

Asymptotic Weyl symmetry and its anomaly in a curved spacetime

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(Received 3 February 2026; accepted 2 March 2026; published 1 April 2026)

We explore an unusual symmetry in a field theory on a specific $(1+1)$ -dimensional curved spacetime, which has an interesting interpretation as an approximate asymptotic Weyl symmetry. Unlike the conventional Weyl symmetry, the boundary term under the variation plays a crucial role in understanding for its anomaly. After converting a two-dimensional field theory on curved spacetime to an inhomogeneous field theory, we obtain the vacuum expectation value of the energy-momentum tensor. Then, we show the existence of an Unruh-like effect in the bubble wall expansion at the zero temperature.

DOI: [10.1103/PhysRevD.113.086001](https://doi.org/10.1103/PhysRevD.113.086001)

I. INTRODUCTION

A classical field theory whose action is invariant under local Weyl transformations exhibits a vanishing trace of the energy-momentum tensor, $T^\mu_\mu = 0$. More precisely, under a Weyl transformation $g_{\mu\nu}(x) \rightarrow e^{2\sigma(x)}g_{\mu\nu}(x)$, given by

$$\delta_\sigma S = \int d^d x \sqrt{-g} \sigma(x) T^\mu_\mu, \quad x \equiv (t, \vec{x}), \quad (1.1)$$

Weyl invariance at the classical level requires $T^\mu_\mu = 0$. At the quantum level, this symmetry is generically broken, leading to the well-known trace anomaly, $\langle T^\mu_\mu \rangle \neq 0$ [1].

On the other hand, the study of symmetries restricted to the asymptotic boundary of spacetime has provided profound insights. In four-dimensional general relativity, the asymptotic symmetry group on the asymptotically flat spacetime is not the Poincaré group but the infinite-dimensional Bondi–van der Burg–Metzner–Sachs group [2–4]. In the context of 2D conformal field theory, the infinite-dimensional Virasoro symmetry is realized as the asymptotic symmetry group of the 3D anti-de Sitter (AdS) bulk [5]. Especially, the asymptotic symmetry analysis in bulk AdS₃ is regarded as the precursor of the modern AdS/CFT correspondence. In these canonical examples, the asymptotic symmetry is understood as an exact remnant or

subgroup of classical bulk symmetry. The quantum anomaly is then interpreted as a modification of this symmetry algebra, such as a central extension.

We investigate the quantum effect of a $(1+1)$ -dimensional scalar field theory, which enjoys an “asymptotic symmetry” but is *not* Weyl-invariant with nonminimal coupling to gravity, described by the action

$$S_{\text{FTCS}} = \int_{\mathcal{M}} d^2 x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m_0^2 \phi^2 - \frac{1}{2} \xi \mathcal{R} \phi^2 \right), \quad (1.2)$$

where m_0 and ξ are constants, and $\mathcal{R}$ denotes the curvature scalar. In our previous study [6], we showed that the asymptotic behavior of $\langle T^\mu_\mu \rangle$ coincides with the well-known universal trace-anomaly form of the conformal scalar field theory. This interesting coincidence suggests the possible existence of a hidden symmetry that has so far remained unexplained. While several works have investigated the generalization of the trace anomaly to non-conformal theories (see, for example, [7]), the present study offers an alternative viewpoint toward understanding this phenomenon, based on the presence of an asymptotic Weyl symmetry (AWS), which will be explained in later sections. The main novelty of the present study is to identify and formulate a new type of “asymptotic symmetry.”

We established in [8,9] that the field theory on curved spacetime (FTCS) (1.2) on a conformally flat metric $ds^2 = e^{2\omega(x)} \eta_{\mu\nu} dx^\mu dx^\nu$ is classically converted to an inhomogeneous field theory (IFT) on $(1+1)$ -dimensional flat spacetime

¹Here, “asymptotic symmetry” is not a symmetry in the usual sense, namely one under which the action is invariant.

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$$S_{\text{IFT}} = \int d^2x \left(-\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m_{\text{eff}}^2(x) \phi^2 \right). \quad (1.3)$$

In this framework, geometric information is encoded into the position-dependent effective mass

$$m_{\text{eff}}^2(x) = e^{2\omega(x)}(m_0^2 + \xi \mathcal{R}). \quad (1.4)$$

We suggested to extend this conversion to the quantum level, allowing for a consistent quantization, e.g., using Hadamard regularization techniques from the FTCS framework to compute correlation functions in the IFT [9].

Since the IFT action (1.3) is defined on a fixed flat background, the AWS cannot act on the metric $\eta_{\mu\nu}$. Moreover, as the scalar field ϕ remains invariant, the symmetry must instead be realized as an internal symmetry acting on the inhomogeneous parameter of the IFT—the effective mass $m_{\text{eff}}^2(x)$. In this sense, the AWS is regarded as a hidden symmetry from the IFT perspective.

This paper is organized as follows. In Sec. II, we review the conversion between FTCS and IFT by extending our previous work to a nonsupersymmetric background. Section III introduces a new asymptotic symmetry, which we refer to as the AWS, and examines its anomalous features. In Sec. IV, we compute the renormalized Hadamard two-point function and the vacuum expectation value (VEV) of the energy–momentum tensor, and we discuss the physical implications of these results. Finally, Sec. V presents concluding remarks and possible future directions.

II. FTCS-IFT CONVERSION BEYOND SUSY BACKGROUNDS

We have investigated the classical conversion between FTCS (1.2) and IFT (1.3) in $(1+1)$ dimensions, and applied this relation to translate the quantum effects of the FTCS into those of the IFT [6,8,9]. In these works, our analysis mainly relied on a supersymmetric background geometry. In the present paper, we instead employ a nonsupersymmetric geometry that contains a single parameter interpolating the previously used supersymmetric geometry and the Minkowski spacetime. Our goal is to focus on the AWS and its associated anomaly, which were overlooked in our previous studies, and to interpret those quantum effects in IFT.

A. Review on conversion between FTCS and IFT

Since the conversion between FTCS and IFT constitutes a key element of our framework, we begin by briefly revisiting its conceptual basis. A common objection to our approach is that two actions with different underlying symmetry groups cannot describe the same physical system. While this argument is reasonable in a generic context, it does not strictly apply to the present case. Our

starting point in converting the FTCS (in a specific conformal gauge) to the IFT is the same classical equation of motion and the same initial conditions for the scalar field ϕ . Consequently, the classical solutions of ϕ are identical in both formulations, even though the actions differ. The discrepancy in their symmetry structures should thus be viewed not as a physical inconsistency, but rather as reflecting distinct interpretations—or domains of applicability—of the same underlying solutions.

Our central claim in previous studies is that, although not fully consistent quantization scheme for the IFT is yet known, the quantization of the IFT can be partially understood through its conversion from the FTCS.² This viewpoint is motivated by the field quantization formalism on the solution space in curved spacetime [10]. Based on the same solution space, we argued that the quantization of the IFT can, at least in part, be inferred from its correspondence with the FTCS.

In the specific scalar field example, the effective mass m_{eff}^2 in the IFT can be decomposed into two contributions m_0^2 and $\xi \mathcal{R}$ in (1.4) from the FTCS perspective. From the viewpoint of the IFT, this division may appear artificial; nevertheless, it serves as a practical prescription for the IFT quantization scheme, rather than as a unique or unambiguous construction.

In particular, one of our proposals in this framework is that the energy-momentum tensor in the IFT can be obtained from that in the FTCS by adopting the same algebraic vacuum state in both formulations, thereby defining a common Fock space. We continue to employ this prescription in the present work and interpret our results accordingly. Although different decompositions of m_{eff}^2 may correspond to distinct quantization schemes, it is plausible that certain physical consequences remain universal and independent of such choices. More concretely, while the algebraic vacuum³ in the IFT and FTCS can be chosen to coincide by construction, this does not necessarily guarantee its physical appropriateness in the IFT framework. However, in the absence of any clear symmetry principle, constraint, or physical argument forbidding this choice, it is reasonable to adopt this identification and explore its physical implications. A complete resolution of this issue lies beyond the scope of the present work and will require further investigation.

²We do not claim that the full quantum IFT is equivalent to the full quantum FTCS, even for the scalar field, since the conversion is performed within a specific conformal gauge. While the FTCS may appear more restricted due to diffeomorphism constraints, the conversion nevertheless captures certain quantum aspects of the IFT. In what follows, we focus on the intersection regime in which quantum features of the IFT can be meaningfully interpreted through this correspondence.

³These states possess considerable freedom and must ultimately be restricted by physical criteria.

In the following, we introduce a one-parameter family of background metrics, specified by α , which continuously interpolates between the Minkowski spacetime ($\alpha = 1$) and the supersymmetric background studied in our previous works [6,9] at $\alpha = 0$. The nonsupersymmetric cases ($0 < \alpha < 1$) are free from the curvature singularity that appears in the left asymptotic limit of $x \rightarrow -\infty$ in the supersymmetric background. Upon converting the FTCS to the IFT description, this geometric modification manifests as a distinctive feature of the scalar IFT—namely, the emergence of two different asymptotic mass parameters, m_ℓ^2 and m_r^2 of a single scalar IFT. Here, m_ℓ denotes the mass in the left asymptotic limit, while m_r represents the mass in the right asymptotic limit. The supersymmetric background in the FTCS corresponds to the special case where $m_\ell^2 = 0$ in the IFT framework.

