

Polyakov loop model with exact static quark determinant in the 't Hooft-Veneziano limit: $U(N)$ case

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I investigate a d -dimensional $U(N)$ Polyakov loop model that includes the exact static determinant with N_f degenerate quark flavor and depends explicitly on the quark mass and chemical potential. In the large N, N_f limit mean field gives the exact solution, and the core of the Polyakov loop model is reduced to a deformed unitary matrix model, which I solve exactly. I compute the free energy, the expectation value of the Polyakov loop, and the quark condensate. The phase diagram of the model and the type of phase transition is investigated and shows it depends on the ratio $\kappa = N_f/N$.

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I. INTRODUCTION

In this paper, we focus on a three-dimensional effective Polyakov loop (PL) model, describing the $(d+1)$ -dimensional $U(N)$ LGT with N_f flavor of staggered fermions at finite baryon density.

The resulting PL model we work with takes the form

$$Z_\Lambda(N, N_f) = \int \prod_x dU(x) \times \exp \left[\beta \sum_{x,\nu} \text{ReTr} U(x) \text{Tr} U^\dagger(x + e_\nu) \right] \times \prod_{f=1}^{N_f} B_q(m_f, \mu_f), \quad (1)$$

where the determinant is taken over group indices. The coupling of this effective model depends on the temporal gauge coupling and can be expressed as

$$\beta \equiv D_{\text{fund}}(\beta_g) = \left(\frac{C_f(\beta_g)}{s_1(I)C_0(\beta_g)} \right)^{N_t}, \quad C_f(\beta) = \sum_{k=-\infty}^{\infty} \det I_{\delta_{i,1}-i+j+k}(\beta_g)_{1 \leq i,j \leq N}. \quad (2)$$

In this model the matrices $U(x)$ play the role of Polyakov loops, the only gauge-invariant operators surviving the integration over spatial gauge fields and over quarks. The integration in (1) is performed with respect to the Haar measure on G .

In case of exact static determinant with N_f degenerated quark flavors (or 1D QCD, see review of dual models [1–3])

$$\prod_{f=1}^{N_f} B_q(m_f, \mu_f) = B_{st}(h_+^f, h_-^f) = A_{st} \det [1 + h_+^f U(x)]^{N_f} [1 + h_-^f U^\dagger(x)]^{N_f}, \quad (3)$$

where

$$A_{st} = h^{-NN_f}, \quad h_\pm = h e^{\pm \mu}, \quad h = e^{-N_t \text{arcsinh } m}. \quad (4)$$

An action of this type describes the interaction between Polyakov loops $\text{Tr} U(x)$ at finite temperature and in the presence of N_f heavy degenerate quark flavors. The parameter h can be related to the mass of the quark (its exact form depends on the kind of lattice fermion) and $\mu = \beta \mu_q$, where μ_q is the chemical potential of the quark.

We solve this model in the 't Hooft-Veneziano limit [4,5]:

$$N \rightarrow \infty, \quad N_f \rightarrow \infty, \quad \kappa = \frac{N_f}{N} = \text{const}, \quad (5)$$

where the mean-field approximation becomes exact (in the spirit of [6,7]). The central analytical object in this limit is a deformed unitary matrix model:

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$$A_{st} \int dU e^{N(g_+ \text{Tr} U + g_- \text{Tr} U^\dagger)} \det[1 + h_+ U]^{N_f} [1 + h_- U^\dagger]^{N_f} \\ \approx A_{st} \int dU e^{N((g_+ + \kappa h_+) \text{Tr} U + (g_- + \kappa h_-) \text{Tr} U^\dagger)},$$

which generalizes the classical unitary matrix model of GWW [8,9]. So far in the context of the PL model, the static fermion determinant was investigated in the heavy quark approximation [10–12].

Our main technical task was to solve this model in the 't Hooft-Veneziano limit. We solve this matrix model exactly at large N and derive analytic expressions for the free energy, the Polyakov loop expectation value, and the baryon density. We determine the whole phase diagram and identify the presence of a third-order phase transition, which switches to a first-order one as $\kappa \rightarrow 0$. This result is a generalization of the classical GWW solution [8,9]. Case $h = 1$ considered in [13–15], $\beta = 0$ in [16]. At finite N this model was considered in [17,18].

This paper is organized as follows. In Sec. II we derive all the additional analytical work including the mean field approach for a large N , representation of $U(N)$ unitary matrix model (which is the core of this PL model), and present the exact solution of this matrix model in the limit of large N, N_f . In Sec. III we apply a solution of the matrix model to obtain a phase diagram of the PL model, expressions for its free energy, and observables like average PL and correlation function. The summary is presented in the final section.

