Two-loop spacelike splitting amplitude for $\mathcal{N}=4$ Super-Yang-Mills theory

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The study of collinear behavior for gauge theories in the spacelike region is of great phenomenological and theoretical importance. We analytically calculate the two-loop spacelike splitting amplitude for the full color $\mathcal{N}=4$ super-Yang-Mills theory. The result is derived by two complementary methods starting from the known amplitude: one is based on a discontinuity analysis, while the other one is based on calculating Feynman integrals analytically in the collinear limit. Our result explicitly shows terms that violate naive factorization. However we show that factorization is restored at the level of color-summed unpolarized squared amplitudes at next-to-next-to-next-to leading order. We conjecture that the two-loop tripole terms in the generalized splitting amplitudes in quantum chromodynamics are identical to what we obtain in $\mathcal{N}=4$ super Yang-Mills theory.

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I. INTRODUCTION

The factorization of perturbative quark-gluon dynamics from nonperturbative hadron dynamics lies at the heart of high-energy collider physics, such as the Large Hadron Collider (LHC). It allows the prediction of hard scattering cross sections, denoted by σ, through the schematic formula

$$\sigma = f \otimes f \otimes \hat{\sigma}, \tag{1}$$

where f represents the parton distribution functions (PDFs) parametrizing the nonperturbative partonic structure of hadrons, and $\hat{\sigma}$ is the partonic cross section calculated from on-shell scattering amplitudes of quarks, gluons, and other Standard Model particles. Building on the factorization theorem, impressive results have been achieved for the precision program at the LHC, examples ranging from next-to-next-to-leading order (NNLO) high multiplicity processes [1–6] to next-to-next-to-next-to-leading order (NNNLO) color singlet production at the LHC [7–21]. Besides the challenges in calculating virtual scattering amplitudes, an essential aspect underlying recent progress is the development of infrared subtraction methods, either in the form of local subtraction or slicing [22–34]. Both approaches rely on the factorization of soft and collinear singularities. For a recent review, we refer the reader to [35].

The predictive power of the factorization theorem in Eq. (1) relies on the universality of the PDFs, meaning that they should be independent of the underlying scattering processes. Given the importance of factorization, it has been studied with great efforts since the early days of quantum chromodynamics (QCD), resulting in a remarkable proof of factorization for unpolarized Drell-Yan production at hadron colliders [36,37]. For other important processes such as jet production at hadron colliders, a rigorous proof of factorization is currently absent.

In recent years, the study of high-energy scattering in extreme kinematics, notably scattering in the transverse-momentum-dependent region, has revealed an interesting possibility of factorization violation [38–40]. In particular, an explicit counterexample has been found starting from one-loop in a toy model for single-spin asymmetry in dihadron production at hadron colliders by Collins and Qiu [41]. They also suggest that a class of active-spectator diagrams at two loops can potentially lead to cross-section-level violation of collinear factorization for unpolarized dihadron production at N^{3}LO. For more inclusive observable, such as hadronic transverse energy E_T [42] and beam thrust [43], factorization violation has also been discussed.

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So far our discussion has been on cross-section-level observable. Since cross section are computed from square of scattering amplitudes, it is natural to ask whether there is collinear factorization violation at amplitude level. This problem was addressed in a seminal paper by Catani, de Florian, and Rodrigo, who show that infrared poles of spacelike splitting amplitudes for multijet production can depend on the color and kinematic information of noncollinear partons at two loops [44], manifestly violating amplitude-level collinear factorization. They argue that after integrating over the collinear phase space, such nonfactorization contributions can lead to process-dependent collinear singularities that cannot be canceled by PDF renormalization, thereby potentially invalidating the universality of PDFs for such processes at sufficiently high order.

