

Schwinger effect with backreaction in 1 + 1D massive QED with a strong external field

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In the presence of a strong electric field, the vacuum is unstable to the production of pairs of charged particles—the Schwinger effect. The created pairs extract energy from the electric field, resulting in nontrivial backreaction. In this paper, we study 1 + 1D massive QED subject to strong external electric fields in a self-consistent and fully quantum manner. We use the bosonized version of the theory, which attains a cosine interaction term in the presence of nonzero fermion mass m . However, the assumption of a strong electric field justifies a perturbative treatment of the cosine interaction, i.e., an expansion in m . We calculate the vacuum expectation value of the electric field to first order in m and show that—surprisingly—it satisfies a classical nonlinear partial differential equation (related to the sine-Gordon equation). We show that the electric field exhibits dissipation-free oscillations (analogous to ordinary plasma oscillations) and calculate the plasma frequency analytically. We also compare to the semiclassical approximation commonly used to study backreaction, showing that it fails to capture the $O(m)$ shift in the plasma frequency.

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I. INTRODUCTION

The mere existence of sufficiently strong electric fields leads to the production of electron-positron pairs due to the Schwinger effect [1,2]. The critical field strength for pair production is given in terms of the electron mass m and elementary charge $e > 0$ by

$$E_c = \frac{m^2 c^3}{e \hbar} \sim 10^{18} \text{ V/m}, \quad (1)$$

below which the effect is exponentially suppressed. Particle creation leads to a natural question concerning the energetics of the mechanism: Where does the energy of those particles come from? Clearly, the electric field itself is the source; thus, the created particles must extract energy from the field and reduce its strength. In addition, the newly born charged particles will both feel and modify the electric field. Understanding these complex dynamics is the problem of Schwinger effect *backreaction*. Backreaction to particle creation is also important in gravitational problems, such as particle production by the expanding universe [3] and Hawking radiation (e.g., [4,5]).

One may study the backreaction to the Schwinger effect using the “semiclassical” approximation, in which the

electric field is treated as a classical field rather than as an operator. Then, the classical field is demanded to be sourced by the expectation value of the current [6–11]. The semiclassical approximation is formally justified in a large- N expansion, but may still provide insight in the context of a single matter field.

Ultimately, the time evolution needs to be derived from full QED. While the general case remains intractable, Chu and Vachaspati [12] showed that backreaction can be treated exactly for a massless fermion in 1 + 1D (the “Schwinger model” [13]) using the method of bosonization [14,15]. They used the fact that 1 + 1D massless QED is “dual” to a free scalar (boson) theory of mass $M = q/\sqrt{\pi}$ (q is the elementary charge in 1 + 1D) [16–18], and showed that, by virtue of the linearity of the field equation, a fully quantum description of the screening process can be obtained by *simply solving a classical Klein-Gordon equation*.

In the massive Schwinger model (1 + 1 QED with fermions of mass m), the bosonized theory is no longer free. The interacting theory requires an explicit regulator to define, and the associated scale Λ actually appears in the boson field equations. If we include an external classical field E_C , then the field equations are

$$\left(\partial_t^2 - \partial_x^2 + \frac{q^2}{\pi} \right) \phi + \Lambda m \sin(\sqrt{4\pi}\phi) = -\frac{q}{\sqrt{\pi}} E_C, \quad (2)$$

with the precise meaning of $\sin(\sqrt{4\pi}\phi)$ depending on how the theory is regulated. The value of Λ should not affect observable predictions, but any attempt to treat (2)

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classically must somehow pick a value for Λ . Prior work has fixed Λ in a derivative expansion [19,20], at a large occupation number [21], or in a WKB approximation [22], or simply left as a free parameter [23,24].

In this paper we will study the massive theory as a perturbation of the massless theory and show that a version of (2) indeed arises to first order in m . Our field operators are defined in the Schrödinger picture, and regularization is done with normal ordering [14]. We assume that at early times the quantum state is the vacuum and the external classical electric field is zero. Then, the external field turns on at some time $t = t_0$ after which it varies arbitrarily. Under these assumptions, we find that, to first order in m , the field expectation value $\langle\phi\rangle$ satisfies the following equation:

$$\left(\partial_t^2 - \partial_x^2 + \frac{q^2}{\pi}\right)\langle\phi\rangle + \frac{e^\gamma m q}{\pi} \sin(\sqrt{4\pi}\langle\phi\rangle) = -\frac{q}{\sqrt{\pi}} E_C, \quad (3)$$

where $\gamma = 0.57721 \dots$ is the Euler-Mascheroni constant. This is the sine-Gordon equation (2) with the particular choice $\Lambda = e^\gamma q/\pi$, and ϕ is now understood as describing the expectation value of the field. We find it highly nontrivial that this classical-looking result arises from a fully quantum treatment. We emphasize in particular that no classical $\hbar \rightarrow 0$ limit has been taken. Instead, Eq. (3) describes a truly quantum phenomenon.

The boson expectation value $\langle\phi\rangle$ is related to the electric field expectation value by

$$\langle E \rangle = E_C + \frac{q}{\sqrt{\pi}} \langle\phi\rangle. \quad (4)$$

Thus, we see that the expectation value of the electric field obeys a simple partial differential equation (3). Notice that no “particles” or “pair creation” are explicitly visible in Eq. (3), and these concepts likewise do not arise in its derivation. We find this advantageous since the fundamental objects in QFT are the fields rather than particles. Equation (3) simply shows the response of local observables (expectation value of E) to the external field. Its solutions can be used to validate the heuristic idea of created particles screening electric fields.

Indeed, this process was beautifully illustrated in the massless case by Chu and Vachaspati [12] in the setting of capacitor breakdown. We revisit this problem in the (perturbatively) massive case and consider a slightly modified setup. We ensure local charge conservation for the external source by “assembling” the capacitor: its plates begin at the same location and then move to their final separation. For the boundary condition of the field, we consider the two cases: (i) the field may freely pass through the plates, and (ii) the field outside the capacitor is strictly zero (as if the plates are mirrors). For the first case, we found agreement with Refs. [12,21,22] that the electric field is screened inside the capacitor. However, for the second

case where the dynamical part of the field cannot spread out to infinity, the field never reaches a static state, but rather continues to oscillate indefinitely.

The frequency of these oscillations is the effective plasma frequency for the QED plasma created by the strong external field. We may define a precise plasma frequency by considering spatially homogeneous external fields. By solving (3) perturbatively, we find that the QED plasma frequency is given to $O(m)$ by

$$\omega = \frac{q}{\sqrt{\pi}} + \frac{m q e^\gamma}{\pi E_C} \cos\left(\frac{2\pi E_C}{q}\right) J_1\left(\frac{2\pi E_C}{q}\right). \quad (5)$$

We also study the problem semiclassically in order to test the validity of the semiclassical approximation. Semiclassical results are available for spatially homogeneous electric fields (at any value of m) [25] and can be compared to our small- m results in full QED. We find that the semiclassical approximation is exact at $m = 0$, but fails at $O(m)$, predicting that there is no correction at this order, in disagreement with our full-QED result. We may say that the semiclassical approximation is qualitatively correct but quantitatively incorrect, in particular failing to capture the $O(m)$ shift in the plasma frequency (5).

As explained in Sec. III, our perturbative treatment of the fermion mass is justified at strong electric field ($E_C \gg m$) and/or strong fermion coupling ($q \gg m$). However, versions of Eq. (2) with the same value $\Lambda = e^\gamma q/\pi$ also arise in other physical regimes [20–22], suggesting a more fundamental significance within the massive Schwinger model. Further research is required to clarify the scope of the quasiclassical description in massive 1 + 1D QED.

This paper is organized as follows. In Sec. II, we review the bosonization of the massive Schwinger model and derive the QED Hamiltonian in terms of the boson field. In Sec. III, we calculate $\hat{\phi}$ to first order in the fermion mass and show that it satisfies Eq. (3). In Sec. IV, we revisit the capacitor breakdown problem and demonstrate that the screening effect is captured by solving Eq. (3). In Sec. V, we consider a spatially homogeneous background field and derive the plasma frequency (5). In Sec. VI, consider the same problem in the semiclassical approximation and compare the results. We conclude in Sec. VII with a brief discussion of interesting future directions.

In this paper, we take $c = \hbar = 1$, and the signature of the metric is $(-, +)$.

II. BOSONIZATION OF 1 + 1D QED

In this section, we derive the QED Hamiltonian in terms of the boson field, closely following Coleman [26].¹ The Lagrangian of 1 + 1D QED in the presence of an

¹In [26], the signature of the metric is opposite to that in our paper.

external classical source J_C^μ satisfying the conservation law $\partial_\mu J_C^\mu = 0$ is given by

$$L = \int dx \left[i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi + qA_\mu \bar{\psi}\gamma^\mu \psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J_C^\mu \right]. \quad (6)$$

The spatial integral is taken from $-\infty$ to ∞ , and γ^μ are the gamma matrices, for which we take the following representation:

$$\gamma^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (7)$$

In 1 + 1D, the electromagnetic field behaves quite differently compared to the 3 + 1D case. First of all, there is no magnetic field, and there is only one component of the electric field. Furthermore, there are no electromagnetic waves since there are no transverse fields. This can be understood in a physical sense by considering a uniformly charged plate in 3 + 1D. The electric field created by the plate is uniform everywhere, and it never decays, no matter how far one goes away from the plate. In addition, the motion of the plate does not affect the electric field at all; i.e., no waves are generated. In the quantum theory, this indicates the nonexistence of photons. This feature of the electromagnetic field in 1 + 1D can be most simply captured by taking the Coulomb gauge $A^\mu(t, x) = (A(t, x), 0)$. We will use this gauge exclusively, and express our final result in terms of the gauge-invariant electric field. The electric field E is then given by

$$E(t, x) = -\partial_x A(t, x). \quad (8)$$

Then, the Lagrangian (6) no longer depends on the time derivative of $A(t, x)$. Variation of Eq. (6) with respect to $A(t, x)$ results in Gauss' law:

$$\partial_x^2 A(t, x) = -(q\psi^\dagger \psi + J_C^t). \quad (9)$$

We express the general solution as

$$A(t, x) = -\frac{q}{2} \int |x - y| \psi^\dagger(t, y) \psi(t, y) dy + A_C(t, x), \quad (10)$$

where A_C is a potential for the classical source, i.e.,

$$J_C^t(t, x) = \partial_x E_C(t, x) = -\partial_x^2 A_C(t, x). \quad (11)$$

Equation (10) shows that $A(t, x)$ at time t is determined only by the fermion field and the classical source at time t without solving the evolution equation, showing the non-dynamical nature of the gauge field.

