

Complete two-loop unrenormalized electroweak corrections to $e^+e^- \rightarrow HZ$

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We compute the complete two-loop unrenormalized electroweak corrections to the Higgsstrahlung process $e^+e^- \rightarrow HZ$ at future Higgs factories. Our result for a given center-of-mass energy is expressed as a piecewise function defined by several deeply expanded power series, which has high precision and can be further manipulated efficiently. We find that the finite part of the unrenormalized two-loop electroweak corrections can contribute a few thousandths, which might indicate their importance to match the precision of future Higgs factories. This work represents the first complete calculation of the unrenormalized two-loop electroweak corrections for processes with four external particles.

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I. INTRODUCTION

The physics of the Higgs boson has become a key area of research in high-energy physics since its discovery a decade ago [1,2]. In the Standard Model (SM) of particle physics, the Higgs boson is considered the direct evidence of the electroweak (EW) spontaneous symmetry breaking based on the Higgs mechanism. However, the current experimental precision cannot exclude the possibility of an exotic Higgs potential deviating from the SM, which is a typical feature in many new physics models. Therefore, the Higgs boson could be the most promising probe for new physics beyond the SM.

The precise measurement of the Higgs boson is the key mission for the next generation of high-energy experiment facilities. In recent years, several proposals for Higgs factories have been put forward, including the International Linear Collider (ILC) [3–5], the Circular

Electron Positron Collider (CEPC) [6,7], and the Future Circular Collider (FCC-ee) [8–10]. Millions of Higgs bosons are expected to be produced at these facilities through processes such as Higgsstrahlung $e^+e^- \rightarrow ZH$, W-boson fusion $e^+e^- \rightarrow \nu_e \bar{\nu}_e H$, and Z-boson fusion $e^+e^- \rightarrow e^+e^- H$. At a typical center-of-mass energy of 240 GeV, the dominant contribution is the Higgsstrahlung process.

Preliminary investigations have shown that, with an expected integrated luminosity of 5.6 ab^{-1} [11], the Higgsstrahlung cross section $\sigma(e^+e^- \rightarrow ZH)$ can be measured with a precision of 0.51% [11]. To match this experimental precision, the relevant theoretical predictions must also be pushed to a similar level. The next-to-leading order (NLO) electroweak (EW) effect has been studied in Refs. [12–14], and more recently, mixed EW-QCD correction effects have been obtained [15–18]. However, the best theoretical uncertainty is still as high as 1%, which is not yet compatible with the experimental accuracy. Therefore, next-to-next-to-leading order (NNLO) EW corrections are crucial for Higgs physics analysis at future Higgs factories. Recently, some integrals involved in two-loop EW corrections have been calculated [19,20], but a complete two-loop calculation is still missing.

Calculating the complete two-loop EW corrections for $2 \rightarrow 2$ processes such as the Higgsstrahlung process is a longstanding challenging task, but it is essential for providing accurate theoretical predictions. On one hand, the 25, 377 Feynman diagrams involved in the calculation

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[21] make the process extremely time-consuming and resource-intensive. On the other hand, the Feynman integrals, which involve six different mass scales, are beyond the reach of current analytical methods (for a review, see Refs. [22,23]). In this paper, we take advantage of recent developments in multiloop calculation techniques and present the first complete two-loop unrenormalized EW corrections for the Higgsstrahlung process $e^+e^- \rightarrow HZ$ at future Higgs factories, which is a crucial step towards the complete NNLO EW corrections for this process.

II. GENERATION OF AMPLITUDES

We consider the two-loop electroweak correction to the process,

$$e^+(k_1) + e^-(k_2) \rightarrow H(k_3) + Z(k_4), \quad (1)$$

where the external momenta satisfy the on shell conditions,

$$k_1^2 = k_2^2 = 0, \quad k_3^2 = m_H^2, \quad k_4^2 = m_Z^2, \quad (2)$$

and momentum conservation,

$$k_1 + k_2 = k_3 + k_4. \quad (3)$$

The Mandelstam variables are defined as usual,

$$s = (k_1 + k_2)^2, \quad t = (k_1 - k_3)^2. \quad (4)$$

We generate the Feynman amplitudes with QGRAF [24,25] and FeynArts [26], with some sample two-loop Feynman diagrams shown in Fig. 1. A detailed classification of all 25377 two-loop diagrams has been provided in Ref [21]. We use in-house *MATHEMATICA* codes to express the interference of two-loop amplitudes with tree-level amplitudes as linear combinations of some scalar integrals. The coefficients of these scalar integrals are rational functions of physical parameters, including the Mandelstam variables s and t , the squared masses of corresponding particles m_t^2 , m_H^2 , m_Z^2 , and m_W^2 , the dimensional regulator $\epsilon = (4 - D)/2$, the EW coupling constant α , etc.