In both cases, the origin of potential factorization breaking comes from the loop corrections to the spacelike collinear limit, where a collinear parton is emitted from the incoming parton. Using soft-collinear effective theory for forward scattering [45], such effects have been related to the exchange of Glauber gluons [46,47]. Explicitly, for a massless scattering process $p_1 + p_2 \to p_3 + p_4 + p_5$, the amplitude in the spacelike collinear limit, $p_2 \| p_3$, factorizes as

$$\mathcal{A}_5(p_1, p_2, p_3, p_4, p_5) \xrightarrow{p_2 \| p_3} \mathbf{Sp} \times \mathcal{A}_4(p_1, \tilde{P}, p_4, p_5). \quad (2)$$

In the usual strict collinear factorization, the splitting amplitude **Sp** depends on the quantum numbers and kinematics of the collinear pair only. However, Ref. [44] found that for spacelike collinear limits such as those in (2), the splitting amplitude necessarily involves the quantum numbers and momenta of noncollinear partons. Due to potentially complicated kinematic and color dependencies, the full factorization-violating splitting amplitudes have not yet been computed, except for their infrared singularities, although the strictly collinear factorized splitting amplitudes are known up to three loops in QCD [48].

Given the importance of the collinear factorization for collider physics, significant effort has been spent examining the consequences of the generalized collinear factorization formula in (2). Interestingly, the pole terms of the amplitude-level factorization violation found in [44] cancel at the squared cross-section level, owing to a form of exponentiation of the infrared singularities [46]. For nonpole terms, the soft limit of the generalized splitting amplitude has been studied in [49], where a cancellation at the cross-section level has also been found. Recently, amplitude-level factorization violation has also been investigated in the context of more than one collinear direction [50].

Despite these efforts, a conclusive statement about the possibility of factorization breaking or not at the cross-section level from spacelike splitting such as (2) is still missing. In this paper, we initiate a systematic study in this direction by computing, for the first time, the *finite* terms in the generalized splitting amplitude in (2) to two loops, as depicted in Fig. 1. We perform the calculation in $\mathcal{N} = 4$ super Yang-Mills theory ($\mathcal{N} = 4$ SYM), a close cousin of QCD that shares similar infrared behavior in perturbative theory. Building upon the remarkable data for high multiplicities amplitudes in the literature [51–56], we systematically develop techniques for the analytic continuation of high-multiplicity amplitudes from timelike collinear kinematics to spacelike collinear kinematics. We believe such techniques will be useful in future investigations of more complicated factorization-breaking configurations.

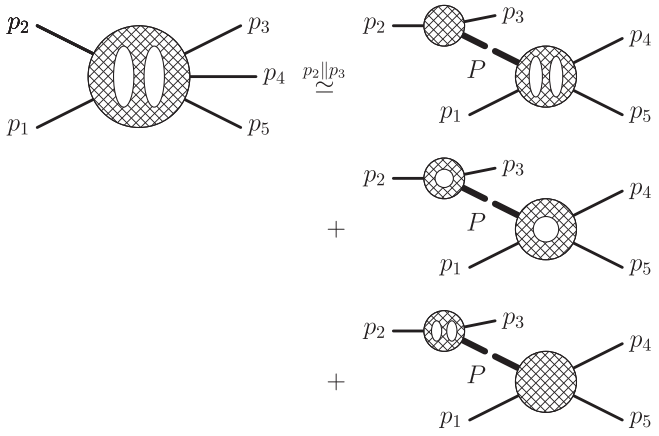


FIG. 1. Schematic form of spacelike collinear factorization for two-loop five-particle scattering. The full factorized expression involves the generalized spacelike splitting amplitude at tree, one- and two loops, as well as the $2 \to 2$ amplitudes from tree to two loops.

While the intermediate steps of our calculation involve lengthy expressions, remarkably, the final generalized splitting amplitudes in $\mathcal{N} = 4$ SYM have a simple form. More importantly, the potential factorization-breaking terms in the generalized splitting amplitude enjoy a partially exponentiated form, causing the factorization-breaking effects to cancel at the cross-section level.

This paper is organized as follows. We first calculate the two-loop five-point massless integrals in the spacelike collinear limit in terms of Goncharov polylogarithms (GPLs) [57] and then assemble the $\mathcal{N} = 4$ two-loop five-point amplitude in this limit. These computations provide a solid ground for the colliner factorization violating discussion. In parallel, we analyze the analytic continuation of five-point scattering amplitudes, and explicitly identify the collinear factorization violating terms. This computation agrees with the result from our master integral computations. Based on these computations, we finally derive the generalized two-loop splitting amplitudes, which constitutes our main result.