The Hamiltonian is obtained by Legendre transforming Eq. (6). Eliminating $A(t, x)$ using Eq. (10) and discarding the surface integral terms as well as a nondynamical term proportional to $A_C \partial_x^2 A_C$, one finds

$$H(t) = \int dx [-i\bar{\psi}\gamma^x \partial_x \psi + m\bar{\psi}\psi + qA_C \psi^\dagger \psi] - \frac{q^2}{4} \iint dx dy |x - y| \psi^\dagger(t, x) \psi(t, x) \psi^\dagger(t, y) \psi(t, y). \quad (12)$$

At this point, everything is classical. The quantum theory in the Schrödinger picture is obtained by promoting the fermion field $\psi(t_0, x)$ at some time t_0 to a time-independent operator $\hat{\psi}(x)$ that satisfies the canonical anticommutation relation,

$$\{\hat{\psi}_\alpha(x), \hat{\psi}_\beta^\dagger(y)\} = \delta_{\alpha\beta} \delta(x - y). \quad (13)$$

We choose the time t_0 at which the Schrödinger picture operator is defined to be equal to the time at which the external classical electric field is turned on. To avoid the infinite zero-point energy we normal-order the Hamiltonian as follows:

$$\hat{H}(t) = \int dx [-i\bar{\hat{\psi}}\gamma^x \partial_x \hat{\psi} + m\bar{\hat{\psi}}\hat{\psi} + qA_C \hat{\psi}^\dagger \hat{\psi}] - \frac{q^2}{4} \iint dx dy :\hat{\psi}^\dagger(x) \hat{\psi}(x) : |x - y| : \hat{\psi}^\dagger(y) \hat{\psi}(y) :. \quad (14)$$

This particular normal ordering makes the Hamiltonian (14) differ from the bare Hamiltonian [Eq. (12) with $\psi(t_0, x) \rightarrow \hat{\psi}(x)$] by

$$\hat{H} = \hat{H}_{\text{bare}} + \mathcal{C}_1 + \hat{Q}\mathcal{C}_2, \quad (15)$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are infinite c -numbers and $\hat{Q}$ is the charge operator,

$$\hat{Q} = \int :\hat{\psi}^\dagger(x) \hat{\psi}(x) : dx. \quad (16)$$

Thus, the Hamiltonians describe the same physics provided that we restrict to states of zero charge. This assumption is consistent with time evolution since $\hat{Q}$ commutes with the Hamiltonian. We will assume zero total charge for the remainder of the paper.

Keep in mind that we are in the Schrödinger picture even though the Hamiltonian $\hat{H}$ depends on time. This is simply due to the fact that the external classical field is time dependent, and the energy of the system is not generally conserved.

Now, we are ready to bosonize this Hamiltonian. The boson field $\hat{\phi}$ and its canonical momentum $\hat{\Pi}$ satisfy the canonical commutation relations,

$$[\hat{\phi}(x), \hat{\Pi}(y)] = i\delta(x - y), \quad (17)$$

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0, \quad (18)$$

$$[\hat{\Pi}(x), \hat{\Pi}(y)] = 0, \quad (19)$$

and the bosonization identities are² (e.g., [26,34,35])

$$:-i\hat{\psi}\gamma^x\partial_x\hat{\psi}: \leftrightarrow N_\mu \left\{ \frac{1}{2}\hat{\Pi}^2 + \frac{1}{2}(\partial_x\hat{\phi})^2 \right\}, \quad (20)$$

$$:\tilde{\psi}\gamma^t\hat{\psi}: \leftrightarrow \frac{1}{\sqrt{\pi}}\partial_x\hat{\phi}, \quad (21)$$

$$:\tilde{\psi}\gamma^x\hat{\psi}: \leftrightarrow -\frac{1}{\sqrt{\pi}}\hat{\Pi}, \quad (22)$$

$$:\tilde{\psi}\hat{\psi}: \leftrightarrow -\frac{e^\gamma\mu}{2\pi}N_\mu \left\{ \cos\sqrt{4\pi}\hat{\phi} \right\}. \quad (23)$$

The symbol N_μ denotes normal ordering with respect to an arbitrary mass scale μ . This procedure is defined by mode expanding $\hat{\phi}(x)$ and $\hat{\Pi}(x)$ as

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega_{k,\mu}}} \left(\hat{a}_{k,\mu} e^{ikx} + \hat{a}_{k,\mu}^\dagger e^{-ikx} \right), \quad (24)$$

$$\hat{\Pi}(x) = -i \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sqrt{\frac{\omega_{k,\mu}}{2}} \left(\hat{a}_{k,\mu} e^{ikx} - \hat{a}_{k,\mu}^\dagger e^{-ikx} \right), \quad (25)$$

where

$$\omega_{k,\mu} = \sqrt{k^2 + \mu^2}. \quad (26)$$

Then, $N_\mu\{\dots\}$ means to rearrange the order of operators by placing all $\hat{a}_{k,\mu}$ to the right of all $\hat{a}_{k,\mu}^\dagger$.

The choice of μ is fully arbitrary. A change in μ shifts the kinetic terms (20) by an infinite c -number and leaves the cosine term (23) invariant [26], so that the dynamics are unmodified. At the level of the substitution rules, there is no natural choice of μ . However, the original theory has mass scales of m and q . Furthermore, the last term in (14) bosonizes to $\int \frac{1}{2\pi} q^2 \hat{\phi}^2$ (dropping boundary terms) indicating the presence of a mass

$$M \equiv \frac{q}{\sqrt{\pi}}. \quad (27)$$

Since we will be treating the theory perturbatively in the fermion mass m , this “meson mass” M is the most natural scale for normal ordering. Adopting this choice, the bosonized Hamilton becomes

$$\hat{H}(t) = \int dx N_M \left\{ \frac{1}{2}\hat{\Pi}(x)^2 + \frac{1}{2}(\partial_x\hat{\phi}(x))^2 + \frac{1}{2}M^2\hat{\phi}(x)^2 + \frac{q}{\sqrt{\pi}}E_C(t,x)\hat{\phi}(x) - \frac{e^\gamma m M}{2\pi} \cos\sqrt{4\pi}\hat{\phi}(x) \right\}. \quad (28)$$

Notice that at $m = 0$, M has a clear physical meaning as the mass of the $\hat{\phi}(x)$ particle. This mass is also present in the cosine self-interaction term. On the other hand, the coefficient $q/\sqrt{\pi}$ in front of $E_C(t,x)$ functions as an electric coupling, so we do not express it in terms of M , even though it has the same numerical value.

It is worth emphasizing that the choice $\mu = M$ of normal ordering mass is purely for convenience. This choice allows us to separate the Hamiltonian (28) into a free part, which evolves as a free field of mass M , and an interaction part that only contains the cosine term. If we had chosen a different value for μ , we would need to evolve the free part with mass μ , and the interaction Hamiltonian would include terms quadratic in $\hat{\phi}$, unnecessarily complicating the calculations.

The Schrödinger-picture field operator $\hat{A}(t,x)$ is obtained by replacing $\psi(t,x)$ with $\hat{\psi}(x)$ in Eq. (10). The electric field operator $\hat{E}(t,x)$ is then given by

$$\hat{E}(t,x) = -\partial_x\hat{A}(t,x) = \frac{q}{\sqrt{\pi}}\hat{\phi}(x) + E_C(t,x). \quad (29)$$

Although the operators $\hat{A}$ and $\hat{E}$ are defined in the Schrödinger picture, they inherit explicit time dependence from the classical field. Equation (29) gives a nice physical meaning to the boson field $\hat{\phi}(x)$: it is the quantum part of the electric field.

Before proceeding, it is worth noting the transformation properties of Eq. (29). In 1 + 1 dimensions, the electric field $E = F_{10}$ is invariant under Lorentz boosts but flips sign under parity transformations, characterizing it as a pseudoscalar. The bosonized field $\hat{\phi}(x)$ also transforms as a pseudoscalar according to Eq. (21).

III. VACUUM EXPECTATION VALUE

We consider the case where the electric field is zero at an early time and turned on at $t = t_0$.³ Before the external field

²The correspondence between the Fermi and boson theories has been extensively studied, and an overwhelming amount of evidence suggests that the two theories are equivalent when these substitutions are made in the Hamiltonian (e.g., [14,15,27–33]).

