

Manipulating Bell nonlocality and entanglement in polarized electron-positron annihilation

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The hyperon-antihyperon pairs produced in electron-positron annihilation as a massive two-qubit quantum system can be used to study the quantum correlations at high energies. This paper is theoretically dedicated to how polarization of lepton beams manipulates the Bell nonlocality and entanglement of hyperon pair systems. The response of the Clauser–Horne–Shimony–Holt parameter, concurrence, and negativity to the polarization degree of the beam is numerically calculated by exploiting the joint spin density matrix of hyperon-antihyperon pairs. Different influences of longitudinal and transverse polarization of beams on entanglement are found and compared. The results provide alternative perspectives for the decay of charmonium to hyperon pairs.

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I. INTRODUCTION

High energy particle physics enables a new avenue of investigating entanglement and Bell nonlocality, two defining characteristics of quantum mechanics which have been convincingly tested in low-energy photonic and atomic systems [1–3]. Elementary particles, e.g., top quarks [4–7] and neutrinos [8], were exploited to witness entanglement in a high energy experiment. Plenty of new proposals, for instance, top-quark and τ -lepton pair at unpolarized hadron and lepton colliders [9] and polarized lepton colliders [10,11], massless quark pair in dihadron pair production at lepton colliders [12], and electron-quark or quark-antiquark pairs at electron-ion colliders [13–16], have been declared as alternative environments to explore quantum entanglement.

From another landscape, maximal entanglement is conjectured as a probe of many interesting physics in hadronic field. Maximal entanglement between helicity states has been previously proposed to constrain quantum electrodynamics and weak mixing angles [17]. Maximal

entanglement of partons seems to provide a new perspective in color confinement and evolution of quantum chromodynamics [18–21] as well as hadronization [22]. Maximal entanglement also facilitates the study of rich spin-dependent proton-proton interactions [23].

The connection between entanglement and (fundamental and approximate) symmetries of particle interactions is enlightening. Entanglement suppression in strong interactions is found to be correlated with approximate spin-flavor symmetries, leading to emergent symmetries in low-energy baryon interactions [24]. Entangled elementary particle systems constrain new physics beyond the Standard Model [9,25]. The parity-violating interactions in spin half bipartite systems is shown to disentangle the entanglement of particle pairs [26]. The influence of spin and lifetime entanglement on the decay of entangled particles is also investigated [27]. The quantum correlated $D\bar{D}$ pairs are used to measure the strong-phase differences between their decay amplitudes [28,29].

Hyperon-antihyperon pairs produced in electron-positron annihilation can be regarded as massive qubit system [30]. Several scenarios for the observation of quantum entanglement at electron-positron colliders are suggested [31–39], leading to the recent experimental claim of the exclusion of local hidden variable theory after addressing several loopholes by the BESIII Collaboration [40]. Although experiments at colliders, in principle, do not rule out all types of hidden variable theories as discussed by several critical investigations [41–47], the study of entanglement of elementary and composite particles is of interest

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from a theoretical perspective. The nontrivial correlation of helicity states of virtual photons or charmonium produced in electron-positron annihilation results from the fact that the helicity of a lepton is effectively conserved in the vertex in the massless limit [48]. This is a feature of quantum electrodynamics related to the chiral nature of the interaction that the chiral eigenstates are the same as the helicity eigenstates for both massless particle and antiparticle spinors. The spin density matrix of hyperon and antihyperon carries forward the correlation of helicity states by virtue of the nonzero relative phase between electro and magnetic form factors in a timelike region. In all previous studies, hyperon-antihyperon pairs are considered to be produced in unpolarized electron-positron collisions. This work will study how the longitudinal and transverse beam polarization manipulates the Bell nonlocality and entanglement of hyperon-antihyperon pairs.

This paper is organized as follows. In Sec. II, the formalism of Bell nonlocality, concurrence, and negativity are introduced based on the spin density matrix incorporating beam polarization effects. The influence of longitudinal and transverse beam polarization is present, respectively, in Secs. III and IV. The hierarchy of quantum correlations and separable states are discussed in Sec. V, and the conclusion is briefly given in Sec. VI.

II. THEORETICAL SETUP

A. Spin density matrix

A pair of hyperon-antihyperon ($Y\bar{Y}$), both of spin half, constitutes a massive 2-qubit system, which can be fully described by a density matrix $\rho_{Y\bar{Y}}$. As a (1,1)-type tensor on

the Hilbert space $\mathcal{H} = \mathcal{H}_Y \otimes \mathcal{H}_{\bar{Y}}$, its Bloch-Fano representation reads [49] as

$$\rho_{Y\bar{Y}} = \frac{1}{4} \left(I^{(4)} + \mathbf{B}^+ \cdot \boldsymbol{\sigma} \otimes I^{(2)} + I^{(2)} \otimes \mathbf{B}^- \cdot \boldsymbol{\sigma} + \sum_{i,j} C_{ij} \sigma_i \otimes \sigma_j \right), \quad (1)$$

$$i, j = 1, 2, 3,$$

where $\mathbf{B}^\pm$ are polarization or Bloch vectors of the hyperon and antihyperon, $\boldsymbol{\sigma}$ is the vector operator combined by three Pauli operators $\{\sigma_x, \sigma_y, \sigma_z\}$, and C_{ij} ($i, j = 1, 2, 3$) is the correlation tensor. Here $I^{(2)}$ and $I^{(4)}$ are 2×2 and 4×4 identity matrices, respectively. Under the basis $\{\sigma_\mu \otimes \sigma_\nu\}$ with $\sigma_0 = I^{(2)}$, Eq. (1) also can be organized into a more compact form:

$$\rho_{Y\bar{Y}} = \frac{1}{4} \Theta_{\mu\nu} \sigma_\mu \otimes \sigma_\nu, \quad \mu, \nu = 0, 1, 2, 3, \quad (2)$$

with $\Theta_{00} = 1$, $\Theta_{i0} = B_i^+$ and $\Theta_{0j} = B_j^-$, and $\Theta_{ij} = C_{ij}$ satisfies $\Theta_{\mu\nu} = \langle \sigma_\mu \otimes \sigma_\nu \rangle = \text{Tr}(\rho_{Y\bar{Y}} \sigma_\mu \otimes \sigma_\nu)$.

In the $e^+e^- \rightarrow \gamma^*/\psi \rightarrow Y\bar{Y}$ process, the produced hyperon and antihyperon move in opposite directions within the center-of-mass (c.m.) frame due to momentum conservation, and the chosen coordinate system is illustrated in Fig. 1. In the case of a longitudinally polarized electron beam, or when both beams are transversely polarized, the non-normalized $\Theta_{\mu\nu}$ reads as [48,50–55]

$$\begin{aligned} \Theta_{\mu\nu} \propto & \begin{pmatrix} 1 + \alpha_\psi \cos^2 \theta & 0 & \beta_\psi \sin \theta \cos \theta & 0 \\ 0 & \sin^2 \theta & 0 & \gamma_\psi \sin \theta \cos \theta \\ -\beta_\psi \sin \theta \cos \theta & 0 & \alpha_\psi \sin^2 \theta & 0 \\ 0 & -\gamma_\psi \sin \theta \cos \theta & 0 & -\alpha_\psi - \cos^2 \theta \end{pmatrix} \\ & + P_T^2 \begin{pmatrix} \alpha_\psi \sin^2 \theta \cos 2\phi & -\beta_\psi \sin \theta \sin 2\phi & -\beta_\psi \sin \theta \cos \theta \cos 2\phi & 0 \\ -\beta_\psi \sin \theta \sin 2\phi & (\alpha_\psi + \cos^2 \theta) \cos 2\phi & -(1 + \alpha_\psi) \cos \theta \sin 2\phi & -\gamma_\psi \sin \theta \cos \theta \cos 2\phi \\ \beta_\psi \sin \theta \cos \theta \cos 2\phi & (1 + \alpha_\psi) \cos \theta \sin 2\phi & (1 + \alpha_\psi \cos^2 \theta) \cos 2\phi & -\gamma_\psi \sin \theta \sin 2\phi \\ 0 & \gamma_\psi \sin \theta \cos \theta \cos 2\phi & -\gamma_\psi \sin \theta \sin 2\phi & -\sin^2 \theta \cos 2\phi \end{pmatrix} \\ & + P_L \begin{pmatrix} 0 & \gamma_\psi \sin \theta & 0 & (1 + \alpha_\psi) \cos \theta \\ \gamma_\psi \sin \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_\psi \sin \theta \\ -(1 + \alpha_\psi) \cos \theta & 0 & -\beta_\psi \sin \theta & 0 \end{pmatrix}, \quad (3) \end{aligned}$$

where θ is the scattering angle between the momenta of the incoming electron and outgoing hyperon, e.g., $\cos \theta = \hat{\mathbf{p}}_e \cdot \hat{\mathbf{p}}_Y$ in Fig. 1. Here $\alpha_\psi \in [-1, 1]$ is the decay parameter of the vector charmonium into the hyperon and antihyperon

