

Nonlinear self-duality in $4p$ dimensions

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Building on the earlier work by Araki and Tanii, Aschieri *et al.*, and Buratti *et al.*, we demonstrate that every model for self-dual nonlinear electrodynamics in four dimensions has a $U(1)$ duality-invariant extension to $4p > 4$ dimensions, and construct new self-dual nonlinear theories for a gauge $(2p - 1)$ -form. We present a family of models for self-dual nonlinear $(2p - 1)$ -form electrodynamics in which the trace of the energy-momentum tensor determines the flow with respect to a duality-invariant deformation parameter. Finally, we propose the interesting problem of computing the so-called induced action in the case that self-dual $(2p - 1)$ -form electrodynamics is coupled to dilaton and axion fields, and the compact duality group $U(1)$ is enhanced to the noncompact group $SL(2, \mathbb{R})$.

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I. INTRODUCTION

As an extension of the seminal work by Gaillard and Zumino [1], the general formalism of duality-invariant models for nonlinear electrodynamics in four dimensions was developed in the mid-1990s [2–5] (see [6,7] for comprehensive reviews). The Gaillard-Zumino-Gibbons-Rasheed (GZGR) approach was generalized to off-shell $\mathcal{N} = 1$ and $\mathcal{N} = 2$ globally [6,8] and locally [9,10] supersymmetric theories. In particular, the first consistent perturbative scheme to construct the $\mathcal{N} = 2$ supersymmetric Born-Infeld action was given in [6] (this approach was further pursued in [11,12]). The GZGR formalism was also extended to higher dimensions [2,13–15].

In Minkowski space of even dimension $d = 4p \equiv 2n$, with p a positive integer, we consider a self-interacting theory of a gauge $(n - 1)$ -form $A_{a_1 \dots a_{n-1}}$ with the property that its Lagrangian, $L(F_{a_1 \dots a_n})$, is a function of the field strength $F_{a_1 \dots a_n} = n \partial_{[a_1} A_{a_2 \dots a_n]} \equiv F_{a(n)}$.¹ We assume that the theory possesses $U(1)$ duality invariance. This means that the Lagrangian is a solution to the self-duality equation [14]

$$\tilde{G}^{a_1 \dots a_n} G_{a_1 \dots a_n} + \tilde{F}^{a_1 \dots a_n} F_{a_1 \dots a_n} \equiv \tilde{G} \cdot G + \tilde{F} \cdot F = 0, \quad (1.1)$$

¹We follow the notation and conventions of [6].

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where we have introduced

$$\tilde{G}^{a_1 \dots a_n}(F) = n! \frac{\partial L(F)}{\partial F_{a_1 \dots a_n}}. \quad (1.2)$$

As usual, the notation $\tilde{F}$ is used for the Hodge dual of F ,

$$\tilde{F}^{a_1 \dots a_n} = \frac{1}{n!} \epsilon^{a_1 \dots a_n b_1 \dots b_n} F_{b_1 \dots b_n}. \quad (1.3)$$

The simplest solution of (1.1) is the free model

$$L_{\text{free}}(F) = -\frac{1}{2n!} F^{a_1 \dots a_n} F_{a_1 \dots a_n}, \quad (1.4)$$

which is a conformal field theory in $d = 2n$ dimensions. It should be pointed out that (1.1) is identically satisfied in $d = 4p + 2$ dimensions.

In four dimensions ($n = 2$), the self-duality equations (1.1) was derived by Gibbons and Rasheed in 1995 [2], and soon after, it was rederived by Gaillard and Zumino [4,5] by applying their formalism developed in 1981 [1]. The latter authors also introduced the term “self-duality equation” for (1.1). In 1983, Bialynicki-Birula [16] proposed a necessary and sufficient condition for $U(1)$ duality invariance of nonlinear electrodynamics with Lagrangian $L(F_{a_1 a_2})$. One can always recast $L(F_{a_1 a_2})$ as a function of two (pseudo) scalar variables, $\mathcal{L}(S, P)$, where S and P denote the electromagnetic invariants defined in (2.4) for $n = 2$. The condition derived in [16] is given by equation (2.5) below and is equivalent to (1.1) for all $U(1)$ duality-invariant models of the form $L(F_{a_1 a_2}) = \mathcal{L}(S, P)$. However, the equation (1.1) is more advantageous for nontrivial generalizations, including: (i) self-dual theories