B. A static background

In [6,9], we studied the FTCS model (1.2) describing a free massive scalar field propagating on a two-dimensional curved background described by the conformally flat metric depending only on x :

$$ds^2 = e^{2\omega(x)}(-dt^2 + dx^2). \quad (2.1)$$

In this case, the curvature scalar can be written as

$$\mathcal{R} = -2e^{-2\omega(x)}\omega''(x). \quad (2.2)$$

Varying the action (1.2) with respect to the scalar field and the metric yields the field equation

$$(-\square + m_0^2 + \xi\mathcal{R})\phi = 0, \quad \square = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu), \quad (2.3)$$

where $\xi > 0$ and $m_0^2 > 0$, and the classical energy-momentum tensor

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}[(\nabla\phi)^2 + m_0^2\phi^2] + \xi(-\nabla_\mu\nabla_\nu + g_{\mu\nu}\nabla^2)\phi^2, \quad (2.4)$$

respectively. In the specific conformal gauge (2.1), the equations of motion of the scalar field becomes

$$(-\eta^{\mu\nu}\partial_\mu\partial_\nu + m_{\text{eff}}^2(x))\phi = 0, \quad (2.5)$$

where $m_{\text{eff}}^2(x)$ has same expression in (1.4). The same equation of motion can also be read from the IFT action (1.3).

C. Nonsupersymmetric spacetime background

Our static metric, which is a one-parameter extension of the supersymmetric metric with choice of $b > 0$, is taken by

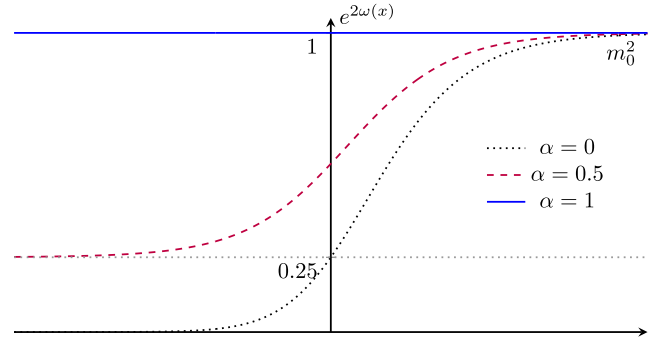


FIG. 1. Plot of $e^{2\omega(x)}$ for $b = 1$ and $\alpha = 0, 0.5, 1$.

$$e^{\omega(x)} = \frac{1 + \alpha e^{-bx}}{1 + e^{-bx}}, \quad 0 \leq \alpha \leq 1. \quad (2.6)$$

The FTCS action (1.2) cannot be supersymmetrized on the background metric (2.6) for $0 < \alpha < 1$. For the details see Appendix B. It would be plausible to assume that the physical quantities would be a smooth or continuous with respect to the parameter α except the limit⁴ $\alpha \rightarrow 0$. In Fig. 1, we plot the conformal factor $e^{2\omega(x)}$ for three representative cases: the supersymmetric background ($\alpha = 0$), an intermediate geometry ($\alpha = 0.5$), and the flat spacetime limit ($\alpha = 1$).

We now summarize the global and qualitative features of $m_{\text{eff}}^2(x)$ arising from the background metric (2.6).

Global behavior: The dimensionless effective mass square profile,

$$\frac{m_{\text{eff}}^2(x)}{m_0^2} = \left(\frac{1 + \alpha e^{-bx}}{1 + e^{-bx}} \right)^2 + \frac{2\xi b^2}{m_0^2} \left(\frac{e^{-bx}}{(1 + e^{-bx})^2} - \frac{\alpha e^{-bx}}{(1 + \alpha e^{-bx})^2} \right), \quad (2.7)$$

interpolates smoothly between two asymptotic limits,

$$\frac{m_{\text{eff}}^2(x)}{m_0^2} \rightarrow \begin{cases} 1 & x \rightarrow +\infty \\ \alpha^2 & x \rightarrow -\infty \end{cases}. \quad (2.8)$$

This position-dependent mass profile exhibits a single smooth transition between the two asymptotic constant mass values, $m_\ell^2 = m_0\alpha^2$ in the left and $m_r^2 = m_0^2$ in the right. From (2.7), one can also see that global bounds of m_{eff}^2 exist for all x . In particular, a sufficient condition for strict positivity of the inhomogeneous mass parameter, $m_{\text{eff}}^2(x)$, is

$$\alpha^2 > \frac{\xi b^2}{2m_0^2}. \quad (2.9)$$

⁴Though the limit $\alpha \rightarrow 0$ may be continuous or smooth in the IFT context, we allow its noncontinuity as a possibility.

This inequality guarantees that the profile remains positive definite throughout the entire spatial domain. Though $m_{\text{eff}}^2(x) < 0$ for some range of x may be an interesting possibility of IFT, we will focus on $\alpha^2 > \frac{\xi b^2}{2m_0^2}$ in the whole range of x in the following.

To figure out the properties of $m_{\text{eff}}^2(x)$, we differentiate (2.7) with respect to x , leading to

$$\begin{aligned} \frac{d}{dx} \left(\frac{m_{\text{eff}}^2}{m_0^2} \right) &= 2be^{-bx} \left[\frac{(1-\alpha)(1+\alpha e^{-bx})}{(1+e^{-bx})^3} \right. \\ &\quad \left. - \frac{\xi b^2}{m_0^2} \left(\frac{1-e^{-bx}}{(1+e^{-bx})^3} - \frac{\alpha(1-\alpha e^{-bx})}{(1+\alpha e^{-bx})^3} \right) \right]. \end{aligned} \quad (2.10)$$

For $0 < \alpha < 1$, the first term in the bracket is always positive, whereas the second term, proportional to ξ , can change its sign within the intermediate region. For small $\xi b^2/m_0^2$ the profile increases monotonically from α^2 to 1. As $\xi b^2/m_0^2$ grows, a shallow minimum and later a weak maximum emerge, reflecting local compensation between the two contributions in (2.7).

Qualitative behavior: For $\xi > 0$ and $m_0^2 > 0$, the number of extrema is controlled by the curvature-induced contribution proportional to $2\xi b^2/m_0^2$ in (2.7):

$$\begin{cases} \frac{\xi b^2}{m_0^2} \leq \alpha^2: & \text{monotonic increase (no extrema),} \\ \alpha^2 < \frac{\xi b^2}{m_0^2} \leq 1: & \text{one minimum in the intermediate region,} \\ \frac{\xi b^2}{m_0^2} > 1: & \text{one minimum and one maximum.} \end{cases} \quad (2.11)$$

Hence, the coupling $\xi b^2/m_0^2$ controls whether additional extrema emerge beyond the smooth interpolation between the two asymptotic masses. These features are illustrated in Fig. 2.

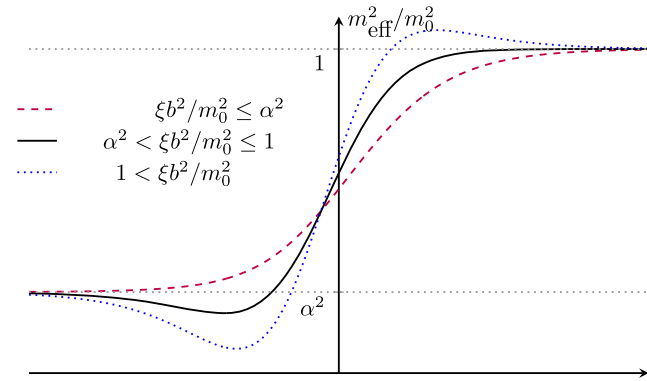


FIG. 2. Plot of m_{eff}^2/m_0^2 for $\alpha = 0.5$ and $\xi b^2/m_0^2 = 0.1, 1, 1.9$.

For the following analysis, we set

$$\frac{\xi b^2}{m_0^2} = 1, \quad (2.12)$$

which provides asymptotic behavior of $m_{\text{eff}}^2(x)$ allowing the AWS. For the overall positivity of $m_{\text{eff}}^2(x)$, as we see in (2.9), we focus on a representative case given by $\frac{1}{2} < \alpha^2 < 1$. See Fig. 3.

III. ASYMPTOTIC WEYL SYMMETRY

The FTCS action (1.2) does not enjoy conventional Weyl symmetry, as is already clear from the nonvanishing trace of the classical energy-momentum tensor,

$$T^\mu{}_\mu = (\xi \square - m_0^2) \phi^2, \quad (3.1)$$

obtained from (2.4). Nevertheless, we shall show that, for a suitable choice of parameters, the theory admits an *asymptotic* version of Weyl symmetry in the right asymptotic region $x \rightarrow +\infty$.