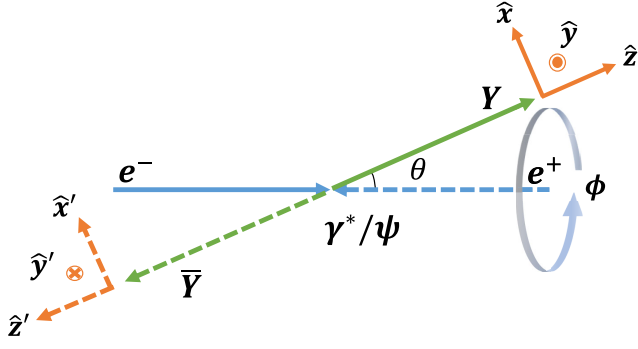


FIG. 1. Process of $e^+e^- \rightarrow \gamma^*/\psi \rightarrow Y\bar{Y}$ in the c.m. frame. The coordinate systems of Y and $\bar{Y}$ are chosen to be of the same chirality with $\{\hat{x}, \hat{y}, \hat{z}\}$ being three axes in the helicity rest frame of Y and $\{\hat{x}', \hat{y}', \hat{z}'\}$ of $\bar{Y}$. For hyperon Y , $\hat{y} = (\hat{p}_e \times \hat{p}_Y)/|\hat{p}_e \times \hat{p}_Y|$, $\hat{z} = \hat{p}_Y$, $\hat{x} = \hat{y} \times \hat{z}$, where $\hat{p}_e$ and $\hat{p}_Y$ are unit vectors along momentum directions of the electron and hyperon, respectively. The axes of antihyperon $\bar{Y}$ hold the transformation: $\{\hat{x}', \hat{y}', \hat{z}'\} = \{\hat{x}, -\hat{y}, -\hat{z}\}$.

pair, $\beta_\psi = \sqrt{1 - \alpha_\psi^2} \sin \Delta\Phi$ and $\gamma_\psi = \sqrt{1 - \alpha_\psi^2} \cos \Delta\Phi$, where $\Delta\Phi \in (-\pi, \pi]$ is the relative phase between electro-magnetic form factors. The parameters of $J/\psi \rightarrow Y\bar{Y}$ measured by the BESIII Collaboration are listed in Table I. The $\rho_{Y\bar{Y}}$ determined by the above is of rank-2.

Throughout our paper, the longitudinal and transverse polarization of beams is separately considered. The Bloch vector $\mathbf{B}^\pm$ can be read from the coefficient matrix in Eq. (3). For longitudinally polarized beams, $\mathbf{B}_L^+$ and $\mathbf{B}_L^-$ are given by

$$\mathbf{B}_L^+ = \frac{1}{\chi_L} \begin{pmatrix} P_L \gamma_\psi \sin \theta \\ -\beta_\psi \sin \theta \cos \theta \\ -P_L(1 + \alpha_\psi) \cos \theta \end{pmatrix},$$

$$\mathbf{B}_L^- = \frac{1}{\chi_L} \begin{pmatrix} P_L \gamma_\psi \sin \theta \\ \beta_\psi \sin \theta \cos \theta \\ P_L(1 + \alpha_\psi) \cos \theta \end{pmatrix}, \quad (4)$$

with normalization coefficient $\chi_L = 1 + \alpha_\psi \cos^2 \theta$, and the correlation tensor $\mathbf{C}_L$ is

$$\mathbf{C}_L = \frac{1}{\chi_L} \begin{pmatrix} \sin^2 \theta & 0 & \gamma_\psi \cos \theta \sin \theta \\ 0 & \alpha_\psi \sin^2 \theta & -P_L \beta_\psi \sin \theta \\ -\gamma_\psi \cos \theta \sin \theta & -P_L \beta_\psi \sin \theta & -\alpha_\psi - \cos^2 \theta \end{pmatrix}. \quad (5)$$

For transversely polarized beams, $\mathbf{B}_T^+$ and $\mathbf{B}_T^-$ are

$$\mathbf{B}_T^+ = \frac{1}{\chi_T} \begin{pmatrix} -P_T^2 \beta_\psi \sin \theta \sin 2\phi \\ \beta_\psi \sin \theta \cos \theta (P_T^2 \cos 2\phi - 1) \\ 0 \end{pmatrix},$$

$$\mathbf{B}_T^- = \frac{1}{\chi_T} \begin{pmatrix} -P_T^2 \beta_\psi \sin \theta \sin 2\phi, \\ \beta_\psi \sin \theta \cos \theta (1 - P_T^2 \cos 2\phi) \\ 0 \end{pmatrix}, \quad (6)$$

with $\chi_T = 1 + \alpha_\psi \cos^2 \theta + P_T^2 \alpha_\psi \sin^2 \theta \cos 2\phi$. The correlation tensor $\mathbf{C}_T$ is

$$\mathbf{C}_T = \frac{1}{\chi_T} \begin{pmatrix} \sin^2 \theta + P_T^2(\alpha_\psi + \cos^2 \theta) \cos 2\phi & -P_T^2(1 + \alpha_\psi) \cos \theta \sin 2\phi & \gamma_\psi \sin \theta \cos \theta (1 - P_T^2 \cos 2\phi) \\ P_T^2(1 + \alpha_\psi) \cos \theta \sin 2\phi & \alpha_\psi \sin^2 \theta + P_T^2(1 + \alpha_\psi \cos^2 \theta) \cos 2\phi & -P_T^2 \gamma_\psi \sin \theta \sin 2\phi \\ -\gamma_\psi \sin \theta \cos \theta (1 - P_T^2 \cos 2\phi) & -P_T^2 \gamma_\psi \sin \theta \sin 2\phi & -\alpha_\psi - \cos^2 \theta - P_T^2 \sin^2 \theta \cos 2\phi \end{pmatrix}. \quad (7)$$

By rotating the coordinate system $\{\hat{x}', \hat{y}', \hat{z}'\} = \{\hat{x}, \hat{y}, \hat{z}\}$, the coefficient matrix $\Theta_{\mu\nu}$ can be symmetrized. Notably, Eqs. (4) and (6) demonstrate that, unlike elementary particles, hyperons acquire transverse

polarization even when produced by unpolarized lepton beams [50].

If $\mathbf{B}^\pm$ has only one nonzero component, by exchanging the axes corresponding to the nonzero transverse

TABLE I. Parameters of $J/\psi \rightarrow Y\bar{Y}$ decay for $Y = \Lambda, \Sigma^+, \Sigma^0, \Xi^-, \Xi^0$ measured by the BESIII Collaboration.

