/2} = 0$ are $\Lambda_{(1/2)^+, \text{RS}} = 1.149$ GeV and $\Lambda_{(3/2)^+, \text{RS}} = 1.230$ GeV.

Including the spin-dependent corrections from V_{SS} , the eigenvalues $E_{M_{1/2}}^i$ and $E_{M_{3/2}}^i$ in this scenario are

$$\begin{aligned} E_{M_{1/2}}^i &= \left(-\frac{2\Delta_1^c}{3} - \frac{3\Delta_2^c}{4}, -\frac{2\Delta_1^c}{3} + \frac{\Delta_2^c}{4}, \frac{\Delta_1^c}{3} + \frac{3\Delta_2^c}{4} \right) = (-149.1, -7.8, 56.9) \text{ MeV}, \\ E_{M_{3/2}}^i &= \left(\frac{\Delta_1^c}{3} - \frac{3\Delta_2^c}{4}, -\frac{2\Delta_1^c}{3} + \frac{\Delta_2^c}{4}, \frac{\Delta_1^c}{3} + \frac{3\Delta_2^c}{4} \right) = (-84.4, -7.8, 56.9) \text{ MeV}, \end{aligned} \quad (3.16)$$

where the first, second, and third entries in the parentheses correspond to $i = 1, 2, 3$, respectively. The masses of the states are obtained after adding $E_{\Sigma_c \bar{D}}$ to the eigenvalues $E_{M_{1/2}}^i$ and $E_{M_{3/2}}^i$ in Eq. (3.16). This implies that for $J^P = (1/2)^-$, there is a state exactly at each of the $\Sigma_c \bar{D}$, $\Sigma_c \bar{D}^*$, and $\Sigma_c^* \bar{D}^*$ thresholds, while for $J^P = (3/2)^-$, there is a state exactly at each of the $\Sigma_c^* \bar{D}$, $\Sigma_c \bar{D}^*$, and $\Sigma_c^* \bar{D}^*$ thresholds. The state with mass 4.321 GeV at the $\Sigma_c \bar{D}$ threshold is identified with $P_{c\bar{c}}(4312)^+$, the state with mass 4.385 GeV at the $\Sigma_c^* \bar{D}$ threshold is identified with $P_{c\bar{c}}(4380)^+$, and the two states with mass 4.462 GeV at the $\Sigma_c \bar{D}^*$ threshold are identified with $P_{c\bar{c}}(4440)^+$ and $P_{c\bar{c}}(4457)^+$, whose J^P quantum numbers could be both $(1/2)^-$ and $(3/2)^-$. There are three states at the $\Sigma_c^* \bar{D}^*$ threshold with mass 4.527 GeV and $J^P = (1/2)^-, (3/2)^-,$ and $(5/2)^-$.

However, the experimental states listed in Table I have masses slightly below the $\Sigma_c \bar{D}$, $\Sigma_c^* \bar{D}$, and $\Sigma_c \bar{D}^*$ thresholds after including spin corrections. The eigenvalues $E_{M_{1/2}}^i$ and $E_{M_{3/2}}^i$ ($i = 1, 2, 3$) of the 3×3 matrices $M_{1/2}$ and $M_{3/2}$ are functions of the eigenenergies $\mathcal{E}_{1/2}$ and $\mathcal{E}_{3/2}$. To reproduce the masses of the four observed pentaquark states within experimental uncertainties (see Table I), we identify two optimal choices for these eigenenergies: $(\mathcal{E}_{1/2}, \mathcal{E}_{3/2}) = (-23 \text{ MeV}, -1 \text{ MeV})$ and $(\mathcal{E}_{1/2}, \mathcal{E}_{3/2}) = (-0.5 \text{ MeV}, -14 \text{ MeV})$, which we denote as Scenario 1 and Scenario 2, respectively. In the following, we discuss both scenarios.

2. Scenario 1: $\mathcal{E}_{1/2} = -23 \text{ MeV}$ and $\mathcal{E}_{3/2} = -1 \text{ MeV}$

The values of the adjoint baryon masses (in the RS scheme) that yield $(\mathcal{E}_{1/2}, \mathcal{E}_{3/2}) = (-23 \text{ MeV}, -1 \text{ MeV})$ are $\Lambda_{(1/2)^+, \text{RS}} = 0.998$ GeV and $\Lambda_{(3/2)^+, \text{RS}} = 1.209$ GeV. Including the spin-dependent corrections from V_{SS} , the eigenvalues $E_{M_{1/2}}^i$, $E_{M_{3/2}}^i$, and $E_{\frac{35}{22}}^{s=1}$ are

$$\begin{aligned} E_{M_{1/2}}^i &= (-158.1, -25.6, 36.7) \text{ MeV}, \\ E_{\frac{35}{22}}^{s=1} &= 55.9 \text{ MeV}, \\ E_{M_{3/2}}^i &= (-93.7, -11.4, 44.8) \text{ MeV}, \end{aligned} \quad (3.17)$$

where the first, second, and third entries in the parentheses correspond to $i = 1, 2, 3$, respectively. The masses of the states, obtained after adding $E_{\Sigma_c \bar{D}}$ to the results in Eq. (3.17), and the assigned J^P quantum numbers are listed in Table IV. The identification with the experimental states is shown in Fig. 2.

The eigenvectors of the matrices $M_{1/2}$ and $M_{3/2}$ in Eqs. (3.14) and (3.15) allow us to express the pentaquark states in Table IV in terms of the multiplet states listed in Table II, which have well-defined heavy-quark-antiquark spin s and LDF angular momentum k :

¹¹We use the RS masses, m_c^{RS} and m_b^{RS} , defined at the renormalon subtraction scale $\nu_f = 1$ GeV and given in Sec. IID 2.

$$\begin{pmatrix} |P_{c\bar{c}}(4312)^+\rangle \\ |P_{c\bar{c}}(4440)^+\rangle \\ |P_{c\bar{c}}(4507)^+\rangle \end{pmatrix} = \begin{pmatrix} 0.541 & 0.327 & 0.775 \\ -0.207 & -0.842 & 0.499 \\ -0.815 & 0.430 & 0.388 \end{pmatrix} \begin{pmatrix} |s=0, k=1/2\rangle_{J^P=(1/2)^-} \\ |s=1, k=1/2\rangle_{J^P=(1/2)^-} \\ |s=1, k=3/2\rangle_{J^P=(1/2)^-} \end{pmatrix}, \quad (3.18)$$

$$\begin{pmatrix} |P_{c\bar{c}}(4380)^+\rangle \\ |P_{c\bar{c}}(4457)^+\rangle \\ |P_{c\bar{c}}(4515)^+\rangle \end{pmatrix} = \begin{pmatrix} 0.646 & -0.484 & 0.589 \\ -0.361 & 0.486 & 0.796 \\ 0.672 & 0.727 & -0.139 \end{pmatrix} \begin{pmatrix} |s=1, k=1/2\rangle_{J^P=(3/2)^-} \\ |s=0, k=3/2\rangle_{J^P=(3/2)^-} \\ |s=1, k=3/2\rangle_{J^P=(3/2)^-} \end{pmatrix}, \quad (3.19)$$

$$|P_{c\bar{c}}(4526)^+\rangle = |s=1, k=3/2\rangle_{J^P=(5/2)^-}. \quad (3.20)$$

3. Scenario 2: $\mathcal{E}_{1/2} = -0.5$ MeV and $\mathcal{E}_{3/2} = -14$ MeV

The values of the adjoint baryon masses (in the RS scheme) that yield $(\mathcal{E}_{1/2}, \mathcal{E}_{3/2}) = (-0.5 \text{ MeV}, -14 \text{ MeV})$ are $\Lambda_{(1/2)^+, \text{RS}} = 1.125 \text{ GeV}$ and $\Lambda_{(3/2)^+, \text{RS}} = 1.152 \text{ GeV}$. Including the spin-dependent corrections from V_{SS} , the eigenvalues $E_{M_{1/2}}^i$, $E_{M_{3/2}}^i$, and $E_{\frac{35}{22}}^{s=1}$ are

$$\begin{aligned} E_{M_{1/2}}^i &= (-158.8, -11.1, 55.0) \text{ MeV}, \\ E_{\frac{35}{22}}^{s=1} &= 42.9 \text{ MeV}, \\ E_{M_{3/2}}^i &= (-94.2, -20.4, 50.8) \text{ MeV}, \end{aligned} \quad (3.21)$$

where the first, second, and third entries in the parentheses correspond to $i = 1, 2, 3$, respectively. The masses of the states, obtained after adding $E_{\Sigma_c \bar{D}}$ to the results in Eq. (3.21), and the assigned J^P quantum numbers are listed in Table V. The identification with the experimental states is shown in Fig. 3.

Similar to Eqs. (3.18)–(3.20), we can express the pentaquark states in Table V as

TABLE IV. Masses and assigned J^P quantum numbers for the charm pentaquark states in scenario 1 listed in order of increasing mass. The first four states are the experimental states shown in Table I. The last three states are below the $\Sigma_c^* \bar{D}$ threshold and have not yet been observed experimentally. All states have isospin $I = 1/2$.

$P_{c\bar{c}}$ state	Mass (MeV)	J^P
$P_{c\bar{c}}(4312)^+$	4312	$(1/2)^-$
$P_{c\bar{c}}(4380)^+$	4376	$(3/2)^-$
$P_{c\bar{c}}(4440)^+$	4444	$(1/2)^-$
$P_{c\bar{c}}(4457)^+$	4458	$(3/2)^-$
$P_{c\bar{c}}(4507)^+$	4507	$(1/2)^-$
$P_{c\bar{c}}(4515)^+$	4515	$(3/2)^-$
$P_{c\bar{c}}(4526)^+$	4526	$(5/2)^-$

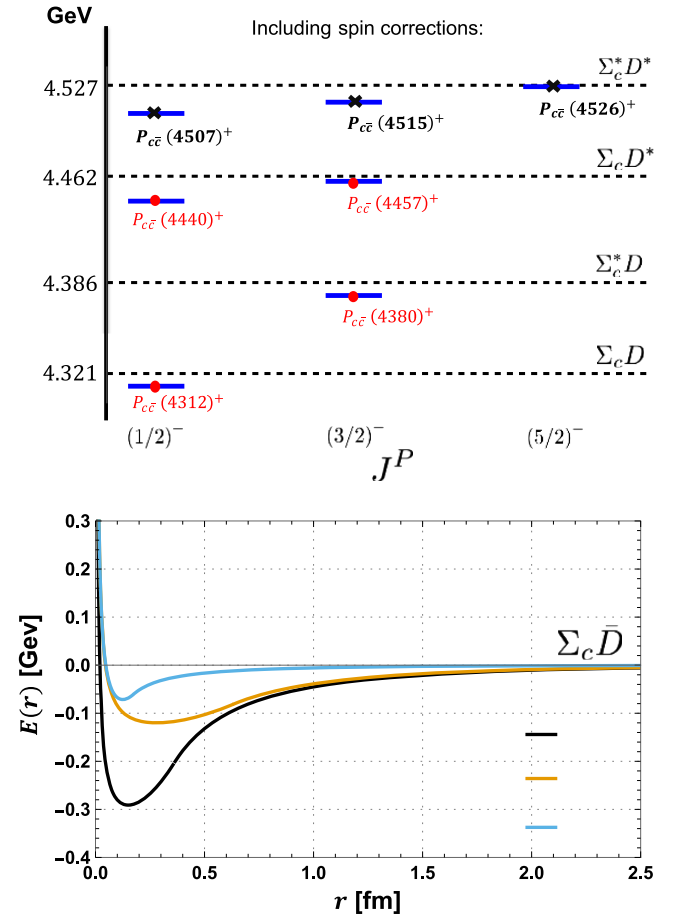


FIG. 2. Top: spectrum of the lowest $c\bar{c}$ pentaquark multiplet in Table II after including spin corrections in Scenario 1. After solving the coupled Schrödinger equations (3.6) and (3.7) and using Eqs. (3.14) and (3.15) to account for spin corrections, the obtained states are represented as horizontal blue bars. The four experimental $P_{c\bar{c}}$ states listed in Table I are shown as red dots, while the three unobserved ones are shown as black crosses. Bottom: assumed BO potential curves $E_{(1/2)_g}$, $E_{(1/2)_g'}$, and $E_{(3/2)_g}$ corresponding to the adjoint baryon masses $\Lambda_{(1/2)^+, \text{RS}} = 0.998 \text{ GeV}$ and $\Lambda_{(3/2)^+, \text{RS}} = 1.209 \text{ GeV}$. Isospin of the states is $I = 1/2$.