in $d = 4p = 2n$ dimensions [2,13–15]; (ii) duality-invariant off-shell supersymmetric theories [6,8]; and (iii) self-dual higher-derivative theories of the form

$$S[F] = \int d^{2n}x L(F, \partial F, \partial^2 F, \dots), \quad F \equiv F_{a(n)}. \quad (1.5)$$

In the presence of higher derivatives, the self-duality equation (1.1) must be replaced with the following equation on the action functional $S[F]$:

$$\int d^{2n}x \{ \tilde{G}^{a_1 \dots a_n} G_{a_1 \dots a_n} + \tilde{F}^{a_1 \dots a_n} F_{a_1 \dots a_n} \} = 0, \quad \tilde{G}^{a_1 \dots a_n}(F) = n! \frac{\delta}{\delta F_{a_1 \dots a_n}} S[F]. \quad (1.6)$$

Here, it is assumed that $S[F]$ is unambiguously defined as a functional of an unconstrained n -form $F_{a(n)}$. In the $n = 2$ case, this equation was proposed for the first time in [6].

Only a few explicit solutions of the self-duality equation (1.1) are known beyond four dimensions. In particular, Ref. [15] introduced Born-Infeld-like actions with $N \geq 1$ complex field strengths (and also with $N > 1$ real field strengths) as a generalization of the Roček-Tseytlin construction [17] for the Maxwell-Goldstone multiplet model [18] for partial $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ supersymmetry breaking [19]. In this letter we will demonstrate, building on the results of [14,15,20], that every model for self-dual nonlinear electrodynamics in four dimensions has a $U(1)$ duality-invariant extension to $4p = 2n > 4$ dimensions and construct new self-dual nonlinear theories for a gauge $(n-1)$ -form. We also present a family of models for self-dual $(n-1)$ -form electrodynamics in which the trace of the energy-momentum tensor determines the flow with respect to a duality-invariant deformation parameter. Finally, we put forward an interesting open problem of computing a so-called induced action in the case that self-dual $(n-1)$ -form electrodynamics is coupled to dilaton and axion fields, and the compact duality group $U(1)$ is enhanced to the non-compact group $SL(2, \mathbb{R})$.

II. DIFFERENT FORMS OF THE SELF-DUALITY EQUATION

It is useful to introduce (anti) self-dual components of F ,

$$F_{a_1 \dots a_n}^\pm = \frac{1}{2} (F_{a_1 \dots a_n} \pm i \tilde{F}_{a_1 \dots a_n}), \quad \tilde{F}^\pm = \mp i F^\pm, \quad (2.1)$$

such that

$$F = F^+ + F^-, \quad \tilde{F} = -iF^+ + iF^-. \quad (2.2)$$

The same definition can be used to introduce the (anti) self-dual components of G . Then, the self-duality equation (1.1) turns into

$$\text{Im}(G^+ \cdot G^+ + F^+ \cdot F^+) = 0. \quad (2.3)$$

We are going to look for a class of $U(1)$ duality-invariant theories of the form²

$$L(F_{a(n)}) = \mathcal{L}(S, P), \quad S = -\frac{1}{2n!} F^{a_1 \dots a_n} F_{a_1 \dots a_n}, \quad P = -\frac{1}{2n!} \tilde{F}^{a_1 \dots a_n} F_{a_1 \dots a_n}. \quad (2.4)$$

In four dimensions, S and P are the only independent invariants of the electromagnetic field. More general invariants of the n -form $F_{a(n)}$ in $d = 2n = 4p > 4$ dimensions exist (see, e.g., [21] for the technical details). However, we restrict our attention to models of the form (2.4). For the theories under consideration, the self-duality equation (2.3) is equivalent to

$$P(\mathcal{L}_S^2 - \mathcal{L}_P^2 - 1) = S \mathcal{L}_S \mathcal{L}_P, \quad (2.5)$$

where $\mathcal{L}_S$ and $\mathcal{L}_P$ denote the derivatives of $\mathcal{L}$ with respect to S and P , respectively. In the case of nonlinear electrodynamics ($n = 2$), Eq. (2.5) was derived for the first time by Bialynicki-Birula [16]. As mentioned in Sec. I, (2.5) is equivalent to the self-duality equation (1.1).

