

Generalized Wigner-Weyl calculus for gauge theory and nondissipative transport

Praveen D. Xavier^{*} and M. A. Zubkov[†]

Physics Department, Ariel University, Ariel



(Received 19 June 2025; accepted 11 August 2025; published 29 September 2025)

We consider a theory of fermions interacting with a (in general, non-Abelian) gauge field. The theory is assumed to be essentially inhomogeneous, which might be provided by nontrivial background fields interacting with both fermions and gauge bosons. For this theory, a version of Wigner-Weyl calculus is developed, in which the Wigner transformation of the fermion Green function belongs to a matrix representation of the gauge group. We demonstrate the power of the proposed formalism through the representation of responses of vector and axial currents to the gauge field strength through the topological invariants composed of the Wigner transformed two-point Green functions. This way, a new family of nondissipative transport phenomena is introduced. In particular, we discuss the non-Abelian versions of the chiral separation effect and of the quantum Hall effect.

DOI: [10.1103/5h5v-jtfk](https://doi.org/10.1103/5h5v-jtfk)

I. INTRODUCTION

There is a long history of using topological invariants in quantum field theory with their applications to both high-energy physics and condensed matter physics. Possibly, the most well-known example is the description of the quantum Hall effect (QHE) via the TKNN invariant called in the names of the authors of [1]. This invariant has been proposed for homogeneous systems in the presence of a constant magnetic field. It is also relevant for the intrinsic anomalous QHE in homogeneous topological insulators without an external magnetic field. The value of the TKNN invariant is robust to smooth changes of the system [2–7]. However, the TKNN invariant, by construction, is given by the integral of the Berry curvature over the occupied energy levels and, as such, it is neither defined for systems with interactions, nor for the case when an inhomogeneity is present. This gap has been filled by more involved constructions, which include topological invariants composed of Green functions. The first step toward this direction has been made for the description of the intrinsic anomalous quantum Hall effect (AQHE) in 2D topological insulators [8,9] (see also Chapter 21.2.1 in [10]), where the Hall conductivity is given by (in units of $e^2/\hbar$)

$$\sigma_H = \frac{\mathcal{N}}{2\pi} \quad (1)$$

with

$$\mathcal{N} = -\frac{\epsilon_{ijk}}{3!4\pi^2} \int d^3p \text{Tr} \left[G(p) \frac{\partial G^{-1}(p)}{\partial p_i} \frac{\partial G(p)}{\partial p_j} \frac{\partial G^{-1}(p)}{\partial p_k} \right]. \quad (2)$$

Here, $G(p)$ is the two-point Green function in momentum space. Topological invariants similar to that of (2) were considered also in condensed matter physics in the description of 3D fermion systems with a Fermi surface. Namely, the topological invariant responsible for the stability of the Fermi surface has the form $N_1 = \text{tr} \oint_C \frac{1}{2\pi i} G(p_0, p) dG^{-1}(p_0, p)$, where C is the closed path in Euclidean momentum space around the Fermi surface [10] in momentum space. The stability of Fermi points is protected by a topological invariant, which has a form even closer to that of (2) [10,11],

$$N_3 = \frac{1}{24\pi^2} \epsilon_{\mu\nu\rho\lambda} \text{tr} \int_S dS^\mu G \partial^\nu G^{-1} G \partial^\rho G^{-1} G \partial^\lambda G^{-1}. \quad (3)$$

In this expression, S is the three-dimensional hypersurface in 4D momentum space. It encloses the Fermi point, and may be infinitely small. Similar topological invariants were used in other chapters of condensed matter physics theory [12–16]. In particular, gapless fermions existing at the edges of topological insulators are protected by such invariants due to the corresponding bulk-boundary correspondence [17,18]. Gapless fermions in the bulk of topological semimetals [10,16] are protected by similar invariants. The same

^{*}Contact author: praveen.xavier@msmail.ariel.ac.il

[†]Contact author: zubkov.mike@gmail.com

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

is true of fermion zero modes living in the core of topological defects in the ^3He superfluid [19]. In high-energy physics theory, similar topological invariants in momentum space have been discussed as well [8,10,20–27].

Although the TKNN invariant and its form in (2) were originally proposed for systems without interactions and without inhomogeneity, it was widely believed that both these expressions remain relevant for models with weak interactions and even in the presence of disorder. The former statement for (2) has been proven within 2 + 1D quantum electrodynamics [28,29]. This proof has been

extended to a more wide class of *homogeneous* models in [30].

For QHE systems in the presence of a constant homogeneous magnetic field, the role of interactions in the description of Hall conductivity has been discussed in many publications [31–34]. It has been found that the QHE conductivity remains robust to the introduction of weak interactions. The case when both interactions and disorder are present is more involved. It was considered in [35–38], where it was found that the Hall conductivity averaged over the system area is given by (1) with

$$\mathcal{N} = -\frac{T\epsilon_{ijk}}{|\mathbf{A}|3!4\pi^2} \int d^3x \int_{\mathcal{M}} d^3p \text{tr} G_W(x, p) \star \frac{\partial Q_W(x, p)}{\partial p_i} \star \frac{\partial G_W(x, p)}{\partial p_j} \star \frac{\partial Q_W(x, p)}{\partial p_k}. \quad (4)$$

In this expression, T is temperature (assumed to be small), $|\mathbf{A}|$ is the system area, $G_W(x, p)$ is the Wigner transform of the one-particle electron Green function $\hat{G}$, and the operator $\hat{Q}$ is the inverse of $\hat{G}$. Correspondingly, Q_W is the Weyl symbol of $\hat{Q}$. By the star we denote the Moyal product. Equation (4) has been obtained for lattice models using the so-called approximate lattice Wigner-Weyl calculus, which works for the case of weak inhomogeneity: when external fields vary slowly at the distance of the order of the lattice spacing.

Wigner-Weyl calculus itself, in its original form, stemmed from the works of Weyl [39] and Wigner [40]. It was developed in works by Groenewold [41] and Moyal [42] in the context of ordinary quantum mechanics. Originally the main idea was to use the Wigner distribution instead of the wave function, and replace operators of physical observables by their Weyl symbols (defined as functions in phase space). The operator products are then replaced by the Moyal (star) products of Weyl symbols [43,44]. Many applications of this formalism to various problems in quantum mechanics were proposed [45,46]—the most well known of those is the case of the anharmonic oscillator. Being modified accordingly, Wigner-Weyl calculus has been applied subsequently to high-energy physics theory as well as to condensed matter physics [47–53]. In particular, this calculus has been used in QCD [54,55], in quantum kinetic theory [56,57], and within noncommutative field theories [58,59]. Several applications have been found to a variety of quantum systems and even to cosmology [60–62].

While a majority of original works were devoted to systems defined in continuous spacetime, applications to solid-state physics require the use of lattice models. Development of Wigner-Weyl calculus for lattice models encountered several problems. Attempts to overcome these problems began a long time ago [63]. The important results in this field of research resulted from the works of Buot [53,64,65], Wooters [66], Leonhardt [67], Kasperowitz [68],

and Ligabó [69]. It is worth mentioning that Wigner-Weyl calculus is intimately related to deformation quantization [70–73].

The approximate version of lattice Wigner-Weyl calculus [35–38] (see also references therein) has been successful in applications to several practical problems related to non-dissipative transport phenomena [74–78]. The general output of these studies is that the response of nondissipative currents to the corresponding external fields is given by topological invariants, which are robust to smooth modifications of the systems.

A precise version of the Wigner-Weyl formalism for lattice models suited for the description of the QHE has been developed recently [79,80]. Within this formalism it is possible to consider the lattice models precisely—without approximation of slow varying external fields. In particular, this formalism allows one to deal with artificial lattices, in which magnetic flux through the lattice cell approaches the order of magnitude of the magnetic flux quantum, and, correspondingly, the Hofstadter butterfly might be observed.

Investigation of nondissipative transport effects has a long history. These effects exist within condensed matter physics and high-energy physics [81–94]. One of such effects is the chiral separation effect (CSE) proposed in [81]. Its essence is the appearance of an axial current directed along the external magnetic field in the presence of nonzero chemical potential in a system of relativistic Dirac fermions. For massless fermions, the axial current in these systems is proportional to the external magnetic field strength F_{ij} and to the ordinary chemical potential μ . It is expected that this effect exists in the quark-gluon plasma phase of quark matter existing at large temperatures and may be observed experimentally in heavy ion collisions [82–85], where the quark-gluon plasma appears in the presence of large magnetic fields [86,95–104].

As it was mentioned above, the responses of nondissipative currents to external fields (i.e., their conductivities) are

in many cases expressed through topological invariants. Quark matter at nonzero baryon chemical potential is an interesting example of such a system, in which—for technical reasons—the lattice numerical simulation cannot be used, while perturbative calculations also cannot give the complete picture of physical phenomena [86,95–104]. Here, we are in a situation when the existing computational methods fail to describe dynamics properly. In this situation, the nondissipative transport phenomena (in particular, the chiral separation effect) can still be investigated analytically. Namely, their conductivities, being topological invariants, are robust to smooth modification of the system. Therefore, if a simple model is found that is connected to real QCD by smooth modification (without phase transition

on the way), the conductivity of interest may be calculated within such a model. Namely, in [105,106] it was shown that in the presence of chiral symmetry, the response of axial current (averaged over the system volume $\mathbf{V}$) to the external magnetic field and to the chemical potential is given by

$$\frac{d}{d\mu} \bar{J}_5^k = \frac{\mathcal{N}}{4\pi^2} \epsilon^{0ijk} F_{ij}, \quad (5)$$

where $\mathcal{N}$ is a topological invariant expressed through the Wigner transformed Green function G_W and Weyl symbol of the Dirac operator Q_W ,

$$\mathcal{N} = \frac{1}{48\pi^2 \mathbf{V}} \int d^3x \int_{\Sigma_0} \text{tr}[\gamma^5 G_W \star dQ_W \star G_W \wedge \star dQ_W \star G_W \star \wedge dQ_W]. \quad (6)$$

The hypersurface Σ_0 in $4D$ momentum space consists of the two hyperplanes $p_4 = \pm\epsilon \rightarrow 0$. The singularities of expression standing inside the integral are situated between these two hyperplanes. γ^5 commutes or anticommutes with Q_W and G_W in a vicinity of the positions of these singularities (the latter represent extension of the notion of Fermi surface to the nonhomogeneous systems). With such a definition, Eq. (6) is a topological invariant.

It is expected that in quark matter there is a phase transition at $T = 0$ and finite μ , but the properties of the hypothetical phases situated there are not known [107]. There may exist at large μ color superconductivity, the quarkyonic phase or a phase with restored chiral symmetry. In the latter case, the results of [106] may be applied directly to the investigation of the CSE in this phase. This part of the phase diagram may be relevant, in particular, for the description of physics inside neutron stars [108].

There is an intimate relation between the chiral anomaly and the CSE [109] as well as the chiral magnetic effect (CME) [82,110–113]. While in thermal equilibrium the CME is absent [91–94,114–117], it is present out of equilibrium—even very close to equilibrium [118]. Moreover, the CME also manifests itself in the magnetoconductivity of relativistic fermionic matter [119] and may be observed experimentally in Weyl and Dirac semimetals [120].

An important arena for the experimental observation of nondissipative transport phenomena is Dirac and Weyl semimetals [88,109,121–126]. In these materials, emergent relativistic invariance is present, and the Dirac fermions emerge. The CSE may be observed experimentally in magnetic Weyl semimetals through the observation of the so-called Weyl orbits [127].

In the abovementioned cases of QHE and CSE, the conductivities are expressed through topological invariants using Wigner-Weyl calculus. In both cases, it is the

response of certain nondissipative currents to the *Abelian* field strength. The present paper is devoted to the extension of the two mentioned effects (QHE and CSE) to the case when the external gauge field is *non-Abelian*. Correspondingly, we generalize the Wigner-Weyl calculus of [36,106,127] (see also references therein) to a form that is suitable for dealing with non-Abelian gauge fields. We accept the definition of the covariant Weyl symbol of the operator proposed a long time ago in [128]. This Weyl symbol transforms covariantly under the gauge transformation, unlike the one adopted in [36,106,127]. This allows us to perform further analysis of non-Abelian gauge theory (rather than the Abelian one) along the lines of [36,106,127]. Notice that the Weyl symbol of [128] (see also references therein) was used widely in certain other applications, in particular during extension of the approach to the systems in curved spacetime [129], and to the chiral kinetic theory [130].

II. STATEMENT OF THE MAIN RESULTS

A. Generalized (or covariant) Wigner-Weyl calculus

We are working with a quantum field theory defined in continuous D -dimensional Euclidean spacetime. Consider the quantum mechanical operator

$$\hat{\pi}_\mu := \hat{p}_\mu - A_\mu(\hat{x}), \quad (7)$$

where $A_\mu(\cdot) \equiv A_\mu^a(\cdot) t^a$ is an external non-Abelian gauge field and $\hat{x}^\mu, \hat{p}_\mu$ are the D -dimensional quantum mechanical position and momentum operators with canonical commutation relations $[\hat{x}^\mu, \hat{p}_\nu] = i\delta_\nu^\mu$ (we are using the system of units with $\hbar = c = 1$). The operators to be considered below are defined in the Hilbert space $\mathcal{H}$ spanned on the eigenvectors of operators $\hat{x}^\mu$ or $\hat{p}_\mu$, denoted correspondingly

by ket vectors $|x\rangle$ or $|p\rangle$. Conjugated space $\mathcal{H}^*$ is spanned on bra vectors $\langle x|$ or $\langle p|$. The scalar products are $\langle x|x'\rangle = \delta^{(D)}(x - x')$, $\langle p|p'\rangle = \delta^{(D)}(p - p')$, and $\langle x|p\rangle = \langle p|x\rangle^* = \frac{1}{(2\pi)^{D/2}} e^{ipx}$. Thus, we deal with the mathematical basis of ordinary quantum mechanics in D -dimensional space. Mostly, we will consider the case $D = 4$, while the conventional quantum mechanics in $3 + 1$ D spacetime deals with $D = 3$.

We define the generalized Weyl symbol (we will also call it the generalized Wigner transform or simply “the Wigner transform”) of an operator $\hat{X}$ as

$$X_W(x, p) = \int dy e^{ipy} \langle x | e^{-\frac{i}{2}y\hat{\pi}} \hat{X} e^{-\frac{i}{2}y\hat{\pi}} | x \rangle. \quad (8)$$

This definition was proposed for the first time in [128] and was used since then in many applications. We are going to propose the specific application of this Weyl symbol, intended to derive the topological expressions of certain nondissipative currents. In general, $\hat{X}$ and X_W are matrices with the matrix dimensions of A . Equation (8) can also be written as follows:

$$X_W(x, p) = \int dy e^{ipy} U(x, x - y/2) \langle x - y/2 | \hat{X} | x + y/2 \rangle U(x + y/2, x). \quad (9)$$

Here,

$$U(w, x) = \text{Pexp} \left(i \int_{x \rightarrow w} dz A(z) \right) \quad (10)$$

is the path-ordered exponential along the straight path from x to w . From this expression, it is obvious that $X_W(x, p)$ is transformed under the gauge transformation $\Omega(x)$ as $X_W(x, p) \rightarrow \Omega(x) X_W(x, p) \Omega(x)^\dagger$, i.e., that it belongs to the adjoint representation. Besides, it is easy to prove the following expression for the trace of an operator:

$$\text{Tr} \hat{X} = \frac{1}{(2\pi)^D} \text{tr}_D \text{tr}_G \int dp dx X_W(x, p). \quad (11)$$

Here and below, by tr_D , tr_G , and tr_H we denote the traces with respect to the spinor indices (and the other internal indices, excluding the gauge ones), gauge indices, and the Hilbert space, respectively. Obviously,

$$\text{Tr} = \text{tr}_D \text{tr}_G \text{tr}_H,$$

and Eq. (11) replaces the trace with respect to Hilbert space by an integral in phase space.

By the star product of the Weyl symbols of two operators, we understand the Weyl symbol of the products of these operators,¹

$$(\hat{X} \hat{Y})_W(x, p) = X_W(x, p) \star Y_W(x, p). \quad (12)$$

Below, we summarize the main properties of the generalized Weyl symbol (or, covariant Weyl symbol) introduced above, and of the corresponding star product:

(1) The inverse (Weyl) transform is

$$\hat{X} = (2\pi)^{-2D} \int dq dy dx dp e^{\frac{i}{2}y(\hat{\pi}-p)} e^{iq(\hat{x}-x)} X_W(x, p) e^{\frac{i}{2}y(\hat{\pi}-p)}. \quad (13)$$

¹The associativity of the star product can be proven as follows: we have proven that $X_W \star Y_W = (\hat{X} \circ \hat{Y})_W$. Therefore, $(X_W \star Y_W) \star Z_W = (\hat{T} \circ \hat{Z})_W$, where $\hat{T}$ is the operator such that $T_W = X_W \star Y_W$ (its uniqueness follows from the fact that the generalized Wigner and Weyl transforms are inverses of each other). But since $(\hat{X} \circ \hat{Y})_W = X_W \star Y_W$, we must have $\hat{T} = \hat{X} \circ \hat{Y}$. So $(X_W \star Y_W) \star Z_W = ((\hat{X} \circ \hat{Y}) \circ \hat{Z})_W$. Similarly, $X_W \star (Y_W \star Z_W) = (\hat{X} \circ (\hat{Y} \circ \hat{Z}))_W$. But because the operator product $\circ$ is associative, we have $(X_W \star Y_W) \star Z_W = X_W \star (Y_W \star Z_W)$.