As already mentioned in Sec. II, we focus on an asymptotically flat background in which the effective mass $m_{\text{eff}}^2(x)$ develops a nearly flat profile in the right asymptotic region $x \rightarrow +\infty$ (see Fig. 2). In the conformally flat metric (2.1), this “nearly flat” behavior is encoded in the asymptotic expansion

$$e^{2\omega(x)} = 1 + g_1 e^{-bx} + \mathcal{O}(e^{-2bx}). \quad (3.2)$$

In our static metric in (2.6), the coefficient g_1 is given by $g_1 = 2(\alpha - 1)$.

A. Classical AWS

Let us consider a Weyl rescaling of the form

$$\begin{aligned} g_{\mu\nu} &\longrightarrow \tilde{g}_{\mu\nu} = e^{2\sigma(t,x)} g_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}, \\ e^{2\sigma(t,x)} &= 1 + a_1 e^{-bx} + \mathcal{O}(e^{-2bx}), \end{aligned} \quad (3.3)$$

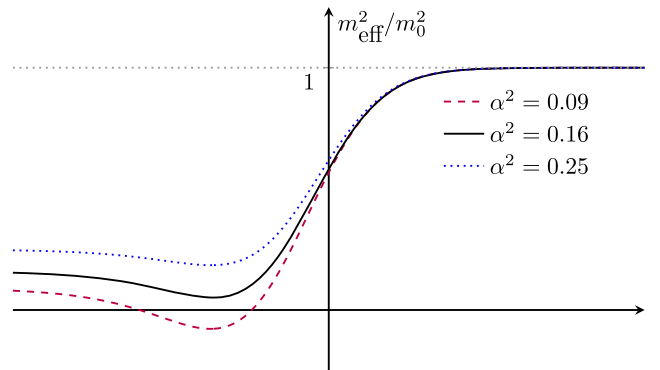


FIG. 3. Plot of m_{eff}^2/m_0^2 for $\alpha = 0.5$ and $\xi b^2/m_0^2 = 0.1, 1, 1.9$.

where a_1 is taken as a constant for simplicity. This induces the variations

$$\begin{aligned}\delta\sqrt{-g} &= a_1 e^{-bx} + \mathcal{O}(e^{-2bx}), \\ \delta(\sqrt{-g}\mathcal{R}) &= -b^2 a_1 e^{-bx} + \mathcal{O}(e^{-2bx}),\end{aligned}$$

and hence the variation of the FTCS action takes the form

$$\delta S_{\text{FTCS}} = -\frac{1}{2} \int_{\mathcal{M}} d^2x [a_1 e^{-bx} (m_0^2 - \xi b^2) \phi^2 + \mathcal{O}(e^{-2bx})]. \quad (3.4)$$

If the condition (2.12) is satisfied, the action is invariant up to the order e^{-bx} .

The above discussion can be recast in terms of the energy-momentum tensor and an associated boundary term. Varying the action with respect to the background metric yields

$$\delta_g S_{\text{FTCS}} = -\frac{1}{2} \int_{\mathcal{M}} d^2x \sqrt{-g} [T_{\mu\nu} \delta g^{\mu\nu} + 2\xi B], \quad (3.5)$$

where the boundary term B is given by

$$\begin{aligned}B &\equiv \nabla_\alpha (\phi^2 \nabla^\alpha (g_{\mu\nu} \delta g^{\mu\nu}) - g_{\mu\nu} \delta g^{\mu\nu} \nabla^\alpha \phi^2) \\ &\quad - \nabla_\mu (\phi^2 \nabla_\nu \delta g^{\mu\nu} - \delta g^{\mu\nu} \nabla_\nu \phi^2).\end{aligned} \quad (3.6)$$

For the AWS transformation (3.3), with $\delta g_{\mu\nu} = a_1 e^{-bx} g_{\mu\nu} + \mathcal{O}(e^{-2bx})$, this reduces to

$$B = (\Box \phi^2 - b^2 \phi^2) a_1 e^{-bx} + \mathcal{O}(e^{-2bx}), \quad (3.7)$$

and then, the variation of the action (3.5) becomes

$$\begin{aligned}\delta_{\text{AWS}} S_{\text{FTCS}} &= -\frac{1}{2} \int_{\mathcal{M}} d^2x \sqrt{-g} \\ &\quad \times [-a_1 e^{-bx} (T^\mu_\mu - \xi (\Box \phi^2 - b^2 \phi^2)) \\ &\quad + \mathcal{O}(e^{-2bx})].\end{aligned} \quad (3.8)$$

Using (3.1), we can see that this expression is nothing but (3.4).

In a Weyl-invariant theory the boundary term does not contribute and T^μ_μ vanishes at the off-shell level. In our nonconformal case, by contrast, the boundary term is essential: the classical trace (3.1) does not vanish even at the on-shell level. However, using (2.12), we can check that the variation of the action given in (3.4) or equivalently (3.8) vanishes up to the order e^{-bx} .

B. AWS and anomaly

A natural question is whether the AWS holds at the quantum level. In this subsection, we argue that the renormalization procedure breaks AWS, inducing a

trace-anomalylike expression to the VEV of the energy-momentum tensor up to $\mathcal{O}(e^{-bx})$.

We perform the analysis in the Euclidean path-integral formalism. The partition function and the generating functional are defined by

$$Z[g] \equiv \int \mathcal{D}\phi e^{-S_{\text{FTCS}}[\phi;g]}, \quad W[g_E] \equiv -\ln Z[g]. \quad (3.9)$$

Using (3.5), we have a generic variation of the generating functional with respect to the metric as

$$\delta_g W = -\frac{1}{2} \int d^2x \sqrt{g} \langle T_{\mu\nu} \delta g^{\mu\nu} + \xi B \rangle_{\text{ren}}. \quad (3.10)$$

If the path-integral measure were invariant under AWS, the right-hand side of (3.10) would vanish up to order e^{-bx} when $\delta g_{\mu\nu}$ is chosen as in (3.3). However, it is well known that the measure is not invariant under a generic Weyl rescaling, and in two dimensions the corresponding anomaly takes the form

$$\delta_\sigma W = \frac{1}{24\pi} \int d^2x \sqrt{g} \mathcal{R} \sigma, \quad \delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}. \quad (3.11)$$

Applying this to the AWS transformation with $2\sigma = a_1 e^{-bx} + \mathcal{O}(e^{-2bx})$, we obtain

$$\langle T^\mu_\mu - \xi (\Box \phi^2 - b^2 \phi^2) \rangle_{\text{ren}} = \frac{1}{24\pi} \mathcal{R} + \mathcal{O}(e^{-2bx}), \quad (3.12)$$

which can be rewritten as the renormalized trace of the energy-momentum tensor,

$$\langle T^\mu_\mu \rangle_{\text{ren}} = \langle \xi (\Box \phi^2 - b^2 \phi^2) \rangle_{\text{ren}} + \frac{1}{24\pi} \mathcal{R} + \mathcal{O}(e^{-2bx}). \quad (3.13)$$

For the tuned parameter choice (2.12) the first term on the right-hand side of (3.13) corresponds to the classical trace (3.1). In the vacuum state that we shall specify in Sec. IV this contribution is parametrically suppressed in the right asymptotic region, so that the renormalized trace is effectively governed by the curvature term alone. Thus, although the model is not conformal, the asymptotic Weyl symmetry ensures that the leading quantum trace takes the same universal form as in the conformal case. We refer to this effect as the *asymptotic Weyl anomaly*: the quantum breaking of an approximate symmetry that holds only asymptotically. In Sec. IV, we shall verify this picture explicitly by computing the renormalized energy-momentum tensor via the Hadamard renormalization method for specifically chosen vacuum.

C. Interpretation in the IFT description

Finally, let us briefly interpret the AWS condition in the IFT framework introduced in Secs. I and II. There we showed that the FTCS model on the conformally flat metric (2.1) is classically converted to an inhomogeneous scalar field theory on flat spacetime, with action (1.3) and position-dependent mass (1.4). In terms of the IFT variables, the effective mass admits the asymptotic expansion

$$m_{\text{eff}}^2(x) = m_0^2 + g_1(m_0^2 - \xi b^2)e^{-bx} + \mathcal{O}(e^{-2bx}). \quad (3.14)$$

Hence the $\mathcal{O}(e^{-bx})$ contribution to $m_{\text{eff}}^2(x)$ vanishes precisely when the AWS condition (2.12) is satisfied. From the IFT perspective, this means that the parameter tuning that allows AWS in the FTCS description corresponds to a highly flattened mass profile in the right asymptotic region. Without our proposal for FTCS-IFT conversion, this symmetry aspect of the tuning would remain obscure in the IFT description. We would like to interpret AWS as a hidden in IFT. Of course, this does not imply that a hidden symmetry necessarily exists in every IFT.