Decay channel	α_ψ	$\Delta\Phi/\text{rad}$	Ref.
$J/\psi \rightarrow \Lambda \bar{\Lambda}$	$0.4748 \pm 0.0022 \pm 0.0031$	$0.748 \pm 0.006 \pm 0.004$	[56–59]
$J/\psi \rightarrow \Sigma^+ \bar{\Sigma}^-$	$-0.5047 \pm 0.0018 \pm 0.0010$	$-0.2744 \pm 0.0033 \pm 0.0010$	[60–62]
$J/\psi \rightarrow \Sigma^0 \bar{\Sigma}^0$	$-0.4133 \pm 0.0035 \pm 0.0077$	$-0.0828 \pm 0.0068 \pm 0.0033$	[63]
$J/\psi \rightarrow \Xi^- \bar{\Xi}^+$	$0.5851 \pm 0.0044 \pm 0.0034$	$1.2205 \pm 0.0159 \pm 0.0056$	[64–66]
$J/\psi \rightarrow \Xi^0 \bar{\Xi}^0$	$0.514 \pm 0.006 \pm 0.015$	$1.168 \pm 0.019 \pm 0.018$	[67,68]

polarization component with the $\hat{z}$ -axis, and afterward diagonalizing the correlation matrix $\mathbf{C}$, the coefficient matrix $\Theta_{\mu\nu}$ always can be written in a simpler form [36]:

$$\Theta_{\mu\nu} \rightarrow \Theta'_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & B_z \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ B_z & 0 & 0 & \lambda_3 \end{pmatrix}, \quad (8)$$

where B_z is the nonzero polarization component of $\mathbf{B}^\pm$, and λ_1 , λ_2 , and λ_3 are the three eigenvalues of the correlation matrix $\mathbf{C}$. With the help of Eq. (2) the density matrix ρ can be expressed in the so-called X state [69]:

$$\begin{aligned} \rho \rightarrow \rho^X &= \frac{1}{4} \Theta'_{\mu\nu} \sigma_\mu \otimes \sigma_\nu \\ &= \frac{1}{4} \begin{pmatrix} 1+2B_z+\lambda_3 & 0 & 0 & \lambda_1-\lambda_2 \\ 0 & 1-\lambda_3 & \lambda_1+\lambda_2 & 0 \\ 0 & \lambda_1+\lambda_2 & 1-\lambda_3 & 0 \\ \lambda_1-\lambda_2 & 0 & 0 & 1-2B_z+\lambda_3 \end{pmatrix}. \end{aligned} \quad (9)$$

Here ρ and ρ^X are local unitary equivalents:

$$\begin{aligned} \rho^X &= (U \otimes V) \rho (U^\dagger \otimes V^\dagger) \\ &= \frac{1}{4} \left(I^{(4)} + B_z \sigma_z \otimes I^{(2)} + I^{(2)} \otimes B_z \sigma_z + \sum_i \lambda_i \sigma_i \otimes \sigma_i \right), \end{aligned} \quad (10)$$

where U and V are local unitary operators acting on the Hilbert spaces of the respective subsystems Y and $\bar{Y}$. Under such local unitary transformations, the measures of Bell nonlocality and quantum entanglement are strictly invariant [70].

For a hyperon–antihyperon system $\rho_{Y\bar{Y}}$, the density matrix can be reduced to an X -state at scattering angles of $\theta = 0$, $\pi/2$, and π . Let $\rho_{Y\bar{Y}}^{P_L}$ be the density matrix in the presence of longitudinal beam polarization and Θ_L be its corresponding coefficient matrix. The parameters $\{B_z, \lambda_i (i = 1, 2, 3)\}$ in the symmetric form Θ'_L , obtained via unitary transformation on Θ_L , are given by

$$\begin{aligned} \theta = \frac{\pi}{2}: B_z &= P_L \gamma_\psi, \\ \lambda_{1,2} &= \pm \sqrt{\alpha_\psi^2 + P_L^2 \beta_\psi^2}, \quad \lambda_3 = 1; \end{aligned} \quad (11)$$

$$\theta = 0, \quad \pi: B_z = -P_L, \quad \lambda_{1,2} = 0, \quad \lambda_3 = 1. \quad (12)$$

Similarly, let $\rho_{Y\bar{Y}}^{P_T}$ denote the density matrix for the case of a transversely polarized beam and Θ_T is its corresponding

coefficient matrix. The parameters $\{B_z, \lambda_i (i = 1, 2, 3)\}$ in the Θ'_T matrix are

$$\begin{aligned} \theta = \frac{\pi}{2}: B_z &= -\frac{P_T^2 \beta_\psi \sin 2\phi}{1 + P_T^2 \alpha_\psi \cos 2\phi}, \\ \lambda_{1,2} &= \pm \frac{\sqrt{\alpha_\psi^2 + 2P_T^2 \alpha_\psi \cos 2\phi + P_T^4 \eta}}{1 + P_T^2 \alpha_\psi \cos 2\phi}, \\ \lambda_3 &= 1; \end{aligned} \quad (13)$$

$$\theta = 0, \quad \pi: B_z = 0, \quad \lambda_{1,2} = \pm P_T^2, \quad \lambda_3 = 1, \quad (14)$$

with $\eta = \gamma_\psi^2 \sin^2 2\phi + \cos^2 2\phi$. Equation (14) corresponds to a special case of the X -state with vanishing polarization ($\mathbf{B}^\pm = \mathbf{0}$), known as the Bell diagonal state (BDS). In the case of unpolarized lepton beams Eqs. (11)–(13) are also reduced to the BDS, as found in literature [36]. A local choice of basis allows such a state to be expressed as a convex combination of the four maximally entangled Bell states [71]:

$$\rho_{Y\bar{Y}}^{\text{BDS}} = \frac{1}{4} \left(I^{(4)} + \sum_i \lambda_i \sigma_i \otimes \sigma_i \right). \quad (15)$$

B. Bell nonlocality and quantum entanglement

Bell nonlocality is a fundamental quantum feature that manifests through violation of Bell-type inequalities [72,73]. Among these, the Clauser–Horne–Shimony–Holt (CHSH) inequality provides an experimentally viable and widely adopted framework [74]:

$$|\langle \hat{A}_1 \hat{B}_1 \rangle + \langle \hat{A}_1 \hat{B}_2 \rangle + \langle \hat{A}_2 \hat{B}_1 \rangle - \langle \hat{A}_2 \hat{B}_2 \rangle| \leq 2, \quad (16)$$

where $\hat{A}_i$ and $\hat{B}_j$ ($i, j = 1, 2$) denote dichotomic observables with possible outcomes $\{0, 1\}$ in two subsystems, respectively. Spin operators are defined as $\hat{A}_i = \boldsymbol{\alpha}_i \cdot \boldsymbol{\sigma}$ and $\hat{B}_j = \boldsymbol{\beta}_j \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_j$ are unit vectors specifying measurement directions for subsystems A and B , respectively. Then the expectation value for any bipartite state ρ is given by

$$\langle \hat{A}_i \hat{B}_j \rangle \equiv \text{Tr}[\rho(\boldsymbol{\alpha}_i \cdot \boldsymbol{\sigma} \otimes \boldsymbol{\beta}_j \cdot \boldsymbol{\sigma})]. \quad (17)$$

Since the measurement directions $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_j$ can be optimized to maximize the violation of the CHSH inequality, the CHSH parameter $\mathcal{B}[\rho]$ is defined as the maximal violation of the inequality with respect to the chosen measurement directions. After substituting Eq. (17) into Eq. (16), it can be expressed in closed algebraic form:

$$\begin{aligned} \mathcal{B}[\rho] &:= \max_{\alpha_i, \beta_j} |\alpha_1 \cdot \mathbf{C} \cdot (\beta_1 + \beta_2) + \alpha_2 \cdot \mathbf{C} \cdot (\beta_1 - \beta_2)| \\ &= 2\sqrt{m}. \end{aligned} \quad (18)$$

Here $\mathbf{C}$ denotes the spin correlation matrix, and m is the sum of two largest eigenvalues of $\mathbf{C}^T \mathbf{C}$ [75], any of which is ≤ 1 . When $\mathcal{B}[\rho]$ lies within $(2, 2\sqrt{2}]$, the bipartite system is Bell nonlocal with $\mathcal{B} = 2\sqrt{2}$ representing the maximal Bell nonlocality.