TABLE V. Masses and assigned J^P quantum numbers for the charm pentaquark states in Scenario 2 listed in order of increasing mass. The first four states are the experimental states shown in Table I. The last three states are below the $\Sigma_c^* \bar{D}^*$ threshold and have not been experimentally observed so far. All states have isospin $I = 1/2$.

$P_{c\bar{c}}$ state	Mass (MeV)	J^P
$P_{c\bar{c}}(4312)^+$	4311	$(1/2)^-$
$P_{c\bar{c}}(4380)^+$	4376	$(3/2)^-$
$P_{c\bar{c}}(4440)^+$	4449	$(3/2)^-$
$P_{c\bar{c}}(4457)^+$	4459	$(1/2)^-$
$P_{c\bar{c}}(4513)^+$	4513	$(5/2)^-$
$P_{c\bar{c}}(4521)^+$	4521	$(3/2)^-$
$P_{c\bar{c}}(4525)^+$	4525	$(1/2)^-$

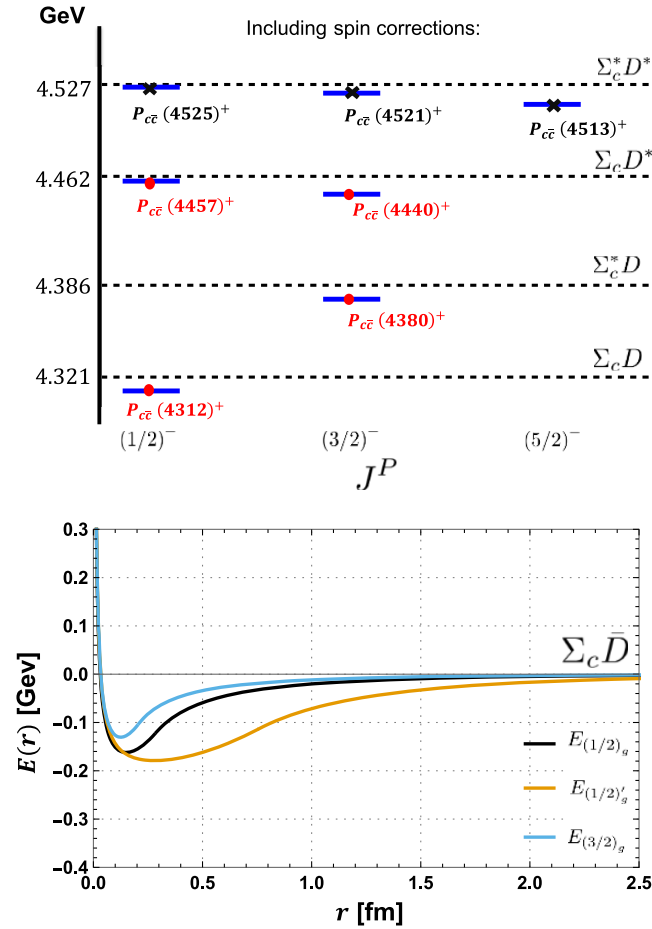


FIG. 3. Top: spectrum of the lowest $c\bar{c}$ pentaquark multiplet in Table II after including spin corrections in Scenario 2. The masses are computed from the adjoint baryon masses $\Lambda_{1/2,\text{RS}} = 1.125$ GeV and $\Lambda_{3/2,\text{RS}} = 1.152$ GeV. The labeling is the same as in Fig. 2. Bottom: assumed BO potential curves $E_{(1/2)_g}$, $E_{(1/2)'_g}$ and $E_{(3/2)_g}$ for Scenario 2 corresponding to the adjoint baryon masses $\Lambda_{(1/2)^+,\text{RS}} = 1.125$ GeV and $\Lambda_{(3/2)^+,\text{RS}} = 1.152$ GeV. Isospin of the states is $I = 1/2$.

$$\begin{pmatrix} |P_{c\bar{c}}(4312)^+\rangle \\ |P_{c\bar{c}}(4457)^+\rangle \\ |P_{c\bar{c}}(4525)^+\rangle \end{pmatrix} = \begin{pmatrix} 0.475 & 0.267 & 0.838 \\ -0.331 & -0.829 & 0.452 \\ -0.815 & 0.492 & 0.305 \end{pmatrix} \times \begin{pmatrix} |s=0, k=1/2\rangle_{J^P=(1/2)^-} \\ |s=1, k=1/2\rangle_{J^P=(1/2)^-} \\ |s=1, k=3/2\rangle_{J^P=(1/2)^-} \end{pmatrix}, \quad (3.22)$$

$$\begin{pmatrix} |P_{c\bar{c}}(4380)^+\rangle \\ |P_{c\bar{c}}(4440)^+\rangle \\ |P_{c\bar{c}}(4521)^+\rangle \end{pmatrix} = \begin{pmatrix} 0.536 & -0.506 & 0.676 \\ -0.314 & 0.624 & 0.716 \\ 0.784 & 0.596 & -0.176 \end{pmatrix} \times \begin{pmatrix} |s=1, k=1/2\rangle_{J^P=(3/2)^-} \\ |s=0, k=3/2\rangle_{J^P=(3/2)^-} \\ |s=1, k=3/2\rangle_{J^P=(3/2)^-} \end{pmatrix}, \quad (3.23)$$

$$|P_{c\bar{c}}(4513)^+\rangle = |s=1, k=3/2\rangle_{J^P=(5/2)^-}. \quad (3.24)$$

Contrary to Scenario 1, to the states $P_{c\bar{c}}(4440)$ and $P_{c\bar{c}}(4457)$ are assigned $J^P = (3/2)^-$ and $J^P = (1/2)^-$, respectively. Moreover, the pentaquark states that are slightly below the $\Sigma_c^* \bar{D}^*$ threshold have slightly different masses. In both scenarios, the pentaquark states with $J^P = (1/2)^-$ and $J^P = (3/2)^-$ are superpositions of heavy-quark pair spin singlet, $s=0$, and triplet, $s=1$, states reflecting a violation of the heavy-quark spin symmetry. Only the pentaquark state with $J^P = (5/2)^-$ has a well-defined heavy-quark spin configuration with $s=1$.

IV. DECAYS

In this section, we examine the decays of the pentaquark states obtained in Scenario 1 [see Eqs. (3.18)–(3.20)] and Scenario 2 [see Eqs. (3.22)–(3.24)], focusing on their decays into J/ψ , η_c , $\Lambda_c \bar{D}$, and $\Lambda_c \bar{D}^*$. We compute the semi-inclusive decay rates of pentaquark states to J/ψ and η_c following Ref. [105], and we evaluate the decay width ratios to $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$ thresholds following Ref. [109].

A. Semi-inclusive decays to J/ψ and η_c

Our aim is to compute the semi-inclusive decay rates of a pentaquark $P_{c\bar{c}}$ decaying into quarkonium states J/ψ and η_c : $P_{c\bar{c}} \rightarrow J/\psi + X$ and $P_{c\bar{c}} \rightarrow \eta_c + X$, where X denotes light hadrons. The energy transfer in the transition $P_{c\bar{c}} \rightarrow Q_n + X$, with $Q_n = \{J/\psi, \eta_c\}$ is ΔE , which is simply the mass difference between the initial pentaquark and the final quarkonium state. The energy gap ΔE is larger than 1 GeV, and thus satisfies the condition $\Delta E \gg \Lambda_{\text{QCD}}$. The $Q\bar{Q}$ pairs in pentaquarks are in a color-octet state, while the $Q\bar{Q}$ pairs in J/ψ or η_c are in a color-singlet state.

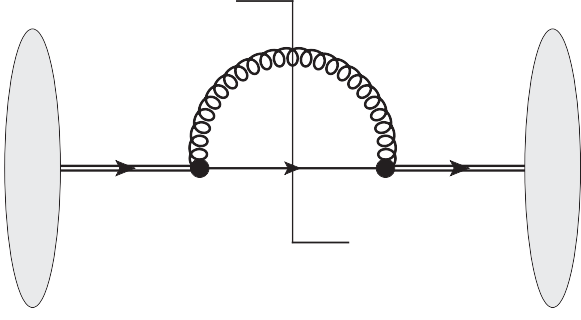


FIG. 4. Pentaquark state $P_{c\bar{c}}$ (gray blobs), and a $Q\bar{Q}$ pair in a color-singlet (single line) and octet (double line) state. The singlet is associated with the quarkonium state. The curly line stands for the energetic gluon emitted in the transition and the black dots for the chromomagnetic-dipole (M1) couplings. The vertical line is the cut. The imaginary part of this self-energy diagram gives the semi-inclusive widths of the transitions $P_{c\bar{c}} \rightarrow J/\psi + X$ and $P_{c\bar{c}} \rightarrow \eta_c + X$, where X denotes light hadrons. The light quark d.o.f. that are part of the pentaquark are treated as spectators and are not displayed.

With such a large energy gap ΔE , the gluon emitted by the heavy quarks in the transition from an octet to a singlet state can be treated in weak-coupling perturbation theory as shown in Fig. 4 and semi-inclusive decay widths may be computed along the lines of Refs. [100,105,106].

In BOEFT, all modes associated with high-energy scales down to and including Λ_{QCD} are integrated out, which means that gluons of energy and momentum of order ΔE should also be integrated out. This leads to an imaginary contribution to the pentaquark BO potential related to the semi-inclusive decay rate by the optical theorem. In the context of quarkonium hybrids, the semi-inclusive decay rates of hybrids to low-lying quarkonium states at leading and subleading power in the inverse of the heavy-quark mass were computed in Refs [100,105,106] using the spectator gluon approximation: the LDF (low-energy gluons) that constitute the hybrid do not interact with the high-energy gluons emitted in the transition from octet to singlet that carry the large energy $\Delta E \gg \Lambda_{\text{QCD}}$. The leading order contribution comes from the spin-conserving semi-inclusive decay induced by the chromoelectric-dipole (E1) coupling of the gluon with the color octet and singlet quark-antiquark pairs, while the subleading order contribution comes from the spin-flipping semi-inclusive decay induced by the chromomagnetic-dipole (M1) coupling of the gluon with the color octet and singlet quark-antiquark pairs [105]. With spin-conserving (or spin-flipping) decays, we mean that the spin of the heavy-quark-antiquark $Q\bar{Q}$ pair remains the same (or changes) from the initial to the final state.

