

Quantum Kerr black hole from matrix theory of quantum gravity

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Recently, a quantum mechanical theory of quantum spaces described by a large N non-commutative coordinates is proposed as a model for quantum gravity [arXiv:2406.01466]. In this paper, we construct Kerr black hole as a rotating noncommutative geometry solution of this theory. Due to rotation, the fuzzy sphere is deformed into a fuzzy ellipsoid, which matches exactly the outer horizon of the Kerr black hole in the Boyer-Lindquist coordinates. Together with a half-filled Fermi sea, the fuzzy solution reproduces the Bekenstein-Hawking entropy as well as the mass and angular momentum of the Kerr black hole. These results provide support that the proposed quantum mechanics of quantum spaces as a model of quantum gravity.

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I. INTRODUCTION

General relativity is a classical dynamical theory of continuum spacetime. Classical spacetime is described by coordinate functions $x^\mu(P)$ defined over the spacetime manifold $\mathcal{M}$ and $P \in \mathcal{M}$ is any point (event) in the spacetime. Einstein theory of spacetime consists of the equipment of a metric on $\mathcal{M}$. One may allow the use of any coordinate systems and impose diffeomorphism invariance as a symmetry. The simplest Lagrangian which is consistent with the diffeomorphism symmetry is the Hilbert-Einstein action. One then show that this dynamical theory of spacetime contains gravity by finding solutions of the theory and by examining their effects on motion of probes described by geodesic motion.

A particular interesting and puzzling solution is the black hole. First it is not a consistent solution as it contains singularity [1], meaning that the description of black holes in terms of a classical spacetime is incomplete. Besides, it exhibits a Bekenstein-Hawking entropy [2,3] whose microscopic origin is unknown. (see however [4] for success in microstates counting for supersymmetric black holes in string theory). Nevertheless, the area dependent nature of the entropy suggests that quantum gravity is holographic [5,6]. Moreover, naive consideration of black hole evolution suggests a loss of quantum information [7–9]. It is

believed that only in a theory of quantum gravity can these problems be properly addressed and resolved.

In a recent paper [10], a formulation of quantum gravity in terms of a quantum mechanics of quantum space was proposed. The proposal was motivated by a number of considerations. Recent studies of black hole in the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [11] has cumulated at the success in obtaining the Page curve behavior [12,13] of the entanglement entropy of the Hawking radiation. However, the origin and nature of island, a new saddle point in the holographic entropy formula in the island proposal [14–17], is not clear from a fundamental point of view. Motivated by the curious feature that island is located just underneath the black hole horizon, we postulated [18] that the quantum states of black hole responsible for the Bekenstein-Hawking entropy are given by a thin shell of Bell degrees of fermionic freedom located at the region just underneath the horizon. This picture was further developed in [19,20] into a bottom-up model of quantum black hole where the energy and entropy of black holes were successfully accounted for.

To make progress, one wants to have a dynamical theory of quantum space where a quantum black hole is obtained as a solution. Our phenomenological model [19,20] suggests the desired theory must contains fermionic degrees of freedom. Fermions are featured in various popular approaches to quantum gravity such as string/M theory [21], AdS/CFT [11], BFSS quantum mechanics [22,23] or IKKT matrix model [24]. Since supersymmetry is always built-in in these setups, the fermion content as well as the supersymmetric action are highly constrained. We argued in [10] that there is actually no essential reason to insist on having supersymmetry if one is interested in quantum

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mechanics. Gathering the ideas discussed above, we proposed in [10] a $SU(N)$ quantum mechanics of non-commutative geometry as a theory of quantum gravity. The theory has a Higgs-like bosonic potential which allows for the construction of nontrivial “Higgs-vacuum.” This is possible since we do not assume supersymmetry. In general, solutions of the theory are given by dynamical noncommutative geometry. In [10], an interesting non-commutative geometric solution of fuzzy sphere was found. Together with a half-filled Fermi sea residing over the fuzzy sphere, the solution possesses energy of a Schwarzschild black hole and a microstates entropy which agrees precisely with the Bekenstein-Hawking entropy expected for a quantum black hole. It was also shown that the interaction energy between two fuzzy spheres has the correct dependence on the Newton constants and masses as required by gravity. It was conjectured that Newton gravity between static sources would be reproduced in the large distance limit.

In this paper, we construct a new time dependent solution of the quantum mechanics. The solution describes a rotating fuzzy ellipsoid. With a half-filled Fermi sea, the solution has a microstate counting that reproduces precisely the Bekenstein-Hawking entropy of a quantum Kerr black hole. Moreover, the angular momentum of the solution matches precisely that of the Kerr black hole and it has an energy matches that of the classical mass of Kerr black hole, with the leading correction at the order of a^4/r_+^4 . We propose that the fuzzy ellipsoid with a half-filled Fermi sea as a quantum description of the Kerr black hole in quantum gravity. Since a rotating black hole is a much more complicated system, the fact that our proposed dynamical theory of quantum space contains also the Kerr black hole provides further support that the proposed quantum mechanics theory as a model of quantum gravity.

