

Nonrelativistic meson masses from the Curci-Ferrari model

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We study the mass spectrum of nonrelativistic mesons composed of charm and bottom quarks within the framework of the Curci-Ferrari model in the Landau gauge, deriving the Hamiltonian from the scattering amplitude of a single massive gluon exchange. We solve the Schrödinger equation for the dominant terms of the Hamiltonian, which include the kinetic energy and a Yukawa-type potential, together with a confining Cornell potential. Other corrections to the energy, such as the spin-dependent terms, are introduced perturbatively. We perform two types of fits to the experimental/lattice data. In the first, we fit the charmonium and bottomonium spectra independently. In the second, we fit all three spectra simultaneously, making use of the one-loop running of the gauge coupling and the gluon mass. Both fits provide a reasonable description of the available data within the present model uncertainties and the latter, being more constrained, favors a nonvanishing gluon mass.

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I. INTRODUCTION

Unraveling the IR regime of QCD is of paramount importance to grasp the mechanisms behind fundamental phenomena such as the spontaneous breaking of chiral symmetry and confinement. In contrast to the UV regime, where QCD displays asymptotic freedom, the standard Faddeev-Popov (FP) perturbative approach is ill-suited for accessing the IR of the theory, due to the presence of a Landau pole. As a consequence, various nonperturbative approaches have been proposed to investigate QCD at low energies.

Among those approaches, lattice simulations have played a pivotal role in elucidating the intricacies of QCD in the IR regime, revealing some unexpected properties. For example, nowadays there is a consensus within the lattice community concerning the IR behavior of the gluon propagator in the Landau gauge; it saturates to a finite value at vanishing momentum, similarly to a massive propagator [1–6]. This particular behavior aligns with findings from functional methods, such as approaches rooted in Schwinger-Dyson equations (SDE) [7–10] and functional renormalization group [11–14] which yield the so called

decoupling solutions. Notably, preceding the confirmation through numerical simulations, such solutions had already been observed in SDE studies [15].

Numerical simulations have also revealed another significant property of low-energy QCD; the coupling between gluons and ghosts features no singular behavior in the IR [3,16,17]. Instead, lattice data suggest that even in the domain of very small energy, the relative order of magnitude of successive terms in perturbation theory should be of the order of 0.2 or 0.3. This finding is remarkable, implying that some sort of perturbation theory might, to some extent, capture the behavior of pure Yang-Mills theory in the IR, contradicting the results from the FP approach, that finds a Landau pole.

This discrepancy can be attributed to the incomplete justification of the FP procedure in the IR, stemming from the presence of Gribov copies [18]. Some approaches, such as the Gribov-Zwanziger (GZ) [19] approach and its refined version [20,21], have attempted to take into account these effects by modifying the FP action and introducing various auxiliary fields.

Another possible route, instead of explaining the emergence of the gluon mass, intends to explore its consequences. One of such approaches is a particular case of a more general kind of theory put forth by G. Curci and R. Ferrari in the 1970s [22,23] that, in Landau gauge, reduces to a gluon mass extension of the standard FP action [24,25]. We will refer to this framework as the Curci-Ferrari (CF) model in Landau gauge.

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The CF action is renormalizable [22]. In addition, as shown in [26], the CF model features certain trajectories of the renormalization flow that are Landau pole free, the so called infrared safe trajectories. Furthermore, in the quenched approximation, the gauge coupling is compatible with a standard perturbative analysis for all momentum scales. By exploiting this property, several correlation functions have been evaluated at one-loop [24,25,27–31], displaying, in general terms, very good agreement with lattice simulations. Moreover, this agreement has improved in cases where two-loop corrections have been included [31–35], which is consistent with the perturbative analysis.

Results of the two-point functions in the presence of quarks support the validity of the perturbative approach within the CF model for the gluon, ghost and quark dressing functions [28,34]. As for the quark mass function, it can be reproduced with a two-loop calculation in the case of heavy quarks, whereas in the case of light quarks the model is able to reproduce lattice data by employing a double expansion, in the pure gauge coupling and in the inverse of the number of colors [34,36,37].

The CF model has also been employed to address phenomena of Yang-Mills/QCD at finite temperature and density [38–45]. In particular, it has been successfully used to investigate the QCD phase diagram [39,40,44,45] and neutron stars phenomenology [43]. More recently, an infrared-safe renormalization scheme has been extended to Minkowskian space-time [46], the model has successfully been used to study confinement [47] as well as parton distribution and fragmentation functions [48]. For a review on the Curci-Ferrari model and its applications, see [49].

Our goal in this work is to begin with the exploration of the mass spectrum of mesons by employing the CF model. To this end, heavy mesons are an excellent starting point, as they feature a rich spectroscopy, see, e.g., [50], and can be described by nonrelativistic approaches. These include nonrelativistic QCD (NRQCD) [51,52], which employs an expansion in the inverse of the quark mass, and potential nonrelativistic QCD (pNRQCD), where degrees of freedom of NRQCD that do not lead to physical states are integrated out [53–55]. Heavy mesons spectra have also been obtained by means of lattice simulations, see, e.g., [56–58] and in the context of Dyson-Schwinger and Bethe-Salpeter equations (BSE), see for instance [59–61].

Another type of approaches, that emerged with the very discovery of charmonium, are the potential models [62–72]. Typically, because of the large masses of the heavy quarks, these models describe the quarkonia as a two-body system governed by the Schrödinger equation, where the quark-antiquark potential can be modeled based on phenomenological or theoretical grounds. One of the simplest models, the Coulomb plus linear potential, is simply based on the asymptotic limits of the interquark potential of QCD. More specifically, it is modeled according to a one-gluon exchange interaction at short distances

and by a linear potential, responsible for confinement, at large distances [63,73,74]. Several variations of this potential are available in the literature, all of them providing a good description of experimental data [75–77]. Reviews on the various theoretical and experimental approaches to heavy mesons can be found in Refs. [77,78].

In this work we aim to make a first exploration on the description of the mass spectrum of heavy mesons by the CF model. In particular, our goal is to determine whether the experimental/lattice data available for the spectra favors a nonzero gluon mass. To this end we will use a nonrelativistic potential model based on a one-massive-gluon exchange potential, which describes the short-distance regime plus a linear potential, consistent with confinement at large distances. Then, by solving the Schrödinger equation and introducing perturbative corrections, we fit the experimental/lattice masses of bound states of charm and bottom quarks. This work complements other approaches, that also have evaluated the impact of some sort of gluon mass on heavy mesons physics, see, e.g., [79–83].

It is clear that the gluon mass would exert a more substantial impact on the spectrum of relativistic mesons. However, in such cases, the spectrum must be derived by employing the BSE [84], rendering the analysis considerably more involved. Additionally, the approximations to be used in the BSE are harder to justify. As a result, in this first exploratory paper we decided to take a simpler approach, leaving the analysis of relativistic mesons for a future work.

The article is organized as follows. In Sec. II we introduce the Curci-Ferrari model in Landau gauge in Minkowski space-time. In Sec. III we present the calculation of the one-massive-gluon exchange potential, which we will incorporate into the Hamiltonian before plugging it into the Schrödinger equation. In Sec. IV we discuss the solutions to the Schrödinger equation as well as the introduction of perturbative corrections. In Sec. V we present the infrared-safe renormalization scheme of the CF model, from which we obtain the running of the gauge coupling and the gluon mass. We present our results in Sec. VI, discussing its relation with the gluon mass as well as the consistency with the nonrelativistic framework. Finally, in Sec. VII we provide the conclusions of this work.

II. THE CURCI-FERRARI MODEL IN LANDAU GAUGE

The CF model [22] in Landau gauge in Minkowski space-time is defined by the following Lagrangian density,

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a} - h^a \partial^\mu A_\mu^a + \frac{m^2}{2} A_\mu^a A^{\mu,a} \\ & - \partial^\mu \bar{c}^a (D_\mu c)^a + \sum_{j=1}^{N_f} \bar{\psi}_j (i\bar{D} - M_j) \psi_j, \end{aligned} \quad (1)$$

where, as usual, we introduced $(D_\mu c)^a = \partial_\mu c^a + g f^{abc} A_\mu^b c^c$ and the field strength tensor $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$. The parameter g refers to the bare gauge coupling and Latin indices are associated with color indices of the SU(3) gauge group. The mass parameters m and M_j are the bare gluon and quark mass for each flavor j , respectively.