The bipartite states can be always described by a density matrix $\rho \in \mathcal{T}_{\mathcal{H}_A \otimes \mathcal{H}_B}$, the set of all $(1,1)$ -type tensors in a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces of subsystems A and B . If the density matrix ρ can be expressed as [76]

$$\rho = \sum_i c_i \rho_A^i \otimes \rho_B^i, \quad (19)$$

with $\sum_i c_i = 1$, $\rho_A^i \in \mathcal{T}_{\mathcal{H}_A}$, and $\rho_B^i \in \mathcal{T}_{\mathcal{H}_B}$, then the state ρ is separable and otherwise it is entangled. For a two-qubit system, ρ is separable if and only if the partially transposed density matrix of the composite system with respect to either subsystem ρ^{T_A} or ρ^{T_B} (labeled as ρ^Γ) is positive semidefinite, known as the positive partial transpose criterion [77,78]. Since ρ^Γ can have at most one negative eigenvalue for any two-qubit state [79], a simplified criterion is that ρ is separable if and only if [80,81]

$$\det(\rho^\Gamma) \geq 0. \quad (20)$$

Since the partial transpose preserves the trace ($\text{Tr} \rho = \text{Tr} \rho^\Gamma = 1$), the presence of negative eigenvalues in ρ^Γ necessitates that the sum of the absolute values of the eigenvalues is strictly greater than 1, leading to the definition of the negativity as an entanglement measure [82]:

$$\mathcal{N}[\rho] = \frac{\sum_i |t_i| - 1}{2}, \quad (21)$$

where $\{t_i\}$ are the eigenvalues of ρ^Γ . For qubit systems, the partial transposition on subsystem A or B is equivalent to applying the transformation $\sigma_y \rightarrow -\sigma_y$ on $\mathcal{H}_A$ or $\mathcal{H}_B$. To provide a more comprehensive characterization of quantum correlations alongside negativity, we also consider Wootters' concurrence, defined as [83]

$$\mathcal{C}[\rho] = \max \{0, k_1 - k_2 - k_3 - k_4\}, \quad (22)$$

where k_i ($i = 1, 2, 3, 4$) denote the eigenvalues (sorted in decreasing order) of the Hermitian matrix $R = \sqrt{\sqrt{\rho} \tilde{\rho} \sqrt{\rho}}$. Here, $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$ represents the spin-flipped state of ρ , corresponding to the transformation $\sigma \rightarrow -\sigma$ on $\mathcal{H}_A$ and $\mathcal{H}_B$.

Both negativity and concurrence are reliable entanglement measures for 2-qubit systems given that

- (i) $\mathcal{C}[\rho] = 0$, $\mathcal{N}[\rho] = 0$ indicate a separable state;
- (ii) $\mathcal{C}[\rho] > 0$, $\mathcal{N}[\rho] > 0$ signify an entangled state; and
- (iii) $\mathcal{C}[\rho] = 1$, $\mathcal{N}[\rho] = \frac{1}{2}$ correspond to maximal entanglement.

Generally, no explicit analytical relation links them. For a special case of the rank-1 density matrix $\rho = |\psi\rangle\langle\psi|$, $\mathcal{C}(|\psi\rangle) = 2\mathcal{N}(|\psi\rangle)$ constitutes one of the necessary conditions for the system to be in a pure state. If $|P_L| = 1$ or $|P_T| = 1$, the density matrix $\rho_{Y\bar{Y}}$ in Eq. (1) reduces to a rank-1 matrix.

Furthermore, the Peres-Horodecki criterion states that an X-state in Eq. (9) is entangled if and only if either $\rho_{22}^X \rho_{33}^X < |\rho_{14}^X|^2$ or $\rho_{11}^X \rho_{44}^X < |\rho_{23}^X|^2$ holds [84] [note that two conditions cannot be simultaneously satisfied [79]]. The Wootters concurrence and negativity of such an X-state are given by [36,69]

$$\mathcal{C}[\rho^X] = 2 \max \left\{ 0, |\rho_{14}^X| - \sqrt{\rho_{22}^X \rho_{33}^X}, |\rho_{23}^X| - \sqrt{\rho_{11}^X \rho_{44}^X} \right\}, \quad (23)$$

$$\begin{aligned} \mathcal{N}[\rho^X] = -\min & \left\{ 0, \frac{\rho_{11}^X + \rho_{44}^X}{2} - \sqrt{\left(\frac{\rho_{11}^X - \rho_{44}^X}{2} \right)^2 + \rho_{23}^X{}^2}, \right. \\ & \left. \times \frac{\rho_{22}^X + \rho_{33}^X}{2} - \sqrt{\left(\frac{\rho_{22}^X - \rho_{33}^X}{2} \right)^2 + \rho_{14}^X{}^2} \right\}, \end{aligned} \quad (24)$$

which yield $\mathcal{C}[\rho_{Y\bar{Y}}^X] = 2\mathcal{N}[\rho_{Y\bar{Y}}^X] = |\lambda_2|$ for $\theta = 0, \pi/2$, and π , recalling that the eigenvalues of $\mathbf{C}^T \mathbf{C}$ derived from Eq. (8) are $\{\lambda_1^2, \lambda_2^2, \lambda_3^2\}$. It is also worth noting that $\mathcal{B}[\rho_{Y\bar{Y}}^X] = 2\sqrt{\lambda_2^2 + \lambda_3^2}$ at $\theta = 0, \pi/2$, and π .

III. LONGITUDINAL BEAM POLARIZATION

In this section, we will discuss the impact of longitudinal polarization of beams on Bell-nonlocality and quantum entanglement. Only the results of $J/\psi \rightarrow Y\bar{Y}$ decay for $Y = \Lambda, \Sigma^+, \Xi^-, \text{ and } \Xi^0$ are shown; see the Appendix for $J/\psi \rightarrow \Sigma^0 \bar{\Sigma}^0$ and $\psi(3686) \rightarrow Y\bar{Y}$. However, the conclusions for $J/\psi \rightarrow \Sigma^0 \bar{\Sigma}^0$ will also be presented in the main text, and any conclusions that do not hold universally across all channels are explicitly noted.

The θ and P_L dependence of the CHSH parameter $\mathcal{B}[\rho_{Y\bar{Y}}^{P_L}]$ is shown in Fig. 2 with the parameters in Table I. Note that $\mathcal{B}(\theta) = \mathcal{B}(\pi - \theta)$ with $\theta \in [0, \pi]$, and $\mathcal{B}(P_L) = \mathcal{B}(-P_L)$ with $P_L \in [-1, 1]$. It can be seen that increasing P_L enhances Bell-nonlocality across the full range of θ with the exception of $\theta = 0$ and π , under which angle $\mathcal{B} \equiv 2$ is independent of P_L [see Eq. (12)]. In Fig. 2, the area between two dashed lines in each panel represents

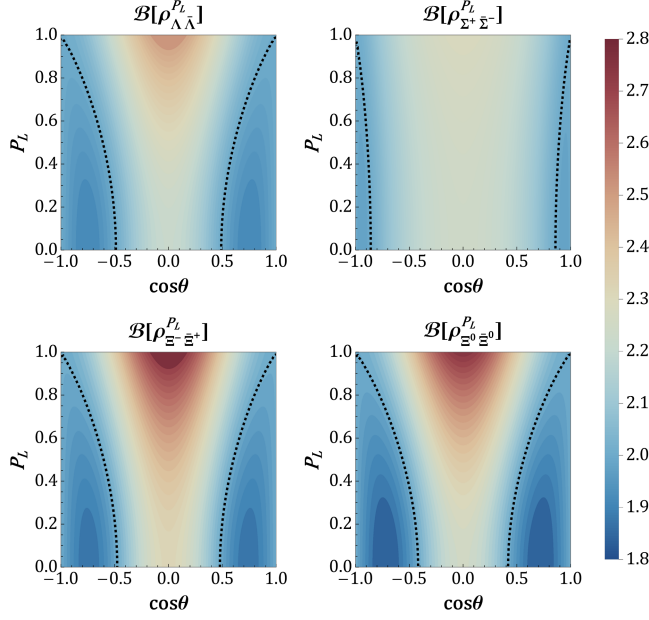


FIG. 2. CHSH parameter $\mathcal{B}[\rho_{YY}^{P_L}]$ as a function of $\cos \theta$ (θ is the scattering angle) and longitudinal beam polarization degree P_L in $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Xi^-, \text{ and } \Xi^0$. The dashed curve is for $\mathcal{B}[\rho_{YY}^{P_L}] = 2$.

