

Comparison of the hadronic vacuum polarization between hadronic τ -decay data and lattice QCD

Noah Allen,¹ Diogo Boito²,³ Maarten Golterman^{1,3}, Kim Maltman^{4,5}, Lucas M. Mansur⁶, and Santiago Peris³

¹*Department of Physics and Astronomy, San Francisco State University, San Francisco, California 94132, USA*

²*Instituto de Física de São Carlos, Universidade de São Paulo, 13566-590 São Carlos, SP, Brazil*

³*Department of Physics and IFAE-BIST, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Barcelona, Spain*

⁴*Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, Canada*

⁵*CSSM, University of Adelaide, Adelaide, South Australia 5005, Australia*

⁶*Institut für Kernphysik, Johannes Gutenberg-Universität Mainz, D-55128 Mainz, Germany*



(Received 3 June 2026; accepted 9 July 2026; published 11 August 2026)

We compare the isospin-one, vector-current hadronic vacuum polarization (HVP) obtained from isospin-symmetric lattice QCD with that obtained from a dispersive representation employing inclusive hadronic τ decay data corrected for isospin breaking. We consider the subtracted HVP evaluated at squared Euclidean momenta ranging from 0.5 GeV² to 12 GeV², together with the light-quark-connected HVP contribution to the muon anomalous magnetic moment and the short-, intermediate- and long-distance RBC/UKQCD window components thereof. Dispersive contributions from the region of hadronic invariant masses above the τ mass are evaluated using perturbative QCD. We also consider dispersive determinations using τ data only for contributions from two-pion, or two-pion and four-pion, modes, and evaluating the remaining contributions using exclusive-mode $e^+e^- \rightarrow$ hadrons cross sections up to about 2 GeV, lessening the dependence on perturbation theory. We find generally good agreement between lattice and τ -based results. However, a comparison of τ -based window-quantity contributions for the two four-pion modes to expectations for those contributions based on the Pais relations and e^+e^- four-pion cross sections, reveals significant differences for the $2\pi^-\pi^+\pi^0$ mode.

DOI: [10.1103/5y9v-c2h6](https://doi.org/10.1103/5y9v-c2h6)

I. INTRODUCTION

Because of the recent high-precision measurement of the anomalous magnetic moment of the muon $a_\mu = (g-2)/2$ [1,2], there has been much interest in the hadronic vacuum polarization (HVP) contribution a_μ^{HVP} , which plays a crucial role in the determination of the Standard Model (SM) expectation for a_μ . The reason is that, at present, the error on a_μ^{HVP} strongly dominates the total error on the SM expectation [3]. Moreover, there are large discrepancies between the estimate for a_μ^{HVP} from lattice QCD and dispersive estimates obtained from the R ratio measured in $e^+e^- \rightarrow$ hadrons(γ) [3], with the exception of those using results from the CMD-3 experiment for the two pion exclusive-mode contribution [4].

These discrepancies can be traced to the exclusive electroproduction mode $e^+e^- \rightarrow \pi^+\pi^-$, for which there are

significant differences in the cross section in the region around the ρ peak. Because this exclusive mode has isospin 1, data for the dispersive analysis can also be obtained from the decay $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$, if one can reliably correct for the differences in isospin breaking (IB) between the two processes. Much effort has gone into this approach; for a partial list of references, see Refs. [5–13] and other references therein.

In this paper, while also considering the question of to what extent information from hadronic τ decays can help provide additional insight, we take a slightly different approach, focusing on the facts that (i) the $I=1$ contribution to a_μ^{HVP} can be obtained from the inclusive vector isovector spectral function obtained from nonstrange vector hadronic τ decays, and (ii) in the isospin limit, the $I=1$ contribution to a_μ^{HVP} is directly related to the light-quark connected (lqc) contribution computed in isospin-symmetric lattice QCD:

$$a_\mu^{\text{HVP,lqc}} = \frac{10}{9} a_\mu^{\text{HVP,I=1}}. \quad (1.1)$$

Our primary goal is to use hadronic τ -decay data in the vector isovector channel, and to compare results based on these data

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

with results obtained from lattice QCD. The quantities we are interested in are $a_\mu^{\text{HVP,lqc}}$, the lqc parts of the window quantities defined in Ref. [14], and values for the $I = 1$ component of the subtracted vacuum polarization $\bar{\Pi}(Q^2)$ as a function of Euclidean momentum Q which contributes to the running of the electromagnetic (EM) coupling α , and for which values were obtained in lattice QCD [15,16]. Results for the lqc intermediate window are of particular interest given that the largest of the discrepancies between pre-CMD-3 dispersive and lattice results was observed for this quantity [17].

Most lattice computations of $a_\mu^{\text{HVP,lqc}}$ are carried out in the “time-momentum” representation [18], in which $a_\mu^{\text{HVP,lqc}}$ is expressed as a weighted integral over the zero-spatial-momentum current-current correlator $C^{I=1}(t)$, where t is the Euclidean time. This correlator can be obtained from a Laplace transform of the inclusive vector isovector spectral function, $\rho_V^{I=1}(s)$, where s is the hadronic invariant mass squared. For squared invariant masses $s \leq m_\tau^2$, $\rho_V^{I=1}(s)$ can be obtained from the differential distributions measured in nonstrange hadronic τ decays. For $s > m_\tau^2$, where no such experimental data are available, we determine $\rho_V^{I=1}(s)$ using perturbation theory (PT) to fifth order in the strong coupling $\alpha_s(m_\tau^2)$ [19],¹ employing a model to estimate uncertainties due to residual quark-hadron duality violations in that region.

For the τ -based input for $\rho_V^{I=1}(s)$ in the region $s \leq m_\tau^2$, we use the results of Ref. [20]. These were obtained by combining τ -decay data from ALEPH [21–23], OPAL [24] and Belle [25] for the two-pion and from ALEPH and OPAL for the four-pion contributions, with the small contributions from other (“residual”) modes obtained using exclusive-mode electroproduction data (for a detailed explanation of how this was done, see Ref. [26]), with the exception of the K^-K^0 mode, for which τ -decay data from Ref. [27] was used.² The resulting $\rho_V^{I=1}(s)$ is the most precise vector isovector inclusive spectral function available at present that does not involve data from e^+e^- to two or four pions. As we will see in this paper, in addition to the well-known discrepancies between $\tau \rightarrow$ hadrons-based and $e^+e^- \rightarrow$ hadrons-based two-pion distributions, there appear to be further discrepancies between τ and electroproduction results for the four-pion modes. In making comparisons between τ - and electroproduction-based results, we will use the 2019 results by Keshavarzi, Nomura, and Teuber (“KNT19”) for the exclusive-mode electroproduction $R(s)$ contributions produced and made available to us by the authors of Ref. [28].

The lattice results we will compare with have been computed in “isoQCD,” which is QCD without EM

corrections, and in the limit $m_u = m_d$, defined such that the pion mass is equal to the neutral pion mass. We will use the definition of isoQCD specified in Ref. [29], which is sufficiently close to the scheme used in Ref. [3] for our purposes. The inclusive τ -decay data we employ have already been corrected by the short-distance electroweak factor S_{EW} [30], but this is not the only source of IB in the τ data, and we will need to correct the τ -based results for these additional IB contributions before comparing with isoQCD lattice results. We note that, while EM and strong IB corrections have been computed or estimated by various lattice-QCD collaborations, these corrections have only been considered for the EM current correlator, and not for the charged $I = 1$, vector current case relevant for τ decays.

This paper is organized as follows. In Sec. II we provide definitions and collect the theory results we need for our analysis. In particular, in Sec. II C we give a detailed description of our treatment of IB in the τ data. In Sec. III we describe in more detail all data we employ, with emphasis on the data for $\tau \rightarrow 2\pi^- \pi^+ \pi^0 \nu_\tau$ and $\tau \rightarrow \pi^- 3\pi^0 \nu_\tau$, which we compare to Pais-relation [31] expectations obtained using $e^+e^- \rightarrow 2\pi^+ 2\pi^-$ and $e^+e^- \rightarrow \pi^+ \pi^- 2\pi^0$ cross section input from Ref. [28]. In Sec. IV we present our results for the comparison between τ -based and lattice results, both for $\bar{\Pi}(Q^2)$ at a range of values between $Q^2 = 0.5$ and 12 GeV^2 (Sec. IV A), and for $a_\mu^{\text{HVP,lqc}}$ and the short-distance, intermediate-distance and long-distance window quantities (Sec. IV B). In Sec. IV C we consider what is changed if only the two-pion data are taken from τ decays, with all other contributions taken from the KNT19 compilation [28] up to the KNT19 exclusive-mode-region endpoint, $s = s_{\text{KNT}} \equiv (1.9375 \text{ GeV})^2$, including those of the two-pion mode between the τ mass squared and the KNT19 endpoint. We also consider the case that in this approach the four-pion data are taken from τ decays instead of from the KNT19 compilation. Section V contains our conclusions. Two Appendices collect some technical details.

II. THEORY INGREDIENTS

A. Definitions

The $I = 1$ subtracted vacuum polarization $\bar{\Pi}^{I=1}(Q^2)$ is dispersively related to the corresponding isovector, vector spectral function $\rho_V^{I=1}(s)$ as

$$\bar{\Pi}^{I=1}(Q^2) = Q^2 \int_{s_{\text{th}}}^{\infty} ds \frac{\rho_V^{I=1}(s)}{s(s + Q^2)}, \quad (2.1)$$

where Q is the Euclidean momentum, and s is the invariant mass squared, with s_{th} the threshold. We decompose $\bar{\Pi}^{I=1}(Q^2)$ as

$$\bar{\Pi}^{I=1}(Q^2) = \bar{\Pi}_{\text{data}}^{I=1}(Q^2) + \bar{\Pi}_{\text{PT}}^{I=1}(Q^2) + \bar{\Pi}_{\text{DV}}^{I=1}(Q^2). \quad (2.2)$$

This breakdown is needed because data are only available up to $s = m_\tau^2$, where m_τ is the τ mass, and we thus have to

¹Only an estimate of the fifth-order term is known; see Eq. (2.10).

²A complete list of references used in estimating residual-mode contributions can be found in Ref. [26].

rely on a theory representation for larger s , composed of a PT part and a duality-violating (DV) part, as will be discussed below. Likewise, we define the Euclidean time correlation function as

$$C^{I=1}(t) = \frac{1}{2} \int_{s_{\text{th}}}^{\infty} ds \sqrt{s} e^{-\sqrt{s}t} \rho_V^{I=1}(s), \quad t > 0, \quad (2.3)$$

with corresponding decomposition

$$C^{I=1}(t) = C_{\text{data}}^{I=1}(t) + C_{\text{PT}}^{I=1}(t) + C_{\text{DV}}^{I=1}(t). \quad (2.4)$$

This is equal to exactly 9/10 times the lqc contribution to the correlation function of the electromagnetic current, computed in lattice QCD in the isospin limit, i.e.,

$$C^{\text{lqc}}(t) = \frac{10}{9} C^{I=1}(t). \quad (2.5)$$

Since we will only be concerned with vector $I = 1$ correlation functions, we will typically drop the superscript $I = 1$ and understand the quantities of interest to be vector, isovector channel ones in the rest of this paper. We will, however, keep the superscript “lqc” where needed.

The main task of this paper is the construction of $\bar{\Pi}(Q^2)$ and $C(t)$ from the vector nonstrange hadronic τ -decay data. Given data $\rho(s_i)$ for the spectral function at values s_i from $s_1 = s_{\text{th}}$ to a maximum value s_N with N the number of data points, the data-based parts of $\bar{\Pi}(Q^2)$ and $C(t)$ are

$$\begin{aligned} \bar{\Pi}_{\text{data}}(Q^2) &= Q^2 \sum_{i=2}^N (s_i - s_{i-1}) \frac{1}{2} \left(\frac{\rho(s_i)}{s_i(s_i + Q^2)} + \frac{\rho(s_{i-1})}{s_{i-1}(s_{i-1} + Q^2)} \right), \\ C_{\text{data}}(t) &= \sum_{i=2}^N (s_i - s_{i-1}) \frac{1}{2} (\sqrt{s_i} e^{-\sqrt{s_i}t} \rho(s_i) + \sqrt{s_{i-1}} e^{-\sqrt{s_{i-1}}t} \rho(s_{i-1})), \end{aligned} \quad (2.6)$$

where $\rho(s_1) = \rho(s_{\text{th}}) = 0$, and we used the trapezoidal rule for carrying out the sums.