In the context of the present study on pentaquarks, we use the results from Ref. [105] and the spectator light-quark approximation: the LDF (three light quarks) in the pentaquark do not interact with the high-energy gluon of energy ΔE . The transition $P_{c\bar{c}} \rightarrow Q_n + X$, with $Q_n = \{J/\psi, \eta_c\}$

proceeds via spin-flipping semi-inclusive decay, as both the initial pentaquark and the final quarkonium states have zero orbital angular momentum ($L_Q = 0$).¹² We use the spin-averaged masses for pentaquark and quarkonium to estimate the decay rates for both Scenarios 1 and 2. The spin-flipping semi-inclusive decay width is given by [105,106]

$$\Gamma(H_m \rightarrow Q_n) = \frac{2\alpha_s(\Delta E)}{9} T^{aj} (T^{aj})^\dagger \Delta E^3, \quad (4.1)$$

with T^{aj} the matrix element

$$T^{aj} = \frac{1}{m_Q} \left[\int d^3r \Psi_{k^P, P=-}^{(N), \alpha \dagger}(\mathbf{r}) \Phi_{(n)}^{Q\bar{Q}}(\mathbf{r}) \right] \langle \chi_{P_{c\bar{c}}} | (S_1^j - S_2^j) | \chi_{Q_n} \rangle, \quad (4.2)$$

where $k^P = (1/2)^+$ or $(3/2)^+$, α denotes the pentaquark vector index, S_1 and S_2 are the spin vectors of the heavy quark and heavy antiquark and $|\chi_{P_{c\bar{c}}}\rangle$ and $|\chi_{Q_n}\rangle$ denote the $Q\bar{Q}$ pair spin state for pentaquark and quarkonium, respectively. The spin-matrix elements are computed in Appendix D of Ref. [105]. The pentaquark wave functions are given by Eq. (3.2) for $k^P = (1/2)^+$ and Eq. (3.3) for $k^P = (3/2)^+$, while $\Phi_{(n)}^{Q\bar{Q}}$ denotes the quarkonium wave function with quantum numbers $n \equiv \{n, J, m_j, l, s\}$.¹³

The $P_{c\bar{c}}$ states are superpositions of both heavy-quark spin $s = 0$ and $s = 1$ states for $J^P = (1/2)^-$ and $J^P = (3/2)^-$. The nonvanishing of the matrix element (4.2) constrains the spin-0 component of $P_{c\bar{c}}$ to decay into the spin-1 final state J/ψ , and the spin-1 component of $P_{c\bar{c}}$ to decay into the spin-0 final state $\eta_c(1S)$. For the spin-0 component, the spin-flipping rate in Eq. (4.1) is multiplied by a factor 3 corresponding to the three polarizations of J/ψ . As evident from Eqs. (3.18), (3.19), (3.22), and (3.23), for the spin-1 component, both $k^P = (1/2)^+$ and $k^P = (3/2)^+$ contribute. The results for the semi-inclusive decay rates are shown in Table VI.

¹²Spin-conserving semi-inclusive decays of $P_{c\bar{c}}$ would instead produce P-wave quarkonium states.

¹³The quarkonium wave functions are obtained from solving the Schrödinger equation with the potential

$$E_{\Sigma_g^+}(r) = \begin{cases} V_s^{\text{RS}}(r, \nu_f), & r < R_{\Sigma_g^+} \\ \sigma r + V'_0, & r \geq R_{\Sigma_g^+} \end{cases},$$

where we use the RS singlet potential $V_s^{\text{RS}}(r)$ up to order α_s^3 in perturbation theory (see, e.g., Ref. [135]), $\sigma = 0.218 \text{ GeV}^2$ is the string tension from the lattice determination in Ref. [128], and $\nu_f = 1 \text{ GeV}$ is the renormalon subtraction scale. The constant $V'_0 = -0.651 \text{ GeV}$ and the matching radius $R_{\Sigma_g^+} = 0.339 \text{ fm}$ are determined from demanding continuity up to the first derivative. To compute the energy difference ΔE , we use the spin-averaged quarkonium masses from PDG [17].

TABLE VI. Semi-inclusive decay rates of pentaquark states decaying into J/ψ or η_c : $P_{c\bar{c}} \rightarrow Q_n + X$, with $Q_n = \{J/\psi, \eta_c\}$, where X denotes light hadrons. The decay rates are computed from Eqs. (4.1) and (4.2). The pentaquark states are denoted by $P_{c\bar{c}}[J^P](\text{mass})$, where the masses are in MeV. We have used the spin-averaged masses for both pentaquark and quarkonium states to estimate the decay rates. For both Scenarios 1 and 2 discussed in Sec. III C, we show the decay rates to J/ψ on the left and decay rates to $\eta_c(1S)$ on the right. The first error comes from assuming that the adjoint baryon masses have an uncertainty of the same order as the gluelump masses for hybrids, which is ± 0.15 GeV [133]. The second error bar is from varying the scale of α_s from $\Delta E/2$ to $2\Delta E$. For the decay rates shown, $\Delta E \gtrsim 1.3$ GeV and $\alpha_s(\Delta E) \lesssim 0.32$. The first four states in each scenario are the experimental states shown in Table I. The pentaquark state $P_{c\bar{c}}[J^P = (5/2)^-]$ decays only to $\eta_c(1S)$.

$P_{c\bar{c}}[J^P](\text{mass}) \rightarrow J/\psi[1^{--}]$	$\Gamma(\text{MeV})$
Scenario 1	
$P_{c\bar{c}}[(1/2)^-](4312)$	$31^{+11}_{-9} {}^{+14}_{-7}$
$P_{c\bar{c}}[(3/2)^-](4380)$	$4^{+1}_{-1} {}^{+2}_{-1}$
$P_{c\bar{c}}[(1/2)^-](4440)$	$5^{+2}_{-1} {}^{+2}_{-1}$
$P_{c\bar{c}}[(3/2)^-](4457)$	$4^{+1}_{-1} {}^{+2}_{-1}$
$P_{c\bar{c}}[(1/2)^-](4507)$	$71^{+26}_{-21} {}^{+32}_{-16}$
$P_{c\bar{c}}[(3/2)^-](4515)$	$8^{+3}_{-2} {}^{+4}_{-2}$
Scenario 2	
$P_{c\bar{c}}[(1/2)^-](4312)$	$3^{+1}_{-1} {}^{+1}_{-1}$
$P_{c\bar{c}}[(3/2)^-](4380)$	$16^{+6}_{-5} {}^{+7}_{-4}$
$P_{c\bar{c}}[(3/2)^-](4440)$	$23^{+9}_{-7} {}^{+11}_{-5}$
$P_{c\bar{c}}[(1/2)^-](4457)$	$1.3^{+0.5}_{-0.4} {}^{+0.6}_{-0.3}$
$P_{c\bar{c}}[(3/2)^-](4521)$	$22^{+8}_{-6} {}^{+10}_{-5}$
$P_{c\bar{c}}[(1/2)^-](4525)$	$8^{+3}_{-2} {}^{+4}_{-2}$
$P_{c\bar{c}}[J^P](\text{mass}) \rightarrow \eta_c(1S)[0^{-+}]$	$\Gamma(\text{MeV})$
Scenario 1	
$P_{c\bar{c}}[(1/2)^-](4312)$	$7^{+2}_{-1} {}^{+2}_{-1}$
$P_{c\bar{c}}[(3/2)^-](4380)$	$17^{+5}_{-4} {}^{+7}_{-3}$
$P_{c\bar{c}}[(1/2)^-](4440)$	$26^{+9}_{-7} {}^{+11}_{-6}$
$P_{c\bar{c}}[(3/2)^-](4457)$	$8^{+2}_{-2} {}^{+3}_{-1}$
$P_{c\bar{c}}[(1/2)^-](4507)$	$7^{+2}_{-2} {}^{+3}_{-2}$
$P_{c\bar{c}}[(3/2)^-](4515)$	$16^{+6}_{-5} {}^{+7}_{-4}$
$P_{c\bar{c}}[(5/2)^-](4526)$	$5^{+2}_{-1} {}^{+2}_{-1}$
Scenario 2	
$P_{c\bar{c}}[(1/2)^-](4312)$	$15^{+5}_{-4} {}^{+6}_{-3}$
$P_{c\bar{c}}[(3/2)^-](4380)$	$11^{+3}_{-3} {}^{+4}_{-2}$
$P_{c\bar{c}}[(3/2)^-](4440)$	$10^{+4}_{-3} {}^{+5}_{-2}$
$P_{c\bar{c}}[(1/2)^-](4457)$	$7^{+2}_{-1} {}^{+2}_{-1}$
$P_{c\bar{c}}[(5/2)^-](4513)$	$21^{+7}_{-6} {}^{+9}_{-5}$

(Table continued)

TABLE VI. (Continued)

$P_{c\bar{c}}[J^P](\text{mass}) \rightarrow \eta_c(1S)[0^{-+}]$	$\Gamma(\text{MeV})$
$P_{c\bar{c}}[(3/2)^-](4521)$	$3^{+1}_{-1} {}^{+1}_{-1}$
$P_{c\bar{c}}[(1/2)^-](4525)$	$3^{+1}_{-1} {}^{+1}_{-0.5}$

The sum of the semi-inclusive decay rates to J/ψ and $\eta_c(1S)$ in Table VI provides a lower bound for the total decay width of the pentaquark state. Particularly, for the $P_{c\bar{c}}(4312)^+$ state, the experimentally measured total width is 10 ± 5 MeV (see Table I) [17]. Our estimate of the lower bound on the total width for $P_{c\bar{c}}(4312)^+$ is 38^{+18}_{-12} MeV in Scenario 1 and 18^{+8}_{-5} MeV in Scenario 2. The central value of the estimate from Scenario 1 is around 3.8 times the experimental value, while the estimate from Scenario 2 is consistent within errors with the experimental determination—thus favoring Scenario 2. Additionally, we get the ratios of the decay rates to $\eta_c(1S)$ and J/ψ as

$$\begin{aligned} \frac{\Gamma(P_{c\bar{c}}(4312)^+ \rightarrow \eta_c(1S) + X)}{\Gamma(P_{c\bar{c}}(4312)^+ \rightarrow J/\psi + X)} &= 0.23^{+0.16}_{-0.09} \quad (\text{Scenario 1}), \\ \frac{\Gamma(P_{c\bar{c}}(4312)^+ \rightarrow \eta_c(1S) + X)}{\Gamma(P_{c\bar{c}}(4312)^+ \rightarrow J/\psi + X)} &= 5.0^{+3.5}_{-2.9} \quad (\text{Scenario 2}). \end{aligned} \quad (4.3)$$

For comparison, in Refs. [41,82], using the heavy quark spin symmetry, the analytic ratio of the exclusive S-wave decay widths for $P_{c\bar{c}}(4312)^+ \rightarrow \eta_c(1S) + p$ and $P_{c\bar{c}}(4312)^+ \rightarrow J/\psi + p$, where p is the proton, was predicted to be

$$\frac{\Gamma(P_{c\bar{c}}(4312)^+ \rightarrow \eta_c(1S) + p)}{\Gamma(P_{c\bar{c}}(4312)^+ \rightarrow J/\psi + p)} = 3.0. \quad (4.4)$$

Although the central value of our semi-inclusive ratio from Scenario 2 in Eq. (4.3) is about 1.7 times the value in Eq. (4.4), the two results are compatible within the quoted uncertainties.