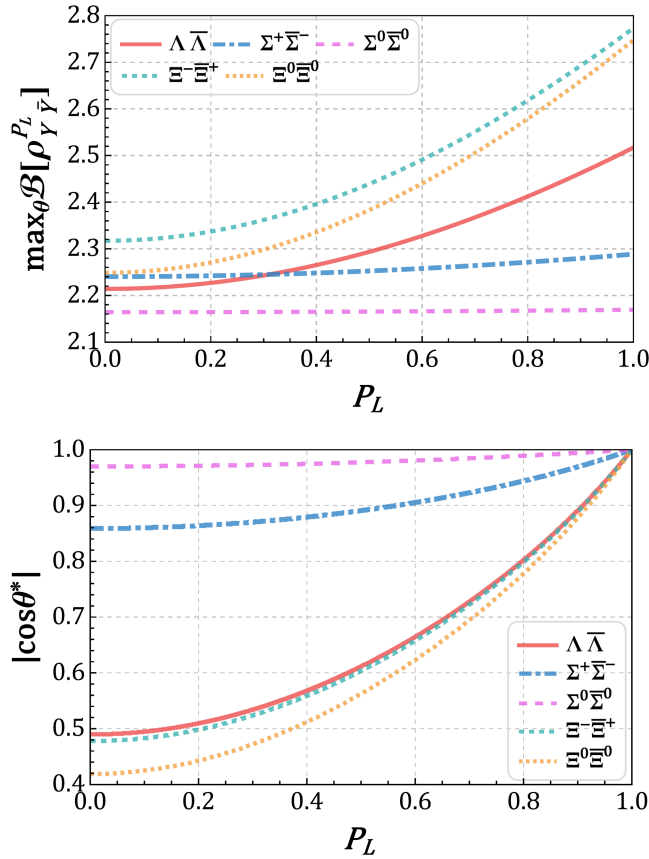


FIG. 3. The $\max_{\theta} \mathcal{B}[\rho_{YY}^{P_L}]$ (upper panel) and $\cos \theta^*$ (lower panel, see main text for definition of θ^*) as functions of P_L in $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Xi^-, \text{ and } \Xi^0$.

Bell-nonlocality $\mathcal{B} \in (2, \max_{\theta} \mathcal{B}]$ with $\max_{\theta} \mathcal{B}$ being the largest CHSH parameter. As can be seen in Eq. (11), the $\max_{\theta} \mathcal{B}$ is present at $\theta = \pi/2$ at any P_L for all channels:

$$\max_{\theta} \mathcal{B} = \mathcal{B}\left(\theta = \frac{\pi}{2}\right) = 2\sqrt{1 + \alpha_{\psi}^2 + P_L^2 \beta_{\psi}^2} < 2\sqrt{2}, \quad (25)$$

which is displayed in the upper panel of Fig. 3. Above relation indicates that the impact of P_L on $\max_{\theta} \mathcal{B}$ is mediated by the parameter β_{ψ} . For the Σ^+ and Σ^0 channels, β_{ψ} is extremely small—a direct consequence of their small $\Delta\Phi$ (see Table I). Therefore, the contribution from the lepton beam polarization is strongly suppressed, rendering $\max_{\theta} \mathcal{B}$ largely insensitive to P_L for these two channels.

Increasing P_L expands the angular coverage of Bell-nonlocality as can also be seen in Fig. 2. The corresponding angle for $\mathcal{B}[\rho_{YY}^{P_L}] = 2$ is defined as the critical angle θ^* , shown in the lower panel of Fig. 3. Here $|\cos \theta^*|$ moves to 1 when P_L is approaching the maximum degree, so that the system of hyperon pairs is always Bell-nonlocal across the range $\theta \in (0, \pi)$ at $P_L = 1$. It is noted that, for Σ^+ and Σ^0 channels, θ^* is already close to zero in the case of unpolarized beams.

Figure 4 shows the θ and P_L dependence of the concurrence $\mathcal{C}[\rho_{YY}^{P_L}]$. The negativity $\mathcal{N}[\rho_{YY}^{P_L}]$ is shown by Fig. 15 in the Appendix. It can be seen that a higher degree of longitudinal polarization leads to enhanced concurrence $\mathcal{C}$ and negativity $\mathcal{N}$ over the full θ range with the exception of $\theta = 0$ and π , from Eqs. (12), (23), and (24), under which angles $\mathcal{C} = 2\mathcal{N} \equiv 0$ means that the states are separable and independent on P_L .

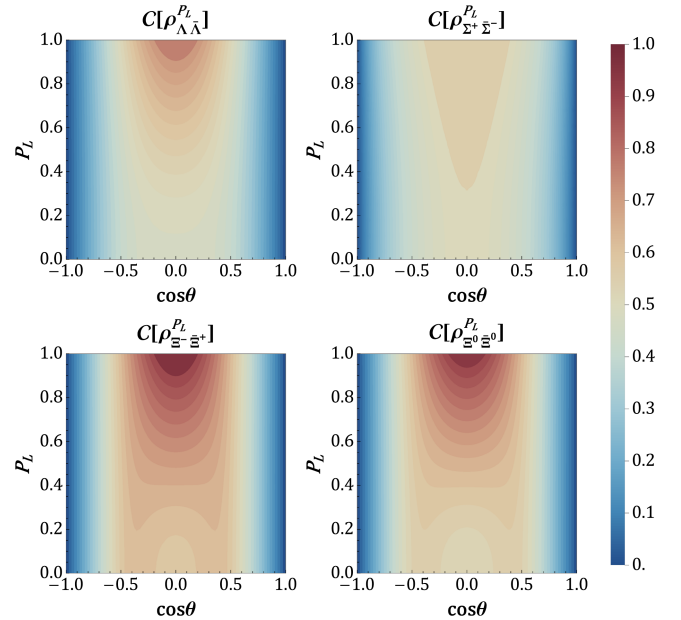


FIG. 4. Concurrence $\mathcal{C}[\rho_{YY}^{P_L}]$ as a function of $\cos \theta$ and P_L in $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Xi^-, \text{ and } \Xi^0$.

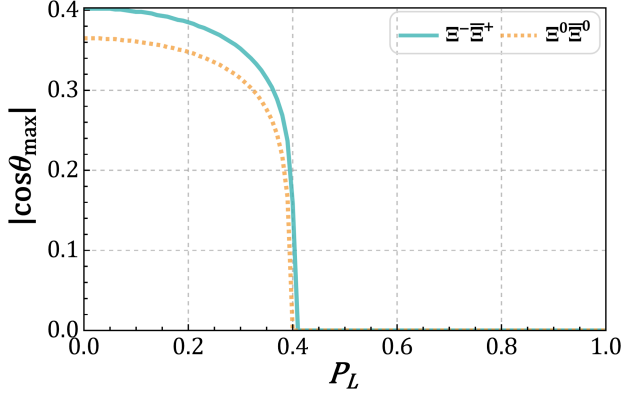


FIG. 5. The $\cos \theta_{\max}$ (see main text for definition of $\theta_{\max}$) as a function of P_L in $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Xi^-$ and Ξ^0 .