The maximum value s_N is (approximately) equal to m_τ^2 .³ For values of $s > s_N$ we use five-loop massless PT [19] with an estimate for the order- α_s^5 coefficient:

$$\begin{aligned} \bar{\Pi}_{\text{PT}}(Q^2) &= Q^2 \int_{s_N}^{\infty} ds \frac{\rho_{\text{PT}}(s)}{s(s + Q^2)}, \\ C_{\text{PT}}(t) &= \frac{1}{2} \int_{s_N}^{\infty} ds \sqrt{s} e^{-\sqrt{s}t} \rho_{\text{PT}}(s), \end{aligned} \quad (2.7)$$

where

$$\rho_{\text{PT}}(s) = \frac{1}{4\pi^2} \left(1 + \sum_{n=1}^5 \sum_{m=1}^n c_{nm} J_m a^n(s; m_\tau^2) \right) \quad (2.8)$$

with $a(s; m_\tau^2) \equiv \alpha_s(s; m_\tau^2)/\pi$ the strong coupling evolved from m_τ^2 to s using five-loop running [32,33]

$$J_m = \pi^{m-1} \sin(\pi m/2) \quad (2.9)$$

and, for QCD with three flavors,

$$\begin{aligned} c_{11} &= 1, & c_{21} &= 1.63982, & c_{31} &= 6.37101, \\ c_{41} &= 49.0757, & c_{51} &= 283 \pm 141, \end{aligned} \quad (2.10)$$

where the values for c_{nm} for $m > 1$, which follow from the renormalization group,⁴ are given in Appendix A. The value of c_{51} has not been calculated, and instead, as indicated in Eq. (2.10) we use an estimate [35,36].⁵ The values of the c_{mn} are those for $N_f = 3$ perturbation theory. While this is appropriate for our comparison with $N_f = 2 + 1$ flavor lattice results from Ref. [15] in Sec. IV A, many of the lattice results we compare with in Sec. IV B have been obtained with $N_f = 2 + 1 + 1$ flavor lattice QCD. However, the effects of including or excluding dynamical charm effects appear to be very small [37,38], and we will ignore them in this paper.

At the τ mass, quark-hadron DVs are still visible in the experimental spectral distribution, and we thus introduce a DV component to the representation of the spectral function above $s = s_N$ to parametrize this nonperturbative effect. We use the model we have employed starting in Ref. [39], based on the theoretical considerations about the asymptotic behavior of the spectral function of Ref. [40].⁶ The explicit form is

$$\rho_{\text{DV}}(s) = e^{-\delta-\gamma s} \sin(\alpha + \beta s), \quad (2.11)$$

where α , β , γ and δ are the “DV” parameters. Both the perturbative part and the DV model will only be employed

³For the dataset employed in Secs. III and IV s_N is slightly smaller than m_τ^2 .

⁴See for example Ref. [34].

⁵For completeness, we include a brief review of the derivation of Eq. (2.8) in Appendix A.

⁶Earlier work can be found in Refs. [41–48].

above $s_N \approx m_\tau^2$. The expressions for $\bar{\Pi}_{\text{DV}}(Q^2)$ and $C_{\text{DV}}(t)$ simply follow by replacing $\rho_{\text{PT}}(s)$ by $\rho_{\text{DV}}(s)$ in Eq. (2.7).

B. Light-quark-connected window quantities

In addition to comparing the vacuum polarization $\bar{\Pi}(Q^2)$ obtained using τ data with lattice results, it is also interesting to compare the lqc part of the HVP contribution a_μ^{HVP} to the anomalous muon magnetic moment and related window quantities [14] obtained using the τ data with results obtained on the lattice in isoQCD.

The lqc contribution to a_μ^{HVP} is given by

$$a_\mu^{\text{HVP,lqc}} = \frac{10}{9} \int_0^\infty dt w(t) C(t), \quad (2.12)$$

where $C(t)$ is defined in Eq. (2.3), and $w(t)$ is the Bernecker-Meyer kernel [18], for which we use the sufficiently precise approximation of Ref. [49], normalized such that for small t

$$w(t) = \frac{1}{18} \alpha_{\text{EM}}^2 m_\mu^2 (t^4 + \mathcal{O}(m_\mu^2 t^2)). \quad (2.13)$$

Window quantities are defined by inserting “window” functions $W(t)$ into Eq. (2.12),

$$a_\mu^{\text{W,lqc}} = \frac{10}{9} \int_0^\infty dt W(t) w(t) C(t). \quad (2.14)$$

Using

$$\theta(t; t', \Delta) = \frac{1}{2} \left(1 + \tanh \left(\frac{t - t'}{\Delta} \right) \right), \quad (2.15)$$

Ref. [14] defined the short-distance (SD), intermediate (W1) and long-distance (LD) window functions as

$$\begin{aligned} W_{\text{SD}} &= 1 - \theta(t; t_0, \Delta), \\ W_{\text{W1}} &= \theta(t; t_0, \Delta) - \theta(t; t_1, \Delta), \\ W_{\text{LD}} &= \theta(t; t_1, \Delta), \end{aligned} \quad (2.16)$$

choosing

$$t_0 = 0.4 \text{ fm}, \quad t_1 = 1 \text{ fm}, \quad \Delta = 0.15 \text{ fm}. \quad (2.17)$$

In Sec. IV we will compare results obtained using the τ -based, IB-corrected spectral functions with lqc lattice results for $a_\mu^{\text{HVP,lqc}}$ and $a_\mu^{\text{W,lqc}}$, with $W = \text{SD, W1 and LD}$.

C. Isospin breaking

In order to compare with lattice results for isospin-symmetric lqc quantities, we need to adjust the τ -data-based parts of $\bar{\Pi}(Q^2)$ and $C(t)$, defined in Eq. (2.6), for IB. The theoretical part above $s = s_N$ is

already in pure isospin-symmetric QCD,⁷ but the data-based part needs to be corrected. A first observation is that, working to first order in IB, IB corrections in the $I = 1$ vacuum polarization are purely EM in origin. This follows because both currents in the lqc diagrams conserve chirality, and thus two mass insertions of m_u or m_d are needed for a nonvanishing IB contribution. If the insertions are on the same quark line, the insertion will be proportional to $m_u^2 + m_d^2$; if they are on different quark lines, they are proportional to $m_u m_d$. These combinations can be rewritten as $(m_u + m_d)^2 \pm (m_u - m_d)^2$, and thus are second order in strong IB. Another way to see this is to observe that the two $I = 1$ currents in the correlator are G -parity positive, while the strong isospin breaking (SIB) mass operator is G -parity negative. A contribution with a single SIB mass insertion thus vanishes identically. We will not explicitly use this observation below, but, for example, it excludes the need to consider the form factor $f_-(s)$ that appears in the differential hadronic τ decay rate [50] if we work to first order in IB.

For the case of the EM current correlator, IB corrections have been computed or estimated on the lattice [16,38,51–53]. However, the breakdown into individual components needed to adjust these IB corrections to the case of the τ -based representation is not available. We thus need to compare with lattice results in isoQCD, and adjust the τ -based representation for IB effects in the τ -decay data. Here we describe how we estimate these effects, including a discussion of our estimate of the associated systematic errors.

We will separate out the two-pion component of the $I = 1$ spectral function, $\rho_V(s) \equiv \rho_V^{I=1}(s)$, and, with the four-pion and other subdominant contributions expected to have only small IB corrections of order 1% or less, we will assume that IB in the remaining part of the spectrum can be neglected for the quantities considered in this paper. Our task is thus to correct the exclusive two-pion-mode contribution to $\rho_V(s)$ obtained from hadronic τ decays for IB. For this, we first need to define what is meant by pure isoQCD, also at low energies. We will take the definition from Ref. [29], which for our purposes is equivalent to taking the “WP25” scheme of Ref. [3]. In this scheme, we set the charged and neutral pion masses to $m_{\pi^{\text{iso}}} = m_{\pi^0} = 135 \text{ MeV}$, and the charged and neutral kaon masses to $m_{K^{\text{iso}}} = 494.6 \text{ MeV}$.⁸

An advantage of the current τ -based approach when it comes to estimating IB corrections is the absence of the mixed-isospin, isoscalar-isovector contributions (such as those produced by $\rho - \omega$ mixing) present in measured $e^+ e^- \rightarrow \text{hadrons}(\gamma)$ cross sections. This makes the task of estimating IB corrections in the current analysis somewhat simpler than that needed to estimate the IB corrections

⁷Above $s = s_N$ the up and down quark masses can be neglected.

⁸Kaon loop contributions barely play a role in the IB corrections to the two-pion spectral function.

required to convert the τ -based two-pion distribution to that measured in $e^+e^- \rightarrow \text{hadrons}(\gamma)$. We will base our approach on the recent study of Ref. [13], starting from the expression for the two-pion contribution to the spectral function

$$\rho_{[\tau^- \rightarrow \pi^- \pi^0 \nu_\tau(\gamma)]}(s) = S_{\text{EW}}^{\pi\pi} \beta_{\pi^- \pi^0}^3(s) |f_+(s)|^2 G_{\text{EM}}(s), \quad (2.18)$$

where $S_{\text{EW}}^{\pi\pi}$ accounts for short-distance electroweak corrections and $G_{\text{EM}}(s)$ accounts for radiative corrections in the decay. The factor $\beta_{\pi^- \pi^0}^3(s)$ is the usual phase-space factor,

$$\beta_{\pi^- \pi^0}(s) = \lambda^{1/2} \left(1, \frac{m_{\pi^-}^2}{s}, \frac{m_{\pi^0}^2}{s} \right), \quad (2.19)$$

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx),$$

and, finally, $f_+(s)$ is the weak pion form factor. To first order in IB (i.e., first order in fine-structure constant α_{EM} and the down-up quark mass difference $m_d - m_u$), Eq. (2.18) is sufficient. In the spectral functions to be used below [20], the inclusive factor S_{EW} [30] has already been removed. Therefore, for the two-pion contribution, the relative additional electroweak correction factor is equal to $S_{\text{EW}}^{\pi\pi}/S_{\text{EW}}$ (see Ref. [54] and references therein).

For the s -dependent EM correction $G_{\text{EM}}(s)$ we employ the dispersive determination of Ref. [13]. We interpolate the tabulated results⁹ using a cubic spline, as shown in Fig. 1. Since $G_{\text{EM}}(s) = 1 + \mathcal{O}(\alpha_{\text{EM}})$, we can neglect the dependence on light quark or pion mass differences in $G_{\text{EM}} - 1$, as such effects will be second order in IB.

The phase-space factor $\beta_{\pi^- \pi^0}^3(s)$ in Eq. (2.18) depends on the pion masses, and we thus include in our correction of $\rho_{[\tau^- \rightarrow \pi^- \pi^0 \nu_\tau(\gamma)]}(s)$ a factor,

$$\frac{\beta_{\pi^0 \pi^0}^3(s)}{\beta_{\pi^- \pi^0}^3(s)}, \quad (2.20)$$

in order to adjust the spectral function to that corresponding to isoQCD.

Similarly, the two-pion form factor $f_+(s)$ depends on the pion masses. Such an IB effect was not considered in Ref. [13], which was focused on a dispersive estimate of the radiative corrections, leaving IB contributions to $f_+(s)$ to future work. Here we investigate this additional source of IB using the chiral-perturbation-theory (ChPT)-inspired

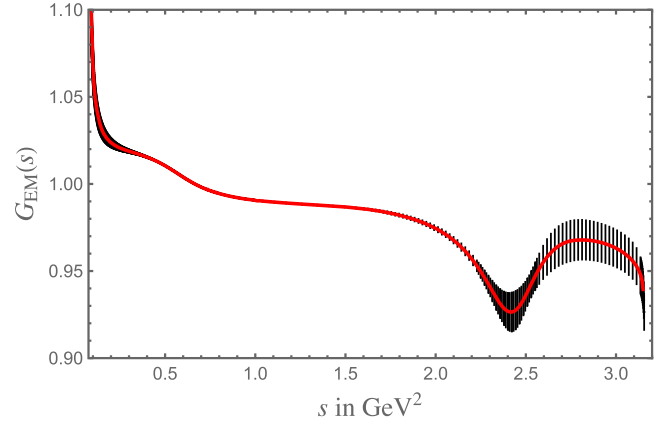


FIG. 1. The radiative correction factor $G_{\text{EM}}(s)$. The black points are the tabulated values provided in Ref. [13]; the red curve is a cubic spline interpolation of these points.

model of Refs. [50,55] for which

$$f_+^{\text{model}}(s) = \frac{m_\rho^2}{m_\rho^2 - s - im_\rho \Gamma_\rho(s)} \exp[2\tilde{H}_{\pi^- \pi^0}(s) + \tilde{H}_{K^- K^0}(s)], \quad (2.21)$$

in which

$$\Gamma_\rho(s) = \frac{m_\rho s}{96\pi f_\pi^2} \left(\beta_{\pi^- \pi^0}^3(s) \theta(s - (m_{\pi^+} + m_{\pi^0})^2) + \frac{1}{2} \beta_{K^- K^0}^3(s) \theta(s - (m_{K^-} + m_{K^0})^2) \right), \quad (2.22)$$

and

$$\tilde{H}_{ij}(s) = \frac{1}{f_\pi^2} \text{Re } h_{ij}(s, m_\rho), \quad (2.23)$$

with $h_{ij}(s, \mu)$, which encapsulates one-loop effects from ChPT, given in Appendix B for masses m_i and m_j .