For the $P_{c\bar{c}}(4380)^+$ state, the experimentally measured total width is 210 ± 90 MeV (see Table I) [17], which implies that it is a very broad state. Our estimate of the lower bound on the total width is 21^{+9}_{-5} MeV in Scenario 1 and 27^{+10}_{-7} MeV in Scenario 2—both significantly below the central experimental value. This discrepancy suggests that the state identified in our analysis might be narrower than the experimentally observed one. However, a definitive conclusion requires narrowing the large uncertainty affecting the measured width and including in the theoretical determination the decay channels to $\Lambda_c \bar{D}^*$ and $\Lambda_c \bar{D}$. For the $P_{c\bar{c}}(4440)^+$ and $P_{c\bar{c}}(4457)^+$ states, the experimentally measured total widths are 21^{+10}_{-11} MeV and $6.4^{+6.0}_{-2.8}$ MeV, respectively (see Table I) [17]. Our estimates of the lower

bounds in both scenarios are consistent with the experimental determinations within errors.

For the pentaquark state with $J^P = (5/2)^-$, $P_{c\bar{c}}(4526)^+$ in Scenario 1 or $P_{c\bar{c}}(4513)^+$ in Scenario 2, the decay proceeds only to an $\eta_c(1S)$ final state, as shown in Table VI. This is because the state has a well-defined heavy-quark spin configuration $s = 1$.

B. Ratios of decays to $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$

In this section, our aim is to compute the ratios of the pentaquark decays to the $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$ thresholds: $P_{c\bar{c}} \rightarrow \Lambda_c \bar{D}$ and $P_{c\bar{c}} \rightarrow \Lambda_c \bar{D}^*$. In the BO formalism, these decays proceed through a coupling or mixing potential that arises between BO potentials sharing the same quantum numbers. In our current work, the pentaquark BO potentials can mix or couple with BO potentials that asymptotically connect to the spin-isospin averaged $\Lambda_c \bar{D}$ threshold—provided they share the same BO quantum numbers. The pentaquark BO quantum numbers are $(1/2)_g$ and $\{(1/2)'_g, (3/2)_g\}$, corresponding to adjoint baryons $k^P = (1/2)^+$ and $k^P = (3/2)^+$, respectively, as shown in Table II. The $\Lambda_c \bar{D}$ threshold has BO quantum number $(1/2)_g$, corresponding to total light-

quark spin parity $k^P = (1/2)^+$ with $k_{qq}^P = 0^+$ (first row in Table III). Consequently, relevant for the present case is the mixing of the pentaquark BO potentials with quantum numbers $(1/2)_g$ and $(1/2)'_g$ with the BO potential $(1/2)_g$ connecting with the $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$ thresholds. Including this mixing modifies the radial Schrödinger equations (3.6) and (3.7), with the mixing potentials appearing as off-diagonal terms in the potential matrix. Following Ref. [85], the Schrödinger equation (3.6) gets modified into the two-channel equation

$$\left[-\frac{1}{m_Q r^2} \partial_r r^2 \partial_r + \frac{1}{m_Q r^2} \begin{pmatrix} (l - \frac{1}{2})(l + \frac{1}{2}) & 0 \\ 0 & (l - \frac{1}{2})(l + \frac{1}{2}) \end{pmatrix} + \begin{pmatrix} E_{(1/2)_g} & V_1^{\text{mix}} \\ V_1^{\text{mix}} & E_{(1/2)'_g} \end{pmatrix} \right] \begin{pmatrix} \psi_{(1/2)}^{(N)} \\ \psi_{\Lambda}^{(N)} \end{pmatrix} = \mathcal{E}_{1/2} \begin{pmatrix} \psi_{1/2, \sigma_P}^{(N)} \\ \psi_{\Lambda}^{(N)} \end{pmatrix}, \quad (4.5)$$

and the Schrödinger equation (3.7) gets modified into the three-channel equation

$$\left[-\frac{1}{m_Q r^2} \partial_r r^2 \partial_r + \frac{1}{m_Q r^2} \begin{pmatrix} l(l-1) + \frac{9}{4} & -\sqrt{3l(l+1) - \frac{9}{4}} & 0 \\ -\sqrt{3l(l+1) - \frac{9}{4}} & l(l+1) - \frac{3}{4} & 0 \\ 0 & 0 & (l + \frac{1}{2})(l + \frac{3}{2}) \end{pmatrix} + \begin{pmatrix} E_{(1/2)'_g} & 0 & V_2^{\text{mix}} \\ 0 & E_{(3/2)_g} & 0 \\ V_2^{\text{mix}} & 0 & E_{\Lambda(1/2)_g}^{\Lambda} \end{pmatrix} \right] \begin{pmatrix} \psi_{(1/2)^{++}}^{(N)} \\ \psi_{(3/2)^+}^{(N)} \\ \psi_{\Lambda}^{(N)} \end{pmatrix} = \mathcal{E}_{3/2} \begin{pmatrix} \psi_{(1/2)^{++}}^{(N)} \\ \psi_{(3/2)^+}^{(N)} \\ \psi_{\Lambda}^{(N)} \end{pmatrix}, \quad (4.6)$$

where $E_{(1/2)_g}^{\Lambda}$ is the BO potential that asymptotically connects to the spin-isospin averaged $\Lambda_c/\Lambda_b - \bar{D}/\bar{B}$ threshold from above, as shown in Fig. 1, without supporting bound states. The mixing is unknown, since it has not yet been measured in lattice QCD. We assume that it is much smaller than the energy gap of order 200 MeV that separates the spin-isospin averaged $\Sigma_c/\Sigma_b - \bar{D}/\bar{B}$ and $\Lambda_c/\Lambda_b - \bar{D}/\bar{B}$ thresholds. Under this assumption, the mixing has only a minimal effect on the spectrum obtained from the Schrödinger equations without mixing (3.6) and (3.7). The decay widths of pentaquark states to (spin-isospin averaged) $\Lambda_c \bar{D}$ thresholds, which proceeds through the mixing potentials, can be obtained from solving the coupled-channel Schrödinger equations (4.5) and (4.6). However due to the lack of lattice QCD information on the mixing potentials V_1^{mix} and V_2^{mix} , we can only provide analytical results for the ratios of the decay widths.

The pentaquark states $P_{c\bar{c}}$ with quantum numbers J^P in Scenario 1 are listed in Table IV and those in Scenario 2 are listed in Table V. Their expressions in terms of the multiplets listed in Table II, characterized by a well-defined heavy-quark–antiquark spin s and LDF angular momentum k , are shown in Eqs. (3.18)–(3.20) and (3.22)–(3.24). The S-wave $\Lambda_c \bar{D}$ threshold has $J^P = (1/2)^-$, while the S-wave $\Lambda_c \bar{D}^*$ threshold has $J^P = \{(1/2)^-, (3/2)^-\}$. Therefore, for the pentaquark states $P_{c\bar{c}}[J^P = (1/2)^-]$ in both Scenarios 1 and 2, the decay to S-wave $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$ is allowed, while, for $P_{c\bar{c}}[J^P = (3/2)^-]$ in both Scenarios 1 and 2, only the decay to S-wave $\Lambda_c \bar{D}^*$ is allowed, while the decay to $\Lambda_c \bar{D}$ requires a D-wave transition. Similarly, the pentaquark states $P_{c\bar{c}}[J^P = (5/2)^-]$ in both scenarios 1 and 2 can decay to both $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$ only through D-wave transitions. Since D-wave decays involve higher angular momentum barriers, they are expected to be suppressed compared to S-wave decays.

Following Ref. [109], the mixing or coupling potential governing the transitions $P_{c\bar{c}}[J^P] \rightarrow \Lambda_c \bar{D}, \Lambda_c \bar{D}^*$ can be written as

$$V_{s,l,\eta}^{J^P} \left(k^P, L_Q \rightarrow \left[\left(\frac{1^+}{2}, 0^+ \right) J_1, \left(\frac{1^-}{2}, \frac{1^+}{2} \right) J_2 \right] J', L'_Q \right) \\ = \sqrt{2} (-1)^{1/2+s+L'_Q+J} \sqrt{\tilde{s} \tilde{l} \tilde{J}_1 \tilde{J}_2 \tilde{J}'} \left\{ \begin{matrix} s & \frac{1}{2} & J' \\ L'_Q & J & l \end{matrix} \right\} \left\{ \begin{matrix} \frac{1}{2} & \frac{1}{2} & s \\ 0 & \frac{1}{2} & \frac{1}{2} \\ J_1 & J_2 & J' \end{matrix} \right\} G_{l,\eta}^P \left(k^P, L_Q \rightarrow \frac{1^+}{2}, L'_Q \right), \quad (4.7)$$

where J_1 and J_2 are the spins of the Λ_c baryon and $\bar{D}$ or $\bar{D}^*$ mesons respectively, $J_1 = 1/2$ for Λ_c and $J_2 = 0$ or 1 for $\bar{D}$ or $\bar{D}^*$, J' is the quantum number of $J' = J_1 + J_2$, whose values are $J' = 1/2$ for $\Lambda_c \bar{D}$ and $J' = \{1/2, 3/2\}$ for $\Lambda_c \bar{D}^*$, L'_Q is the quantum number denoting the relative angular momentum between the baryon Λ_c and the meson $\bar{D}$ or $\bar{D}^*$, whose values are $L'_Q = 0$ for S-wave and $L'_Q = 2$ for D-wave, and $\tilde{J} = 2J + 1$. The curly brackets $\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$ and

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{matrix} \right\}$$

are the Wigner $6j$ symbol and Wigner $9j$ symbol, respectively. The quantum numbers in the parenthesis and square brackets on the lhs of Eq. (4.7) have been summed over and the resulting quantum numbers are written to the right.¹⁴ The superscript on V in Eq. (4.7) are quantum numbers J^P , which is exactly conserved between the pentaquark states $P_{c\bar{c}}$ and $\Lambda_c \bar{D}$ or $\Lambda_c \bar{D}^*$ thresholds. The transition amplitude $G_{l,\eta}^P$, which depends on the combined angular momentum l , can be expressed as

$$G_{l,\eta}^P \left(k^P, L_Q \rightarrow \left(0^+, \frac{1^+}{2} \right) \frac{1}{2}, L'_Q \right) = (-1)^{L_Q+L'_Q} \sum_{\lambda=\mp 1/2} \left\langle \begin{matrix} k & l \\ \lambda & -\lambda \end{matrix} \middle| \begin{matrix} L_Q \\ 0 \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{1}{2} & l \\ \lambda & -\lambda \end{matrix} \middle| \begin{matrix} L'_Q \\ 0 \end{matrix} \right\rangle g_{\lambda,\eta} \left(k^P \rightarrow \left(0^+, \frac{1^+}{2} \right) \right), \quad (4.8)$$

where $\langle \begin{matrix} j_1 & j_2 \\ m_1 & m_2 \end{matrix} | \begin{matrix} j \\ m \end{matrix} \rangle$ denotes Clebsch-Gordon coefficients, and λ is the eigenvalue of $\mathbf{K} \cdot \hat{\mathbf{r}}$ that for $\Lambda_c \bar{D}$ or $\Lambda_c \bar{D}^*$ is $\lambda = \pm 1/2$, corresponding to total light-quark spin parity $(1/2)^+$. The quantity $g_{\lambda,\eta}$ is the transition amplitude between the light-quark state in the pentaquark and the spin-isospin averaged $\Lambda_c \bar{D}$ threshold in the static limit. This quantity can, in principle, be computed in lattice QCD, although it is currently unknown.¹⁵ Since $g_{\lambda,\eta}$ depends solely on the light-quark states in the pentaquark and the spin-isospin averaged $\Lambda_c \bar{D}$ threshold, it is the same for all pentaquark states with the same light-quark quantum numbers k^P . Using Eq. (4.8), we get for the S-wave transitions

$$G_{\frac{1}{2},+}^- \left(\frac{1^+}{2}, 0 \rightarrow \left(0^+, \frac{1^+}{2} \right) \frac{1}{2}, 0 \right) = \frac{1}{2} \left[g_{1/2,+} \left(\frac{1^+}{2} \rightarrow \left(0^+, \frac{1^+}{2} \right) \right) + g_{-1/2,+} \left(\frac{1^+}{2} \rightarrow \left(0^+, \frac{1^+}{2} \right) \right) \right], \\ G_{\frac{3}{2},+}^- \left(\frac{3^+}{2}, 0 \rightarrow \left(0^+, \frac{1^+}{2} \right) \frac{1}{2}, 0 \right) = 0. \quad (4.9)$$