IV. QUANTUM ENERGY-MOMENTUM TENSOR

As emphasized in Sec. II, the background metric is asymptotically flat on both sides for $0 < \alpha \leq 1$. In this section, we focus on the right asymptotic region $x \rightarrow +\infty$ within the parameter regime (2.12), where the AWS discussed in Sec. III is realized. Our aim is to compute the renormalized energy-momentum tensor in a specified vacuum state and to explicitly confirm the asymptotic Weyl anomaly previously derived in a formal manner in Sec. III.

A. Renormalized Hadamard two-point function

We begin by analyzing the FTCS action (1.2) on the background geometry (2.6), focusing on the regime in which AWS is present in the right asymptotic limit. To investigate quantum effects in this region, we compute the VEV of the energy-momentum tensor. Following the approach developed in our previous work [6], we employ the Hadamard regularization method to construct the renormalized two-point function.

As a first step, we evaluate the VEV of the product of two field operators at a point x_ϵ , which is located slightly away from the right asymptotic region. For concreteness, we approximate

$$\omega_R^\epsilon(\phi(x)\phi(x')) \simeq {}_\text{R}^{\epsilon}\langle 0|\phi(x)\phi(x')|0\rangle_\text{R}^\epsilon, \quad (4.1)$$

where ω_R^ϵ denotes the Hadamard state defined at x_ϵ . The state $|0\rangle_\text{R}^\epsilon$ represents an approximate vacuum, specified by the annihilation conditions associated with the operators b_k and c_ω defined at x_ϵ . For further details, see (2.52), (2.53) and Appendix B of [6]. Figure 4 provides an additional plot illustrating the situation under consideration. In practice, if

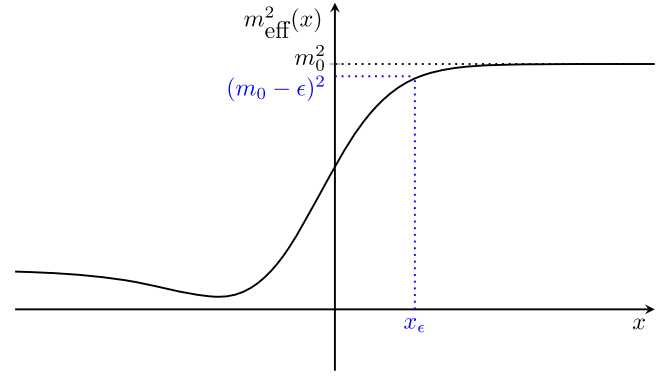


FIG. 4. Plot of the effective mass profile $m_{\text{eff}}^2(x)$ for $\alpha = 0.5$ and $\xi b^2/m_0^2 = 1$, satisfying the condition (2.12). From the expression in (2.7), one finds that $\epsilon \sim \mathcal{O}(e^{-2bx_\epsilon})$. Hence, the contribution of c_ω is subleading and can be safely neglected in our analysis.

one keeps (4.1) only up to terms of order e^{-bx} , the contribution from the c_ω sector can be neglected. In what follows we therefore evaluate the right-hand side of (4.1) using only the b_k operators.⁵

In order to apply the Hadamard regularization method to the VEV of the two-point function discussed above, we first compute the (positive-frequency) Wightman function $G^+(x, x')$ defined by

$$(\square - m_0^2 - \xi \mathcal{R})G^+(x, x') = -\frac{1}{\sqrt{-g}}\delta(t-t')\delta(x-x'). \quad (4.2)$$

The Wightman function can be expressed in terms of mode functions as

$$G^+(x, x') = \int d\mu_\eta u_\eta(x) u_\eta^*(x'), \quad (4.3)$$

where $d\mu_\eta$ denotes the measure associated with the mode index η , and $u_\eta(x)$ satisfies the field equation (2.3) or (2.5). Exploiting time-translation invariance, we write

$$u_\eta(x) = \frac{1}{\sqrt{2\omega}} v_k^{(\mp)}(x) = e^{-i\omega t} \phi_\omega^{(\mp)}(x), \quad (4.4)$$

where the spatial mode function $\phi_\omega^{(\mp)}(x)$ satisfies

$$\left(-\frac{d^2}{dx^2} + m_{\text{eff}}^2(x)\right) \phi_\omega^{(\mp)}(x) = \omega^2 \phi_\omega^{(\mp)}(x) \quad (4.5)$$

with $m_{\text{eff}}^2(x)$ given in (2.7). In the right asymptotic region the quantized scalar field can be written as [6]

⁵For more rigorous canonical approach, we can adopt the Weyl-Titchmarsh-Kodaira theory [11,12].

$$\phi(x) \simeq \int_0^\infty \frac{dk}{\sqrt{2\pi}\sqrt{2\omega}} \sum_{i=\pm} [b_k^{(i)} v_k^{(i)}(x) + (b_k^{(i)})^\dagger (v_k^{(i)}(x))^*], \quad (4.6)$$

where the frequency is given by $\omega = \sqrt{k^2 + m_0^2}$. Using (4.6) in the state $|0\rangle_R^\epsilon$, the Wightman function becomes

$$G_\epsilon^+(x, x') \simeq {}_\epsilon \langle 0 | \phi(x) \phi(x') | 0 \rangle_R^\epsilon \simeq \int_0^\infty \frac{dk}{4\pi\omega} \sum_{i=\mp} v_k^{(i)}(x) (v_k^{(i)}(x'))^*, \quad (4.7)$$

where we used the commutation relation

$$[b_k^{(i)}, (b_{k'}^{(j)})^\dagger] \simeq \delta^{ij} \delta(k - k'). \quad (4.8)$$

We now evaluate the right-hand side of (4.7) using (4.4) and (4.5). Under the condition (2.12), the coefficient of the e^{-bx} term in the mode functions $\phi_\omega^{(\mp)}(x)$ vanishes, and their asymptotic form near the right region simplifies to

$$\phi_\omega^{(\mp)}(x) = \frac{1}{\sqrt{2\omega}} \left(1 - \frac{(1-\alpha)(1+5\alpha^2)m_0^2}{4b(b \mp ik)} e^{-2bx} + \mathcal{O}(e^{-3bx}) \right) e^{\pm ikx}. \quad (4.9)$$

Some comments are in order. For the supersymmetric background with $\alpha = 0$, an analytic solution to (4.5) is known [9]. For $0 < \alpha < 1$, no analytic solution is available, and we instead obtain an approximate solution in the vicinity of the right asymptotic region,

$$\phi_\omega^{(\mp)}(x) = \frac{1}{\sqrt{2\omega}} \left[1 - \frac{2(1-\alpha)m_0^2 \left(1 - \frac{\xi b^2}{m_0^2} \right)}{b(b \mp 2ik)} e^{-bx} - \frac{(1-\alpha)m_0^2 \left(-3 + \alpha + \frac{4\xi b^2}{m_0^2} (1+\alpha) - \frac{4(1-\alpha)m_0^2}{b(b \mp 2ik)} \left(1 - \frac{\xi b^2}{m_0^2} \right)^2 \right)}{4b(b \mp ik)} e^{-2bx} + \mathcal{O}(e^{-3bx}) \right] e^{\pm ikx}. \quad (4.10)$$

Without the condition (2.12), the term of the order e^{-bx} in (4.10) survives. Then, our approximation scheme breaks down.

Substituting the mode functions $v_k^{(i)}$ obtained from (4.4) and (4.9) into (4.7), we obtain an unrenormalized expression for $G_\epsilon^+(x, x')$, which will be renormalized using the Hadamard procedure. In $(1+1)$ dimensions, the short-distance behavior of the Wightman function is universally fixed and takes the Hadamard form

$$G^+(x, x') = \frac{1}{4\pi} (V(x, x') \ln(\mu^2 \sigma(x, x')) + W(x, x'; \mu)), \quad (4.11)$$

where $V(x, x')$ and $W(x, x'; \mu)$ are smooth biscalars, finite in the coincidence limit $x' \rightarrow x$, and can be expanded in powers of the Synge function $\sigma(x, x')$,

$$V(x, x') = \sum_{n=0}^{+\infty} V_n(x, x') \sigma^n(x, x'),$$

$$W(x, x'; \mu) = \sum_{n=0}^{+\infty} W_n(x, x'; \mu) \sigma^n(x, x'). \quad (4.12)$$

The function $V(x, x')$ is fixed entirely by the background geometry together with the field equation, whereas

$W(x, x'; \mu)$ incorporates both geometric contributions and the dependence on the particular quantum state. The constant μ , which has dimensions of mass, is introduced to render the argument of the logarithm dimensionless and is normalized so that the Minkowski vacuum carries vanishing vacuum energy. The Hadamard coefficients $V_n(x, x')$ are obtained by solving recursion relations along the geodesic connecting x and x' . By contrast, for the coefficients $W_n(x, x'; \mu)$ the leading term $W_0(x, x'; \mu)$ is not fixed by these recursion relations and encodes the choice of quantum state. For further discussion, see [13] or Appendix A of [6]. Here, $G_\epsilon^+(x, x')$ in (4.7) is valid only around x_ϵ only, whereas $G^+(x, x')$ in (4.11) applies throughout the entire spacetime region.