In order to correct the τ -data-based parts of Eq. (2.6) to the world of isoQCD, we thus need to also multiply $\rho_{[\tau^- \rightarrow \pi^- \pi^0 \nu_\tau(\gamma)]}(s)$ by the ratio of the form factor (2.21) evaluated with both pion masses equal to the neutral pion mass and both kaon masses equal to the kaon mass of isoQCD, to that evaluated with “real-world” pion and kaon masses. The complete correction factor is thus

$$F_{\text{IB}}(s) = \frac{S_{\text{EW}}}{S_{\text{EW}}^{\pi\pi}} \frac{1}{G_{\text{EM}}(s)} \frac{\beta_{\pi^0 \pi^0}^3(s)}{\beta_{\pi^- \pi^0}^3(s)} \frac{|f_+^{\text{model}}(s)[m_{\pi^-} \rightarrow m_{\pi^0}; m_{K^-, K^0} \rightarrow m_{K^{\text{iso}}}]|^2}{|f_+^{\text{model}}(s)[\text{real world}]|^2}. \quad (2.24)$$

⁹This table can be found at <https://arxiv.org/src/2511.07507v2/anc>.

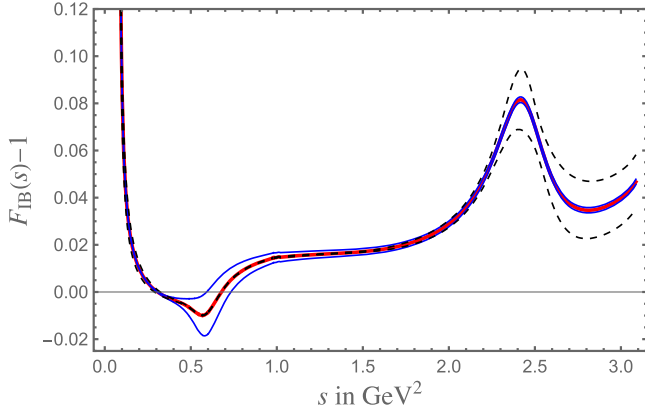


FIG. 2. The correction factor for isospin breaking, Eq. (2.24), minus 1. The central value is shown in red, with the error band from Fig. 1 in black (dashed thin curves) and the error band from varying Γ_ρ in Eq. (2.22) by a factor $1 \pm 2\alpha_{\text{eEM}}/\pi$ in blue (solid thin curves).

The factor S_{EW} in the numerator accounts for the fact that all spectral data employed in this paper have already been corrected by the factor S_{EW} , so only the relative difference between S_{EW} and $S_{\text{EW}}^{\pi\pi}$ still needs to be accounted for.

Finally, the threshold $s_{\text{th}} = 4m_{\pi^0}^2$ for $I = 1$ hadronic τ decays in isoQCD is lower than the threshold $s_{\text{th}} = (m_{\pi^-} + m_{\pi^0})^2$ in the real world, and we need to correct for this. For such small s , this correction can be reliably calculated using the phase space factor $\beta_{\pi^0\pi^0}^3(s)$ and the form factor of Eq. (2.21) evaluated with both pion masses set equal to m_{π^0} . It turns out that the thus-obtained isoQCD spectral function matches very well to the lowest data points, after they have been corrected as described above. It also turns out that this threshold correction is so small that it can be safely neglected.

The use of the model of Ref. [50] for the form factor $f_+(s)$ is the weakest step in modeling IB corrections. To explore the impact of possible changes in the form used for the s -dependent width, we have replaced the factor $m_\rho s$ by $m_\rho^2 \sqrt{s}$, following the form for Γ_ρ used in Ref. [56]; we find that this has no significant numerical impact. We have also followed the prescription of Ref. [3] for modeling unknown EM corrections to the $\rho\pi\pi$ coupling which affect ρ decays by multiplying Γ_ρ by the factor $1 \pm 2\alpha_{\text{EM}}/\pi$. This is done only in the denominator of Eq. (2.24), and not in the numerator, as the latter corresponds to the world of isoQCD in which there are no EM corrections and thus also no uncertainties due to EM effects. It turns out that this is a significant effect, as discussed in Ref. [3], and we will use the size of this effect as an estimate of the systematic error associated with using Eq. (2.24) in Sec. IV below. We show the IB correction factor, together with the errors from G_{EM} and from the prescription above, in Fig. 2. In addition, in order to account for other IB effects not covered in the discussion above (such as EM corrections to the ρ mass),

we will add (in quadrature) an additional systematic error of 50% of the total IB correction itself, on the quantities to be computed in Sec. IV. We have checked, for example, that this covers the effect of varying the ρ mass independently in the numerator and denominator of Eq. (2.24) within the range allowed by the errors on the ρ^0 and ρ^+ masses given in Ref. [57].

III. DATA

In this section, we will specify our data input. We will be using the results from Ref. [20] for the vector $I = 1$ exclusive-mode spectral function contributions, reviewed in more detail in Sec. III A for the two-pion and inclusive cases, and in Sec. III B for the four-pion case. Here we summarize other input employed in our analysis.

The main parameter values we use are

$$\begin{aligned}\alpha_s(m_\tau^2) &= 0.317(6), \\ m_\tau &= 1.77693 \text{ GeV}, \\ \alpha_{\text{EM}} &= \frac{1}{137.036}, \\ m_\mu &= 0.105658 \text{ GeV}.\end{aligned}\quad (3.1)$$

Here $\alpha_s(m_\tau^2)$ is taken from Ref. [58], evolved to the τ mass, and quantities shown without errors are so precisely known that their errors do not affect our analysis.

For IB corrections, we take the additional parameter values to be

$$\begin{aligned}m_{\pi^-} &= 0.13957039 \text{ GeV}, \\ m_{\pi^0} &= 0.1349768 \text{ GeV}, \\ m_{K^-} &= 0.493677 \text{ GeV}, \\ m_{K^0} &= 0.497611 \text{ GeV}, \\ m_{K^{\text{iso}}} &= 0.4946 \text{ GeV}, \\ m_\rho &= 0.775 \text{ GeV}, \\ f_\pi &= 0.09221 \text{ GeV}, \\ S_{\text{EW}} &= 1.0201, \\ S_{\text{EW}}^{\pi\pi} &= 1.0233,\end{aligned}\quad (3.2)$$

where all values come from Ref. [57] except $m_{K^{\text{iso}}}$ which comes from Ref. [58], S_{EW} which is taken from Ref. [30], and $S_{\text{EW}}^{\pi\pi}$, for which we follow the discussion of Ref. [54]. Strictly speaking the pion mass in isoQCD is defined to be 0.135 GeV in Ref. [58], but the difference with m_{π^0} given above is too small to have any effect. Errors on all these values are negligibly small for our purposes.

We will also need values for the parameters in Eq. (2.11). We take these from a fit of the $w = 1$ moment [20] at $s_0^{\text{min}} = 1.5747 \text{ GeV}^2$ fixing $\alpha_s(m_\tau^2) = 0.317$:

$$\begin{aligned}
\delta &= 3.3857, \\
\gamma &= 0.62575 \text{ GeV}^{-2}, \\
\alpha &= -0.20412, \\
\beta &= 3.2111 \text{ GeV}^{-2}.
\end{aligned} \tag{3.3}$$

The uncertainties on these values are not small, but they are not very important, as we will take the difference of the effect of including DVs with these parameters and omitting DVs completely as the systematic error on the DV contribution.

A. Hadronic τ -decay data

Our hadronic spectral data for the $I = 1$ vector spectral function are obtained by combining data from a range of experiments. First, for the $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ spectral distribution, we combine data from ALEPH [21–23], OPAL [24] and Belle [25], as described in detail in Ref. [20].¹⁰ We use the 2π spectral distribution to estimate IB corrections, as described in Sec. II C. Then, ALEPH and OPAL data for the decays $\tau^- \rightarrow 2\pi^- \pi^+ \pi^0 \nu_\tau$ and $\tau^- \rightarrow \pi^- 3\pi^0 \nu_\tau$ are combined into a single 4π spectral function; this is again described in detail in Ref. [20]. In Sec. III B below we will compare the mode-by-mode ALEPH and OPAL four-pion spectral distributions with those implied by four-pion electroproduction data, which are related by isospin symmetry as expressed through the Pais relations [cf. Eq. (3.4a)].

Contributions from the remaining (“residual”) modes have also all been determined from experimental data, as described in Ref. [26]; none of the contributions from these modes is obtained from Monte-Carlo simulation. This represents an improvement over the ALEPH and OPAL treatments of contributions from these modes, where, lacking such experimental input, Monte Carlo estimates had to be employed. The sum of the two-pion, four-pion and residual-mode spectral distributions obtained in Ref. [20] yields the inclusive spectral function to be used in the data parts of Eqs. (2.1) and (2.3).

We show the inclusive spectral function as well as the two-pion, combined four-pion and combined residual spectral distributions in Fig. 3. The four-pion exclusive-mode spectral function contributions will be shown in the next subsection.

B. Spectral distributions for the four-pion modes

With the recent focus on the contribution from the two-pion mode to a_μ^{HVP} [3], discrepancies between the spectral distributions for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ and $e^+ e^- \rightarrow \pi^+ \pi^-$ have been studied extensively (see for example Ref. [61]).

¹⁰The inclusion of CLEO data [59,60] was investigated in an appendix of Ref. [20], neglecting the unknown systematic correlations, and the impact of CLEO data was found to be small.

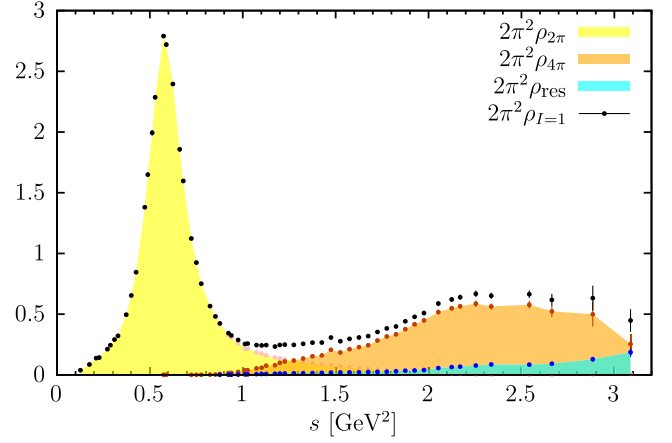


FIG. 3. Rescaled inclusive $I = 1$ spectral function $2\pi^2\rho(s)$, the two-pion mode $2\pi^2\rho_{2\pi}(s)$ (yellow area), the four-pion modes $2\pi^2\rho_{4\pi}(s)$ (orange area), and the residual modes $2\pi^2\rho_{\text{res}}(s)$ (light-blue area) from vector hadronic τ decays, from Ref. [20]. The normalization shown is such that the parton-model value of the rescaled inclusive spectral function is $1/2$.

Here we consider the two four-pion modes in more detail, comparing contributions to $\rho_V(s)$ obtained from the $\tau^- \rightarrow \pi^- 3\pi^0 \nu_\tau$ and $\tau^- \rightarrow 2\pi^- \pi^+ \pi^0 \nu_\tau$ decay distributions to those implied by the $I = 1$, EM spectral function contributions obtained from measured $e^+ e^- \rightarrow 2\pi^+ 2\pi^-$ and $e^+ e^- \rightarrow \pi^+ \pi^- 2\pi^0$ cross sections. With exact isospin symmetry, the EM expectations for the τ exclusive-mode contributions are given by the Pais relations [31,62]

$$\begin{aligned}
\rho_V[\tau, 2\pi^- \pi^+ \pi^0](s) &= \frac{1}{2} \rho_V[\text{EM}, 2\pi^- 2\pi^+](s) \\
&\quad + \rho_V[\text{EM}, \pi^- \pi^+ 2\pi^0](s),
\end{aligned} \tag{3.4a}$$

$$\rho_V[\tau, \pi^- 3\pi^0](s) = \frac{1}{2} \rho_V[\text{EM}, 2\pi^- 2\pi^+](s). \tag{3.4b}$$

The quantities on the right-hand side of these equations differ by a factor of 2 from the usual EM four-pion spectral function contributions, as the $I = 1$ part of the EM current is $1/\sqrt{2}$ times the neutral-current partner of the charged vector ud current.

It is unknown how to systematically correct these relations for IB. However, since the support of the four-pion spectral distributions has little overlap with the $\rho - \omega$ narrow-resonance region, one expects these relations to hold to better than 1% (the typical size of IB corrections) also when IB is taken into account.