II. SPACELIKE COLLINEAR LIMIT OF TWO-LOOP FIVE-POINT MASSLESS FEYNMAN INTEGRALS

In this section, we calculate all two-loop five-point massless master integrals in the spacelike collinear region, up to weight 4 in terms of GPLs. The result will be used for calculating the $\mathcal{N}=4$ two-loop five-point amplitude and then the two-loop splitting function in this limit. Although those integrals were calculated in [51,58] and their expressions have been implemented in the package PentagonMI [52], we find that new simple expression can be obtained by calculating them directly in the collinear limit.

The $2\to 3$ scattering for massless particles is characterized by the five Mandelstam variables s_{12}, s_{23}, s_{34}, s_{45} and s_{15}. The $12\to 345$ physical region is,

$$s_{12}>0,\quad s_{23}<0,\quad s_{34}>0,\quad s_{45}>0,\quad s_{15}<0, \tag{3}$$

and $\mathrm{Im}(\epsilon_5)>0$, where $\epsilon_5\equiv 4i\varepsilon_{\mu_1\mu_2\mu_3\mu_4}p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_4^{\mu_4}$.

There are three types of two-loop five-point massless Feynman integral families, namely pentagon-box, hexagon-box and double pentagon. Each family has $5!=120$ permutations. The corresponding integral basis with uniform transcendentality (UT), canonical differential equation was obtained in Refs. [51,58–60]. There are 31 letters for the integrals [61]. In summary, there are $3\times 120=360$ differential equation systems. The boundary values and the iterative integral form were calculated in [51]. The analytic boundary value at a point

$$X_0\colon\ \{s_{12},s_{23},s_{34},s_{45},s_{15}\}=\{3,-1,1,1,-1\} \tag{4}$$

is available in Ref. [52]. Furthermore in the $12\to 345$ scattering region, up to weight 2, all master integrals are obtained in terms of classical polylogarithms, while the weight 3 and 4 parts are expressed as one-fold integrals for the fast numeric evaluation [52].

Our goal is to get all two-loop five-point massless integrals up to the weight 4 in terms of polylogarithms, in the spacelike collinear region. Without loss of generality, we consider $2\|3$, and a generic point P in this region can be parametrized as,

$$\{s_{12},s_{23},s_{34},s_{45},s_{15}\}=\{sz,-4\delta^2,(1-z)xs,s,xs+c\delta\} \tag{5}$$

with the parameter range $s>0$, $z>1$, $x<0$, $\delta\to 0$.

We solve the canonical differential equations along the integration path illustrated in Fig. 2. It is important to get the boundary value at a point in that region. This point is chosen as,

$$X_1\colon\{s_{12},s_{23},s_{34},s_{45},s_{15}\}=\{4,-4\delta^2,1,2,-1\} \tag{6}$$

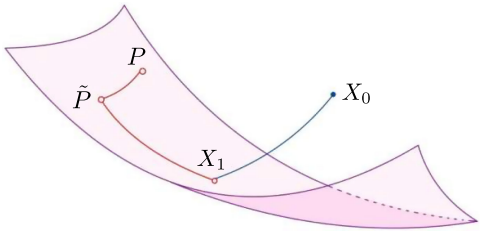


FIG. 2. Path for solving the canonical differential equation of two-loop five-point integrals in the spacelike region. One starts from the base point X_0, and integrates along a contour to a special point X_1 on the $2\|3$ collinear region. Two further integrations are taken to reach the generic collinear region point P.

where δ is a small positive number. Boundary value at X_1 can be determined by solving the canonical differential equation in λ, along a curve from X_0 to X_1,

$$\{s_{12},s_{23},s_{34},s_{45},s_{15}\}=\left\{\frac{4}{\lambda^2+1},\frac{-4\lambda^2}{\lambda^2+1},1,\frac{2-2\lambda^2}{\lambda^2+1},-1\right\} \tag{7}$$

The base point X_0 corresponds to $\lambda=1/\sqrt{3}$, while X_1 corresponds to $\lambda\to\delta$. It is straightforward to solve the differential equation along this curve to obtained boundary values at X_1, which are polynomials of $\log\delta$.