The renormalized Wightman function around x_ϵ is then given by

$$G_{\text{ren}}^+(x, x'; \mu) = \frac{1}{4\pi} W(x, x'; \mu) = G_\epsilon^+(x, x') - \frac{1}{4\pi} V(x, x') \ln[\mu^2 \sigma(x, x')]. \quad (4.13)$$

Our aim is to compute the VEV of the energy-momentum tensor near the right asymptotic limit using $G_{\text{ren}}^+(x, x'; \mu)$.

Since $G_e^+(\mathbf{x}, \mathbf{x}')$ in the right-hand side of (4.13) can already be obtained from (4.7) and (4.9), which is simply given in terms of the Minkowski Synge function σ_M by $K_0(m\sqrt{2\sigma_M})/2\pi$ in our approximation scheme (See [6]), the remaining task is to determine $V(\mathbf{x}, \mathbf{x}')$ and $\sigma(\mathbf{x}, \mathbf{x}')$ in the right asymptotic limit.

In the metric (2.6), the spacelike Synge function is given by

$$\begin{aligned} 2\sigma(\mathbf{x}, \mathbf{x}') &= -(\Delta t)^2 + (\Delta x)^2 \\ &\quad - 2(1-\alpha)e^{-\frac{b}{2}(x+x')} \left[-(\Delta t)^2 + (\Delta x)^2 \right. \\ &\quad \left. - \frac{b^2}{24}(\Delta t)^2(\Delta x)^2 + \frac{b^2}{24}(\Delta x)^4 \right] + \dots, \end{aligned} \quad (4.14)$$

where $\Delta t \equiv t' - t$ and $\Delta x \equiv x' - x$, and the ellipsis denotes higher-order terms involving combinations of Δt and Δx . The procedure for computing the Synge function is described in detail in Appendix A. In particular, we have obtained the general form of the Synge function with a conformal factor $e^{2\omega} = e^{2\omega(t,x)}$ for several leading terms in $(1+1)$ dimensions.

Without imposing the condition (2.12), we now compute the biscalar $V(\mathbf{x}, \mathbf{x}')$ up to $\mathcal{O}(e^{-bx})$. The first two Hadamard coefficients [13] are

$$\begin{aligned} V_0(\mathbf{x}, \mathbf{x}') &= -1 - \frac{1}{12} \mathcal{R}_{\alpha\beta}(\mathbf{x}) \nabla_x^\alpha \sigma \nabla_{x'}^\beta \sigma + \mathcal{O}(\sigma^{3/2}), \\ V_1(\mathbf{x}, \mathbf{x}') &= -\frac{m_0^2}{2} - \frac{\mathcal{R}(\mathbf{x})}{4} \left(2\xi - \frac{1}{3} \right) + \mathcal{O}(\sigma^{1/2}). \end{aligned} \quad (4.15)$$

Substituting the metric (2.6) and the Synge function (4.14) into (4.15), we obtain

$$\begin{aligned} V_0 &= -1 - \frac{b^2}{12} (1-\alpha) (-(\Delta t)^2 + (\Delta x)^2) e^{-bx} + \mathcal{O}(e^{-2bx}), \\ V_1 &= -\frac{m_0^2}{2} - \frac{b^2}{2} \left(2\xi - \frac{1}{3} \right) (1-\alpha) e^{-bx} + \mathcal{O}(e^{-2bx}), \end{aligned} \quad (4.16)$$

where we used the Ricci tensor components in the metric (2.6),

$$\begin{aligned} \mathcal{R}_{tt} &= -b^2(1-\alpha)e^{-bx} + \mathcal{O}(e^{-2bx}), \\ \mathcal{R}_{xx} &= b^2(1-\alpha)e^{-bx} + \mathcal{O}(e^{-2bx}), \\ \mathcal{R}_{tx} &= 0, \quad \mathcal{R} = 2b^2(1-\alpha)e^{-bx} + \mathcal{O}(e^{-2bx}). \end{aligned} \quad (4.17)$$

From these results, we obtain

$$V(\mathbf{x}, \mathbf{x}') = -1 - \frac{m_0^2}{4} (-(\Delta t)^2 + (\Delta x)^2) \quad (4.18)$$

$$\begin{aligned} &+ \frac{1}{2} (m_0^2 - \xi b^2) (1-\alpha) (-(\Delta t)^2 + (\Delta x)^2) e^{-bx} \\ &+ \mathcal{O}(e^{-2bx}). \end{aligned} \quad (4.19)$$

Combining (4.7), (4.9), (4.14), and (4.18), we finally obtain the renormalized Wightman function under the AWS condition (2.12) up to $\mathcal{O}(e^{-bx})$,

$$\begin{aligned} G_{\text{ren}}^+(\mathbf{x}, \mathbf{x}'; \mu) &= \frac{1}{4\pi} \left(1 + \frac{m_0^2}{2} \sigma_M \right) \ln \left(\frac{\mu^2}{2m_0^2} \right) \\ &\quad + \frac{1}{2\pi} (-\gamma + \ln 2) + \frac{m_0^2}{4\pi} (1 - \gamma + \ln 2) \sigma_M \\ &\quad - \frac{1-\alpha}{2\pi} \left(1 + \frac{b^2}{24} (\Delta x)^2 + \frac{m_0^2}{2} \sigma_M \right) e^{-b\frac{x+x'}{2}} + \dots, \end{aligned} \quad (4.20)$$

where $2\sigma_M \equiv -(\Delta t)^2 + (\Delta x)^2$ and the ellipsis denotes higher-order terms in Δt , Δx , and $e^{-b(x+x')/2}$.

B. Quantum energy-momentum tensor and trace anomaly

We now compute the quantum energy-momentum tensor using the renormalized Wightman function $G_{\text{ren}}^+(\mathbf{x}, \mathbf{x}'; \mu)$ obtained in the previous subsection IV A. One of our aims is to determine, through explicit calculations, whether the VEV of T^μ_μ with respect to the vacuum state $|0\rangle_R^\epsilon$ defined at x_ϵ vanishes asymptotically, or instead acquires an anomalous contribution.

1. VEV of the energy-momentum tensor

To obtain the VEV of $T_{\mu\nu}$, we apply the point-splitting method and use the renormalized Hadamard function introduced in (4.13) and (4.20). For the vacuum $|0\rangle_R^\epsilon$, we have

$$\langle T_{\mu\nu}(\mathbf{x}; \mu) \rangle_R^\epsilon = \lim_{x' \rightarrow x} \mathcal{T}_{\mu\nu} G_{\text{ren}}^+(\mathbf{x}, \mathbf{x}'; \mu), \quad (4.21)$$

where the relevant bidifferential operator in $(1+1)$ dimensions is

$$\mathcal{T}_{\mu\nu} = (1 - 2\xi) \partial_\mu \partial_{\nu'} + g_{\mu\nu'} \left[\left(2\xi - \frac{1}{2} \right) g^{\alpha\beta'} \partial_\alpha \partial_{\beta'} - \frac{1}{2} m_0^2 + 2\xi g^{\alpha\beta} \nabla_\alpha \nabla_{\beta'} \right] - 2\xi \delta_\mu^{\mu'} \partial_{\mu'} \partial_{\nu'} + \frac{1}{4} g_{\mu\nu'} P_x, \quad (4.22)$$

with $P_x = -\square_x + m_0^2 + \xi \mathcal{R}$. As discussed in [14,15], this last term in the right-hand side of (4.22) is not included in the standard treatment [16], since it does not appear in the classical tensor (2.4). If the last term in (4.22) is omitted, the VEV of

energy-momentum tensor is no longer covariantly conserved. For more details in our specific setup, see [6].

Inserting (4.20) into (4.21) and simplifying the expression using the condition (2.12), we obtain

$$\langle T_{tt} \rangle_R^\epsilon = \frac{m_0^2}{4\pi} \left[\frac{1}{2} - \gamma + \ln 2 + \frac{1}{2} \ln \left(\frac{\mu^2}{2m_0^2} \right) \right] [1 - 2(1 - \alpha)e^{-bx}] - \frac{b^2}{12\pi} (1 - \alpha)e^{-bx} + \mathcal{O}(e^{-2bx}), \quad (4.23)$$

$$\langle T_{xx} \rangle_R^\epsilon = \frac{m_0^2}{4\pi} \left[-\frac{1}{2} + \gamma - \ln 2 - \frac{1}{2} \ln \left(\frac{\mu^2}{2m_0^2} \right) \right] [1 - 2(1 - \alpha)e^{-bx}] + \mathcal{O}(e^{-2bx}), \quad (4.24)$$

$$\langle T_{tx} \rangle_R^\epsilon = \langle T_{xt} \rangle_R^\epsilon = \mathcal{O}(e^{-2bx}). \quad (4.25)$$

It is worth to note that all terms with the coefficient ξ cancel among themselves in the above results. Since the scalar curvature has the form given in (4.17), the expressions above can be interpreted as an expansion valid in the regime where the curvature is small, namely in the limit $x \rightarrow \infty$. One can show that the VEV of the energy-momentum tensor given in (4.23)–(4.25) is covariantly conserved up to $\mathcal{O}(e^{-bx})$, irrespective of the value of μ . Note also that the above results reduce to those of our previous supersymmetric case [6] when $\alpha = 0$ and $\xi = 1/4$ (i.e., $m_0 = b/2$).