A comparison of the τ -based ALEPH/OPAL four-pion results and corresponding Pais relation expectations is shown in Fig. 4. The Pais-relation expectations have been evaluated using the KNT19 four-pion exclusive-mode distribution results provided to us by the authors of Ref. [28]. The points labeled “ALEPH” and “OPAL” are

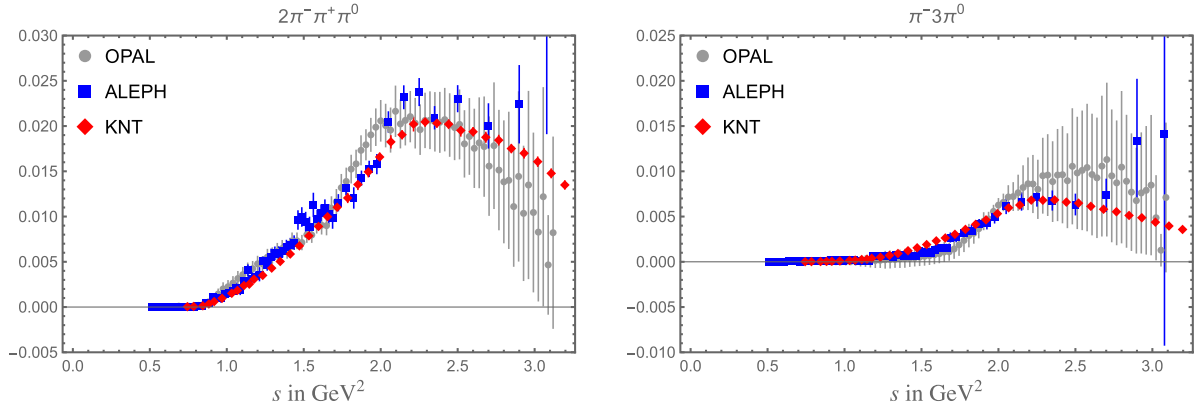


FIG. 4. $2\pi^-\pi^+\pi^0$ (left panel) and $\pi^-3\pi^0$ (right panel) ALEPH (blue) and OPAL (gray) τ -based spectral functions, compared with the e^+e^- -based combinations on the right-hand sides of Eqs. (3.4a) and (3.4b), obtained using KNT19 data (red).

the ALEPH and OPAL versions of the left-hand sides of Eq. (3.4a) and those labeled “KNT19” the KNT19-based versions of the right-hand sides. In Fig. 5 we show the relative differences between the ALEPH and OPAL four-pion spectral distributions and the KNT19-based Pais relation expectations after interpolation of the ALEPH and OPAL results to the s values of the KNT19 set. These relative differences are defined as follows. Given

a τ -based distribution $\rho_\tau^{\text{exp,mode}}(s)$ and the corresponding KNT19-based expectation, $\rho_{e^+e^-}^{\text{mode}}(s)$, where “exp” is ALEPH or OPAL, and “mode” is $2\pi^-\pi^+\pi^0$ or $\pi^-3\pi^0$, we define the ratios

$$R^{\text{exp,mode}}(s) = \frac{\rho_\tau^{\text{exp,mode}}(s) - \rho_{e^+e^-}^{\text{mode}}(s)}{\frac{1}{2}(\rho_\tau^{\text{exp,mode}}(s) + \rho_{e^+e^-}^{\text{mode}}(s))}. \quad (3.5)$$

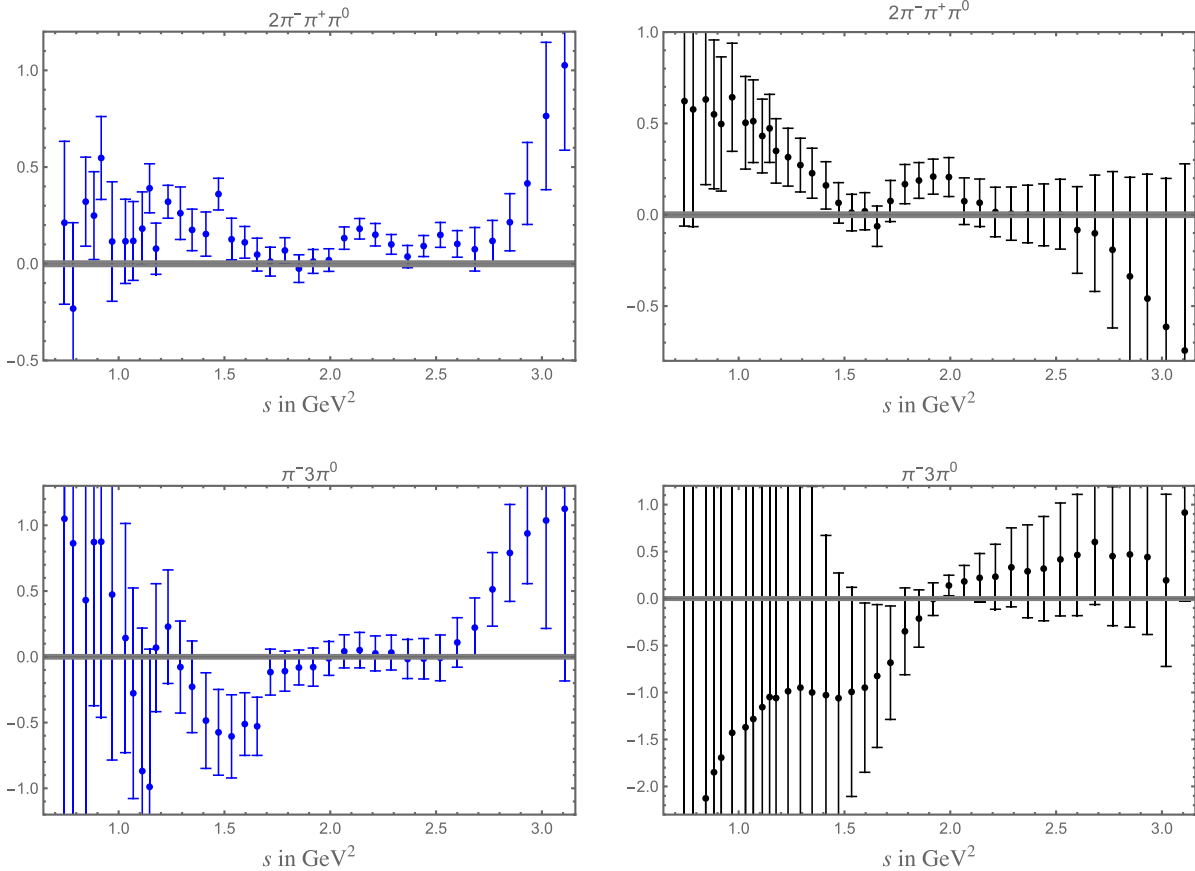


FIG. 5. The ratios $R^{\text{exp,mode}}(s)$ defined in Eq. (3.5), for experiments ALEPH (left) and OPAL (right) and modes $2\pi^-\pi^+\pi^0$ (top) and $\pi^-3\pi^0$ (bottom). The gray horizontal band shows the region of values between -0.01 and $+0.01$.

TABLE I. Integrals over the τ data for the $2\pi^-\pi^+\pi^0$ spectral distribution and over the KNT19 EM spectral distributions in the combination shown on the right-hand side of Eq. (3.4a) with SD, W1, LD and HVP weights, in units of 10^{-10} . The τ data have been linearly interpolated to the KNT19 grid. The data are integrated from 0.743906 GeV^2 to 3.01983 GeV^2 , which avoids extrapolation of the τ data.

Dataset	SD	W1	LD	HVP
ALEPH $1\pi^0$	5.37(19)	17.06(44)	5.157(92)	27.58(70)
OPAL $1\pi^0$	4.77(36)	15.96(69)	5.23(25)	26.0(1.1)
KNT19 $1\pi^0$	4.63(12)	14.80(39)	4.45(13)	23.88(63)

TABLE II. Integrals over the τ data for the $\pi^-3\pi^0$ spectral distribution and over the KNT19 EM spectral distribution on the right-hand side of Eq. (3.4b) with SD, W1, LD and HVP weights, in units of 10^{-10} . The τ data have been linearly interpolated to the KNT19 grid. The data are integrated from 0.883468 GeV^2 to 3.01983 GeV^2 , which avoids extrapolation of the τ data.

Dataset	SD	W1	LD	HVP
ALEPH $3\pi^0$	1.59(22)	4.52(49)	1.076(97)	7.18(79)
OPAL $3\pi^0$	1.61(56)	4.5(1.0)	0.98(14)	7.1(1.5)
KNT19 $3\pi^0$	1.361(18)	4.164(56)	1.104(16)	6.629(90)

If the difference between $\rho_{\tau}^{\text{exp,mode}}(s)$ and $\rho_{e^+e^-}^{\text{mode}}(s)$ is explained by IB, we expect the corresponding ratio to lie between approximately -0.01 and $+0.01$, shown in Fig. 5 by the gray horizontal bands centered at the horizontal axis.

These figures show that there are significant discrepancies between the τ -based data and e^+e^- based data, in particular for the $2\pi^-\pi^+\pi^0$ mode for which errors are smaller. This may not be immediately obvious from Fig. 5, in part because the figure does not show correlations between different points.

In Tables I [for Eq. (3.4a)] and II [for Eq. (3.4b)] we show the results of the integrals with weights defined in Eq. (2.14) for the SD, W1 and LD windows as well as the

integral with $W = 1$ in Eq. (2.14) for the four-pion spectral distributions from ALEPH and OPAL, compared with the same integrals over the KNT19 EM combinations on the right-hand side of these equations. From these tables, we conclude that there are significant differences for the $2\pi^-\pi^+\pi^0$ mode between the ALEPH and KNT19 data, ranging between 3.3σ for the SD window to 4.4σ for the LD window. For the OPAL data, which has larger errors, the differences are less pronounced, varying between 0.4σ and 2.8σ for the SD and LD windows. For the $\pi^-3\pi^0$ mode there is good agreement between the τ -based and e^+e^- -based window values. We note, however, that the τ -based window values have much larger relative errors for the $\pi^-3\pi^0$ mode than for the $2\pi^-\pi^+\pi^0$ mode. The results of Tables I and II are illustrated in Fig. 6, where we plotted the τ -based values shown in the tables normalized to the KNT19 values.

IV. COMPARISONS WITH LATTICE QCD

In this section, we compare τ -based dispersive results with results from lattice QCD.

Sections IV A and IV B contain the comparisons for $\bar{\Pi}(Q^2)$, and the window quantities and $a_{\mu}^{\text{HVP,lqc}}$, respectively, with dispersive contributions from the τ kinematic region $s \leq m_{\tau}^2$ evaluated using inclusive τ vector isovector data. The lattice results in Sec. IV A come primarily from the Mainz Collaboration [15], with a few results also available from the BMW Collaboration [16]. Recently, preliminary work on $\bar{\Pi}(Q^2)$ was also reported by the Fermilab-MILC Collaboration [63], but no quantitative results are available yet for comparison. The lattice results in Sec. IV B are taken from Ref. [3]. In Sec. IV C we perform alternate comparisons in which τ dispersive input to the data-based parts of the various dispersive integrals is restricted to either exclusive two-pion- or exclusive two-pion- and four-pion-mode contributions [20]. Contributions from all other exclusive modes, as well as from two-pion and four-pion contributions in the region $s > m_{\tau}^2$, are obtained using KNT19 data [28] up to the KNT19 endpoint s_{KNT} , with the perturbative form of the inclusive spectral

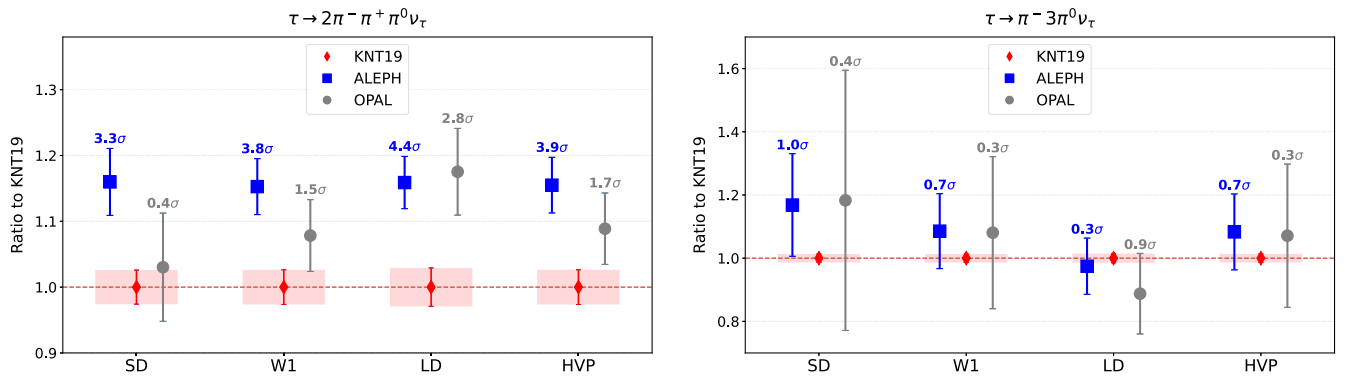


FIG. 6. Visual illustration of Tables I and II. All results normalized to KNT19 central values.

TABLE III. Comparison of τ -based $\bar{\Pi}(Q^2)$ values (which include the IB correction shown in the third column) with the lattice values corresponding to the $\bar{\Pi}_M(Q^2)$ results from Table 5 of Ref. [15]. All values in columns 2, 3 and 4 are in units of 10^{-5} . The errors in the second column are the error without IB corrections, the uncertainty on the IB correction, and the total error in square brackets; see the main text. The results of Ref. [15] have been multiplied by 2 to account for the isospin relation $\rho_{e^+e^-}^{I=1} = \frac{1}{2}\rho_V$. The pull is defined as the difference between the lattice and τ -based values, normalized by total error in the second column and the error in the fourth column added in quadrature.