(2)

$$\begin{aligned}
X_W \star Y_W &= (2\pi)^{-4D} \int dw e^{ipw} \int dq dy dz dk dq' dy' dz' dk' e^{-i(yk+qz+y'k'+q'z')} \\
&\quad \times \langle x | e^{-\frac{i}{2}w\hat{\pi}} e^{\frac{i}{2}y\hat{\pi}} e^{iq\hat{x}} X_W(z, k) e^{\frac{i}{2}y'\hat{\pi}} e^{iq'\hat{x}} Y_W(z', k') e^{\frac{i}{2}y'\hat{\pi}} e^{-\frac{i}{2}w\hat{\pi}} | x \rangle \\
&= (2\pi)^{-2D} \int dy dk dy' dk' e^{-iy(k-p)-iy'(k'-p)} \\
&\quad \times U(x, x - (y + y')/2) U(x - (y + y')/2, x - y'/2) X_W(x - y'/2, k) U(x - y'/2, x + (y - y')/2) \\
&\quad \times U(x + (y - y')/2, x + y/2) Y_W(x + y/2, k') U(x + y/2, x + (y + y')/2) U(x + (y + y')/2, x). \quad (14)
\end{aligned}$$

(3)

$$X_W(x, p) \star Y_W(x, p) = \left(e^{\frac{i}{2}(\bar{\partial}_{p_1} + \bar{\partial}_{p_2})\bar{D}_x} e^{-\frac{i}{2}\bar{\partial}_{p_1}\bar{D}_x} X_W(x, p_1) e^{-\frac{i}{2}\bar{D}_x\bar{\partial}_{p_1}} e^{-\frac{i}{2}\bar{\partial}_{p_2}\bar{D}_x} Y_W(x, p_2) e^{-\frac{i}{2}\bar{\partial}_{p_2}\bar{D}_x} e^{\frac{i}{2}(\bar{\partial}_{p_1} + \bar{\partial}_{p_2})\bar{D}_x} \right) 1|_{p_1=p_2=p}. \quad (15)$$

(4) For any operators $\hat{X}$ and $\hat{Y}$, we have

$$\text{tr}_G \text{tr}_H \hat{X} \hat{Y} = \text{tr}_G \frac{1}{(2\pi)^D} \int dp dx X_W(x, p) \star Y_W(x, p) = \text{tr}_G \frac{1}{(2\pi)^D} \int dp dx X_W(x, p) Y_W(x, p), \quad (16)$$

provided that the above integrals are convergent. Here, tr_G and tr_H are traces with respect to the gauge indices and the Hilbert space, respectively.

(5) Let $D_\mu \equiv \partial_\mu - iA_\mu$ be the gauge covariant derivative and consider the operator $Q(x, -iD)$ of the form of

$$Q(x, -iD) = \sum_{|\alpha| \leq m} o_\alpha(x) \circ (-iD)^\alpha \quad (17)$$

with integer number m and the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $|\alpha| = \sum_i \alpha_i$. By $(-iD)^\alpha$ we denote the product of derivatives $(-iD)^\alpha = \Pi_\mu (-iD)_\mu^{\alpha_\mu}$. By $o_\alpha(x) \circ (-iD)^\alpha$ we denote

$$o_\alpha(x) \circ (-iD)^\alpha = \frac{1}{2^{|\alpha|}} \{ \dots \{ o_\alpha(x), (-iD_1) \} \dots (-iD_1) \} (-iD_2) \} \dots (-iD_2) \} \dots (-iD_4) \}. \quad (18)$$

Here, $(-iD_i)$ is encountered α_i times. For each α , $o_\alpha(x)$ is a matrix-valued function of coordinates. This is a matrix both in spinor space and in internal space, but it is a singlet in the gauge group of the gauge field A . We assume that the given operator is Hermitian, and $\{Q(x, -iD), \gamma^5\} = 0$. In chiral representation of gamma matrices, we have

$$Q(x, -iD) = \begin{pmatrix} 0 & \hat{O} \\ \hat{O}^\dagger & 0 \end{pmatrix}. \quad (19)$$

We require that $\hat{O}$ is an elliptic operator.

We also denote $\hat{Q} := Q(\hat{x}, \hat{\pi})$. Then, the following exact relationship holds:

$$Q_W(x, p) \equiv (\hat{Q})_W(x, p) = Q(x, p). \quad (20)$$

Here, $Q(x, p)$ is understood as

$$Q(x, p) = \sum_{|\alpha| \leq m} o_\alpha(x) p^\alpha, \quad (21)$$

that is, we simply substitute to Eq. (17) p instead of $-iD$ in order to obtain the above expression.²

B. Wigner-Weyl gauge theory

In four-dimensional Euclidean spacetime, consider an external, time-independent non-Abelian gauge field, $A_\mu(x)$ (with gauge group G), minimally coupled to fermions (in the fundamental representation), where the partition function is

$$Z = \int D[\bar{\psi}, \psi] e^S, \quad (22)$$

$$S = - \int d^4x \bar{\psi}(x) Q(x, -iD) \psi(x),$$

where $D_\mu \equiv \partial_\mu - iA_\mu$ is the gauge covariant derivative and $Q(x, -iD)$ is defined by (17).

We will show that the Wigner transformed Green functions obey the following properties:

- (1) The Wigner transform of the Green function G_W obeys the generalized Groenewold equation

$$G_W(x, p) \star Q(x, p) = 1. \quad (23)$$

- (2) The non-Abelian vector current (which belongs to the adjoint representation of the gauge group) is defined as the Noether current corresponding to the gauge transformation

$$\psi(x) \rightarrow e^{i\alpha(x)} \psi(x), \quad (24)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{-i\alpha(x)}, \quad (25)$$

where $\alpha(x)$ belongs to the Lie algebra of the gauge group. Recall that we denote by tr_D the trace with respect to the indices other than those corresponding to the given gauge group.

In terms of the Wigner transform of the Green function, G_W , the average vector current is given by

$$\langle J_\mu(x) \rangle = -\text{tr}_D \int \frac{d^4p}{(2\pi)^4} G_W \partial_{p_\mu} Q. \quad (26)$$

- (3) The axial current is defined as the Noether current corresponding to the axial gauge transformation

$$\psi(x) \rightarrow e^{i\alpha(x)\gamma^5} \psi(x), \quad (27)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{i\alpha(x)\gamma^5}. \quad (28)$$

The averaged axial current is given by

$$\langle J_\mu(x) \rangle = -\frac{1}{2} \text{tr}_D \int \frac{d^4p}{(2\pi)^4} G_W \partial_{p_\mu} [Q, \gamma^5]. \quad (29)$$

- (4) To distinguish between the x dependence arising from the inhomogeneity (caused by the background fields) vs from the gauge field, we introduce the function $G_W(x, p, z)$ (the z dependence arises from the dependence of $\star$ on the gauge field, while the x dependence is the consequence of the background inhomogeneity). Namely, $G_W(x, p, z)$ is the solution of

²We can also consider the more complicated situation, when an operator is defined through the formal series

$$Q(x, -iD) = \lim_{m \rightarrow \infty} \sum_{|\alpha| \leq m} o_\alpha(x) \circ (-iD)^\alpha.$$

In order to give the precise meaning to the above formal series, let us reorder using conventional commutation relations of these operators the terms in the $\circ$ product in such a way that all functions of x^{ν_i} [including $A(x)$ and its derivatives] stand on the left, while ordinary derivatives ∂_{μ_j} stand on the right,

$$Q(x, -iD) = \sum_\alpha \tilde{o}_\alpha(x) (-i\partial)^\alpha.$$

Here, $\tilde{o}_\alpha(x)$ is a tensor that is made of $o_\alpha(x)$ and $A(x)$ with its derivatives using commutation relations between $A(x)$ and D_{μ_j} . Now, we give the meaning to these formal series as follows. We determine the mixed matrix elements

$$\langle x | Q(x, -iD) | p \rangle = \sum_\alpha \tilde{o}_\alpha(x) p^\alpha.$$

This series is understood as the definition of the analytical function of x^ν and p_μ in a certain (possibly, very small) vicinity of $p = 0$, that is, we assume that the given series is convergent in a certain vicinity of this point in momentum space. The whole function at the arbitrary values of x and p is then understood as its analytical continuation.

$$\begin{aligned}
1 &= (2\pi)^{-2D} \int dy dk dy' dk' e^{-iy(k-p) - iy'(k'-p)} Q(x - y'/2, k) \\
&\times U(z, z - (y + y')/2) U(z - (y + y')/2, z - y'/2) U(z - y'/2, z + (y - y')/2) \\
&\times U(z + (y - y')/2, z + y/2) G_W(x + y/2, k', z + y/2) U(z + y/2, z + (y + y')/2) U(z + (y + y')/2, z), \quad (30)
\end{aligned}$$

and (30) is to be compared with (14). One can see that the difference is that in (30), the dependence of the parallel transporters on x is replaced by the dependence on z . Therefore, it is obvious that $G_W(x, p) = G_W(x, p, x)$.

Note that $Q(x, p)$ has no z dependence. The solution of the Groenewold equation may be expanded in powers of the covariant derivative D_z ,

$$G_W(x, p, z) = \sum_{n \geq 0} G^{(n)}(x, p, z). \quad (31)$$

Here, $G^{(n)}(x, p, z)$ contains n powers of D_z . To lowest order in D_z , the Groenewold equation becomes

$$Q \star G^{(0)} = 1, \quad (32)$$

where $\star = e^{\frac{i}{2}(\vec{\partial}_x \vec{\partial}_p - \vec{\partial}_p \vec{\partial}_x)}$ is the ordinary Moyal product. In the next orders, we have $G^{(1)} = 0$ and

$$\begin{aligned}
G^{(2)}(x, p, z) \\
= -\frac{i}{2} G^{(0)}(x, p) \star \partial_{p_\mu} Q(x, p) \star \partial_{p_\nu} G^{(0)}(x, p) F_{\mu\nu}(z), \quad (33)
\end{aligned}$$

where $F_{\mu\nu}$ is non-Abelian gauge field strength.

C. Non-Abelian nondissipative transport

We use Pauli-Villars regularization in order to subtract from the observable quantities the unphysical contributions of the ultraviolet region in momentum space. For the details of this regularization see, for example, Ref. [131]. It substitutes the expression for the thermodynamic potential (T is temperature) $\Omega = -T \log Z$ by

$$\Omega_{\text{reg}} = -T \sum_i C_i \log Z(M_i), \quad (34)$$

where $Z(M_i)$ is calculated as (22) with Q substituted by $Q_{(M_i)}(i\partial_p, p - A(i\partial_p))$, such that the nonzero parameter M_i provides the appearance of a gap (i.e., a mass for the fermion). Coefficients C_i obey the conditions

$$\sum_i C_i = 0, \quad \sum_i C_i M_i^2 = 0, \quad C_0 = 1. \quad (35)$$

These conditions provide, in particular, vanishing contributions of the ultraviolet momentum space region to the thermodynamical potential of Eq. (34). This occurs because

at $|p| \gg M$, the contributions of the Pauli-Villars regulators are equal to the contribution of the original fermions. Their sum vanishes because $\sum_i C_i = 0$. At the same time, the contributions of Pauli-Villars regulators from the region with $|p| \ll M$ vanish, so that only those of the original fermions remain. Thus, the procedure of Pauli-Villars regularization (in the absence of interfermion interactions) results in the subtraction of unphysical artifacts originated in ultraviolet within the nonregularized fermionic field theory. Notice that the same job is done, for example, by lattice regularization. Let us define

$$N_3(x) = -\frac{1}{48\pi^2} \int_{\Sigma_0} \text{tr}_D(\gamma^5 G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ), \quad (36)$$

where the hypersurface Σ_0 in $4D$ momentum space consists of the two hyperplanes $p_4 = \pm\epsilon \rightarrow 0$. The singularities of expression standing inside the integral are situated between these two hyperplanes. The positions of these singularities generalize the notion of the Fermi surface to the nonhomogeneous systems. By $dG^{(0)}$ and dQ we denote the one-forms, being the external derivatives of $Q((\vec{x}, 0)(\vec{p}, \pm\epsilon))$ and $G((\vec{x}, 0)(\vec{p}, \pm\epsilon))$ correspondingly considered as functions of spatial momenta $\vec{p} = (p_1, p_2, p_3)$ with a fixed value of $\vec{x}$.

(1) The non-Abelian version of the CSE.

Suppose that the $3 + 1D$ fermionic system with Fermi surface is in the presence of an external non-Abelian magnetic field [i.e., when only the components $F_{ij}(x)$ are nonzero with $i, j = 1, 2, 3$]. We suppose that in a small vicinity of the Fermi surface, chiral symmetry is present. Then, provided that the field strength varies slowly compared to $N_3(x)$, we obtain, for the response of axial current to chemical potential,

$$\frac{d}{d\mu} \bar{J}_i^{(5)} = \frac{1}{4\pi^2} \epsilon_{ijk} N_3 F_{jk}. \quad (37)$$

Here, N_3 is given by

$$\begin{aligned}
N_3 &= -\frac{1}{48\pi^2 V} \int d^3x \\
&\times \int_{\Sigma_0} \text{tr}_D(G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ \gamma^5), \quad (38)
\end{aligned}$$

where V is the overall volume of the system. Notice that Eqs. (37) and (38) are precisely the same for non-Abelian and Abelian cases. The only difference is the meaning of the field strength and of the axial current (Abelian or non-Abelian). In the Abelian case, this representation has been obtained previously (see Ref. [106] and references therein). Now, we generalize this result to the case of the non-Abelian gauge fields. In Secs. VIB 1 and VIB 2, we consider the particular cases of systems, in which the value of N_3 is equal to the number of Dirac fermions interacting with the given non-Abelian gauge field. In a general case, N_3 is a topological invariant, i.e., it is robust to smooth modifications of the system. This allows to calculate it for rather complicated systems, if it is known that they may be transformed smoothly to the simpler ones.

Notice that in the case of an inhomogeneous system, the notion of a conventional Fermi surface strictly speaking is not defined. The same refers also to the systems with interactions. For the case of an interacting system, the notion of the Fermi surface can be defined, however, as the position of the poles of the Green function. The extension of the notion of the Fermi surface to the case of the system with inhomogeneity is also possible. In our present paper, we consider the position (in momentum space) of the singularities of expression standing inside the integral in Eq. (2.32) as the position of the Fermi surface. These positions form a two-dimensional surface in momentum space depending on the coordinate x . At each value of x , the three-dimensional hypersurface Σ_0 embraces this two-dimensional “Fermi surface.”

As for the notion of the mass gap, it has the clear meaning in the systems we are considering. Namely, if for any value of chemical potential there exists the “Fermi surface” defined above, then there is no gap in the given system. If for the values of chemical potential within the interval between the two values μ_1 and μ_2 the “Fermi surface” is absent, then we can say that this interval represents the gap for the given system.

- (2) The non-Abelian analog of the QHE in the case of a $3 + 1$ D system.

Let us consider the case when the fermion is massive, and in Euclidean spacetime its propagator does not have poles. Besides, we require that the external gauge field strength is varying slowly. The response of the spatial components of the vector current (averaged over the system volume) to non-Abelian *electric* field strength (with nonzero $F_{4i} = -iE_i$ components) receives the form

$$\bar{J}_i^{v,QHE} = -\frac{1}{4\pi^2} \epsilon_{ijk} \tilde{M}^{*(v)j} E_k \quad (39)$$

with

$$\tilde{M}^{*(v)i} = -\frac{1}{V24\pi^2} \left[\int d^3x \wedge dp^i \wedge \text{tr}_D \right. \\ \left. \times (G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ) \right]_{\text{reg}}. \quad (40)$$

In this expression, the integration is over the product of the spatial part of coordinate space ($d^3x \equiv dx^1 \wedge dx^2 \wedge dx^3$) and the four-dimensional momentum space. The subscript “reg” means that the unphysical contribution of the Dirac sea should be subtracted from the considered quantity. Such a contribution does not present in some cases, as demonstrated below, when the particular examples of the quantum systems are considered (Sec. VII C). However, if such a contribution is present, the Pauli-Villars regularization provides their cancellation. It is worth mentioning that vector $\tilde{M}^{*(v)i}$ is related to the vector of intrinsic magnetic moment $\mathcal{M}$ of the fermion system (see Appendix B),

$$\frac{\partial}{\partial \mu} \mathcal{M}_i = \frac{1}{4\pi^2} \tilde{M}^{*(v)i}. \quad (41)$$

As for the case of the CSE considered above, the expressions for the quantum Hall effect given by Eqs. (39) and (40) are identical for the Abelian and non-Abelian external gauge field. The Abelian case has been considered before (see Ref. [132]). Now, we generalize this result to the non-Abelian case using the developed formalism of covariant Wigner-Weyl calculus. Notice that strictly speaking, Eq. (40) does not represent a topological invariant. However, in many cases it may be represented as an integral of a topological invariant (see below, Sec. VII C).

- (3) The non-Abelian analog of the QHE in the case of a gapped $2 + 1$ D system.