Following the procedure adopted in [6], we require that the VEV of the energy-momentum tensor vanish in the Minkowski vacuum $|0\rangle_M$, i.e.,

$${}_M\langle 0|T_{\mu\nu}|0\rangle_M = 0. \quad (4.26)$$

In the Minkowski vacuum, the VEV of the energy-momentum tensor is

$$\begin{aligned} \langle T_{tt} \rangle_M &= \frac{m_0^2}{4\pi} \left[\frac{1}{2} - \gamma + \ln 2 + \frac{1}{2} \ln \left(\frac{\mu^2}{2m_0^2} \right) \right], \\ \langle T_{xx} \rangle_M &= \frac{m_0^2}{4\pi} \left[-\frac{1}{2} + \gamma - \ln 2 - \frac{1}{2} \ln \left(\frac{\mu^2}{2m_0^2} \right) \right], \\ \langle T_{tx} \rangle_M &= \langle T_{xt} \rangle_M = 0. \end{aligned} \quad (4.27)$$

Thus, in order to achieve the requirement (4.26), we must choose

$$\mu = \frac{m_0}{\sqrt{2}} e^{\gamma - \frac{1}{2}}. \quad (4.28)$$

Substituting this value of μ into (4.23)–(4.25) yields the asymptotic form

$$\begin{aligned} \langle T_{tt} \rangle_R^\epsilon &= -\frac{b^2}{12\pi} (1 - \alpha)e^{-bx} + \mathcal{O}(e^{-2bx}), \\ \langle T_{xx} \rangle_R^\epsilon &= \mathcal{O}(e^{-2bx}), \\ \langle T_{tx} \rangle_R^\epsilon &= \langle T_{xt} \rangle_R^\epsilon = \mathcal{O}(e^{-2bx}). \end{aligned} \quad (4.29)$$

This result represents the quantum effects of the state $|0\rangle_R^\epsilon$ defined at x_ϵ , under the condition that the VEV of the

energy-momentum tensor vanishes in the Minkowski vacuum. As shown in (2.6), the background metric reduces to Minkowski space when $\alpha = 1$, which is consistent with (4.29). According to our proposal [6,8,9], we can identify

$$\langle T_{\mu\nu} \rangle_{\text{IFT}}^\epsilon = \langle T_{\mu\nu} \rangle_R^\epsilon. \quad (4.30)$$

2. Asymptotic Weyl anomaly

We now connect the formal AWS analysis of Sec. III with the explicit results of the previous subsections, in order to exhibit the asymptotic trace anomaly in the right asymptotic region. Using (4.29) together with the curvature scalar in (4.17), we find

$$\begin{aligned} g^{\mu\nu} \langle T_{\mu\nu}(x; \mu) \rangle_R^\epsilon &= \frac{1}{12\pi} b^2 (1 - \alpha)e^{-bx} + \mathcal{O}(e^{-2bx}) \\ &= \frac{1}{24\pi} \mathcal{R} + \mathcal{O}(e^{-2bx}), \end{aligned} \quad (4.31)$$

which is precisely the asymptotic anomaly form anticipated in Sec. III.

Within the Hadamard renormalization scheme, a general two-dimensional scalar field with the nonminimal coupling in (1.2) satisfies the local relation [13]

$$g^{\mu\nu} \langle T_{\mu\nu}(x; \mu) \rangle = (\xi \square - m_0^2) \delta G_{\text{ren}}^+(x, x; \mu) + \frac{1}{24\pi} \mathcal{R}, \quad (4.32)$$

where $\delta G_{\text{ren}}^+ \equiv G_{\text{ren}}^+ - G_{\text{ren},M}^+$.⁶ Here, the renormalized Wightman function in Minkowski spacetime $G_{\text{ren},M}^+$ is given by $-\frac{1}{4\pi}$. In our setup the renormalized Wightman function (4.20) obeys the relation

$$\delta G_{\text{ren}}^+(x, x; \mu) = -\frac{1}{2\pi} (1 - \alpha)e^{-bx} + \mathcal{O}(e^{-2bx}), \quad (4.33)$$

so that the state-dependent term $(\xi \square - m_0^2) \delta G_{\text{ren}}^+$ vanishes up to $\mathcal{O}(e^{-2bx})$ in the right asymptotic region. Consequently, the purely geometric curvature term in (4.32) provides the leading contribution to $\langle T_{\mu}^\mu \rangle$. This gives

⁶We adopt the notation δG_{ren}^+ (rather than G_{ren}^+ as used in Ref. [6]) to improve clarity and avoid potential confusion.

an explicit realization of the asymptotic Weyl anomaly for the class of vacua considered here and confirms the formal argument of Sec. III.

C. Unruh-like effect in the bubble-wall expansion and IFT implication

Finally, we comment on the physical interpretation of the energy-momentum tensor and its implications in the IFT description. The result that $\langle T_{tt} \rangle_R^\epsilon < 0$ in (4.29) suggests that an observer located slightly away from the right asymptotic region measures a negative local energy density with respect to the Minkowski vacuum. By analogy with the Rindler case, this can be interpreted as indicating that such an observer would detect a thermallike distribution of particles associated with the field, whereas the right Minkowski vacuum remains empty for inertial observers at spatial infinity. Therefore, we interpreted this result as an Unruh-like effect. See [6,8,9].

In studies of first-order electroweak phase transitions, the growth of bubbles has long been analyzed [6,11,12,17–22]. Within thick-wall descriptions, models employing spatially varying masses are standard, and we adopt the IFT with the mass profile given in (2.7). Here we focus solely on the vacuum contribution to the resistance acting on the wall. Writing the total pressure as $P_{\text{tot}} = -\Delta V + \Delta P$, only the latter can survive in the wall rest frame, and in our zero-temperature model the classical contribution vanishes. Thus the relevant pressure is purely quantum. This bubble wall expansion can be modeled by our scalar IFT [6,11,12]. Then, using the identification (4.30), near the Higgs-side region of the wall, we can see that the pressure difference becomes

$$\Delta P_\epsilon = \langle T_{xx} \rangle_{\text{IFT}}^\epsilon - \langle T_{xx} \rangle_{\text{IFT}}^H, \quad (4.34)$$

with the Higgs-side contribution being zero in our prescription, where $\langle T_{xx} \rangle_{\text{IFT}}^H \equiv \langle T_{xx} \rangle_M$. From the asymptotic form of $\langle T_{xx} \rangle_R^\epsilon$ in (4.29), we obtain⁷

$$\Delta P_\epsilon = \mathcal{O}(e^{-2bx}). \quad (4.35)$$

Therefore, from the IFT perspective, although the quantum vacuum produces an Unruh-like contribution as the leading effect (of order e^{-bx}), the positive (or negative) pressure that would oppose (or enhance) the expansion of the bubble wall vanishes at leading order. In other words, the effect of the pressure difference ΔP begins to contribute only at least at the next-to-leading order (e^{-2bx}). More detailed analysis of this issue will be deferred to future work.

⁷In our previous work [6], due to a computational oversight, we reported a positive pressure contribution at the leading order (e^{-bx}). This has been corrected in the recent revision.

V. CONCLUSIONS

Modern understanding of high energy physics is based on quantum field theory (QFT), which is regarded as an effective theory of the more fundamental theory, string/M-theory. Nevertheless, a (effective) field theory has quite versatile applicability and makes various conceptual and practical achievements. Therefore, it is still meaningful to explore various extensions of QFT. In this regard, one of the crucial ingredients of relativistic QFT is the existence of a rather large global symmetry: Poincaré symmetry. However, in some low energy regimes, there are situations lacking a large global symmetry. For instance, there would be background fields which can be regarded as nondynamical. The well-known example of these situations is QFT with a fixed nondynamical curved background spacetime. In this case the algebraic approach has been adopted to overcome various conceptual issues, reconciling the apparent coordinate dependence of the choice of Hilbert spaces. IFT explored in our recent works [6,8,9] is another example, which have a similar property with FTCS. But, the absence of general covariance renders IFT to be more difficult to understand its quantum aspects, even though it would have some applications in various low energy regimes.