II. KERR METRIC IN GENERAL RELATIVITY

In general relativity, a black hole with mass M and angular momentum J is described by the Kerr metric. The Kerr metric can be written in a number of different ways. See for example [25] for a brief introduction to the Kerr spacetime. In the Boyer-Lindquist coordinates, the Kerr metric reads

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar \sin^2\theta}{\rho^2} dt d\phi + \frac{\Sigma}{\rho^2} \sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2, \quad (1)$$

where

$$\begin{aligned} \rho^2 &:= r^2 + a^2 \cos^2\theta, & \Delta &:= r^2 - 2Mr + a^2, \\ \Sigma &:= (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta. \end{aligned} \quad (2)$$

The Boyer-Lindquist coordinates is a generalization of the Schwarzschild coordinates which allows the use of spherical coordinates. In the asymptotic region $r \rightarrow \infty$, we have

$$\begin{aligned} ds^2 = & -\left[1 - \frac{2M}{r} + O\left(\frac{1}{r^3}\right)\right] dt^2 \\ & - \left[\frac{4Mas \sin^2\theta}{r} + O\left(\frac{1}{r^3}\right)\right] d\phi dt \\ & + \left[1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right] [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \end{aligned} \quad (3)$$

This confirms that M is the mass and $J = aM$ is the angular momentum of the black hole spacetime. It is interesting to note that for $M = 0$, the metric (1) is flat and reads

$$ds^2 = -dt^2 + \frac{r^2 + a^2 \cos^2\theta}{r^2 + a^2} dr^2 + (r^2 + a^2 \cos^2\theta) d\theta^2 + (r^2 + a^2) \sin^2\theta d\phi^2. \quad (4)$$

This is in fact the standard Minkowski space written using the “oblate spherical” coordinates, which is related to the Cartesian coordinates with the coordinate transformation

$$\begin{aligned} x &= \sqrt{r^2 + a^2} \sin\theta \cos\phi, & y &= \sqrt{r^2 + a^2} \sin\theta \sin\phi, \\ z &= r \cos\theta. \end{aligned} \quad (5)$$

It is clear that, for $a \neq 0$, the oblate spherical coordinates (r, θ, ϕ) is different from the standard polar coordinates for 3-dimensional Euclidean space.

The Kerr metric is a stationary, axially symmetric and asymptotically flat vacuum solution to the Einstein equation. It has the Killing vectors $t^\alpha = \partial x^\alpha / \partial t$ and $\phi^\alpha = \partial x^\alpha / \partial \phi$. The metric is invariant under simultaneous inversion of $t \rightarrow -t$ and $\phi \rightarrow -\phi$, which is consistent with the fact that time reversal of a rotating object reverse the rotation of the object. The metric reveals a curvature singularity $R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \sim 1/\rho^{12}$ at $\rho = 0$. In addition the metric has a coordinate singularity at $\Delta = 0$. For $a \leq M$, the roots of $\Delta = 0$ are real. The larger solution

$$r_+ = M + \sqrt{M^2 - a^2} \quad (6)$$

denotes the position of the outer horizon. That the surfaces $r = r_\pm$ are event horizon can be seen from the fact that they are null there: $g^{\alpha\beta} \partial_\alpha r \partial_\beta r = g^{rr} = \Delta/\rho^2 = 0$. Examination of the null geodesics near the surface shows that they are indeed horizons. Viewed in the Cartesian coordinates (5), the horizon is an elliptical surface

$$\frac{x^2 + y^2}{r_+^2 + a^2} + \frac{z^2}{r_+^2} = 1. \quad (7)$$

We also remark that the outer horizon is a Killing horizon since the time like Killing vector $\xi^\alpha := t^\alpha + \omega_H \phi^\alpha$ becomes null at $r = r_+$, where

$$\omega_H = \frac{a}{r_+^2 + a^2} \quad (8)$$

is the angular velocity of the black hole.

For the Kerr metric, the horizon area is given by

$$A = 4\pi(r_+^2 + a^2), \quad (9)$$

which give rises to the Bekenstein-Hawking entropy

$$S = \frac{A}{4G} = \frac{\pi(r_+^2 + a^2)}{G}. \quad (10)$$

The mass of the Kerr black hole, written in terms of r_+ , is

$$M = \frac{r_+^2 + a^2}{2Gr_+} \quad (11)$$

and the angular momentum J

$$J = a \frac{r_+^2 + a^2}{2Gr_+}. \quad (12)$$

We will show below that these properties (10), (11), and (12) of the Kerr black hole are reproduced by a rotating solution in our proposed quantum mechanics of quantum space.