The response of the vector current (averaged over the system volume), in the gapped $2 + 1$ D system, to the non-Abelian electric field strength is (for the case of an external field varying slowly)

$$\bar{J}_i^{v,QHE} = \frac{1}{2\pi} \epsilon_{ij} M_3 E_j \quad (42)$$

with

$$M_3 = -\frac{1}{S24\pi^2} \\ \times \left[\int d^2x \int \text{tr}_D (G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ) \right]_{\text{reg}}. \quad (43)$$

Here, S is the area of the system, and the integration is over the product of the two-dimensional spatial coordinate space and three-dimensional momentum space. As for the three-dimensional case, in this expression the subscript “reg” means that the unphysical contribution of the Dirac sea should be subtracted. Equations (42) and (43) are precisely the same for ordinary (Abelian) QHE. For that case, these results were obtained in [133]. Now, we generalize the result of [133] to the case of the non-Abelian electric field and non-Abelian Hall current using the developed machinery of covariant Wigner-Weyl calculus. Notice that the above defined quantity M_3 is a topological invariant, which reflects the fact that the QHE conductivity (both Abelian and non-Abelian) is robust to the smooth modifications of the system.

Thus, summarized above are the main results of this paper. Their proofs are given in the forthcoming sections.

In the present paper, we consider the response of axial and vector currents to the external non-Abelian gauge field strength. At the same time, the system contains inhomogeneity of another source. We call the sources of this extra inhomogeneity “the background fields.” We do not consider the response to these fields. In particular, our Dirac operator Q entering Eqs. (38), (40), and (43) (the expressions responsible for the CSE and QHE) contains these background fields, which provide its dependence on x . At the same time, the external gauge field does not enter this Q .

III. GENERALIZED WIGNER-WEYL CALCULUS FOR THE GAUGE THEORY

A. Generalized Wigner transform

We consider the theory in D -dimensional Euclidean space with $A_\mu(\cdot) \equiv A_\mu^a(\cdot)t^a$ that is an external non-Abelian gauge field. The construction used in the present paper is based on the notions of $\hat{x}^\mu$ and $\hat{p}_\mu$ that are the D -dimensional position and momentum operators, and the covariant momentum operator $\hat{\pi}_\mu = \hat{p}_\mu - A_\mu(\hat{x})$. Definitions related to these operators and the corresponding Hilbert space of states are given above in Sec. II A.

The generalized Weyl symbol or covariant Weyl symbol of an operator $\hat{X}$ is understood as the generalized/covariant Wigner transform of its matrix elements (hereafter simply “the Wigner transform”),

$$X_W(x, p) = \int dy e^{ipy} \langle x | e^{-\frac{i}{2}y\hat{\pi}} \hat{X} e^{-\frac{i}{2}y\hat{\pi}} | x \rangle. \quad (44)$$

In general, $\hat{X}$ and X_W are matrices with the matrix dimensions of A . Notice that the definition of Eq. (44) was proposed for the first time in [134].

Below, we will prove that the inverse (Weyl) transform is

$$\hat{X} = (2\pi)^{-2D} \int dq dy dx dp e^{\frac{i}{2}y(\hat{\pi}-p)} e^{iq(\hat{x}-x)} X_W(x, p) e^{\frac{i}{2}y(\hat{\pi}-p)}. \quad (45)$$

Proof. Substituting (45) in the rhs of (44), we obtain

$$(2\pi)^{-2D} \int dy e^{ipy} \int dq' dy' dx' dp' \langle x | e^{-\frac{i}{2}y\hat{\pi}} e^{\frac{i}{2}y'\hat{\pi}} e^{iq'\hat{x}} X_W(x', p') e^{\frac{i}{2}y'\hat{\pi}} e^{-\frac{i}{2}y\hat{\pi}} | x \rangle e^{-i(q'x' + y'p')}. \quad (46)$$

Now, we use the following fact³ [see Eq. (3.10) of [128]]:

$$\exp(yD)\psi(x) = U(x, x+y)\psi(x+y), \psi^\dagger(x) \exp(y\tilde{D}^*) = \psi^\dagger(x+y)U(x+y, x), \quad (50)$$

³To prove this we note that

$$\exp(yD) = \lim_{N \rightarrow \infty} \prod_{n=1}^N (1 + (y/N)D). \quad (47)$$

But we can write

$$1 + (y/N)D = (1 + (y/N)\partial_x)(1 - i(y/N)A(x)) = \mathcal{T}(y/N)(1 - i(y/N)A(x)) \quad (48)$$

[up to $\mathcal{O}(N^{-2})$ terms], where $\mathcal{T}(\varepsilon)$ is the translation operator: $\mathcal{T}(\varepsilon)\psi(x) = \psi(x + \varepsilon)$. So,

$$\exp(yD)\psi(x) = \lim_{N \rightarrow \infty} \left(\prod_{n=1}^N \mathcal{T}(y/N)(1 - i(y/N)A(x)) \right) \psi(x), \quad (49)$$

which is easy to see gives the required result when we use the definition (52) for $U(x, x+y)$.

where $\psi(x)$ is any element of the fundamental representation and D_μ is the gauge covariant derivative: $D_\mu \equiv \partial_\mu - iA_\mu(x)$, $D_\mu^* \equiv \partial_\mu + iA_\mu(x)$, and

$$U(w, x) = \text{Pexp} \left(i \int_{x \rightarrow w} dz A(z) \right) \quad (51)$$

is the path-ordered exponential along the straight path from x to w . Its definition is

$$U(w, x) = \lim_{N \rightarrow \infty} f_N(N) f_N(N-1) \dots f_N(0), \quad (52)$$

where

$$f_N(n) = \left(1 + \frac{i}{N} (w-x)^\mu A_\mu \left(x + \frac{n}{N} (w-x) \right) \right). \quad (53)$$

From the definition, it is clear that $A(x)$ stands on the right, while $A(w)$ stands on the left, and that U is unitary and satisfies $U(x, w)^{-1} = U(w, x)$. If $\phi(x)$ lies in the adjoint representation instead, then the analog of (50) is

$$\exp(y\mathcal{D})\phi(x) = U(x, x+y)\phi(x+y)U(x+y, x). \quad (54)$$

We denote here the covariant derivative in the adjoint representation by $\mathcal{D}$ to distinguish it from the one in the fundamental representation.

In bra-ket notation, Eq. (50) reads

$$e^{-iy\hat{\pi}}|x\rangle = |x+y\rangle U(x+y, x) \quad (55)$$

and

$$\langle x|e^{iy\hat{\pi}} = U(x, x+y)\langle x+y|. \quad (56)$$

Therefore,

$$\begin{aligned} \langle x|e^{-\frac{i}{2}y\hat{\pi}}e^{\frac{i}{2}y'\hat{\pi}}e^{iq'\hat{x}}X_W(x', p')e^{\frac{i}{2}y'\hat{\pi}}e^{-\frac{i}{2}y\hat{\pi}}|x\rangle \\ = e^{iq'x}X_W(x', p')\delta(y-y'). \end{aligned} \quad (57)$$

Then, it becomes easy to verify that (46) reduces to $X_W(x, p)$, as required.

Conversely, substituting (44) in the rhs of (45), we obtain

$$(2\pi)^{-D} \int dq dy dx e^{\frac{i}{2}y\hat{\pi}} e^{iq(\hat{x}-x)} \langle x|e^{-\frac{i}{2}y\hat{\pi}}\hat{X}e^{-\frac{i}{2}y\hat{\pi}}|x\rangle e^{\frac{i}{2}y\hat{\pi}}. \quad (58)$$

We can write $\langle x|e^{-\frac{i}{2}y\hat{\pi}}\hat{X}e^{-\frac{i}{2}y\hat{\pi}}|x\rangle$ as

$$U(x, x-y/2)\langle x-y/2|\hat{X}|x+y/2\rangle U(x+y/2, x). \quad (59)$$

Inserting a complete set of states afore (58), we get

$$\int dy dx |x-y/2\rangle \langle x| \langle x-y/2|\hat{X}|x+y/2\rangle U(x+y/2, x) e^{\frac{i}{2}y\hat{\pi}}. \quad (60)$$

Inserting a complete set of states after the expression, we get

$$\int dy dx |x-y/2\rangle \langle x-y/2|\hat{X}|x+y/2\rangle \langle x+y/2|. \quad (61)$$

It is evident that this is $\hat{X}$, as required.

We note that (44) can also be written as follows:

$$\begin{aligned} X_W(x, p) &= \int dy e^{ipy} U(x, x-y/2) \\ &\times \langle x-y/2|\hat{X}|x+y/2\rangle U(x+y/2, x). \end{aligned} \quad (62)$$

From this expression it is obvious that $X_W(x, p)$ is transformed under the gauge transformation $\Omega(x)$ as $X_W(x, p) \rightarrow \Omega(x)X_W(x, p)\Omega(x)^\dagger$, i.e., that it belongs to the adjoint representation.

Now, consider the generalized Wigner transform of the composition of two operators using (45) twice in (44),

$$\begin{aligned} (\hat{X}\hat{Y})_W(x, p) &= (2\pi)^{-4D} \int dw e^{ipw} \int dq dy dz dk dq' dy' dz' dk' e^{-i(yk+qz+y'k'+q'z')} \\ &\times \langle x|e^{-\frac{i}{2}w\hat{\pi}}e^{\frac{i}{2}y\hat{\pi}}e^{iq\hat{x}}X_W(z, k)e^{\frac{i}{2}y\hat{\pi}}e^{\frac{i}{2}y'\hat{\pi}}e^{iq'\hat{x}}Y_W(z', k')e^{\frac{i}{2}y'\hat{\pi}}e^{-\frac{i}{2}w\hat{\pi}}|x\rangle \\ &= (2\pi)^{-2D} \int dy dk dy' dk' e^{-iy(k-p)-iy'(k'-p)} \\ &\times U(x, x-(y+y')/2)U(x-(y+y')/2, x-y'/2)X_W(x-y'/2, k)U(x-y'/2, x+(y-y')/2) \\ &\times U(x+(y-y')/2, x+y/2)Y_W(x+y/2, k')U(x+y/2, x+(y+y')/2)U(x+(y+y')/2, x) \\ &= X_W \star Y_W, \end{aligned} \quad (63)$$

where in the final line we have defined the generalized star product.

B. Generalized Moyal product

Recall that $D = \partial - iA$. Now, let us show that the generalized star product may be written as

$$X_W(x, p) \star Y_W(x, p) = \left(e^{\frac{i}{2}(\tilde{\partial}_{p_1} + \tilde{\partial}_{p_2})\tilde{D}_x} e^{-\frac{i}{2}\tilde{\partial}_{p_1}\tilde{D}_x} X_W(x, p_1) e^{-\frac{i}{2}\tilde{D}_x\tilde{\partial}_{p_1}} e^{-\frac{i}{2}\tilde{\partial}_{p_2}\tilde{D}_x} Y_W(x, p_2) e^{-\frac{i}{2}\tilde{\partial}_{p_2}\tilde{D}_x} e^{\frac{i}{2}(\tilde{\partial}_{p_1} + \tilde{\partial}_{p_2})\tilde{D}_x} \right) \times 1 \Big|_{p_1=p_2=p}. \quad (64)$$

We can trivially write $X_W(x, p) = (2\pi)^{-D} \int dk dy e^{-iy(k-p)} X_W(x, k)$ and $Y_W(x, p) = (2\pi)^{-D} \int dk' dy' e^{-iy'(k'-p)} Y_W(x, k')$. Then,

$$\begin{aligned} X_W(x, p) \star Y_W(x, p) &= (2\pi)^{-2D} \int dy dk dy' dk' e^{-iy(k-p) - iy'(k'-p)} \\ &\quad \times e^{-\frac{y'+y}{2}\tilde{D}_x} e^{\frac{y}{2}\tilde{D}_x} X_W(x, k') e^{\frac{y'}{2}\tilde{D}_x} Y_W(x, k') e^{\frac{1}{2}y'\tilde{D}_x} e^{-\frac{y+y'}{2}\tilde{D}_x} \times 1. \end{aligned} \quad (65)$$

Now we use the following sequence of relations:

$$\begin{aligned} &e^{-\frac{y'+y}{2}\tilde{D}_x} e^{\frac{y}{2}\tilde{D}_x} X_W(x, k) e^{\frac{y'}{2}\tilde{D}_x} \\ &Y_W(x, k') e^{\frac{1}{2}y'\tilde{D}_x} e^{-\frac{y+y'}{2}\tilde{D}_x} \times 1 \\ &= e^{-\frac{y'+y}{2}\tilde{D}_x} e^{\frac{y}{2}\tilde{D}_x} X_W(x, k) e^{\frac{y'}{2}\tilde{D}_x} e^{\frac{y'}{2}\tilde{D}_x} \\ &Y_W(x, k') e^{\frac{1}{2}y'\tilde{D}_x} U\left(x, x - \frac{y+y'}{2}\right) \\ &= e^{-\frac{y'+y}{2}\tilde{D}_x} e^{\frac{y}{2}\tilde{D}_x} X_W(x, k) e^{\frac{y'}{2}\tilde{D}_x} e^{\frac{y'}{2}\tilde{D}_x} \\ &Y_W(x, k') U(x, x + y'/2) U\left(x + y'/2, x - \frac{y}{2}\right) \\ &= e^{-\frac{y'+y}{2}\tilde{D}_x} e^{\frac{y}{2}\tilde{D}_x} X_W(x, k) e^{\frac{y'}{2}\tilde{D}_x} U(x, x + y'/2) \\ &Y_W(x + y'/2, k') U(x + y'/2, x + y') U\left(x + y', x - \frac{y-y'}{2}\right) \\ &= e^{-\frac{y'+y}{2}\tilde{D}_x} e^{\frac{y}{2}\tilde{D}_x} X_W(x, k) U(x, x + y/2) U(x + y/2, x + (y+y')/2) \\ &Y_W(x + (y' + y)/2, k') U(x + (y' + y)/2, x + y' + y/2) U\left(x + y' + y/2, x + \frac{y'}{2}\right) \\ &= e^{-\frac{y'+y}{2}\tilde{D}_x} U(x, x + y/2) X_W(x + y/2, k) U(x + y/2, x + y) U(x + y, x + y + y'/2) \\ &Y_W(x + y + y'/2, k') U(x + y + y'/2, x + y' + y) U\left(x + y' + y, x + \frac{y'+y}{2}\right) \\ &= U(x, x - (y + y')/2) U(x - (y + y')/2, x - y'/2) X_W(x - y'/2, k) \\ &U(x - y'/2, x + y/2 - y'/2) U(x + y/2 - y'/2, x + y/2) \\ &Y_W(x + y/2, k') U(x + y/2, x + y'/2 + y/2) U(x + y'/2 + y/2, x). \end{aligned} \quad (66)$$

One can also derive the following representation:

$$\begin{aligned} X_W(x, p) \star Y_W(x, p) &= (2\pi)^{-2D} \int dy dk dy' dk' e^{-iy(k-p) - iy'(k'-p)} \\ &\quad \times e^{-\frac{y'+y}{2}\tilde{D}_x} e^{\frac{y}{2}\tilde{D}_x} e^{\frac{y'}{2}\tilde{D}_x} \left(e^{-\frac{y'}{2}\tilde{D}_x} X_W(x, k) \right) e^{-\frac{y'}{2}\tilde{D}_x} e^{\frac{y}{2}\tilde{D}_x} e^{\frac{y'}{2}\tilde{D}_x} e^{-\frac{1}{2}y'\tilde{D}_x} \left(e^{\frac{y}{2}\tilde{D}_x} Y_W(x, k') \right) e^{\frac{1}{2}y'\tilde{D}_x} e^{\frac{y}{2}\tilde{D}_x} e^{-\frac{y+y'}{2}\tilde{D}_x} \times 1, \end{aligned} \quad (67)$$

where $\mathcal{D}$ is the covariant derivative in adjoint representation.

Another useful representation is

$$X_W(x, p) \star Y_W(x, p) = \quad (68)$$

$$(2\pi)^{-2D} \int dy dk dy' dk' e^{-iy(k-p) - iy'(k'-p)} \quad (69)$$

$$W(x, y, y') \left(e^{\frac{(y+y')}{2} \bar{D}_x} e^{-y' \bar{D}_x} e^{-\frac{y}{2} \bar{D}_x} X_W(x, k) \right) \times \left(e^{\frac{(y+y')}{2} \bar{D}_x} e^{-\frac{y'}{2} \bar{D}_x} Y_W(x, k') \right), \quad (70)$$

where $W(x, y, y')$ is the following triangular Wilson loop:

$$W(x, y, y') = U(x, x - (y + y')/2) U(x - (y + y')/2, x + y/2 - y'/2) \times U(x + y/2 - y'/2, x + y'/2 + y/2) U(x + y'/2 + y/2, x). \quad (71)$$

This representation allows us to derive the “generalized Poisson bracket”: one has to expand (70) to linear order in y and y' (to this order the Wilson loop is just 1) and write polynomials in y and y' as momentum derivatives using integration by parts, giving

$$X_W \star Y_W = X_W Y_W - \frac{i}{2} (\partial_{p_\mu} X_W \mathcal{D}_\mu Y_W - \mathcal{D}_\mu X_W \partial_{p_\mu} Y_W) + \text{higher-order terms.} \quad (72)$$

So, compared to the usual Poisson bracket, the difference is that the positional partial derivative is replaced with the covariant derivative.

IV. NOETHER PROCEDURE IN TERMS OF WIGNER-WEYL CALCULUS FOR AXIAL AND VECTOR CURRENTS

A. Partition function and the action in terms of Wigner-Weyl calculus

We consider the 3 + 1D theory after Wick rotation to the Euclidean spacetime. All external fields in the original theory do not depend on time, which allows us to deal with the technique of equilibrium theory. Among other external fields, we distinguish the non-Abelian gauge field $A_\mu(x)$ (with gauge group G). It is coupled minimally to fermions from the fundamental representation. The partition function of the theory is given above in Eq. (22). The notions of covariant derivative $D_\mu \equiv \partial_\mu - iA_\mu$ and Dirac operator $Q(x, -iD)$ are defined in Secs. II A and II B. The “inhomogeneity” in this model is contained in the explicit x dependence in $Q(x, -iD)$.