A high-precision PSLQ computation determines the coefficients of $\log\delta$ for all boundary values at X_1 as the combinations of constants used in Ref. [52] for the boundary value at X_0. Therefore the boundary values at X_1 are simplified to a compact form. With analytic boundary values, consider two steps to evaluate all two-loop five-point massless integrals at a generic spacelike collinear point P,

(i) Step 1. Integrate from X_1 to a point $\tilde{P}$ in the spacelike collinear region with $c=0$,

$$\{s_{12},s_{23},s_{34},s_{45},s_{15}\}=\{sz,-4\delta^2,(1-z)xs,s,xs\} \tag{8}$$

Factorize out s by the dimensional analysis, and the original 31 letters are reduced to 10 rational letters in the limit $\delta\to 0$.

$$x,1+x,-1+z,z,-1-x+z,x+z,1+xz,\\ -1+z+xz,-1-x+xz,-x+z+xz \tag{9}$$

Then we solve the differential equation analytically along a path from X_1 to $\tilde{P}$, by integrating x from $-\frac{1}{2}$ and z from 2. The dependence on s is restored via dimensional analysis. The result is written in GPLs.

(ii) Step 2. The dependence on c needs a special treatment. We carry out the integration for the variable c to get the master integrals evaluated at

a generic point P in the spacelike collinear region, from the result at $\tilde{P}$ in the previous step. The differential equation in c contains the square root from ϵ_5. The following substitution allows us to rationalize ϵ_5

$$c \equiv \frac{-8s^2x(1+x)yz\sqrt{-s^3x(1+x)(-1+z)z}}{1+y^2}, \quad (10)$$

Here the new variable y would be related to the cross ratios z_I and $\bar{z}_I$ from (26) as follows,

$$\left(\frac{i+y}{i-y}\right)^4 = \frac{z_4 z_5}{\bar{z}_4 \bar{z}_5} \quad (11)$$

The physical scattering condition in (3) implies that y is real and close to zero. Next, we integrate over y along the real axis from the origin, where the 31 letters are reduced to four rational letters,

$$1+y, \qquad -1+y, \qquad i+y, \qquad -i+y \quad (12)$$

The integration in y is thus straightforward. We note that in the limit $\delta \to 0$, only a subset of nonplanar Feynman integrals depend on y, while all planar integrals do not.

Finally we get all two-loop five-point master integrals for all the permutations, in terms of GPLs, up to weight 4, for generic points in the spacelike collinear region. This result is verified with the numerics from package PentagonMI [52], in the limit $\delta \to 0$.

To demonstrate this section's computation, we list more details in the Supplemental Material [62]. For example, consider the $1 \to 2$, $2 \to 3$, $3 \to 5$, $4 \to 1$, $5 \to 4$ permutation of the nonplanar double pentagon integral family with the notation in Ref. [52]. The corresponding 108 integrals' values at X_1 are given in the Supplemental Material [62] as `DP23514_X1.txt` which depends on $\log\delta$. After the step 1, the dependence on s, z, x is calculated and the resulting expressions are given in `DP23514_step1.txt`. The final analytic result of those integrals for an arbitrary point P in the spacelike collinear limit, after the step 2, is provided in `DP23514_step2.txt`.

III. TWO-LOOP FIVE-POINT $\mathcal{N}=4$ SCATTERING AMPLITUDE IN THE SPACELIKE COLLINEAR REGION

The full-color two-loop five-point $\mathcal{N}=4$ scattering amplitude can be constructed from the integrand in Ref. [63]. This amplitude has the following color decomposition,

$$\mathcal{A}_5^{(2)} = \sum_{\lambda=1}^{12}\left(N_c^2 A_\lambda^{(2,0)} + A_\lambda^{(2,2)}\right)T_\lambda + \sum_{\lambda=13}^{22}\left(N_c A_\lambda^{(2,1)}\right)T_\lambda \quad (13)$$

where the corresponding color basis has 12 single trace and 10 double trace functions [64],

$$T_1 = [\mathrm{Tr}(12345) - \mathrm{Tr}(15432)], \ldots$$
$$T_{13} = \mathrm{Tr}(12)[\mathrm{Tr}(345) - \mathrm{Tr}(543)], \ldots \quad (14)$$