In the context of the investigation of the IR regime of QCD, the CF model is strongly motivated by the decoupling behavior of the gluon propagator in lattice simulations [3,5,6,85,86]. In this respect, the CF model is the simplest renormalizable deformation of the standard FP action which, by explicitly adding a gluon mass term, is able to account for these results [24,25]. Since the gluon field is massive, the tree level gluon propagator in momentum space reads

$$G_{\mu\nu}^{ab}(q) = \frac{-i\delta^{ab}}{q^2 - m^2 + i\epsilon} \left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right). \quad (2)$$

Before ending this section it is important to stress that the gluon mass term is introduced at the level of the gauge-fixed action and, up to date, has not been derived from first principles. We believe, however, that this term should capture, at least partially, the effect of the Gribov copies in the IR.¹ Whatever the gluon mass origin may be, we will consider the CF model as a phenomenological tool, which has shown to provide very good results while introducing a minimal amount of modeling.

III. ONE-MASSIVE GLUON EXCHANGE POTENTIAL

The BSE [84] provides a covariant relativistic formalism to describe two-particle bound states by including a kernel that encompasses all two-particle-irreducible interactions between the constituents. A rigorous (semi)analytic description of the quarkonium spectrum typically requires solving the BSE. However, in the nonrelativistic limit, as the works in the context of potential models have demonstrated [62–68,70–72], heavy quarkonium is well-described by the simpler Schrödinger equation.

To evaluate the potential, given the exploratory nature of this work, we will restrict ourselves to next-to-leading order corrections within the one-gluon exchange approximation. A rigorous analysis regarding all next-to-leading order diagrams in the CF framework goes beyond the scope of the present work.²

¹See, for instance, Ref. [87] where the gauge fixing procedure introduced by the authors, which averages over the Gribov copies, lifts the Gribov degeneracy and leads to a massive extension of the FP action. However, the relation between this term and the CF gluon mass is not clear.

²Such analysis should include a detailed study of the order of each diagram contributing to the meson masses, putting emphasis on the enhancement of the diagrams due to the presence of complex poles.

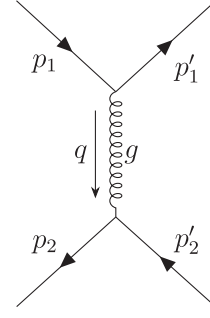


FIG. 1. Tree-level diagram in QCD for quark-antiquark state transition.

Thus, we derive the quark-antiquark interaction potential from the scattering amplitude of a one-gluon exchange diagram from the CF model. Following the Born approximation [88] this potential is, up to a prefactor, given by the Fourier transform of the scattering amplitude depicted in Fig. 1. This diagram represents a tree-level process in which a quark with initial momentum p_1 interacts with an antiquark of momentum p_2 through the exchange of a gluon carrying momentum q . Primed quantities indicate the final momentum of the particles involved. Such a potential will be composed of a leading Yukawa term and relativistic next-to-leading order corrections, including spin-dependent interactions, as we will see.

Applying the Feynman rules to the tree-level diagram shown in Fig. 1, we obtain the scattering matrix (S-matrix) of the process:

$$i\mathcal{M} = [\bar{u}(p_1') c_1'^{\dagger} (ig\gamma^\mu t^a) c_1 u(p_1)] G_{\mu\nu}^{ab}(q) [\bar{v}(p_2) c_2^{\dagger} (ig\gamma^\nu t^b) c_2' v(p_2')]. \quad (3)$$

Here,

$$u(p_1) = \sqrt{2m} \begin{pmatrix} \left(1 - \frac{\mathbf{p}_1^2}{8M_1^2}\right) w_1 \\ \frac{\boldsymbol{\sigma}_1 \cdot \mathbf{p}_1}{2M_1} w_1 \end{pmatrix}, \quad \bar{u}(p_1') = \sqrt{2M_1} \left(w_1'^{\dagger} \left(1 - \frac{\mathbf{p}_1'^2}{8M_1^2}\right) - w_1'^{\dagger} \frac{\boldsymbol{\sigma}_1 \cdot \mathbf{p}_1'}{2M_1} \right), \quad (4)$$

are the Dirac spinors associated with quarks, and

$$v(p_2') = \sqrt{2M_2} \begin{pmatrix} \frac{\boldsymbol{\sigma}_2 \cdot \mathbf{p}_2'}{2M_2} w_2' \\ \left(1 - \frac{\mathbf{p}_2'^2}{8M_2^2}\right) w_2' \end{pmatrix}, \quad \bar{v}(p_2) = \sqrt{2M_2} \left(w_2^{\dagger} \frac{\boldsymbol{\sigma}_2 \cdot \mathbf{p}_2}{2M_2} - w_2^{\dagger} \left(1 - \frac{\mathbf{p}_2^2}{8M_2^2}\right) \right), \quad (5)$$

are those associated with antiquarks. Due to momentum conservation the initial and final quantities are related by $\mathbf{p}_1' = \mathbf{p}_1 - \mathbf{q}$ and $\mathbf{p}_2' = \mathbf{p}_2 + \mathbf{q}$.

There are several ways to determine the component q_0 of the gluon momentum. One of them is to consider that the external particles are on shell. This choice is specially useful when dealing with scattering amplitudes. In contrast, as for bound states a more natural choice is $q_0 = 0$. This is because the location of bound states poles in the truncated two-particle Green's function is independent of the relative energies of the constituents [89].³

Finally, the amplitude (3) takes the form

$$i\mathcal{M} = g^2 \frac{4}{3} \frac{i4M_1M_2}{(\mathbf{q}^2 + m^2)} \left\{ -1 + \left(\frac{1}{M_1^2} + \frac{1}{M_2^2} \right) \frac{\mathbf{q}^2}{8} \right. \\ + \frac{i\boldsymbol{\sigma}_1 \cdot (\mathbf{q} \times \mathbf{p}_1)}{4M_1^2} + \frac{i\boldsymbol{\sigma}_2 \cdot (\mathbf{q} \times \mathbf{p}_2)}{4M_2^2} \\ + \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{M_1M_2} - \frac{(\mathbf{p}_1 \cdot \mathbf{q})(\mathbf{p}_2 \cdot \mathbf{q})}{M_1M_2\mathbf{q}^2} \\ - \frac{i\boldsymbol{\sigma}_1 \cdot (\mathbf{q} \times \mathbf{p}_2)}{2M_1M_2} - \frac{i\boldsymbol{\sigma}_2 \cdot (\mathbf{q} \times \mathbf{p}_1)}{2M_1M_2} \\ \left. - \frac{(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\mathbf{q}^2}{4M_1M_2} + \frac{(\boldsymbol{\sigma}_1 \cdot \mathbf{q})(\boldsymbol{\sigma}_2 \cdot \mathbf{q})}{4M_1M_2} \right\}. \quad (6)$$

As already mentioned, the interaction potential is found by taking the Fourier transform of the amplitude (6) (without the factor $i4M_1M_2$). The details of this calculation are shown in Appendix A. As a result, we obtain

$$H = \sum_{i=0}^9 H_i, \quad (7)$$

where

$$H_0 = \frac{1}{2\mu} p^2 - g^2 \frac{e^{-mr}}{3\pi r} \quad (8)$$

and $\mu \equiv \frac{M_1M_2}{M_1+M_2}$ is the reduced mass of the meson. Besides,

$$H_1 = -\left(\frac{1}{M_1^3} + \frac{1}{M_2^3} \right) \frac{\mathbf{p}^4}{8}, \quad (9)$$

$$H_2 = -\left(\frac{1}{M_1^2} + \frac{1}{M_2^2} \right) \frac{g^2 m^2 e^{-mr}}{24\pi r}, \quad (10)$$

$$H_3 = \frac{g^2}{6} \left(\frac{1}{M_1^2} + \frac{1}{M_2^2} \right) \delta^3(r), \quad (11)$$

$$H_4 = g^2(mr + 1) \frac{e^{-mr}}{3\pi r^3} \left(\frac{1}{2M_1^2} + \frac{1}{M_2M_1} \right) \mathbf{s}_1 \cdot \mathbf{L} \\ + g^2(mr + 1) \frac{e^{-mr}}{3\pi r^3} \left(\frac{1}{2M_2^2} + \frac{1}{M_2M_1} \right) \mathbf{s}_2 \cdot \mathbf{L}, \quad (12)$$

$$H_5 = \frac{g^2}{3\pi M_1M_2m^2} \left[\frac{\mathbf{p}^2}{r^3} - \frac{3(\mathbf{p} \cdot \mathbf{r})^2}{r^5} \right], \quad (13)$$

$$H_6 = \frac{g^2 e^{-mr}}{3\pi M_1M_2m^2} \left\{ -\frac{\mathbf{p}^2}{r^3} (mr + 1) \right. \\ \left. + \frac{3(\mathbf{p} \cdot \mathbf{r})^2}{r^5} \left(\frac{m^2 r^2}{3} + mr + 1 \right) \right\}, \quad (14)$$

$$H_7 = \frac{8}{9} \frac{g^2}{M_1M_2} (\mathbf{s}_1 \cdot \mathbf{s}_2) \delta^3(r), \quad (15)$$

$$H_8 = -\frac{g^2 e^{-mr}}{3\pi M_1M_2} \frac{\mathbf{s}_1 \cdot \mathbf{s}_2}{r^3} (m^2 r^2 + mr + 1), \quad (16)$$

$$H_9 = \frac{g^2 e^{-mr}}{3\pi M_1M_2} \frac{(\mathbf{s}_1 \cdot \mathbf{r})(\mathbf{s}_2 \cdot \mathbf{r})}{r^5} \left(\frac{m^2 r^2}{3} + mr + 1 \right). \quad (17)$$

As we will work in the center-of-mass frame of the system, the momenta of the particles (quark and antiquark) satisfy $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$. The orbital angular momentum is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the spin operators of the quark and antiquark (represented by $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively) are related to the Pauli matrices as follows:

$$\mathbf{s}_1 = \frac{\boldsymbol{\sigma}_1}{2}, \quad \mathbf{s}_2 = -\frac{\boldsymbol{\sigma}_2}{2}. \quad (18)$$

The Hamiltonian (7), associated with the two-body problem, agrees with that obtained by Gromes in [90] for a two-fermion system of equal masses. We note that the spin terms are of Breit-type, as can be seen in Refs. [91–93] as well as in an example of Breit interaction, provided in Ref. [94].