III. KERR BLACK HOLE FROM QUANTUM MECHANICS OF QUANTUM SPACE

Consider the proposal of [10] for a $SU(N)$ quantum mechanical theory of 3-dimensional quantized space defined by the Lagrangian

$$L = \text{tr} \left[\frac{1}{2M_0} \dot{X}^{a2} + \frac{M_P}{N^2} ([X^a, X^b]^2 + 4X^{a2}) + i\psi^\dagger \dot{\psi} - a_2 \frac{M_P}{N^2} \psi^\dagger \sigma^a X^a \psi \right] - a_3 r_X M_P, \quad (13)$$

or the Hamiltonian

$$H = \text{tr} \left[\frac{M_0}{2} P^{a2} - \frac{M_P}{N^2} ([X^a, X^b]^2 + 4X^{a2}) + a_2 \frac{M_P}{N^2} \psi^\dagger \sigma^a X^a \psi \right] + a_3 r_X M_P. \quad (14)$$

Here X_{mn}^a , $a = 1, 2, 3$ represent the coordinates of a 3-dimensional quantized space, σ^a are the Pauli matrices and $\psi_{mn}, \psi_{mn}^\dagger$ are two fermionic coordinates. The theory has $SU(N)$ symmetry as well as a flavor $SO(3)$ symmetry.

We do not assume supersymmetry, and so the choice of variables as well as the form of the Lagrangian is not dictated by symmetry. Instead, we took a bottom up approach in the art of phenomenological model building. The relative coefficient between the mass term and the commutator square term has been chosen to be a factor of 4. This can be achieved by rescaling the X 's. The Lagrangian is characterized by a single mass scale M_P called the Planck mass, which appears in front of the commutator square term with a unit coefficient. The Yukawa coupling and bosonic kinetic term have dimensionful coefficients which we specify in terms of M_P through the coefficient a_2 and a_0 where $M_0 = a_0^2 M_P$. In addition, we have included a topological action term specified by a_3 . Here r_X is the rank of the matrix $\Gamma := [X^a, X^b]^2$. The r_X term does not affect the equation of motion, but measures the energy of space due to non-commutativity. We have shown in [10] that the reproduction of the energy and entropy of the Schwarzschild black hole requires to identify the Planck mass and the Planck length in terms of the Newtonian constant G as

$$M_P = \frac{1}{\gamma} \sqrt{\frac{2}{\pi G}}, \quad l_P = \sqrt{\frac{2G}{\pi}}, \quad (15)$$

where $\gamma := a_2 - 1 + 2a_3$ and the coefficients a_2, a_3 are constrained such that $\gamma > 0$. In this paper, we show that these relations are unmodified (as required) and the rotating fuzzy sphere solution of the model matches well the shape, size, energy, entropy and the angular momentum of the Kerr black hole.

Let us consider dynamical solution to the theory. The classical equation of motion for a bosonic matrix configuration is given by

$$-\ddot{X}^a + \frac{4a_0^2 M_P^2}{N^2} ([X^b, [X^a, X^b]] + 2X^a) = 0. \quad (16)$$

The Kerr black hole is a stationary solution of general relativity with axial symmetry. Suppose the rotation is around the z -axis, then it is natural to consider a new basis of coordinates (X^+, X^-, X^3) where

$$X^\pm := \frac{1}{\sqrt{2}} (X^1 \pm iX^2). \quad (17)$$

In the new basis, the equations of motion reads

$$-\ddot{X}^+ + \frac{4a_0^2 M_P^2}{N^2} ([X^+, [X^+, X^-]] + [X^3, [X^+, X^3]] + 2X^+) = 0, \quad (18)$$

$$-\ddot{X}^3 + \frac{4a_0^2 M_P^2}{N^2} ([X^-, [X^3, X^+]] + [X^+, [X^3, X^-]] + 2X^3) = 0. \quad (19)$$

The equation of motion for X^- is not independent and can be obtained from the one (18) of X^+ . For rotation, let us consider the ansatz

$$X^\pm(t) = e^{\pm i\omega t} X^\pm(0), \quad X^3 \text{ independent of } t \quad (20)$$

which is appropriate for axial symmetric rotational motion. The equation of motion becomes

$$[X^+, [X^+, X^-]] + [X^3, [X^+, X^3]] + 2c^2 X^+ = 0, \quad (21)$$

$$[X^-, [X^3, X^+]] + [X^+, [X^3, X^-]] + 2X^3 = 0, \quad (22)$$

where $c^2 := 1 + N^2\omega^2/(8a_0^2 M_P^2)$ and the equations are written for time t . The equations (21) and (22) can be solved by utilizing the spin $j = (N-1)/2$ representation of $SU(2)$ in the Cartan-Weyl basis. They satisfy the commutation relations