This result implies that, among the components in Eqs. (3.18)–(3.20) and (3.22)–(3.24), those involving LDF with $k = 3/2$, i.e., $|s = 1, k = 3/2\rangle_{J^P=(1/2)^-}$, $|s = 0, k = 3/2\rangle_{J^P=(3/2)^-}$, and so on, do not contribute to the S-wave decays of $P_{c\bar{c}}$ states to $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$. A direct consequence is that $\Gamma[P_{c\bar{c}}[J^P = (5/2)^-] \rightarrow \Lambda_c \bar{D}, \Lambda_c \bar{D}^*] = 0$ for S-wave transitions, in agreement with the earlier discussion that such decays proceed only via D-wave and are thus suppressed.

Only the transition amplitude $G_{1/2,+}^-$ in Eq. (4.9) contributes to the decays of $P_{c\bar{c}}$ states to S-wave $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$ states. Therefore, it cancels out in the ratio of decay rates. Following Ref. [109], the ratio of decay widths for the pentaquark states $P_{c\bar{c}}$ with $J^P = (1/2)^-$ is given by

¹⁴In the notation $[(\frac{1^+}{2}, 0^+) J_1, (\frac{1^-}{2}, \frac{1^+}{2}) J_2] J'$ appearing in Eq. (4.7), the first parenthesis identifies the Λ_c , whose heavy-quark spin $1/2$ and light-quark spin 0 have been summed to give the total spin J_1 , which is written to the right of the parenthesis. Similarly, the second parenthesis identifies the $\bar{D}$ or $\bar{D}^*$, whose heavy-quark spin $1/2$ and light-quark spin $1/2$ have been summed to give the total spin J_2 , which is written to the right of the parenthesis. The sum of J_1 and J_2 gives J' , which is the quantum number of the S-wave $\Lambda_c \bar{D}$ or $\Lambda_c \bar{D}^*$ system written to the right of the square bracket.

¹⁵The transition rate has been computed only in the case of string breaking (avoided crossing) for the case of quarkonium and lowest meson-antimeson thresholds in lattice QCD in Refs. [136,137].

$$\frac{\Gamma(P_{c\bar{c}}[J^P = (1/2)^-] \rightarrow \Lambda_c \bar{D})/v_{\Lambda_c \bar{D}}}{\Gamma(P_{c\bar{c}}[J^P = (1/2)^-] \rightarrow \Lambda_c \bar{D}^*)/v_{\Lambda_c \bar{D}^*}} = \frac{\left| aV_{0, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}; J' = \frac{1}{2}) + bV_{1, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}; J' = \frac{1}{2}) \right|^2}{\sum_{J'=1/2}^{3/2} \left| aV_{0, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}^*; J') + bV_{1, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}^*; J') \right|^2}, \quad (4.10)$$

where the phase space factor for each final-state channel with orbital angular momentum L_Q' is proportional to $v_\alpha^{2L_Q'+1}$, α being either $\Lambda_c \bar{D}^*$ or $\Lambda_c \bar{D}$ and v_α the velocity of each of the outgoing heavy hadrons in the c.m. frame. The velocity v_α is determined through energy and momentum conservation by the masses of the initial pentaquark $P_{c\bar{c}}$, final baryon Λ_c and meson $\bar{D}^*$ or $\bar{D}$. The factors a and b in Eq. (4.10) are the elements of the 3×3 matrices in Eqs. (3.18) and (3.22): $a =_{J^P=(1/2)^-} \langle s=0, k=1/2 | P_{c\bar{c}}[J^P = (1/2)^-] \rangle$ and $b =_{J^P=(1/2)^-} \langle s=1, k=1/2 | P_{c\bar{c}}[J^P = (1/2)^-] \rangle$. For simplicity, in Eq. (4.10), we have used a compact notation for the coupling potential in Eq. (4.7), as follows:

$$\begin{aligned} V_{0, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}^*; J') &= V_{0, \frac{1}{2}, +}^{\frac{1}{2}-} \left(\frac{1}{2}^+, 0 \rightarrow \left[\left(\frac{1}{2}^+, 0^+ \right) \frac{1}{2}, \left(\frac{1}{2}^-, \frac{1}{2}^+ \right) 1 \right] J', 0 \right), \\ V_{1, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}^*; J') &= V_{1, \frac{1}{2}, +}^{\frac{1}{2}-} \left(\frac{1}{2}^+, 0 \rightarrow \left[\left(\frac{1}{2}^+, 0^+ \right) \frac{1}{2}, \left(\frac{1}{2}^-, \frac{1}{2}^+ \right) 1 \right] J', 0 \right), \\ V_{0, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}; J' = \frac{1}{2}) &= V_{0, \frac{1}{2}, +}^{\frac{1}{2}-} \left(\frac{1}{2}^+, 0 \rightarrow \left[\left(\frac{1}{2}^+, 0^+ \right) \frac{1}{2}, \left(\frac{1}{2}^-, \frac{1}{2}^+ \right) 0 \right] \frac{1}{2}, 0 \right). \end{aligned} \quad (4.11)$$

Similarly, the ratio of decay widths for the pentaquark states $P_{c\bar{c}}$ with $J^P = (3/2)^-$ can be obtained by setting $a = 0$ and $b =_{J^P=(3/2)^-} \langle s=1, k=1/2 | P_{c\bar{c}}[J^P = (3/2)^-] \rangle$ in Eq. (4.10), the matrix elements given by Eqs. (3.19) and (3.23), and substituting $V_{1, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}; J' = \frac{1}{2})$ with $V_{1, \frac{1}{2}, +}^{\frac{3}{2}-}(\Lambda_c \bar{D}; J' = \frac{1}{2})$ and $V_{1, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}^*; J')$ with $V_{1, \frac{1}{2}, +}^{\frac{3}{2}-}(\Lambda_c \bar{D}^*; J')$.

The results for the numerator and denominator of the ratio in the rhs of Eq. (4.10) for different pentaquarks $P_{c\bar{c}}$ decaying into S-wave $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$ are shown in Fig. 5 for Scenario 1 and in Fig. 6 for Scenario 2. For a given pentaquark state, $P_{c\bar{c}}$, from the entries in Figs. 5 and 6, it follows that

$$\begin{aligned} \frac{\Gamma^{(1)}(P_{c\bar{c}}(4440)^+ \rightarrow \Lambda_c \bar{D})/v_{\Lambda_c \bar{D}}}{\Gamma^{(1)}(P_{c\bar{c}}(4440)^+ \rightarrow \Lambda_c \bar{D}^*)/v_{\Lambda_c \bar{D}^*}} &= 1.09, & \frac{\Gamma^{(2)}(P_{c\bar{c}}(4440)^+ \rightarrow \Lambda_c \bar{D})/v_{\Lambda_c \bar{D}}}{\Gamma^{(2)}(P_{c\bar{c}}(4440)^+ \rightarrow \Lambda_c \bar{D}^*)/v_{\Lambda_c \bar{D}^*}} &= 0, \\ \frac{\Gamma^{(1)}(P_{c\bar{c}}(4457)^+ \rightarrow \Lambda_c \bar{D})/v_{\Lambda_c \bar{D}}}{\Gamma^{(1)}(P_{c\bar{c}}(4457)^+ \rightarrow \Lambda_c \bar{D}^*)/v_{\Lambda_c \bar{D}^*}} &= 0, & \frac{\Gamma^{(2)}(P_{c\bar{c}}(4457)^+ \rightarrow \Lambda_c \bar{D})/v_{\Lambda_c \bar{D}}}{\Gamma^{(2)}(P_{c\bar{c}}(4457)^+ \rightarrow \Lambda_c \bar{D}^*)/v_{\Lambda_c \bar{D}^*}} &= 0.62, \\ \frac{\Gamma^{(1)}(P_{c\bar{c}}(4507)^+ \rightarrow \Lambda_c \bar{D})/v_{\Lambda_c \bar{D}}}{\Gamma^{(1)}(P_{c\bar{c}}(4507)^+ \rightarrow \Lambda_c \bar{D}^*)/v_{\Lambda_c \bar{D}^*}} &= 2.53, & \frac{\Gamma^{(2)}(P_{c\bar{c}}(4525)^+ \rightarrow \Lambda_c \bar{D})/v_{\Lambda_c \bar{D}}}{\Gamma^{(2)}(P_{c\bar{c}}(4525)^+ \rightarrow \Lambda_c \bar{D}^*)/v_{\Lambda_c \bar{D}^*}} &= 3.28, \end{aligned} \quad (4.12)$$

$J^P = (1/2)^-$			
	$P_{c\bar{c}}(4312)^+$	$P_{c\bar{c}}(4440)^+$	$P_{c\bar{c}}(4507)^+$
$\Lambda_c \bar{D}$	0	3.52	5.48
$\Lambda_c \bar{D}^*$	3.59	3.24	2.17
$J^P = (3/2)^-$			
	$P_{c\bar{c}}(4380)^+$	$P_{c\bar{c}}(4457)^+$	$P_{c\bar{c}}(4515)^+$
$\Lambda_c \bar{D}$	0	0	0
$\Lambda_c \bar{D}^*$	3.74	1.17	4.09
$J^P = (5/2)^-$			
	$P_{c\bar{c}}(4526)^+$		
$\Lambda_c \bar{D}$	0		
$\Lambda_c \bar{D}^*$	0		

FIG. 5. Scenario 1: rows marked $\Lambda_c \bar{D}$ show $\left| aV_{0, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}; J' = \frac{1}{2}) + bV_{1, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}; J' = \frac{1}{2}) \right|^2 / G_{\frac{1}{2}, +}^-(\frac{1}{2}^+, 0 \rightarrow (0^+, \frac{1}{2}^+) \frac{1}{2}, 0)^2$ and rows marked $\Lambda_c \bar{D}^*$ show $\sum_{J'=1/2}^{3/2} \left| aV_{0, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}^*; J') + bV_{1, \frac{1}{2}, +}^{\frac{1}{2}-}(\Lambda_c \bar{D}^*; J') \right|^2 / G_{\frac{1}{2}, +}^-(\frac{1}{2}^+, 0 \rightarrow (0^+, \frac{1}{2}^+) \frac{1}{2}, 0)^2$, i.e., the numerator and denominator of the ratio in the rhs of Eq. (4.10), for the different states listed in the columns. The ratios of the different entries provide all the decay widths ratios that follow from Eq. (4.10).