The largest negativity with respect to θ , labeled as $\max_{\theta} \mathcal{N}$, locates at $\theta = \pi/2$ across the full P_L range for all channels. The largest concurrence with respect to θ , denoted as $\max_{\theta} \mathcal{C}$, lies

- (a) at $\theta = \pi/2$ across the full P_L range for Λ , Σ^+ , and Σ^0 channels; and
- (b) at the angles of $\theta = \theta_{\max}$ and $\pi - \theta_{\max}$ when P_L is small, and moves to $\theta = \pi/2$ as P_L increases to certain values for Ξ^- and Ξ^0 channels.

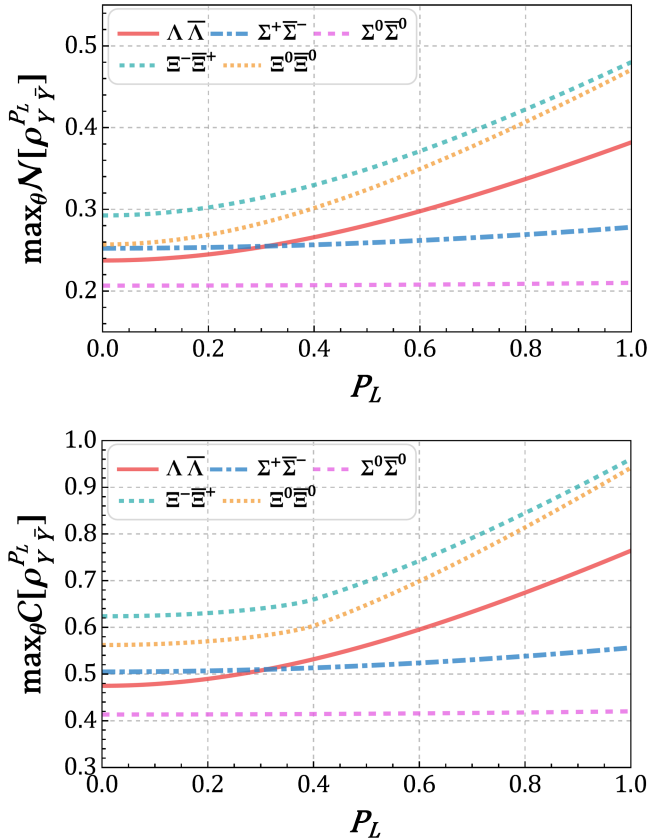


FIG. 6. The $\max_{\theta} \mathcal{N}[\rho_{Y\bar{Y}}^{P_L}]$ (upper panel) and $\max_{\theta} \mathcal{C}[\rho_{Y\bar{Y}}^{P_L}]$ (lower panel) as functions of P_L in $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Lambda$, Σ^+ , Σ^0 , Ξ^- , and Ξ^0 .

For the latter case, $|\cos \theta_{\max}|$ is plotted as a function of P_L in Fig. 5. It can be found that, at $P_L \gtrsim 0.4$, $\max_{\theta} \mathcal{C}$ is shifted to $\theta = \pi/2$ for both channels. From Eqs. (11), (23), and (24), one has

$$\mathcal{C}\left(\theta = \frac{\pi}{2}\right) = 2\mathcal{N}\left(\theta = \frac{\pi}{2}\right) = \sqrt{\alpha_{\psi}^2 + P_L^2 \beta_{\psi}^2}, \quad (26)$$

As shown in Fig. 6, $\max_{\theta} \mathcal{N}$ and $\max_{\theta} \mathcal{C}$ increase monotonically with increasing P_L for all channels. They both reach the largest extrema at $P_L = 1$, but not maximal entanglement since $\max_{\theta} \mathcal{C} < 1$ persists. For the exact same reason, $\max_{\theta} \mathcal{N}$ and $\max_{\theta} \mathcal{C}$ of Σ^+ and Σ^0 channels are insensitive to P_L , as is the case for $\max_{\theta} \mathcal{B}$.

IV. TRANSVERSE BEAM POLARIZATION

The situation for transverse polarization is more complicated considering that it introduces the dependence on the azimuthal angle ϕ with respect to the scattering plane. Figure 7 displays the P_T and ϕ dependence of the largest CHSH parameter with respect to θ , labeled as $\max_{\theta} \mathcal{B}[\rho_{Y\bar{Y}}^{P_T}]$, with a note of $\mathcal{B}(\theta) = \mathcal{B}(\pi - \theta)$ with $\theta \in [0, \pi]$, $\mathcal{B}(P_T) = \mathcal{B}(-P_T)$ with $P_T \in [-1, 1]$, and $\mathcal{B}(\phi) = \mathcal{B}(\pi - \phi)$, $\mathcal{B}(\phi) = \mathcal{B}(\pi + \phi)$ with $\phi \in [0, \pi]$. Here $\max_{\theta} \mathcal{B}$ appears at $\theta = 0, \pi$ or $\theta = \pi/2$ when $0 \leq P_T < 1$, which can be derived from Eq. (13):

$$\mathcal{B}(\theta = 0, \pi) = 2\sqrt{1 + P_T^4}, \quad (27)$$

$$\mathcal{B}\left(\theta = \frac{\pi}{2}\right) = \frac{2\sqrt{1 + \alpha_{\psi}^2 + 4P_T^2 \alpha_{\psi} \cos 2\phi + P_T^4 \varepsilon}}{1 + P_T^2 \alpha_{\psi} \cos 2\phi}, \quad (28)$$

with $\varepsilon = (1 + \alpha_{\psi}^2) \cos^2 2\phi + \gamma_{\psi}^2 \sin^2 2\phi$. When $P_T = 1$, $\max_{\theta} \mathcal{B}$ occurs at

- (a) $\theta = 0, \pi$ and $\theta = \pi/2$ simultaneously if $\phi = \pi/2 + k\pi$ ($k = 0, 1$),
- (b) the full θ range if $\phi = k\pi$ ($k = 0, 1, 2$), and
- (c) $\theta = 0, \pi$ or $\theta = \pi/2$ as determined by Eqs. (27) and (28) otherwise.

The dashed curve $\mathcal{B}(\theta = 0, \pi) = \mathcal{B}(\theta = \pi/2)$ in Fig. 7 divides each panel into two regions. For the region above the dashed curve, $\max_{\theta} \mathcal{B}$ occurs at $\theta = 0, \pi$ and attains maximal violation of the CHSH inequality for arbitrary ϕ at $P_T = 1$ from Eq. (27). For the region below the dashed curve, $\max_{\theta} \mathcal{B}$ occurs at $\theta = \pi/2$ and holds the maximal violation for $\phi = k\pi/2$ ($k = 0, 1, 2, 3, 4$) at $P_T = 1$ from Eq. (28). Furthermore, the maximal violation is attained across the full θ range for $\phi = k\pi$ ($k = 0, 1, 2$) at $P_T = 1$. It is found for both regions that $\max_{\theta} \mathcal{B} \geq 2$ across full ϕ coverage, and $P_T = 1$ drives $\max_{\theta} \mathcal{B}$ to approach $2\sqrt{2}$ at any fixed ϕ .