$$[T^+, T^-] = T^3, \quad [T^3, T^\pm] = \pm T^\pm \quad (23)$$

and the Casimir relation

$$T^+ T^- + T^- T^+ + T_3^2 = \frac{N^2 - 1}{4} \mathbf{1}. \quad (24)$$

It is easy to see that

$$X^\pm(t) = e^{\pm i\omega t} T^\pm, \quad X^3(t) = c_3 T^3, \quad c_3 = \sqrt{1 + \frac{N^2\omega^2}{4a_0^2 M_P^2}} \quad (25)$$

solve the equation of motion. As $c_3 \geq 1$, we can introduce the definition

$$\cos \beta = 1/c_3, \quad (26)$$

which is a measures of the angular velocity. Let us next introduce the dimensional coordinates. To motivate the noncommutative version of (5), note that the matrices

$$\hat{T}^a := \frac{2}{\sqrt{N^2 - 1}} T^a \quad (27)$$

plays the role of the directional cosines ($\sin \theta \cos \phi$, $\sin \theta \sin \phi$, $\cos \theta$) in the noncommutative case. As a result, we adopt

$$Y^\pm = 2l_P X^\pm, \quad Y^3 = 2l_P X^3/c_3^2 \quad (28)$$

as the dimensionful Cartesian coordinates in the oblate spherical coordinate form. Here the normalization factor has been chosen such that, when written in terms of the “directional cosines” (27), we have

$$Y^\pm = l_P \sqrt{N^2 - 1} e^{\pm i\omega t} \hat{T}^\pm, \quad Y^3 = l_P \sqrt{N^2 - 1} \cos \beta \hat{T}^3. \quad (29)$$

This agrees with the form of (5) if the parameters R and a are extracted from our solution as

$$\sqrt{a^2 + R^2} = l_P \sqrt{N^2 - 1}, \quad R = l_P \sqrt{N^2 - 1} \cos \beta \quad (30)$$

and note the fact that our solution corresponds to the rotation $\phi = \omega t$. In terms of Y 's, our solution satisfies the commutation relations

$$[Y^+, Y^-] = \frac{2R/\cos \beta}{\sqrt{N^2 - 1}} Y^3, \quad [Y^3, Y^\pm] = \pm \frac{2R \cos \beta}{\sqrt{N^2 - 1}} Y^\pm \quad (31)$$

and the constraint

$$\frac{Y_1^2 + Y_2^2}{R^2 + a^2} + \frac{Y_3^2}{R^2} = 1. \quad (32)$$

Note that the relations (32), (31) are time independent and they define an oblate fuzzy ellipsoid for $\omega \neq 0$. It is interesting that the shape (32) of our solution agrees precisely with (7) of the Kerr black hole. This provides a first check of our claim that the rotating fuzzy ellipsoid solution provides a description of the quantum Kerr black hole. Note also that (30) takes on the simple form

$$R = Nl_P \cos \beta, \quad a = Nl_P \sin \beta \quad (33)$$

in the leading order of large N limit, which we will assume from now on.

Over the geometry of the fuzzy ellipsoid, the Hamiltonian of the system reads

$$H = H_B + H_F, \quad (34)$$

where H_B is the bosonic part of the Hamiltonian and H_F is the part of the Hamiltonian. It is

$$H_B = (2a_3 - 1) \frac{NM_P}{2} + \frac{N^3\omega^2}{12a_0^2 M_P}, \quad (35)$$

where we have used $\text{tr} T_3^2 = \frac{1}{3} \text{tr} T^2 = \frac{N^3}{12}$. The fermionic Hamiltonian $H_F = a_2 \frac{M_P}{N^2} \psi^\dagger \sigma^a X^a \psi$ is more interesting. Due to the anisotropy introduced by the coordinates (25), we can write

$$H_F = H_F^0 + h_F, \quad (36)$$

as an isotropic part

$$H_F^0 = a_2 \frac{M_P}{N^2} \psi^\dagger \sigma^a T^a(t) \psi \quad (37)$$

plus a term h_F

$$h_F = a_2(c_3 - 1) \frac{M_P}{N^2} \psi^\dagger \sigma^3 T^3 \psi := \psi^\dagger \tilde{h}_F \psi, \quad (38)$$

where in (37),

$$T^\pm(t) := e^{\pm i\omega t} T^\pm, \quad T^3(t) := T^3 \quad (39)$$

denote the generators of $SU(2)$ under a rotation of angle ωt around the z -axis; and we have introduced the matrix

$$\tilde{h}_F := a_2(c_3 - 1) \frac{M_P}{N^2} \sigma^3 T^3 \quad (40)$$

in (38). The term h_F is treated as a perturbation and is of the order of a^2/R^2 for slow rotation. The unperturbed Hamiltonian H_F^0 can be diagonalized as in [10] and give rises to a Fermi sea. In fact, the matrix $K := \sigma^a T^a(t)$ admits the eigendecomposition

$$K_{(m\alpha)(n\beta)} = \frac{N}{2} \sum_{p=1}^N (\mathcal{U}_{m\alpha}^p \mathcal{U}_{n\beta}^{p\dagger} - \mathcal{V}_{m\alpha}^p \mathcal{V}_{n\beta}^{p\dagger}), \quad (41)$$