³In QED₂, a constant E_C is sometimes referred to as a “ θ term” (e.g., [26]). Here, setting $E_C = 0$ before $t = t_0$ indicates that we consider the case of external fields where $\theta = 0$.

is turned on, we assume that the state is in the vacuum state $|\Omega\rangle$ defined to be the lowest energy eigenvector of the Hamiltonian (28) at $t < t_0$. Since the bosonized Hamiltonian (28) is considered to be equivalent to the original fermion Hamiltonian (14), the vacuum state $|\Omega\rangle$ defined on the boson side should also be the vacuum state on the fermion side. After turning on the field ($t > t_0$), $|\Omega\rangle$ evolves to a state $|t; \Omega\rangle$ according to the Schrödinger equation:

$$i \frac{d}{dt} |t; \Omega\rangle = \hat{H}(t) |t; \Omega\rangle. \quad (30)$$

Since the $\hat{\phi}(x)$ is the scaled electric field due to Eq. (29), the expectation value of $\hat{\phi}(x)$ evaluated on $|t; \Omega\rangle$ can capture the time evolution of the vacuum as a response to the external field; thus, our goal is to compute

$$\langle \phi \rangle(t, x) \equiv \langle t; \Omega | \hat{\phi}(x) | t; \Omega \rangle. \quad (31)$$

A. Interaction picture

We now split the Hamiltonian (28) into a free part (which includes the classical source) and an interaction part (the cosine term). To explicitly write down the vacuum $|\Omega\rangle$, we introduce an adiabatic switching function that turns off the interaction at early times.

We define the total Hamiltonian $\hat{H}_T(t)$ by

$$\hat{H}_T(t) = \hat{H}_0(t) + \hat{H}_{\text{int}}(t), \quad (32)$$

where

$$\hat{H}_0(t) = \hat{H}_{\text{ff}} + \hat{H}_E(t) \quad (33)$$

and each piece is given by

$$\hat{H}_{\text{ff}} = \int dx N_M \left[\frac{1}{2} \hat{\Pi}(x)^2 + \frac{1}{2} (\partial_x \hat{\phi}(x))^2 + \frac{1}{2} M^2 \hat{\phi}(x)^2 \right], \quad (34)$$

$$\hat{H}_E(t) = \int dx \frac{q}{\sqrt{\pi}} E_C(t, x) \hat{\phi}(x), \quad (35)$$

$$\hat{H}_{\text{int}}(t) = \int dx N_M \left[-\frac{e^\gamma m f(t) M}{2\pi} \cos \sqrt{4\pi} \hat{\phi}(x) \right]. \quad (36)$$

$\hat{H}_{\text{ff}}$ represents the “free-free” part of the Hamiltonian in the sense that this Hamiltonian describes a free massive particle with mass M with no interaction or electric field. The function $f(t)$ in $\hat{H}_{\text{int}}(t)$ is chosen to vanish at an early time and to reach $f = 1$ before $t = t_0$ (the time at which the external classical field is turned on). In addition, we demand that the transition of $f(t)$ from 0 to 1 is

infinitesimally slow. This procedure allows us to write down the expression of $|\Omega\rangle$ in terms of operators and states defined in the interaction picture as explained below.

In the infinite past, there is no external electric field and the interaction term is turned off due to the $f(t)$; thus, the total Hamiltonian is given by

$$\hat{H}_T(-\infty) = \hat{H}_{\text{ff}}. \quad (37)$$

The free-free Hamiltonian $\hat{H}_{\text{ff}}$ defines a vacuum state $|0, M\rangle$ in the usual way, i.e., by

$$\hat{a}_{k,M} |0, M\rangle = 0 \quad (38)$$

for all k with $\hat{a}_{k,M}$ defined in (24) and (25). This provides the initial value for the true vacuum state according to (37). Since $f(t)$ turns on very slowly, we may evolve the vacuum using the adiabatic approximation. In particular, the true vacuum $|\Omega\rangle$ can be approximated by evolving $|0, M\rangle$ from the infinite past to time t_0 using the Hamiltonian (28),

$$|\Omega\rangle \simeq \hat{U}_T(t_0, -\infty) |0, M\rangle. \quad (39)$$

Here, $\simeq$ means that two expressions differ only by a phase, and $\hat{U}_T(t, t')$ is the evolution operator for $\hat{H}_T$, defined by Eq. (A1).

The interaction picture is obtained by evolving the operators with the free time evolution operator $\hat{U}_0(t, t')$ and evolving the state with the interaction evolution operator $\hat{U}_{\text{int}}(t, t')$. Since our Hamiltonian depends on time, the theory is slightly modified from the most familiar form, and for the reader's convenience, we review the relevant equations in Appendix A.

The interaction picture operators $\hat{\phi}^I(t, x)$ and $\hat{\Pi}^I(t, x)$ are defined by [see Eq. (A4)]

$$\hat{\phi}^I(t, x) = \hat{U}_0^\dagger(t, t_0) \hat{\phi}(x) \hat{U}_0(t, t_0), \quad (40)$$

$$\hat{\Pi}^I(t, x) = \hat{U}_0^\dagger(t, t_0) \hat{\Pi}(x) \hat{U}_0(t, t_0). \quad (41)$$

The above definition ensures the following conditions:

$$\hat{\phi}^I(t_0, x) = \hat{\phi}(x), \quad (42)$$

$$\hat{\Pi}^I(t_0, x) = \hat{\Pi}(x). \quad (43)$$

Using Eqs. (17)–(19), one can show that these operators satisfy the equal-time commutation relation

$$[\hat{\phi}^I(t, x), \hat{\Pi}^I(t, y)] = i\delta(x - y), \quad (44)$$

$$[\hat{\phi}^I(t, x), \hat{\phi}^I(t, y)] = 0, \quad (45)$$

$$[\hat{\Pi}^I(t, x), \hat{\Pi}^I(t, y)] = 0. \quad (46)$$

Then, from Eq. (A5) and using Eqs. (44)–(46), we have the following operator equations for $\hat{\phi}^I(t, x)$ and $\hat{\Pi}^I(t, x)$:

$$\frac{\partial}{\partial t} \hat{\phi}^I(t, x) = -i[\hat{\phi}^I(t, x), \hat{H}_0^I(t)] = \hat{\Pi}^I(t, x), \quad (47)$$

$$\begin{aligned} \frac{\partial}{\partial t} \hat{\Pi}^I(t, x) &= -i[\hat{\Pi}^I(t, x), \hat{H}_0^I(t)] \\ &= \partial_x^2 \hat{\phi}^I(t, x) - \mu^2 \hat{\phi}^I(t, x) - \frac{q}{\sqrt{\pi}} E_C(t, x). \end{aligned} \quad (48)$$

The explicit solutions to these equations are given by

$$\hat{\phi}^I(t, x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2\omega_{k,M}}} (\hat{a}_{k,M} e^{-i\omega_{k,M}(t-t_0)+ikx} + \hat{a}_{k,M}^\dagger e^{i\omega_{k,M}(t-t_0)-ikx}) + \phi_{\text{cl}}(t, x), \quad (49)$$

$$\hat{\Pi}^I(t, x) = -i \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sqrt{\frac{\omega_{k,M}}{2}} (\hat{a}_{k,M} e^{-i\omega_{k,M}(t-t_0)+ikx} - \hat{a}_{k,M}^\dagger e^{i\omega_{k,M}(t-t_0)-ikx}) + \partial_t \phi_{\text{cl}}(t, x), \quad (50)$$

where $\omega_{k,M}$ is defined by Eq. (26). The classical part $\phi_{\text{cl}}(t, x)$ satisfies the Klein-Gordon equation

$$(\partial_t^2 - \partial_x^2 + M^2)\phi_{\text{cl}}(t, x) = -\frac{q}{\sqrt{\pi}} E_C(t, x), \quad (51)$$

with the condition

$$\phi_{\text{cl}}(t, x) = 0 = \partial_t \phi_{\text{cl}}(t, x) \quad \text{for } t \leq t_0. \quad (52)$$

This condition is a consequence of the fact that $\hat{\phi}^I(t, x)$ and $\hat{\Pi}^I(t, x)$ need to satisfy the conditions (42) and (43) as well as the fact that $E_C(t, x)$ vanishes at $t \leq t_0$. Equivalently, ϕ_{cl} is the retarded solution to Eq. (51).

The free evolution operator before the electric field turns on only changes $|0, M\rangle$ by a phase, i.e.,

$$\hat{U}_0(t_0, -\infty)|0, M\rangle \simeq |0, M\rangle. \quad (53)$$

The expression of the state $|t; \Omega\rangle$ is then

$$\begin{aligned} |t; \Omega\rangle &= \hat{U}_T(t, t_0)|\Omega\rangle \\ &\simeq \hat{U}_0(t, t_0)\hat{U}_{\text{int}}(t, -\infty)|0, M\rangle, \end{aligned} \quad (54)$$

using (39) and (A7). Then, the expression of $\langle\phi\rangle = \langle t; \Omega|\hat{\phi}(x)|t; \Omega\rangle$ can be written in the interaction picture by

$$\langle\phi\rangle = \langle 0, M|\hat{U}_{\text{int}}^\dagger(t, -\infty)\hat{\phi}^I(t, x)\hat{U}_{\text{int}}(t, -\infty)|0, M\rangle, \quad (55)$$

with $\hat{U}_{\text{int}}(t, t')$ given by Eq. (57).

To proceed further, an important observation is that the normal ordering $N_M\{\dots\}$ and the operation to obtain the interaction picture operators commute; i.e., for a function $F(\hat{\phi})$ of polynomials of $\hat{\phi}$ (not necessarily evaluated at the same spatial points), we have

$$\hat{U}_0^\dagger(t, t_0)N_M\{F(\hat{\phi})\}\hat{U}_0(t, t_0) = N_M\{F(\hat{\phi}^I)\}. \quad (56)$$

This allows us to write $\hat{U}_{\text{int}}(t, t')$ fully in terms of $\hat{\phi}^I(t, x)$ while keeping the structure of normal ordering intact. In particular, from (A9) and (36) the Dyson series solution is

$$\begin{aligned} \hat{U}_{\text{int}}(t, t') &= T \exp \left(i \int_{t'}^t ds \int dx N_M \left\{ \frac{e' m M f(s)}{2\pi} \cos \sqrt{4\pi} \hat{\phi}^I(s, x) \right\} \right), \end{aligned} \quad (57)$$

where T denotes time ordering. Note that Eq. (57) is a shorthand notation for a Taylor series in m and will provide an asymptotic approximation as $m \rightarrow 0$ as long as each term gives a finite expression. We carry out the expansion to the first order in m and find a finite result for $\langle\phi\rangle$.