Duality-invariant models of the form (2.4) have a long history. In 1999, Araki and Tani [14] studied $U(N)$ duality-invariant theories³ of N gauge $(n-1)$ -forms with the field strengths $F_{a(n)}^I$, $I = 1, \dots, N$. They restricted their discussion to Lagrangians

$$L(F_{a(n)}^I) = \mathcal{L}(S^{IJ}, P^{IJ}), \quad S^{IJ} := -\frac{1}{2n!} F^I \cdot F^J, \quad P^{IJ} := -\frac{1}{2n!} F^I \cdot \tilde{F}^J, \quad (2.6)$$

and they used perturbation theory to demonstrate the existence of $U(N)$ duality-invariant theories possessing the weak-field limit

$$\mathcal{L} = S^{II} + \mathcal{O}(F^4). \quad (2.7)$$

No closed-form solution was given in [14]. Born-Infeld-like actions with N field strengths $F_{a(n)}^I$ in $d = 2n$ dimensions were studied in Zumino and collaborators in [15]. In the $N = 1$ case, their Born-Infeld $(n-1)$ -form

²In the $d = 2n = 4p + 2$ case, the invariant P is identically equal to zero.

³If the dimension of spacetime is even, $d = 2n$, the maximal duality group for a system of N gauge $(n-1)$ -forms depends on the dimension of spacetime. The duality group is $U(N)$ if $n = 2p$, and $O(N) \times O(N)$ if $n = 2p + 1$ [14] (see, e.g., Sec. 8 of [6] for a review). In the presence of scalars, the maximal duality group is $Sp(2N, \mathbb{R})$ if $d = 4p$ and $SO(N, N)$ if $d = 4p + 2$. The fact that the maximal duality group depends on the dimension of spacetime was discussed in the mid-1980s [22,23] and also in the late 1990s [24,25].

electrodynamics agreed with the one sketched by Gibbons and Rasheed [2].⁴ The same model was also derived in [26] using Hamiltonian techniques and later in [20]. In fact, Buratti, Lechner and Melotti [20] also arrived at the equation (2.5) and the conclusion that every model for self-dual nonlinear electrodynamics admits an extension to higher dimensions.⁵

III. EXAMPLES OF SELF-DUAL THEORIES IN $4p$ DIMENSIONS

Our discussion implies that every model for self-dual nonlinear electrodynamics in four dimensions admits an extension to self-dual $(n-1)$ -form nonlinear electrodynamics in $2n = 4p > 4$ dimensions. In particular, associated with the Born-Infeld Lagrangian [28] is the Born-Infeld $(n-1)$ -form electrodynamics

$$L_{\text{BI}}(F_{a(n)}) = T - \sqrt{T^2 - 2TS - P^2}, \quad (3.1)$$

with T a duality-invariant parameter. This theory was introduced by several groups [2, 15, 20, 26].

Next, associated with the ModMax theory, which was discovered by Bandos, Lechner, Sorokin and Townsend [29] (and soon rederived, in a simpler setting, by Kosyakov [30]), is the $\text{U}(1)$ duality-invariant $(n-1)$ -form electrodynamics [31]

$$L_{\text{MM}}(F_{a(n)}) = S \cosh \gamma + \sqrt{S^2 + P^2} \sinh \gamma, \quad (3.2)$$

which is conformal in $d = 2n = 4p$ dimensions. The original ModMax theory is a unique conformal and $\text{U}(1)$ duality-invariant theory in four dimensions. Its higher-dimensional extension (3.2) is not unique. There exist more general self-dual and conformal models for $(n-1)$ -form electrodynamics. This can be readily seen by making use of the auxiliary-field formulation of [32] which is a generalization of the $d = 4$ approach developed by Ivanov and Zupnik [33–35]. In this formulation for $\text{U}(1)$ duality-invariant nonlinear $(n-1)$ -form electrodynamics, the Lagrangian $L(F, V)$ depends on an unconstrained rank- n antisymmetric tensor $V_{a_1 \dots a_n}$ and has the form