II. PL MODEL AND ONE-SITE INTEGRAL

A. Mean field method for large N in d dimension

The partition function of the $U(N)$ and $SU(N)$ PL model with local nearest-neighbor interaction can be rewritten in the large- N, N_f limit as (see [7])

$$Z_\Lambda(N, N_f) = \int \prod_x dU(x) \exp \left[\beta \sum_{x,\mu} \text{ReTr} U(x) \text{Tr} U^\dagger(x + e_\mu) \right] A_{st} \\ \prod_x \det[1 + h_+ U(x)]^{N_f} [1 + h_- U^\dagger(x)]^{N_f} = \left[e^{-NN_f \frac{\beta d}{\kappa} \text{Reww}^\dagger} \sum_{q=-\infty}^{\infty} \Xi_q(g_\pm, h_\pm) \right]^{L^d}. \quad (6)$$

In the joint limit of large- N , large- N_f , the mean field approach becomes exact, provided that the ratio $N_f/N = \kappa$ is kept finite. The sum over q is then replaced by an integral under $q \rightarrow u/N_f$. The resulting core of the Polyakov loop model with the exact static fermion determinant takes the form of a deformed unitary matrix model

$$\sum_{q=-\infty}^{\infty} \Xi_q(g_\pm, h_\pm) = A_{st} \int dU e^{N(g_+ \text{Tr} U + g_- \text{Tr} U^\dagger)} \det[1 + h_+ U]^{N_f} [1 + h_- U^\dagger]^{N_f} = \quad (7)$$

$$= e^{NN_f F_{SU(N)}} = \int_{-\infty}^{\infty} du e^{NN_f S_{\text{eff}}[g_\pm, h_\pm; u]}, \quad (8)$$

where we denote

$$w = \left\langle \frac{1}{N} \text{Tr} U \right\rangle, \quad w^\dagger = \left\langle \frac{1}{N} \text{Tr} U^\dagger \right\rangle, \quad g_+ = bw^\dagger, \quad g_- = bw, \quad (9)$$

and $b = \beta d$.

In the large N, N_f limit, the mean field approach becomes exact:

$$Z_\Lambda(N, N_f) = [e^{NN_f (-\frac{g_+ g_-}{\kappa b} + F_{SU(N)})}]^{L^d} = [e^{NN_f f_{SU(N)}}]^{L^d}, \quad (10)$$

where we introduce $S_{\text{eff}} \equiv S$.

The mean-field equations (with $b = \beta d$) read

$$\frac{g_-}{\kappa b} = \frac{\partial}{\partial g_+} S_{\text{eff}}[\kappa, g_\pm, h_\pm; u], \quad \frac{g_+}{\kappa b} = \frac{\partial}{\partial g_-} S_{\text{eff}}[\kappa, g_\pm, h_\pm; u], \quad (11)$$

together with an additional condition reflecting the $SU(N)$ nature of the model (the extremum equation in u):

$$\frac{\partial}{\partial u} S_{\text{eff}}[\kappa, g_{\pm}, h_{\pm}; u] = 0. \quad (12)$$

B. Formulation of deformed unitary matrix model

The variables g_+ and g_- play a technical role and attain their physical values only after embedding the effective one-site model into the full d -dimensional PL model. At this stage it is useful to redefine $g_{\pm} \rightarrow g_{\pm} e^{\mp \mu}$ so that $h_{\pm} = h$, although we keep the notation $h_{\pm}$ whenever needed.

We can reformulate (7) with the help of parametrization of the group matrix U using its eigenvalues as

$$\begin{aligned} \Xi(g_{\pm}, h_{\pm}) &= A_{st} \prod_{n=1}^N \sum_{k_n, l_n=0}^{N_f} h_+^{k_n} h_-^{l_n} \binom{N_f}{k_n} \binom{N_f}{l_n} \\ &\times \sum_{r_n, s_n=0}^{\infty} \frac{(Ng_+)^{r_n}}{r_n!} \frac{(Ng_-)^{s_n}}{s_n!} Q, \end{aligned} \quad (13)$$

where

$$\begin{aligned} Q &= \int_{SU(N)} dU \prod_{n=1}^N e^{i(k_n - l_n + r_n - s_n)\phi_n} \\ &= \sum_{q=-\infty}^{\infty} \det_{1 \leq i, j \leq N} \delta_{i-j+k_i-l_j+r_i-s_j+q, 0} \end{aligned}$$

and with help of determinant (that linked with skew Schur function, see [19]):

$$\begin{aligned} \Xi(g_{\pm}, h_{\pm}) &= A_{st} \sum_{q=-\infty}^{\infty} \prod_{n=1}^N \sum_{r_n, s_n=0}^{\infty} \frac{(Ng_+)^{r_n}}{r_n!} \frac{(Ng_-)^{s_n}}{s_n!} \\ &\times \sum_{l_n=0}^{N_f} h_+^{l_n - r_n + s_n + q} h_-^{l_n} \binom{N_f}{l_n} \det_{1 \leq i, j \leq N} \\ &\times \begin{pmatrix} N_f \\ -i + j + l_i - r_i + s_j + q \end{pmatrix}. \end{aligned} \quad (14)$$

All the details needed and another formulation will be found in the Appendix A 1.

C. Solution of $U(N)$ deformed unitary matrix model and its phase diagram

All calculations were performed as calculations of the corresponding determinants (see details in A 2) with generalization of particular cases for finite N . All these formulas work for $SU(N)$, so we take $q = 0$ to get the case $U(N)$ ($S_{\text{eff}}[\kappa, g_{\pm}, h_{\pm}; 0] \equiv F$). After exponentiation of the series, we take the 't Hooft-Veneziano limit and obtain a strict expansion of the free energy as small h series that have only three different terms [up to g^2 as we see in (15)] and small combined m and $g_{\pm}$ series that contain infinite many terms (A8) (see more details in Sec. A 3).