The MHV partial amplitude can be expanded further over 6 Parke-Taylor factors. Furthermore, by the integration-by-parts (IBP) reduction,

$$A^{(2,k)} = \sum_{i=1}^{6}\sum_j c_{ij}\mathrm{PT}_i I_j \quad (15)$$

where each I_j is an element of the UT integral basis mentioned in the previous section. Coefficients c_{ij}'s are rational numbers. The Parke-Taylors factors are,

$$\mathrm{PT}_1 = \mathrm{PT}(12345), \qquad \mathrm{PT}_2 = \mathrm{PT}(12354),$$
$$\mathrm{PT}_3 = \mathrm{PT}(12453), \qquad \mathrm{PT}_4 = \mathrm{PT}(12534),$$
$$\mathrm{PT}_5 = \mathrm{PT}(13425), \qquad \mathrm{PT}_6 = \mathrm{PT}(15423) \quad (16)$$

with the definition,

$$\mathrm{PT}(i_1i_2i_3i_4i_5) = \frac{\delta^8(Q)}{\langle i_1i_2\rangle\langle i_2i_3\rangle\langle i_3i_4\rangle\langle i_4i_5\rangle\langle i_5i_1\rangle} \quad (17)$$

where $\delta^8(Q)$ is the Dirac delta function for the superspace. The two-loop five-point $\mathcal{N}=4$ amplitude's symbol expression was calculated in [58,65]. On the other hand, the analytic two-loop (and three-loop) four-point $\mathcal{N}=4$ super-Yang-Mills amplitude is in Ref. [66].

With our analytic result of two-loop five-point master integrals, in terms of GPLs, in the spacelike collinear limit $2\|3$, it is straightforward to assemble the $\mathcal{N}=4$ amplitude (13) in this region. We also consider the ratio between Parke-Taylor factors in this limit,

$$\left\{\frac{\mathrm{PT}_1}{\mathrm{PT}_1}, \ldots, \frac{\mathrm{PT}_6}{\mathrm{PT}_1}\right\} \underset{\delta\to 0}{\sim} \left\{1, \frac{-x}{x+1}, 0, 0, 0, 1\right\} \quad (18)$$

with the parametrization defined in (5). The result is organized as a Laurent series in ϵ up to ϵ^0.

The analytic result for $\mathcal{N}=4$ amplitude in the spacelike collinear limit, consists of GPLs. However, it can be further simplified to an expression of classical polylogarithms only. We achieved the simplification by first rewriting the amplitude in terms of harmonic polylogarithms (HPLs) and then decomposing them into Lyndon words with the aid of the package Polylogtools [67]. The dependence on the parameter y is only through the power of the function $\log(\frac{i-y}{i+y})$.

The amplitude's analytic expression is provided in the Supplemental Material [62] as `TwoLoopAmplitude_SpaceLike_Collinear.txt`.

IV. DISCONTINUITY OF THE FIVE-POINT $\mathcal{N}=4$ SCATTERING AMPLITUDE

Gauge theory amplitudes factorize in a universal way in the timelike collinear regime [68–71]. In this section we describe a procedure to obtain the spacelike collinear limit of the five-point amplitudes from the timelike regime via analytic continuation. This approach is complementary to the method we exploit in previous sections.

Assuming timelike splitting, the (color-stripped) five-point amplitude strictly factorizes [72–75]

$$A_5|_{\text{T.L.}}(2,3,\ldots)\xrightarrow{2\|3}\text{Split}_{-\lambda}(z;2,3)A_4(P^\lambda,\cdots) \quad (19)$$

In the collinear kinematic space, the amplitude is governed by a universal splitting function depending on the momentum fraction variable z ranging from 0 to 1. Analytic continuation of the splitting function into the spacelike regime ($z>1$) is obstructed by the ambiguity in the sign of its imaginary part [74].