In operator notation, the action can be written as

$$S = \text{tr}_D \text{tr}_G \text{tr}_H (\hat{Q} \hat{\rho}), \quad (73)$$

where $\hat{Q} := Q(\hat{x}, \hat{\pi})$ [recall $\hat{\pi} := \hat{p} - A(\hat{x})$] and $\langle x | \hat{\rho} | y \rangle := \psi(x) \bar{\psi}(y)$, and tr_D , tr_G , and tr_H are traces with respect to the spinor indices (and the other internal indices, excluding the gauge ones), gauge indices, and the Hilbert space, respectively.

We note the following exact relationship:

$$Q_W(x, p) \equiv (\hat{Q})_W(x, p) = Q(x, p), \quad (74)$$

which one can verify by taking the Weyl transform of $Q(x, p)$ using [97]. This formula means that Q_W is independent of the gauge field and acts as the identity matrix on the fundamental representation.

B. Vector current

Consider the following non-Abelian, local gauge transformation:

$$\psi(x) \rightarrow e^{i\alpha(x)} \psi(x), \quad (75)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{-i\alpha(x)}, \quad (76)$$

where $\alpha(x)$ belongs to the Lie algebra.

The variation of the action under an infinitesimal version of (75) and (76) is

$$\delta S = \text{tr}_D \text{tr}_G \text{tr}_H (\delta \hat{Q} \hat{\rho}), \quad (77)$$

$$\delta \hat{Q} = [\hat{Q}, i\alpha(\hat{x})]. \quad (78)$$

Using the trace, this can be written as

$$\delta S = -\text{tr}_D \text{tr}_G \text{tr}_H (i\alpha(\hat{x}) [\hat{Q}, \hat{\rho}]). \quad (79)$$

Using the fact that $\text{tr}_G \text{tr}_H (\hat{X} \hat{Y}) = \text{tr}_G \text{tr}_\Gamma (X_W Y_W)$ [where $\text{tr}_\Gamma \equiv \int d^4x \int (2\pi)^{-4} d^4p$ is the trace with respect to the phase space], we get

$$\delta S = \text{tr}_G \int d^4x \alpha(x) \Gamma(x), \quad (80)$$

$$\Gamma(x) = -i \text{tr}_D \int \frac{d^4p}{(2\pi)^4} (Q \star \rho_W - \rho_W \star Q). \quad (81)$$

Using (63) (the definition of the star) and integrating over p we get

$$\Gamma(x) = -i \text{tr}_D \int (2\pi)^{-8} d^4y d^4k d^4k' e^{-iy(k-k')} U(x, x + y/2) \times (Q(x + y/2, k) \rho_W(x + y/2, k') - \rho_W(x + y/2, k) Q(x + y/2, k')) U(x + y/2, x). \quad (82)$$

Using (54), we can write (82) as

$$\Gamma(x) = -i\text{tr}_D \int (2\pi)^{-8} d^4 y d^4 k d^4 k' e^{-iy(k-k')} e^{y\mathcal{D}/2} (Q(x, k) \rho_W(x, k') - \rho_W(x, k) Q(x, k')). \quad (83)$$

Integrating by parts we get

$$\Gamma(x) = -i\text{tr}_D \int (2\pi)^{-8} d^4 y d^4 k d^4 k' e^{-iy(k-k')} (e^{-i\partial_k \mathcal{D}/2} (Q(x, k) \rho_W(x, k')) - e^{i\partial_{k'} \mathcal{D}/2} (\rho_W(x, k) Q(x, k'))). \quad (84)$$

Expanding in powers of the covariant derivative we get

$$\Gamma = +2i\text{tr}_D \int \frac{d^4 p}{(2\pi)^4} \sum_{k=0}^{\infty} \frac{(i/2)^{2k+1}}{(2k+1)!} [\Pi_{i=1\dots 2k+1} \mathcal{D}_{\mu_i}] (\rho_W [\Pi_{i=1\dots 2k+1} \partial_{p_{\mu_i}}] Q) \quad (85)$$

and

$$\Gamma = \mathcal{D}_{\mu} J_{\mu} \quad (86)$$

$$J_{\mu} = -\text{tr}_D \int \frac{d^4 p}{(2\pi)^4} \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (2k+1)!} [\Pi_{i=1\dots 2k} \mathcal{D}_{\mu_i}] (\rho_W [\Pi_{i=1\dots 2k} \partial_{p_{\mu_i}}] \partial_{p_{\mu}} Q). \quad (87)$$

The expression for the vector current is simplified considerably in two important cases: if

$$\partial_{p_{\mu_1}} \dots \partial_{p_{\mu_n}} Q(x, p) = 0 \quad \text{for } n \geq 3 \quad (88)$$

or if we can neglect

$$\frac{1}{4^k (2k+1)!} \mathcal{D}_{\mu_1} \dots \mathcal{D}_{\mu_{2k}} \partial_{p_{\mu_1}} \dots \partial_{p_{\mu_{2k}}} \partial_{p_{\mu_0}} Q(x, p) \quad \text{for } k \geq 1, \quad (89)$$

compared to $\partial_{p_{\mu_0}} Q(x, p)$ because the inhomogeneity is sufficiently weak, i.e., the higher derivatives of the background fields entering the Dirac operator may be neglected.⁴ In both these cases, we arrive at

$$J_{\mu} = -\text{tr}_D \int \frac{d^4 p}{(2\pi)^4} \rho_W \partial_{p_{\mu}} Q. \quad (90)$$

We can come to the same formula in a different way. Namely, let us express the action as

$$\delta S = -i\text{tr}_D \text{tr}_G \text{tr}_H (\hat{\rho} [\alpha(\hat{x}), \hat{Q}]). \quad (91)$$

Using the fact that $\text{tr}_G \text{tr}_H (\hat{X} \hat{Y}) = \text{tr}_G \text{tr}_\Gamma (X_W Y_W)$ [where $\text{tr}_\Gamma \equiv \int d^4 x \int (2\pi)^{-4} d^4 p$ is the trace with respect to the phase space], we get

$$\delta S = -i\text{tr}_D \text{tr}_G \int d^4 x \frac{d^4 p}{(2\pi)^4} \rho_W(x, p) (\alpha(x) \star Q(x, p) - Q(x, p) \star \alpha(x)). \quad (92)$$

Using (63) (the definition of the star) we get

⁴In order to better understand the nature of the second case mentioned above, let us consider the situation in which $Q(x, p) \sim f(x) \frac{\sin(pa)}{a}$. Here, the function $f(x)$ models the background fields, while the parameter a represents the scale specific to the Dirac operator (if the background fields were constant). Then, Eq. (108) gives $\frac{d^{2k}}{d^2 x} f(x) a^{2k} \sin(pa)$, which is to be compared with $f(x) \sin(pa)$. This way, we arrive at the condition $\frac{d^{2k}}{d^2 x} f(x) a^{2k} \ll f(x)$, i.e., that the function $f(x)$ varies slowly enough that its variation at the distance of the order of a is to be neglected.

$$\begin{aligned}
\delta S &= -i \text{tr}_D \text{tr}_G \int d^4x \frac{d^4p}{(2\pi)^4} \rho_W(x, p) \left((2\pi)^{-2D} \int dy dk dy' dk' e^{-iy(k-p) - iy'(k'-p)} \right. \\
&\quad \times U(x, x - (y + y')/2) U(x - (y + y')/2, x - y'/2) \alpha(x - y'/2) U(x - y'/2, x + (y - y')/2) \\
&\quad \times U(x + (y - y')/2, x + y/2) Q(x + y/2, k') U(x + y/2, x + (y + y')/2) U(x + (y + y')/2, x) - (\alpha(x) \leftrightarrow Q(x, p)) \Big) \\
&= -i \text{tr}_D \text{tr}_G \int d^4x \frac{d^4p}{(2\pi)^4} \rho_W(x, p) \left((2\pi)^{-D} \int dy' dk' e^{-iy'(k'-p)} \right. \\
&\quad \times U(x, x - y'/2) \alpha(x - y'/2) U(x - y'/2, x) Q(x, k') - (\alpha(x) \leftrightarrow Q(x, p)) \Big) \\
&= -i \text{tr}_D \text{tr}_G \int d^4x \frac{d^4p}{(2\pi)^4} \rho_W(x, p) (2\pi)^{-D} \int dy' dk' e^{-iy'(k'-p)} ([e^{-y'^D/2} \alpha(x)] Q(x, k') - Q(x, k') [e^{y'^D/2} \alpha(x)]). \tag{93}
\end{aligned}$$

Now let us assume that $\alpha(x)$ is varying smoothly, so that we can neglect the derivatives higher than the first ones. Then, we arrive at

$$\begin{aligned}
\delta S &= -i \text{tr}_D \text{tr}_G \int d^4x \alpha(x) \mathcal{D}_\mu \int \frac{d^4p}{(2\pi)^4} (2\pi)^{-D} \int dy' dk' e^{-iy'(k'-p)} \rho_W(x, p) y'^\mu Q(x, k') \\
&= + \text{tr}_D \text{tr}_G \int d^4x \alpha(x) \mathcal{D}_\mu \int \frac{d^4p}{(2\pi)^4} (2\pi)^{-D} \int dy' dk' \partial_{k'_\mu} e^{-iy'(k'-p)} \rho_W(x, p) Q(x, k') \\
&= - \text{tr}_D \text{tr}_G \int d^4x \alpha(x) \mathcal{D}_\mu \int \frac{d^4p}{(2\pi)^4} \rho_W(x, p) \partial_{p_\mu} Q(x, p). \tag{94}
\end{aligned}$$

The expectation value of the current is

$$\langle J_\mu \rangle = - \text{tr}_D \int \frac{d^4p}{(2\pi)^4} G_W \partial_{p_\mu} Q, \tag{95}$$

where $G_W := \langle \rho_W \rangle$. We also have $G_W = (\hat{G})_W$, where $\hat{G} = \hat{Q}^{-1}$, i.e.,

$$\hat{Q} \hat{G} = 1. \tag{96}$$

C. Axial current

Now consider the following non-Abelian, local chiral transformation:

$$\psi(x) \rightarrow e^{i\alpha(x)\gamma^5} \psi(x), \tag{97}$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{i\alpha(x)\gamma^5}, \tag{98}$$

where $\alpha(x)$ is in the Lie algebra.

The variation of the action under an infinitesimal version of (97) and (98) is

$$\delta S = \text{tr}_D \text{tr}_G \text{tr}_H (\delta \hat{Q} \hat{\rho}), \tag{99}$$

$$\delta \hat{Q} = \{\hat{Q}, i\alpha(\hat{x})\gamma^5\}. \tag{100}$$

Using the trace, this can be written as

$$\delta S = \text{tr}_D \text{tr}_G \text{tr}_H (i\alpha(\hat{x})\gamma^5 \{\hat{Q}, \hat{\rho}\}). \tag{101}$$

Using the fact that $\text{tr}_G \text{tr}_H(\hat{X} \hat{Y}) = \text{tr}_G \text{tr}_\Gamma(X_W Y_W)$ [where $\text{tr}_\Gamma \equiv \int d^4x \int (2\pi)^{-4} d^4p$ is the trace with respect to the phase space], we get

$$\delta S = \text{tr}_G \int d^4x \alpha(x) \Gamma(x), \quad (102)$$

$$\Gamma(x) = i \text{tr}_D \gamma^5 \int \frac{d^4p}{(2\pi)^4} (Q \star \rho_W + \rho_W \star Q). \quad (103)$$

Using (63) (the definition of the star) and integrating over p we get

$$\begin{aligned} \Gamma(x) = i \text{tr}_D \gamma^5 \int (2\pi)^{-8} d^4y d^4k d^4k' e^{-iy(k-k')} U(x, x+y/2) \\ \times (Q(x+y/2, k) \rho_W(x+y/2, k') + \rho_W(x+y/2, k) Q(x+y/2, k')) U(x+y/2, x). \end{aligned} \quad (104)$$

Using (54), we can write (104) as

$$\Gamma(x) = i \text{tr}_D \gamma^5 \int (2\pi)^{-8} d^4y d^4k d^4k' e^{-iy(k-k')} e^{y\mathcal{D}/2} (Q(x, k) \rho_W(x, k') + \rho_W(x, k) Q(x, k')). \quad (105)$$

Integrating by parts we get

$$\Gamma(x) = i \text{tr}_D \gamma^5 \int (2\pi)^{-8} d^4y d^4k d^4k' e^{-iy(k-k')} (e^{-i\partial_k \mathcal{D}/2} (Q(x, k) \rho_W(x, k')) + e^{i\partial_{k'} \mathcal{D}/2} (\rho_W(x, k) Q(x, k'))). \quad (106)$$

Expanding in powers of the covariant derivative and assuming that $Q(x, p)$ is such that

$$\partial_{p_{\mu_1}} \dots \partial_{p_{\mu_n}} Q(x, p) = 0 \quad \text{for } n \geq 3 \quad (107)$$

or

$$\left| \frac{1}{4^k (2k+1)!} \mathcal{D}_{\mu_1} \dots \mathcal{D}_{\mu_{2k}} \partial_{p_{\mu_1}} \dots \partial_{p_{\mu_{2k}}} \partial_{p_{\mu_0}} Q(x, p) \right| \ll |\partial_{p_{\mu_0}} Q(x, p)| \quad \text{for } k \geq 1, \quad (108)$$

we get

$$\Gamma = i \text{tr}_D \gamma^5 \int \frac{d^4p}{(2\pi)^4} \{Q, \rho_W\} + \mathcal{D}_\mu J_\mu^{(5)}, \quad (109)$$

$$J_\mu^{(5)} = \frac{1}{2} \text{tr}_D \gamma^5 \int \frac{d^4p}{(2\pi)^4} (\partial_{p_\mu} Q \rho_W - \rho_W \partial_{p_\mu} Q). \quad (110)$$

The same expression appears if we assume that function $\alpha(x)$ varies slowly.

Let us suppose for a moment that $\{\gamma^5, Q\} = 0$ (i.e., classical chiral symmetry—it is actually broken by regularization). Then, we get

$$\Gamma = \mathcal{D}_\mu J_\mu^{(5)}, \quad (111)$$

$$J_\mu^{(5)} = -\text{tr}_D \gamma^5 \int \frac{d^4p}{(2\pi)^4} \rho_W \partial_{p_\mu} Q. \quad (112)$$

When the equations of motion are satisfied, then $\delta S = 0$, which, since $\alpha(x)$ is arbitrary, leads to

$$\Gamma(x) = 0 \quad (\text{when equations of motion (EOMs) are satisfied}). \quad (113)$$

Using (111), we get the classical conservation law

$$\mathcal{D}_\mu J_\mu^{(5)} = 0 \quad (\text{when EOMs are satisfied}). \quad (114)$$

This equation is broken due to the chiral anomaly, which becomes obvious when an ultraviolet regularization that does not respect chiral symmetry is applied. The expectation value of the current is

$$\langle J_\mu^{(5)} \rangle = -\frac{1}{2} \text{tr}_D \int \frac{d^4 p}{(2\pi)^4} G_W \partial_{p_\mu} [Q, \gamma^5]. \quad (115)$$

Recall that if the Dirac operator contains extra indices (in addition to the Dirac spinor index and the index of the given gauge group of the field A), we assume that the symbol tr_D contains both these indices and the spinor ones.

V. ITERATIVE SOLUTION OF THE GENERALIZED GROENEWOLD EQUATION

The Wigner transform of the identity operator is 1. Taking the Wigner transform of (96) and using (63) and (74) then gives

$$\begin{aligned} 1 &= (2\pi)^{-2D} \int dy dk dy' dk' e^{-iy(k-p)-iy'(k'-p)} Q(x-y'/2, k) \\ &\quad \times U(x, x-(y+y')/2) U(x-(y+y')/2, x-y'/2) U(x-y'/2, x+(y-y')/2) \\ &\quad \times U(x+(y-y')/2, x+y/2) G_W(x+y/2, k') U(x+y/2, x+(y+y')/2) U(x+(y+y')/2, x). \end{aligned} \quad (116)$$

To distinguish between the x dependence arising from the inhomogeneity vs from the gauge field, we introduce the function $G_W(x, p, z)$ (the z dependence arises from the gauge field while the x dependence from the inhomogeneity), which is the solution of

$$\begin{aligned} 1 &= (2\pi)^{-2D} \int dy dk dy' dk' e^{-iy(k-p)-iy'(k'-p)} Q(x-y'/2, k) \\ &\quad \times U(z, z-(y+y')/2) U(z-(y+y')/2, z-y'/2) U(z-y'/2, z+(y-y')/2) \\ &\quad \times U(z+(y-y')/2, z+y/2) G_W(x+y/2, k', z+y/2) U(z+y/2, z+(y+y')/2) U(z+(y+y')/2, z). \end{aligned} \quad (117)$$

Comparing with (116), it is easy to see that $G_W(x, p) = G_W(x, p, x)$. Note that $Q(x, p)$ has no z dependence. Using (67), we rewrite (117) as

$$1 = (2\pi)^{-2D} \int dy dk dy' dk' e^{-iy(k-p)-iy'(k'-p)} Q(x-y'/2, k) e^{-\frac{1}{2}(y+y')D_z} e^{yD_z} e^{\frac{y'}{2}D_z} G_W(x+y/2, k', z) e^{\frac{y'}{2}D_z} e^{-\frac{1}{2}(y+y')D_z} \times 1, \quad (118)$$

where D_z is the gauge covariant derivative with respect to z —it acts on the right—and 1 means the function of z that is equal to 1 and thus does not depend on z at all. We will expand (118) in powers of D_z and expand G_W as

$$G_W(x, p, z) = \sum_{n \geq 0} G^{(n)}(x, p, z), \quad (119)$$

where $G^{(n)}(x, p, z)$ contains n powers of D_z . We will suppose that $G^{(0)}(x, p, z) \equiv G^{(0)}(x, p)$ is independent of $A(z)$, so we can take it to be proportional to the identity matrix in gauge space.