where $\mathcal{U}_{n\beta}^p, \mathcal{V}_{n\beta}^p (p = 1, \dots, N)$ are eigenvectors of K , which now become time-dependent. They satisfy the orthonormality condition and the completeness relation

$$\mathcal{U}_{m\alpha}^{p\dagger} \mathcal{U}_{m\alpha}^q = \mathcal{V}_{m\alpha}^{p\dagger} \mathcal{V}_{m\alpha}^q = \delta^{pq}, \quad \mathcal{U}_{m\alpha}^{p\dagger} \mathcal{V}_{m\alpha}^q = 0, \quad (42)$$

$$\mathcal{U}_{m\alpha}^p \mathcal{U}_{n\beta}^{p\dagger} + \mathcal{V}_{m\alpha}^p \mathcal{V}_{n\beta}^{p\dagger} = \delta_{mn} \delta_{\alpha\beta}, \quad (43)$$

which are however time independent and identical to before. Introducing the fermionic oscillators

$$\xi_k^p := \mathcal{U}_{n\beta}^{p\dagger} \psi_{nk\beta}, \quad \chi_k^{p\dagger} := \mathcal{V}_{n\beta}^{p\dagger} \psi_{nk\beta}, \quad (44)$$

and after subtracting away a constant due to normal ordering, H_F^0 can be written in the diagonal form

$$H_F^0 = a_2 \frac{M_P}{2N} \left(\sum_{p,k=1}^N \xi_k^{p\dagger} \xi_k^p + \chi_k^{p\dagger} \chi_k^p \right) \quad (45)$$

and has the eigenstates

$$|\Psi_{k_1 \dots k_r l_1 \dots l_s}^{p_1 \dots p_r q_1 \dots q_s}\rangle := \xi_{k_1}^{p_1\dagger} \dots \xi_{k_r}^{p_r\dagger} \chi_{l_1}^{q_1\dagger} \dots \chi_{l_s}^{q_s\dagger} |0\rangle, \quad (46)$$

where $|0\rangle$ is the fermionic Fock vacuum. Denoting r, s the occupation number for the ξ -type and χ -type fermionic oscillators respectively, it reads

$$H_F^0 = a_2 \frac{M_P}{2N} (n + N^2), \quad n := r + s - N^2, \quad (47)$$

where $n = -N^2, \dots, 0, \dots, N^2$ specifies the energy level within the Fermi sea. The lowest level $n = -N^2$ corresponds to an empty Fermi sea, while the highest level $n = N^2$ corresponds to a completely filled Fermi sea.

We can divided the space of eigenstates of H_F^0 according to their corresponding level n . In general, the set of eigenstates at level n is given by

$$\mathbb{V}_n := \{|i\rangle, i = 1, \dots, \Omega_n\}, \quad \Omega_n = \binom{2N^2}{N^2 + n} \quad (48)$$

where $|i\rangle$ are those states (46) with $r + s = N^2 + n$ and Ω_n is the total number of states at the level n . As in [10], one see that the Bekenstein entropy bound [26] is satisfied for the fuzzy configurations with $n \geq 0$ and the bound is saturated at $n = 0$. This suggests that the $n = 0$ configuration, being most compactly packed, should be identified with a black hole. Let us therefore consider the configuration of a half-filled ($n = 0$) Fermi sea. We have

$$\Omega_0 = 2^{2N^2} \quad (49)$$

in the leading order of large N . Since classically we do not make observation of the quantum states, this give rises to a coarse grained entropy $S = \log_2 \Omega_0$:

$$S = 2N^2 = \frac{\pi(R^2 + a^2)}{G}. \quad (50)$$

This is precisely the Bekenstein-Hawking entropy of a Kerr black hole if we identify R with r_+ of the outer horizon radius of the Kerr black hole. To make this identification, we need to show that R satisfies the relation (11) for a given mass M of the Kerr black hole. For this, we need to include the energy of the system of Fermi oscillators in the fuzzy background, which we will do next.

For the half-filled Fermi sea, we have

$$H_F^0 = a_2 \frac{M_P N}{2} \quad (51)$$

and we are left with the contribution to the energy due to the perturbation h_F . On this, we may use the perturbation theory for degenerate energy level to compute the correction to the $n = 0$ energy level. This requires the knowledge of the matrix elements $\langle i|h_F|j\rangle$ and its eigenvalues, where $|i\rangle$ are the unperturbed states in $\mathbb{V}_0$. However, physically the set of microstates is not observed. It is more meaningful to consider the average corrections over the set of microstates. Since the system is in isolation, the set of microstates form a microcanonical ensemble. It was shown in the Appendix of [10] that the ensemble average of the corrections