$$L(F, V) = \frac{1}{n!} \left\{ \frac{1}{2} F \cdot F + V \cdot V - 2V \cdot F \right\} + L_{\text{int}}(V), \quad (3.3)$$

with $V \cdot F := V^{a_1 \dots a_n} F_{a_1 \dots a_n}$. The self-duality equation (1.1) is equivalent to the manifest $\text{U}(1)$ invariance of the

⁴In the $N = 1$ case, the self-duality equations used in [15] reduce to (2.5).

⁵We derived this result as a by-product of similar observations for $\text{U}(1)$ duality-invariant higher-spin models in four dimensions [27].

self-interaction $L_{\text{int}}(V) = L_{\text{int}}(V_+, V_-)$,

$$L_{\text{int}}(e^{i\varphi} V_+, e^{-i\varphi} V_-) = L_{\text{int}}(V_+, V_-), \quad \varphi \in \mathbb{R}. \quad (3.4)$$

The theory with Lagrangian (3.3) is conformal if the self-interaction is a homogeneous function of V of degree $+2$,

$$L_{\text{int}}(\sigma V) = \sigma^2 L_{\text{int}}(V). \quad (3.5)$$

We point out that the $\text{U}(1)$ duality-invariant theories of the form (2.4) correspond to those auxiliary-field models (3.3) in which the self-interaction $L_{\text{int}}(V)$ is given by $L_{\text{int}}(V_+, V_-) = f(V_+ \cdot V_+ V_- \cdot V_-)$, with $f(x)$ being a real function of one variable. The self-coupling corresponding to the $(n-1)$ -form ModMax theory (3.2) proves to be $L_{\text{int}}(V) \propto \sqrt{V_+ \cdot V_+ V_- \cdot V_-}$ (see [36] for the four-dimensional analysis). It is clear that in $d = 2n = 4p > 4$ dimensions there exists more general solutions of the equations (3.4) and (3.5).

Finally, we introduce a higher-dimensional generalization of the algorithm proposed in [37] to generate $\text{U}(1)$ duality-invariant models. Specifically, if the Lagrangian (2.4) is a solution of (2.5), then the following model

$$L_\gamma(F_{a(n)}) := \mathcal{L}(\Omega, P), \quad \Omega := S \cosh \gamma + \sqrt{S^2 + P^2} \sinh \gamma \quad (3.6)$$

is also a solution of the self-duality equation (2.5). This construction leads to new self-dual models for $(n-1)$ -form electrodynamics. Applying this recipe to (3.1) leads to a higher-dimensional analog of the ModMax-Born theory introduced in [31].

IV. ENERGY-MOMENTUM FLOWS

In this paper we have demonstrated that every model for self-dual nonlinear electrodynamics in four dimensions has a $\text{U}(1)$ duality-invariant extension to $4p > 4$ dimensions.⁶ Beyond four dimensions, there exist more general self-dual theories, as follows from the known generating formulations for such theories [20, 32, 38]. The important point is that in $4p \equiv 2n > 4$ dimensions there exist more general invariants of $F_{a(n)}$ (see [21] for the technical details) than the invariants S and P given in (2.4).