In what follows, we will make use of the fact that the triangular Wilson loop

$$\begin{aligned} &U(z, z-(y+y')/2) U(z-(y+y')/2, z+(y-y')/2) U(z+(y-y')/2, z+(y+y')/2) U(z+(y+y')/2, z) \\ &= e^{-\frac{1}{2}(y+y')D_z} e^{yD_z} e^{y'D_z} e^{-\frac{1}{2}(y+y')D_z} \times 1 \end{aligned} \quad (120)$$

can be expanded in powers of D_z as

$$1 - \frac{i}{2} y^\mu y'^\nu F_{\mu\nu}(z) - \frac{i}{12} (y^\alpha - y'^\alpha) y^\mu y'^\nu \mathcal{D}_\alpha F_{\mu\nu} - \frac{i}{48} (y^\alpha y^\beta + y'^\alpha y'^\beta) y^\mu y'^\nu \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu}(z) - \frac{1}{8} y^\mu y'^\nu y^\alpha y'^\beta F_{\mu\nu}(z) F_{\alpha\beta}(z) + \mathcal{O}(D_z^5). \quad (121)$$

Now, to lowest order in D_z , Eq. (118) becomes

$$Q \star G^{(0)} = 1, \quad (122)$$

where $\star = e^{\frac{i}{2}(\vec{\partial}_x \vec{\partial}_p - \vec{\partial}_p \vec{\partial}_x)}$ is the ordinary star product. Expanding (118) to order 1 in D_z proves that $G^{(1)} = 0$. Expanding to order 2 gives

$$0 = Q \star G^{(2)} + \frac{i}{2} \partial_{p_\mu} Q \star \partial_{p_\nu} G^{(0)} F_{\mu\nu}(z), \quad (123)$$

where we have made use of the fact that $[D_\mu, D_\nu] = -iF_{\mu\nu}$, whose solution is

$$\begin{aligned} G^{(2)}(x, p, z) \\ = -\frac{i}{2} G^{(0)}(x, p) \star \partial_{p_\mu} Q(x, p) \star \partial_{p_\nu} G^{(0)}(x, p) F_{\mu\nu}(z). \end{aligned} \quad (124)$$

We will define

$$\Gamma_{\mu\nu} := -\frac{i}{4} G^{(0)} \star \partial_{p_\mu} Q \star \partial_{p_\nu} G^{(0)}. \quad (125)$$

So,

$$G_W = G^{(0)} + \Gamma_{\mu\nu} F_{\mu\nu} + O(D^3). \quad (126)$$

VI. AXIAL CURRENT IN PAULI-VILLARS REGULARIZATION AND NON-ABELIAN CSE

A. General considerations

In this paper, we use the Pauli-Villars regularization (for the detailed definitions, see above in Sec. II C). The grand thermodynamic potential is usually defined as $\Omega = -T \log Z$ (where T is temperature). In the given regularization, this expression is modified as

$$\Omega = -T \sum_i C_i \log Z(M_i). \quad (127)$$

Here, $Z(M_i)$ is calculated as (22) with Q substituted by $Q_{(M_i)}(i\partial_p, p - A(i\partial_p))$, such that the nonzero parameter M_i provides the appearance of a gap (i.e., a mass for the fermion). The coefficients C_i obey the conditions of Eq. (35). These conditions provide the cancellation of ultraviolet divergencies. All macroscopic quantities may be calculated as combinations of Ω and its derivatives with respect to various parameters. Following the previous steps, we come, then, to the conclusion that in the regularized theory, the axial current is

$$J_\mu^{(5)a} \equiv -\frac{1}{2} \sum_i C_i \int_{\mathcal{M}} \frac{dp}{(2\pi)^4} \text{tr}_D \text{tr}_G (G_{(M_i),W} \partial_{p_\mu} [Q_{(M_i)}, t^a \gamma^5]). \quad (128)$$

Here, t^a is the generator of the gauge group under consideration. We substitute here (126) and obtain

$$J_\mu^{(5)a} = S_\mu(x) \text{tr}_G t^a + M_{\mu\alpha\beta}(x) \text{tr}_G (t^a F_{\alpha\beta}), \quad (129)$$

$$S_\mu(x) := \frac{1}{2} \sum_i C_i \int \frac{dp}{(2\pi)^4} \text{tr}_D ([\gamma^5, \partial_{p^\mu} Q] G^{(0)}), \quad (130)$$

$$M_{\mu\alpha\beta}(x) := \frac{1}{2} \sum_i C_i \int \frac{dp}{(2\pi)^4} \text{tr}_D ([t^a \gamma^5, \partial_{p^\mu} Q] \Gamma_{\alpha\beta}). \quad (131)$$

We use (124) and obtain

$$\begin{aligned} M_{\mu\alpha\beta}(x) := & -\frac{i}{8} \sum_i C_i \int \frac{dp}{(2\pi)^4} \text{tr}_D \\ & \times ([\gamma^5, \partial_{p^\mu} Q] G^{(0)} \star \partial_{p^\alpha} Q \star \partial_{p^\beta} G^{(0)}). \end{aligned} \quad (132)$$

We define the conductivity of the non-Abelian chiral separation effect as the response of the spatial components of the axial current to both the external magnetic field strength and to the chemical potential,

$$\frac{\partial}{\partial \mu} J_i^{\text{CSE}} \equiv \frac{\partial}{\partial \mu} M_{ijk} \epsilon_{jkl} B_l. \quad (133)$$

Chemical potential appears as the shift of p_4 : $ip_4 \rightarrow ip_4 + \mu$. Therefore,

$$\begin{aligned} \frac{\partial}{\partial \mu} M_{ijk}(x) = & -\frac{1}{8} \sum_i C_i \int \frac{dp}{(2\pi)^4} \frac{\partial}{\partial p_4} \text{tr}_D \\ & \times ([\gamma^5, \partial_{p^i} Q] G^{(0)} \star \partial_{p^j} Q \star \partial_{p^k} G^{(0)}). \end{aligned} \quad (134)$$

Due to the presence of poles in the expression standing inside the integral, the integral is not zero, but it is reduced to integrals over the hyperplanes Σ_{p_4} for $p_4 = \pm\epsilon, \pm\infty$, $\epsilon \rightarrow 0$,

$$\begin{aligned} \frac{\partial}{\partial \mu} M_{ijk}(x) = & -\frac{1}{8} \sum_i C_i \sum_{p_4=\pm\epsilon, \pm\infty} \int_{\Sigma_{p_4}} \frac{dp_1 dp_2 dp_3}{(2\pi)^4} \text{tr}_D \\ & \times ([\gamma^5, \partial_{p^i} Q] G^{(0)} \star \partial_{p^j} Q \star \partial_{p^k} G^{(0)}). \end{aligned} \quad (135)$$

At this point we use the following properties of the given system: (i) it is chiral close to the Fermi surface, i.e., at low energies (this does not concern the Pauli-Villars massive fields), and (ii) it is chiral in the ultraviolet, i.e., at large momenta (this concerns also the Pauli-Villars fields). Using the properties of (35) we come to the conclusion that the contributions of the integrals at $\Sigma_{\pm\infty}$ cancel each other, the Pauli-Villars regulators disappear from the above expression, and we are left with the integral over $\Sigma_{\pm\epsilon}$ for the original massless field only. The latter integrals can be reduced to integrals over the hypersurfaces $p_4 = \pm\epsilon$ that

enclose the poles of the expression standing inside the integral,

$$\frac{\partial}{\partial \mu} \langle J_i^{(CSE)} \rangle = \sigma_{CSE} B_i. \quad (136)$$

Here, $B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$ is the color-magnetic field, averaging on the left-hand side is over the whole system volume, while

$$\sigma_{CSE} = \frac{N_3}{2\pi^2} \quad (137)$$

and

$$N_3 = -\frac{1}{V48\pi^2} \sum_{p_4=\pm\epsilon} \int d^3x \int_{\Sigma_{p_4}} \text{tr}_D \times (G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ \gamma^5). \quad (138)$$

The expression in the above expression is a topological invariant due to the presence of integration over the three-dimensional momenta $\vec{p}$ and coordinates $\vec{x}$. There exists also the representation of Eq. (138) in the form of an integral over the compact hypersurface.⁵

⁵In particular, we can represent

$$N_3 = -\frac{1}{V48\pi^2} \int d^3x \int_{\Sigma_0} \text{tr}_D (\tilde{G}^{(0)} \star d\tilde{Q} \star \wedge d\tilde{G}^{(0)} \star \wedge d\tilde{Q} \gamma^5).$$

In transition from Eq. (138) to this expression, we perform a smooth modification of the Dirac operator: $Q((\vec{p}, \pm\epsilon), x) \rightarrow Q((\vec{p}, \pm\omega(\vec{p})), x) \equiv \tilde{Q}(\vec{p}, \vec{x})$. Here, $\omega(\vec{p})$ is a function, which describes the dependence of p_4 on $\vec{p}$ on the hypersurface Σ_0 . The latter hypersurface in momentum space for any x embraces the singularities of the expression standing inside the integral. For the case of a homogeneous system, these singularities appear along the Fermi surface. In the inhomogeneous case, the geometrical place in momentum space of such singularities depends on x and generalizes the notion of the Fermi surface. In turn, in transition from Eq. (138) we deform also the Green function as $G^{(0)}(\vec{p}, \pm\epsilon, \vec{x}) \rightarrow \tilde{G}^{(0)}(\vec{p}, \vec{x})$, where the latter function obeys the *transformed* Groenewold equation

$$\tilde{Q}(\vec{p}, \vec{x}) e^{\frac{i}{2}(\vec{\partial}_x \vec{\partial}_p - \vec{\partial}_p \vec{\partial}_x)} \tilde{G}^{(0)}(\vec{p}, \vec{x}) = 1.$$

The given smooth modifications $Q \rightarrow \tilde{Q}$ and $G^{(0)} \rightarrow \tilde{G}^{(0)}$ do not result in a jump of the value of N_3 because defined this way, it is the topological invariant, embracing the singularities of expression standing in the integral in Eq. (138). Notice that in [105,106], such a transformation of the two hyperplanes to hypersurface Σ_0 of more general form was mentioned as well. However, this was not done with sufficient care—the above-mentioned understanding of the Moyal product and the proper modification of the Groenewold equation for $\tilde{G}^{(0)}$ were not mentioned. Apart from this, the results of [105,106] for the Abelian CSE remain correct.

B. Illustrative examples

1. Massless Dirac fermions in the presence of background $U(1)$ gauge field and background vierbein

For the purpose of illustration let us consider first the model of massless Dirac fermions ψ in the presence of external $SU(N)$ gauge field A and background fields, the $U(1)$ gauge field C and vierbein e_a^μ .

Here, we consider the systems with Dirac operator $\hat{Q}$ of the form

$$\hat{Q} = \gamma^a \{ |e| e_a^\mu (\hat{p}_\mu - A_\mu - C_\mu) \} / 2 - m |e|. \quad (139)$$

The vierbein $e_a^\mu(x)$ depends on coordinates, and $|e| = \det^{-1}[e_a^\mu]$ is the inverse determinant of matrix with components $e_a^\mu(x)$. This is the general form of the action for Dirac fermions in the presence of Riemann-Cartan gravity (with vanishing spin connection but with nonzero torsion) and $U(1)$ gauge field B . For the Pauli-Villars regulators, the values of m are finite and are assumed to be large, while the original Dirac operator contains vanishing mass $m = 0$. Consider the case when the background fields are constant, and the Wigner transform of $\hat{Q}$ (at $A = 0$) can be calculated and decomposed in terms of the gamma matrices as

$$Q(p) = \gamma^a |e| e_a^\mu (p_\mu - C_\mu) - m |e| = -\tilde{m} + \ell,$$

$$\text{where } \ell_a = |e| e^{b\mu} (\delta_{ab} (p_\mu - C_\mu)), \quad \tilde{m} = m |e|. \quad (140)$$

The inverse is, therefore,

$$G^{(0)}(p) = Q^{-1}(p) = c + \not{r}$$

$$c = X^{-1} \tilde{m}$$

$$r_a = X^{-1} \ell_a,$$

$$\text{where } X = \tilde{m}^2 + \ell^2. \quad (141)$$

The conductivity of the non-Abelian chiral separation effect is given by Eq. (137) with N_3 of Eq. (138) given by $N_3 = 1$ if the vierbein e_a^μ is nondegenerate. For the derivation, see [106]. The calculation of Eq. (138) is reduced to that of [106] when matrix e_a^μ is continuously deformed to a unit matrix. If the number of Dirac fermions is multiplied and equals n , then $N_3 = n$. Since the value of N_3 is a topological invariant, its value is not changed when the background fields C and e are slowly varying.

2. N_f Dirac fermions interacting with dynamical $SU(N_c)$ gauge fields

Let us now consider the more involved case, when $C \in SU(N_c)$ is dynamical gauge field while the number of the flavors of the Dirac fermions that experienced the background field A and the dynamical field C is N_f (that is, the total number of Dirac fermions is $N \times N_c \times N_f$).

We consider the case when $N \times N_f > 11N_c/2$, in which the theory does not possess chiral symmetry breaking [135]. (Say, we can take $N_c = 2$, $N = 3$, $N_f = 4$.) We also assume weak coupling of fermions to C . In this situation, again, the topological invariant can be calculated using Eq. (138), where we insert the interacting Green function instead of $G^{(0)}$, and its inverse instead of Q . Again, using the technique of [106], we obtain $N_3 = N_c \times N_f$.

VII. VECTOR CURRENT AND NON-ABELIAN QHE

A. Non-Abelian QHE in 3+1D systems

For the vector current, the above procedure may be modified accordingly, and we obtain, instead of (129),

$$J_\mu = S_\mu^v(x) + M_{\mu\alpha\beta}^v(x) F_{\alpha\beta} \quad (142)$$

with

$$M_{\mu\alpha\beta}^{(v)}(x) := \frac{i}{4} \sum_i C_i \int \frac{dp}{(2\pi)^4} \text{tr}_D (\partial_{p^\mu} Q G^{(0)} \star \partial_{p^\alpha} Q \star \partial_{p^\beta} G^{(0)})$$

$$S_\mu^v(x) = \sum_i C_i \int \frac{dp}{(2\pi)^4} \text{tr}_D (\partial_{p^\mu} Q G^{(0)}). \quad (143)$$

According to the Bloch theorem [136], the first term in (142) for $\mu \neq 0$ vanishes because it is averaged over the spatial volume (total electric current vanishes in equilibrium), and we are left with the response of the spatial current to the external field strength,

$$J_\mu = \epsilon_{\nu\mu\alpha\beta} M^{*(v)\nu}(x) F_{\alpha\beta} \quad (144)$$

with

$$M^{*(v)\nu}(x) := \frac{i}{2 \times 3!} \epsilon^{\nu\mu\alpha\beta} \sum_i C_i \int \frac{dp}{(2\pi)^4} \text{tr}_D$$

$$\times (\partial_{p^\mu} Q G^{(0)} \star \partial_{p^\alpha} Q \star \partial_{p^\beta} G^{(0)}). \quad (145)$$

Let us consider the case of the external field varying slowly. The response of spatial components of the vector current (averaged over the system volume) to non-Abelian *electric* field strength (with nonzero $F_{4i} = -iE_i$ components) receives the form

$$\bar{J}_i^{v,QHE} = -\frac{1}{4\pi^2} \epsilon_{ijk} \tilde{M}^{*(v)j} E_k \quad (146)$$

with

$$\tilde{M}^{*(v)i} = 2iM^{*(v)i} = -\frac{1}{V24\pi^2} \sum_i C_i \int d^3x \wedge dp^i \wedge \text{tr}_D$$

$$\times (G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ). \quad (147)$$

In this expression, $d^3x \equiv dx^1 \wedge dx^2 \wedge dx^3$, the integral is over the product of three-dimensional spatial coordinate space and four-dimensional momentum space, and the sum over Pauli-Villars regulators is important in order to cancel the ultraviolet divergencies. In particular, in this way, the divergent contribution of the Dirac sea is cancelled. At the same time, the vector current is an observable quantity, and its value cannot depend on the regularization. Below, we will consider particular examples of the models that possess non-Abelian QHE. We will see that in all considered cases, the meaning of regulators in Eq. (156) is only to subtract the divergent contribution of the Dirac sea (if any). We, therefore, are able to simplify the above expression,

$$\tilde{M}^{*(v)i} = 2iM^{*(v)i} = -\frac{1}{V24\pi^2}$$

$$\times \left[\int d^3x \wedge dp^i \wedge \text{tr}_D (G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ) \right]_{\text{reg}}. \quad (148)$$

Here, subscript “reg” means that the divergent contribution of the Dirac sea is to be subtracted from the considered quantity. This may particularly be achieved if we add the sum over the Pauli-Villars regulators, as in Eq. (157).