IV. SOLUTIONS TO THE SCHRÖDINGER EQUATION AND PERTURBATIVE CORRECTIONS

The Hamiltonian (7) is nonconfining. To take into account confinement, we can add a linear potential [73,95] of the type

$$V_b = br, \quad (19)$$

which is referred to as the Cornell potential. The parameter b is known as the string tension. Thus, to determine the mass spectrum of quarkonium, first we solve the Schrödinger equation numerically, with a Hamiltonian given by

³Furthermore, in [89] it is shown that, provided that the kernel of the BSE is static, the truncated two-particle Green's function is actually independent of q_0 .

$$H_S = \frac{1}{2\mu} \mathbf{p}^2 - g^2 \frac{e^{-mr}}{3\pi r} + br, \quad (20)$$

where the first two terms correspond to the leading contributions to the Hamiltonian (7), while the third term is the nonperturbative Cornell potential. As we are working with a central potential, the states in the position representation read

$$\psi_{n\ell m}(r) = R_{n\ell}(r) Y_{\ell}^m(\theta, \phi), \quad (21)$$

where $Y_{\ell}^m(\theta, \phi)$ are spherical harmonics with eigenvalues $\ell(\ell+1)$ of $\mathbf{L}^2$ (where $\ell = 0, 1, 2, \dots$) and eigenvalues m of L_z . Now, we can write the radial part of the Schrödinger equation (in analogy to the case of the hydrogen atom) as

$$\frac{-1}{2\mu r^2} \frac{d}{dr} \left(r^2 \frac{dR_{n\ell}}{dr} \right) + \left[\frac{\ell(\ell+1)}{2\mu r^2} + V(r) \right] R_{n\ell} = E_{n\ell} R_{n\ell}, \quad (22)$$

where ℓ is the angular momentum quantum number and n is the index associated with the energy level.

In order to express Eq. (22) in terms of dimensionless quantities for distance and energy, we redefine the radial variable as $\rho = \frac{r}{a_0}$, with $a_0 = \frac{3\pi}{g^2\mu}$, and the energy eigenvalues as $\lambda_{n\ell} = \sqrt{-\frac{E_{n\ell}}{E_I}}$, with $E_I = \frac{g^4\mu}{2(3\pi)^2}$. As a result, Eq. (22) transforms into

$$\left[\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} + \frac{2e^{-\tilde{m}\rho}}{\rho} + \tilde{b}\rho - \lambda_{n\ell}^2 \right] u_{n\ell}(\rho) = 0, \quad (23)$$

with $\tilde{b} = \frac{ba_0}{E_I}$ and $\tilde{m} = ma_0$. The function $u_{n\ell}$ is defined in terms of $R_{n\ell}(r)$ as $u_{n\ell}(r) = rR_{n\ell}(r)$, and the condition at the origin is $u_{n\ell}(0) = 0$. Eq. (23) can be solved numerically to find the eigenstates and eigenenergies of H_S . The numerical solutions of the equation will give us eigenstates and eigenenergies of H_S . The corrections to the energy can be calculated as

$$E_1 = \langle n_0 | H_P | n_0 \rangle, \quad (24)$$

where $|n_0\rangle$ are the eigenstates of H_S and $H_P = \sum_{i=1}^9 H_i$, where the addends H_i were defined in Eqs. (9) to (17). We will work with states with $n \in \{1, 2\}$ and $\ell \in \{0, 1\}$, where n denotes the radial quantum number in the spectroscopic convention, $n = n_r + 1$, with n_r the number of nodes of the radial wave function. The pairs $(n, \ell) = (1, 0), (2, 0), (1, 1), (2, 1)$ thus correspond to the $1S, 2S, 1P$, and $2P$ states, respectively. In order to compute the correction E_1 , the Hamiltonian H_P must be diagonalizable in the basis $\{|n_0\rangle\}$. The details regarding the choice of the appropriate basis for diagonalizing the perturbative terms can be found in [96,97]. The evaluation of E_1 is performed

numerically, after calculating the Fourier transform of H_P , as explained in Appendix B.

V. RENORMALIZATION GROUP

Our results for the spectra will depend on the string tension, b , the quark masses, M_c and M_b , the gauge coupling, g , and the gluon mass m . To describe the whole spectrum of mesons composed of charm and bottom quarks with the same set of parameters, we need to take into account the dependence on the scale of g and m . In the case of the string tension, we will treat it as a constant, as is typically done in the context of potential models. The running of the quark masses is not relevant to our analysis, as we are working with the quark pole masses.

Within the CF model, the dependence of the coupling and the gluon mass on the energy scales has already been derived from the evaluation of the two-point functions of the model, both at one- and two-loop order [25,28,32,34]. To derive these running parameters, first we need to introduce the renormalization, or Z -, factors for the fields,

$$\begin{aligned} A_B^{a,\mu} &= \sqrt{Z_A} A^{a,\mu}, & c_B^a &= \sqrt{Z_c} c^a, & \bar{c}_B^a &= \sqrt{Z_c} \bar{c}^a, \\ \psi_B &= \sqrt{Z_\psi} \psi, & \bar{\psi}_B &= \sqrt{Z_\psi} \bar{\psi}, \end{aligned} \quad (25)$$

and the parameters

$$g_B = Z_g g, \quad m_B^2 = Z_m m^2, \quad M_B = Z_M M. \quad (26)$$

In the above notation, bare and renormalized quantities differ by the label 'B', which denotes the former. In the end, there are six independent renormalization factors that are determined through the following renormalization conditions:

$$\begin{aligned} G^{-1}(p = \tilde{\mu}) &= m^2 + \tilde{\mu}^2, & F(p = \tilde{\mu}) &= 1, \\ Z(p = \tilde{\mu}) &= 1, & M(p = \tilde{\mu}) &= M, \end{aligned} \quad (27)$$

and

$$Z_g \sqrt{Z_A} Z_c = 1, \quad Z_m^2 Z_A Z_c = 1. \quad (28)$$

The conditions provided in (27) impose that the gluon propagator, G , the ghost dressing function, F , and the form factors associated with the quark propagator,

$$S(p) = Z(p) \frac{i\not{p} + M(p)}{p^2 + M^2(p)},$$

acquire their tree-level form at the renormalization scale $p = \tilde{\mu}$. In the case of the function $M(p)$, we impose that it equals the quark mass at the renormalization scale. The conditions from (28) extend two nonrenormalization theorems of the CF model to the finite parts of the renormalization factors. The set of six conditions outlined above

defines an infrared safe renormalization (IRS) scheme, meaning that the coupling does not feature a Landau pole in the infrared. In order to find the running of g and m , it is convenient to introduce the anomalous dimensions:

$$\gamma_{m^2} \equiv \frac{d \ln Z_{m^2}}{d \ln \tilde{\mu}}, \quad \gamma_g \equiv \frac{d \ln Z_g}{d \ln \tilde{\mu}}, \quad (29)$$

and the β -functions,

$$\beta_{m^2} \equiv \frac{dm^2}{d \ln \tilde{\mu}}, \quad \beta_g \equiv \frac{dg}{d \ln \tilde{\mu}}. \quad (30)$$

The dependence of the parameters on the scale is encoded in the β -functions, that relate to the anomalous dimensions as [34]

$$0 = \gamma_{m^2} + \frac{\beta_{m^2}}{m^2} = \gamma_g + \frac{\beta_g}{g}. \quad (31)$$

In addition, in the case of the IRS, the β -functions for the coupling and the gluon mass obey

$$\begin{aligned} \beta_g &= g(\gamma_A/2 + \gamma_c), \\ \beta_{m^2} &= m^2(\gamma_A + \gamma_c). \end{aligned} \quad (32)$$