A few years ago, a remarkable result was established [39] for self-dual nonlinear electrodynamics ($n = 2$). Given a one-parameter family of $\text{U}(1)$ duality-invariant theories, $L(F; g)$, with g a duality-invariant parameter, the Lagrangian obeys the $T\bar{T}$ -like flow equation

$$\frac{\partial}{\partial g} L = \mathfrak{F}(T_{ab}), \quad (4.1)$$

⁶Many years ago it was shown [6, 8] that every model for self-dual nonlinear electrodynamics possesses an $\mathcal{N} = 1$ supersymmetric duality-invariant extension.

for some function $\mathfrak{F}$ of the energy-momentum tensor T_{ab} . This theorem extends several explicit examples considered earlier in the literature in the context of $T\bar{T}$ deformations [40–42]. The quoted result of [39] has been extended to nonlinear chiral theories in six dimensions [43]. In ten dimensions, however, there is a flow in a chiral nonlinear theory which is not generated by the energy-momentum tensor [44], and a similar conclusion is expected in $4p$ dimensions. It is natural to wonder whether the relation (4.1) holds for the family of self-dual theories considered in this paper. The answer appears to be negative in general. However, below we present a family of models for self-dual $(n-1)$ -form electrodynamics in which the trace of the energy-momentum tensor determines the flow with respect to a duality-invariant deformation parameter.

The energy-momentum tensor corresponding to $\mathcal{L}(S, P)$, Eq. (2.4), is

$$T_{ab} = \mathcal{L}_S T_{ab}^{(\text{free})} + \eta_{ab}(\mathcal{L} - S\mathcal{L}_S - P\mathcal{L}_P), \quad (4.2)$$

where

$$T_{ab}^{(\text{free})} = \frac{1}{(n-1)!} \left(F_a \cdot F_b - \frac{1}{d} \eta_{ab} F \cdot F \right),$$

$$F_a \cdot F_b := F_{ac_1 \dots c_{n-1}} F_b{}^{c_1 \dots c_{n-1}} \quad (4.3)$$

is the energy-momentum tensor of the free model (1.4) (see, e.g., [45]). Since (1.4) is conformal, the corresponding energy-momentum tensor is traceless,

$$\eta^{ab} T_{ab}^{(\text{free})} = 0. \quad (4.4)$$

Making use of this property, for the energy-momentum tensor (4.2) we get

$$\Theta := \eta^{ab} T_{ab} = d(\mathcal{L} - S\mathcal{L}_S - P\mathcal{L}_P). \quad (4.5)$$

Conformal theories are characterized by the condition $\Theta = 0$.

Now, we should make two observations. First, if $L(F_{a(n)})$ is a solution of the self-duality Eq. (1.1), then the following combination

$$L - \frac{1}{2n!} F \cdot \tilde{G} \quad (4.6)$$

is a duality-invariant observable [14].⁷ This means that it is invariant under the infinitesimal $\text{SO}(2)$ duality transformation

$$\delta F = \lambda G, \quad \delta G = -\lambda F. \quad (4.7)$$

⁷In the four-dimensional case ($n = 2$), this result was derived by Gaillard and Zumino [1,4,5].

In case $L(F_{a(n)})$ is of the form (2.4), one finds

$$L - \frac{1}{2n!} F \cdot \tilde{G} = \mathcal{L} - S\mathcal{L}_S - P\mathcal{L}_P = \frac{1}{d} \Theta. \quad (4.8)$$

Second, we recall the observation given in [6], which is if $L(F_{a(n)})$ is a solution of the self-duality equation (1.1), then

$$L^{(g)}(F_{a(n)}) := \frac{1}{g^2} L(gF_{a(n)}), \quad g \in \mathbb{R}^+, \quad (4.9)$$

is also a solution of the self-duality equation (1.1), with g being a duality-invariant parameter.⁸ According to [4,5,14], the operator $\partial \mathcal{L}^{(g)} / \partial g$ is duality invariant. We now restrict our attention to the case of $L(F_{a(n)})$ of the form (2.4). Then, a direct calculation gives

$$\frac{\partial L^{(g)}}{\partial g} = -\frac{2}{g} \left\{ L^{(g)} - \frac{1}{2n!} F \cdot \tilde{G}^{(g)} \right\}. \quad (4.10)$$

The relations (4.5) and (4.10) lead to the flow equation

$$\frac{\partial L^{(g)}}{\partial g} = -\frac{2}{gd} \Theta^{(g)}, \quad (4.11)$$

which is a generalization of the $d = 4$ result derived in [39]. In the case of the Born-Infeld $(n-1)$ -form electrodynamics, Eq. (3.1) was obtained in [46].