It is well-known that in dimensional regularization, the anticommutation relation $\{\gamma^\mu, \gamma_5\} = 0$ and the cyclicity of the Dirac trace cannot be satisfied simultaneously. In practice, we prefer to keep the anticommutation relation not only to simplify the computation, but also because using a nonanticommuting γ_5 with the other Dirac matrices γ^μ leads to “spurious anomalies” which violate chiral symmetry and hence gauge invariance [27]. In this case, one must impose the relevant Ward-Takahashi and Slavnov-Taylor identities order by order to keep the renormalizability of the perturbation theory, which makes practical calculations much more difficult and tedious. However, using an anticommuting γ_5 necessitates a specific

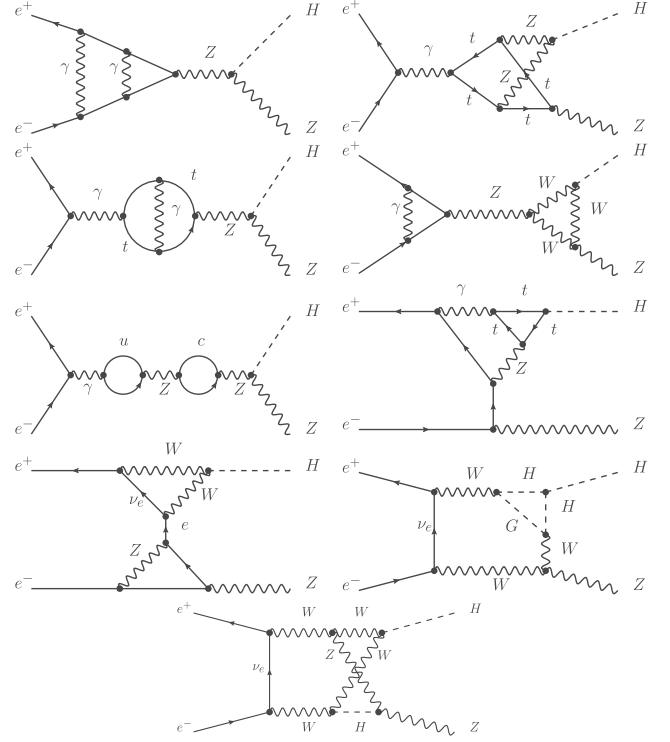


FIG. 1. Sample Feynman diagrams of the two-loop electroweak correction for the process $e^+e^- \rightarrow HZ$.

prescription for reading fermion loops, since different reading points may lead to different results. We therefore adopt the KKS-scheme [28–30], which resolves this ambiguity by defining the final results as the average over all possible reading points. These reading points originate from both the head and the tail of a nonsinglet-type three-point fermion-axial-coupling subgraph that encompasses the maximal one-particle-irreducible loop correction [30,31]. Furthermore, as the Standard Model underlying our calculation is anomaly-free, no additional renormalization is required to restore the symmetries, at least to the order considered here [30,31].

III. INTEGRAL REDUCTION AND BOUNDARY CONDITION

Totally we obtain about 3×10^4 Feynman integrals, which are then grouped into 372 integral families. We decompose our target integrals into linear combinations of a smaller set of so-called master integrals using integration-by-parts (IBP) reduction [32]. To accomplish the decomposition, we first use LiteRed [33] and FiniteFlow [34] to generate and solve the system of IBP identities based on the Laporta’s algorithm [35] over a finite field. Around 200 numerical samplings are sufficient to construct the block-triangular relations for the target integrals proposed in Refs. [36,37]. The block-triangular relations are analytic in the variables t and ϵ , while the masses of the particles and s are fixed as rational numbers. By making

full use of these block-triangular relations, we are able to efficiently generate a large number of samplings (approximately 10^4) and eventually reconstruct the reduction coefficients. This strategy significantly reduces the computation time compared to reduction without the use of block-triangular relations. We further use symmetries among families to reduce the full set of integrals to a single minimal basis of master integrals. Working with this minimal basis allows us to analyze the properties of all master integrals in a uniform manner and avoid unnecessary duplication of computations. In total, we obtained 7675 master integrals which constitute the minimal basis for the problem.