In this work, we used the results from the CF model at one-loop order [28], that read

$$\begin{aligned} \gamma_A &= -\frac{g^2 N_c}{192\pi^2 t^3} \left(t(34t - 175t + 6) - 2t^5 \ln(t) \right. \\ &\quad + 2(t+1)^2(2t^3 - 11t^2 + 20t - 3) \ln(t+1) \\ &\quad + 2t^{3/2}\sqrt{t+4}(t^3 - 9t^2 + 20t - 36) \\ &\quad \times \ln\left(\frac{\sqrt{t+4} - \sqrt{t}}{\sqrt{t+4} + \sqrt{t}}\right) \Bigg) + \sum_{i=1}^{N_f} \frac{g^2 T_f}{\pi^2} \left\{ \frac{t_i - 6}{6t_i} \right. \\ &\quad \left. - \frac{2}{\sqrt{t_i^3(t_i+4)}} \ln\left(\frac{\sqrt{t_i+4} - \sqrt{t_i}}{\sqrt{t_i+4} + \sqrt{t_i}}\right) \right\} \end{aligned} \quad (33)$$

and

$$\begin{aligned} \gamma_c &= -\frac{g^2 N_c}{32\pi t^2} (2(t+1)t - t^3 \ln(t) \\ &\quad + (t+1)^2(t-2) \ln(t+1)), \end{aligned} \quad (34)$$

where $N_c = 3$ is the number of colors, $T_f = 1/2$, $N_f = 3 + 1$ the number of quark flavors, $t^2 = \tilde{\mu}^2/m^2$ and $t_i^2 = \tilde{\mu}^2/M_i^2$, with m the gluon mass and M_i the quark mass. We worked with four active quark flavors: up, down, strange, and charm. As the masses of the first three are much smaller than the mass of the charm, we considered them massless. As for the charm quark, M_c , we determined its value by fitting the experimental values of the mass spectrum of charmonium and charm-bottom mesons.

VI. RESULTS

Our goal is to fit the experimental/lattice data of the spectrum of quarkonia, composed exclusively of charm and bottom quarks and their respective antiquarks, by employing the CF model. In addition, we aim to identify the region of the parameter space that displays compatibility with the data.⁴ We will focus specifically on the gluon mass, with the objective of determining whether the case of a non-vanishing gluon mass is preferred over the massless case. For the fits, we restrict the analysis to ground states, $n = 1$, with angular momentum $\ell = 0$ and $\ell = 1$, as we expect the nonrelativistic approximation to work better in the case of these quantum numbers. For completeness, we also provide predictions for the case $n = 2$.

In what follows, we present the results of the spectra for two kinds of fits. In the first fit, which we denote as FIT I, we fit the spectra of charmonium and bottomonium independently, since both spectra correspond to two different physical scales. This type of fit is common in the literature, see, e.g., [70,101,102]. In the second fit, referred to as FIT II, we take a more rigorous footing, by employing the same set of parameters for both charmonium and bottomonium. As for the gauge coupling and the gluon mass, we take into account their evolution with the scale in accordance with the renormalization group equations.

A. FIT I: Bottomonium and charmonium fitted independently

In this section we show the results for the simplest possible fit, i.e., we fit the charmonium and bottomonium spectra independently, without taking into account that the gluon mass and the gauge coupling for charmonium and bottomonium scales are connected through the renormalization group equations.

We begin by computing the meson spectra for a range of values of the parameters of the model, $\{b, M, g, m\}$, where b is the string tension, M the quark mass (bottom/charm), g the gauge coupling and m the gluon mass. Once the spectrum is known for a given set of parameters, we compute the error with respect to the data as

$$\chi = \sqrt{\frac{1}{N} \sum_{i=1}^N \left(\frac{M_i^{\text{CF}} - M_i^{\text{exp}}}{M_i^{\text{exp}}} \right)^2}, \quad (35)$$

⁴The experimental data is extracted from [98]. In the case of the meson B_c^* , no experimental determination of its mass was available at the time this analysis was performed, and we therefore compared it with the result from lattice simulations [99]. Our results do not change significantly when excluding this case from the fit. While this work was being completed, the ATLAS Collaboration reported the first observation of the B_c^{*+} meson, with a mass of $6339.0 \pm 1.4^{+1.0}_{-1.4} \pm 0.3$ MeV [100], in agreement with the lattice value employed here and compatible with our prediction.

where the index i runs over the number of heavy mesons of the type $q\bar{q}$, and M_i^{CF} and M_i^{exp} are model and experimental meson masses, respectively.

To compute the uncertainty of our results, we note that a meson mass from the CF model, M^{CF} , can be expanded as

$$M^{\text{CF}} = M^{(0)} + \lambda M^{(1)} + \lambda^2 M^{(2)} + \mathcal{O}(\lambda^3), \quad (36)$$

where $\lambda \equiv \frac{g^2 N_c}{16\pi^2}$ is the small parameter of the perturbative expansion. As a consequence, we estimate the uncertainty of our results simply as $\lambda^2 M^{\text{exp}}$, where M^{exp} is the experimental value of the meson mass in question.⁵ From now on, any uncertainty quoted for our results refers to $\lambda^2 M^{\text{exp}}$. Parameter-fitting errors and model systematic uncertainties are never included.

As for bottomonium, we implemented the following parameter scans:

$$\begin{aligned} 1.6 \leq g \leq 2.1, \\ 0.01 \text{ GeV} \leq m \leq 0.60 \text{ GeV}, \\ 0.15 \text{ GeV}^2 \leq b \leq 0.40 \text{ GeV}^2, \\ 4.5 \text{ GeV} \leq M_b \leq 4.8 \text{ GeV}, \end{aligned} \quad (37)$$

with the steps, $dg = 0.05$, $dm = 0.02 \text{ GeV}$, $db = 0.01 \text{ GeV}^2$ and $dM_b = 0.05$. Once the coarse minimum was located, the scan was refined in its vicinity by halving the steps in g , m and the quark masses. The same procedure was applied to FIT II.

The values of the parameters that minimize χ for the bottomonium spectrum are $\{b, M_b, g, m\} = \{0.33 \text{ GeV}^2, 4.575 \text{ GeV}, 1.75, 40 \text{ MeV}\}$, corresponding to an error of $\chi = 0.00014$. The results of this fit along with the corresponding experimental values are displayed in Table. I, where we observe that in all cases the results of the model are compatible with the experimental data.

For charmonium, we scanned the same ranges of the coupling, gluon mass, and string tension as for bottomonium. In the case of the charm mass, we considered

$$1.15 \text{ GeV} \leq M_c \leq 1.45 \text{ GeV}, \quad (38)$$

with a step of $dM_c = 0.05 \text{ GeV}$. The values that best fit the data are $\{b, M_c, g, m\} = \{0.26 \text{ GeV}^2, 1.250 \text{ GeV}, 1.925, 30 \text{ MeV}\}$, with a global error of $\chi = 0.00152$. The results of this fit and the comparison with the experimental values are shown in Table. II. Similarly to bottomonium, all of our results agree with experimental/lattice values within error bars.

⁵Since the relevant dynamical scale for quarkonium is the binding energy and not the full meson mass, this uncertainty should be taken as a conservative order-of-magnitude estimate rather than a full computation of the theoretical uncertainty.

TABLE I. Experimental [98] and calculated masses (this work) for $b\bar{b}$ states by employing FIT I.

State	Particle name	Experimental mass (GeV)	Calculated mass (this work) (GeV)
1^1S_0	$\eta_b(1S)$	9.3987(20)	9.3978(318)
1^3S_1	$\Upsilon(1S)$	9.46040(10)	9.4601(320)
1^3P_0	$\chi_{b0}(1P)$	9.85944(42)(31)	9.8636(334)
1^3P_1	$\chi_{b1}(1P)$	9.89278(26)(31)	9.8903(335)
1^1P_1	$h_b(1P)$	9.8993(8)	9.8992(335)
1^3P_2	$\chi_{b2}(1P)$	9.91221(26)(31)	9.9116(336)

TABLE II. Experimental [98] and calculated masses (this work) for $c\bar{c}$ states by employing FIT I.

State	Particle name	Experimental mass (GeV)	Calculated mass (this work) (GeV)
1^1S_0	$\eta_c(1S)$	2.9841(4)	2.9790(148)
1^3S_1	$J/\psi(1S)$	3.096900(6)	3.0984(153)
1^3P_0	$\chi_{c0}(1P)$	3.41471(30)	3.4246(169)
1^3P_1	$\chi_{c1}(1P)$	3.51067(5)	3.5017(174)
1^1P_1	$h_c(1P)$	3.52537(14)	3.5274(175)
1^3P_2	$\chi_{c2}(1P)$	3.55617(7)	3.5632(176)

Finally, we note that the χ error associated with charmonium is an order of magnitude larger than that of bottomonium. This is consistent with the nonrelativistic approximation, which we expect to work better for heavier mesons.