V. SELF-DUALITY AND QUANTUM THEORY

Models for $(n-1)$ -form electrodynamics in $d = 4p = 2n > 4$, including the self-dual theories studied above, are of some interest from the quantum mechanical point of view. First, irrespective of duality invariance, any model of the form $L(F_n)$, with $F_n = dA_{n-1}$, is a reducible gauge theory, following the terminology of [47].⁹ Covariant quantization of such theories cannot be carried out using the famous Faddeev-Popov approach [48], which is used for Yang-Mills theories. One should instead make use of alternative approaches [49–54], which are applicable when quantizing reducible Abelian gauge theories, such as gauge p -forms, or the more powerful Batalin-Vilkovisky quantization [47].

In the case of self-dual $(n-1)$ -form electrodynamics in $2n = 4p > 4$ dimensions, the quantisation aspects get entangled with dynamical ones. As a natural generalisation of the dilaton-axion coupling to self-dual nonlinear

⁸The Born-Infeld $(n-1)$ -form electrodynamics (3.1) is an example of such a theory with $T = g^{-2}$.

⁹Its stage of reducibility is $n-2$. The field strength $F_n = dA_{n-1}$ is invariant under the gauge transformation $\delta A_{n-1} = d\zeta_{n-2}$. Here, the variation δA_{n-1} remains unchanged if the gauge parameter is changed as $\zeta_{n-2} \rightarrow \zeta_{n-2} + d\zeta_{n-3}$ and so on. As a result, there emerge $n-2$ stages of reducibility.

electrodynamics [3,4], the $U(1)$ duality group can be enhanced to the non-compact group $SL(2, \mathbb{R})$ by coupling the gauge field A_{n-1} to the dilaton φ and axion $\mathbf{a}$ [13,14]. In particular, associated with the free $(n-1)$ -form electrodynamics (1.4) is the interacting model¹⁰

$$L(F, \tau, \bar{\tau}) = -\frac{1}{2n!} e^{-\varphi} F^{a_1 \dots a_n} F_{a_1 \dots a_n} + \frac{1}{2n!} \mathbf{a} \tilde{F}^{a_1 \dots a_n} F_{a_1 \dots a_n},$$

$$\tau = \mathbf{a} + i e^{-\varphi}. \quad (5.1)$$

The complex field $\tau = \mathbf{a} + i e^{-\varphi}$ is dimensionless and parametrizes the Hermitian symmetric space $SL(2, \mathbb{R})/SO(2)$ realized as the Poincaré upper half-plane. The duality group $SL(2, \mathbb{R})$ acts on τ by fractional linear transformations

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}). \quad (5.2)$$

At the classical level, the theory (5.1) is conformal and possesses $SL(2, \mathbb{R})$ duality invariance. Both symmetries prove to be anomalous in the quantum theory. However, if one integrates out the potential A_{n-1} in the path integral, it is known that the logarithmically divergent part of the one-loop effective action (the so-called induced action) should be a local functional of τ and $\bar{\tau}$ that is conformal and $SL(2, \mathbb{R})$ invariant. This induced action has not yet been computed for $n > 2$, and its derivation is an interesting open problem. In addition, to the best of our knowledge, covariant quantisation of the gauge theory (5.1) has not been carried out for $n > 2$.

In four dimensions ($n = 2$), the induced action was computed by Osborn [55] and rederived later in [56] in the framework of induced $\mathcal{N} = 4$ conformal supergravity. In curved space, its general form is

$$L_{\text{induced}}^{(n=2)} = \frac{1}{2(\text{Im } \tau)^2} \left[\mathcal{D}^2 \tau \mathcal{D}^2 \bar{\tau} - 2 \left(R^{ab} - \frac{1}{3} \eta^{ab} R \right) \nabla_a \tau \nabla_b \bar{\tau} \right]$$

$$+ \frac{1}{12(\text{Im } \tau)^4} \times [\alpha \nabla^a \tau \nabla_a \tau \nabla^b \bar{\tau} \nabla_b \bar{\tau} + \beta \nabla^a \tau \nabla_a \bar{\tau} \nabla^b \tau \nabla_b \bar{\tau}], \quad (5.3)$$