After some algebra, we have the next exact expression for the free energy $U(N)$ matrix model in the case $g_+ = g_- = g$ [and we denote $F_{U(N)} \equiv F_{1,2}[\kappa, g_{\pm}, h_{\pm}]$ depending on the region]:

$$F_1[\kappa, g, h] = -\log h - \kappa \log(1 - h^2) + 2gh + \frac{g^2}{\kappa}, \quad (15)$$

$$\begin{aligned} F_2[\kappa, g, h] &= \frac{2g}{\kappa} + \frac{\bar{g}(2\bar{g}^2 + 2(\bar{g} + 1)\kappa - 5\bar{g} + 2)}{\kappa(2\bar{g} - 2\kappa - 1)} - \frac{(\kappa + 1)^2 \log(1 - 2\bar{g} + \kappa)}{\kappa} \\ &+ \frac{\log(1 - 2\bar{g})}{2\kappa} + \frac{(2\kappa + 1)^2 \log(1 - 2\bar{g} + 2\kappa)}{2\kappa} + \frac{(2\kappa + 1) \log(\cosh^2(\frac{m}{2}))}{2\kappa} - \kappa \log 4\kappa, \end{aligned} \quad (16)$$

where

$$\begin{aligned} \bar{g} &\equiv \bar{g}(G, \kappa) = \frac{1}{3}(1 + G + \kappa) - \frac{(1 - 2G + \kappa)^2}{6\sqrt{3}P} - \frac{\sqrt{3}P}{6} \\ P &= -8G^3 + 12G^2(\kappa + 1) + 3G(7\kappa^2 - 4\kappa - 2) + (\kappa + 1)^3 \\ &+ 3\kappa \sqrt{3G(-16G^3 + 24G^2(\kappa + 1) + 3G(\kappa - 2)(5\kappa + 2) + 2(\kappa + 1)^3)}, \end{aligned} \quad (17)$$

where $G = g \cosh^2(\frac{m}{2})$.

This solution relies on the property that, almost everywhere in the series expansion, the variables m and g appear only through the combination $G = g \cosh^2 \frac{m}{2}$ as can be seen in (A8). This property allows one to bypass the need to use the

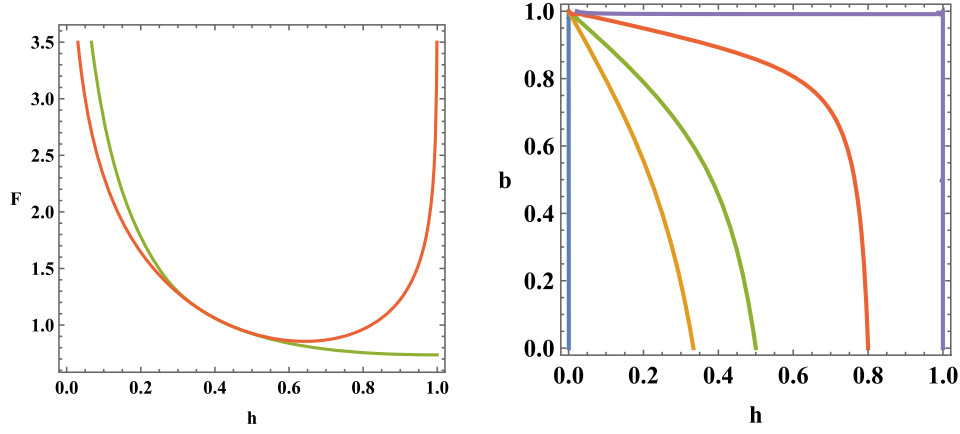


FIG. 1. Free energies of the PL model as function of h at $\kappa = 1/2$ and $b = 5/12$ where red - f_1 , green - f_2 (left panel), and phase diagram of PL model in $h - b$ coordinates with fixed $\kappa = 0, 1/8, 1/2, 1, \infty$ (corresponding colors counted from violet to blue) (right panel).

method of orthogonal polynomials [20]. We apply a method of reconstruction that is kind of algebraic lifting from the critical line. The reconstruction proceeds as follows. First, we need to know the critical line exactly [see (21)]—which is determined as a series in g and then resummed to the exact expression $h_{cr} = \frac{1-2g}{\xi-2g}$ ($\xi = 2\kappa + 1$). Second, consider the following property: both free energies become identical on the critical line:

$$F_2\left[\kappa, \bar{g}, \frac{1-2\bar{g}}{\xi-2\bar{g}}\right] = F_1\left[\kappa, \bar{g}, \frac{1-2\bar{g}}{\xi-2\bar{g}}\right]. \quad (18)$$

Thus, the equality on the critical line provides the exact value of the F_2 restricted to the phase boundary (because we know F_1 exactly). We will see below that $\bar{g} = \bar{g}(G, \kappa)$ is the solution of a cubic algebraic equation. Third, if we want to reconstruct the true dependence of F_2 , we make use of the fact that the property $F_2[\kappa, g, h] = F_2[\kappa, G]$ is almost true. That means that the next identity holds after projection:

$$G = \bar{g} \cosh^2\left(\frac{m}{2}\right) \Big|_{m \rightarrow -\log\left[\frac{1-2\bar{g}}{\xi-2\bar{g}}\right]} = \frac{\bar{g}(2\bar{g} - \kappa - 1)^2}{(2\bar{g} - 1)(2\bar{g} - 2\kappa - 1)} = g \cosh^2 \frac{m}{2}. \quad (19)$$

This relation provides the algebraic map between the invariant combination G and the parameter g on the critical line. This holds for all sides of equality (however, $F_1[\kappa, g, h] = F_1[\kappa, G]$ is not true at all). Therefore we only need to invert the relation (19) and express $\bar{g}$ as a function of G . Applying this substitution to F_1 yields the exact expression for F_2 . The only obstruction to expressing the result entirely in terms of G comes from two terms in the free energy which do not combine into the invariant combination G that “spoil” the property $F_2[\kappa, g, h] = F_2[\kappa, G]$ as we see in (A8). These terms are therefore separated explicitly, while the remaining part depends only on G .