B. Calculation of $\langle\phi\rangle$

It is helpful to review a few facts before computing the field expectation value. First, we have the normal-ordering identity

$$[e^{\hat{A}}, \hat{B}] = [\hat{A}, \hat{B}]e^{\hat{A}}, \quad (58)$$

which holds for any operators $\hat{A}$ and $\hat{B}$ whose commutator is a c -number. This in particular implies that

$$\begin{aligned} &[N_M\{\cos \sqrt{4\pi} \hat{\phi}^I(t_1, y_1)\}, \hat{\phi}^I(t_2, x_2)] \\ &= -\sqrt{4\pi} [\hat{\phi}^I(t_1, y_1), \hat{\phi}^I(t_2, x_2)] N_M\{\sin \sqrt{4\pi} \hat{\phi}^I(t_1, y_1)\}. \end{aligned} \quad (59)$$

Another useful fact is that for any composite operators $F(\hat{\phi}^I(t, x), \hat{\Pi}^I(t, x))$ formed from polynomials of $\hat{\phi}^I(t, x)$ and $\hat{\Pi}^I(t, x)$, we have

$$\begin{aligned} \langle 0, M | N_M \{ F(\hat{\phi}^I(t, x), \hat{\Pi}^I(t, x)) \} | 0, M \rangle \\ = F(\phi_{\text{cl}}(t, x), \partial_t \phi_{\text{cl}}(t, x)). \end{aligned} \quad (60)$$

To check, notice that the states on both sides annihilate all the operator parts of $\hat{\phi}^I(t, x)$ and $\hat{\Pi}^I(t, x)$, leaving only the classical parts.

Finally, we note that the retarded Green function for the Klein-Gordon equation may be expressed in terms of the interaction-picture fields as (e.g., [36])

$$G_{\text{ret}}(\Delta t, \Delta x) = i\theta(\Delta t)[\hat{\phi}^I(t, x), \hat{\phi}^I(t', y)], \quad (61)$$

where $\Delta t = t - t'$, $\Delta x = x - y$, and $\theta(t)$ is the step function. The retarded green function satisfies

$$(\partial_t^2 - \partial_x^2 + M^2)G_{\text{ret}}(t, x) = \delta(t)\delta(x) \quad (62)$$

and is explicitly expressed as (e.g., [37,38])

$$\begin{aligned} G_{\text{ret}}(\Delta t, \Delta x) &= \frac{\theta(\Delta t)}{\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega_{k,M}} \sin(\omega_{k,M}\Delta t) e^{ik\Delta x} \\ &= \frac{1}{2} \theta(\Delta t) \theta(\Delta t^2 - \Delta x^2) J_0\left(M\sqrt{\Delta t^2 - \Delta x^2}\right). \end{aligned} \quad (63)$$

With the ingredients assembled, the calculation of $\langle \phi \rangle$ is straightforward,

$$\begin{aligned} \langle \phi \rangle &= \langle 0, M | \hat{\phi}^I(t, x) | 0, M \rangle + \int_{-\infty}^t dt' \int dy \left(-i \frac{e' m M f(t')}{2\pi\sqrt{\pi}} \right) \langle 0, M | [N_M \{ \cos \sqrt{4\pi} \hat{\phi}^I(t', y) \}, \hat{\phi}^I(t, x)] | 0, M \rangle \\ &= \phi_{\text{cl}}(t, x) + \int_{-\infty}^t dt' \int dy \left(i \frac{e' m M f(t')}{2\pi\sqrt{\pi}} \right) \sqrt{4\pi} [\hat{\phi}^I(t', y), \hat{\phi}^I(t, x)] \langle 0, M | N_M \{ \sin \sqrt{4\pi} \hat{\phi}^I(t', y) \} | 0, M \rangle \\ &= \phi_{\text{cl}}(t, x) + \int_{-\infty}^{\infty} dt' \int dy \left(-\frac{e' m q}{\pi} \right) \sin \left[\sqrt{4\pi} \phi_{\text{cl}}(t', y) \right] G_{\text{ret}}(t - t', x - y). \end{aligned} \quad (64)$$

The first line follows from (55) and (57), while the second and third lines use (59) and (61), respectively. In the third line we are able to set $f(t') = 1$ since $\phi_{\text{cl}}(t, x) = 0$ at $t < t_0$. Equation (64) is the causal perturbative solution to Eq. (3), justifying the claim that $\langle \phi \rangle$ obeys this equation to $O(m)$.

C. Validity of the m expansion

The perturbative expression (64) for $\langle \phi \rangle$ is likely to provide a good approximation as long as the perturbation is numerically small compared to the background term ϕ_{cl} . The comparison may be made more easily at the level of the differential equation (3). For this purpose we will also restore factors of $\hbar$, noting that ϕ is dimensionless, while q and E_C have the same units and qE_C has units of force. Equation (3) then becomes

$$\left(\partial_t^2 - \partial_x^2 + \frac{q^2}{\hbar\pi} \right) \langle \phi \rangle + \frac{e' m q}{\hbar^{3/2}\pi} \sin \left(\sqrt{4\pi} \langle \phi \rangle \right) = -\frac{q}{\hbar\sqrt{\pi}} E_C. \quad (65)$$

The nonanalyticity in $\hbar$ indicates that effects predicted by Eq. (65) cannot be captured in the limit of $\hbar \rightarrow 0$.

For the mass expansion to be valid, we require that the sine term of Eq. (65) is much smaller than at least

one other term. There are two cases where this assumption is justified.

The first case is the *strong-field limit*. If the field has typical strength $E_C \sim \mathcal{E}$, then the condition $\mathcal{E} \gg m/\hbar^{1/2}$ guarantees that the sine term in Eq. (65) is small compared to the electric field. However, since E_C is “turned on” from zero starting at time $t = t_0$, there may be times when it is not small compared to the sine term. We expect that if the transition is sufficiently rapid, then the approximation will remain valid, and we may see this in the equations as follows. If $\mathcal{T}$ denotes the timescale for E_C to reach $\mathcal{E}$, then we expect $\mathcal{T}$ to set the timescale of variation for $\langle \phi \rangle$ during this period. Using $\partial_t^2 \langle \phi \rangle \sim \langle \phi \rangle / \mathcal{T}^2$, we find the condition $e' m q / \hbar^{3/2} \ll \pi / \mathcal{T}^2$ for the sine term to be small compared to $\partial_t^2 \langle \phi \rangle$. Thus, a regime of validity for our approximation is

$$\mathcal{E} \gg \frac{m}{\hbar^{1/2}}, \quad \mathcal{T} \ll \sqrt{\frac{\hbar^{3/2}}{mq}}. \quad (66)$$

The second case where small m treatment is justified is the *fermion strong coupling limit*. We see directly in Eq. (3) that the sine term is small compared to the mass term $q^2 \langle \phi \rangle / (\hbar\pi)$ provided

$$q \gg \frac{m}{\hbar^{1/2}}. \quad (67)$$

This regime brings out the strong-weak nature of the duality; the strong coupling $q \gg m/\hbar^{1/2}$ on the fermion side translates to weak coupling on the boson side, enabling perturbative calculations like ours.

Notice that the conditions of validity (66) and (67) cannot be satisfied if one takes $\hbar \rightarrow 0$. This is the sense in which our result may be considered “fully quantum.”

D. Conservation of energy

Before proceeding, we note that Eq. (3) admits a conserved energy when the external field E_C is time independent. We express this quantity as

$$\mathcal{E} = \mathcal{E}_{\text{particle}} + \mathcal{E}_{\text{field}}, \quad (68)$$

where

$$\begin{aligned} \mathcal{E}_{\text{particle}} = \int \left[\frac{1}{2} (\partial_t \langle \phi \rangle)^2 + \frac{1}{2} (\partial_x \langle \phi \rangle)^2 \right. \\ \left. - \frac{e^\gamma m M}{2\pi} \cos \sqrt{4\pi} \langle \phi \rangle \right] dx \end{aligned} \quad (69)$$

and

$$\mathcal{E}_{\text{field}} = \int \frac{1}{2} \langle E \rangle^2 dx = \int \frac{1}{2} \left(E_C + \frac{q}{\sqrt{\pi}} \langle \phi \rangle \right)^2 dx, \quad (70)$$

noting Eq. (4). Notice that $\mathcal{E}_{\text{field}}$ is just the “classical” field energy associated with the expected value $\langle E \rangle$ of the electric field; likewise, we may think of $\mathcal{E}_{\text{particle}}$ as a fictitious classical matter field energy. The conservation of $\mathcal{E}$ represents an exchange between field and particle degrees of freedom.

In Appendix B we show that after adding back in the nondynamical term $\frac{1}{2} E_C^2$ to (28), the expectation value of the Hamiltonian is equal to $\mathcal{E}$ to first order in m , i.e.,

$$\langle \hat{H} \rangle = \mathcal{E}. \quad (71)$$

This establishes the fact that $\mathcal{E}$ is truly the energy of the system. However, this simplicity does not extend to the division into particle and field: the “energies” $\mathcal{E}_{\text{particle}}$ and $\mathcal{E}_{\text{field}}$ are not equal to the expectation values of their quantum counterparts.

IV. CAPACITOR BREAKDOWN REVISITED

As shown in [12,21,22], bosonized 1 + 1D QED can be used to study the breakdown of a capacitor when its electric field is strong. For the massless case, Ref. [12] considered this problem by placing two oppositely charged plates at a separate distance and letting the field evolve according to Eq. (3) with zero fermion mass. The massive case was considered in Ref. [21] by solving Eq. (3) assuming a large boson occupation to justify the classical treatment.

Reference [22] developed a method to accurately track the long-term evolution of the field for both massless and massive cases.