For the density of non-Abelian charge, we obtain in a similar way the response to the external *magnetic* non-Abelian field B_i ,

$$\bar{\rho}^v = iJ_4 = \frac{1}{4\pi^2} \tilde{M}^{*(v)i} B_i. \quad (149)$$

In addition to Eqs. (149) and (146), one can derive the response of the non-Abelian current to the (magnetic) field strength,

$$\bar{J}_i^{v(B)} = \frac{1}{4\pi^2} \tilde{M}^{*(v)4} B_i. \quad (150)$$

Notice that $\tilde{M}^{*(v)4}$ is the same quantity that enters the expression for ordinary electric current. Due to the Bloch theorem, the latter is absent in equilibrium. Therefore, $\tilde{M}^{*(v)4}$ should vanish as well as the current of (150).

It is worth mentioning that if the fermions under consideration are electrically charged, the vector $\tilde{M}^{*(v)i}$ is related to the intrinsic magnetic moment $\mathcal{M}$ of the fermion system (see Appendix B and⁶ Sec. 3 of [132]),

$$\frac{1}{4\pi^2} \tilde{M}^{*(v)i} = \frac{\partial}{\partial \mu} \mathcal{M}_i. \quad (151)$$

⁶Notice that in [132], there is a mistake in the sign of the magnetic moment.

Therefore, we have

$$\begin{aligned}\vec{J}^{v,QHE} &= \left[\vec{E} \times \frac{\partial}{\partial \mu} \vec{\mathcal{M}} \right], \\ \rho^{v,QHE} &= \left(\vec{B} \frac{\partial}{\partial \mu} \vec{\mathcal{M}} \right).\end{aligned}\quad (152)$$

B. Non-Abelian QHE in 2+1D

Let us consider the 2 + 1D gapped system. Again, as in 3 + 1D, for the vector current we obtain, as in (146),

$$J_\mu^v = S_\mu^v(x) + M_{\mu\alpha\beta}^v(x) F_{\alpha\beta} \quad (153)$$

with

$$\begin{aligned}M_{\mu\alpha\beta}^v(x) &:= \frac{i}{4} \sum_i C_i \int \frac{dp}{(2\pi)^3} \text{tr}_D (\partial_{p^\mu} Q G^{(0)} \star \partial_{p^\alpha} Q \star \partial_{p^\beta} G^{(0)}), \\ S_\mu^v(x) &= \sum_i C_i \int \frac{dp}{(2\pi)^3} \text{tr}_D (\partial_{p^\mu} Q G^{(0)}).\end{aligned}\quad (154)$$

The response of the vector current (spatial components averaged over the system volume) to the non-Abelian *electric* field strength is (for the case of external field varying slowly)

$$\vec{J}_i^{v,QHE} = \frac{1}{2\pi} \epsilon_{ij} M_3^v E_j \quad (155)$$

with

$$\begin{aligned}M_3^v &= -\frac{1}{S 24\pi^2} \sum_i C_i \int dx^1 \wedge dx^2 \wedge \text{tr}_D \\ &\times (G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ).\end{aligned}\quad (156)$$

Here, S is the area of the sample assumed to be large. Integration is over the product of the two-dimensional spatial part of coordinate space and three-dimensional momentum space. As above, in the case of the four-dimensional system, the only meaning of the regulator fields is to subtract the divergent unphysical contribution (if any) of the Dirac sea. That is why we formulate Eq. (156) as

$$M_3^v = -\frac{1}{S 24\pi^2} \left[\int d^2x \wedge \text{tr}_D (G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ) \right]_{\text{reg}}, \quad (157)$$

where $d^2x \equiv dx^1 \wedge dx^2$, and the subscript “reg” means the abovementioned subtraction of an unphysical contribution. One can see that in our approach, this subtraction appears automatically if one uses the Pauli-Villars regularization. However, the same result, of course, will be valid in any other regularization. In particular, in lattice regularization there is the peculiar mechanism of the subtraction of the Dirac sea contributions. For the details in the applications to graphene, see [137].

The response of the non-Abelian charge density to external *magnetic* field strength B is

$$\bar{\rho}^v = \frac{1}{2\pi} M_3^v B. \quad (158)$$

In a similar way to that of Appendix B, one can prove that the magnetic moment $\mathcal{M}$ of the 2D system is related to M_3^v as

$$\frac{\partial \mathcal{M}}{\partial \mu} = \frac{1}{2\pi} M_3^v. \quad (159)$$

Therefore, we represent the Hall current and the (non-Abelian) charge density as

$$\begin{aligned}\vec{J}^{v,QHE} &= {}^* \vec{E} \frac{\partial}{\partial \mu} \mathcal{M}, \\ \rho^{v,QHE} &= B \frac{\partial}{\partial \mu} \mathcal{M},\end{aligned}\quad (160)$$

where $*$ is the 2D Hodge duality operation.

C. Illustrative examples

1. Two-dimensional systems

- (1) Let us consider the case of a two-dimensional nonrelativistic system. Fermionic fields ψ interact with external non-Abelian $SU(N)$ gauge field A , and are in the presence of the Abelian $U(1)$ gauge field C . The field strength of C is of the magnetic type, i.e., we suppose that the only nonzero components of $C_{ij} = \partial_i C_j - \partial_j C_i$ are those with $i, j = 1, 2$, and that $C_{12} = H = \text{const}$. We then consider $\hat{Q}$ of the form

$$\hat{Q} = -\partial_\tau + \mu - A_0 - C_0 + \frac{\vec{D}^2}{2M}, \quad \vec{D} = \vec{\partial} - i\vec{A} - i\vec{C}. \quad (161)$$

Here, $\tau = x^3$ is an “imaginary” time coordinate. By $\vec{D}$ we denote the two-dimensional (spatial) covariant

derivative. M is the mass of the fermions, while μ is chemical potential. Following the calculations presented in [133] (Appendices E–G), we obtain that the value of M_3^v does not need a regularization (i.e., it remains convergent), and we obtain

$$M_3^v = -\frac{1}{S24\pi^2} \int d^2x \wedge \text{tr}_D \times (G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ) = n, \quad (162)$$

where n is the number of occupied Landau levels (of the field H), i.e., the number of those levels situated below μ .

- (2) Let us modify the case considered above adding Coulomb interactions between fermions with a weak coupling constant. Weak interactions within perturbation theory cannot change the value of topological invariant M_3^v , as was proven in [138]. We come to the conclusion that the system exists in the integer QHE states with the same value of M_3^v as without interactions. Notice that the nonperturbative effects of interactions may result in the fractional QHE, also in the non-Abelian case. However, we do not consider this possibility here.
- (3) Now let us consider the example in which the regularization in Eq. (156) is important. Namely, we will be interested in the 2D system with relativistic symmetry. As above, the fermionic fields ψ interact with external non-Abelian $SU(N)$ gauge field A , and are in the presence of the nontrivial vierbein field e_a^μ and the Abelian $U(1)$ gauge field C . The field strength of C is again of the magnetic type, i.e., the only nonzero components of the field strength $C_{ij} = \partial_i C_j - \partial_j C_i$ are those with $i, j = 1, 2$, and $C_{12} = H = \text{const}$. Operator $\hat{Q}$ has the form

$$\begin{aligned} \hat{Q} &= \frac{1}{2} \sigma^a (|e| e_a^\mu D_\mu + D_\mu |e| e_a^\mu) - m|e|, \\ \vec{D} &= \vec{\partial} - i\vec{A} - i\vec{C}, \\ D_3 &= \partial_3 - A_0 - C_0 + \mu. \end{aligned} \quad (163)$$

Here, $\tau = x^3$ is an “imaginary” time coordinate. By $\vec{D}$ we denote the two-dimensional (spatial) covariant derivative. m is the mass of the fermions, while μ is chemical potential. Consider first the case of constant vierbein. Again, following the calculations presented in [133] (Appendices E–G), we obtain

$$M_3^v = -\frac{1}{S24\pi^2} \int d^2x \wedge \text{tr}_D \times (G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ) = n. \quad (164)$$

Now, n is the number of the occupied Landau levels (of the field H), i.e., the number of those

levels situated below μ . However, the number of those levels is always infinite and, therefore, the above expression is divergent. When we take into account the Pauli-Villars regulators, the value of M_3^v becomes

$$M_3^v = -\frac{1}{S24\pi^2} \sum_i C_i \int d^2x \wedge \text{tr}_D \times (G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ) = \sum_i C_i n_i. \quad (165)$$

Here, n_i ($i = 1, 2, \dots$) is the number of occupied Landau levels for the Pauli-Villars regulators with masses $M_i \gg m$, while $n_0 = n$ is the number of occupied Landau levels for the field ψ itself. Suppose that $\mu > 0$. The number of Landau levels situated below zero does not depend on M_i, m (and thus on i) for $i = 0, 1, \dots$. Because of the condition $\sum_i C_i = 0$, their contributions to M_3^v cancel each other. At the same time, the number of occupied Landau levels above zero for the regulators vanishes. We come to the conclusion that

$$M_3^v = n_{>0},$$

where $n_{>0}$ is the number of occupied levels of ψ situated above zero. One can easily find that for $\mu < 0$, the answer will be

$$M_3^v = n_{<0},$$

where $n_{<0}$ is the number of vacant Landau levels of ψ situated below zero.

- (4) The case considered in the previous item may be complicated by adding the slow varying vierbein e_a^μ and the Coulomb interactions between the fermions with a weak coupling constant. As above, using the results of [138] we come to the conclusion that the results of Sec. III are not modified in this case as well.

2. 3 + 1D systems

- (1) Now let us turn to the 3 + 1D system. We start from the effective low-energy field theory of Weyl semimetals. In the proposed model, the fermionic fields ψ interact with external non-Abelian $SU(N)$ gauge field A . We do not concretize here the origin of this field. Let us just notice that it might appear as an emergent gauge field caused by elastic deformations. We then consider $\hat{Q}$ of the form

$$\begin{aligned}\hat{Q} &= \gamma^\mu iD_\mu + \gamma^\mu \gamma^5 b_\mu, \\ \vec{D} &= \vec{\partial} - i\vec{A}, \\ D_4 &= \partial_4 + A_0 - \mu.\end{aligned}\quad (166)$$

Here, x^4 is an “imaginary” time coordinate. By $\vec{D}$ we denote the three-dimensional (spatial) covariant derivative. μ is chemical potential, while vector $b_\mu = (\vec{b}, 0)$ does not depend on coordinates. Vector $\tilde{M}^{*(v)i}$ of Eq. (156) may be calculated using the formalism developed in [133], see also [139]. Namely, we can represent

$$\tilde{M}^{*(v)i} = \sum_j C_j \int dp_i \tilde{N}_3^{(j)(i)}(P) \quad (167)$$

with

$$\begin{aligned}\tilde{N}_3^{(j)(i)}(P) &= -\frac{1}{V24\pi^2} \int d^3x \\ &\times \int_{\Sigma^{(i)}(P)} \text{tr}_D(G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ).\end{aligned}\quad (168)$$

Here, the integral is over the hyperplane $\Sigma^{(i)}(P)$ in momentum space orthogonal to the i th axis. The i th component of the coordinates of all points on this hyperplane are equal to P . Without loss of generality, let us consider the case when vector $\vec{b}$ is directed along the third axis. The value of $\tilde{N}_3^{(0)(3)}(P)$ may be calculated easily using the technique developed in [133]. It is given by

$$\tilde{N}_3^{(0)(3)}(P) = \begin{cases} 1, & P \in (-|\vec{b}|, |\vec{b}|) \\ 0, & \text{otherwise.} \end{cases} \quad (169)$$

The other components of $\tilde{N}_3^{(j)(i)}(P)$ vanish (including those that correspond to the Pauli-Villars regulators). We arrive at the conclusion that

$$\tilde{M}^{*(v)i} = 2b_i$$

and the non-Abelian QHE results in vector current

$$\vec{J}_i^{v(B)} = -\frac{\epsilon_{ijk}}{2\pi^2} b_j E_k. \quad (170)$$

- (2) Now let us consider the case of the system similar to that considered above, but with an extra external $U(1)$ gauge field C of magnetic type (i.e., $C_{ij} = \frac{1}{2}\epsilon_{ijk}H_k$, where H is the magnetic field). Operator $\hat{Q}$ now has the form

$$\begin{aligned}\hat{Q} &= \gamma^\mu iD_\mu - M_j, \\ \vec{D} &= \vec{\partial} - i\vec{A} - i\vec{C}, \\ D_4 &= \partial_4 - \mu + A_0 + C_0.\end{aligned}\quad (171)$$

Here, M_j is the mass parameter (its value M for the Pauli-Villars regulators is assumed to be large, its value $M_0 = m$ for the original fermions may be small), and κ is constant. For simplicity, let us consider the case when $\vec{H} \parallel \vec{b}$ while $b^4 = 0$. In this model, the Landau levels are formed. The component of momentum along $\vec{b}$ remains a good quantum number. We represent again

$$\tilde{M}^{*(v)i} = \sum_j C_j \int dp_i \tilde{N}_3^{(j)(i)}(P) \quad (172)$$

with

$$\begin{aligned}\tilde{N}_3^{(j)(i)}(P) &= -\frac{1}{V24\pi^2} \int d^3x \\ &\times \int_{\Sigma^{(i)}(P)} \text{tr}_D(G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ).\end{aligned}\quad (173)$$

As above, we direct the third axis along vector $\vec{b}$. Then for the vierbein not depending on coordinates, the values of $\tilde{N}_3^{(j)(3)}(P)$ may be calculated within an auxiliary 2 + 1D system as the topological invariant

$$\begin{aligned}\tilde{N}_3^{(j)(3)}(P) &= -\frac{1}{S24\pi^2} \int d^2x \\ &\times \int \text{tr}_D(\tilde{G} \star d\tilde{Q} \star \wedge d\tilde{G} \star \wedge d\tilde{Q})\end{aligned}\quad (174)$$

with the 2 + 1D Dirac operator of the form

$$\begin{aligned}\hat{Q} &= \gamma^\mu iD_\mu - \gamma^3 P - M, \quad \vec{D} = \vec{\partial} - i\vec{C}, \\ D_4 &= \partial_4 - \mu, \quad \mu = 1, 2, 4.\end{aligned}\quad (175)$$

Its Weyl symbol obeys $\tilde{Q}_W \star \tilde{G}_W = 1$. Here, P plays the role of parameter. Energy levels of the n th level are given by

$$\begin{aligned}E_0^{(j)} &= \pm \sqrt{P^2 + M_j^2}, \\ E_n^{(j)} &= \text{sign}(n) \sqrt{P^2 + M_j^2 + 2|n||B|}, \quad n \in \mathbb{Z}, n \neq 0.\end{aligned}\quad (176)$$

The topological invariant responsible for the QHE is equal to the number of occupied Landau levels [to those with $E^{(j)}(P) \leq \mu$],

$$\tilde{N}_3^{(j)(3)}(P) = \sum_n \theta(\mu - E_n^{(j)}(P)). \quad (177)$$

The components of $\tilde{N}_3^{(j)(i)}(P)$ with $i \neq 3$ vanish. For the purpose of illustration, let us consider the case $0 < \mu \ll \sqrt{2|H|}$ and $m = 0$. Then, the contributions of Landau levels with negative energy for all j are equal (and divergent). These divergent contributions to $\tilde{M}^{*(v)3}$ cancel each other due to the condition $\sum_j C_j = 0$. We arrive at the contribution of the upper branch of the lowest Landau level of the original fermion for $P \in (-\mu, +\mu)$,

$$\tilde{M}^{*(v)3} = 2\mu. \quad (178)$$

Notice that the expression of Eq. (177) is equal to the number of the occupied energy levels situated below the chemical potential. This is the integer number, which is robust to the smooth modification of the system. However, the total expression for the $3 + 1$ D QHE conductivity of the considered system is given by Eq. (178). It is given by an integral of the topological invariant of Eq. (177) times dP . The result is proportional to the chemical potential, and is not topological invariant.

VIII. CONCLUSIONS AND DISCUSSION

In the present paper, we proposed a new version of Wigner-Weyl calculus. It extends the constructions of [106,133,138] to the case when the Wigner transformed two-point fermion Green function belongs to the adjoint representation of the gauge group. In turn, the latter construction utilizes the notion of the covariant Weyl symbol of the operator proposed a long time ago in [134]. Basic properties of the so-constructed Wigner-Weyl calculus remain similar to those of the Wigner-Weyl calculus of [36,106,133]. However, there are important differences that allow us to express responses of vector and axial currents to the non-Abelian field strength through topological invariants.

Using the developed machinery, we calculated responses of axial and vector non-Abelian currents to the external non-Abelian field strength. The former led us to the non-Abelian version of the CSE, while the latter resulted in the non-Abelian version of the QHE. We considered systems with Dirac operator $\hat{Q}$ of rather general form. An important property of the systems considered is their inhomogeneity, which may be caused by spatially varying external fields, including those that represent the macroscopic motion of the system [140]. The obtained results for the non-Abelian CSE and non-Abelian QHE formally repeat the previously known results [106,132,133] on the corresponding Abelian effects. In particular, the conductivities of these phenomena are expressed through the topological invariants that are

robust to smooth modifications of the systems. Thus, in the present paper we extend the calculation of conductivities of the CSE and QHE to the non-Abelian systems using the developed machinery of covariant Wigner-Weyl calculus. We considered several examples of particular models, in which the discussed effects are present. Within these models we calculated the corresponding conductivities, which illustrate our general findings.

The more complicated systems with (in general) non-local and inhomogeneous Dirac operators $\hat{Q}$ may be considered as well. The only restriction on such systems is that the fermion action is to be quadratic in the fermion field. In order to consider even more complicated theories, an interaction with extra fields may be introduced, and the latter fields are to be treated as background ones.