After removing these two non-invariant contributions, the remaining expression can be reconstructed by inverting the algebraic relation (19) and finally restoring these two terms in the full expression for the free energy

$$F_2[\kappa, g, h] = F_1\left[\kappa, \bar{g}(G, \kappa), \frac{1-2\bar{g}(G, \kappa)}{\xi-2\bar{g}(G, \kappa)}\right] - \frac{2\bar{g}(G, \kappa)}{\kappa} - \frac{(2\kappa+1) \frac{(2\bar{g}(G, \kappa)-\kappa-1)^2}{(2\bar{g}(G, \kappa)-1)(2\bar{g}(G, \kappa)-2\kappa-1)}}{2\kappa} + \frac{2g}{\kappa} + \frac{(2\kappa+1) \log(\cosh^2 \frac{m}{2})}{2\kappa}. \quad (20)$$

The critical line is defined by

$$h_{cr} = \frac{1-2g}{\xi-2g}, \quad \text{or } m = \mp \log \left[\frac{1-2g}{\xi-2g} \right]. \quad (21)$$

that provides the third-order phase transition (see Fig. 1, left panel). The third derivative of $\Delta F = F_2[\kappa, g, h] - F_1[\kappa, g, h]$ at the phase transition exhibits the finite jump

$$\Delta F_h'''|_{h=h_{cr}} = \frac{(1-2g+2\kappa)^6}{4\kappa^2(2\kappa^2+3(1-2g)^2\kappa-(2g-1)^3)}.$$

III. FREE ENERGY OF THE PL MODEL IN MEAN FIELD APPROXIMATION

The mean field equation for $F_1[g, h; 0]$ gives

$$g_{MF} = \frac{b\kappa h}{1-b}. \quad (22)$$

Free energy of the confinement phase

$$f_{U(N)}^1 = -\log h + b \frac{\kappa h^2}{1-b} - \kappa \log(1-h^2). \quad (23)$$

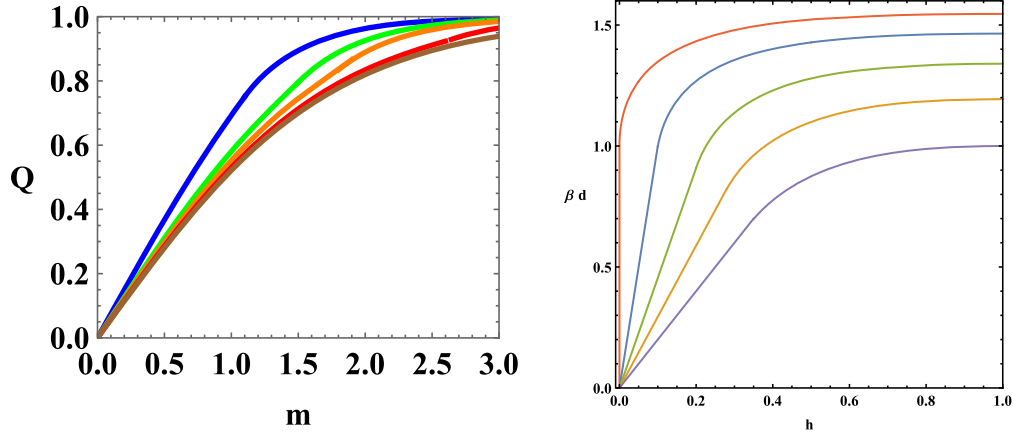


FIG. 2. Quark condensate vs m $\kappa = 1$ ($b = 0, 0.5, 0.66, 0.85, 1$) (left panel) and the average PL in $h - b$ coordinates at $\kappa = 1$ ($b = 0, 0.36, 0.52, 0.8, 1$) (left panel). Colors counted from top to bottom.

For the second free energy we arrive at the next mean field equation:

$$\frac{g_{\text{MF}}}{\kappa b} = \frac{1}{2} \frac{\partial}{\partial g} F_2[\kappa, g, h] = \frac{1}{g_{\text{MF}}} \frac{\bar{g}(2\bar{g}^2 - 3\bar{g} + \kappa + 1)}{\kappa(2\bar{g} - 2\kappa - 1)} + \frac{1}{\kappa}, \quad (24)$$

where $\bar{g} = \bar{g}(G, \kappa)$ is as in (17) and $G = g_{\text{MF}} \cosh^2(\frac{m}{2})$. Free energy of

$$f_{U(N)}^2 = -\frac{g_{\text{MF}}^2}{\kappa b} + F_2[\kappa, g_{\text{MF}}, h], \quad (25)$$