We compute the minimal set of master integrals using differential equations (DEs) [38] based on power series expansion [39,40]. The DEs for the master integrals with respect to t are constructed using the IBP reduction method discussed earlier. The boundary conditions (BCs), chosen at $t/m_t^2 = -1/2$, are fixed using the auxiliary mass flow method [20,41–43] implemented in AMFlow [44]. Specifically, we use the “mass” mode [20] to treat internal masses (m_W^2, m_Z^2, m_H^2) as our auxiliary mass parameter, allowing us to make great use of the auxiliary mass flow method while keeping the number of master integrals unchanged. We defer the detailed discussion of the auxiliary flow method in the Appendix.

IV. SOLVING DIFFERENTIAL EQUATIONS

With the BCs fixed, we solve the DE for the minimal basis with respect to t/m_t^2 in order to obtain piecewise functions throughout the physical region, following an approach similar to those adopted in public packages such as DiffExp [45], SeaSyde [46], and line [47]. In the present case, however, the multiscale nature of the process leads to a large number of singularities, making the continuation procedure particularly challenging. For example, with $\sqrt{s} = 240$ GeV, singularities in the t/m_t^2 plane are shown in Fig. 2.

By investigating the asymptotic behaviors near singularities, with the help of DE and BCs, we find that only a few of them are physical while all the others are either spurious or nonphysical. Ideally, the convergent radius of a series expansion is limited only by physical singularities. In practice, however, spurious and nonphysical singularities can still affect numerical stability; see, for example, the Appendix of Ref. [48]. Since multiloop multiscale problems typically contain many such singularities, expanding around them is both inefficient and unnecessary. It is therefore important to understand their impact and develop strategies to mitigate their effects when solving the DEs.

A. Bypassing spurious and nonphysical singularities

The singularities appearing in the DEs can be classified into three types; 1) spurious, 2) nonphysical, and 3)

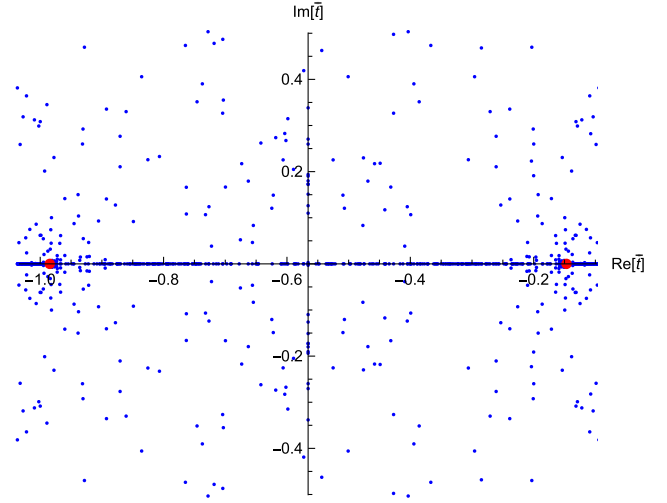


FIG. 2. The distribution of singularities of the minimal basis is shown as a function of $\bar{t} = t/m_t^2$ with $\sqrt{s} = 240$ GeV. Blue dots represent all singularities extracted from the differential equations. Two red dots on the real axis are the boundary points of the physical region.

physical, according to the behavior of the master integrals at those points. Physical singularities are singular on the physical Riemann sheet, appearing either as poles or branch cuts. A typical example is the production threshold. nonphysical singularities are regular on the physical sheet but become singular on other Riemann sheets; a well-known example is the pseudothreshold. Spurious singularities, on the other hand, are regular on all Riemann sheets. They do not reflect intrinsic properties of the master integrals and typically arise from the choice of master integrals, such as dimensional-regulator-dependent singularities.

We find that spurious singularities can always be bypassed by performing matching outside the singularity and increasing the working precision. In particular, consider a series expansion around x_0 . By matching the series at a nearby regular point x_b and keeping sufficiently high numerical precision during intermediate steps, one can effectively bypass spurious singularities lying inside the circle centered at x_0 with radius $R_b = |x_b - x_0|$. Here matching means determining the independent coefficients of the series expansion at x_0 by evaluating the expansion at x_b and matching it to the boundary condition provided there.