$J^P = (1/2)^-$			
	$P_{c\bar{c}}(4312)^+$	$P_{c\bar{c}}(4457)^+$	$P_{c\bar{c}}(4525)^+$
$\Lambda_c \bar{D}$	0	2.74	6.26
$\Lambda_c \bar{D}^*$	2.67	4.42	1.91
$J^P = (3/2)^-$			
	$P_{c\bar{c}}(4380)^+$	$P_{c\bar{c}}(4440)^+$	$P_{c\bar{c}}(4521)^+$
$\Lambda_c \bar{D}$	0	0	0
$\Lambda_c \bar{D}^*$	2.59	0.89	5.53
$J^P = (5/2)^-$			
	$P_{c\bar{c}}(4513)^+$		
$\Lambda_c \bar{D}$	0		
$\Lambda_c \bar{D}^*$	0		

FIG. 6. Scenario 2: same as in Fig. 5.

where the superscripts (1) or (2) denote results in Scenario 1 or 2, respectively. We recall that the state $P_{c\bar{c}}(4440)^+$ is assigned $J^P = (1/2)^-$ in Scenario 1, while in Scenario 2 it has $J^P = (3/2)^-$; the reverse holds for $P_{c\bar{c}}(4457)^+$. The state $P_{c\bar{c}}(4507)^+$ in Scenario 1 and the state $P_{c\bar{c}}(4525)^+$ in Scenario 2 are close to the $\Sigma_c^* \bar{D}^*$ threshold with $J^P = (1/2)^-$.

In Ref. [41], considering that the states $P_{c\bar{c}}(4440)^+$ or $P_{c\bar{c}}(4457)^+$ are exactly at the $\Sigma_c \bar{D}^*$ threshold with quantum numbers $J^P = (1/2)^-$ (Scenario 0 in Sec. III C), the analytically predicted ratio of S-wave decay rates turns out to be

$$\frac{\Gamma(P_{c\bar{c}}[J^P = (1/2)^-] \rightarrow \Lambda_c \bar{D})/v_{\Lambda_c \bar{D}}}{\Gamma(P_{c\bar{c}}[J^P = (1/2)^-] \rightarrow \Lambda_c \bar{D}^*)/v_{\Lambda_c \bar{D}^*}} = \frac{3}{4}. \quad (4.13)$$

Our results in Eq. (4.12) yield a ratio of 1.09 in Scenario 1 and 0.62 in Scenario 2. The result of Eq. (4.13) is therefore closer to the one of Scenario 2, where the $P_{c\bar{c}}(4440)^+$ and $P_{c\bar{c}}(4457)^+$ are assigned with $J^P = (3/2)^-$ and $J^P = (1/2)^-$, respectively.

Considering Scenario 2 in Fig. 6, for $J^P = (1/2)^-$ states, we observe that the $P_{c\bar{c}}(4312)^+$ only decays into $\Lambda_c \bar{D}^*$, which is consistent with the prediction from the molecular picture [41].¹⁶ For $P_{c\bar{c}}(4457)^+$, the relative partial decay width to a $\Lambda_c \bar{D}^*$ final state is about 1.66 times larger than that of $P_{c\bar{c}}(4312)^+$, which implies that $P_{c\bar{c}}(4457)^+$ exhibits a slightly broader width into the $\Lambda_c \bar{D}^*$ channel, consistent with its higher mass and larger available phase space.¹⁷ For $P_{c\bar{c}}(4525)^+$, the dominant S-wave decay is to $\Lambda_c \bar{D}$, which is 2.29 times larger than that of $P_{c\bar{c}}(4457)^+$ into the same channel. The large total S-wave width suggests that the

$P_{c\bar{c}}(4525)^+$ is likely a relatively broad resonance compared to lower-lying states such as $P_{c\bar{c}}(4312)^+$ and $P_{c\bar{c}}(4457)^+$. Also, accounting for the decays of $P_{c\bar{c}}(4525)^+$ to J/ψ and $\eta_c(1S)$ from Table VI, the broad width may explain why this state has not yet been observed experimentally.

For $J^P = (3/2)^-$ states, we observe that all the states only decay to S-wave $\Lambda_c \bar{D}^*$, as discussed at the beginning of this section. For $P_{c\bar{c}}(4380)^+$, the relative partial decay rate for decays into $\Lambda_c \bar{D}^*$ is of the same order as that of $P_{c\bar{c}}(4312)^+$ (2.59 vs 2.67 in Fig. 6), which suggests that $P_{c\bar{c}}(4380)^+$ has a total width comparable to the one of $P_{c\bar{c}}(4312)^+$. Also including decays to J/ψ and $\eta_c(1S)$ from Table VI, our analysis suggests that this state is narrower than the broad structure seen in experiments, in agreement with Refs. [28,29]. For $P_{c\bar{c}}(4440)^+$, the relative partial decay rate for decays into $\Lambda_c \bar{D}^*$ is about three times smaller than that of $P_{c\bar{c}}(4380)^+$. Moreover, from Table VI, we see that this state may have significant decay modes to J/ψ and $\eta_c(1S)$. For $P_{c\bar{c}}(4521)^+$, the relative partial decay rate for decays into $\Lambda_c \bar{D}^*$ is 2.14 times larger than that of $P_{c\bar{c}}(4380)^+$, which suggests that this state is likely a relatively broad resonance compared to lower-lying states such as $P_{c\bar{c}}(4380)^+$ and $P_{c\bar{c}}(4440)^+$. Also accounting for the decays of $P_{c\bar{c}}(4525)^+$ to J/ψ and $\eta_c(1S)$ from Table VI, the broad nature of the resonance could be the reason why this state has not been seen in experiments.

The $J^P = (5/2)^-$ state $P_{c\bar{c}}(4513)^+$ can decay into $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$ only via D-wave transitions, which are suppressed compared to S-wave decays, indicating that the state may have a narrower width compared to other pentaquark states in Fig. 6. Moreover, from Table VI, we observe that this state only decays to $\eta_c(1S)$ with a width of 21 MeV (in Scenario 2). The narrow width for decays into $\Lambda_c \bar{D}$, $\Lambda_c \bar{D}^*$ and the fact that the state decays to $\eta_c(1S)$ and not to J/ψ clearly makes the detection of $P_{c\bar{c}}(4513)^+$ a challenging enterprise.

¹⁶In the molecular picture, the $P_{c\bar{c}}(4312)^+$ is a $\Sigma_c \bar{D}$ molecule. There is no $DD\pi$ vertex to account for the transition to $\Lambda_c \bar{D}$ [41].

¹⁷The total S-wave width of $P_{c\bar{c}}(4457)^+$ is around 2.7 times larger than the total S-wave width of $P_{c\bar{c}}(4312)^+$. However, their measured total widths are similar (see Table I).

V. PREDICTIONS FOR BOTTOM PENTAQUARKS $P_{b\bar{b}}$

As discussed in the previous section, Scenario 2 is favored based on the analyses of the charm pentaquark semi-inclusive decay widths into J/ψ and $\eta_c(1S)$, and decay patterns into S-wave $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$ thresholds. In this section, we provide predictions for the bottom pentaquark states $P_{b\bar{b}}$ adopting Scenario 2.

The values of the adjoint baryon masses (in the RS scheme) tuned on the charm pentaquark case are $\Lambda_{(1/2)^+, \text{RS}} = 1.125$ GeV and $\Lambda_{(3/2)^+, \text{RS}} = 1.152$ GeV. Solving the Schrödinger equations (3.6) and (3.7) with these inputs for the adjoint baryon masses appearing in the potentials (2.11) and with $m_b^{\text{RS}} = 4.863$ GeV yields $(\mathcal{E}_{1/2}, \mathcal{E}_{3/2}) = (-21 \text{ MeV}, -59 \text{ MeV})$. Including the spin-dependent corrections from V_{SS} , the eigenvalues $E_{M_{1/2}}^i$, $E_{M_{3/2}}^i$, and $E_{\frac{3}{2}\frac{3}{2}}^{s=1}$ corresponding to the $J^P = (1/2)^-$, $(3/2)^-$, and $(5/2)^-$ states are

$$\begin{aligned} E_{M_{1/2}}^i &= (-97.7, -28.4, -5.5) \text{ MeV}, \\ E_{\frac{3}{2}\frac{3}{2}}^{s=1} &= -41.0 \text{ MeV}, \\ E_{M_{3/2}}^i &= (-79.1, -58.1, -12.7) \text{ MeV}, \end{aligned} \quad (5.1)$$

where the first, second, and third entries in the parentheses correspond to $i = 1, 2, 3$ respectively. The masses of the states are obtained after adding $E_{\Sigma_b \bar{B}} = 11.140$ GeV to the results in Eq. (5.1), and the assigned J^P quantum numbers are shown in Fig. 7. The eigenvectors of the matrices $M_{1/2}$

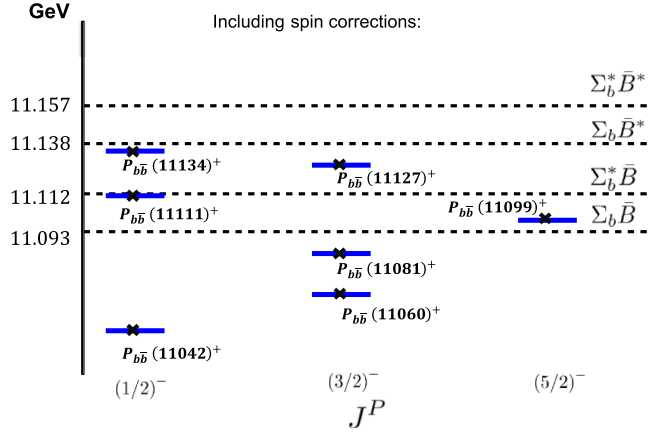


FIG. 7. Predicted spectrum of the lowest-lying bottom pentaquark states $P_{b\bar{b}}$ and their assigned J^P quantum numbers in Scenario 2, which the analysis of charm pentaquarks favors. The masses of the states, in units of MeV, are indicated in the parentheses next to each state. The states, after solving the coupled Schrödinger equations (3.6) and (3.7) and using Eqs. (3.14) and (3.15) to account for spin corrections, are represented by horizontal blue bars; the black crosses on the horizontal bars denote that the states have not been seen yet in experiments. The masses are computed from the adjoint baryon masses $\Lambda_{(1/2)^+, \text{RS}} = 1.125$ GeV and $\Lambda_{(3/2)^+, \text{RS}} = 1.152$ GeV. Isospin is $I = 1/2$ for all the states.