In all areas we have a third-order phase transition except line $b = 1$ where we have switching to the first order. In the limit of $\kappa \rightarrow 0$ type of phase transition also switch to the first order (see Fig. 1, right panel). That is all in the spirit of the PL lattice model in the 't Hooft-Veneziano limit (which is solvable with the help of the famous GWW model solution) but in our case we obtained a slightly more complicated expression for the critical line:

$$h_{cr}(b, \xi) = -\frac{(b-3)\xi + 2 + \sqrt{((b-3)\xi + 2)^2 + 8(b-1)(\xi-1)}}{4(\xi-1)}. \quad (26)$$

Average PL and quark condensate.
Quark condensate defined as

$$Q = \frac{\partial f_{U(N)}^{1,2}}{\partial m}.$$

Average PL:

$$W(1, \mu) = W^\dagger(1, -\mu) = \begin{cases} \frac{\kappa h e^{-\mu}}{1-b}, & h < h_{cr}, \\ \kappa g_{\text{MF}} e^{-\mu}, & h_{cr} < h < 1/h_{cr}, \\ \frac{\kappa/h e^{-\mu}}{1-b}, & h > 1/h_{cr}, \end{cases} \quad (27)$$

where $g_{\text{MF}} = g[b, \kappa, h]$ is the solution of Eq. (24).

Chiral symmetry is restored only at $m = 0$ (see Fig. 2).

IV. SUMMARY AND PERSPECTIVES

The $U(N)$ PL model that takes into account exact static quark determinant with N_f degenerate flavors of quarks investigated in the mean-field approximation. First, we solved the deformed unitary matrix model. Second, apply this solution as the core that is needed to represent the mean field of the PL model.

Let us briefly summarize our main results:

- (i) We study next the deformed unitary matrix model (that is core of the earlier mentioned PL lattice model)

$$\Xi(g_{\pm}, h_{\pm}) = A_{st} \int dU e^{N(g_+ \text{Tr} U + g_- \text{Tr} U^\dagger)} \times \det[1 + h_+ U]^{N_f} [1 + h_- U^\dagger]^{N_f}.$$

We present a solution of such models in the 't Hooft-Veneziano limit [4,5]: $g \rightarrow 0, N \rightarrow \infty, N_f \rightarrow \infty$ such that the product $g^2 N$ and the ratio $N_f/N = \kappa$ are kept fixed (g is the coupling constant).

- (ii) Obtained expansion in small h and small m . We find an exact solution of this model for the $U(N)$ case in the 't Hooft-Veneziano limit and for all m and $g_+ = g_-$. I established third-order phase transition with critical line $h = \frac{1-2g}{2\kappa+1-2g}$.

- (iii) The model around $h = 0$ reproduces the GWW solution in terms of g [8,9]:

$$F_1 = -\log h + \frac{1}{\kappa}(g^2) + 2gh + O(h^2),$$

$$F_2 = -\log h + \frac{1}{\kappa}\left(2g - \frac{3}{4} - \frac{1}{2}\log 2g\right) + \frac{(4g-1)}{2g}h + O(h^2).$$

At $g = 0$, the model reduces to a one-dimensional QCD [16].

- (iv) Critical line of the general case $g_+ \neq g_-$ (see (A11)):

$$h = \frac{1 - g_+ - g_-}{\xi - g_+ - g_-} + \frac{2(g_+ - g_-)^2 \kappa^3}{(1 + k)(1 + 2k)^3} \times \left(1 + \frac{(g_+ + g_-)(8\kappa + 5)}{(\kappa + 1)(2\kappa + 1)} + O(g^2)\right).$$

That guaranties a third-order phase transition not only for $g_+ = g_-$ case, but also for the general one. When we put $\kappa \rightarrow 0$ we switch the order of the phase transition to the first one.

- (v) The solution of this matrix model could serve as the generation functional for the more sophisticated PL model that takes into account any powers of spin $(\text{Tr}U)^r$, and as the approach behind the mean field one.

The PL model was investigated in the mean field approximation (in the limit large N, N_f this approach becomes the exact):

- (i) I obtained the free energy and the critical line given by (26). It can be summarized briefly as follows. The model in the limit of large N, N_f has two phases. When $h = 0$ (infinitely heavy quarks) one finds a first-order phase transition. It exhibits a third-order phase transition GWW type.

- (ii) Limit $\kappa \rightarrow 0$ drive to domination of only one phase (“confinement”) F_1 with switching of the phase transition to the first order. In the opposite limit $\kappa \rightarrow \infty$ survives the other phase (“deconfinement”) without phase transition.
- (iii) We calculate some local observables, such as the average PL and the quark condensate.

As a natural extension of the present work, I plan to consider the more involved $SU(N)$ case, which includes nonzero baryon density. As is well known, the large- N limits of $U(N)$ and $SU(N)$ theories are not identical [21], which manifests itself through the emergence of baryon density. The appearance of nonzero baryon density signals a phase transition at finite chemical potential and can be viewed as a continuation of the $U(N)$ case (in simpler models usually only one such critical line exists [12]). In our model with the exact quark determinant an additional critical line appears when fermion saturation is reached; see also our discussion of 1D QCD in Ref. [16].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX

1. Different representation of the deformed unitary matrix model

Collection of all representations.