B. FIT II: Bottomonium and charmonium fitted simultaneously

A more consistent way of performing these fits should take into account that the gluon mass and the gauge coupling, evaluated at the typical charmonium and bottomonium scales, are related through the renormalization group equations. To this end, we fit the data by scanning the parameter space for the gauge coupling, g , and the gluon mass, m , at the charmonium reduced mass, $\mu_{c\bar{c}} = M_c/2$. With these parameters, we can compute the mass spectrum of charmonium. In order to find the mass spectrum of charm-bottom mesons and bottomonium, we employed the values of g and m evaluated at their respective reduced masses, $\mu_{cb} = M_c M_b / (M_c + M_b)$ and $\mu_{b\bar{b}} = M_b/2$.

To fit the data we computed χ for the following range of parameters:

$$\begin{aligned} 1.7 \leq g(\mu_{c\bar{c}}) \leq 2.9, \\ 0.01 \text{ GeV} \leq m(\mu_{c\bar{c}}) \leq 0.85 \text{ GeV}, \\ 0.15 \text{ GeV}^2 \leq b \leq 0.35 \text{ GeV}^2, \\ 4.5 \text{ GeV} \leq M_b \leq 4.7 \text{ GeV}, \\ 1.10 \text{ GeV} \leq M_c \leq 1.35 \text{ GeV}, \end{aligned} \quad (39)$$

TABLE III. Experimental [98] and calculated masses (this work) for $c\bar{c}$ states by employing FIT II. States with $n = 1$ are fitted, states with $n = 2$ are predictions of the model.

State	Particle name	Experimental mass (GeV)	Calculated mass (this work) (GeV)
1^1S_0	$\eta_c(1S)$	2.9841(4)	2.9810(200)
1^3S_1	$J/\psi(1S)$	3.096900(6)	3.0900(207)
1^3P_0	$\chi_{c0}(1P)$	3.41471(30)	3.4363(228)
1^3P_1	$\chi_{c1}(1P)$	3.51067(5)	3.5245(238)
1^1P_1	$h_c(1P)$	3.52537(14)	3.4993(235)
1^3P_2	$\chi_{c2}(1P)$	3.55617(7)	3.5422(238)
2^1S_0	$\eta_c(2S)$	3.6377(9)	3.6280(243)
2^3S_1	$\psi(2S)$	3.686097(11)	3.7298(247)

TABLE IV. Experimental masses from [98]. As for B_c^* , the value is a result of lattice simulations [99]. The calculated masses (this work) were obtained by employing the FIT II. States with $n = 1$ are fitted, states with $n = 2$ are predictions of the model.

State	Particle name	Experimental—lattice mass (GeV)	Calculated mass (this work) (GeV)
1^1S_0	B_c^+	6.27447(32)	6.2664(400)
1^3S_1	B_c^*	6.331(4)(6)	6.3267(404)
2^1S_0	$B_c^\pm(2S)$	6.8712(10)	6.8421(439)

with steps, $dg = 0.05$, $dm = 0.02$ GeV, $db = 0.01$ GeV², $dM_b = 0.05$, and found its minimum (see Fig. 2). The values that best fit the data are $\{b, M_c, M_b, g(\mu_{c\bar{c}}), m(\mu_{c\bar{c}})\} = \{0.28 \text{ GeV}^2, 1.125 \text{ GeV}, 4.550 \text{ GeV}, 2.075, 840 \text{ MeV}\}$, with $\mu_{c\bar{c}} = M_c/2 = 562.5$ MeV, with an error of $\chi = 0.00177$. At the bottomonium scale, $\mu_{b\bar{b}} = M_b/2 = 2.275$ GeV, the values of the gauge coupling and the gluon mass are $g(\mu_{b\bar{b}}) = 1.92$ and $m(\mu_{b\bar{b}}) = 810$ MeV, respectively.

Our results are collected in Tables III–V. We note that in all cases, with the exception of $\psi(2S)$ and marginally $h_c(1P)$, these are compatible with the data, when taking into account the uncertainty of our calculations. This can be observed very clearly in Figs. 3–5, where the experimental data is within the error bars of our calculation. In Tables III–V and Figs. 3–5 we have included the model predictions for the $n = 2$ states. All of them are in agreement with the experimental data, with the exception of the state $\psi(2S)$. Remarkably, all four $n = 2$ states for bottomonium are reproduced within error bars which, again, seems to signal that the nonrelativistic approximation works better in this case, as expected.

Concerning the compatibility of the parameters that best fit the data with the values obtained in other works, we find that, in the case of M_c and M_b , the former is slightly lower and the latter slightly higher than most values reported in the literature, see [98] and references therein. As for the string tension, b , it is somewhat higher than usually

TABLE V. Experimental [98] and calculated masses (this work) for $b\bar{b}$ states by employing FIT II. In the case of $\eta_b(2S)$, the experimental value was extracted from [103]. States with $n = 1$ are fitted, states with $n = 2$ are predictions of the model.

State	Particle name	Experimental mass (GeV)	Calculated mass (this work) (GeV)
1^1S_0	$\eta_b(1S)$	9.3987(20)	9.4139(457)
1^3S_1	$\Upsilon(1S)$	9.46040(10)	9.4877(460)
1^3P_0	$\chi_{b0}(1P)$	9.85944(42)(31)	9.8564(479)
1^3P_1	$\chi_{b1}(1P)$	9.89278(26)(31)	9.8870(481)
1^1P_1	$h_b(1P)$	9.8993(8)	9.8787(481)
1^3P_2	$\chi_{b2}(1P)$	9.91221(26)(31)	9.8952(482)
2^1S_0	$\eta_b(2S)$	9.9990(35)	9.9749(486)
2^3S_1	$\Upsilon(2S)$	10.0234(5)	10.0201(487)
2^3P_0	$\chi_{b0}(2P)$	10.2325(4)(5)	10.2539(497)
2^3P_1	$\chi_{b1}(2P)$	10.255446(22)(50)	10.2757(498)
2^1P_1	$h_b(2P)$	10.2598(5)(11)	10.2835(499)
2^3P_2	$\chi_{b2}(2P)$	10.26865(22)(50)	10.2922(499)

reported [77,104], but still compatible with some studies, see, for example, [105].

Contour plots of the error χ as a function of the gluon mass, the gauge coupling and the string tension are displayed in Fig. 2. We observe that the region of the parameter space with minimum χ corresponds to a nonzero gluon mass. Furthermore, we note that as the gluon mass decreases, the error becomes larger. As a result, we can conclude that FIT II clearly favors the massive over the massless gluon case. This is not the case for FIT I, where fitting each spectrum independently allows the quark masses and the string tension to largely absorb the effect of the gluon mass, leaving m poorly constrained. This is consistent with the results of other works, such as [72,83,106]. The datasets employed to generate the contour plots as well as the values of the mass spectra are provided in [107].

Before ending this section, it is important to test the self-consistency of our approach by providing an estimate of the velocities of the valence quarks of the mesons. As discussed in [101,108], the mean square velocity can be extracted from the kinetic energy, leading to

$$\langle v^2 \rangle = \frac{1}{2\mu} (E - \langle V^{(0)}(r) \rangle). \quad (40)$$

In this evaluation, we restrict our calculation to the potential $V^{(0)}(r) = -g^2 \frac{e^{-mr}}{4\pi r} + br$, where μ is the reduced mass of the system. Of course, in this estimate we are not taking into account the corrections from Eqs. (9) to (17). Thus, we will provide estimates for the velocity of the valence quarks just in the cases of charmonium, bottomonium, and charm-bottom mesons, with $\ell = 0$ and $\ell = 1$. The results are displayed in Table. VI. We observe that the mean square