Because of the absence of a large global symmetry, there is no consensus how to quantize IFT in generic situations. Though we have presented a certain way to accomplish the quantization procedures in a specific setup [6,8,9], the complete answer to the quantization remains as an open issue. In this paper, we continue to explore the quantization of IFT and provide its interpretation. The main results in this paper are two folded. First, we establish the existence of a hidden approximate asymptotic Weyl-like symmetry, designated as AWS, in a specific FTCS, and provide its interpretation within the IFT framework. This observation explains our previous result on the anomaly expression of the trace of energy-momentum tensor VEV even in a certain nonconformal IFT, as shown in (4.29) and (4.31). A crucial element for this result is the choice of a physically appropriate vacuum. In Sec. IV, we have provided how to choose approximately such a vacuum by using canonical formalism [See the paragraph below (4.1)]. From the algebraic viewpoint, this corresponds to choosing the vacuum such that the first term in (4.32) should vanish at the relevant order. In this regard, we would like to emphasize again that AWS is not exact but just asymptotic, which is valid only for a specific profile of the background metric.

Second, we address concerns in the FTCS framework, regarding the validity of Hadamard renormalization method in a spacetime with a null singularity. Since our one-parameter extension of the metric for $\alpha \neq 0$ admits no singularity, we eliminated some doubts about the applicability of the Hadamard renormalization. This in turn shows us that the physical interpretation of our results in IFT is more reliable.

Now, we would like to provide some comments about IFT under consideration with respect to another inhomogeneous theory with a different perspective. In a recent work [23–26], the so-called driven inhomogeneous CFT [27,28] is explored in conjunction with a curved spacetime metric and the AdS/CFT correspondence. In this work, the inhomogeneity is introduced into the global Hamiltonian while local conformal operator structure is preserved. In other words, a Hamiltonian is deformed by a inhomogeneous function while a tensor operator or a corresponding state can still be classified by a conformal weight. It is interesting to note that the interpretation of the driven inhomogeneous CFT in terms of CFT on a certain curved background is quite similar to our case. Furthermore, the choice of a specific renormalization scheme is emphasized in [26]. This is also similar with our case in the sense that a specific renormalization scheme is required to convert FTCS to IFT. However, there is one crucial difference. In the case of inhomogeneous CFT, there is a large conformal symmetry allowing the famous state-operator correspondence. Therefore, there seems no issue on the choice of Hilbert space in the conversion from the driven inhomogeneous CFT to CFT on curved spacetime. In some sense, the large symmetry allows a preferred choice of vacuum in Hilbert space. This is not the case in our examples and one of the main issues in our approach. In our previous paper [6,8,9], we have provided one way to choose an observer-dependent (global) vacuum in the context of the algebraic approach, which is also adopted in this paper. All these results tell us that a specific renormalization scheme with a careful choice of a certain vacuum is quite crucial to understand the quantization of IFT.

There are various directions to pursue in the future. One of urgent directions is to understand the more concrete nature of AWS. It would be very interesting whether it exists in other field theory. Since our observation about this new type symmetry is a bit nonconventional in the sense that it is just asymptotically valid, it requires more clarification. It would also be interesting to analyze the left asymptotic region, where reflection modes play an essential role. Another issue is related to fermionic IFT, which would have some applications in condensed matter physics. Especially, the fermionic zero modes seems to exist when it is realized as a superpartner of scalar IFT, which is also needed to be explored. The finite temperature physics is still needed to be explored in the context of IFT, which may be related to nonequilibrium physics and the driven inhomogeneous CFT.

ACKNOWLEDGMENTS

This work was supported in part by the National Research Foundation of Korea (NRF) grant with Grants No. RS-2019-NR040081 (O. K., J. H., and S.-A. P.), No. RS-2023-00208047 (J. H.), No. RS-2023-00208011

(S.-H. Y.), and in part by NRF grant funded by the Korea government (MSIT) No. RS-2025-00553127 (O. K., J. H., S.-A. P., and S.-H. Y.).

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: SYNGE FUNCTION

In this Appendix, we provide some details about properties of Synge function relevant in the main text. To compute the renormalized (positive-frequency) Wightman function and the VEV of energy-momentum tensor, we need the short-distance expansion of Synge function in terms of coordinates. Following Refs. [29,30], we define Synge function as a bitensor along the geodesic connecting two points $\mathbf{x}'$ and $\mathbf{x}$

$$\sigma(\mathbf{x}', \mathbf{x}) \equiv \frac{1}{2}(\lambda_1 - \lambda_0) \int_{\lambda_0}^{\lambda_1} d\lambda g_{\mu\nu} t^\mu t^\nu, \quad t^\mu \equiv \frac{dx^\mu}{d\lambda}, \quad (\text{A1})$$

where λ_0 denotes the affine parameter for the “base point” $\mathbf{x}$ and λ_1 does for the field point $\mathbf{x}'$. It is straightforward to see that this function is one half of the signed squared geodesic distance for two points $\mathbf{x}'$ and $\mathbf{x}$. One of the essential property of Synge function is

$$\sigma_\mu \sigma^\mu = \pm 2\sigma, \quad \sigma_\mu \equiv \nabla_\mu \sigma. \quad (\text{A2})$$

In the following, we would like to derive the following expansion of Synge function:

$$\sigma(\mathbf{x}', \mathbf{x}) = \sum_{m=0}^{\infty} \frac{1}{m!} \zeta_{\mu_1 \mu_2 \dots \mu_m}(\mathbf{x}) \Delta x^{\mu_1} \Delta x^{\mu_2} \dots \Delta x^{\mu_m}, \quad (\text{A3})$$

where $\Delta x^\mu \equiv (\mathbf{x}' - \mathbf{x})^\mu$ and $\zeta_{\mu_1 \mu_2 \dots \mu_m} = \zeta_{(\mu_1 \mu_2 \dots \mu_m)}$ is defined recursively by⁸

$$\begin{aligned} 2(1 - \ell) \zeta_{\mu_1 \dots \mu_\ell} &= -2\ell \zeta_{\mu_1 \dots \mu_{\ell-1}, \mu_\ell} \\ &+ \sum_{k=2}^{\ell-2} \binom{\ell}{k} g^{\alpha\beta} (\zeta_{(\mu_1 \dots \mu_k, |\alpha} - \zeta_{\mu_1 \dots \mu_k |\alpha}) \\ &\times (\zeta_{\mu_{k+1} \dots \mu_\ell, |\beta} - \zeta_{\mu_{k+1} \dots \mu_\ell |\beta}). \end{aligned} \quad (\text{A4})$$

with the initial data

$$\zeta = 0, \quad \zeta_\mu = 0, \quad \zeta_{\mu\nu} = g_{\mu\nu}. \quad (\text{A5})$$

Derivation of the recursion relation. Using $\sigma_\alpha \sigma^\alpha = \pm 2\sigma$, we go ahead as follows. First, note that

⁸The parenthesis for indices denotes totally symmetrization of indices.

$$\sigma_\alpha = \sum_{n=1}^{\infty} \frac{1}{n!} (\zeta_{\mu_1 \mu_2 \dots \mu_n, \alpha} - \zeta_{\mu_1 \mu_2 \dots \mu_n \alpha}) \Delta x^{\mu_1} \dots \Delta x^{\mu_n}, \quad (\text{A6})$$

where one may note that $\zeta_{\mu, \alpha} = 0$. Here, ' denotes the covariant derivative as $\zeta_{\mu, \alpha} \equiv \nabla_\alpha \zeta_\mu$. Therefore, the first term in the right-hand side is given by $-\zeta_{\mu\alpha} \Delta x^\mu$. Squaring this expression, we have

$$\begin{aligned} \sigma_\alpha \sigma^\alpha &= \sum_{m,n=1}^{\infty} \frac{1}{m!n!} g^{\alpha\beta} (\zeta_{\mu_1 \mu_2 \dots \mu_m, \alpha} - \zeta_{\mu_1 \mu_2 \dots \mu_m \alpha}) (\zeta_{\mu_1 \mu_2 \dots \mu_n, \beta} - \zeta_{\mu_1 \mu_2 \dots \mu_n \beta}) \Delta x^{\mu_1} \dots \Delta x^{\mu_m} \Delta x^{\mu_1} \dots \Delta x^{\mu_n}, \\ &= 2 \sum_{\ell=1}^{\infty} \frac{1}{\ell!} g^{\alpha\beta} (-\zeta_{\mu\alpha}) (\zeta_{\nu_1 \nu_2 \dots \nu_\ell, \beta} - \zeta_{\nu_1 \nu_2 \dots \nu_\ell \beta}) \Delta x^{\mu} \Delta x^{\nu_1} \dots \Delta x^{\nu_\ell} \\ &\quad + \sum_{m,n=2}^{\infty} \frac{1}{m!n!} g^{\alpha\beta} (\zeta_{\mu_1 \mu_2 \dots \mu_m, \alpha} - \zeta_{\mu_1 \mu_2 \dots \mu_m \alpha}) (\zeta_{\mu_1 \mu_2 \dots \mu_n, \beta'} - \zeta_{\mu_1 \mu_2 \dots \mu_n \beta}) \Delta x^{\mu_1} \dots \Delta x^{\mu_m} \Delta x^{\mu_1} \dots \Delta x^{\mu_n}, \end{aligned}$$