During this procedure, large intermediate numerical values can appear, scaling as $(R_b/R_s)^N$, where R_s denotes the distance from the expansion point to the nearest spurious singularity and N is the truncation order. These large factors are artifacts of solving DE and cancel in the final series expansion. An empirical estimate for the additional working precision required is $2N \log_{10}(R_b/R_s)$. Note that the numerical precision decrease rapidly outside the matching point, scaling as $N \log_{10}(|x - x_0|/R_b)$.

Whether a nonphysical singularity can be bypassed in the same way as a spurious singularity depends on whether the cancellation that renders the master integrals regular at that point is stable under truncation and region separation. In particular, the cancellation is protected if:

- (i) each master integral either contains the full structure responsible for the cancellation or does not contain it at all; and
- (ii) the cancellation does not mix contributions from different structures.

When these conditions are satisfied, the nonphysical singularity can be bypassed. The first condition can be understood as follows. Consider a nonphysical singularity $x = 1/3$ arising from $\frac{\log(3-2x)}{1-3x}$. If a master integral contains only part of this structure, namely the numerator, while the factor $\frac{1}{1-3x}$ appears explicitly in the DE, then cancellation between numerator and denominator is spoiled by the truncation involved in expanding $\frac{1}{1-3x}$ when solving the DE. If this condition is not satisfied, it can be restored by a basis transformation derived from the DE (see Appendix for an example). The second condition concerns series expansions around singular points. For instance, the expansion of $\frac{2 \log 2 + \log(x)/(1-x)}{1-2x}$ around $x = 0$ cannot bypass $x = 1/2$. The reason is that the regularity at $x = 1/2$ relies on a cancellation between two distinct structures; a logarithmic term and a Taylor series. When performing a series expansion around $x = 0$, the point $x = 1/2$ becomes a nonremovable singularity that limits the convergence radius of both contributions. There is no way to reorganize these two series into a single series that remains convergent in the region $x > 1/2$. In other words, the convergence radius of a singular series expansion is determined by the smallest convergence radius among all its structures.

B. Construction of the piecewise function

After discussing the properties of the series expansions, we now describe their application to the present problem. In the center-of-mass frame, the Mandelstam variable t is related to the Higgs boson scattering angle by

$$t = \frac{m_H^2 + m_Z^2}{2} - \frac{s}{2}(1 - \beta \cos \theta),$$

$$\beta = \frac{\sqrt{((s - (m_Z + m_H)^2)(s - (m_H - m_Z)^2))}}{s}. \quad (5)$$

We adopt the following numerical configuration:

$$\frac{s}{m_t^2} = \frac{83}{43}, \quad \frac{m_H^2}{m_t^2} = \frac{12}{23}, \quad \frac{m_Z^2}{m_t^2} = \frac{23}{83}, \quad \frac{m_W^2}{m_t^2} = \frac{14}{65}, \quad (6)$$

which correspond to $\sqrt{s} = 240$ GeV, $m_t = 172.69$ GeV [49], $m_H = 124.7$ GeV, $m_Z = 90.906$ GeV, and $m_W = 80.145$ GeV. The corresponding physical region under

the numerical configuration is $[-\frac{46436}{82087} - \frac{2\sqrt{295523395}}{82087}, -\frac{46436}{82087} + \frac{2\sqrt{295523395}}{82087}]^1$.

Our strategy for constructing the piecewise function in the physical region is combining the series expansions at the two endpoints with regular point expansions in the interior of the interval. We begin with the BCs at $t/m_t^2 = -1/2$. We choose the regular expansion point $t/m_t^2 = -0.264566$ such that the distance between the expansion point and the boundary point equals one half of the distance between the expansion point and the right endpoint. We then perform the series expansion and evaluate the master integrals at $t/m_t^2 = -0.186088$, which serves as the boundary condition for the next expansion. This procedure is repeated until the distance between the newly obtained boundary point and the right endpoint becomes smaller than half of the convergence radius of the endpoint expansion. At that stage, the endpoint series expansion can be computed with the desired precision. The left endpoint is treated in the same manner.

In total, we choose the following 19 points of t/m_t^2 :

$$\begin{aligned} &\{-0.984472, -0.984462, -0.984314, -0.983871, \\ &\quad -0.982542, -0.978554, -0.966590, -0.930698, \\ &\quad -0.823024, -0.500000, -0.264566, -0.186088, \\ &\quad -0.159929, -0.151209, -0.148303, -0.147334, \\ &\quad -0.147011, -0.146903, -0.146893\} \end{aligned} \quad (7)$$

to divide the physical region $(-0.984536, -0.146849)$ into 20 pieces. We note that nontrivial analytic structures arise only at the endpoints of physical region, while the interior is described by regular Taylor expansions. During the numerical evaluations, ϵ is set to a series of fixed small values and the ϵ dependence is reconstructed at the amplitude level as proposed in Refs. [43,44]. In particular, we solve the DE at $\epsilon = 10^{-4} + n \times 10^{-6}$, $n = 1, 2, \dots, 8$, using BCs evaluated with 60-digit precision.