Our fully quantum calculation requires the external electric field to vanish in the asymptotic past. To “turn on” the electric field in a manner compatible with local conservation of charge, we take our capacitor plates to begin at the same spatial location and then separate. In particular, the plates start at $x = 0$ and move in the opposite direction with velocity $v < 1$ until they reach a final separation of $2L$. This can be modeled by setting the background field to be

$$E_C = \begin{cases} 0, & t < 0, \\ E_0 \theta(vt - x) \theta(vt + x), & 0 < t < L/v, \\ E_0 \theta(L + x) \theta(L - x), & t > L/v. \end{cases} \quad (72)$$

(We set the time that the electric field turns on to be $t_0 = 0$.) The charge density associated with this electric field is

$$\begin{aligned} \rho_C = \partial_x E_C \\ = \begin{cases} 0, & t < 0, \\ -E_0 \delta(vt - x) + E_0 \delta(vt + x), & 0 < t < L/v, \\ -E_0 \delta(L - x) + E_0 \delta(L + x), & t > L/v. \end{cases} \end{aligned} \quad (73)$$

Thus, the background profile (72) corresponds to a positively charged plate moving to the left until it reaches $x = -L$, together with a negatively charged plate moving to the right until it reaches $x = +L$.

The previous work [12,21,22] did not impose any boundary conditions at the capacitor plates. The field can then become nonzero outside, which may be interpreted as allowing the created particles passing through the plates. Another option we consider is to impose $\langle \phi \rangle = 0$ at the plates, which can be thought of as causing fields and particles to reflect off the plates, as if they are impenetrable mirrors. In each case, we solved Eq. (3) numerically. Since q has the dimension of mass as we can see in Eq. (27), we use it as a reference and set other parameters as $m/q = 0.1$, $E_0/q = 1$, $v = 0.8$, $qL = 20$. We use a standard fourth-order Runge-Kutta method to solve the PDE (3).

Figure 1 shows the case of the penetrable plates. In this case, the time dependence eventually ceases as the dynamical part of the field spreads out to infinity. The final static state of the electric field can be found by solving Eq. (3) assuming no time dependence, taking the external electric field to be in its final state, with the capacitor fully assembled. By expanding $\langle \phi \rangle$ perturbatively in m as

$$\langle \phi \rangle \approx \phi_0 + \phi_1, \quad (74)$$

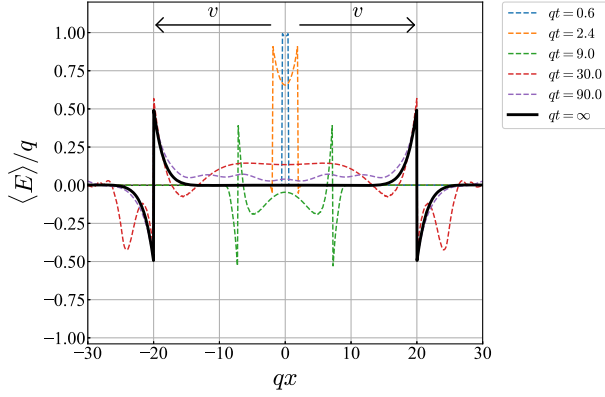


FIG. 1. The expectation value of the electric field $\langle E \rangle = E_C + q\langle\phi\rangle/\sqrt{\pi}$. Two charged plates start at $x = 0$ and move in opposite directions with velocity $v = 0.8$. The plates are stopped when they reach $qx = \pm 20$. The electric field eventually reaches a static state as all the dynamical part spreads out to infinity. The static solution (black line) shows exponential screening of the plates from both sides. We set $m/q = 0.1$, $E_0/q = 1.0$.

and plugging this into Eq. (3), we find

$$\phi_0(x) = \begin{cases} \frac{E_0}{M} (e^{-ML} \cosh(Mx) - 1), & |x| \leq L, \\ -\frac{E_0}{M} \sinh(ML) e^{-M|x|}, & |x| > L, \end{cases} \quad (75)$$

and

$$\phi_1(x) = -\frac{e^\gamma m}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-M|x-y|} \sin \sqrt{4\pi} \phi_0(y) dy. \quad (76)$$

The black line in Fig. 1 shows this analytic solution. For the final static state, we can see from Eq. (75) that the field decays exponentially away from the plates with a characteristic length scale of $1/M$. One may say that a “cloud of particles” surrounds each plate, screening its electric field. This reproduces the results of [12] and derives the first correction in fermion mass.

Figure 2 shows the alternative case of the impenetrable plates, where particles “bounce off.” In this case, the field keeps oscillating and never settles down to a static state. This may be attributed to conservation of energylike quantity (68). As one can see in Fig. 2, the field is dynamical even after the plates reach the final separation. This is the same behavior as a classical plasma confined to a box with no dissipation, and the oscillations are known as plasma oscillations.

V. PLASMA FREQUENCY FOR QED

A classical plasma consists of point charges of both signs of charge interacting with the self-consistent electromagnetic field. In this context, local separation between oppositely charged particles results in an electrostatic

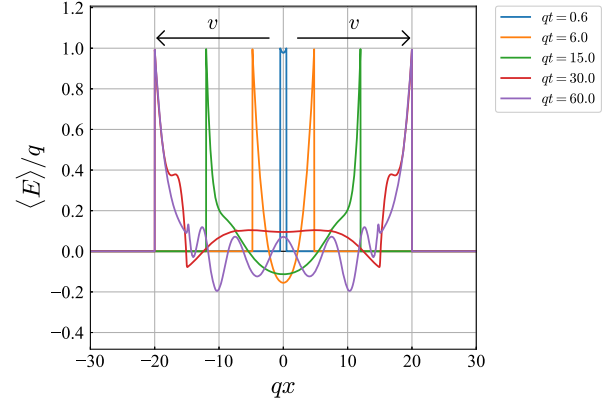


FIG. 2. Same as Fig. 1, except for impenetrable places ($\langle\phi\rangle$ is forced to vanish outside the plates). In this case, the electric field inside the capacitor never reaches a static state; the dynamical part of the field bounces back and forth inside the capacitor.

restoring force that produces oscillations. The electric field reacts to the motion and also oscillates at the same frequency. The QED plasma produced by capacitor breakdown also displays these electric field oscillations, although the underlying mechanism may be quite different. In this section, we derive the frequency of the oscillations and also exhibit an effective potential formulation.

For this purpose, we consider a spatially homogeneous external electric field that turns on at $t = t_0$ to the constant value E_C :

$$E_C(t) = E_C \theta(t). \quad (77)$$

We also assume that $\langle\phi\rangle$ is spatially homogeneous, so that Eq. (3) becomes an ordinary differential equation,

$$\frac{d^2}{dt^2} \langle\phi\rangle + M^2 \langle\phi\rangle = -\frac{e^\gamma m q}{\pi} \sin(\sqrt{4\pi} \langle\phi\rangle) - \frac{q}{\sqrt{\pi}} E_C, \quad (78)$$

holding for $t > 0$. Since there is no external field for $t < 0$, we set $\langle\phi\rangle = 0 = \partial_t \langle\phi\rangle$ for $t < 0$.

At zeroth order in m , Eq. (78) is a simple harmonic oscillator. The constant external field E_C just shifts the equilibrium position, and the plasma frequency is just $\omega = M$. To find the first correction, we use a standard perturbative approach to a nonlinear ODE known as the Lindstedt-Poincaré method (see [39,40] for details). We expand the frequency as

$$\omega \approx M + \omega_1, \quad (79)$$

where ω_1 is order m . We introduce a new time parameter $\tau = \omega t$ and expand $\langle\phi\rangle$ as

$$\langle\phi\rangle(t) \approx \phi_0(\tau) + \phi_1(\tau), \quad (80)$$

where $\phi_1(\tau)$ is the order m term. Note that τ in Eq. (80) is not expanded in m . By changing the variable from t to τ , Eq. (78) becomes

$$\omega^2 \langle \phi \rangle'' + M^2 \langle \phi \rangle = -\frac{e^\gamma m q}{\pi} \sin\left(\sqrt{4\pi} \langle \phi \rangle\right) - \frac{q}{\sqrt{\pi}} E_C, \quad (81)$$

where $'$ denotes the τ derivative. Plugging Eqs. (79) and (80) into Eq. (81) and extracting the zeroth-order terms in m , we find

$$M^2 \phi_0'' + M^2 \phi_0 = -\frac{q E_C}{\sqrt{\pi}}. \quad (82)$$

The solution to this equation satisfying $\phi_0(0) = 0 = \partial_t \phi_0(0)$ is given by

$$\phi_0 = \frac{\sqrt{\pi} E_C}{q} (\cos \tau - 1). \quad (83)$$

The first-order equation from Eq. (81) is

$$\phi_1'' + \phi_1 = -\frac{e^\gamma m}{q} \sin\left(\sqrt{4\pi} \phi_0\right) - \frac{2\sqrt{\pi} \omega_1}{q} \phi_0''. \quad (84)$$

Using Eq. (83) and Jacobi-Anger expansion

$$\cos(z \cos \theta) = J_0(z) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(z) \cos(2k\theta), \quad (85)$$

$$\sin(z \cos \theta) = 2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(z) \cos((2k+1)\theta), \quad (86)$$

where $J_k(x)$ is the k th Bessel function of the first kind, Eq. (84) becomes

$$\begin{aligned} \phi_1'' + \phi_1 &= \left(-\frac{2e^\gamma m}{q} \cos\left(\frac{2\pi E_C}{q}\right) J_1\left(\frac{2\pi E_C}{q}\right) + \frac{2\pi \omega_1 E_C}{q^2} \right) \cos \tau \\ &\quad - \frac{e^\gamma m}{q} \sum_{k=1}^{\infty} (-1)^k J_{2k+1}\left(\frac{2\pi E_C}{q}\right) \cos[(2k+1)\tau] \\ &\quad + \frac{e^\gamma m}{q} \left(J_0\left(\frac{2\pi E_C}{q}\right) \right. \\ &\quad \left. + \sum_{k=1}^{\infty} (-1)^k J_{2k}\left(\frac{2\pi E_C}{q}\right) \cos[2k\tau] \right). \end{aligned} \quad (87)$$

For $\phi_1(\tau)$ to remain periodic instead of growing indefinitely, the first term of the right-hand side (rhs) of Eq. (87) needs to vanish. This condition leads to the determination of the correction to the frequency as

$$\omega_1 = \frac{m q e^\gamma}{\pi E_C} \cos\left(\frac{2\pi E_C}{q}\right) J_1\left(\frac{2\pi E_C}{q}\right). \quad (88)$$

Combining Eq. (88) with Eq. (79), we obtain Eq. (5).