One can see that our expressions for the non-Abelian QHE and CSE look identical to those of the Abelian effects. However, there is the huge difference behind this. In order to prove that the resulting expressions for the CSE and QHE conductivity are so simple, we had to develop the complicated machinery presented in Secs. III–V. Technically, the difference between the Abelian and non-Abelian cases is important. To give an analogy, one can say that this difference is as the difference between the motion of a point particle in special relativity (Minkowski space-time) and general relativity (Riemannian spacetime). The action for the particle looks the same (mass times geodesic distance along the worldline). However, behind this there is a nontrivial metric in the case of Riemannian geometry, and the presence of Christoffel symbols in the equation of motion. To proceed, the machinery of Secs. III–V is the main result of the present paper. Its complexity and beauty is the main reason for writing the present paper. Notice, however, that a certain piece of an Abelian version of this formalism was developed earlier in [128]. In Secs. VI and VII, we present only two particular fields where this machinery may be applied. As expected, there should be many other fields of mathematical physics where it can be applied as well. However, already here we see that the results obtained with the aid of the developed formalism are strikingly simple in spite of the complexity of the considered problems. We obtain that the conductivities of QHE in $2 + 1$ D and CSE in $3 + 1$ D are given by the topological invariants, while the conductivity of the $3 + 1$ D QHE is given by an expression that itself might not be a topological invariant, but is related intimately to the topological invariants (similar to the case of the Abelian QHE in Weyl semimetals—see Sec. VII C 2).

One of the motivations of the present research was to look for the topological meaning of the topological invariant N_3 of Eq. (138). It enters expression for the conductivity of the chiral separation effect (both Abelian and non-Abelian). We investigated the case of background fields that may remain inhomogeneous at infinity and, therefore, carry nontrivial topology. Namely, the response of the axial current to the external magnetic field and to the

chemical potential is given by the same topological invariant of (138). It is expected that this non-Abelian CSE is related to the non-Abelian chiral anomaly in a similar way to the relation between the ordinary CSE and the Abelian chiral anomaly. Establishment of this relation, however, remains out of the scope of the present paper, and will be considered in a forthcoming publication.

Apart from the academic interest, the proposed non-Abelian CSE may appear in real high-energy physics. A situation when the non-Abelian magnetic field is present is realized, in particular, inside the non-Abelian vortices. Such vortices exist, in particular, in quark matter in the hypothetical color superconductor phases (see Refs. [141,142] and references therein). The closed strings containing color-magnetic flux exist in quark matter in these phases. The non-Abelian CSE then predicts the existence of a non-Abelian axial current along such strings. This may cause certain secondary effects that might play a role in the physics of quark matter at large baryon chemical potentials.

The second row of (152) predicts the appearance of a non-Abelian charge density in the case when both the color-magnetic field and intrinsic magnetic moment coexist. This may take place inside non-Abelian vortices in the color-superconductor phase when an intrinsic magnetic moment is present in quark matter. The latter may be caused, say, by the ordinary magnetic field coexisting within the non-Abelian vortices with the color-magnetic field. It may also be caused by the macroscopic motion of quark matter.

In addition to investigation of the non-Abelian axial currents, we considered the response of the vector non-Abelian current to the external (electric) non-Abelian gauge field. In 3 + 1D systems, this response is given by an expression, which is given by an integral of a topological invariant, and it results in the non-Abelian version of the QHE. The emergent non-Abelian gauge fields can occur in condensed matter systems [10]—including both ultracold atoms and real solids, see also [143–145] and references therein. In particular, the emergent non-Abelian gauge fields may appear in Dirac/Weyl semimetals in the presence of elastic deformations. The illustrative example presented in Sec. VII C 2 (item 1) demonstrates that in such systems, the non-Abelian QHE is possible.

The two-dimensional non-Abelian QHE may, in principle, be observed in the 2D Weyl semimetals (see Ref. [145] and references therein).

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: EUCLIDEAN CONVENTIONS

In the present paper, we use the following conventions concerning vector indices. In relativistic theory, there are two types of indices: upper and lower. For the gauge

potential, these two indices are related by

$$A^\mu = \eta^{\mu\nu} A_\nu, \quad \mu, \nu = 0, 1, 2, 3,$$

where $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Consequently, lowering of spatial Minkowski space indices is followed by a change of sign, while lowering of the 0th component is not followed by a change of sign.

In Euclidean space, lowering and lifting of indices does not result in any change of the corresponding components of a vector,

$$A^i = A_i, \quad i = 1, 2, 3, 4.$$

According to our conventions, Euclidean components of a vector are related to their Minkowskian components as follows:

$$A^i = A_i = A^\mu = -A_\mu, \quad \mu = i = 1, 2, 3,$$

but

$$A^4 = A_4 = -iA^0 = -iA_0$$

in spite of the following relation between components of the coordinates:

$$x^4 = x_4 = +ix^0 = +ix_0.$$

This is because the Matsubara frequency $\omega = p^4$ is related to the energy $\mathcal{E} = p^0$ as $p_4 = \omega = -i\mathcal{E} = -ip_0$. With this identification, we obtain for the Dirac operator of a simple system with Lagrangian depending on momentum only

$$Q(\omega, \vec{p}) = i\omega - \mathcal{H}(\vec{p}),$$

with the plus sign in front of $i\omega$. Then,

$$Q(p_4 - A_4, \vec{p} - \vec{A}) = i\omega - A_0 - \mathcal{H}(\vec{p} - \vec{A}),$$

as it should.

In particular, the above conventions are valid for components of the four-vector of electromagnetic potential. For the field strength F we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu.$$

We define Euclidean field strength using a similar expression,

$$F^{ij} = F_{ij} = -F_{\mu\nu} = \frac{\partial}{\partial x^i} A_j - \frac{\partial}{\partial x^j} A_i,$$

$$\mu = i = 1, 2, 3, \quad \nu = j = 1, 2, 3.$$

Correspondingly, $F_{ij} = \epsilon_{ijk} B_k$, where $\vec{B}$ is the magnetic field.

Mixed spatial-time components are

$$F^{4j} = F_{4j} = \frac{\partial}{\partial x^4} A_j - \frac{\partial}{\partial x^j} A_4 \quad (\text{A1})$$

$$= i\partial_{x^0} A_\nu + i\partial_{x^\nu} A_0, \quad \nu = j = 1, 2, 3, \quad (\text{A2})$$

and $F_{4j} = -iE_j$ since we do not consider a vector potential depending on time.

APPENDIX B: MAGNETIC MOMENT AND TOPOLOGICAL INVARIANT RESPONSIBLE FOR THE QHE

We can define the magnetic moment of the system of charged fermions as the response of the thermodynamical potential to the external $U(1)$ magnetic field. We assume here that the external electric $U(1)$ field is absent. At the same time, the given $U(1)$ is assumed to be a subgroup of the gauge group considered in the main text. We do not consider in this appendix the Pauli-Villars regularization. Extension of the results obtained below to the case when this regularization is added is straightforward.

Let us calculate the response of the partition function to the chemical potential using the developed Wigner-Weyl formalism,

$$\begin{aligned} \delta \log Z &= -i \int d^4x \frac{d^4p}{(2\pi)^4} \text{tr}_D \text{tr}_G G_W \partial_{p_4} Q \delta\mu \\ &= -i \int d^4x \frac{d^4p}{(2\pi)^4} \text{tr}_D \text{tr}_G G_W(x, p) \star \partial_{p_4} Q(x, p) \delta\mu. \end{aligned} \quad (\text{B1})$$

In the second line of the above expression, we introduced the star product.

Next, we calculate the response to the external $U(1)$ field strength $\delta\mathcal{F}_{ij} = \epsilon_{ijk} \mathcal{B}_k$, where $\mathcal{B}$ is the $U(1)$ magnetic field. This results in

$$\delta \log Z = \frac{1}{2} \epsilon_{ijk} \int d^4x \frac{d^4p}{(2\pi)^4} \text{tr}_D G_W^{(0)} \star \partial_{p_i} Q \star G_W^{(0)} \star \partial_{p_j} Q \star G_W^{(0)} \star \partial_{p_k} Q \star G_W^{(0)} \star \partial_{p_4} Q \delta\mu \delta\mathcal{B}_k.$$

The integration over imaginary time x^4 goes from 0 to inverse temperature $1/T \rightarrow \infty$. The response of the thermodynamical potential to variation of the $U(1)$ magnetic field is given by

$$\delta\Omega = -\vec{\mathcal{M}} \delta\vec{\mathcal{B}}.$$

The relation between the thermodynamical potential Ω and partition function $\log Z = -\Omega/T$ allows us to derive an expression for the derivative of the magnetic moment $\vec{\mathcal{M}}$ with respect to the chemical potential,

$$\frac{\partial}{\partial\mu} \mathcal{M}^i = \frac{1}{2V} \epsilon_{ijk} \int d^3r \frac{d^4p}{(2\pi)^4} \text{tr}_D G_W^{(0)} \star \partial_{p_j} Q \star G_W^{(0)} \star \partial_{p_k} Q \star G_W^{(0)} \star \partial_{p_4} Q. \quad (\text{B2})$$

Above, V is the overall volume of the sample. We can rewrite this expression as follows:

$$\frac{\partial}{\partial\mu} \mathcal{M}^i = \frac{1}{4\pi^2} \tilde{M}^{*(v)i}, \quad (\text{B3})$$

where $\tilde{M}^{*(v)i}$, given by Eq. (156), is a topological quantity in the case that the integrand is free of singularities.

- [1] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall conductance in a two-dimensional periodic potential, *Phys. Rev. Lett.* **49**, 405 (1982).
- [2] J. E. Avron, R. Seiler, and B. Simon, Homotopy and quantization in condensed matter physics, *Phys. Rev. Lett.* **51**, 51 (1983).
- [3] Eduardo Fradkin, *Field Theories of Condensed Matter Physics* (Addison Wesley Publishing Company, Reading, MA, 1991).
- [4] Y. Hatsugai, Topological aspects of the quantum Hall effect, *J. Phys. Condens. Matter* **9**, 2507 (1997).
- [5] Xiao-Liang Qi, Taylor L. Hughes, and Shou-Cheng Zhang, Topological field theory of time-reversal invariant insulators, *Phys. Rev. B* **78**, 195424 (2008).
- [6] Ralph M. Kaufmann, Dan Li, and Birgit Wehefritz-Kaufmann, Notes on topological insulators, *Rev. Math. Phys.* **28**, 1630003 (2016).
- [7] David Tong, Lectures on the quantum Hall effect, [arXiv:1606.06687](https://arxiv.org/abs/1606.06687).
- [8] K. Ishikawa and T. Matsuyama, Magnetic field induced multi component QED in three-dimensions and quantum Hall effect, *Z. Phys. C* **33**, 41 (1986).
- [9] G. E. Volovik, An analog of the quantum Hall effect in a superfluid ^3He film, *Sov. Phys. JETP* **67**, 9 (1988) [*Zh. Eksp. Teor. Fiz.* **94**, 123 (1988)].
- [10] G. E. Volovik, *The Universe in a Helium Droplet* (Clarendon Press, Oxford, 2003).
- [11] T. Matsuyama, Quantization of conductivity induced by topological structure of energy momentum space in generalized QED in three-dimensions, *Prog. Theor. Phys.* **77**, 711 (1987).
- [12] M. Z. Hasan and C. L. Kane, Topological insulators, *Rev. Mod. Phys.* **82**, 3045 (2010).
- [13] Xiao-Liang Qi and Shou-Cheng Zhang, Topological insulators and superconductors, *Rev. Mod. Phys.* **83**, 1057 (2011).
- [14] G. E. Volovik, Topology of quantum vacuum, [arXiv:1111.4627](https://arxiv.org/abs/1111.4627).
- [15] G. E. Volovik, Quantum phase transitions from topology in momentum space, in *Quantum Analogues: From Phase Transitions to Black Holes and Cosmology*, edited by W. G. Unruh and R. Schutzhold, Springer Lecture Notes in Physics Vol. 718 (Springer, New York, 2007), pp. 31–73.
- [16] G. E. Volovik, Topological invariants for standard model: From semi-metal to topological insulator, *JETP Lett.* **91**, 55 (2010).
- [17] V. Gurarie, Single-particle Green's functions and interacting topological insulators, *Phys. Rev. B* **83**, 085426 (2011).
- [18] A. M. Essin and V. Gurarie, Bulk-boundary correspondence of topological insulators from their Green's functions, *Phys. Rev. B* **84**, 125132 (2011).
- [19] G. E. Volovik, Topological superfluids, *J. Exp. Theor. Phys.* **129**, 618 (2019).
- [20] H. B. Nielsen and M. Ninomiya, Absence of neutrinos on a lattice: (II). Intuitive topological proof, *Nucl. Phys.* **B193**, 173 (1981).
- [21] H. B. Nielsen and M. Ninomiya, Absence of neutrinos on a lattice: (I). Proof by homotopy theory, *Nucl. Phys.* **B185**, 20 (1981).
- [22] H. So, Induced topological invariants by lattice fermions in odd dimensions, *Prog. Theor. Phys.* **74**, 585 (1985).
- [23] D. B. Kaplan, Method for simulating chiral fermions on the lattice, *Phys. Lett. B* **288**, 342 (1992).
- [24] M. F. L. Golterman, K. Jansen, and D. B. Kaplan, Chern-Simons currents and Chiral fermions on the lattice, *Phys. Lett. B* **301**, 219 (1993).
- [25] P. Hořava, Stability of Fermi surfaces and K -theory, *Phys. Rev. Lett.* **95**, 016405 (2005).
- [26] Michael Creutz, Four-dimensional graphene and chiral fermions, *J. High Energy Phys.* **04** (2008) 017.
- [27] David B. Kaplan and Sichun Sun, Spacetime as a topological insulator: Mechanism for the origin of the fermion generations, *Phys. Rev. Lett.* **108**, 181807 (2012).
- [28] S. Coleman and B. Hill, No more corrections to the topological mass term in QED_3 , *Phys. Lett.* **159B**, 184 (1985).
- [29] T. Lee, The absence of radiative corrections from higher-order loops to the topological mass in $(2 + 1)$ -dimensional electrodynamics, *Phys. Lett. B* **171**, 247 (1986).
- [30] C. X. Zhang and M. A. Zubkov, Influence of interactions on the anomalous quantum Hall effect, *J. Phys. A* **53**, 195002 (2021).
- [31] Ryogo Kubo, Hiroshi Hasegawa, and Natsuki Hashitsume, Quantum theory of galvanomagnetic effect. I. Basic considerations, *J. Phys. Soc. Jpn.* **14**, 56 (1959).
- [32] Qian Niu, D. J. Thouless, and Yong-Shi Wu, Quantized Hall conductance as a topological invariant, *Phys. Rev. B* **31**, 3372 (1985).
- [33] B. L. Altshuler, D. Khmel'nitzkii, A. I. Larkin, and P. A. Lee, Magnetoresistance and Hall effect in a disordered two-dimensional electron gas, *Phys. Rev. B* **22**, 5142 (1980).
- [34] B. L. Altshuler and A. G. Aronov, *Electron-Electron Interaction in Disordered Systems*, edited by A. L. Efros and M. Pollak (Elsevier, North Holland, Amsterdam, 1985).
- [35] C. X. Zhang and M. A. Zubkov, Hall conductivity as the topological invariant in the phase space in the presence of interactions and a nonuniform magnetic field, *JETP Lett.* **110**, 487 (2019).
- [36] C. X. Zhang and M. A. Zubkov, Influence of interactions on integer quantum Hall effect, *Ann. Phys. (N.Y.)* **444**, 169016 (2022).
- [37] M. A. Zubkov and Xi Wu, Topological invariant in terms of the Green functions for the quantum Hall effect in the presence of varying magnetic field, *Ann. Phys. (N.Y.)* **418**, 168179 (2020).
- [38] M. Suleymanov, M. Selch, M. A. Zubkov, and C. X. Zhang, Hall conductivity as the topological invariant in magnetic Brillouin zone, *Phys. Rev. B* **107**, 245105 (2023).
- [39] H. Weyl, Quantenmechanik und Gruppentheorie, *Z. Phys.* **46**, 1 (1927).
- [40] E. P. Wigner, On the quantum correction for thermodynamic equilibrium, *Phys. Rev.* **40**, 749 (1932).
- [41] H. J. Groenewold, On the principles of elementary quantum mechanics, *Physica (Utrecht)* **12**, 405 (1946).
- [42] J. E. Moyal, Quantum mechanics as a statistical theory, *Math. Proc. Cambridge Philos. Soc.* **45**, 99 (1949).
- [43] S. T. Ali and M. Englis, Quantization methods: A guide for physicists and analysts, *Rev. Math. Phys.* **17**, 391 (2005).