This ambiguity can be traced back to the sign of $i\pi$'s generated in different discontinuity channels, e.g.

$$\ln\frac{s_{3I}}{s_{PI}}\cong\ln|1-z|+i\pi\theta(z-1)\text{sign}(s_{PI}) \quad (20)$$

which relies on the signature of leg I, referring to any of the initial-state or final-state noncollinear legs in the scattering process. To resolve this issue we propose a path for analytic continuation for the full amplitudes, which starts from a generic point Y_0 away from the collinear kinematic space where $\delta=0$. The path is parametrized by a variable

$$\tau\equiv s_{13}/(s_{13}+s_{12}) \quad (21)$$

where subscript 1 labels the unique incoming noncollinear leg in the five-point scattering process. Note that a generic kinematic point can be parametrized by $\{\tau,s,\delta,c,x\}$ whereas in the collinear limit τ reduces to $1-z$. Starting from a point on the positive real τ-axis, the analytic continuation path Γ contains three segments

$$\Gamma_1:\ \tau\to\tau e^{-2\pi i},\qquad \Gamma_2:\delta\to 0,\qquad \Gamma_3:\tau\to-|\tau|-i0.$$

Γ_1 is a residue contour encircling the origin of the real τ-axis, picking up the monodromy around $\tau=0$. Wrapping the contour along Γ_1, the discrepancy between the signature of noncollinear momenta has been compensated. In the next step we send δ to zero, landing onto the a point Y_1 in the timelike collinear region. In the final step, the contour Γ_3 lies within the collinear kinematic space. It goes from the positive to the lower side of negative real τ-axis, reaching a point Y_2 in the spacelike collinear region. The full path for the analytic continuation is illustrated in Fig. 3.

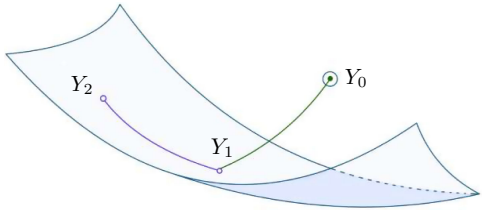


FIG. 3. Path for analytic continuation of the five-point $\mathcal{N}=4$ amplitude. We start from a generic point Y_0 with $\tau>0$ and pick up the monodromy around $\tau=0$ before taking the $2\|3$ collinear limit. Then we travel from a timelike region point $Y_1(\tau>0)$ to the spacelike region point $Y_2(\tau<0)$.

Analytic continuation along the path Γ allows us to compute the difference between the five-point amplitudes in the timelike v.s. spacelike collinear regime, which define the factorization breaking terms. Schematically we have

$$A_5|_{\text{S.L.}}\cong A_5|_{\text{T.L.}}+\text{disc}_\Gamma[A_5] \quad (22)$$

A shortcut to analyzing the discontinuity is to work with the symbol of the amplitudes. We start with the symbol of the one- and two-loop five-point amplitudes [58,65], promoting them to the maximally iterated coproduct by the restoring the $i\pi$'s associated with its first entry [76]. Then we perform analytic continuation on the coproduct following the path Γ and compare with [49]. This allows us to determine the two-loop factorization breaking terms [see Eq. (25)] up to potential $\pi^3\times\mathcal{O}(z-1)$ corrections. We find agreement with the explicit computation of two-loop five-point master integrals.

V. TWO-LOOP GENERALIZED SPLITTING AMPLITUDES

As a result of the analysis in the previous sections, we arrive at the main result of this paper. We find that full-color two-loop amplitudes have the following generalized factorization form in the collinear limit where $p_a\cong(1-z)P$, $p_b\cong zP$,

$$\mathcal{A}_5(p_a,p_b,p_i,p_j,p_k)\xrightarrow{a\|b}\mathbf{Sp}\times\mathcal{A}_4(P,p_i,p_j,p_k). \quad (23)$$

Here **Sp** are the generalized splitting amplitudes proposed by Catani *et al.* [44]. The four-dimensional helicity (FDH) scheme is used here for regularization, according to Ref. [74], to maintain supersymmetry. We present the one- and two-loop splitting amplitudes explicitly through $\mathcal{O}(\epsilon^0)$. The divergent terms at two loops were already known, but the finite terms are new.