Note that the first and second lines in the right-hand side of the above last equality can be reshuffled as

$$\begin{aligned} \text{1st} &= -2 \sum_{\ell=1}^{\infty} \frac{1}{\ell!} (\zeta_{\mu_1 \mu_2 \dots \mu_\ell, \mu_{\ell+1}} - \zeta_{\mu_1 \mu_2 \dots \mu_\ell \mu_{\ell+1}}) \Delta x^{\mu_1} \dots \Delta x^{\mu_\ell} \Delta x^{\mu_{\ell+1}} \\ &= -2 \sum_{\ell=1}^{\infty} \frac{1}{\ell!} (\zeta_{(\mu_1 \mu_2 \dots \mu_\ell, \mu_{\ell+1})} - \zeta_{\mu_1 \mu_2 \dots \mu_\ell \mu_{\ell+1}}) \Delta x^{\mu_1} \dots \Delta x^{\mu_\ell} \Delta x^{\mu_{\ell+1}}, \end{aligned} \quad (\text{A7})$$

$$\begin{aligned} \text{2nd} &= \sum_{m,n=2}^{\infty} \frac{1}{m!n!} g^{\alpha\beta} (\zeta_{\mu_1 \mu_2 \dots \mu_m, \alpha} - \zeta_{\mu_1 \mu_2 \dots \mu_m \alpha}) (\zeta_{\mu_1 \mu_2 \dots \mu_n, \beta} - \zeta_{\mu_1 \mu_2 \dots \mu_n \beta}) \Delta x^{\mu_1} \dots \Delta x^{\mu_m} \Delta x^{\mu_1} \dots \Delta x^{\mu_n} \\ &= \sum_{\ell=4}^{\infty} \sum_{k=2}^{\ell-2} \frac{1}{(\ell-k)!k!} g^{\alpha\beta} (\zeta_{(\mu_1 \dots \mu_k, \alpha} - \zeta_{\mu_1 \dots \mu_k \alpha}) (\zeta_{\mu_{k+1} \dots \mu_\ell, \beta'} - \zeta_{\mu_{k+1} \dots \mu_\ell \beta}) \Delta x^{\mu_1} \dots \Delta x^{\mu_\ell} \end{aligned} \quad (\text{A8})$$

Using the relation $\sigma_\alpha \sigma^\alpha = 2\sigma$, we should identify

$$\text{1st} + \text{2nd} = 2\sigma = 2 \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \zeta_{\mu_1 \dots \mu_\ell} \Delta x^{\mu_1} \dots \Delta x^{\mu_\ell}. \quad (\text{A9})$$

Using the reshuffled expressions in (A7) and (A8) in the above equation and collecting the same order in $\Delta x^{\mu_1} \dots \Delta x^{\mu_\ell}$, we finally obtain (for $\ell > 2$) the recursion relation (A4).

Now, it is straightforward to derive

$$\zeta_{\mu\nu\rho} = 3g_{\alpha(\rho} \Gamma_{\mu\nu)}^\alpha, \quad (\text{A10})$$

and

$$\zeta_{\mu\nu\alpha\beta} = \frac{4}{3} \partial_{(\beta} \zeta_{\mu\nu\alpha)} - \frac{1}{6} g^{\rho\sigma} (\partial_\rho \zeta_{\mu\nu} - \zeta_{\rho(\mu\nu)}) (\partial_{|\sigma|} \zeta_{\alpha\beta}) - \zeta_{|\sigma| \alpha\beta}). \quad (\text{A11})$$

In our two-dimensional case, it is sufficient to keep the terms up to the order of $m = 4$ in computing the Hadamard renormalization.

For the one-parameter extended metric in the main text

$$ds^2 = \left(\frac{1 + \alpha e^{-bx}}{1 + e^{-bx}} \right)^2 = 1 - 2(1 - \alpha)e^{-bx} + \mathcal{O}(e^{-bx}), \quad (\text{A12})$$

we have

$$\begin{aligned} \zeta_{txx} &= \zeta_{xtx} = \zeta_{xtt} = -b(1 - \alpha)e^{-bx} + \mathcal{O}(e^{-2bx}), \\ \zeta_{xxx} &= \frac{be^{-bx}}{(1 + e^{-bx})^3} = 3b(1 - \alpha)e^{-bx} + \mathcal{O}(e^{-2bx}), \end{aligned}$$

and

$$\begin{aligned} \zeta_{txxx} &= \zeta_{xtxx} = \zeta_{txxt} = \zeta_{xttx} = \zeta_{xttt} = \zeta_{xttt} = \frac{2b^2}{3} (1 - \alpha)e^{-bx} + \mathcal{O}(e^{-2bx}), \\ \zeta_{xxxx} &= -4b^2(1 - \alpha)e^{-bx} + \mathcal{O}(e^{-2bx}). \end{aligned}$$

Then, finally we have

$$2\sigma(\mathbf{x}', \mathbf{x}) = -(\Delta t)^2 + (\Delta x)^2 - 2(1 - \alpha)e^{-\frac{b}{2}(x+x')} \left[-(\Delta t)^2 + (\Delta x)^2 - \frac{b^2}{24}(\Delta t)^2(\Delta x)^2 + \frac{b^2}{24}(\Delta x)^4 \right] + \dots,$$

which is used in the main text. Note that one can obtain the Synge function expansion of the supersymmetric background metric in [6] by taking $\alpha = 0$.

Just for a reference, we present the result of Synge function expansion for a generic conformal case in $(1 + 1)$ dimensions

$$ds^2 = e^{2\omega(t,x)}(-dt^2 + dx^2). \quad (\text{A13})$$

In this case, we have

$$2\sigma(\mathbf{x}', \mathbf{x}) = (-\Delta t^2 + \Delta x^2)e^{2\omega} \left[1 + \dot{\omega}\Delta t + \omega'\Delta x + \frac{1}{3}(\dot{\omega}' + 2\dot{\omega}\omega')\Delta t\Delta x + \frac{1}{6}(\ddot{\omega} + 2\omega'^2)\Delta t^2 + \frac{1}{6}(\omega'' + 2\dot{\omega}^2)\Delta x^2 + \dots \right], \quad (\text{A14})$$

where $\dot{\omega} \equiv \frac{\partial}{\partial t}$.

APPENDIX B: NONSUPERSYMMETRIC BACKGROUNDS IN FTCS

In this work, we consider the static metric given in (2.6). For $\alpha = 0$, this metric reduces to the supersymmetric background on the FTCS that we analyzed in our previous paper [9], whereas for $\alpha = 1$ it becomes the flat metric. Therefore, the parameter α provides a continuous interpolation between the supersymmetric background and the flat background. Since the flat background can also be regarded as a supersymmetric background, the static metric (2.6) preserves supersymmetry at $\alpha = 0$ with $\mathcal{N} = (\frac{1}{2}, \frac{1}{2})$ or at $\alpha = 1$ with $\mathcal{N} = (1, 1)$. However, the FTCS (1.2) we are interested in generically breaks supersymmetry completely on the static background (2.6) for $0 < \alpha < 1$. The reason is as follows.

A systematic framework for constructing supersymmetric field theories on curved backgrounds was developed in [31]. In the rigid limit $M_{\text{Pl}} \rightarrow \infty$, the gravity multiplet becomes nondynamical, whereas the matter multiplets remain dynamical and continue to exhibit global supersymmetry provided certain consistency conditions are satisfied. Within this setup, the requirement that the supersymmetry variation of the gravitino vanish leads to the generalized Killing spinor equation

$$\nabla_\mu \epsilon(\mathbf{x}) = M_\mu(\mathbf{x})\epsilon(\mathbf{x}), \quad (\text{B1})$$

where ϵ is the supersymmetry parameter and $M_\mu(\mathbf{x})$ is a background-dependent matrix acting on spinor indices, determined entirely by the bosonic components of the nondynamical gravity multiplet. See also [32] for $\mathcal{N} = (\frac{1}{2}, \frac{1}{2})$ for general space-dependent parameters in $(1 + 1)$ -dimensional Minkowski space.

In our previous work [9], we considered a special class of backgrounds admitting supersymmetric FTCS solutions, characterized by $M_\mu = \frac{1}{2}f(\mathcal{R})\gamma_\mu$. For the particular choice $f(\mathcal{R}) = (\xi/m_0)\mathcal{R}$, the bosonic sector of the FTCS reduces to the action in (1.2). In this setup, for the conformally flat metric (2.1), the warp factor $\omega(x)$ satisfies the differential equation

$$\omega'' + \frac{m_0}{2\xi}e^\omega\omega' = 0, \quad (\text{B2})$$

and provides a solution to the generalized Killing spinor equation (B1), preserving $\mathcal{N} = (\frac{1}{2}, \frac{1}{2})$ supersymmetry. It is straightforward to verify that the modified metric (2.6) fails to satisfy the condition (B2) for generic values of α , except in the case $\alpha = 0$. Consequently, the FTCS action (1.2) cannot be supersymmetrized on the background geometry (2.6) for $0 < \alpha < 1$.

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