Q^2 (GeV ²)	$\bar{\Pi}(Q^2)$	IB corr	$2 \times \bar{\Pi}_M(Q^2)$	Pull
0.5	4490(20)(22)[30]	-29(22)	4634(42)	2.8
1	6366(29)(31)[42]	-39(31)	6534(38)	3.0
2	8345(40)(37)[55]	-48(37)	8528(46)	2.6
3	9503(48)(40)[63]	-51(40)	9688(52)	2.3
4	10318(53)(42)[68]	-53(42)	10530(40)	2.7
5	10946(57)(43)[72]	-54(43)	11162(40)	2.6
6	11456(61)(44)[75]	-55(44)	11674(42)	2.5
7	11886(63)(44)[77]	-56(44)	12102(44)	2.4
8	12257(65)(45)[79]	-56(45)	12470(46)	2.3
9	12583(67)(45)[81]	-57(45)	12796(48)	2.3
12	13377(71)(46)[84]	-58(46)	13586(52)	2.1

function, supplemented by our estimate of possible DV corrections, used above that. In this subsection, we will also evaluate the four-pion contributions to the window quantities and $a_\mu^{\text{HVP,lqc}}$ with τ -based input on the left-hand sides and KNT19 input on the right-hand sides. This will provide additional quantitative measures of the impact of the observed discrepancies between τ -based and e^+e^- -based measurements of the four-pion spectral distributions on quantities of interest in the determination of $a_\mu^{\text{HVP,lqc}}$.

A. Subtracted vacuum polarization $\bar{\Pi}(Q^2)$

In Table III we compare results for $\bar{\Pi}(Q^2)$ from the τ -based dispersive representation (2.1) to those obtained on the lattice by the Mainz Collaboration [15], for values of Q^2 between 0.5 GeV² and 12 GeV². The latter are both continuum extrapolated and finite-volume corrected.

The errors on the data-based contributions to the dispersive results were obtained from the full covariance matrix of the input inclusive spectral function.

Perturbative contributions were evaluated using the input $\alpha_s(m_\tau^2)$ specified in Eq. (3.1), with errors equal to the quadrature sum of (i) the error produced by propagating the uncertainty in Eq. (3.1) through the computation, and (ii) a systematic error equal to the maximum variation produced by either turning off the $\mathcal{O}(\alpha_s^5)$ term in Eq. (2.8) altogether, or varying c_{51} within the bounds specified in Eq. (2.10). The shift that results from varying c_{51} is, for all Q^2 , largest when c_{51} takes on its lowest allowed value in this range.

DV contributions were evaluated using the input parameter values shown in Eq. (3.3). A 100% systematic

uncertainty, equal to the shift produced by turning off DVs altogether, is assigned to these estimates.

There are no correlations between the data, PT and DV sources of error.

The central values for $\bar{\Pi}(Q^2)$ obtained from the dispersive representation of Eqs. (2.1) and (2.2) are shown in the second column of the table. The IB corrections listed in the third column have already been applied. The first error on the $\bar{\Pi}(Q^2)$ entries in the second column is that obtained by adding in quadrature the errors on the data, PT and DV parts. The second error shown is the systematic uncertainty associated with IB corrections, described in Sec. II C above; this error is also shown in the third column. The total error is obtained by summing these two errors in quadrature, and is shown in square brackets in the second column. The fourth column shows the corresponding lattice values, obtained by multiplying the results of Ref. [15] by a factor of 2 to match our normalization convention.

Finally, the last column of the table shows the tension between the τ -based representation and the results of Ref. [15], quantified as a pull for each value of Q^2 , obtained by taking the difference and dividing by the quadrature sum of the total errors in the second and fourth columns. We see that there is a 2.1 – 3.0 σ tension, with the τ -based results lying systematically below the lattice values.

The BMW Collaboration, in Ref. [16], reported a lattice value,

$$\bar{\Pi}(Q^2 = 1 \text{ GeV}^2) = 0.06453(42), \quad (4.1)$$

where we assigned the finite-volume correction between the reference volume and infinite volume entirely to the lqc part, and multiplied by 9/10 to convert from the BMW lqc normalization to the $I = 1$ normalization. The BMW result differs from our 1 GeV² result by 1.5 σ . A value for $\bar{\Pi}(Q^2 = 10 \text{ GeV}^2) - \bar{\Pi}(Q^2 = 1 \text{ GeV}^2)$ was also reported in Ref. [16]:

$$\bar{\Pi}(Q^2 = 10 \text{ GeV}^2) - \bar{\Pi}(Q^2 = 1 \text{ GeV}^2) = 0.06545(40), \quad (4.2)$$

where we again adjusted to the $I = 1$ normalization used in this paper. Our value at 10 GeV² is $\bar{\Pi}(10 \text{ GeV}^2) = 0.12875(68)(46)[82]$, which includes the IB correction $-0.00057(46)$, in the notation of Table III. However, without a more detailed understanding of the correlations between the systematic uncertainties on our estimates of the IB corrections at different s , the strength of the correlations between the uncertainties on the associated IB corrections to $\bar{\Pi}(Q^2)$ at different Q^2 is unclear. However, one obtains the smallest uncertainty for the IB correction to the dispersive version of the difference in Eq. (4.2) if one assumes the errors on the IB corrections at 1 GeV² and

TABLE IV. Comparison of τ -based quantities $a_\mu^{X,\text{lqc}}$ for $X = \text{SD, W1, LD}$ and HVP with the WP25 average values [3]. For the two different lattice HVP averages, see Ref. [3]. All entries in units of 10^{-10} . See the main text.

Mode	SD	W1	LD	HVP
τ -data	25.61(28)	192.36(88)	406.2(2.5)	624.2(3.2)
PT + DV	22.26(21)	14.58(39)	0.669(38)	37.52(63)
EM IB	-0.13(11)	-1.40(1.1)	-2.2(1.9)	-3.7(3.1)
lqc total	47.74(37)	205.55(1.5)	404.7(3.1)	657.9(4.5)
WP25	48.123(83)	206.97(41)	406.0(4.9)	659.5(4.7) (Avg. A) 661.1(5.0) (Avg. B)

10 GeV^2 are fully (and positively) correlated. In this case we find a dispersive result that differs by 0.61σ from the lattice result in Eq. (4.2). If instead we assume the IB errors at 1 GeV^2 and 10 GeV^2 to be uncorrelated, this difference is reduced to 0.44σ .

From Table III, it can be seen that the errors obtained for the τ -based representation are competitive with those obtained on the lattice, especially for lower Q^2 values, where the data part of $\bar{\Pi}(Q^2)$ constitutes a larger part of the total. For $Q^2 = 0.5 \text{ GeV}^2$, the data part is about 91% of the total and the perturbative part, i.e., the contribution to $\bar{\Pi}(Q^2)$ above $s \approx m_\tau^2$ [cf. Eq. (2.7)] about 9%. For $Q^2 = 12 \text{ GeV}^2$ these percentages are about 68% and 32%, respectively. For all values of Q^2 , the DV part is only a fraction of a percent. In addition, the magnitude of the IB correction, and the corresponding systematic error, increase with increasing Q^2 , though the size of the IB correction relative to $\bar{\Pi}(Q^2)$ decreases with increasing Q^2 , as one would expect, varying from 0.65% at $Q^2 = 0.5 \text{ GeV}^2$ to 0.43% at $Q^2 = 12 \text{ GeV}^2$.

B. Light-quark connected window quantities

In this subsection, we compare the quantities defined in Eqs. (2.12) and (2.14) obtained from the τ -based representation for $C(t)$ of Eq. (2.3) to the average lattice values provided in Ref. [3], which is based on the work of Refs. [14,16,38,51–53,64–76]. Figures showing the individual results from these references will be shown in Sec. IV C.

The results are shown in Table IV. The row “ τ -data” shows our results, without IB corrections, and with errors obtained analogously to those in Table III. The row “WP25” shows the lattice averages from Ref. [3], with two consistent averages for $a_\mu^{\text{HVP,lqc}}$ (for explanation of lattice averaging procedures, we refer to Ref. [3]). Our results are quoted using the definition of isoQCD of Ref. [29], which differs slightly from the scheme of Ref. [3]. However, within current errors the two definitions are sufficiently close for this comparison.

Clearly, the τ -based and lattice results are consistent, with the largest relative difference, which occurs for the SD window, equal to 1.0σ . We note that for the LD window and the full HVP, the τ -based results have somewhat smaller

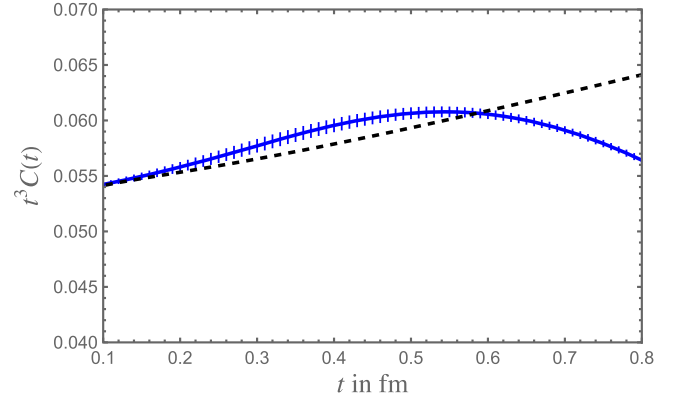


FIG. 7. Comparison of $t^3 C(t)$ between the τ -based representation (blue) and QCD perturbation theory (black, dashed) with $\alpha_s(m_\tau^2) = 0.317$.

errors, while the τ -based SD and W1 values are not as precise as the corresponding lattice results.

Since large- t contributions to the SD window are strongly suppressed by the corresponding window functions, it is interesting to compare the τ -based values for the SD window quantity with that obtained in PT only, in pure QCD with massless up and down quarks [77]. For the SD window, we find

$$a_\mu^{\text{SD,lqc,PT}} = 47.0(1.2) \times 10^{-10}. \quad (4.3)$$

This value is about 0.5σ lower than the nonperturbative, τ -based value shown in Table IV, with the perturbative error in Eq. (4.3) being much larger than that in the table. It is somewhat surprising that PT gives a good value for the SD window, as that window has a significant contribution, about 36% (cf. Table V), from the 2π mode, which includes the ρ peak region.

Finally, we compare $C(t)$ from our τ -based representation with pure PT. As $C(t) \sim 1/t^3$ for $t \rightarrow 0$, we show the combination $t^3 C(t)$ in Fig. 7, with the τ -based representation in blue and the purely perturbative version in black, in the range between 0.1 and 0.8 fm.¹¹ Small nonperturbative effects are already visible in the region between 0.2 and 0.6 fm.

C. Comparisons based on exclusive boldmathtau-decay mode contributions

In the previous subsection, the isospin-limit lqc contributions to the RBC/UKQCD windows and to the total HVP were obtained from the inclusive $I = 1 \tau \rightarrow \text{hadrons} + \nu_\tau$ spectral function, supplemented with PT and DV contributions for $s > 3.087 \text{ GeV}^2$. In a series of previous papers, using the strategy of Refs. [79,80], we obtained data-driven, results for the lqc and the

¹¹This figure is inspired by Fig. 7 of Ref. [52].

strange-plus-light-quark-disconnected ($s + \text{lqd}$) contributions to the same window quantities and to the total HVP from e^+e^- data [17,78,81], using pre-CMD-3 combined exclusive-mode KNT19 distributions provided by the authors of Ref. [28]. Results for the lqc and $s + \text{lqd}$ contribution to the total HVP [78–80] were also obtained using publicly available electroproduction results from Davier *et al.* [82]. Given the present discrepancies in the $e^+e^- \rightarrow \pi^+\pi^-$ cross section results from different experiments [3], an interesting exercise carried out in Ref. [78] was to replace the pre-CMD-3 KNT19 combination of $e^+e^- \rightarrow \pi^+\pi^-$ cross sections with the recent CMD-3 results [4], where available. We showed that, using the new CMD-3 results, the data driven lqc and $s + \text{lqd}$ contributions in the isospin limit agree well with the lattice results.

It is therefore interesting to reobtain the lqc contribution to the RBC/UKQCD windows and the total HVP by replacing the pre-CMD-3 KNT19 $e^+e^- \rightarrow \pi^+\pi^-$ combination with the τ -based 2π spectral distribution where available (i.e., up to $s = 3.087 \text{ GeV}^2$), keeping all other contributions from the KNT19 compilation used in our previous work [78] unchanged. This allows us to

investigate whether using the τ -based 2π spectral distribution leads to agreement between isospin-symmetric data-driven and lattice results, as was already observed to happen when the new CMD-3 two-pion cross section results were employed. In addition, given the tensions observed in Sec. III B between the τ -based and e^+e^- -based four-pion mode distributions, it is of interest to obtain τ -based results for the lqc windows where both the 2π and the 4π mode distributions from the KNT19 compilation are replaced by their τ -based counterparts in the energy region up to the τ mass.