- (i) Small g expansion around 1D QCD (each order is the corresponding averaged spins of 1D QCD as the Schur skew function):

$$\sum_q \Xi_q(g_{\pm}, h_{\pm}) = A_{st} \sum_q e^{Nq\mu} \sum_{r,s=0} \frac{(Ng_+)^r (Ng_-)^s}{r!s!} \sum_{l,l'} h_+^l h_-^{l'} \sum_{\nu \vdash r, \mu \vdash s} d(\nu) d(\mu) \sum_{\sigma \vdash r+l} s_{\sigma/\nu}(1^{N_f}) s_{N^q \sigma/\mu}(1^{N_f}) \delta_{r+l, s+l'+qN}. \quad (\text{A1})$$

- (ii) Small m expansion. Expanding in the most symmetrical way: the partition function (14) can be reexpanded by small m expansion.

$$\sum_q \Xi_q(g_{\pm}, h_{\pm}) = \sum_q e^{Nq\mu} Z_0^{N, N_f}(q) \sum_{r,s=0}^{\infty} \frac{(Ng_+)^r (Ng_-)^s}{r!s!} \sum_{n=0}^{\infty} A_{r,s}(q, n) m^{2n}. \quad (\text{A2})$$

General structure of the expansion near $h = 1$:

$$\begin{aligned} \prod_{n=1}^N e^{-l_n m} &= \prod_{n=1}^N \left(1 - l_n m + \frac{l_n^2}{2!} m^2 - \frac{l_n^3}{3!} m^3 + \dots \right) = 1 - \sum_{k=1}^N l_k m \\ &\quad + \frac{1}{2!} \left(\sum_{k=1}^N l_k^2 + \sum_{k \neq r=1}^N l_k l_r \right) m^2 - \frac{1}{3!} \left(\sum_{k=1}^N l_k^3 + 3 \sum_{k \neq r=1}^N l_k^2 l_r + \sum_{k \neq r \neq s=1}^N l_k l_r l_s \right) m^3 + \dots \\ &= 1 + \frac{1}{2!} T_1 m^2 + \frac{1}{4!} \left(T_2 + \sum_{k \neq r=1}^N T_1 T_1 \right) m^4 + O(m^6). \end{aligned}$$

We choose $l \rightarrow 2l - N_f + a$ and $a = -r_i + s_j - i + j + q$ to make the most efficient way of calculation.

$$Z = e^{-(2l - N_f + a)m} = 1 - (2l - N_f + a)m + \frac{(2l - N_f + a)^2}{2} m^2 - \frac{(2l - N_f + a)^3}{3!} m^3 + \dots$$

Making summation (denoting)

$$\begin{aligned} \sum_{l=0}^{N_f} \binom{N_f}{l} \binom{N_f}{l+a} &= \binom{2N_f}{N_f+a} = T_0, \\ \sum_{l=0}^{N_f} \binom{N_f}{l} \binom{N_f}{l+a} (2l - N_f + a) &= 0, \\ \frac{1}{T_0} \sum_{l=0}^{N_f} \binom{N_f}{l} \binom{N_f}{l+a} \frac{(2l - N_f + a)^2}{2!} &= -\frac{(a^2 - N_f^2)}{2(2N_f - 1)} = T_1, \\ \sum_{l=0}^{N_f} \binom{N_f}{l} \binom{N_f}{l+a} \frac{(2l - N_f + a)^3}{3!} &= 0, \\ \frac{1}{T_0} \sum_{l=0}^{N_f} \binom{N_f}{l} \binom{N_f}{l+a} \frac{(2l - N_f + a)^4}{4!} &= \frac{(a^2 - n^2)(4n + 3(a^2 - n^2))}{4!(2n - 3)(2n - 1)} = T_2, \end{aligned}$$

and so on. Calculation of determinants:

$$\det_{1 \leq i, j \leq N} T_0 = \frac{G[N + 2N_f + 1]G[N + 1]G[N_f - q + 1]G[N_f + q + 1]}{G[2N_f + 1]G[N + N_f - q + 1]G[N + N_f + q + 1]} = Z_0^{N, N_f}(q),$$

where $G(x)$ is the Burness G function, and

$$A_{0,0}(q, 1) = \frac{1}{Z_0^{N, N_f}(q)} \sum_{k=1}^N \det_{1 \leq i, j \leq N} [T_1^{i,j} \delta_{k,i} + (1 - \delta_{k,i})] T_0^{ij} = \frac{N(N + 2N_f)(N_f^2 - q^2)}{2(4N_f^2 - 1)},$$

and so on. For calculation (as example) $A_{1,1}(q, n)$ we need in $A_{0,0}(q, n)$ make a shift $-i + j \rightarrow -i + \delta_{i,1} - \delta_{j,1}$ (as a “sources” dictated Young tableau) and, in general, $A_{r,s}(q, n)$ this looks like a sum by all partition of both r (associated with some $\sum_p t_p \delta_{j,p}$) and s (associated with some $\sum_p t'_p \delta_{i,p}$).