As can be seen in Fig. 7, the largest value of $\max_{\theta} \mathcal{B}[\rho_{Y\bar{Y}}^{P_T}]$, here labeled as $\max_{\theta, \phi} \mathcal{B}[\rho_{Y\bar{Y}}^{P_T}]$, is attained at

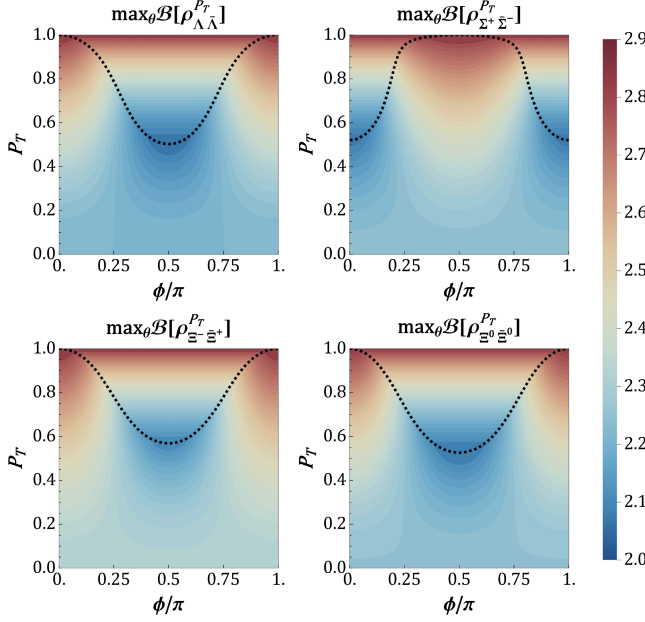


FIG. 7. The $\max_{\theta} B[\rho_{Y\bar{Y}}^{P_T}]$ as a function of azimuthal angle ϕ and transverse beam polarization degree P_T in $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Xi^-,$ and Ξ^0 . The dashed curve is for $B(\theta = 0, \pi) = B(\theta = \pi/2)$.

specific angles ϕ with P_T held constant. As shown in upper panel of Fig. 8, $\max_{\theta, \phi} B$ is always enhanced with the increase of P_T , and occurs at specific angles of (a) $\phi = k\pi$ ($k = 0, 1, 2$) for $\Lambda, \Xi^-,$ and Ξ^0 channels; and (b) $\phi = \pi/2 + k\pi$ ($k = 0, 1$) for Σ^+, Σ^0 channels. Here $\max_{\theta, \phi} B$ is attained at different azimuthal angles ϕ depending on the specific decay channel. This is primarily because the P_T -dependence of $\max_{\theta} B$ is dominated by the term $P_T^2 \alpha_{\psi} \cos 2\phi$ [see Eq. (28)]. Specifically, the parameter α_{ψ} for the $J/\psi \rightarrow \Sigma^+ \bar{\Sigma}^+$ and $\Sigma^0 \bar{\Sigma}^0$ channels is negative, which is opposite to those of the $\Lambda \bar{\Lambda}, \Xi^- \bar{\Xi}^-$, and $\Xi^0 \bar{\Xi}^0$ channels; this sign difference directly shifts the optimal angle ϕ . This distinct behavior is visually reflected by the different trajectories of the dashed curves in Fig. 7. Meanwhile, since the absolute magnitude of α_{ψ} is comparable across all considered channels, their overall sensitivities to P_T remain remarkably similar.

Violation of the CHSH inequality across the full θ range can be achieved by increasing the transverse beam polarization degree P_T . Therefore, a minimal P_T required to achieve $B[\rho_{Y\bar{Y}}^{P_T}] \geq 2$ across full θ range, labeled as $\min P_T$, can be defined as a function of ϕ , as shown in lower panel of Fig. 8. Here $\min P_T$ increases with the increase of $\phi \in [0, \pi/2]$ for the $\Lambda, \Xi^-,$ and Ξ^0 channels, opposite to the trends of the Σ^+ and Σ^0 channels. Once again, this behavioral divergence traces back to the aforementioned sign flip of α_{ψ} among channels.

The largest concurrence and negativity with respect to θ , labeled as $\max_{\theta} C[\rho_{Y\bar{Y}}^{P_T}]$ and $\max_{\theta} \mathcal{N}[\rho_{Y\bar{Y}}^{P_T}]$, respectively, can

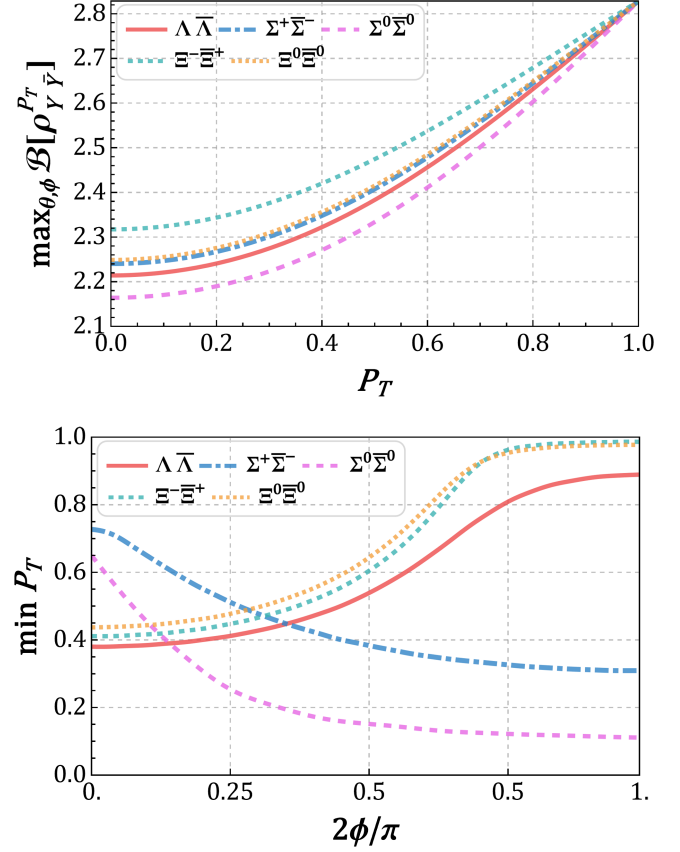


FIG. 8. The $\max_{\theta, \phi} B[\rho_{Y\bar{Y}}^{P_T}]$ as a function of P_T (upper panel) and the minimal transverse polarization $\min P_T(\phi)$ required to achieve $B[\rho_{Y\bar{Y}}^{P_T}] = 2$ as a function of ϕ (lower panel) for all five channels.

be similarly defined. The behavior of $\max_{\theta} C$ in Fig. 9 and $\max_{\theta} \mathcal{N}$ in the Appendix as a function of P_T and ϕ is similar to that of $\max_{\theta} B$. When $0 \leq P_T < 1$, the position of $\max_{\theta} C$ is found at the following:

- (a) For $\Lambda, \Sigma^-,$ and Σ^0 channels: either at $\theta = 0, \pi$ for the region above the dashed curve or at $\theta = \pi/2$ for the region below the dashed curve.
- (b) For Ξ^- and Ξ^0 channels: either at $\theta = 0, \pi$ for the region above the dashed curve or at $\cos \theta = 0.368$ (for Ξ^-) and 0.403 (for Ξ^0) for the region below the dashed curve.

For the position of $\max_{\theta} \mathcal{N}$, case a is applied for all channels. When $P_T = 1$, $\max_{\theta} C$ and $\max_{\theta} \mathcal{N}$ locate at the same position as the situation of $\max_{\theta} B$.

From Eqs. (13), (14), (23), and (24),

$$C(\theta = 0, \pi) = 2\mathcal{N}(\theta = 0, \pi) = P_T^2, \quad (29)$$

$$C\left(\theta = \frac{\pi}{2}\right) = 2\mathcal{N}\left(\theta = \frac{\pi}{2}\right) = \frac{\sqrt{\alpha_{\psi}^2 + 2P_T^2 \alpha_{\psi} \cos 2\phi + P_T^4 \eta}}{1 + P_T^2 \alpha_{\psi} \cos 2\phi}, \quad (30)$$

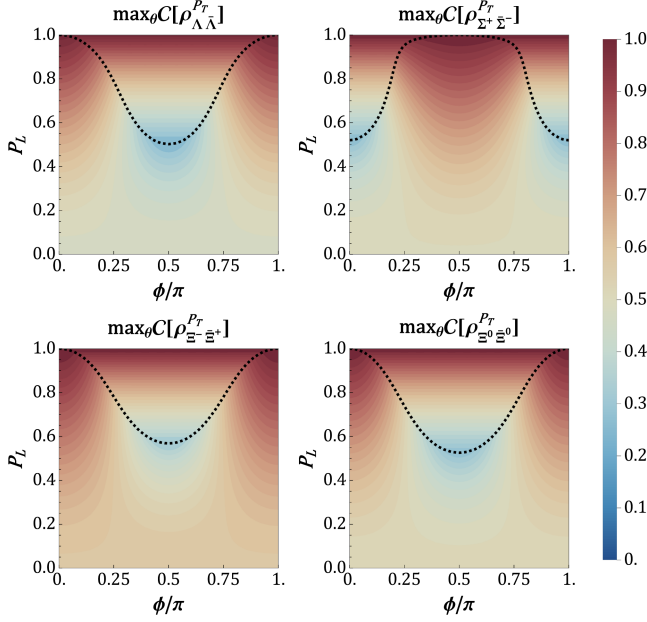


FIG. 9. The $\max_{\theta} C[\rho_T]$ as a function of ϕ and P_T in $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Xi^-, \text{ and } \Xi^0$. See main text for a detailed explanation of the dashed lines.