In this strategy, we assume that regular point expansions can always bypass singularities within the physical region. This assumption is reasonable given the physical interpretation of the scattering angle and is confirmed in calculations. We did not attempt to further improve the convergence behavior at the endpoints through basis transformations, since the current segmentation is already sufficiently efficient. Nevertheless, we verified the validity of this approach across several individual families.

As an additional check, we evaluated the master integrals at $t/m_t^2 = -1/3$ using AMFlow and found agreement.

¹We implemented a differential-equation solver based on the DESolver module of AMFlow to handle square roots. The package is available at <https://gitlab.com/guan.xin/desolver>.

²For brevity, we list floating-point numbers here, while exact numerical values are used in the actual calculation.

Although the points $t/m_t^2 = -1/2$ and $t/m_t^2 = -1/3$ lie within the same segment, this evolution constitutes an important consistency check.³ It simultaneously verifies the correctness of the DE, the BCs, and the numerical continuation through regular points while bypassing spurious and/or nonphysical singularities. We also performed several internal consistency checks:

- (1) Evaluating a random point close to the boundary condition point using two adjacent series expansions and verifying the relative uncertainty satisfies the required precision;
- (2) Computing the convergence radius of each series expansion and confirming that it is consistent with the predefined segmentation; and
- (3) Generating two sets of series expansions with different truncation orders and checking that the precision of lower-order solution matches the expected accuracy.

Combining all ingredients, we obtain amplitudes as deep power series expansions in the physical region. After fitting the ϵ -dependence, the coefficient of the leading pole (ϵ^{-4}) is accurate to 28 digits, while the finite term is accurate to about 10 digits.⁴

To summarize the computational cost, the reduction of integrals in each family along with the construction of DE for the minimal basis with respect to t/m_t^2 took about $\mathcal{O}(10^4)$ CPU hours. The calculation of BCs took about 1.5×10^4 CPU hours, and the piecewise functions of master integrals were obtained in 3×10^3 CPU hours.

Numerical Results.—The two-loop electroweak contribution, denoted as $\mathcal{A}^{(2)}$, is defined as

$$\mathcal{A}^{(n)} = \frac{1}{4} \sum_{\text{spin}} \Re(\mathcal{M}^{(n)} \mathcal{M}^{(0)*}), \quad (8)$$

where $\mathcal{M}^{(n)}$ is the n -loop amplitude, $\mathcal{M}^{(0)}$ is the tree level amplitude, and $\frac{1}{4}$ means averaging over the spins of $e^\pm$.

As a benchmark, we provide numerical results under the numeric configuration in Eq. (6) and the following kinematics $\frac{t}{m_t^2} = -\frac{5}{7}$, i.e., $t = -21301.3 \text{ GeV}^2$. By choosing the regularization scale $\mu = m_Z$ and defining a conventional $\overline{\text{MS}}$ factor $S_\epsilon = (4\pi e^{-\gamma_E})^\epsilon$, we obtain the following results with 10-digit precision:

³AMFlow solves differential equations in the auxiliary parameter η . In the evolution from $\eta = \infty$ to $\eta = 0$, all singularities of the η -differential system, including spurious ones, are treated as genuine singularities when constructing the piecewise integration path. Consequently, the integration never attempts to cross them. The issue of crossing spurious or nonphysical singularities is specific to differential equations formulated directly in kinematic variables.