2. Small h and m expansion for arbitrary N, N_f

Small h expansion

The partition function reads

$$\Xi(g_{\pm}, h \approx 0) = A_{st} \sum_{q=0}^{N_f} e^{Nq\mu} h_+^{Nq} C_{N,N_f}(q) \left(Y_0 + N \left(\frac{g_+}{h} Y_{11} + g_- h Y_{12} \right) + N^2 (g_+^2/h^2 Y_{21} + g_- g_+ Y_{22} + g_-^2 h^2 Y_{23}) + O(g_{\pm}^3) \right), \quad (\text{A3})$$

where

$$\begin{aligned} C_{N,N_f}(q) &= \det_{1 \leq k, l \leq N} \begin{pmatrix} N_f \\ -k + l + q \end{pmatrix} = \frac{G(N+1)G(N+N_f+1)G(q+1)G(N_f+1-q)}{G(N_f+1)G(N+q+1)G(N+N_f+1-q)}, \\ Y_0(q) &= 1 + h^2 N N_f \frac{N_f - q}{N + q} + h^4 \frac{N N_f (N_f - q) [(N^2 - 1)(N_f^2 + 1) - q(N_f - N + q)(N N_f - 1)]}{2(N + q - 1)(N + q)(N + q + 1)} + O(h^6) \\ Y_{11}(q) &= \frac{Nq}{N + N_f - q} + h^2 N_f \frac{N(Nq(N_f - q) + N_f + N)}{(N + N_f - q)(N + q)} \\ &\quad + h^4 \left[\frac{N_f(N_f + 1)}{2} \frac{(N - 1)N(N_f - q)(2N_f + 2N + Nq(N_f - q + 1))}{2(N + N_f - q)(N + q)(N + q - 1)} \right. \\ &\quad \left. + \frac{N_f(N_f - 1)}{2} \frac{N(N + 1)(N_f - q)(2N_f + 2N + Nq(N_f - q - 1))}{2(N + N_f - q)(N + q)(N + q + 1)} \right] + O(h^6) \\ Y_{12}(q) &= \frac{N(N_f - q)}{N + q} + h^2 \frac{N N_f (N_f - q)(N_f(N(N + q) - 1) - Nq(N + q))}{(N + q - 1)(N + q)(N + q + 1)} + O(h^4) \\ Y_{21}(q) &= \frac{Nq(N_f(Nq - 1) + Nq(N - q))}{2(N_f + N - q - 1)(N_f + N - q)(N_f + N - q + 1)} + O(h^2) \\ Y_{22}(q) &= \frac{N(Nq(N_f - q) + N_f + N)}{(N + q)(N_f + N - q)} + O(h^2) \\ Y_{23}(q) &= \frac{N(N_f - q)(N_f(N(N + q) - 1) - Nq(N + q))}{2(N + q - 1)(N + q)(N + q + 1)} + O(h^2). \end{aligned} \quad (\text{A4})$$

Small mass expansion:

$$\begin{aligned} \Xi(g_{\pm}, m \approx 0) &= \sum_{q=0}^{N_f} Z_0^{N,N_f}(q) [H_0(q) + N(g_+ H_1(-q) + g_- H_1(q)) \\ &\quad + N^2 (g_+^2 H_2(q) + g_+ g_- H_{12}(q) + g_-^2 H_2(-q)) + O(g_{\pm}^3)], \end{aligned} \quad (\text{A5})$$

where

$$\begin{aligned} Z_0^{N,N_f}(q) &= \det_{1 \leq k, l \leq N} \begin{pmatrix} 2N_f \\ N_f - k + l + q \end{pmatrix} = \frac{G[N + 2N_f + 1]G[N + 1]G[N_f - q + 1]G[N_f + q + 1]}{G[2N_f + 1]G[N + N_f - q + 1]G[N + N_f + q + 1]}. \\ H_0(q) &= 1 + \frac{st}{(4N_f^2 - 1)} \left[\frac{m^2}{2!} \left(\frac{3(s+2)(t-2)}{4N_f^2 - 9} - 2 \right) \right. \\ &\quad + \frac{m^4}{4!} \left(\frac{20(s+2)(t-2)(s+6)(t-6)}{(4N_f^2 - 25)(4N_f^2 - 9)} + \frac{10st(s+2)(t-2)}{(4N_f^2 - 9)(4N_f^2 - 1)} \right. \\ &\quad \left. \left. - \frac{15st(s+6)(t-6)}{(4N_f^2 - 25)(4N_f^2 - 1)} - \frac{5(s+6)(t-6)}{4N_f^2 - 25} - \frac{45(s+2)(t-2)}{4N_f^2 - 9} + \frac{20st}{4N_f^2 - 1} + 16 \right) + O(m^6) \right], \end{aligned} \quad (\text{A6})$$

$$\begin{aligned}
H_1(q) &= \frac{N(N_f + q)}{N_f + N - q} \left(1 + \frac{m^2}{2} \left(\frac{(ts - 2(2N_f + N)(N_f - q))}{4N_f^2 - 1} + 1 \right) + O(m^4) \right), \\
H_2(q) &= \frac{N(N_f + q)(N^2(N_f + q) + t - 2N_f)}{2(N_f + N - q - 1)(N_f + N - q)(N_f + N - q + 1)} + O(m^2), \\
H_{12}(q) &= \frac{N^2 t + s}{(N_f + N - q)(N_f + N + q)} - 1 + O(m^2), \\
s &= N(N + 2N_f), \quad t = N_f^2 - q^2.
\end{aligned} \tag{A7}$$

Both expansions were performed up to h^{18} , g^9 and m^{12} , g^9 orders, correspondingly.