and $M_{3/2}$ in Eqs. (3.14) and (3.15) allow us to express the pentaquark states as superpositions of the multiplet states listed in Table II, which have well-defined heavy-quark–antiquark spin s and LDF angular momentum k , as follows:

$$\begin{pmatrix} |P_{b\bar{b}}(11042)^+\rangle \\ |P_{b\bar{b}}(11111)^+\rangle \\ |P_{b\bar{b}}(11134)^+\rangle \end{pmatrix} = \begin{pmatrix} 0.318 & 0.153 & 0.936 \\ -0.522 & -0.796 & 0.307 \\ -0.791 & 0.586 & 0.173 \end{pmatrix} \begin{pmatrix} |s=0, k=1/2\rangle_{J^P=(1/2)^-} \\ |s=1, k=1/2\rangle_{J^P=(1/2)^-} \\ |s=1, k=3/2\rangle_{J^P=(1/2)^-} \end{pmatrix}, \quad (5.2)$$

$$\begin{pmatrix} |P_{b\bar{b}}(11060)^+\rangle \\ |P_{b\bar{b}}(11081)^+\rangle \\ |P_{b\bar{b}}(11127)^+\rangle \end{pmatrix} = \begin{pmatrix} 0.309 & -0.503 & 0.807 \\ -0.177 & 0.804 & 0.568 \\ 0.934 & 0.319 & -0.160 \end{pmatrix} \begin{pmatrix} |s=1, k=1/2\rangle_{J^P=(3/2)^-} \\ |s=0, k=3/2\rangle_{J^P=(3/2)^-} \\ |s=1, k=3/2\rangle_{J^P=(3/2)^-} \end{pmatrix}, \quad (5.3)$$

$$|P_{b\bar{b}}(11099)^+\rangle = |s=1, k=3/2\rangle_{J^P=(5/2)^-}. \quad (5.4)$$

Our predictions for the spin-flipping semi-inclusive decay widths into $\Upsilon(1S)$ and $\eta_b(1S)$ of the pentaquark states $P_{b\bar{b}}$ shown in Fig. 7, computed using Eqs. (4.1) and (4.2) along with Eqs. (5.2)–(5.4), are summarized in Table VII. The transition is spin-flipping as the nonvanishing of the matrix element in Eq. (4.2) constrains the spin-0 component of $P_{b\bar{b}}$ to decay into the spin-1 final state $\Upsilon(1S)$ and the spin-1 component of $P_{b\bar{b}}$ to decay into the spin-0 final state $\eta_b(1S)$. The sum of the decay widths to $\Upsilon(1S)$ and $\eta_b(1S)$ sets a lower bound on the total decay width of the $P_{b\bar{b}}$ state. Compared to the semi-inclusive decays in the charm sector given in Table VI, the semi-inclusive rates in the bottom sector are smaller due to $1/m_Q$ suppression from the heavier bottom mass. The pentaquark state $P_{b\bar{b}}[J^P=(5/2)^-](11099)$ only decays to η_b as this state has a well-defined heavy-quark spin $s=1$, as seen from Eq. (5.4).

TABLE VII. Predictions of the semi-inclusive decay rates of pentaquark states $P_{b\bar{b}}$ decaying into $\Upsilon(1S)$ or $\eta_b(1S)$ in Scenario 2: $P_{b\bar{b}} \rightarrow Q_n + X$, with $Q_n = \{\Upsilon(1S), \eta_b(1S)\}$, where X denotes light hadrons. The decay rates are computed from Eqs. (4.1) and (4.2). The pentaquark states are denoted by $P_{b\bar{b}}[J^P](\text{mass})$, where the masses are in MeV. We have used the spin-averaged masses for both pentaquark and quarkonium states to estimate the decay rates. We show the decay rates to $\Upsilon(1S)$ on the left and the decay rates to $\eta_b(1S)$ and on the right. The pentaquark state $P_{b\bar{b}}[J^P = (5/2)^-]$ decays only to η_b .

$P_{b\bar{b}}[J^P](\text{mass}) \rightarrow \Upsilon(1S)[1^{--}]$	$\Gamma(\text{MeV})$
Scenario 2	
$P_{b\bar{b}}[(1/2)^-](11042)$	$1^{+0.4}_{-0.3} {}^{+0.5}_{-0.3}$
$P_{b\bar{b}}[(3/2)^-](11060)$	$2^{+1}_{-1} {}^{+1}_{-0.5}$
$P_{b\bar{b}}[(3/2)^-](11081)$	$6^{+2}_{-2} {}^{+2}_{-2}$
$P_{b\bar{b}}[(1/2)^-](11111)$	$3^{+1}_{-1} {}^{+1}_{-1}$
$P_{b\bar{b}}[(3/2)^-](11127)$	$1^{+0.3}_{-0.2} {}^{+0.4}_{-0.2}$
$P_{b\bar{b}}[(1/2)^-](11134)$	$8^{+2}_{-2} {}^{+3}_{-2}$
$P_{b\bar{b}}[J^P](\text{mass}) \rightarrow \eta_b(1S)[0^{-+}]$	$\Gamma(\text{MeV})$
Scenario 2	
$P_{b\bar{b}}[(1/2)^-](11042)$	$3^{+1}_{-1} {}^{+1}_{-1}$
$P_{b\bar{b}}[(3/2)^-](11060)$	$2^{+1}_{-1} {}^{+1}_{-0.4}$
$P_{b\bar{b}}[(3/2)^-](11081)$	$1^{+0.3}_{-0.3} {}^{+0.4}_{-0.2}$
$P_{b\bar{b}}[(5/2)^-](11099)$	$3^{+1}_{-1} {}^{+1}_{-1}$
$P_{b\bar{b}}[(1/2)^-](11111)$	$3^{+1}_{-1} {}^{+1}_{-1}$
$P_{b\bar{b}}[(3/2)^-](11127)$	$4^{+1}_{-1} {}^{+1}_{-1}$
$P_{b\bar{b}}[(1/2)^-](11134)$	$1.5^{+0.4}_{-0.3} {}^{+0.6}_{-0.2}$

The results for the numerator and denominator of the ratio in the rhs of Eq. (4.10) obtained using Eqs. (5.2)–(5.4) for different pentaquarks $P_{b\bar{b}}$ decaying into S-wave $\Lambda_b \bar{B}$ and $\Lambda_b \bar{B}^*$ are shown in Fig. 8 for Scenario 2. For a given pentaquark state, $P_{b\bar{b}}$, from the entries in Fig. 8, it follows

that

$$\frac{\Gamma(P_{b\bar{b}}(11111)^+ \rightarrow \Lambda_b \bar{B})/v_{\Lambda_b \bar{B}}}{\Gamma(P_{b\bar{b}}(11111)^+ \rightarrow \Lambda_b \bar{B}^*)/v_{\Lambda_b \bar{B}^*}} = 0.25,$$

$$\frac{\Gamma(P_{b\bar{b}}(11134)^+ \rightarrow \Lambda_b \bar{B})/v_{\Lambda_b \bar{B}}}{\Gamma(P_{b\bar{b}}(11134)^+ \rightarrow \Lambda_b \bar{B}^*)/v_{\Lambda_b \bar{B}^*}} = 5.30. \quad (5.5)$$

If the state $P_{b\bar{b}}(11111)^+$ with $J^P = (1/2)^-$ were exactly at the $\Sigma_b \bar{B}^*$ threshold (Scenario 0 in Sec. III C), then the ratio of decay widths would be given by Eq. (4.13). Our results yield a ratio of 0.25 [see Eq. (5.5)], which is three times smaller than Eq. (4.13). This deviation reflects the fact that bottom pentaquark states are more deeply bound, as it follows from the values of the binding energies $\mathcal{E}_{1/2}$ and $\mathcal{E}_{3/2}$.

Considering Fig. 8, for $J^P = (1/2)^-$ states, we observe that $P_{b\bar{b}}(11042)^+$ decays most often into $\Lambda_b \bar{B}^*$. For $P_{b\bar{b}}(11111)^+$, which has dominant decay into $\Lambda_b \bar{B}^*$, the relative partial decay width to a $\Lambda_b \bar{B}^*$ final state is about 5.86 times larger than that of $P_{b\bar{b}}(11042)^+$ implying that the state has much broader width into the $\Lambda_c \bar{B}^*$ channel. For $P_{b\bar{b}}(11134)^+$, the dominant S-wave decay is into $\Lambda_b \bar{B}$, whose width is 4.45 times larger than that of $P_{b\bar{b}}(11111)^+$ into the same channel. The large total S-wave width for both $P_{b\bar{b}}(11111)^+$ and $P_{b\bar{b}}(11134)^+$ suggests that both states are likely to manifest as relatively broad resonances compared to $P_{b\bar{b}}(11042)^+$. For $J^P = (3/2)^-$ states, we observe that all the states decay only to S-wave $\Lambda_b \bar{B}^*$ as discussed at the beginning of this section. For $P_{b\bar{b}}(11060)^+$, the relative partial decay rate for decays into $\Lambda_b \bar{B}^*$ is slightly smaller (0.7 times) than that of $P_{b\bar{b}}(11042)^+$. For $P_{b\bar{b}}(11081)^+$, the relative partial decay rate for decays into $\Lambda_b \bar{B}^*$ is about three times smaller than that of $P_{b\bar{b}}(11060)^+$. Moreover, accounting for the decays to $\Upsilon(1S)$ and $\eta_b(1S)$ in Table VII, both $P_{b\bar{b}}(11060)^+$ and $P_{b\bar{b}}(11081)^+$ are possibly narrow pentaquark states. For $P_{b\bar{b}}(11127)^+$, the relative partial decay rate for decays into $\Lambda_b \bar{B}^*$ is around nine times larger than that of $P_{b\bar{b}}(11060)^+$, which suggests that this state will likely be a relatively much broader resonance compared to

$J^P = (1/2)^-$			
	$P_{b\bar{b}}(11042)^+$	$P_{b\bar{b}}(11111)^+$	$P_{b\bar{b}}(11134)^+$
$\Lambda_b \bar{B}$	0.01	1.65	7.35
$\Lambda_b \bar{B}^*$	1.11	6.50	1.38
$J^P = (3/2)^-$			
	$P_{b\bar{b}}(11060)^+$	$P_{b\bar{b}}(11081)^+$	$P_{b\bar{b}}(11127)^+$
$\Lambda_b \bar{B}$	0	0	0
$\Lambda_b \bar{B}^*$	0.86	0.28	7.86
$J^P = (5/2)^-$			
	$P_{b\bar{b}}(11099)^+$		
$\Lambda_b \bar{B}$	0		
$\Lambda_b \bar{B}^*$	0		

FIG. 8. Same as in Fig. 6 but for $P_{b\bar{b}}$ states decaying into S-wave $\Lambda_b \bar{B}$ and $\Lambda_b \bar{B}^*$ in Scenario 2.

$P_{b\bar{b}}(11060)^+$ and $P_{b\bar{b}}(11081)^+$. The $J^P = (5/2)^-$ state $P_{b\bar{b}}(11099)^+$ can decay into $\Lambda_b \bar{B}$ and $\Lambda_b \bar{B}^*$ only via D-wave transitions, which are suppressed compared to S-wave decays, indicating that the state will have a narrower width compared to other pentaquark states in Fig. 8. Accounting for the decay to $\eta_b(1S)$ from Table VII, further supports that this state may probably be a narrow resonance.

VI. SUMMARY AND CONCLUSIONS

In this work, we have presented a QCD-based description of quarkonium pentaquark $Q\bar{Q}qqq$ states both in the charm and bottom sector. The description is based on the Born–Oppenheimer nonrelativistic effective field theory framework, whose underlying principle is the systematic factorization of the dynamics of the heavy quarks and the three light quarks based on the hierarchy $\Lambda_{\text{QCD}} \gg m_Q v^2$. The BOEFT leads to coupled-channel Schrödinger equations (3.6) and (3.7) for describing the pentaquarks, as derived in Ref. [85]. These equations require pentaquark BO potentials (static energies) as inputs. They appear in the diagonal entries of the potential matrix in the Schrödinger equations (3.6) and (3.7). In Sec. II B, we write down the pentaquark BO potentials at short distances. At short distances, they are the sum of a repulsive color octet potential, adjoint baryon masses Λ_κ and nonperturbative terms of $\mathcal{O}(r^2)$. Currently, there are no lattice QCD computations available for either the adjoint baryon spectrum or the pentaquark BO potentials. To facilitate future lattice studies, we give the expressions of the light-quark operators for the lowest adjoint baryons with quantum numbers $\kappa = \{(1/2)^+, I = 1/2\}$ and $\{(3/2)^+, I = 1/2\}$ in Eqs. (2.6) and (2.7) and the expression of the gauge-invariant interpolating operators in Eq. (2.3). The two-point correlation functions of the interpolating operators can be expressed in terms of generalized Wilson loops, computed in lattice QCD and eventually used to extract the pentaquark BO potentials.