Compared to our previous determinations of the isospin-symmetric lqc windows, the τ -based results have the advantage of being free from leading-order strong IB, as already discussed in Sec. II C. In comparing with the inclusive τ -based results of the previous section, one difference is in the switch point at which one transitions from using data to using PT, supplemented by DV contributions. When using KNT19 data, this switch point occurs at a higher energy, $s = s_{\text{KNT}} = 3.754 \text{ GeV}^2$. Other small differences appear in the treatment of contributions from subdominant modes. In the case of the inclusive

TABLE V. Isospin symmetric lqc contribution to the RBC/UKQCD windows and to the total HVP. Here, the 2π mode contributions are obtained from our combined $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ spectral distribution [20] (“ τ -based 2π ”), while KNT19 input [28], as in Ref. [78], is used for the contributions from all other modes. “KNT19 2π tail” refers to the 2π contribution from the region of s between 3.087 GeV^2 and s_{KNT} , while “Unamb.” and “Amb.” refer to the sums of G -parity unambiguous- and ambiguous-mode distributions, respectively. Results in units of 10^{-10} .

Mode	SD	W1	LD	HVP
τ -based 2π	17.152(66)	165.93(65)	398.5(2.5)	581.6(3.0)
KNT19 2π tail	0.0535(40)	0.0964(72)	0.00897(67)	0.159(12)
Unamb. non 2π	9.41(21)	26.77(58)	7.02(17)	43.19(95)
Amb. modes	0.75(32)	1.88(61)	0.613(91)	3.2(1.0)
PT + DV	20.28(10)	11.06(16)	0.346(11)	31.68(28)
EM IB	−0.13(11)	−1.4(1.1)	−2.2(1.9)	−3.7(3.1)
lqc total	47.51(42)	204.3(1.5)	404.3(3.1)	656.2(4.5)

TABLE VI. Isospin symmetric lqc contribution to the RBC/UKQCD windows and to the total HVP. Here, the 2π and the 4π mode contributions are obtained from our combined $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$, $\tau^- \rightarrow \pi^- 3\pi^0 \nu_\tau$, and $\tau^- \rightarrow 2\pi^- \pi^+ \pi^0 \nu_\tau$ spectral distributions [20] (“ τ -based $2\pi + 4\pi$ ”), while contributions from all other modes are obtained using combined e^+e^- KNT19 input [28], as in Ref. [78]. “KNT19 2π tail” and “KNT19 4π tail” refer to the 2π - and 4π -mode contributions from the region of s between 3.087 GeV^2 and s_{KNT} . “Unamb.” and “Amb.” refer to the sums of G -parity unambiguous- and ambiguous-mode contributions, respectively. Results in units of 10^{-10} .

Mode	SD	W1	LD	HVP
τ -based $2\pi + 4\pi$	24.49(28)	189.17(87)	405.4(2.5)	619.1(3.2)
KNT19 2π tail	0.0535(40)	0.0964(72)	0.0090(67)	0.159(12)
KNT19 4π tail	1.177(24)	2.122(44)	0.1981(42)	3.498(73)
Unamb. non $2\pi/4\pi$	1.383(98)	3.17(19)	0.578(23)	5.13(32)
Amb. modes	0.75(32)	1.88(61)	0.613(91)	3.2(1.0)
PT + DV	20.28(10)	11.06(16)	0.346(11)	31.68(28)
EM IB	−0.13(11)	−1.4(1.1)	−2.2(1.9)	−3.7(3.1)
lqc total	48.00(46)	206.1(1.5)	405.0(3.1)	659.1(4.5)

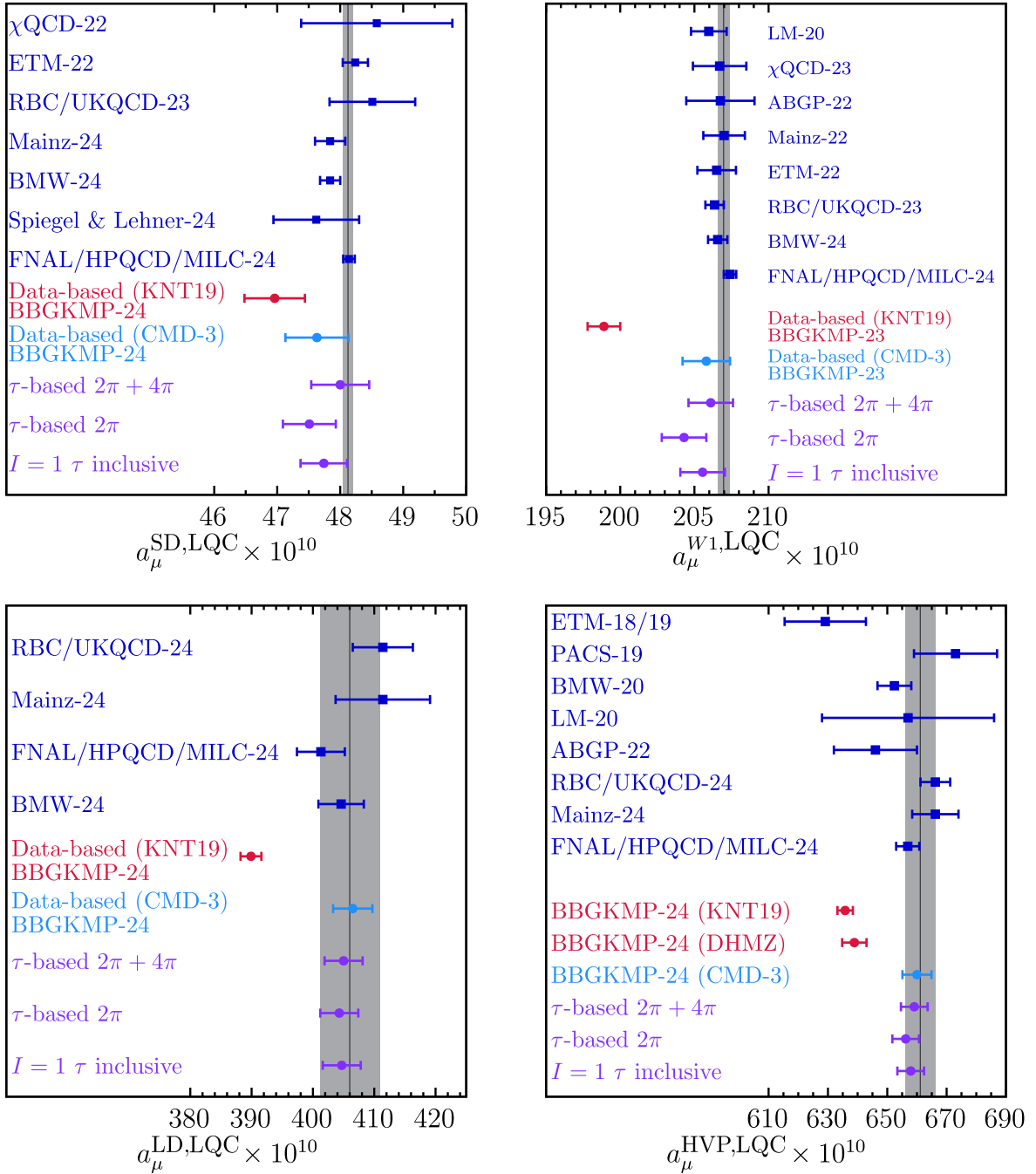


FIG. 8. Data-driven results for isospin-symmetric lqc contribution to the RBC/UKQCD windows and to the total HVP compared with lattice determinations from Refs. [16,38,51–53,64–74,76] and the WP25 lattice averages (gray bands). Results based exclusively on e^+e^- data are labeled “BBGKMP-24” [78]. The τ -based results correspond to those of Tables IV–VI.

τ -based $I = 1$ spectral function, *BABAR* results for $\tau \rightarrow K^- \bar{K}^0 \nu_\tau$ [27] were used to obtain the $K\bar{K}$ contribution. For the $K\bar{K}\pi$ mode, where both vector and axial vector contributions exist, the vector component was obtained using conserved vector current (CVC) and the $I = 1$ vector part of the $e^+e^- \rightarrow K\bar{K}\pi$ distribution produced by *BABAR*’s Dalitz-plot-based separation of the $I = 0$ and $I = 1$ components [83]. Contributions from all other modes are obtained from CVC-related e^+e^- cross sections,

resorting to a maximally conservative separation, assigning $50\% \pm 50\%$ of the total to each of $I = 1$ and $I = 0$ for G -parity ambiguous modes, such as $K\bar{K}2\pi$. In the evaluation based on e^+e^- cross sections, apart from the modes already considered in the construction of the inclusive $I = 1$ τ -based spectral function, the higher switch point means that small contributions from additional modes such as $e^+e^- \rightarrow p\bar{p}$ and $e^+e^- \rightarrow n\bar{n}$, having ambiguous isospin and present only at higher s , have to be included as well. For these

ambiguous modes the maximally conservative isospin separation is again used [79,80].

In Table V, we show the results where only the 2π KNT19 contribution is replaced by the contribution from the combined $\tau \rightarrow \pi^- \pi^0 \nu_\tau$ spectral distribution. Since the input τ spectral function ends at $s = 3.087 \text{ GeV}^2$ a small contribution from the region between 3.087 GeV^2 and the KNT19 exclusive-mode endpoint, $s_{\text{KNT}} = 3.754 \text{ GeV}^2$, obtained using KNT19 results, has to be added. The contributions from all other modes are obtained exactly as in Ref. [78] and include the contributions from all $G = +1$, G -parity unambiguous, non- 2π modes, and our assessments of the vector channel parts of all G -parity-ambiguous-mode contributions, supplemented by PT and DV contributions above the KNT19 endpoint. The EM IB correction is the same as the one shown in Table IV. Comparing these results with the WP25 averages shown in Table IV, we see that the replacement of (pre-CMD-3) KNT19 2π input with τ -based 2π input leads to a good agreement with the lattice results for all four quantities considered.

Next, we replace the KNT19 input for the 2π and two 4π modes with that obtained using our combined $\tau \rightarrow \pi^- \pi^0 \nu_\tau$, $\tau \rightarrow \pi^- 3\pi^0 \nu_\tau$, and $\tau \rightarrow 2\pi^- \pi^+ \pi^0 \nu_\tau$ spectral distributions [20]. The results of this exercise are shown in Table VI and are again compatible, within errors, with the WP25 lattice averages. The values shown in Tables V and VI share several common ingredients and are, therefore, positively correlated. This is especially true for the LD window, for which more than 98% of the total comes from the 2π mode. Small differences between the two strategies are thus potentially significant. This includes the discrepancies found in Sec. III B between the τ and e^+e^- 4π -mode contributions. We estimate the correlation between the results for the LD window in Table V and that in Table VI to be 99.66%, which makes the two results incompatible at the 3.2σ level. The other windows are compatible within 2.3σ or better. Results obtained with the inclusive $I = 1$ spectral function, Table IV, lie between those of Tables V and VI and are compatible with them, even in the presence of strong correlations (especially for the LD window).

In Fig. 8, we show the lqc results from Table VI compared with several recent lattice determinations, the WP25 averages (gray bands), as well as the previously obtained e^+e^- -based determinations of the lqc windows [78]. It is clear that the pre-CMD-3 e^+e^- -based results (in red) are in poor agreement with lattice results, especially for the W1 window, while those produced using either CMD-3 e^+e^- 2π input (in blue) or our new τ -based input (in purple) are all in good agreement with the lattice determinations of the same quantities.

V. CONCLUSION

In this paper, we considered the isospin-1 component of the HVP, which is relevant for high-precision computations

of quantities such as the muon anomalous magnetic moment and the hadronic contribution $\Delta\alpha_{\text{had}}$ to the running of the EM coupling α . We compared values obtained dispersively using $I = 1$ hadronic τ -decay data with those obtained in lattice QCD. Specifically, we considered the subtracted HVP $\bar{\Pi}(Q^2)$, as well as the lqc contributions to the SD, W1 and LD window quantities and a_μ^{HVP} itself. As shown in Fig. 8, generally good agreement was found between the $a_\mu^{\text{HVP,lqc}}$ and lqc window quantities produced using the τ -based representation and lattice results for these same quantities reported and averaged in Ref. [3]. We note that lattice errors are smaller for the SD and W1 window quantities, while the situation is reversed for the LD window and $a_\mu^{\text{HVP,lqc}}$. For $\bar{\Pi}(Q^2)$, with $0.5 \text{ GeV}^2 \leq Q^2 \leq 12 \text{ GeV}^2$, we found τ -based values which lie $2 - 3\sigma$ below those of Ref. [15], and are in better agreement with the smaller number of results reported in Ref. [16].

In order to make the comparisons, the τ -data-based quantities had to be corrected for IB, which we assumed to be dominated by those of the two-pion mode. While radiative corrections appear to be under control [13], IB also affects the form factor $f_+(s)$, for which no model-independent analysis of IB effects exists at present. For this form factor, we relied on the ChPT-inspired model of Ref. [50] with what we believe to be conservative errors, following recommendations from Ref. [3].

Finally, we compared the four-pion spectral distributions obtained using τ data with expectations for those distributions produced using electroproduction data and the isospin-based Pais relations. Particularly interesting comparisons based on these results are between the four-pion contributions to the SD, W1 and LD window quantities, where we find significant differences between the τ -based $2\pi^- \pi^+ \pi^0$ exclusive-mode contributions and their e^+e^- -based counterparts. These differences do not readily show up in our all-contribution totals for these lqc quantities, as the $2\pi^- \pi^+ \pi^0$ differences, though themselves statistically significant when all correlations are taken into account, are smaller than the errors on these totals.