$$\langle h_F \rangle_{\text{En}} := \frac{1}{\Omega_0} \sum_{|i\rangle \in \mathbb{V}_0} \langle i | h_F | i \rangle \quad (52)$$

is given by

$$\langle h_F \rangle_{\text{En}} = \text{tr}(K \tilde{h}_F). \quad (53)$$

This holds in general for any form of $\tilde{h}_F$. For the present case with (38), only the T^3 component of K contributes and so we get

$$\langle h_F \rangle_{\text{En}} = a_2 \frac{c_3 - 1}{6} N M_P. \quad (54)$$

We remark that in the current case of a fuzzy ellipsoid background, the eigenvalues of the fermionic Hamiltonian and hence the ensemble average (52) can be determined exactly; and (54) can be obtained consequently from a slow rotation expansion. We will demonstrate this in the Appendix of this paper. Adding (51), (54), to (35) and expressing it in terms of macroscopic quantities R and a , we have

$$E = \frac{R^2 + a^2}{2GR} f(\beta), \quad f(\beta) := \cos \beta \left(1 + \frac{2d}{3} \tan^2 \beta \right), \quad (55)$$

where $d := (1 + a_2/4)/\gamma$. It is $f = 1$ for $\beta = 0$. For $\beta \ll 1$, f admits an expansion in terms of $\tan^2 \beta$. This corresponds to an expansion of small angular momentum or small curvature since $\tan^2 \beta = a^2/R^2$. In order not to reproduce the Kerr mass, considered as a small a^2/R^2 expansion of the Schwarzschild mass, the leading term $\tan^2 \beta$ of $f(\beta)$ in (55) should vanish. This happens when $d = 3/4$, or equivalently

$$a_2 + 3a_3 = \frac{7}{2}. \quad (56)$$

This implies that f has the small β expansion

$$f = 1 + \frac{1}{8} \tan^4 \beta + \dots \quad (57)$$

and we obtain the result (11) of general relativity (GR)

$$M = \frac{R^2 + a^2}{2GR} \quad (58)$$

in the leading order of small a^2/R^2 . It is remarkable that the Yukawa term in the Lagrangian (13) not only give rises to the needed microstates to account for the entropy of quantum black hole, but also makes contribution to the energy of the fuzzy system, without which the mass of the GR Kerr black hole would not be obtained. We note that in the case of the Schwarzschild black hole [10], the

Hamiltonian over the fuzzy geometry can be treated exactly and our large N quantum mechanics reproduces precisely the GR result for the Schwarzschild mass. On the contrary for the present case of rotating black hole, a perturbative treatment is needed to handle the Hamiltonian over the fuzzy ellipsoid geometry and the result (58) was obtained at the order of small a^2/R^2 . To check it at the next order, the second order perturbative result and the $\sin^4 \beta$ contribution of (57) should add up to zero at the order of a^4/R^4 . In general, to reproduce (58) for arbitrary angular momentum a/R , some nontrivial cancellation among different orders of perturbation theory should happen. This suggests the presence of integrability [27] and is a nontrivial test of our proposal.

The Kerr black hole also carries an angular momentum (12). To facilitate the comparison with the quantum mechanics prediction, let us rewrite it in terms of the matrix model quantities. Using (33) and (26), we obtain

$$J_{GR} = \frac{\omega N^3}{2\pi a_0 M_P}. \quad (59)$$

Let us check it against our rotating solution. In the quantum mechanics, the rotational symmetry give rises to the angular momentum generators

$$L^a = \epsilon_{abc} \text{tr} X^b P^c, \quad S^a = \text{tr} \psi^\dagger \frac{\sigma^a}{2} \psi. \quad (60)$$

For the rotating fuzzy sphere solution, it is easy to obtain

$$L^3 = \frac{\omega N^3}{6a_0^2 M_P}, \quad L^{1,2} = 0. \quad (61)$$

As for the spin, since the half-filled Fermi sea is not observed in GR, we consider the ensemble average $\langle S^a \rangle_{\text{En}}$. It is then easy to use (53) for $\tilde{h}_F = \sigma^a/2$ to obtain

$$\langle S^a \rangle_{\text{En}} = 0. \quad (62)$$

Therefore the angular momentum of the fuzzy sphere is entirely given by the orbital L_3 . It is interesting to note that (59) and (61) takes has the same dependence in N and a precise agreement is possible if

$$a_0 = \frac{\pi}{3}. \quad (63)$$

All in all, the proposed quantum mechanics with $a_0 = \pi/3$ and (56) contains solutions that match well with the GR description of the Schwarzschild and the Kerr black holes. In addition the black hole entropy is reproduced by a microstate counting. It is important to find out if the Newtonian potential between Schwarzschild black holes can be reproduced with a further constraint on the coefficients of the model.