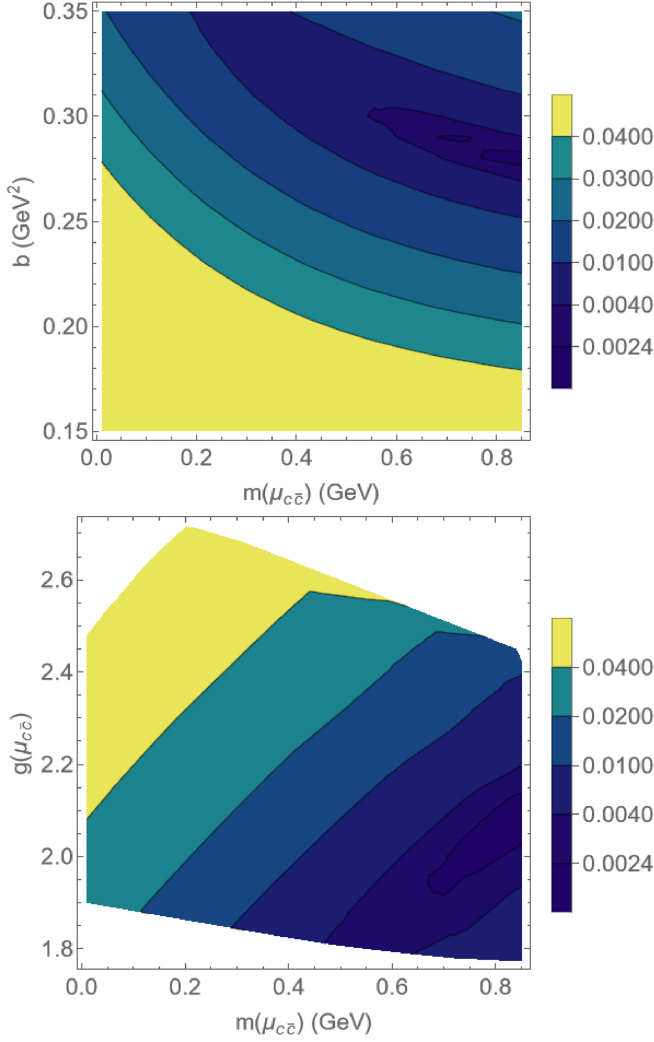


FIG. 2. Contour lines of the error χ . The parameters employed were $\{g(\mu_{c\bar{c}}), M_c, M_b\} = \{2.075, 1.125 \text{ GeV}, 4.550 \text{ GeV}\}$ (top) and $\{b, M_c, M_b\} = \{0.28 \text{ GeV}^2, 1.125 \text{ GeV}, 4.550 \text{ GeV}\}$ (bottom). The datasets used to generate these plots are provided in [107].

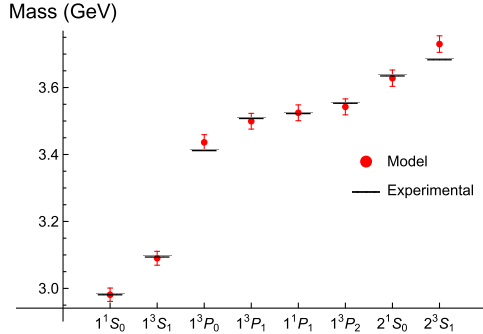


FIG. 3. Charmonium mass spectrum from experimental [98] results and this work—FIT II. The states with $n = 2$ are not taken into account when performing the fit and, as a result, are a prediction of the model.

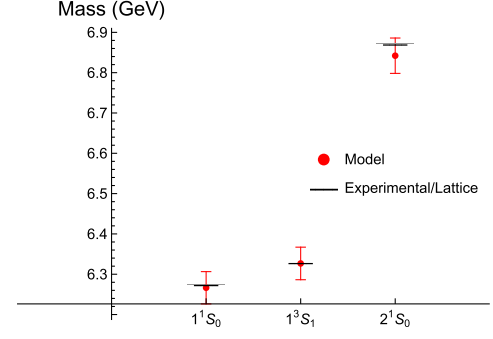


FIG. 4. Charm-bottom mesons mass spectrum. In the case of B_c the black line corresponds to an experimental result [98], whereas as for B_c^* corresponds to a lattice result [99]. Red points correspond to the results of this work—FIT II.

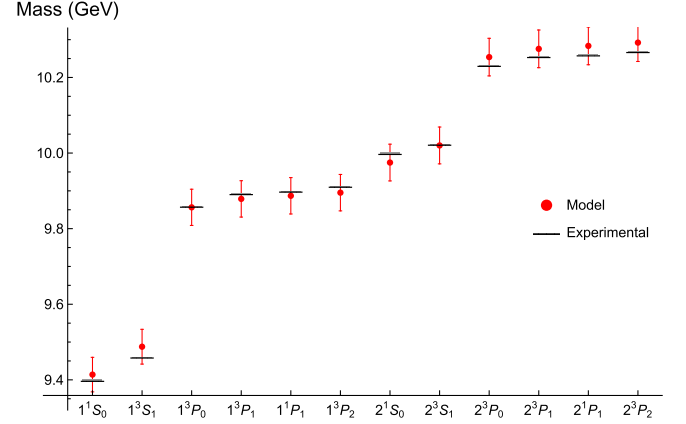


FIG. 5. Bottomonium mass spectrum from experimental [98,103] results and this work—FIT II. The states with $n = 2$ are not taken into account when performing the fit and, as a result, are a prediction of the model.

velocity of the bottom quark is smaller than that of the charm quark, in consistency with the nonrelativistic approach. Moreover, these values are also compatible with those obtained by other potential models [108].

TABLE VI. Mean square velocity of the valence quarks for charmonium, charm-bottom mesons and bottomonium.

Meson	$n\ell$	$\langle v^2 \rangle$
Charmonium	1S	0.2596
	1P	0.3227
	2S	0.3992
Charm-bottom mesons	1S	Bottom: 0.0806 charm: 0.3261
	2S	Bottom: 0.1152 charm: 0.4660
Bottomonium	1S	0.0779
	1P	0.0728
	2S	0.0895
	2P	0.0969

VII. CONCLUSIONS

We evaluated the mass spectrum of charmonium, bottomonium, and charm-bottom mesons by employing the CF model. As a first exploratory work within this framework, we restricted ourselves to the investigation of the spectrum of heavy mesons, as these states admit a nonrelativistic description. We derived the Hamiltonian of the model by employing the one-gluon exchange approximation, supplemented with a confining Cornell potential.

To fit the experimental/lattice data, we solved the Schrödinger equation for the ground state, $n = 1$, restricting the analysis to states with angular momentum $\ell = 0$ and $\ell = 1$, where the nonrelativistic approximation is expected to be more reliable. Other corrections, such as spin-orbit terms, were introduced perturbatively. The parameters of the model, *i.e.*, the coupling, the gluon mass, the string tension and bottom and charm quark masses, were determined by fitting the experimental/lattice data for the spectra. We performed two kinds of fits, denoted as FIT I and FIT II.

As for FIT I, we fitted bottomonium and charmonium spectra independently. In this case, our results provided a reasonable description of the experimental/lattice data. In consistency with the nonrelativistic approximation, the fit of bottomonium spectrum was better than that of charmonium.

In the case of FIT II, we fitted the spectrum of bottomonium, charmonium and charm-bottom mesons simultaneously, by taking into account the one-loop running of the gauge coupling and the gluon mass, as computed in [25]. This fit also featured high compatibility with the available data. Furthermore, to check the consistency of FIT II, we also evaluated the states with $n = 2$. In all cases but $\psi(2S)$, the spectrum was reproduced within error bars, supporting the validity of the model and the nonrelativistic approximation.

Concerning the gluon mass, FIT II clearly favored the case of a nonvanishing gluon mass, which is in line with the results of other works, see, for instance, [79,82,83,106,109–111]. Nevertheless, in order to clearly rule out the massless case, further studies are needed, such as the evaluation of the decay rate of heavy mesons.

In general terms, our results show that the CF model is able to describe the heavy meson spectra by employing a nonrelativistic approximation, with a discrepancy that is of the same order as the one observed in alternative potential models. A rigorous analysis of the accuracy of the results, including the compatibility of the gluon mass with the experimental/lattice data, requires a detailed investigation of the nonrelativistic limit of the BSE within the CF model, which we plan to do in a future work.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [107].

APPENDIX A: FOURIER TRANSFORM

All the terms that appear in scattering amplitude (6) correspond to one of the following forms:

$$\begin{aligned} \text{A: } & \frac{1}{\mathbf{q}^2 + m^2}, & \text{D: } & \frac{(\mathbf{a} \cdot \mathbf{q})(\mathbf{b} \cdot \mathbf{q})}{\mathbf{q}^2 + m^2}, \\ \text{B: } & \frac{\mathbf{q}}{\mathbf{q}^2 + m^2}, & \text{E: } & \frac{(\mathbf{a} \cdot \mathbf{q})(\mathbf{b} \cdot \mathbf{q})}{\mathbf{q}^2(\mathbf{q}^2 + m^2)}. \\ \text{C: } & \frac{\mathbf{q}^2}{\mathbf{q}^2 + m^2}, \end{aligned}$$