ACKNOWLEDGMENTS

D. B. and M. G. would like to thank Andreas Maier, M. G. and S. P. Pere Masjuan, and M. G. Vincenzo Cirigliano and Alessandro Conigli for discussions. D. B. would like to thank the IFAE and UAB for hospitality. N. A. and M. G. are supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics, under Award No. DE-SC0013682. M. G. is also supported as a Severa Ochoa visitor to the IFAE. D. B.'s work was supported by the São Paulo Research Foundation (FAPESP) Grant No. 2021/06756-6 and by CNPq Grant No. 303553/2025-1. K. M. is supported by a grant from the Natural Sciences and Engineering Research Council of Canada. L. M. M. is supported by the Deutsche

Forschungsgemeinschaft (DFG, German Research Foundation)–Project No. 514321794 (CRC1660: Hadron and Nuclei as discovery tools). S. P. is supported by the Spanish Ministerio de Ciencia e Innovación, Grants No. PID2020–112965 GB-I00 and No. PID2023–146142NB-I00, under Grant No. CEX2024–001441-S funded by MICIU/AEI/10.13039/501100011033, and by the Departament de Recerca i Universitats from Generalitat de Catalunya to the Grup de Recerca 00649 (Codi: 2021 SGR 00649). IFAE is partially funded by the CERCA program of the Generalitat de Catalunya.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: DERIVATION OF EQ. (2.8)

We start from the (unsubtracted) vacuum polarization as a function of s ,

$$\Pi(s) = -\frac{1}{4\pi^2} \left(c_{00} + \log \frac{-s}{\mu^2} + \sum_{n=1}^{\infty} a^n(\mu^2) \sum_{m=0}^n c_{nm} \log^m \frac{-s}{\mu^2} \right), \quad (\text{A1})$$

where $a(\mu^2) = \alpha_s(\mu^2)/\pi$, and the independent coefficients,

$$\begin{aligned} c_{22} &= -1.125, \\ c_{32} &= -5.6896, & c_{33} &= 1.6875, \\ c_{42} &= -33.0914, & c_{43} &= 15.8016, & c_{44} &= -2.84766, \\ c_{52} &= -299.177, & c_{53} &= 129.578, & c_{54} &= -40.6161, & c_{55} &= 5.12578. \end{aligned} \quad (\text{A5})$$

APPENDIX B: FORMULAS

The one-loop function $h_{ij}(s, \mu)$ is given by

$$\begin{aligned} h_{ij}(s, \mu) &= \frac{1}{12s} \lambda(s, m_i^2, m_j^2) \bar{J}(s, m_1, m_2) + \frac{1}{18(4\pi)^2} (s - 3(m_i^2 + m_j^2)) \\ &\quad - \frac{1}{12} \left(\frac{2(m_i^2 + m_j^2) - s}{m_i^2 - m_j^2} (L_i(\mu) - L_j(\mu)) - 2(L_i(\mu) + L_j(\mu)) \right), \end{aligned} \quad (\text{B1})$$

where

$$L_i(\mu) = -\frac{m_i^2}{(4\pi)^2} \log \frac{m_i^2}{\mu^2}, \quad (\text{B2})$$

and

c_{n1} , are given in Eq. (2.10). The values of the remaining c_{nm} , $m > 1$, which follow from RG considerations, are given in Eq. (A5) below. The spectral function is defined as ($\epsilon > 0$)

$$\rho(s) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} (\Pi(s + i\epsilon) - \Pi(s - i\epsilon)). \quad (\text{A2})$$

Using the fact that $\Pi(s)$ is analytic in the complex plane except on the positive real axis, and setting $\mu^2 = s$, this can be written as

$$\begin{aligned} \rho(s) &= -\frac{1}{2\pi i} \oint_{|z|=s} dz \frac{d\Pi(z)}{dz} \\ &= \frac{1}{8\pi^3 i} \oint_{|x|=1} \frac{dx}{x} \left(1 + \sum_{n=1}^{\infty} a^n(s) \sum_{m=1}^n m c_{nm} \log^{m-1}(-x) \right) \\ &= \frac{1}{4\pi^2} \left(1 + \sum_{n=1}^{\infty} a^n(s) \sum_{m=1}^n c_{nm} J_m \right) \end{aligned} \quad (\text{A3})$$

using

$$\frac{m}{2\pi i} \oint_{|x|=1} \frac{dx}{x} \log^{m-1}(-x) = J_m, \quad m > 0, \quad \text{integer}, \quad (\text{A4})$$

with J_m given in Eq. (2.9).

For completeness, we list the values of c_{nm} for $n \geq m > 1$:

$$\begin{aligned}
\bar{J}(s, m_1, m_2) &= -\frac{1}{16\pi^2} \int_0^1 dx \log \frac{(1-x)m_1^2 + xm_2^2 - x(1-x)s}{(1-x)m_1^2 + xm_2^2} \\
&= \frac{1}{32\pi^2} \left(2 + \left(\frac{m_1^2 - m_2^2}{s} - \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \right) \log \frac{m_2^2}{m_1^2} + \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{s} \right. \\
&\quad \times \left(\log \frac{m_1^2 - m_2^2 - s + \lambda^{1/2}(s, m_1^2, m_2^2)}{\lambda^{1/2}(s, m_1^2, m_2^2)} + \log \frac{-m_1^2 + m_2^2 - s + \lambda^{1/2}(s, m_1^2, m_2^2)}{\lambda^{1/2}(s, m_1^2, m_2^2)} \right. \\
&\quad \left. \left. - \log \frac{m_1^2 - m_2^2 + s + \lambda^{1/2}(s, m_1^2, m_2^2)}{\lambda^{1/2}(s, m_1^2, m_2^2)} - \log \frac{-m_1^2 + m_2^2 + s + \lambda^{1/2}(s, m_1^2, m_2^2)}{\lambda^{1/2}(s, m_1^2, m_2^2)} \right) \right). \quad (\text{B3})
\end{aligned}$$

One has to be careful with combining the logarithms in this expression, as both the numerators and denominators can have complex phases. Here the logarithm is defined to have a branch cut discontinuity from $-\infty$ to 0, and the expressions above need to be evaluated at $s + i\epsilon$ with $\epsilon > 0$ in the limit $\epsilon \rightarrow 0$.

-
- [1] G. W. Bennett *et al.* (Muon g-2 Collaboration), Final report of the muon E821 anomalous magnetic moment measurement at BNL, *Phys. Rev. D* **73**, 072003 (2006).
- [2] D. P. Aguillard *et al.* (Muon g-2 Collaboration), Measurement of the positive muon anomalous magnetic moment to 127 ppb, *Phys. Rev. Lett.* **135**, 101802 (2025).
- [3] R. Aliberti, T. Aoyama, E. Balzani, A. Bashir, G. Benton, J. Bijnens, V. Biloshytskyi, T. Blum, D. Boito, M. Bruno *et al.*, The anomalous magnetic moment of the muon in the Standard Model: An update, *Phys. Rep.* **1143**, 1 (2025).
- [4] F. V. Ignatov *et al.* (CMD-3 Collaboration), Measurement of the $e^+e^- \rightarrow \pi^+\pi^-$ cross section from threshold to 1.2 GeV with the CMD-3 detector, *Phys. Rev. D* **109**, 112002 (2024).
- [5] M. Davier, A. Hoecker, A. M. Lutz, B. Malaescu, and Z. Zhang, Tensions in $e^+e^- \rightarrow \pi^+\pi^-(\gamma)$ measurements: The new landscape of data-driven hadronic vacuum polarization predictions for the muon $g-2$, *Eur. Phys. J. C* **84**, 721 (2024).
- [6] M. Davier, A. Hoecker, G. Lopez Castro, B. Malaescu, X. H. Mo, G. Toledo Sanchez, P. Wang, C. Z. Yuan, and Z. Zhang, The discrepancy between tau and e^+e^- spectral functions revisited and the consequences for the muon magnetic anomaly, *Eur. Phys. J. C* **66**, 127 (2010).
- [7] J. A. Miranda and P. Roig, New τ -based evaluation of the hadronic contribution to the vacuum polarization piece of the muon anomalous magnetic moment, *Phys. Rev. D* **102**, 114017 (2020).
- [8] P. Masjuan, A. Miranda, and P. Roig, τ data-driven evaluation of Euclidean windows for the hadronic vacuum polarization, *Phys. Lett. B* **850**, 138492 (2024).
- [9] G. L. Castro, A. Miranda, and P. Roig, Isospin breaking corrections in 2π production in tau decays and e^+e^- annihilation: Consequences for the muon g-2 and conserved vector current tests, *Phys. Rev. D* **111**, 073004 (2025).
- [10] G. Colangelo, M. Hoferichter, B. Kubis, and P. Stoffer, Isospin-breaking effects in the two-pion contribution to hadronic vacuum polarization, *J. High Energy Phys.* **10** (2022) 032.
- [11] M. Hoferichter, G. Colangelo, B. L. Hoid, B. Kubis, J. R. de Elvira, D. Schuh, D. Stamen, and P. Stoffer, Phenomenological estimate of isospin breaking in hadronic vacuum polarization, *Phys. Rev. Lett.* **131**, 161905 (2023).
- [12] M. Davier, B. Malaescu, and Z. Zhang, Data-based form factor corrections between the two-pion τ and e^+e^- spectral functions, [arXiv:2504.13789](https://arxiv.org/abs/2504.13789).
- [13] G. Colangelo, M. Cottini, M. Hoferichter, and S. Holz, Improved calculation of radiative corrections to $\tau \rightarrow \pi\pi\nu_\tau$ decays, *Phys. Rev. Lett.* **136**, 101903 (2026) [arXiv:2510.26871](https://arxiv.org/abs/2510.26871); G. Colangelo, M. Cottini, M. Hoferichter, and S. Holz, Radiative corrections to $\tau \rightarrow \pi\pi\nu_\tau$, *J. High Energy Phys.* **02** (2026) 181.
- [14] T. Blum, P. A. Boyle, V. Gülpers, T. Izubuchi, L. Jin, C. Jung, A. Jüttner, C. Lehner, A. Portelli, and J. T. Tsang (RBC Collaboration and UKQCD Collaboration), Calculation of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment, *Phys. Rev. Lett.* **121**, 022003 (2018).
- [15] A. Conigli, D. Djukanovic, G. von Hippel, S. Kuberski, H. B. Meyer, K. Miura, K. Ottnad, A. Risch, and H. Wittig, Precision lattice calculation of the hadronic contribution to the running of the electroweak gauge couplings, [arXiv:2511.01623](https://arxiv.org/abs/2511.01623).
- [16] S. Borsanyi, Z. Fodor, J. N. Guenther, C. Hoelbling, S. D. Katz, L. Lellouch, T. Lippert, K. Miura, L. Parato, K. K. Szabo *et al.*, Leading hadronic contribution to the muon magnetic moment from lattice QCD, *Nature (London)* **593**, 51 (2021).
- [17] G. Benton, D. Boito, M. Golterman, A. Keshavarzi, K. Maltman, and S. Peris, Data-driven determination of the light-quark connected component of the intermediate-window contribution to the muon $g-2$, *Phys. Rev. Lett.* **131**, 251803 (2023).
- [18] D. Bernecker and H. B. Meyer, Vector correlators in lattice QCD: Methods and applications, *Eur. Phys. J. A* **47**, 148 (2011).
- [19] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Order α_s^4 QCD corrections to Z and τ decays, *Phys. Rev. Lett.* **101**, 012002 (2008).