$$\mathbf{Sp}^{(1)} = \left[\frac{\mu^2 z}{s_{ab}(1-z)}\right]^{\epsilon}\left\{2N_c\bar{r}_S^{(1)}(z+i0) + \mathbf{T}_a\cdot\mathbf{T}_{\text{in}}(2\pi i)c_1(\epsilon)\frac{1}{\epsilon}\right\}\mathbf{Sp}^{(0)}, \tag{24}$$

$$\begin{aligned}\mathbf{Sp}^{(2)} = \left[\frac{\mu^2 z}{s_{ab}(1-z)}\right]^{2\epsilon}\Bigg\{&4N_c^2\bar{r}_S^{(2)}(z+i0)\\ &+ N_c\mathbf{T}_a\cdot\mathbf{T}_{\text{in}}(2\pi i)\left[c_2(\epsilon)\frac{1}{\epsilon^3} + c_1^2(\epsilon)\left(-\frac{2}{\epsilon^2}\ln z + \frac{2}{\epsilon}\ln z\ln\left(\frac{z}{z-1}\right) - 2\text{Li}_3\left(1-\frac{1}{z}\right) - \ln(z)\ln^2\left(\frac{z}{z-1}\right)\right)\right]\\ &+ \sum_{I\in\text{outgoing}}[\mathbf{T}_a\cdot\mathbf{T}_{\text{in}}, \mathbf{T}_a\cdot\mathbf{T}_I](2\pi i)\left[\left(\frac{1}{2\epsilon^2} - \frac{1}{2}\zeta_2\right)(\ln|z_I|^2 + i\pi) + \frac{1}{6}\left(\ln^2\frac{z_I}{\bar{z}_I} + 4\pi^2\right)\ln\frac{z_I}{\bar{z}_I} + 2\zeta_3\right]\\ &+ \sum_{I\in\text{outgoing}}\{\mathbf{T}_a\cdot\mathbf{T}_{\text{in}}, \mathbf{T}_a\cdot\mathbf{T}_I\}(2\pi^2)\left[\frac{1}{2\epsilon^2} - \frac{1}{2}\zeta_2\right]\Bigg\}\mathbf{Sp}^{(0)}.\end{aligned} \tag{25}$$

We adopt the color space formalism for the color charge operators [77]. The subscript in labels an incoming hard particle, and I labels any of the outgoing hard particle. $(z_I, \bar{z}_I)$ are the shorthand notation for the collinear limit of the cross ratios,

$$z_I = \frac{\langle ab\rangle\langle \text{in}I\rangle}{\langle \text{in}a\rangle\langle bI\rangle}, \qquad \bar{z}_I = \frac{[ab][\text{in}I]}{[\text{in}a][bI]}, \qquad a\|b. \tag{26}$$

In the physical scattering regime, z_I and $\bar{z}_I$ are a complex conjugate pair. In the collinear limit, $z_I, \bar{z}_I \to 0$, but the ratio $z_I/\bar{z}_I$ is kept finite.

The remaining terms in Eq. (25) are best explained by considering, without loss of generality, the five-point kinematics specified in Eq. (3) where $b = 2, a = 3, \text{in} = 1$, $I \in \{4, 5\}$. In that case, the magnitudes of $\{z_4, z_5\}$ are given in terms of the kinematic variables $\{s, \delta, z, x\}$ introduced in Eq. (5) as follows

$$|z_4|^2 = \frac{4\delta^2(1+x)}{sz(1-z)x}, \qquad |z_5|^2 = \frac{4\delta^2 x}{sz(1-z)(1+x)} \tag{27}$$

The phases of $\{z_4, z_5\}$ are parametrized by the variable y introduced in Eq. (10). Their values differ by a factor of π, which can be specified in terms of GPLs defined as $G_a(y) \equiv \int_0^y \frac{dt}{t-a}$,

$$\begin{aligned}\frac{1}{2i}\ln\frac{z_4}{\bar{z}_4} &= -iG_{-i}(y) + iG_i(y) - \frac{\pi}{2},\\ \frac{1}{2i}\ln\frac{z_5}{\bar{z}_5} &= -iG_{-i}(y) + iG_i(y) + \frac{\pi}{2}\end{aligned} \tag{28}$$

In addition, the constants $c_L(\epsilon)$'s that appear in Eqs. (24) and (25) are specified in the following,

$$\begin{aligned}c_1(\epsilon) &= -e^{\gamma_E\epsilon}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\\ c_2(\epsilon) &= [c_1(\epsilon)]^2\frac{\pi\epsilon}{\tan(\pi\epsilon)} + \epsilon^2 f(\epsilon)c_1(2\epsilon).\end{aligned} \tag{29}$$

where $f(\epsilon) = (\psi(1-\epsilon) - \psi(1))/\epsilon$ with ψ the digamma function.