where ∇_a is the torsion-free Lorentz-covariant derivative,¹¹

$$\mathcal{D}^2 \tau := \hat{\nabla}^a \nabla_a \tau, \quad \hat{\nabla}_a v^\tau = \left(\nabla_a + \frac{i}{\text{Im } \tau} \nabla_a \right) v^\tau,$$

$$\hat{\nabla}_a v^{\bar{\tau}} = \left(\nabla_a - \frac{i}{\text{Im } \tau} \nabla_a \right) v^{\bar{\tau}}, \quad (5.4)$$

and α and β are numerical parameters. Here, $\hat{\nabla}_a$ is the standard σ -model covariant derivative corresponding to the Kähler metric on the Poincaré upper half-plane

¹⁰The second term in (5.1) is identically zero in the case that $d = 4p + 2$.

¹¹As usual, $\nabla_a = e_a^m(x) \partial_m + \frac{1}{2} \omega_a^{bc}(x) M_{bc}$, where e_a^m is the inverse vielbein, ω_a^{bc} is the Lorentz connection, and M_{bc} is the Lorentz generator. The algebra of covariant derivatives is $[\nabla_a, \nabla_b] = \frac{1}{2} R_{ab}^{cd} M_{cd}$, with R_{ab}^{cd} being the Riemann tensor.

$$ds^2 = \frac{1}{(\text{Im } \tau)^2} d\tau d\bar{\tau} = 2g_{\tau\bar{\tau}} d\tau d\bar{\tau}, \quad (5.5)$$

with the Christoffel symbols being $\Gamma_{\tau\tau}^\tau = i(\text{Im } \tau)^{-1}$ and $\Gamma_{\bar{\tau}\bar{\tau}}^{\bar{\tau}} = -i(\text{Im } \tau)^{-1}$. The functional $\int d^4 x e L_{\text{induced}}^{(n=2)}$ is manifestly $SL(2, \mathbb{R})$ invariant and also proves to be Weyl invariant. The Weyl invariance follows from the fact that the Fradkin-Tseytlin (FT) operator [57],

$$\Delta_4 = (\nabla^a \nabla_a)^2 + 2\nabla^a \left(R_{ab} \nabla^b - \frac{1}{3} R \nabla_a \right), \quad (5.6)$$

is conformal.¹²

In the $d = 4$ case, the simple model (5.1) may be generalized to a system of n Abelian vector fields coupled to $\frac{1}{2}n(n+1)$ complex scalars parametrizing the Hermitian symmetric space $Sp(2n, \mathbb{R})/U(n)$. Such a theory is conformal invariant and possesses the maximal noncompact duality group $Sp(2n, \mathbb{R})$. The corresponding induced action, obtained by integrating out the vector fields, was computed in [60]. The induced action, which determines the logarithmically divergent part of the one-loop effective action, is conformal and $Sp(2n, \mathbb{R})$ invariant.

Beyond four dimensions, $n > 2$, the induced action corresponding to (5.1) should also be $SL(2, \mathbb{R})$ and Weyl invariant. This means that the effective Lagrangian $L_{\text{induced}}^{(n>2)}$ describes a higher-derivative nonlinear σ -model with leading part

$$L_{\text{induced}}^{(n>2)} = \frac{1}{2(\text{Im } \tau)^2} [(\hat{\nabla}^2)^{n/2-1} \mathcal{D}^2 \tau] (\hat{\nabla}^2)^{n/2-1} \mathcal{D}^2 \bar{\tau} + \dots \quad (5.7)$$

The existence of such a Lagrangian is guaranteed by the existence of a conformal differential operator in $d = 2n$ dimensions of the form [61] (see also [62–64] and references therein),

$$\Delta_d = (\nabla^a \nabla_a)^{d/2} + \dots, \quad (5.8)$$

where the ellipsis denotes terms of lower order in the covariant derivatives.¹³

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¹²This operator was re-discovered by Paneitz in 1983 [58] and Riegert in 1984 [59].

¹³In six dimensions, the operator Δ_6 was constructed in [65,66].

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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