3. Straight expansions of free energies in the ‘t Hooft-Veneziano limit:

Case $g_+ = g_- \equiv g$

$$\begin{aligned}
F_2[\kappa, g, m] &= F_{g=0}[\kappa, m] - \frac{4g(\xi - 1)}{\xi^2 - 1} \\
&\quad - (\xi - 1) \left\{ \frac{G}{(\xi - 1)^2(\xi + 1)} - \frac{G^2 \xi^2}{2(\xi - 1)^2(\xi + 1)^4} + \frac{G^3 \xi^2}{3(\xi + 1)^7} \right. \\
&\quad - \frac{G^4 \xi^2(\xi^2 - 4\xi + 1)}{4(\xi + 1)^{10}} + \frac{G^5 \xi^2(\xi^4 - 10\xi^3 + 22\xi^2 - 10\xi + 1)}{5(\xi + 1)^{13}} \\
&\quad \left. - \frac{G^6 \xi^2(\xi^6 - 18\xi^5 + 88\xi^4 - 150\xi^3 + 88\xi^2 - 18\xi + 1)}{6(\xi + 1)^{16}} + O(G^7) \right\},
\end{aligned} \tag{A8}$$

where

$$F_{g=0}[\kappa, m] = \frac{\xi \log(4 \cosh^2(\frac{m}{2}))}{\xi - 1} - \frac{(\xi + 1)^2 \log(\xi + 1) - 2\xi^2 \log \xi + (\xi - 1)^2 \log(\xi - 1)}{2(\xi - 1)}$$

and $G = \frac{g}{8} \cosh^2(\frac{m}{2})$, $\xi = 2\kappa + 1$.

Generation functional for the free energy is moderately simple and gives us all this expansion:

$$GF[g, \kappa] = \frac{\partial}{\partial g} F_2[\kappa, g, h = 1] - \frac{2}{\kappa} = \frac{1}{g} \frac{2\bar{g}(2\bar{g}^2 - 3\bar{g} + \kappa + 1)}{\kappa(2\bar{g} - 2\kappa - 1)}, \tag{A9}$$

where $\bar{g} = \bar{g}(G, \kappa)$ is as in (17) and $m = 0$.

And generation functional meets next functional equation

$$GF[g, \xi] = 8 - GF[g/\xi, 1/\xi]$$

and differential equation of third order:

$$\begin{aligned}
&g^3 \frac{\partial^3 GF}{\partial g^3} + 3g^2(\xi - 1) \frac{\partial^3 GF}{\partial g^2 \partial \xi} + 3g(\xi - 1)\xi \frac{\partial^3 GF}{\partial g \partial^2 \xi} + (\xi^2 - 1)\xi \frac{\partial^3 GF}{\partial \xi^3} \\
&+ 6 \left(g^2 \frac{\partial^2 GF}{\partial g^2} + g(2\xi - 1) \frac{\partial^2 GF}{\partial \xi \partial g} + \xi^2 \frac{\partial^2 GF}{\partial \xi^2} + g \frac{\partial GF}{\partial g} + \xi \frac{\partial GF}{\partial \xi} \right) = 0.
\end{aligned} \tag{A10}$$

General case $g_+ \neq g_-$

In case of $U(N)$ and $g_+ \neq g_-$ we have next expansion for both free energy (with re-summation by m):

$$F_1[\kappa, g_+, g_-, m] = -\log h - \kappa \log(1 - h^2) + (g_+ + g_-)h + \frac{g_+ g_-}{\kappa},$$

$$F_2[\kappa, g_+, g_-, m] = F_{g=0}[\kappa, m] + \frac{1+2k}{k} \log 4M + \frac{(g_+ + g_-)}{k} \left(1 - \frac{M}{1+k}\right) \\ + (g_+^2 + g_-^2) \frac{(2k+1)((2k+1)M^2 - (1+k)^2 M)}{4k(k+1)^4} + g_+ g_- \frac{(2k+1)M}{k(k+1)^2} \\ + \frac{(g_-^3 + g_+^3)k(2k+1)(3(k+1)^2 M^2 - 4(2k+1)M^4)}{3(k+1)^7} - \frac{(g_+ g_-^2 + g_-^2 g_+)k(2k+1)M^2}{(k+1)^5} + O(g_+^4, g_-^4),$$

where $M = \cosh^2 \frac{m}{2}$. The critical line is calculated by equalizing F_1 to F_2 and expanding $h = e^{-m}$ in series by g_+, g_- :

$$e^{\mp m} = \frac{1 - g_+ - g_-}{\xi - g_+ - g_-} + \frac{2(g_+ - g_-)^2 \kappa^3}{(1+k)(1+2k)^3} \left(1 + \frac{(g_+ + g_-)(8\kappa + 5)}{(\kappa + 1)(2\kappa + 1)} \right. \\ - \frac{(g_-^2 + g_+^2)(\kappa^4 + 15\kappa^3 - 31\kappa^2 - 48\kappa - 15) - 2g_- g_+(\kappa^4 + 3\kappa^3 + 41\kappa^2 - 48\kappa - 15)}{(\kappa + 1)^2(2\kappa + 1)^2} \\ + \frac{(g_+ g_-^2 + g_-^2 g_+)(-96\kappa^4 + 352\kappa^3 + 795\kappa^2 + 504\kappa + 105)}{(\kappa + 1)^3(2\kappa + 1)^3} \\ \left. + \frac{(g_-^3 + g_+^3)(-224\kappa^4 - 96\kappa^3 + 205\kappa^2 + 168\kappa + 35)}{(\kappa + 1)^3(2\kappa + 1)^3} + O(g_+^4, g_-^4) \right), \quad (A11)$$

that guarantee third-order phase transition (as we understand from first terms of expansion). The solution of the mean field equation is the same as in the restricted case but formally includes the chemical potential.

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