For term (A) we calculate the Fourier transform by first integrating in the angular variables in spherical coordinates:

$$\begin{aligned} & \int \frac{d^3q}{(2\pi)^3} \frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 + m^2} \\ &= \int_0^\infty \frac{dq}{(2\pi)^3} \frac{q^2}{q^2 + m^2} \int d\Omega \sin\theta e^{-iqr\cos\theta} \\ &= \frac{1}{ir(2\pi)^2} \int_0^\infty dq \frac{q}{q^2 + m^2} (e^{-iqr} - e^{+iqr}) \\ &= -\frac{1}{ir(2\pi)^2} \int_{-\infty}^\infty dq \frac{q}{q^2 + m^2} e^{iqr}, \end{aligned} \quad (\text{A1})$$

where $\mathbf{q}$ is the module $|\mathbf{q}| = q$. For the radial integral we use the residue theorem. We consider the upper half-plane which corresponds to the set of complex numbers with positive imaginary part where the only pole is $q = im$. Then, we have

$$\begin{aligned} \int \frac{d^3q}{(2\pi)^3} \frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 + m^2} &= \frac{-1}{ir(2\pi)^2} \int_{-\infty}^\infty dq \frac{q}{q^2 + m^2} e^{-iqr} \\ &= 4\pi^2 i \text{Res}(q = im) \end{aligned} \quad (\text{A2})$$

$$= \frac{e^{-mr}}{4\pi r}, \quad (\text{A3})$$

which corresponds to the Yukawa potential. Note that terms (B) and (C) can be written as derivatives of this potential with respect to r :

$$\begin{aligned} \int \frac{d^3 q}{(2\pi)^3} \frac{\mathbf{q} e^{-i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 + m^2} &= i\nabla_r \int \frac{d^3 q}{(2\pi)^3} \frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 + m^2} \\ &= i\nabla_r \left(\frac{e^{-mr}}{4\pi r} \right) \\ &= -i \left(m + \frac{1}{r} \right) \frac{e^{-mr}}{4\pi r^2} \mathbf{r}, \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} \int \frac{d^3 q}{(2\pi)^3} \frac{\mathbf{q}^2 e^{-i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 + m^2} &= -\nabla_r^2 \int \frac{d^3 q}{(2\pi)^3} \frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 + m^2} \\ &= -\nabla^2 \left(\frac{e^{-mr}}{4\pi r} \right) \\ &= -\frac{m^2 e^{-mr}}{4\pi r} + \delta^3(r). \end{aligned} \quad (\text{A5})$$

For term D we have

$$\begin{aligned} &\int \frac{d^3 q}{(2\pi)^3} \frac{(\mathbf{a} \cdot \mathbf{q})(\mathbf{q} \cdot \mathbf{b})}{\mathbf{q}^2 + m^2} e^{-i\mathbf{q}\cdot\mathbf{r}} \\ &= -(\mathbf{a} \cdot \nabla)(\mathbf{b} \cdot \nabla) \left(\frac{e^{-mr}}{4\pi r} \right) \\ &= \frac{(\mathbf{a} \cdot \mathbf{b})}{3} \delta^3(r) - \left\{ \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} \left(\frac{m^2 r^2}{3} + mr + 1 \right) \right. \\ &\quad \left. - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3} (mr + 1) \right\} \frac{e^{-mr}}{4\pi}. \end{aligned} \quad (\text{A7})$$

Finally, we can rewrite the denominator of term (E) as

$$\frac{1}{\mathbf{q}^2(\mathbf{q}^2 + m^2)} = \frac{1}{m^2 \mathbf{q}^2} - \frac{1}{m^2(\mathbf{q}^2 + m^2)}, \quad (\text{A8})$$

so we end up with a term identical to (D) and another similar one but with $m \rightarrow 0$. Taking this limit, we obtain

$$\begin{aligned} T_1 &= -\frac{4\mu^2}{8} \left(\frac{1}{M_1^3} + \frac{1}{M_2^3} \right) \left\{ E_0^2 \int_0^\infty |R_{n\ell}(r)|^2 r^2 dr + E_0 \frac{g^2}{3\pi} \int_0^\infty e^{-mr} |R_{n\ell}(r)|^2 r dr - E_0 b_0 \int_0^\infty |R_{n\ell}(r)|^2 r^3 dr \right. \\ &\quad \left. + \frac{4g^4}{9\pi^2} \int_0^\infty e^{-2mr} |R_{n\ell}(r)|^2 dr - \frac{g^2 b_0}{3\pi} \int_0^\infty e^{-mr} |R_{n\ell}(r)|^2 r^2 dr + b_0^2 \int_0^\infty |R_{n\ell}(r)|^2 r^4 dr \right\}. \end{aligned} \quad (\text{B4})$$

Next term,

$$T_2 = \left\langle -\left(\frac{1}{M_1^2} + \frac{1}{M_2^2} \right) \frac{g^2 m^2 e^{-mr}}{24\pi r} \right\rangle, \quad (\text{B5})$$

is a function of r , so is diagonal in the same basis of states as the previous one. Its expectation value is

$$T_2 = -\left(\frac{1}{M_1^2} + \frac{1}{M_2^2} \right) \frac{g^2 m^2}{24\pi} \int_0^\infty e^{-mr} |R_{n\ell}(r)|^2 r dr. \quad (\text{B6})$$

$$\int \frac{d^3 q}{(2\pi)^3} \frac{(\mathbf{a} \cdot \mathbf{q})(\mathbf{q} \cdot \mathbf{b})}{\mathbf{q}^2(\mathbf{q}^2 + m^2)} e^{-i\mathbf{q}\cdot\mathbf{r}} \quad (\text{A9})$$

$$\begin{aligned} &= \frac{1}{4\pi} \left\{ \frac{\mathbf{a} \cdot \mathbf{b}}{r^3} - \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} \right\} \\ &\quad + \left\{ \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} \left(\frac{m^2 r^2}{3} + mr + 1 \right) \right. \\ &\quad \left. - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3} (mr + 1) \right\} \frac{e^{-mr}}{4\pi}. \end{aligned} \quad (\text{A10})$$

APPENDIX B: PERTURBATIVE CORRECTIONS

In this section, we calculate the corrections, T_i , corresponding to the expectation values of the terms H_i that constitute the expression (24).

Let us begin by calculating the correction to the kinetic term

$$T_1 = \left\langle -\left(\frac{1}{M_1^3} + \frac{1}{M_2^3} \right) \frac{\mathbf{p}^4}{8} \right\rangle. \quad (\text{B1})$$

which is diagonal in the basis $|n, \ell, m, s_1, s_2, m_1, m_2\rangle$ of energy eigenstates, orbital angular momentum, and spin operators of the two particles. We need to calculate the diagonal matrix elements in this basis. From expression (8), we have

$$\mathbf{p}^2 = 2\mu \left(H_0 + \frac{g^2 e^{-mr}}{3\pi r} - b_0 r \right). \quad (\text{B2})$$

Hence,

$$T_1 = -\frac{\mu^2}{2} \left(\frac{1}{M_1^3} + \frac{1}{M_2^3} \right) \left\langle \left(H_0 + \frac{g^2 e^{-mr}}{3\pi r} - b_0 r \right)^2 \right\rangle. \quad (\text{B3})$$

Since the terms we want to calculate depend only on r , and the spherical harmonics are normalized, we have that

By integrating the delta function against the modulus of the wave function, we obtain the third correction term,

$$T_3 = \left\langle \frac{g^2}{6} \left(\frac{1}{M_1^2} + \frac{1}{M_2^2} \right) \delta^3(r) \right\rangle \quad (\text{B7})$$

$$= \frac{g^2}{6} \left(\frac{1}{M_1^2} + \frac{1}{M_2^2} \right) |\psi_{n\ell}(0)|^2. \quad (\text{B8})$$

Since the wave functions for states with $\ell \neq 0$ are zero at the origin, this correction only appears for states with $\ell = 0$. In the ground state, we have

$$T_3 = \frac{g^2}{24\pi} \left(\frac{1}{M_1^2} + \frac{1}{M_2^2} \right) |R_{10}(0)|^2. \quad (\text{B9})$$

The spin-orbit term is

$$T_4 = g^2 \left\langle (mr+1) \frac{e^{-mr}}{3\pi r^3} \left[\frac{1}{2M_1^2} + \frac{1}{M_2 M_1} \right] \mathbf{s}_1 \cdot \mathbf{L} + (mr+1) \frac{e^{-mr}}{3\pi r^3} \left[\frac{1}{2M_2^2} + \frac{1}{M_2 M_1} \right] \mathbf{s}_2 \cdot \mathbf{L} \right\rangle. \quad (\text{B10})$$

Let us define the spin-sum operator as $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$. The operator $\mathbf{S}^2$ has eigenvalues $S(S+1)$, with $S = 0, 1$. For states with $S = 0$, the correction T_4 is zero.

In the spin-sum basis, the operators $\mathbf{s}_1 \cdot \mathbf{L}$ and $\mathbf{s}_2 \cdot \mathbf{L}$ contribute equally. We will refer to either of the two operators as $\mathbf{s} \cdot \mathbf{L}$, such that $\mathbf{s} \cdot \mathbf{L} = \frac{(\mathbf{s}_1 + \mathbf{s}_2)}{2} \cdot \mathbf{L}$.