- [44] F. A. Berezin and M. A. Shubin, in *Colloquia Mathematica Societatis Janos Bolyai* (North-Holland, Amsterdam, 1972), p. 21.
- [45] T. L. Curtright and C. K. Zachos, Quantum mechanics in phase space, *Asia Pac. Phys. Newsl.* **1**, 37 (2012).
- [46] C. Zachos, D. Fairlie, and T. Curtright, *Quantum Mechanics in Phase Space* (World Scientific, Singapore, 2005).
- [47] L. Cohen, Generalized phase-space distribution functions, *J. Math. Phys. (N.Y.)* **7**, 781 (1966).
- [48] G. S. Agarwal and E. Wolf, Calculus for functions of noncommuting operators and general phase-space methods in quantum mechanics. I. Mapping theorems and ordering of functions of noncommuting operators, *Phys. Rev. D* **2**, 2161 (1970).
- [49] E. C. G. Sudarshan, Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams, *Phys. Rev. Lett.* **10**, 277 (1963).
- [50] R. J. Glauber, Coherent and incoherent states of the radiation field, *Phys. Rev.* **131**, 2766 (1963).
- [51] Kôdi Husimi, Some formal properties of the density matrix, *Proc. Phys. Math. Soc. Jpn.* **24**, 264 (1940).
- [52] K. E. Cahill and R. J. Glauber, Ordered expansions in boson amplitude operators, *Phys. Rev.* **177**, 1882 (1969).
- [53] Felix A. Buot, *Nonequilibrium Quantum Transport Physics in Nanosystems* (World Scientific, Singapore, 2009).
- [54] C. Lorce and B. Pasquini, Quark Wigner distributions and orbital angular momentum, *Phys. Rev. D* **84**, 014015 (2011).
- [55] H. T. Elze, M. Gyulassy, and D. Vasak, Transport equations for the QCD quark Wigner operator, *Nucl. Phys. B* **706**, 276 (1986).
- [56] F. Hebenstreit, R. Alkofer, and H. Gies, Schwinger pair production in space and time-dependent electric fields: Relating the Wigner formalism to quantum kinetic theory, *Phys. Rev. D* **82**, 105026 (2010).
- [57] E. Calzetta, S. Habib, and B. L. Hu, Quantum kinetic field theory in curved space-time: Covariant Wigner function and Liouville-Vlasov equation, *Phys. Rev. D* **37**, 2901 (1988).
- [58] C. Bastos, O. Bertolami, N. Costa Dias, and J. Nuno Prata, Weyl-Wigner formulation of noncommutative quantum mechanics, *J. Math. Phys. (N.Y.)* **49**, 072101 (2008).
- [59] O. F. Dayi and L. T. Kelleyane, Wigner functions for the Landau problem in noncommutative spaces, *Mod. Phys. Lett. A* **17**, 1937 (2002).
- [60] S. Habib and R. Laflamme, Wigner function and decoherence in quantum cosmology, *Phys. Rev. D* **42**, 4056 (1990).
- [61] S. Chapman and U. W. Heinz, HBT correlators: Current formalism versus Wigner function formulation, *Phys. Lett. B* **340**, 250 (1994).
- [62] M. V. Berry, Semi-classical mechanics in phase space: A study of Wigner's function, *Phil. Trans. R. Soc. A* **287**, 0145 (1977).
- [63] Julian Schwinger, Unitary operator bases, *Proc. Natl. Acad. Sci. U.S.A.* **46**, 570 (1960).
- [64] Felix A. Buot, Method for calculating $\text{Tr}H^n$ in solid-state theory, *Phys. Rev. B* **10**, 3700 (1974).
- [65] Felix A. Buot, Quantum superfield theory and lattice Weyl transform in nonequilibrium quantum transport physics, *Quantum Matter* **2**, 247 (2013).
- [66] William K. Wootters, A Wigner-function formulation of finite-state quantum mechanics, *Ann. Phys. (N.Y.)* **176**, 1 (1987).
- [67] Ulf Leonhardt, Quantum-state tomography and discrete Wigner function, *Phys. Rev. Lett.* **74**, 4101 (1995).
- [68] P. Kasperkovitz and M. Peev, Wigner-Weyl formalisms for toroidal geometries, *Ann. Phys. (N.Y.)* **230**, 21 (1994).
- [69] Marilena Ligabò, Torus as phase space: Weyl quantization, dequantization, and Wigner formalism, *J. Math. Phys. (N.Y.)* **57**, 082110 (2016).
- [70] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, Deformation theory and quantization. I. Deformations of symplectic structures, *Ann. Phys. (N.Y.)* **111**, 61 (1978).
- [71] Maxim Kontsevich, Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* **66**, 157 (2003).
- [72] Giovanni Felder and Boris Shoikhet, Deformation quantization with traces, *Lett. Math. Phys.* **53**, 75 (2000).
- [73] V. G. Kupriyanov and D. V. Vassilevich, Star products made (somewhat) easier, *Eur. Phys. J. C* **58**, 627 (2008).
- [74] M. A. Zubkov and Z. V. Khaidukov, Topology of the momentum space, Wigner transformations, and a chiral anomaly in lattice models, *JETP Lett.* **106**, 166 (2017) [*Pis'ma Zh. Eksp. Teor. Fiz.* **106**, 166 (2017)].
- [75] M. N. Chernodub and M. A. Zubkov, Scale magnetic effect in quantum electrodynamics and the Wigner-Weyl formalism, *Phys. Rev. D* **96**, 056006 (2017).
- [76] Z. V. Khaidukov and M. A. Zubkov, Chiral separation effect in lattice regularization, *Phys. Rev. D* **95**, 074502 (2017).
- [77] M. A. Zubkov, Momentum space topology of QCD, *Ann. Phys. (N.Y.)* **393**, 264 (2018).
- [78] M. A. Zubkov, Absence of equilibrium chiral magnetic effect, *Phys. Rev. D* **93**, 105036 (2016).
- [79] I. V. Fialkovsky and M. A. Zubkov, Elastic deformations and Wigner-Weyl formalism in graphene, *Symmetry* **12**, 317 (2020).
- [80] M. A. Zubkov, Discrete Wigner-Weyl calculus for the finite lattice, *J. Phys. A* **56**, 395201 (2023).
- [81] Max A. Metlitski and Ariel R. Zhitnitsky, Anomalous axion interactions and topological currents in dense matter, *Phys. Rev. D* **72**, 045011 (2005).
- [82] D. E. Kharzeev, The chiral magnetic effect and anomaly-induced transport, *Prog. Part. Nucl. Phys.* **75**, 133 (2014).
- [83] D. E. Kharzeev, J. Liao, S. A. Voloshin, and G. Wang, Chiral magnetic effect in high-energy nuclear collisions—A status report, *Prog. Part. Nucl. Phys.* **88**, 1 (2016).
- [84] D. E. Kharzeev, Chern-Simons current and local parity violation in hot QCD matter, *Nucl. Phys. A* **830**, 543 (2009).
- [85] L. P. Csernai, V. K. Magas, and D. J. Wang, Flow vorticity in peripheral high energy heavy ion collisions, *Phys. Rev. C* **87**, 034906 (2013).
- [86] V. A. Miransky and I. A. Shovkovy, Quantum field theory in a magnetic field: From quantum chromodynamics to graphene and Dirac semimetals, *Phys. Rep.* **576**, 1 (2015).
- [87] K. Landsteiner, E. Megias, and F. Pena-Benitez, Anomalous transport from Kubo formulae, *Lect. Notes Phys.* **871**, 433 (2013).
- [88] S. Parameswaran, T. Grover, D. Abanin, D. Pesin, and A. Vishwanath, Probing the chiral anomaly with nonlocal

- transport in Weyl semimetals, *Phys. Rev. X* **4**, 031035 (2014).
- [89] E. V. Gorbar, V. A. Miransky, I. A. Shovkovy, and P. O. Sukhachov, Chiral separation and chiral magnetic effects in a slab: The role of boundaries, *Phys. Rev. B* **92**, 245440 (2015).
- [90] V. A. Miransky and I. A. Shovkovy, Quantum field theory in a magnetic field: From quantum chromodynamics to graphene and Dirac semimetals, *Phys. Rep.* **576**, 1 (2015).
- [91] S. N. Valgushev, M. Puh, and P. V. Buividovich, Chiral magnetic effect in finite-size samples of parity-breaking Weyl semimetals, [arXiv:1512.01405](https://arxiv.org/abs/1512.01405).
- [92] P. V. Buividovich, M. Puh, and S. N. Valgushev, Chiral magnetic conductivity in an interacting lattice model of parity-breaking Weyl semimetal, *Phys. Rev. B* **92**, 205122 (2015).
- [93] P. V. Buividovich, Spontaneous chiral symmetry breaking and the chiral magnetic effect for interacting Dirac fermions with chiral imbalance, *Phys. Rev. D* **90**, 125025 (2014).
- [94] P. V. Buividovich, Anomalous transport with overlap fermions, *Nucl. Phys.* **B925**, 218 (2014).
- [95] K. Fukushima and T. Hatsuda, The phase diagram of dense QCD, *Rep. Prog. Phys.* **74**, 014001 (2011).
- [96] A. V. Smilga, Physics of thermal QCD, *Phys. Rep.* **291**, 106 (1997).
- [97] K. Rajagopal and F. Wilczek, The condensed matter physics of QCD, in *At The Frontier of Particle Physics* (World Scientific, Singapore, 2001), pp. 2061–2151.
- [98] D. H. Rischke, The quark-gluon plasma in equilibrium, *Prog. Part. Nucl. Phys.* **52**, 197 (2004).
- [99] M. G. Alford, A. Schmitt, K. Rajagopal, and T. Schafer, Color superconductivity in dense quark matter, *Rev. Mod. Phys.* **80**, 1455 (2008).
- [100] R. S. Hayano and T. Hatsuda, Hadron properties in the nuclear medium, *Rev. Mod. Phys.* **82**, 2949 (2010).
- [101] M. Huang, QCD phase diagram at high temperature and density, [arXiv:1001.3216](https://arxiv.org/abs/1001.3216).
- [102] J. O. Andersen, W. R. Naylor, and A. Tranberg, Phase diagram of QCD in a magnetic field: A review, *Rev. Mod. Phys.* **88**, 025001 (2016).
- [103] K. Fukushima and C. Sasaki, The phase diagram of nuclear and quark matter at high baryon density, *Prog. Part. Nucl. Phys.* **72**, 99 (2013).
- [104] A. R. Zhitnitsky, QCD as a topologically ordered system, *Ann. Phys. (N.Y.)* **336**, 462 (2013).
- [105] M. Suleymanov and M. A. Zubkov, Chiral separation effect in nonhomogeneous systems, *Phys. Rev. D* **102**, 076019 (2020).
- [106] M. A. Zubkov and Ruslan A. Abramchuk, Effect of interactions on the topological expression for the chiral separation effect, *Phys. Rev. D* **107**, 094021 (2023).
- [107] Yu. A. Simonov and M. A. Trusov, Vacuum phase transition at nonzero baryon density, *Phys. Lett. B* **650**, 36 (2007).
- [108] G. B. Cook, S. L. Shapiro, and S. A. Teukolsky, Rapidly rotating neutron stars in general relativity: Realistic equations of state, *Astrophys. J.* **424**, 823 (1994).
- [109] A. A. Zyuzin and A. A. Burkov, Topological response in Weyl semimetals and the chiral anomaly, *Phys. Rev. B* **86**, 115133 (2012).
- [110] A. Vilenkin, Equilibrium parity-violating current in a magnetic field, *Phys. Rev. D* **22**, 3080 (1980).
- [111] D. E. Kharzeev, The chiral magnetic effect and anomaly-induced transport, *Prog. Part. Nucl. Phys.* **75**, 133 (2014).
- [112] D. E. Kharzeev and H. J. Warringa, Chiral magnetic conductivity, *Phys. Rev. D* **80**, 034028 (2009).
- [113] D. T. Son and N. Yamamoto, Berry curvature, triangle anomalies, and chiral magnetic effect in Fermi liquids, *Phys. Rev. Lett.* **109**, 181602 (2012).
- [114] M. A. Zubkov, Wigner transformation, momentum space topology, and anomalous transport, *Ann. Phys. (N.Y.)* **373**, 298 (2016).
- [115] M. M. Vazifeh and M. Franz, Electromagnetic response of Weyl semimetals, *Phys. Rev. Lett.* **111**, 027201 (2013).
- [116] N. Yamamoto, Generalized Bloch theorem and chiral transport phenomena, *Phys. Rev. D* **92**, 085011 (2015).
- [117] Chitradip Banerjee, Meir Lewkowicz, and Mikhail A. Zubkov, Equilibrium chiral magnetic effect: Spatial inhomogeneity, finite temperature, interactions, *Phys. Lett. B* **819**, 136457 (2021).
- [118] Chitradip Banerjee, Meir Lewkowicz, and Mikhail A. Zubkov, Chiral magnetic effect out of equilibrium, *Phys. Rev. D* **106**, 074508 (2022).
- [119] H. B. Nielsen and M. Ninomiya, Adler-Bell-Jackiw anomaly and Weyl fermions in crystal, *Phys. Lett.* **130B**, 389 (1983).
- [120] Q. Li, D. E. Kharzeev, C. Zhang, Y. Huang, I. Pletikoscic, A. V. Fedorov, R. D. Zhong, J. A. Schneeloch, G. D. Gu, and T. Valla, Chiral magnetic effect in ZrTe 5, *Nat. Phys.* **12**, 550 (2016).
- [121] M. Vazifeh and M. Franz, Electromagnetic response of Weyl semimetals, *Phys. Rev. Lett.* **111**, 027201 (2013).
- [122] Y. Chen, S. Wu, and A. Burkov, Axion response in Weyl semimetals, *Phys. Rev. B* **88**, 125105 (2013).
- [123] Y. Chen, D. Bergman, and A. Burkov, Weyl fermions and the anomalous Hall effect in metallic ferromagnets, *Phys. Rev. B* **88**, 125110 (2013).
- [124] S. T. Ramamurthy and T. L. Hughes, Patterns of electromagnetic response in topological semi-metals, *Phys. Rev. B* **92**, 085105 (2015).
- [125] Pallab Goswami and Sumanta Tewari, Axionic field theory of (3 + 1)-dimensional Weyl semi-metals, *Phys. Rev. B* **88**, 245107 (2013).
- [126] Chao-Xing Liu, Peng Ye, and Xiao-Liang Qi, Chiral gauge field and axial anomaly in a Weyl semimetal, *Phys. Rev. B* **87**, 235306 (2013).
- [127] M. A. Zubkov, Weyl orbits as probe of chiral separation effect in magnetic Weyl semimetals, *J. Phys. Condens. Matter* **36**, 415501 (2024).
- [128] D. Vasak, M. Gyulassy, and H. T. Elze, Quantum transport theory for Abelian plasmas, *Ann. Phys. (N.Y.)* **173**, 462 (1987).
- [129] Frank Antonsen, Quantum theory in curved spacetime using the Wigner function, *Phys. Rev. D* **56**, 920 (1997).
- [130] Jiunn-Wei Chen, Shi Pu, Qun Wang, and Xin-Nian Wang, Berry curvature and four-dimensional monopoles in the relativistic chiral kinetic equation, *Phys. Rev. Lett.* **110**, 262301 (2013).
- [131] Andrei Alekseevich Slavnov, Pauli–Villars regularization for non-Abelian gauge theories, *Teor. Mat. Fiz.* **33**, 210 (1977).

- [132] M. A. Zubkov, Weyl orbits as probe of chiral separation effect in magnetic Weyl semimetals, *J. Phys. Condens. Matter* **36**, 415501 (2024).
- [133] M. A. Zubkov and Xi Wu, Topological invariant in terms of the green functions for the quantum Hall effect in the presence of varying magnetic field, *Ann. Phys. (N.Y.)* **418**, 168179 (2020).
- [134] David Vasak, Miklos Gyulassy, and Hans-Thomas Elze, Quantum transport theory for Abelian plasmas, *Ann. Phys. (N.Y.)* **173**, 462 (1987).
- [135] Thomas Appelquist, John Terning, and L. C. R. Wijewardhana, Zero temperature chiral phase transition in SU(N) gauge theories, *Phys. Rev. Lett.* **77**, 1214 (1996).
- [136] C. X. Zhang and M. A. Zubkov, Note on the Bloch theorem, *Phys. Rev. D* **100**, 116021 (2019).
- [137] Yasuhiro Hatsugai, Takahiro Fukui, and Hideo Aoki, Topological analysis of the quantum hall effect in graphene: Dirac-fermi transition across van hove singularities and edge versus bulk quantum numbers, *Phys. Rev. B* **74**, 205414 (2006).
- [138] C. X. Zhang and M. A. Zubkov, Influence of interactions on integer quantum Hall effect, *Ann. Phys. (N.Y.)* **444**, 169016 (2022).
- [139] J. Nissinen and G. E. Volovik, Elasticity tetrads, mixed axial-gravitational anomalies, and $(3 + 1)$ -d quantum Hall effect, *Phys. Rev. Res.* **1**, 023007 (2019).
- [140] Maik Selch, Ruslan A. Abramchuk, and M. A. Zubkov, Effective Lagrangian for the macroscopic motion of fermionic matter, *Phys. Rev. D* **109**, 016003 (2024).
- [141] Minoru Eto, Toshiaki Fujimori, Muneto Nitta, Keisuke Ohashi, and Norisuke Sakai, Dynamics of non-Abelian vortices, *Phys. Rev. D* **84**, 125030 (2011).
- [142] Minoru Eto, Yuji Hirono, Muneto Nitta, and Shigehiro Yasui, Vortices and other topological solitons in dense quark matter, *Prog. Theor. Exp. Phys.* **2014**, 012D01 (2014).
- [143] Jayantha P. Vyasankere and Vijay B. Shenoy, Bound states of two spin-1/2 fermions in a synthetic non-Abelian gauge field, *Phys. Rev. B* **83**, 094515 (2011).
- [144] Luca Lepori, Ion Cosma Fulga, Andrea Trombettoni, and Michele Burrello, Double Weyl points and Fermi arcs of topological semimetals in non-Abelian gauge potentials, *Phys. Rev. A* **94**, 053633 (2016).
- [145] Yaowu Guo, Zhi Lin, Jia-Qiang Zhao, Jie Lou, and Yan Chen, Two-dimensional tunable Dirac/Weyl semimetal in non-Abelian gauge field, *Sci. Rep.* **9**, 18516 (2019).