⁴The precision of the leading coefficient can be inferred from the absolute value of the ϵ^{-5} coefficient, whose true value vanishes. The precision decreases by about four digits per order due to the choice of the numerical value of ϵ .

$$\frac{\mathcal{A}^{(0)}}{\alpha^2} = 74.56296959 + \mathcal{O}(\epsilon), \quad (9)$$

$$\begin{aligned} \frac{2\mathcal{A}^{(1)}}{\alpha^3 S_\epsilon} &= -23.73413036\epsilon^{-2} + 553.6798599\epsilon^{-1} \\ &\quad - 1485.561791 + \mathcal{O}(\epsilon), \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{2\mathcal{A}^{(2)}}{\alpha^4 S_\epsilon^2} &= 1.888702084\epsilon^{-4} - 92.13508899\epsilon^{-3} \\ &\quad + 845.6490242\epsilon^{-2} + 3118.230629\epsilon^{-1} \\ &\quad + 5139.518437 + \mathcal{O}(\epsilon), \end{aligned} \quad (11)$$

where the factor of 2 is included to account for both the direct and complex conjugate contributions. We note that the current result depends on our choice of Feynman gauge and γ_5 scheme. Divergences should be canceled if all other contributions at the same perturbative order are taken into account, including ultraviolet counterterms and real emissions. Nevertheless, from the finite part we find that the two-loop EW corrections contribute around 0.4%, which points to their potential importance to match the precision of future Higgs factories.

In the attached ancillary file, a piecewise function of t/m_t^2 is provided at $\sqrt{s} = 240 \text{ GeV}$. The function is defined by several deeply expanded power series divided by the points in Eq. (7), which enables us to achieve a high precision at any given value of t . The result can be used for future studies. For other values of $\sqrt{s}$, results can be obtained similarly.

As a validation of our results, the master-integral check at $t/m_t^2 = -\frac{1}{3}$ has already been discussed above. Furthermore, our results for the double box diagrams were compared with those in Ref. [19] and found to be in agreement.

V. SUMMARY

In this paper we present the first complete calculation of the unrenormalized two-loop electroweak corrections for the Higgsstrahlung process $e^+e^- \rightarrow HZ$ at the future Higgs factories. Our result, for a given $\sqrt{s}$, is expressed as a piecewise function defined by several deeply expanded power series. These results have very high precision and can be used efficiently in future studies. This work is successfully accomplished thanks to the application of many state-of-the-art techniques. In particular, we develop an efficient method to construct piecewise solutions of the multiscale differential equations while bypassing removable singularities. Our results constitute the first complete calculation of the unrenormalized two-loop electroweak correction for processes with four external particles.

The present calculation constitutes a major step toward the complete NNLO electroweak corrections for $H + Z$ production at future Higgs factories. In the near future, by

also calculating the ultraviolet counterterms (e.g., in Ref. [50]) and real-emission corrections, the complete NNLO electroweak corrections for $H + Z$ production can be achieved. Our results suggest that the NNLO electroweak corrections can reach the precision demanded by future Higgs factories.

The approach employed in this Letter is also applicable for higher-order electroweak corrections for other processes, such as $e^+e^- \rightarrow W^+W^-$. These corrections will be important for precision measurements and the understanding of electroweak symmetry breaking.

Note added. When our paper was being finalized, a preprint appeared [51] which presents the fermion-loop contributions to the same process as ours. For the double-box fermion-loop topologies, whose amplitudes are finite, we find exact numerical agreement with the results of Ref. [51]. A direct comparison of the remaining contributions is not possible due to the use of different regulators.

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APPENDIX A: AN EXAMPLE OF AMFLOW

To illustrate how to evaluate Feynman integrals at a phase-space point, we take the integral family defined by the last Feynman diagram shown in Fig. 1 as an example, which is one of the most complex Feynman integral families. The integrals in this family can be expressed as

$$I_{\nu_1 \dots \nu_9} = \int \prod_{i=1}^2 \frac{d^D \ell_i}{i\pi^{D/2}} \frac{\mathcal{D}_8^{-\nu_8} \mathcal{D}_9^{-\nu_9}}{\mathcal{D}_1^{\nu_1} \dots \mathcal{D}_7^{\nu_7}}, \quad (\text{A1})$$

where inverse propagators can be chosen as

$$\begin{aligned} \mathcal{D}_1 &= \ell_1^2, & \mathcal{D}_2 &= (\ell_1 + k_1)^2 - m_W^2, & \mathcal{D}_3 &= (\ell_1 - k_2)^2 - m_W^2, \\ \mathcal{D}_4 &= (\ell_1 - \ell_2)^2 - m_H^2, & \mathcal{D}_5 &= (\ell_1 - \ell_2 + k_4)^2 - m_Z^2, \\ \mathcal{D}_6 &= (\ell_2 - k_2)^2 - m_W^2, & \mathcal{D}_7 &= (\ell_2 + k_1 - k_4)^2 - m_W^2, \\ \mathcal{D}_8 &= (\ell_1 - \ell_2 + k_1)^2, & \mathcal{D}_9 &= (\ell_1 - \ell_2 + k_2)^2, \end{aligned} \quad (\text{A2})$$

with the last two being irreducible scalar products introduced for completeness. During the computation, the masses of the particles and the Mandelstam variable s and t are fixed as rational numbers. This means that the integrals only depend on ϵ .