The leading-order plasma frequency $\omega = M = q/\sqrt{\pi}$ depends only on the coupling q . The first-correction (88) displays dependence on the strength of the electric field, both through the small parameter m/E_C and via the finite ratio E_C/q . Notice that there are special values of E_C/q where the correction ω_1 vanishes entirely. It is notable that ω_1 is purely real: there is no dissipation at this order. This fact can be attributed to the conserved quantity discussed in Sec. III D and was already observed in our numerical study of capacitor breakdown in Sec. IV.

The effect of a nonzero mass m can be nicely visualized by considering a potential $V(\phi)$. First, let us write $\langle \phi \rangle \rightarrow \phi$ to simplify the notation. Then, Eq. (78) can be put in the following form:

$$\partial_t^2 \phi = -\frac{\partial V(\phi)}{\partial \phi}, \quad (89)$$

where the potential $V(\phi)$ is given by

$$V(\phi) = \frac{1}{2} M^2 \phi^2 + \frac{q}{\sqrt{\pi}} E_C \phi - \frac{e^\gamma m M}{2\pi} \cos \sqrt{4\pi} \phi. \quad (90)$$

Figure 3 shows the potential $V(\phi)$ for both massless and massive cases. The overall parabolic shape of the potential comes from the first term of Eq. (90), and the shift of the minimum of $V(\phi)$ is due to the electric field E_C . The main difference between the massless and massive is the cosine term in Eq. (90), which can be seen as adding a slight wiggle to the parabolic line in Fig. 3. The oscillating

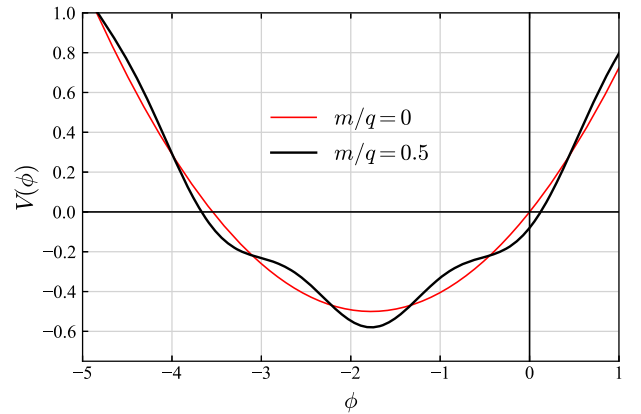


FIG. 3. The potential $V(\phi)$ given in Eq. (90) in the case of constant E_C . The massless and massive cases are shown by the red and black lines. Under this potential, ϕ oscillates back and forth around its minimum. For the massless case, the frequency is given by $\omega = M$. For the massive case, the potential is modified due to the third term in Eq. (90), and the frequency is corrected to be Eq. (5). We set $m/q = 0.5$, and $E_C/q = 1$ for this plot.

behavior of ϕ is analogous to the one-dimensional (1D) mechanics problem. From this perspective, Eq. (89) effectively describes a particle trapped in a potential (90). Since any bounded motion in 1D is periodic [41], ϕ oscillates back and forth in this potential and the dynamics of $\langle\phi\rangle$ are perfectly periodic.

VI. SEMICLASSICAL APPROXIMATION

In this section, we discuss the validity of the semiclassical approximation (e.g., [8,9,42,43]). We first derive the prediction of the semiclassical approximation for the problem under consideration and then compare to the full QED result.

In the semiclassical approximation, the electric field is no longer treated as an operator but just a classical field. This classical field creates a nonzero current $\langle J \rangle_{\text{ren}}$ through the Schwinger mechanism. If the current can be expressed as a functional of the classical field, i.e., $\langle J \rangle_{\text{ren}}[E]$, then the natural Maxwell equation is

$$-\partial_t E = \langle J \rangle_{\text{ren}}[E] + J_{\text{ext}}, \quad (91)$$

where J_{ext} is the current associated with any external electric field $-\partial_t E_{\text{ext}} = J_{\text{ext}}$.⁴ Equation (91) is the semiclassical approximation. Since the renormalized current in general depends on the whole past history of E , the semiclassical equation is an integro-differential equation (or worse).

We can obtain a definite equation in the case of spatially homogeneous electric fields, where the current expectation value may be derived from the adiabatic regularization method (e.g., [25,44–46]). Following [25], the renormalized current density is given by

$$\langle J \rangle_{\text{ren}} = -\frac{q^2}{\pi} A(t) + \frac{q}{2\pi} \int_{-\infty}^{\infty} dk \left[|h_k^I|^2 - |h_k^{II}|^2 + \frac{k}{\omega} \right], \quad (92)$$

where $h_k^I(t)$ and $h_k^{II}(t)$ are mode functions satisfying the following ODE:

$$\frac{d}{dt} \begin{pmatrix} h_k^I(t) \\ h_k^{II}(t) \end{pmatrix} = i \begin{pmatrix} k - qA(t) & -m \\ -m & -k + qA(t) \end{pmatrix} \begin{pmatrix} h_k^I(t) \\ h_k^{II}(t) \end{pmatrix}. \quad (93)$$

Here, the gauge is taken to be $A^\mu(t) = (0, A(t))$. Equation (93) ensures that the following normalization condition is constant in time:

$$|h_k^I|^2 + |h_k^{II}|^2 = 1. \quad (94)$$

To compare the semiclassical approximation with our QED result at linear order in m , we need the expression of the current Eq. (92) to this order. We consider an identical setup to our full QED calculation, where the electric field is turned on at $t = t_0$. Prior to $t = t_0$, the mode functions are those of the free theory,

$$h_k^I(t < t_0) = \sqrt{\frac{\omega_{k,m} - k}{2\omega_{k,m}}} e^{-i\omega_{k,m}(t-t_0)}, \quad (95)$$

$$h_k^{II}(t < t_0) = -\sqrt{\frac{\omega_{k,m} + k}{2\omega_{k,m}}} e^{-i\omega_{k,m}(t-t_0)}, \quad (96)$$

where $\omega_{k,m}$ is given by Eq. (26). Expanding in m gives

$$h_k^I(t < t_0) \approx \theta(-k) e^{ik(t-t_0)} + \theta(k) \frac{m}{2k} e^{-ik(t-t_0)}, \quad (97)$$

$$h_k^{II}(t < t_0) \approx \theta(k) e^{-ik(t-t_0)} + \theta(-k) \frac{m}{2k} e^{ik(t-t_0)}, \quad (98)$$

where $\theta(k)$ is a step function. We expand $h_k^I(t)$ and $h_k^{II}(t)$ in m as

$$h_k^I(t) = x^{(0)}(t) + x^{(1)}(t) + \dots, \quad (99)$$

$$h_k^{II}(t) = y^{(0)}(t) + y^{(1)}(t) + \dots, \quad (100)$$

where $x^{(n)}$ and $y^{(n)}$ are order m^n . At $t < t_0$, Eqs. (99) and (100) have to reduce to Eqs. (97) and (98); thus, we have the following conditions at $t < t_0$:

$$x^{(0)}(t) = \theta(-k) e^{ik(t-t_0)}, \quad (101)$$

$$y^{(0)}(t) = -\theta(k) e^{-ik(t-t_0)}, \quad (102)$$

$$x^{(1)}(t) = \theta(k) \frac{m}{2k} e^{-ik(t-t_0)}, \quad (103)$$

$$y^{(1)}(t) = \theta(-k) \frac{m}{2k} e^{ik(t-t_0)}. \quad (104)$$

By plugging Eqs. (99) and (100) into Eq. (93) and collecting terms order by order, we obtain the hierarchy of equations. From the zeroth order, we have

$$\dot{x}^{(0)} = i(k - qA(t))x^{(0)}, \quad (105)$$

$$\dot{y}^{(0)} = i(k - qA(t))y^{(0)}. \quad (106)$$

For the first order,

$$\dot{x}^{(1)} = i(k - qA(t))x^{(1)} + imy^{(0)}, \quad (107)$$

$$\dot{y}^{(1)} = i(k - qA(t))y^{(1)} + imx^{(0)}. \quad (108)$$

⁴Previously in the paper, the external field current was denoted E_C for “classical”; here we use the label “ext” since all electric fields are classical in the semiclassical approximation.