- [20] D. Boito, A. Eiben, M. Golterman, K. Maltman, L. M. Mansur, and S. Peris, Strong coupling from hadronic $\tau \rightarrow \pi^- \pi^0 \nu_\tau$ from Belle, *Phys. Rev. D* **111**, 074010 (2025).
- [21] R. Barate *et al.* (ALEPH Collaboration), Measurement of the spectral functions of axial-vector hadronic tau decays and determination of $\alpha_s(M_\tau^2)$, *Eur. Phys. J. C* **4**, 409 (1998).
- [22] S. Schael *et al.* (ALEPH Collaboration), Branching ratios and spectral functions of tau decays: Final ALEPH measurements and physics implications, *Phys. Rep.* **421**, 191 (2005).
- [23] M. Davier, A. Hoecker, B. Malaescu, C. Z. Yuan, and Z. Zhang, Update of the ALEPH non-strange spectral functions from hadronic τ decays, *Eur. Phys. J. C* **74**, 2803 (2014).
- [24] K. Ackerstaff *et al.* (OPAL Collaboration), Measurement of the strong coupling constant α_s and the vector and axial-vector spectral functions in hadronic tau decays, *Eur. Phys. J. C* **7**, 571 (1999).
- [25] M. Fujikawa *et al.* (Belle Collaboration), High-statistics study of the $\tau^- \rightarrow \pi^- \pi^0 \nu(\tau)$ decay, *Phys. Rev. D* **78**, 072006 (2008).
- [26] D. Boito, M. Golterman, K. Maltman, S. Peris, M. V. Rodrigues, and W. Schaaf, Strong coupling from an improved τ vector isovector spectral function, *Phys. Rev. D* **103**, 034028 (2021).
- [27] J. P. Lees *et al.* (BABAR Collaboration), Measurement of the spectra function for the $\tau^- \rightarrow K^- K_S \nu_\tau$ decay, *Phys. Rev. D* **98**, 032010 (2018).
- [28] A. Keshavarzi, D. Nomura, and T. Teubner, $g - 2$ of charged leptons, $\alpha(M_Z^2)$ and the hyperfine splitting of muonium: A new data-based analysis, *Phys. Rev. D* **101**, 014029 (2020).
- [29] Y. Aoki *et al.* (Flavour Lattice Averaging Group (FLAG)), FLAG review 2024, *Phys. Rev. D* **113**, 014508 (2026).
- [30] J. Erler, Electroweak radiative corrections to semileptonic tau decays, *Rev. Mex. Fis.* **50**, 200 (2004).
- [31] A. Pais, The many pi-meson problem, *Ann. Phys. (Amsterdam)* **9**, 548 (1960).
- [32] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Five-loop running of the QCD coupling constant, *Phys. Rev. Lett.* **118**, 082002 (2017).
- [33] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren, and A. Vogt, The five-loop beta function of Yang-Mills theory with fermions, *J. High Energy Phys.* **02** (2017) 090.
- [34] M. Jamin, Contour-improved versus fixed-order perturbation theory in hadronic τ decays, *J. High Energy Phys.* **09** (2005) 058.
- [35] M. Beneke and M. Jamin, α_s and the τ hadronic width: Fixed-order, contour-improved and higher-order perturbation theory, *J. High Energy Phys.* **09** (2008) 044.
- [36] D. Boito, P. Masjuan, and F. Olani, Higher-order QCD corrections to hadronic τ decays from Padé approximants, *J. High Energy Phys.* **08** (2018) 075.
- [37] M. Cè, A. Gérardin, G. von Hippel, H. B. Meyer, K. Miura, K. Ottnad, A. Risch, T. San José, J. Wilhelm, and H. Wittig, The hadronic running of the electromagnetic coupling and the electroweak mixing angle from lattice QCD, *J. High Energy Phys.* **08** (2022) 220.
- [38] S. Kuberski, M. Cè, G. von Hippel, H. B. Meyer, K. Ottnad, A. Risch, and H. Wittig, Hadronic vacuum polarization in the muon $g - 2$: The short-distance contribution from lattice QCD, *J. High Energy Phys.* **03** (2024) 172.
- [39] D. Boito, O. Catà, M. Golterman, M. Jamin, K. Maltman, J. Osborne, and S. Peris, A new determination of α_s from hadronic τ decays, *Phys. Rev. D* **84**, 113006 (2011).
- [40] D. Boito, I. Caprini, M. Golterman, K. Maltman, and S. Peris, Hyperasymptotics and quark-hadron duality violations in QCD, *Phys. Rev. D* **97**, 054007 (2018).
- [41] E. C. Poggio, H. R. Quinn, and S. Weinberg, Smearing the quark model, *Phys. Rev. D* **13**, 1958 (1976).
- [42] O. Catà, M. Golterman, and S. Peris, Possible duality violations in τ decay and their impact on the determination of α_s , *Phys. Rev. D* **79**, 053002 (2009).
- [43] B. Blok, M. A. Shifman, and D. X. Zhang, An illustrative example of how quark-hadron duality might work, *Phys. Rev. D* **57**, 2691 (1998); **59**, 019901(E) (1999).
- [44] I. I. Y. Bigi, M. A. Shifman, N. Uraltsev, and A. I. Vainshtein, Heavy flavor decays, OPE and duality in two-dimensional 't Hooft model, *Phys. Rev. D* **59**, 054011 (1999).
- [45] M. A. Shifman, Quark Hadron Duality (2021), pp. 1447–1494.
- [46] M. Golterman, S. Peris, B. Phily, and E. de Rafael, Testing an approximation to large- N_c QCD with a toy model, *J. High Energy Phys.* **01** (2002) 024.
- [47] O. Catà, M. Golterman, and S. Peris, Duality violations and spectral sum rules, *J. High Energy Phys.* **08** (2005) 076.
- [48] O. Catà, M. Golterman, and S. Peris, Unraveling duality violations in hadronic tau decays, *Phys. Rev. D* **77**, 093006 (2008).
- [49] M. Della Morte, A. Francis, V. Gülpers, G. Herdoíza, G. von Hippel, H. Horch, B. Jäger, H. B. Meyer, A. Nyffeler, and H. Wittig, The hadronic vacuum polarization contribution to the muon $g - 2$ from lattice QCD, *J. High Energy Phys.* **10** (2017) 020.
- [50] V. Cirigliano, G. Ecker, and H. Neufeld, Isospin violation and the magnetic moment of the muon, *Phys. Lett. B* **513**, 361 (2001).
- [51] C. Alexandrou *et al.* (Extended Twisted Mass Collaboration), Lattice calculation of the short and intermediate time-distance hadronic vacuum polarization contributions to the muon magnetic moment using twisted-mass fermions, *Phys. Rev. D* **107**, 074506 (2023).
- [52] T. Blum *et al.* (RBC Collaboration and UKQCD Collaboration), Update of Euclidean windows of the hadronic vacuum polarization, *Phys. Rev. D* **108**, 054507 (2023).
- [53] A. Bazavov *et al.* (MILC Collaboration, Fermilab Lattice Collaboration and HPQCD Collaboration), Hadronic vacuum polarization for the muon $g - 2$ from lattice QCD: Complete short and intermediate windows, *Phys. Rev. D* **111**, 094508 (2025).
- [54] V. Cirigliano, M. Hoferichter, and N. Valori, Pion β decay and $\tau \rightarrow \pi \pi \nu_\tau$ beyond leading logarithms, *arXiv:2602.11253*.
- [55] F. Guerrero and A. Pich, Effective field theory description of the pion form-factor, *Phys. Lett. B* **412**, 382 (1997).
- [56] M. Davier, A. Hoecker, G. Lopez Castro, B. Malaescu, X. H. Mo, G. Toledo Sanchez, P. Wang, C. Z. Yuan, and Z. Zhang, The discrepancy between tau and e^+e^- spectral

- functions revisited and the consequences for the muon magnetic anomaly, *Eur. Phys. J. C* **66**, 127 (2010).
- [57] S. Navas *et al.* (Particle Data Group), Review of particle physics, *Phys. Rev. D* **110**, 030001 (2024).
- [58] Y. Aoki *et al.* (Flavour Lattice Averaging Group (FLAG)), FLAG review 2024, *Phys. Rev. D* **113**, 014508 (2026).
- [59] T. Coan *et al.* (CLEO Collaboration), Measurement of α_s from tau decays, *Phys. Lett. B* **356**, 580 (1995).
- [60] S. Anderson *et al.* (CLEO Collaboration), Hadronic structure in the decay $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$, *Phys. Rev. D* **61**, 112002 (2000).
- [61] M. Davier, A. Hoecker, A. M. Lutz, B. Malaescu, and Z. Zhang, Tensions in $e^+e^- \rightarrow \pi^+\pi^-(\gamma)$ measurements: The new landscape of data-driven hadronic vacuum polarization predictions for the muon $g-2$, *Eur. Phys. J. C* **84**, 721 (2024).
- [62] R. Alemany, M. Davier, and A. Hoecker, Improved determination of the hadronic contribution to the muon ($g-2$) and to $\alpha(M_Z^2)$ using new data from hadronic tau decays, *Eur. Phys. J. C* **2**, 123 (1998).
- [63] A. del Pino, D. A. Clarke, C. DeTar, A. X. El-Khadra, E. Gámiz, S. Gottlieb, A. V. Grebe, L. Hostetler, W. I. Jay, A. S. Kronfeld *et al.*, Hadronic contributions to $\alpha(Q^2)$ and $\sin^2_W(Q^2)$ from spectral reconstruction of lattice-QCD data, *Proc. Sci. LATTICE2025* (2025) 518 [arXiv:2604.22349].
- [64] D. Giusti, F. Sanfilippo, and S. Simula, Light-quark contribution to the leading hadronic vacuum polarization term of the muon $g-2$ from twisted-mass fermions, *Phys. Rev. D* **98**, 114504 (2018).
- [65] E. Shintani and Y. Kuramashi (PACS Collaboration), Hadronic vacuum polarization contribution to the muon $g-2$ with $2+1$ flavor lattice QCD on a larger than $(10\text{ fm})^4$ lattice at the physical point, *Phys. Rev. D* **100**, 034517 (2019).
- [66] D. Giusti and S. Simula, Lepton anomalous magnetic moments in Lattice QCD + QED, *Proc. Sci. LATTICE2019* (2019) 104.
- [67] C. Lehner and A. S. Meyer, Consistency of hadronic vacuum polarization between lattice QCD and the R-ratio, *Phys. Rev. D* **101**, 074515 (2020).
- [68] G. Wang, T. Draper, K.-F. Liu, and Y.-B. Yang (chiQCD Collaboration), Muon $g-2$ with overlap valence fermions, *Phys. Rev. D* **107**, 034513 (2023).
- [69] C. Aubin, T. Blum, M. Golterman, and S. Peris, Muon anomalous magnetic moment with staggered fermions: Is the lattice spacing small enough?, *Phys. Rev. D* **106**, 054503 (2022).
- [70] M. Cè, A. Gérardin, G. von Hippel, R. J. Hudspith, S. Kuberski, H. B. Meyer, K. Miura, D. Mohler, K. Ottnad, P. Srijit *et al.*, Window observable for the hadronic vacuum polarization contribution to the muon $g-2$ from lattice QCD, *Phys. Rev. D* **106**, 114502 (2022).
- [71] A. Boccaletti, S. Borsanyi, M. Davier, Z. Fodor, F. Frech, A. Gerardin, D. Giusti, A. Y. Kotov, L. Lellouch, T. Lippert *et al.*, High precision calculation of the hadronic vacuum polarisation contribution to the muon anomaly, *Nature (London)* **653**, 373 (2026).
- [72] S. Spiegel and C. Lehner, High-precision continuum limit study of the HVP short-distance window, *Phys. Rev. D* **111**, 114517 (2025).
- [73] T. Blum *et al.* (RBC Collaboration and UKQCD Collaboration), Long-distance window of the hadronic vacuum polarization for the muon $g-2$, *Phys. Rev. Lett.* **134**, 201901 (2025).
- [74] D. Djukanovic, G. von Hippel, S. Kuberski, H. B. Meyer, N. Miller, K. Ottnad, J. Parrino, A. Risch, and H. Wittig, The hadronic vacuum polarization contribution to the muon $g-2$ at long distances, *J. High Energy Phys.* **04** (2025) 098.
- [75] C. Alexandrou *et al.* (Extended Twisted Mass Collaboration), Strange and charm quark contributions to the muon anomalous magnetic moment in lattice QCD with twisted-mass fermions, *Phys. Rev. D* **111**, 054502 (2025).
- [76] A. Bazavov *et al.* (Fermilab Lattice Collaboration, HPQCD Collaboration, and MILC Collaboration), Hadronic vacuum polarization for the muon $g-2$ from lattice QCD: Long-distance and full light-quark connected contribution, *Phys. Rev. Lett.* **135**, 011901 (2025).
- [77] K. G. Chetyrkin and A. Maier, Massless correlators of vector, scalar and tensor currents in position space at orders α_s^3 and α_s^4 : Explicit analytical results, *Nucl. Phys. B* **844**, 266 (2011).
- [78] G. Benton, D. Boito, M. Golterman, A. Keshavarzi, K. Maltman, and S. Peris, Data-driven results for light-quark connected and strange-plus-disconnected hadronic $g-2$ short- and long-distance windows, *Phys. Rev. D* **111**, 034018 (2025).
- [79] D. Boito, M. Golterman, K. Maltman, and S. Peris, Data-based determination of the isospin-limit light-quark-connected contribution to the anomalous magnetic moment of the muon, *Phys. Rev. D* **107**, 074001 (2023).
- [80] D. Boito, M. Golterman, K. Maltman, and S. Peris, Evaluation of the three-flavor quark-disconnected contribution to the muon anomalous magnetic moment from experimental data, *Phys. Rev. D* **105**, 093003 (2022).
- [81] G. Benton, D. Boito, M. Golterman, A. Keshavarzi, K. Maltman, and S. Peris, Data-driven estimates for light-quark-connected and strange-plus-disconnected hadronic $g-2$ window quantities, *Phys. Rev. D* **109**, 036010 (2024).
- [82] M. Davier, A. Hoecker, B. Malaescu, and Z. Zhang, A new evaluation of the hadronic vacuum polarisation contributions to the muon anomalous magnetic moment and to $\alpha_s(m_Z^2)$, *Eur. Phys. J. C* **80**, 241 (2020); **80**, 410(E) (2020).
- [83] B. Aubert *et al.*, (BABAR Collaboration), Measurements of $e^+e^- \rightarrow K^+K^-\eta$, $K^+K^-\pi^0$ and $K_S^0K^\pm\pi^\mp$ cross sections using initial state radiation events, *Phys. Rev. D* **77**, 092002 (2008).