We use the “mass” mode [20] to insert an auxiliary mass parameter η into propagators $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_6$, and $\mathcal{D}_7$. By doing so, we simplify the integrals in the large mass limit [53,54] to some factorized one-loop integrals and vacuum integrals. Specifically, two subgraphs contribute in this limit. In the first region, ℓ_1 , ℓ_2 , and $(\ell_1 - \ell_2)$ scale as $\sqrt{\eta}$. Since external momenta and masses are of order unity, the contribution from this region is obtained by rescaling ℓ_i and performing a Taylor expansion in η . For example, the second propagator becomes

$$\begin{aligned} \frac{1}{\mathcal{D}_2} &\rightarrow \frac{1}{(\sqrt{\eta}\ell_1 + k_1)^2 - m_W^2 - \eta} \\ &= \frac{1}{\eta(\ell_1^2 - 1)} - \frac{2\ell_1 \cdot k_1}{\eta^{3/2}(\ell_1^2 - 1)^2} + \dots \end{aligned} \quad (\text{A3})$$

For $I_{1111111100}$, the leading term of the integrand contributes η^{D-7} with coefficient,

$$\frac{1}{\ell_1^2(\ell_1^2 - 1)^2(\ell_1 - \ell_2)^4(\ell_2^2 - 1)^2}, \quad (\text{A4})$$

which corresponds to a vacuum integral. Half-integer powers appearing in higher-order terms vanish because integrals of odd powers of loop momenta are zero. In the second region, both loop momenta scale as $\sqrt{\eta}$, while their difference is of order unity. It is convenient to shift the loop momentum as $\ell_2 \rightarrow \ell_1 - \ell'_2$ so that $\ell_1 \sim \sqrt{\eta}$, $\ell'_2 \sim 1$. For $I_{1111111100}$, the leading term of the Taylor expansion contributes $\eta^{\frac{D}{2}-5}$ with coefficient,

$$\frac{1}{\ell_1^2(\ell_1^2 - 1)^4(\ell_2'^2 - m_H^2)(\ell_2' + k_4)^2 - m_Z^2}, \quad (\text{A5})$$

which factorizes into the product of two one-loop integrals.

Assume that the simplified integrals are known for the first few terms of the large mass expansion, we now describe the auxiliary mass flow. The DE provides recursion relations that express higher-order expansion coefficients in terms of lower-order ones. The number of independent coefficients is typically equal to the number of master integrals (and can be smaller if degeneracies occur at singular expansion points). We solve the recursion relations at $\eta = \infty$ derived from the DE with respect to η , to obtain a high-order series expansions (e.g., 200 terms). The auxiliary mass is then numerically continued from $\eta = \infty$ to $\eta = 0$ using the η differential equation, which yields the physical result. During the continuation, the step size is chosen such that the evaluation point is always within half of the convergent radius of the series expansion. The truncation uncertainty is therefore approximately $(1/2)^{200} \approx 10^{-60}$. The continuation path is restricted to the lower half of the complex η plane in order to respect the Feynman prescription. This completes one flow, leaving only the simplified integrals in the large mass limit to be determined. In practice, the auxiliary mass flow method can be applied again to compute those remaining integrals.

We evaluate Feynman integrals with ϵ set to small numerical values, which provide boundary conditions for

differential equations in t . The evaluation of integrals in other families is similar in nature.

For reference, we fit the ϵ dependence at the level of master integrals and provide numerical results for the corner integral of the top sector in Eq. (A1), evaluated under the numeric configuration specified in Eq. (6) and $\frac{t_r}{m_t^2} = -\frac{5}{7}$, with 10-digit precision,

$$\begin{aligned} I_{1111111100} = & (0.3520383331 - 16.25324597i) \\ & + (10.99884113 - 64.23184545i)\epsilon \\ & + (32.18027541 - 134.3145818i)\epsilon^2 \\ & + (45.36688226 - 198.4594358i)\epsilon^3 \\ & + (27.95770697 - 234.3936106i)\epsilon^4 \\ & + \mathcal{O}(\epsilon^5). \end{aligned} \quad (\text{A6})$$