Finally, we finish the paper with some discussions. We remark that in our quantum mechanics, we have, for finite N , quantum spaces and quantum gravity corrections are given by an $1/N$ expansion. It is interesting to understand how the large N expansion give rises to the low energy effective expansion of gravity and beyond. It is also interesting to explore if there is some noncommutative geometric way to express these quantum corrections. In string theory, the entropy of certain class of extremal black hole has been reproduced beautifully by a counting of Bogomolny-Prasad-Sommerfield (BPS) states in a weakly coupled gauge theory [4]. There the gauge system becomes a black hole in the strong coupling regime and the counting is protected by supersymmetry. Our counting is different. We do not rely on the use of supersymmetry or duality. Instead, we employ noncommutative geometry to model quantum gravity and we counted the fermionic microstates that makes up the black hole. Let us also comment on the black hole thermodynamics. Although our fuzzy solutions are classically stable, quantum mechanically there can be tunneling between them. We suspect that this gives the Hawking radiation and give rises to the black hole thermodynamics. There are also other interesting problems. It is interesting to studying the form factor [28,29] for our quantum mechanics and study how chaotic behavior of black hole arises in the large N limit. It is interesting to exact spin dependent force between black holes from the quantum mechanics. It is also interesting to think about how the ergosphere of Kerr black hole appears in our description. A probe analysis may help to reveal it. We leave these interesting problems for future consideration.

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APPENDIX: ENSEMBLE AVERAGE FROM THE EXACT EIGENVALUES

In the main text of the paper, we have obtained the ensemble average (54) for the Hamiltonian h_F by treating it as a perturbation. In this appendix, we show that the eigenvalues of the total Hamiltonian $H_F = H_F^0 + h_F$ can be determined exactly and the result (54) can be obtained from an average of H_F over the ensemble of microstates.

Consider in general adjoint fermions $\psi_{m\alpha}$ of $SU(N)$ and a Hamiltonian

$$H := \text{tr} \psi^\dagger \tilde{H} \psi, \quad (\text{A1})$$

where $(\tilde{H})_{(m\alpha)(n\beta)}$ is an arbitrary matrix. Suppose $\tilde{H}$ admits the eigenvectors $\mathcal{U}^{p+}, \mathcal{V}^{p-}$ that corresponds to positive and negative eigenvalues $\lambda_{p+}, \lambda_{p-}$ respectively. We have the

eigendecomposition of $\tilde{H}$

$$(\tilde{H})_{(m\alpha)(n\beta)} = \sum_{p+} \lambda_{p+} \mathcal{U}_{m\alpha}^{p+} \mathcal{U}_{n\beta}^{p+\dagger} + \sum_{p-} \lambda_{p-} \mathcal{V}_{m\alpha}^{p-} \mathcal{V}_{n\beta}^{p-\dagger}. \quad (\text{A2})$$

The eigenvectors satisfy the orthonormality condition

$$\mathcal{U}_{m\alpha}^{p+\dagger} \mathcal{U}_{m\alpha}^{q+} = \delta^{p+q+}, \quad \mathcal{V}_{m\alpha}^{p-\dagger} \mathcal{V}_{m\alpha}^{q-} = \delta^{p-q-}, \quad \mathcal{U}_{m\alpha}^{p+\dagger} \mathcal{V}_{m\alpha}^{q-} = 0 \quad (\text{A3})$$

and the completeness relation

$$\sum_{p+} \mathcal{U}_{m\alpha}^{p+} \mathcal{U}_{n\beta}^{p+\dagger} + \sum_{p-} \mathcal{V}_{m\alpha}^{p-} \mathcal{V}_{n\beta}^{p-\dagger} = \delta_{mn} \delta_{\alpha\beta}. \quad (\text{A4})$$

Introduce the fermionic oscillators defined by

$$\xi_k^{p+} := \mathcal{U}_{n\beta}^{p+\dagger} \psi_{nk\beta}, \quad \chi_k^{p-} := \mathcal{V}_{n\beta}^{p-\dagger} \psi_{nk\beta}, \quad (\text{A5})$$

H takes the diagonal form

$$H = \sum_{p+,k} \lambda_{p+} \xi_k^{p+\dagger} \xi_k^{p+} - \sum_{p-,k} \lambda_{p-} \chi_k^{p-\dagger} \chi_k^{p-}. \quad (\text{A6})$$

Here we have chosen a normal ordering prescription such that $:\chi \chi^\dagger := -\chi^\dagger \chi$ etc. H has the eigenstates

$$|\Psi_{k_1 \dots k_r, l_1 \dots l_s}^{p_1 \dots p_r, q_1 \dots q_s}\rangle := \xi_{k_1}^{p_1\dagger} \dots \xi_{k_r}^{p_r\dagger} \chi_{l_1}^{q_1\dagger} \dots \chi_{l_s}^{q_s\dagger} |0\rangle, \quad (\text{A7})$$

where $|0\rangle$ is the Fock vacuum. Let us consider the set of states (A7) that corresponds to a half-filled Fermi sea (HFS)

$$\mathbb{V}_{\text{HFS}} := \{|i\rangle, i = 1, \dots, \Omega_0\}, \quad (\text{A8})$$

where $|i\rangle$ are those states (A7) with $r + s = N^2$ and $\Omega_0 = \binom{2N^2}{N^2}$ is the number of states that satisfy this condition. We are interested in the ensemble average defined by

$$\langle f \rangle_{\text{En}} := \frac{1}{\Omega_0} \sum_{|i\rangle \in \mathbb{V}_0} \langle i | f | i \rangle. \quad (\text{A9})$$