The generalized splitting amplitudes **Sp** apply to the case of spacelike splitting, where an incoming particle b emits an outgoing collinear particle a. Setting $\mathbf{T}_a = 0$, **Sp** reduces to the color-singlet timelike splitting amplitude, which is strictly factorized. Up to two-loop order, the factorization violating effects are associated with a color dipole $\mathbf{T}_a\cdot\mathbf{T}_{\text{in}}$, as well as two types of color tripole built from commutator and anticommutator between dipole operators. The structure of ϵ-poles in the dipole and tripole functions follows from IR exponentiation, which has been discussed in [44]. Their *finite* part is new.

Let us discuss this result. In the L-loop generalized splitting amplitude, the color singlet function $\bar{r}_S^{(L)}$ is the universal splitting kernel [74]

$$\bar{r}_S^{(L)}(z) = \left[\frac{s_{ab}(1-z)}{\mu^2 z}\right]^{L\epsilon} r_S^{(L),\mathcal{N}=4}(z, s_{ab}) \tag{30}$$

They could be obtained in the case of timelike splitting, then analytically continued to the region where z is greater than one (but carries a small positive imaginary part).

The 1- and 2-loop dipole functions are purely imaginary, and equal to minus 2 times the imaginary part of the splitting kernel $\bar{r}_S^{(1)}(z+i0)$ and $\bar{r}_S^{(2)}(z+i0)$.

The tripole functions are demonstrated in the last two lines in Eq. (25). They correlate the collinear particle a with both an incoming and an outgoing hard particle, which appear for the first time at the level of two-loop five-point amplitudes, originating from the nonplanar topologies.

Intriguingly, their expressions are *identical to* the two-loop tripole soft gluon emission factors in the spacelike

collinear regime [49,78]. We observe that the two-loop tripole functions have no explicit dependence on the z-variable which defines the fractions of momenta carried by the collinear pair. In retrospect, therefore they could have been fixed by comparing to the soft-collinear ($z \to 1$) limit of the five-point amplitudes. Given the universality of gauge-theory scattering amplitudes in the soft limit and the principle of leading transcendentality [79], we speculate that the two-loop tripole terms in the generalized splitting amplitudes in QCD are *identical to* what we obtain in $\mathcal{N} = 4$ super Yang-Mills theory.

Our explicit results for the spacelike collinear splitting amplitudes resolves the potential issue with factorization violation in physical processes at NNNLO. We show for the first time that the dipole terms in **Sp** evaluate to a pure phase and cancel at the level of squared amplitudes, whereas the tripole terms coincide with the prediction from soft-collinear limit. Hereby we demonstrate that factorization violating effects in the spacelike collinear limit cancel at the level of color summed squared amplitudes at NNNLO in $\mathcal{N} = 4$ super Yang-Mills theory, going beyond the previous considerations from infrared-pole factorization [46] and soft gluon factorization [49].

VI. CONCLUSION AND OUTLOOK

In this paper, we initiated a systematic study of amplitude-level spacelike collinear factorization breaking effects. We obtained for the first time two-loop spacelike generalized splitting amplitudes in $\mathcal{N} = 4$ SYM. As a close cousin of QCD, this theory shares many similarities in the infrared limit, and therefore provides useful reference to future study for actual QCD processes. Furthermore, based on the leading transcendentality principle, we expect the results obtained here also predict the leading transcendental part of the generalized spacelike splitting amplitude in QCD.

We showed that while factorization violation does exist at the amplitude level, those terms cancel at the unpolarized squared amplitude for NNNLO corrections. On the other hand, for polarized observable such as the single-spin asymmetry considered in [41], our results provide concrete input for studying its factorization violation at two loops.

Our results demonstrate for the first time that factorization violation can be studied using the remarkable data of high-multiplicity scattering amplitudes. In order to do so, we developed a systematic methods for both taking the spacelike collinear limit of pentagon functions, as well as for taking discontinuity from integrated amplitudes. These methods will be valuable when considering other processes. For example, they may be used to test our conjecture about the spacelike QCD splitting amplitudes.

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