APPENDIX B: AN EXAMPLE OF BYPASSING NONPHYSICAL SINGULARITIES

We give a simple example to illustrate how to bypass nonphysical singularities. Consider the DE system,

$$A(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{x} & 0 & 0 & 0 \\ \frac{3}{3x-2} & 0 & 0 & 0 \\ \frac{2(-1+8x-21x^2+21x^3-6x^4-4x \log 2+22x^2 \log 2-36x^3 \log 2+18x^4 \log 2)}{(x-1)x(2x-1)^2(3x-2)(3x-1)} & \frac{3-4x}{(x-1)^2(2x-1)^2} & \frac{3}{(3x-1)^2} & 0 \end{pmatrix},$$

satisfied by the four master integrals,

$$\left\{ 1, \log(x), \log(2-3x), \frac{\log(2-3x)}{1-3x} + \frac{2 \log 2 + \frac{\log(x)}{1-x}}{1-2x} \right\}$$

which we denote by $f_1, \dots, f_4$.⁵

It is clear that both $1/3$ and $1/2$ are nonphysical singularities for f_4 . However, if we follow the same strategy used for spurious singularities and expand at $x = 3/10$, with BCs imposed at $x = 4/10$, then the nonphysical singularity $1/3$ lies between the expansion point and the matching point, and a substantial loss of numerical precision occurs. For example, with a 200-term expansion and sufficiently accurate BCs, one obtains only about 17 digits of precision. This precision can be estimated as $N \log_{10}(R_r/R_b) - N \log_{10}(R_b/R_s)$, where R_r denotes the true convergence radius determined by physical

singularities, while the factor (R_b/R_s) describes the precision loss caused by crossing the nonphysical singularity at x_s . If $R_b < R_s$, the second term should be omitted. In the present example, f_4 does not have a homogeneous contribution in the DE, so its truncation error is entirely determined by the other three integrals. Moreover, the only nontrivial integral involving the denominator $(3x-1)$ is f_3 . We therefore have $R_r = \frac{2}{3} - \frac{3}{10} = \frac{11}{30}$, $R_s = \frac{1}{3} - \frac{3}{10} = \frac{1}{30}$, $R_b = \frac{4}{10} - \frac{3}{10} = \frac{1}{10}$. As discussed in the main context, this behavior can be understood from the fact that the master integral f_3 violates the first condition, as it contains only part of the cancellation structure. The role of matching is to express the unknown coefficients in the series expansion in terms of BCs. In the present case, the coefficient matrix already loses precision before the BCs are substituted. By contrast, if we choose $\tilde{f}_3 = \frac{\log(3-2x)}{1-3x}$ as a master integral, then the cancellation is entirely encoded in the BCs. The coefficient matrix obtained from solving the DE no longer suffers from precision loss, and the nonphysical singularity at $x = 1/3$ can be bypassed effectively.

⁵For illustration purposes, we choose four master integrals, although they are linearly related.

The situation for $x = 1/2$ is similar to that for $x = 1/3$. Actually, instead of inspecting the analytic form of the master integrals, which is generally impossible in practical computations—we can infer the required basis transformation directly from DE. Since the expansion around $x = 1/2$ is regular, we can multiply both sides of the DE by $(x - 1/2)^2$ and then take the limit $x \rightarrow 1/2$. This yields a relation among the BCs,

$$f_2 + f_1 \log(2)|_{x=1/2} = 0. \quad (\text{B1})$$

Motivated by this relation, we define a new master integral,

$$\tilde{f}_2 = \frac{f_2 + f_1 \log(2)}{x - 1/2} \quad (\text{B2})$$

so that the cancellation is explicitly encoded in the BCs.

With the new set of master integrals $\{f_1, \tilde{f}_2, \tilde{f}_3, f_4\}$, we find that both $1/3$ and $1/2$ can be bypassed when expanding around either a regular point or a nonphysical singularity. For example, an expansion around $3/10$ with BCs imposed at $4/10$ now achieves approximately $200 \cdot \log_{10}(\frac{3/10}{1/10}) \approx 95$ digits of precision, limited only by the truncation error of $\tilde{f}_2$. In practice, it is convenient to solve the constraints at several nonphysical singularities such that the more complicated integrals are dressed by simpler ones. One can construct examples in which higher-order poles or higher-degree irreducible polynomial denominators are required for the cancellation, implying that such structures may also be needed in the redefinition of the master integrals.

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