In this work, we propose to identify the four experimentally observed hidden-charm pentaquark states $P_{c\bar{c}}$ listed in Table I with states in the lowest multiplet of the BO potentials $(1/2)_g$ and $\{(1/2)'_g, (3/2)_g\}$. These potentials connect smoothly to the $\Sigma_c \bar{D}$ threshold at long distances due to BO quantum number conservation [85] and match to the spectrum of the adjoint baryons $(1/2)^+$ and $(3/2)^+$, respectively, at short distances. The form of the potentials at intermediate distances is unknown, as they have not been computed in lattice QCD. At leading order in the $1/m_Q$ expansion, solving the Schrödinger equations (3.6) and (3.7) with these BO potentials yields a degenerate (spin-averaged) set of states independent of the heavy-quark spin S (see Table II). Since the BO potentials for the $Q\bar{Q}$ pentaquark at large r evolve smoothly into the heavy-light baryon-antimeson thresholds in the static limit and because the spin-dependent interaction that leads to spin splitting in

the heavy-light baryon-antimeson threshold appears at order $1/m_Q$, in Sec. III B, we account for the spin splittings in the pentaquark multiplet in Table II by choosing a spin interaction that resembles the one responsible for the spin interaction in the heavy-light baryon-antimeson states [see Eq. (3.8)]. Besides the four states already observed, our assumed form (at intermediate distances) for the $(1/2)_g$ BO potential connecting with the $\Sigma_c \bar{D}$ threshold at large distances and the $(1/2)^+$ adjoint baryon at short distances supports three additional states, which are near the $\Sigma_c^* \bar{D}^*$ threshold. Seven states are also predicted by the molecular picture based on the heavy-quark spin symmetry in Refs. [27–29,37]. In the absence of lattice data, however, other choices for the behaviour of the BO potentials at intermediate distances are possible. If the $(1/2)_g$ BO potential connecting with the $\Lambda_c \bar{D}$ threshold would support bound states, then three additional low-lying pentaquark states in the charm sector could exist. Ten low-lying pentaquark states, some having positive parity, are predicted in compact pentaquark models such as in Ref. [60]. Because there is no experimental evidence of states near the $\Lambda_c \bar{D}$ threshold, we have assumed that the BO potential that connects to that threshold falls off monotonically from above like in Fig. 1 and hence does not support bound states. Another possibility is that the $(1/2)_g$ BO potential connecting to the $\Sigma_c \bar{D}$ threshold at large distances and the $(1/2)^+$ adjoint baryon at short distances do not support bound states. This case, which brings the number of predicted low-lying pentaquarks to four, has been investigated in Ref. [131]. In BOEFT, all the ground-state pentaquark states have negative parity. The positive-parity pentaquark states arise from orbital excitations with heavy-quark orbital angular momentum $L_Q = 1$ (P-wave) and therefore lie at higher energies than the ground-state pentaquarks.

By treating the adjoint baryon masses $\Lambda_{(1/2)^+}$ and $\Lambda_{(3/2)^+}$ as free parameters fitted on the masses of the four observed $P_{c\bar{c}}$ states after including spin corrections, we find two viable scenarios, Scenario 1 and Scenario 2, whose spectra and J^P assignments are summarized in Tables IV and V, and in Figs. 2 and 3. Notably, in Scenario 2, our predicted masses for the three unobserved $P_{c\bar{c}}$ states show good agreement with the molecular model predictions in Refs. [28,29], even though BOEFT makes no *a priori* assumption of a molecular configuration. The agreement emerges from the complex QCD dynamics encoded in the solutions of the Schrödinger equations (3.6) and (3.7). In particular, the BOEFT can be closer or farther away from the molecular picture results, depending on the values of the adjoint baryon masses and the corresponding eigenenergies coming from the diagonalization of the matrices in Eqs. (3.14) and (3.15). It is therefore of crucial importance to calculate on the lattice the adjoint baryon masses $\Lambda_{(1/2)^+}$ and $\Lambda_{(3/2)^+}$. After including spin corrections, the resulting

$P_{c\bar{c}}$ states are superpositions of multiplet states (Table II) with the heavy-quark–antiquark pair in both spin $s = 0$ and $s = 1$ configurations, as given by Eqs. (3.18)–(3.20) and Eqs. (3.22)–(3.24) for Scenarios 1 and 2, respectively. In Sec. IV A, we compute semi-inclusive decay rates into J/ψ and $\eta_c(1S)$ following Ref. [105]. Spin-flip transitions mediate the decays: the spin-0 component of $P_{c\bar{c}}$ decays to J/ψ and the spin-1 component of $P_{c\bar{c}}$ decays to $\eta_c(1S)$. Spin-conserving transitions of $P_{c\bar{c}}$ would instead lead to P-wave quarkonium states, and hence do not contribute. The sum of these rates provides a lower bound on the total width of the pentaquark state. Results for both scenarios are summarized in Table VI. In particular, for $P_{c\bar{c}}(4312)^+$, which has $J^P = (1/2)^-$ in both scenarios, comparing our lower bound estimate with the total width favors Scenario 2. Scenario 2 assigns $P_{c\bar{c}}(4440)^+$ to $J^P = (3/2)^-$ and $P_{c\bar{c}}(4457)^+$ to $J^P = (1/2)^-$. This J^P assignment agrees with the one in Refs. [28,29,43,44,58,134]. In Scenario 2, the state $P_{c\bar{c}}(4513)^+$ with $J^P = (5/2)^-$ decays only to $\eta_c(1S)$.

In Sec. IV B, we present model-independent predictions for the decay ratios of $P_{c\bar{c}}$ states into S-wave $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$ following Ref. [109]. These transitions are governed by a single transition amplitude [see Eqs. (4.8) and (4.9)], which, in principle, is calculable in lattice QCD. The transition amplitude cancels in the decay width ratios, as shown in Eq. (4.10). We find that the $P_{c\bar{c}}$ states with $J^P = (3/2)^-$ decay only to S-wave $\Lambda_c \bar{D}^*$ and that the decay to $\Lambda_c \bar{D}$ requires a D-wave transition. Also the $P_{c\bar{c}}$ states with $J^P = (5/2)^-$ can decay to both $\Lambda_c \bar{D}$ and $\Lambda_c \bar{D}^*$ only through a D-wave. Our results are summarized in Eq. (4.12) and Figs. 5 and 6. Again, we find that Scenario 2 is favored. For $P_{c\bar{c}}(4380)^+$, to which a now obsolete LHCb analysis attributed a broad width [13], we find that the decay rate into S-wave $\Lambda_c \bar{D}^*$ is of the same order as that of $P_{c\bar{c}}(4312)^+$ (see Fig. 6). Also including the decay widths to J/ψ and $\eta_c(1S)$ from Table VI, our analysis suggests that $P_{c\bar{c}}(4380)^+$ is a narrow state, in agreement with the studies in Refs. [28,29].

In Sec. V, based on Scenario 2, we report our predictions for the bottom pentaquark $P_{b\bar{b}}$ states. The adjoint baryon masses and the BO static potentials are independent of the heavy-quark mass. Unlike in the molecular model, which cannot be easily extended from the charm to the bottom sector, in the BOEFT it is enough to solve the Schrödinger equations (3.6) and (3.7) using the bottom quark mass and the BO static potentials in Eq. (2.11), with the adjoint baryon masses $\Lambda_{(1/2)^+}$ and $\Lambda_{(3/2)^+}$ fixed in the charm sector. The results for the masses of the lowest seven $P_{b\bar{b}}$ states along with their J^P quantum number assignments are shown in Fig. 7. The states are more deeply bound than in the charm sector, as reflected by the values of the binding energies. Our predictions for the semi-inclusive decays into $\Upsilon(1S)$ and $\eta_b(1S)$ are given in Table VII, while model-independent results for the relative decay rates into S-wave $\Lambda_b \bar{B}$ and $\Lambda_b \bar{B}^*$ are shown in Fig. 8. We find that for the pentaquark state $P_{b\bar{b}}(11111)^+$ with $J^P = (1/2)^-$, the ratio

of the decay rates into S-wave $\Lambda_b \bar{B}$ and $\Lambda_b \bar{B}^*$ is different from the molecular picture prediction in Ref. [41].

Finally, we emphasize that definitive, model-independent predictions for hidden-heavy pentaquarks, both $P_{c\bar{c}}$ and $P_{b\bar{b}}$, need several lattice QCD inputs, most importantly the adjoint baryon spectrum and the pentaquark BO potentials. The computation of the adjoint baryon spectrum, and in particular the adjoint baryon masses $\Lambda_{(1/2)^+}$ and $\Lambda_{(3/2)^+}$, may confirm or correct our choices for these quantities. The computation of the BO potentials would establish whether the ground-state multiplet contains ten, seven, or just four $P_{c\bar{c}}$ and $P_{b\bar{b}}$ states. Obviously, the experimental confirmation or discovery of new $P_{c\bar{c}}$ and $P_{b\bar{b}}$ states is the most awaited and needed information to constrain the different scenarios for hidden-heavy pentaquarks in the framework of the BOEFT presented in this and related works.

Notably, the LHCb Collaboration recently reported the observation of a new pentaquark state $P_{c\bar{c}}(4337)^+$ in the decay $B_s^0 \rightarrow J/\psi p \bar{p}$, seen as a structure in the $J/\psi p$ invariant-mass distribution with mass 4337_{-4}^{+7} MeV and width 29_{-18}^{+30} MeV [138]. If this state has $J^P = (1/2)^+$, which currently has the highest statistical significance of 3.7σ [138], then within the BOEFT framework it could correspond to a P-wave resonance state in the BO potential $(1/2)_g$ that dip slightly below the $\Lambda_c \bar{D}$ threshold, then cross it again and asymptotically connects to the $\Lambda_c \bar{D}$ threshold from above. Additionally, no $P_{c\bar{c}}$ states with quartet isospin $I = 3/2$ have yet been observed experimentally. In the BOEFT, the $I = 3/2$ pentaquark states are described by a single-channel Schrödinger equation with BO potential $(1/2)_g$ matching to an adjoint baryon with quantum number $k^P = (1/2)^+$ and $I = 3/2$ at short distances [85]. A comprehensive investigation of these states will be pursued in future work.

The nonstrange and strange charm pentaquark states $P_{c\bar{c}}$ and P_{cs} will be searched and studied through the hadroproduction channels $pp \rightarrow ppJ/\psi$ and $pp \rightarrow pK^+\Lambda J/\psi$ at the upcoming CBM experiment at the GSI-FAIR facility [139]. The direct observation of existing and new pentaquark states through hadroproduction channels has the potential to provide new valuable insights into the formation and internal structure of these states.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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