As a result, we have

$$\langle H \rangle_{\text{En}} = \sum_{p+} \lambda_{p+} n(p+) - \sum_{p-} \lambda_{p-} n(p-), \quad (\text{A10})$$

where $n(p_\pm)$ denotes the occupation number for the eigenvalue level $p_\pm$

$$n(p+) := \sum_k \langle \xi_k^{p+\dagger} \xi_k^{p+} \rangle_{\text{En}}, \quad n(p-) := \sum_k \langle \chi_k^{p-\dagger} \chi_k^{p-} \rangle_{\text{En}}. \quad (\text{A11})$$

For an isolated system, the occurrence probability is the same for all energy levels, therefore

$$n(p_+) = n(p_-) = \frac{N}{2}. \quad (\text{A12})$$

This is obtained by recalling that the total number of p_+ and p_- is $2N$ in the large N limit and that we have the constraint

$$\sum_{p_+} n(p_+) + \sum_{p_-} n(p_-) = N^2 \quad (\text{A13})$$

for a system of microstates that filled half of the Fermi sea. As a result, we obtain

$$\langle H \rangle_{\text{En}} = \frac{N}{2} \left(\sum_{p_+} \lambda_{p_+} - \sum_{p_-} \lambda_{p_-} \right). \quad (\text{A14})$$

This result holds generally independent of the form $\tilde{H}$ of the Hamiltonian. For our QM, $\tilde{H} = \frac{a_2 M_P}{N^2} \sigma^a X^a$. For the single fuzzy sphere background, $\lambda_{p_+} = -\lambda_{p_-} = \frac{a_2 M_P}{N^2} \cdot \frac{N}{2}$ and so $\langle H \rangle_{\text{En}} = a_2 M_P N/2$. For the displayed fuzzy spheres background, the same result is obtained since the relative displacement of the fuzzy spheres induces a magnetic dipole term whose ensemble average is zero [10].

Let us now consider the rotating fuzzy sphere system. In this case, consider $K = \sigma^a T^a(t) + (c_3 - 1)\sigma_3 T^3$ where T^a are the spin $j = (N - 1)/2$ representation of $SU(2)$ as specified by (23), (24) and $T^a(t)$ as defined in (39) denotes the generators of $SU(2)$ under a rotation. It is easy to see that K is diagonalized by the angular momentum states $|j, m\rangle$, $m = -(N - 1)/2 \dots, (N - 1)/2$. We obtain the eigenvalues

$$\lambda_+ = c_3 \frac{N-1}{2} \quad \text{and} \quad -\frac{c_3}{2} + \frac{1}{2} \sqrt{N^2 + (c_3^2 - 1)(2m+1)^2},$$

$$m = -\frac{N-1}{2}, \dots, \frac{N-1}{2}, \quad (\text{A15})$$

$$\lambda_- = -\frac{c_3}{2} - \frac{1}{2} \sqrt{N^2 + (c_3^2 - 1)(2m-1)^2},$$

$$m = -\frac{N-1}{2} + 1, \dots, \frac{N-1}{2}. \quad (\text{A16})$$

There are in total $N + 1$ positive eigenvalues (λ_+) and $N - 1$ negative ones (λ_-). Note that

$$\sum \lambda_+ + \sum \lambda_- = 0, \quad (\text{A17})$$

reflecting the fact that K is traceless. Note also that in the undeformed case $c_3 = 1$, the eigenvalues reduce to their correct limit: $\lambda_+ = (N - 1)/2$, $\lambda_- = -(N + 1)/2$.

With the set (A15) of eigenvalues, we can now compute the ensemble value $\langle H \rangle_{\text{En}}$ using (A14). Expanding (A15) in the limit of small $(c_3 - 1)$, we obtain in the leading limit of large N and small $(c_3 - 1)$,

$$\sum \lambda_+ - \sum \lambda_- = N^2 + (c_3 - 1) \frac{N^2}{3}. \quad (\text{A18})$$

Reinserting back the proportional coefficient $\frac{a_2 M_P}{N^2}$, we final obtain

$$\langle H \rangle_{\text{En}} = \frac{a_2 M_P N}{2} + a_2 M_P N \frac{c_3 - 1}{6}. \quad (\text{A19})$$

This agrees precisely with (51) and (54), in which the latter was obtained from (53) without computing the exact eigenvalues.

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