We can solve Eqs. (105) and (106) immediately and find

$$x^{(0)}(t) = \theta(-k)e^{i\chi(t,t_0)}, \quad (109)$$

$$y^{(0)}(t) = -\theta(k)e^{-i\chi(t,t_0)}, \quad (110)$$

where $\chi(t, t_0)$ is defined as

$$\chi(t, t_0) = \int_{t_0}^t (k - qA(t'))dt'. \quad (111)$$

It is easy to check that Eqs. (109) and (110) indeed satisfy Eqs. (101) and (102) at $t < t_0$, respectively. The calculation for the first-order term is the same, and the solutions to Eqs. (107) and (108) satisfying Eqs. (103) and (104) are given by

$$x^{(1)}(t) = -im\theta(k)e^{i\chi(t,t_0)}\left(\int_{t_0}^t e^{-2i\chi(s,t_0)}ds - \frac{1}{2ik}\right), \quad (112)$$

$$y^{(1)}(t) = im\theta(-k)e^{-i\chi(t,t_0)}\left(\int_{t_0}^t e^{2i\chi(s,t_0)}ds + \frac{1}{2ik}\right). \quad (113)$$

Expanding the integrand of (92) to order m gives

$$|h_k^I|^2 - |h_k^{II}|^2 + \frac{k}{\omega} \approx |x^{(0)}|^2 - |y^{(0)}|^2 + \frac{k}{|k|} + 2\text{Re}(x^{(0)*}x^{(1)} - y^{(0)*}y^{(1)}). \quad (114)$$

However, the rhs actually vanishes. The first line vanishes after using the explicit expressions (109) and (110), while the second line vanishes because $x^{(0)}$ (109) has support only for positive k , while $x^{(1)}$ (112) has support only for negative k (and similarly for y). Thus, accurate to $O(m)$, we find

$$\langle J \rangle_{\text{ren}} = -\frac{q^2}{\pi}A(t), \quad (115)$$

This is just the massless result, so the semiclassical approximation predicts *no correction* to order m . Taking a time derivative of (91) and using (115) as well as $E = -\partial_t A$ and $J_{\text{ext}} = -\partial_t E_{\text{ext}}$, we find

$$\left(\partial_t^2 + \frac{q^2}{\pi}\right)E = \partial_t^2 E_{\text{ext}}. \quad (116)$$

This is the semiclassical prediction, consistent to order m , in the case of spatially homogeneous electric fields. By contrast, the fully quantum result (3) and (4) in this case predicts⁵

⁵We let $E \rightarrow \hat{E}$ and $E_C \rightarrow E_{\text{ext}}$ to match the notation of this section and emphasize that the electric field is an operator in the fully quantum treatment.

$$\left(\partial_t^2 + \frac{q^2}{\pi}\right)\langle \hat{E} \rangle + \frac{e^2 m q^2}{\pi^{3/2}} \sin \frac{2\pi(\langle \hat{E} \rangle - E_{\text{ext}})}{q} = \partial_t^2 E_{\text{ext}}. \quad (117)$$

Comparing (116) and (117), we see that the semiclassical approximation agrees exactly with the fully quantum expectation value at $m = 0$, but misses the sine term at $O(m)$. In particular, it fails to capture the shift in plasma frequency discussed in Sec. V. The exactness of the semiclassical approximation for a massless fermion in a spatially homogeneous electric field was noticed previously in [43,47]. Here we demonstrate the precise manner in which it fails at the first order in m .

VII. OUTLOOK

We conclude by mentioning some open questions and interesting future directions.

Our main result is that the expectation value of the electric field obeys a partial differential equation through $O(m)$. An obvious question is whether such a simple description continues to hold at higher order—will $\langle E \rangle$ still obey a PDE, or will there be nonlocal effects? If the perturbation series is smooth in m , then one can answer this question by evaluating higher-order terms in the Dyson series (55) and (57). On the other hand, in the semiclassical approximation there are $m^2 \log m$ corrections that require an alternative approach [48], and similar difficulties may present in full QED. If a PDE still appears at the next order, it raises the intriguing possibility that $\langle E \rangle$ actually obeys local dynamics in the exact theory.

Another feature of our $O(m)$ result is the lack of dissipation, in the sense that oscillations do not damp out (they can only decay due to spreading out). Mathematically, this feature stems from the conserved energy $\mathcal{E}$ (68) associated with the PDE. Physically, we may say that energy is exchanged between the electric field and the matter, oscillating back and forth indefinitely. However, the semiclassical approximation for finite fermion mass does show a secular decrease in the strength of the electric field [7–11,42], which can be regarded as a permanent transfer of energy from the field to the matter. It is not clear at present whether this electric field dissipation is an artifact of the semiclassical approach, or whether the lack of dissipation in our result is an artifact of the first-order expansion in m .

It is difficult to predict the answer in advance, since we lack a clear physical picture of the origin of the oscillations. In particular, it would be desirable to have more clarity about the extent to which “collective particle behavior” is a useful physical picture. Can the dynamics of 1 + 1D strong-field QED be profitably understood in terms of a large number of point particles interacting with a field that can create and move them, or do these results demand abandonment of this cherished idea? Further research is required to clarify the issue.

Another interesting future direction is comparison with lattice simulations of the theory. In particular, Ref. [49] observed field oscillations attributed to pair production. A larger parameter study with longer-time evolution would allow a direct comparison with the perturbative results we report.

It is also interesting to ask about the evolution of the field when the initial state is not the vacuum. In the massless case studied in [12], the field equation is linear, and the expectation value of the field in any state always satisfies the classical Klein-Gordon equation. By contrast, our perturbative result for the massive case applies only to evolution from the vacuum state. A particularly interesting case is when the initial state is mixed. By choosing a thermal state as the initial condition, one may be able to capture familiar plasma effects, such as Landau damping, in the setting of 1 + 1D massive QED.

Finally, 1 + 1D massive QED might give some insight into astrophysical phenomena around pulsars and black holes surrounded by a strong magnetic field. Since the ultrastrong magnetic field forces charged particles to move along the direction of the magnetic field, treating the matter field as 1 + 1D becomes justifiable [24,50–52]. Moreover, the modification of the Coulomb law in the presence of a strong magnetic field [53] suggests that some kind of dimensional reduction may also occur in the electromagnetic sector. If an effective 1 + 1D theory indeed arises in strong magnetic fields, then the bosonization approach offers the attractive possibility of capturing complicated plasma dynamics with a simple PDE. Indeed, the polar cap discharge for pulsars [54–57] and analogous gap formation and screening in black hole magnetospheres [58–60] are astrophysical examples of electric field breakdown due to pair creation. It would be fascinating to study these important astrophysical processes as 1 + 1D QED phenomena.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: INTERACTION PICTURE FOR A SYSTEM WITH A TIME-VARYING HAMILTONIAN

In this appendix, we briefly review time-dependent perturbation theory using the interaction picture.

Although this perturbation method is almost identical to the standard one [36,61,62], the explicit time dependence of the Hamiltonian slightly modifies the definitions of some operators.

We consider a Schrödinger-picture Hamiltonian $\hat{H}_T(t)$ that may have explicit time dependence, as in Eq. (32). The physical state $|\psi(t)\rangle$ in the Schrödinger picture evolves with time, and its time evolution can be written as a linear transformation of $|\psi(t)\rangle$ by the unitary time evolution operator $\hat{U}_T(t, t')$ through $|\psi(t)\rangle = \hat{U}_T(t, t')|\psi(t')\rangle$. $\hat{U}_T(t, t')$ satisfies the Schrödinger equation

$$i \frac{\partial}{\partial t} \hat{U}_T(t, t') = \hat{H}_T(t) \hat{U}_T(t, t') \quad (\text{A1})$$

with the condition $\hat{U}_T(t, t) = \hat{1}$. $\hat{H}_T(t)$ can be split into the “free” part and the interaction part:

$$\hat{H}_T(t) = \hat{H}_0(t) + \hat{H}_{\text{int}}(t). \quad (\text{A2})$$

Using $\hat{H}_0(t)$, the “free” evolution operator $\hat{U}_0(t, t')$ is defined as the solution to

$$i \frac{\partial}{\partial t} \hat{U}_0(t, t') = \hat{H}_0(t) \hat{U}_0(t, t'), \quad (\text{A3})$$

with $\hat{U}_0(t, t) = \hat{1}$. Given a Schrödinger-picture operator $\hat{O}^S(t)$ (possibly with explicit time dependence), the interaction-picture version $\hat{O}^I(t)$ is defined as

$$\hat{O}^I(t) = \hat{U}_0^\dagger(t, t_0) \hat{O}^S(t) \hat{U}_0(t, t_0), \quad (\text{A4})$$

where t_0 is an arbitrary reference time. The time evolution of $\hat{O}^I(t)$ is then given by the Heisenberg equation that only involves the “free” part of the theory,

$$\frac{\partial}{\partial t} \hat{O}^I(t) = \frac{1}{i} [\hat{O}^I(t), \hat{H}_0^I(t)] + \left(\frac{\partial \hat{O}^S(t)}{\partial t} \right)^I, \quad (\text{A5})$$

where

$$\left(\frac{\partial \hat{O}^S(t)}{\partial t} \right)^I \equiv \hat{U}_0^\dagger(t, t_0) \frac{\partial \hat{O}^S(t)}{\partial t} \hat{U}_0(t, t_0). \quad (\text{A6})$$

The “interaction” time evolution operator $\hat{U}_{\text{int}}(t, t')$ is defined by

$$\hat{U}_{\text{int}}(t, t') = \hat{U}_0^\dagger(t, t_0) \hat{U}_T(t, t') \hat{U}_0(t', t_0) \quad (\text{A7})$$

and satisfies $\hat{U}_{\text{int}}(t, t) = \hat{1}$. By taking the time derivative of Eq. (A7) and using Eqs. (A3) and (A5), the time evolution of $\hat{U}_{\text{int}}(t, t')$ is obtained as

$$i \frac{\partial}{\partial t} \hat{U}_{\text{int}}(t, t') = \hat{H}_{\text{int}}^I(t) \hat{U}_{\text{int}}(t, t'). \quad (\text{A8})$$

The formal solution to Eq. (A8) can be expressed as the Dyson series,

$$\hat{U}_{\text{int}}(t, t') = T \exp \left(-i \int_{t'}^t \hat{H}_{\text{int}}^I(s) ds \right), \quad (\text{A9})$$

where T is the time-ordering operator.