We can also define the total angular momentum as $\mathbf{J} = \mathbf{S} + \mathbf{L}$, taking eigenvalues between $|\ell - S|$ and $\ell + S$. Note that for $S = 1$ and $\ell = 1$, $J = \{0, 1, 2\}$. We can write

$$(\mathbf{s}_1 + \mathbf{s}_2) \cdot \mathbf{L} = \frac{1}{2} (\mathbf{J}^2 - \mathbf{S}^2 - \mathbf{L}^2).$$

This operator is diagonal in the basis of the sum of momenta, and it has eigenvalues $1/2(J(J+1) - S(S+1) - \ell(\ell+1))$, that are zero if $\ell = 0$,

$$T_4 = \frac{g^2}{24\pi} \left(\left[\frac{2M_1 + M_2}{M_2 M_1^2} \right] + \left[\frac{2M_2 + M_1}{M_1 M_2^2} \right] \right) \times (J(J+1) - 4) \int_0^\infty \frac{(mr+1)}{r} e^{-mr} |R_{n1}(r)|^2 dr, \quad (\text{B11})$$

where $n = 1, 2$ corresponds to the 1P and 2P states, respectively. Note that the previous terms, those which do not depend on spin, are also diagonal in the total angular momentum basis.

Now, let us consider

$$T_5 = \left\langle \frac{g^2}{3\pi M_1 M_2 m^2} \left[\frac{\mathbf{p}^2}{r^3} - \frac{3(\mathbf{p} \cdot \mathbf{r})^2}{r^5} \right] \right\rangle. \quad (\text{B12})$$

In the first term of (B12) we can express $\mathbf{p}^2$ using H_0 as we did in (B2). For the term in $(\mathbf{p} \cdot \mathbf{r})^2$ we can write the momentum $\mathbf{p}$ as $\mathbf{p} = -i\nabla$ (considering that it should only act on the wave function). This leads to

$$T_5 = \frac{g^2}{3\pi M_1 M_2 m^2} \left\{ 2\mu \left(E_0 \int_0^\infty \frac{|R_{n\ell}(r)|^2}{r} dr + \frac{g^2}{3\pi} \int_0^\infty \frac{e^{-mr}}{r^2} |R_{n\ell}(r)|^2 dr - b_0 \int_0^\infty |R_{n\ell}(r)|^2 dr \right) + 3 \int_0^\infty \frac{R_{n\ell}(r)}{r} R_{n\ell}''(r) dr \right\}, \quad (\text{B13})$$

where $R_{n\ell}'' = \partial_r^2 R_{n\ell}(r)$. Analogously, the term

$$T_6 = \left\langle -\frac{g^2 e^{-mr}}{3\pi M_1 M_2 m^2} \left[\frac{\mathbf{p}^2}{r^3} (mr+1) - \frac{3(\mathbf{p} \cdot \mathbf{r})^2}{r^5} \left(\frac{m^2 r^2}{3} + mr+1 \right) \right] \right\rangle \quad (\text{B14})$$

takes the form

$$T_6 = \frac{-g^2}{3\pi M_1 M_2 m^2} \left[2\mu \left(E_0 \int_0^\infty \frac{(mr+1)e^{-mr}}{r} |R_{n\ell}(r)|^2 dr + \frac{g^2}{3\pi} \int_0^\infty \frac{(mr+1)e^{-mr}}{r^2} |R_{n\ell}(r)|^2 dr - b_0 \int_0^\infty e^{-mr} (mr+1) |R_{n\ell}(r)|^2 dr \right) + \int_0^\infty \frac{(m^2 r^2 + 3mr+3)}{r} e^{-mr} R_{n\ell} R_{n\ell}'' dr \right]. \quad (\text{B15})$$

The spin terms,

$$T_7 = \left\langle \frac{8}{9} \frac{g^2}{M_1 M_2} (\mathbf{s}_1 \cdot \mathbf{s}_2) \delta^3(r) \right\rangle \quad (\text{B16})$$

and

$$T_8 = \left\langle -\frac{g^2 (\mathbf{s}_1 \cdot \mathbf{s}_2) e^{-mr}}{3\pi M_1 M_2 r^3} (m^2 r^2 + mr+1) \right\rangle, \quad (\text{B17})$$

can be written in the spin sum basis considering

$$\mathbf{s}_1 \cdot \mathbf{s}_2 = \frac{1}{2} [\mathbf{S}^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2].$$

This operator is diagonal, with eigenvalues $\frac{1}{2}[S(S+1) - \frac{3}{2}]$, since the eigenvalues of both $\mathbf{s}_1^2$ and $\mathbf{s}_2^2$ are $\frac{3}{4}$. Then, in this basis,

$$T_7 = \frac{g^2}{9\pi M_1 M_2} \left[S(S+1) - \frac{3}{2} \right] |R_{n\ell}(0)|^2 \quad (\text{B18})$$

and

$$T_8 = -\frac{g^2}{6\pi M_1 M_2} \left[S(S+1) - \frac{3}{2} \right] \times \int_0^\infty dr \frac{e^{-mr}}{r} (m^2 r^2 + mr + 1) |R_{n\ell}(r)|^2. \quad (\text{B19})$$

Finally, we have the term

$$T_9 = \left\langle \frac{-g^2 e^{-mr}}{3\pi M_1 M_2} \frac{(\mathbf{s}_1 \cdot \mathbf{r})(\mathbf{s}_2 \cdot \mathbf{r})}{r^5} \left(\frac{m^2 r^2}{3} + mr + 1 \right) \right\rangle \quad (\text{B20})$$

For two operators $\mathbf{a}$ and $\mathbf{b}$ that commute with each other and commute with the orbital angular momentum and with $\mathbf{r}$, it holds that

$$\langle (\mathbf{a} \cdot \mathbf{b}) r^2 f(r) - 3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r}) f(r) \rangle = -\frac{\langle r^2 f(r) \rangle}{(2\ell+3)(2\ell-1)} \langle 2\mathbf{L}^2(\mathbf{a} \cdot \mathbf{b}) - 3(\mathbf{a} \cdot \mathbf{L})(\mathbf{b} \cdot \mathbf{L}) - 3(\mathbf{b} \cdot \mathbf{L})(\mathbf{a} \cdot \mathbf{L}) \rangle.$$

We can apply this relation to the spin operators and express the products of the spins in terms of the total spin as $\mathbf{s}_1 \cdot \mathbf{s}_2 = \frac{1}{2}[\mathbf{S}^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2]$ and the products $\mathbf{S} \cdot \mathbf{L}$ in terms of the sum of momenta $\mathbf{J} = \mathbf{L} + \mathbf{S}$. This way, we obtain,

$$\langle (\mathbf{s}_1 \cdot \mathbf{s}_2) r^2 f(r) - 3(\mathbf{s}_1 \cdot \mathbf{r})(\mathbf{s}_2 \cdot \mathbf{r}) f(r) \rangle = \frac{-\langle r^2 f(r) \rangle}{(2\ell+3)(2\ell-1)} \left\langle \mathbf{S}^2 \mathbf{L}^2 - \frac{3}{4}(\mathbf{J}^2 - \mathbf{S}^2 - \mathbf{L}^2) - \frac{3}{4}(\mathbf{J}^2 - \mathbf{S}^2 - \mathbf{L}^2)^2 \right\rangle.$$

The operator on the right-hand side of the last expression is diagonal in the angular momentum basis. The mean value of the term on the left-hand side proportional to $(\mathbf{s}_1 \cdot \mathbf{r})(\mathbf{s}_2 \cdot \mathbf{r})$ takes the form

$$\langle 3(\mathbf{s}_1 \cdot \mathbf{r})(\mathbf{s}_2 \cdot \mathbf{r}) f(r) \rangle = \frac{-\langle r^2 f(r) \rangle}{(2\ell+3)(2\ell-1)} \left[S(S+1)\ell(\ell+1) - \frac{3}{4}[J(J+1) - S(S+1) - \ell(\ell+1)] - \frac{3}{4}[J(J+1) - S(S+1) - \ell(\ell+1)]^2 \right] - 4 \left[S(S+1) - \frac{3}{2} \right] \langle r^2 f(r) \rangle.$$

We can finally write the correction T_9 as

$$T_9 = \frac{g^2}{3\pi M_1 M_2} \left\{ \frac{1}{(2\ell+3)(2\ell-1)} \left[S(S+1)\ell(\ell+1) - \frac{3}{4}[J(J+1) - S(S+1) - \ell(\ell+1)] - \frac{3}{4}[J(J+1) - S(S+1) - \ell(\ell+1)]^2 \right] - 4 \left[S(S+1) - \frac{3}{2} \right] \right\} \int_0^\infty dr \frac{e^{-mr}}{r} \left(\frac{m^2 r^2}{3} + mr + 1 \right) |R_{n\ell}(r)|^2. \quad (\text{B21})$$

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