APPENDIX B: ENERGY EXPECTATION VALUE

In this appendix, we show that the conserved quantity (68) is equal to the expectation value of the Hamiltonian operator (28), after the nondynamical part $\frac{1}{2}E_C^2$ is added back in. Using Eq. (56) to commute $\hat{U}_0(t, t')$ and the normal ordering $N_M\{\cdots\}$, the Hamiltonian operator in the interaction picture becomes

$$\begin{aligned} \hat{H}^I(t) = \int dx N_M \left\{ \frac{1}{2} (\hat{\Pi}^I)^2 + \frac{1}{2} (\partial_x \hat{\phi}^I)^2 + \frac{1}{2} M^2 (\hat{\phi}^I)^2 \right. \\ \left. + \frac{q}{\sqrt{\pi}} E_C \hat{\phi}^I + \frac{1}{2} E_C^2 - \frac{e^\gamma m M}{2\pi} \cos \sqrt{4\pi} \hat{\phi}^I \right\}, \quad (\text{B1}) \end{aligned}$$

with $\frac{1}{2}E_C^2$ added back in. After using Eqs. (49) and (50) for the first three terms of Eq. (B1), one may split Eq. (B1) into the following:

$$\hat{H}^I(t) = \hat{H}_1^I + \hat{H}_2^I + \hat{H}_3^I + \hat{H}_4^I, \quad (\text{B2})$$

where each piece is defined by

$$\hat{H}_1^I = \int \frac{dk}{2\pi} \omega_{k,M} \hat{a}_{k,M}^\dagger \hat{a}_{k,M}, \quad (\text{B3})$$

$$\begin{aligned} \hat{H}_2^I = \int dx [\partial_t \hat{\phi}^I \partial_t \phi_{\text{cl}} + \partial_x \hat{\phi}^I \partial_x \phi_{\text{cl}} + M^2 \hat{\phi}^I \phi_{\text{cl}}] \\ - \int dx \left[\frac{1}{2} (\partial_t \phi_{\text{cl}})^2 + \frac{1}{2} (\partial_x \phi_{\text{cl}})^2 + \frac{1}{2} M^2 \phi_{\text{cl}}^2 \right], \quad (\text{B4}) \end{aligned}$$

$$\hat{H}_3^I = \int dx \left(\frac{q}{\sqrt{\pi}} \hat{\phi}^I E_C + \frac{1}{2} E_C^2 \right), \quad (\text{B5})$$

$$\hat{H}_4^I = \int N_M \left\{ -\frac{e^\gamma m M}{2\pi} \cos \sqrt{4\pi} \hat{\phi}^I \right\} dx. \quad (\text{B6})$$

Now, we evaluate the expectation value of Eqs. (B3)–(B6) piece by piece. To see the expectation value of Eq. (B3), consider the expectation value of $\hat{a}_{k,M}^\dagger \hat{a}_{k,M}$. Using Eqs. (54), (56), (59), and (60) and keeping to first order, we find

$$\begin{aligned} \langle \hat{a}_{k,M}^\dagger \hat{a}_{k,M} \rangle &= \langle 0, M | \hat{U}_{\text{int}}^\dagger(t, -\infty) \hat{a}_{k,M}^\dagger \hat{a}_{k,M} \hat{U}_{\text{int}}(t, -\infty) | 0, M \rangle \\ &\approx \langle 0, M | \hat{a}_{k,M}^\dagger \hat{a}_{k,M} | 0, M \rangle \\ &\quad + \int_{-\infty}^t dt' \int dy \left(-i \frac{e^\gamma m M f(t')}{2\pi \sqrt{\pi}} \right) \\ &\quad \times \langle 0, M | [N_M \{ \cos \sqrt{4\pi} \hat{\phi}^I(t', y) \}, \hat{a}_{k,M}^\dagger \hat{a}_{k,M}] | 0, M \rangle = 0, \quad (\text{B7}) \end{aligned}$$

since both zeroth and first-order terms vanish due to $\hat{a}_{k,M} | 0, M \rangle = 0$ (equivalently $\langle 0, M | \hat{a}_{k,M}^\dagger = 0$). This leads to

$$\langle H_1 \rangle = 0. \quad (\text{B8})$$

For the computation of $\langle H_2 \rangle$, notice that operator $\hat{U}_{\text{int}}(t, -\infty)$ does not depend on the spatial coordinate. Thus, using Eq. (55), we find

$$\partial_x \langle \phi \rangle = \langle \partial_x \hat{\phi}^I \rangle. \quad (\text{B9})$$

For the time derivative of $\langle \phi \rangle$, one may use Eqs. (55) and (A8), and then

$$\begin{aligned} \partial_t \langle \phi \rangle &= \langle \partial_t \hat{\phi}^I \rangle + i \langle [\hat{H}_{\text{int}}^I(t), \hat{\phi}^I(t, x)] \rangle \\ &= \langle \partial_t \hat{\phi}^I \rangle, \quad (\text{B10}) \end{aligned}$$

where the commutator of $\hat{H}_{\text{int}}^I$ and $\hat{\phi}^I(t, x)$ vanishes due to the equal time commutation relation (45). Using these results and the fact that $\langle \phi \rangle = \phi_{\text{cl}} + O(m)$, the expectation value of Eq. (B3) to first order in m becomes

$$\langle \hat{H}_2 \rangle = \int dx \left(\frac{1}{2} (\partial_t \langle \phi \rangle)^2 + \frac{1}{2} (\partial_x \langle \phi \rangle)^2 + \frac{1}{2} M^2 \langle \phi \rangle^2 \right). \quad (\text{B11})$$

Furthermore, the expectation value of Eq. (B5) is simply

$$\langle \hat{H}_3 \rangle = \int dx \left(\frac{q}{\sqrt{\pi}} \langle \phi \rangle E_C + \frac{1}{2} E_C^2 \right), \quad (\text{B12})$$

since $\hat{H}_3^I$ is only linear in $\hat{\phi}^I$. For Eq. (B6), notice that this term is already at order m ; thus, we obtain

$$\langle \hat{H}_4 \rangle = \int \left(-\frac{e^\gamma m M}{2\pi} \cos \sqrt{4\pi} \langle \phi \rangle \right) dx. \quad (\text{B13})$$

Combining Eqs. (B8) and (B11)–(B13) and comparing the result to the energy $\mathcal{E}$ defined in Eq. (68), one may conclude $\langle \hat{H} \rangle = \mathcal{E}$ to first order in m , as presented in Eq. (71) in the main text. In other words, the conserved quantity derived in Sec. III D is genuinely the energy expectation value of the system.

Even though the total energy $\mathcal{E}$ can be identified as the total energy expectation, the particle part (69) and field part (70) are not the expectation value of their quantum counterparts. Similar to Eq. (68), which was split into Eqs. (69) and (70), one may split

$$\hat{H} = \hat{H}_{\text{particle}} + \hat{H}_{\text{field}}, \quad (\text{B14})$$

where

$$\hat{H}_{\text{particle}} = \int N_M \left\{ \frac{1}{2} \hat{\Pi}^2 + \frac{1}{2} (\partial_x \hat{\phi})^2 - \frac{e' m M}{2\pi} \cos \sqrt{4\pi} \hat{\phi} \right\} dx \quad (\text{B15})$$

and

$$\hat{H}_{\text{field}} = \int \frac{1}{2} \hat{E}^2 dx = \int \frac{1}{2} N_M \left\{ \left(E_C + \frac{q}{\sqrt{\pi}} \hat{\phi} \right)^2 \right\} dx. \quad (\text{B16})$$

However, the direct calculation to order m using Eqs. (54), (56), (59), (60), (64) shows

$$\begin{aligned} \langle N_M \{ \hat{\phi}^2 \} \rangle &= \langle 0, M | \hat{U}_{\text{int}}^\dagger(t, -\infty) N_M \{ (\hat{\phi}^I)^2 \} \hat{U}_{\text{int}}(t, -\infty) | 0, M \rangle \\ &= \phi_{\text{cl}}(t, x)^2 + \int_{-\infty}^t dt' \int dy \left(-i \frac{e' m M f(t')}{2\pi\sqrt{\pi}} \right) 2\phi_{\text{cl}}(t, x) \langle 0, M | \left[N_M \left\{ \cos \sqrt{4\pi} \hat{\phi}^I(t', y) \right\}, \hat{\phi}_{\text{ff}}^I(t, x) \right] | 0, M \rangle + \Delta \\ &= \langle \phi \rangle^2 + \Delta, \end{aligned} \quad (\text{B17})$$

where $\phi_{\text{ff}}^I(t, x)$ is defined by

$$\phi_{\text{ff}}^I(t, x) = \phi^I(t, x) - \phi_{\text{cl}} \quad (\text{B18})$$

and Δ is

$$\Delta = \int_{-\infty}^t dt' \int dy \left(-i \frac{e' m M f(t')}{2\pi\sqrt{\pi}} \right) \langle 0, M | [N_M \{ \cos \sqrt{4\pi} \hat{\phi}^I(t', y) \}, N_M \{ \hat{\phi}_{\text{ff}}^I(t, x)^2 \}] | 0, M \rangle. \quad (\text{B19})$$

However, Δ is in general nonzero, since there is no obvious annihilation of $\langle 0, M |$ or $| 0, M \rangle$ by $\hat{a}_{k,M}^\dagger$ and $\hat{a}_{k,M}$ in Eq. (B19). In other words, Eq. (B17) leads to

$$\langle N_M \{ \hat{\phi}^2 \} \rangle \neq \langle \phi \rangle^2. \quad (\text{B20})$$

An immediate consequence of this result is

$$\langle \hat{H}_{\text{field}} \rangle \neq \mathcal{E}_{\text{field}}, \quad (\text{B21})$$

which can be quickly verified by taking the expectation value of Eq. (B16) and comparing it to Eq. (70) using Eq. (B20). Since $\langle \hat{H} \rangle = \mathcal{E}$, Eq. (B21) also implies

$$\langle \hat{H}_{\text{particle}} \rangle \neq \mathcal{E}_{\text{particle}}. \quad (\text{B22})$$

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