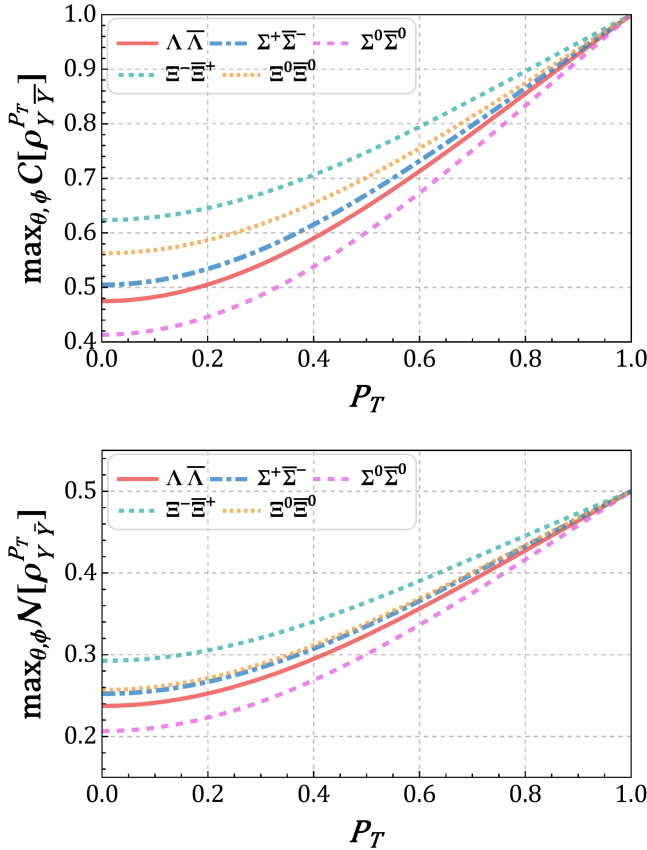


FIG. 10. The $\max_{\theta, \phi} C[\rho_{Y\bar{Y}}^{P_T}]$ (upper panel) and $\max_{\theta, \phi} \mathcal{N}[\rho_{Y\bar{Y}}^{P_T}]$ (lower panel) as functions of P_T in $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Xi^-, \text{ and } \Xi^0$.

with $\eta = \gamma_{\psi}^2 \sin^2 2\phi + \cos^2 2\phi$. From Eq. (29), maximal entanglement is attained at $\theta = 0$ and π for arbitrary ϕ if $P_T = 1$. From Eq. (30), maximal entanglement is attained at $\theta = \pi/2$ for $\phi = k\pi/2$ ($k = 0, 1, 2, 3, 4$) if $P_T = 1$. Furthermore, maximal entanglement occurs across the full θ range for $\phi = k\pi$ ($k = 0, 1, 2$) at $P_T = 1$.

Similar to $\max_{\theta, \phi} \mathcal{B}$, $\max_{\theta, \phi} \mathcal{C}$ and $\max_{\theta, \phi} \mathcal{N}$ can be defined as the largest value of $\max_{\theta} \mathcal{C}$ and $\max_{\theta} \mathcal{N}$ at specific angles ϕ with P_T held constant, respectively. Here $\max_{\theta, \phi} \mathcal{C}$ and $\max_{\theta, \phi} \mathcal{N}$, locating at the same azimuthal angle ϕ as that of $\max_{\theta, \phi} \mathcal{B}$, increase monotonically as the increase of P_T as shown in Fig. 10. Their behavior closely mirrors that of $\max_{\theta, \phi} \mathcal{B}$, as their P_T -dependence is similarly governed by the aforementioned term $P_T^2 \alpha_{\psi} \cos 2\phi$. For $P_T = 1$, the system can always attain maximal entanglement under the proper selection of ϕ and θ angles, as also can be seen in Fig. 9 where $\max_{\theta} \mathcal{C} = 2 \max_{\theta} \mathcal{N} = 1$ at any fixed ϕ at $P_T = 1$.

V. DISCUSSION

A. Hierarchy of quantum correlations

Bell nonlocality and quantum entanglement satisfy the following hierarchy with respect to quantumness:

Bell nonlocality $\subset$ quantum entanglement.

Quantitatively, this fundamental hierarchy is manifested through the relationship between their respective measures. For a two-qubit system, the CHSH parameter characterizing Bell nonlocality is strictly upper-bounded by the concurrence [85]

$$\mathcal{B}[\rho] \leq 2\sqrt{1 + C^2[\rho]}, \quad (31)$$

of which the equality is saturated iff (if and only if) the singular values of the corresponding correlation matrix $\mathbf{C}$ are $\{1, C[\rho], C[\rho]\}$ [85]. It shall be noted that the singular values of real and symmetric matrix $\mathbf{C}$ correspond to the absolute values of its eigenvalues.

Furthermore, an upper bound of Wootters' concurrence is given in terms of the polarization of subsystems [86,87]:

$$C[\rho] \leq \min \left\{ \sqrt{1 - \|\mathbf{B}^+\|^2}, \sqrt{1 - \|\mathbf{B}^-\|^2} \right\}, \quad (32)$$

with $\|\mathbf{B}\| = \sqrt{\text{Tr}(\mathbf{B}^T \mathbf{B})}$. The bound is saturated iff ρ is a pure state. The above relation reveals that the maximum achievable entanglement of the system is bounded by the polarization of its subsystem. Specifically, the upper bound of entanglement of the system decreases as the polarization of its subsystem

increases. By combining Eqs. (31) and (32), the polarization of the subsystems imposes an upper bound on the CHSH parameter of the system:

$$\mathcal{B}[\rho] \leq \min \left\{ 2\sqrt{2 - \|\mathbf{B}^+\|^2}, 2\sqrt{2 - \|\mathbf{B}^-\|^2} \right\}. \quad (33)$$

The above inequality becomes an equality if and only if Eqs. (31) and (32) are both saturated. As a special case, Eq. (33) is trivially saturated for elementary fermion pairs, which are always maximally entangled with vanishing polarization ($\|\mathbf{B}^\pm\| = 0$) of subsystems [10,11].

For hyperon-antihyperon pairs without CP violation as considered here, the correlation matrixes $\mathbf{C}$ in Eqs. (5) and (7) are real and symmetric after a rotation of the coordinate system. Using the results from Sec. II, one can easily prove that the singular values of the correlation matrix $\mathbf{C}$ of $\rho_{Y\bar{Y}}^X$ at $\theta = 0, \pi/2$, and π are $\{1, |\lambda_2|, |\lambda_2|\} = \{1, \mathcal{C}[\rho], \mathcal{C}[\rho]\}$, thus saturating Eq. (31) irrespective of the beam polarization, as shown in Secs. III and IV. Furthermore, from Eqs. (4) and (6), $\|\mathbf{B}^+\| = \|\mathbf{B}^-\|$ always holds and

$$\|\mathbf{B}_L\| = \frac{1}{\chi_L} \sqrt{\beta_\psi^2 \sin^2 \theta \cos^2 \theta + P_L^2 (\beta_\psi^2 \cos^2 \theta + \gamma_\psi^2)}, \quad (34)$$

$$\|\mathbf{B}_T\| = \frac{\beta_\psi \sin \theta}{\chi_T} \sqrt{(1 - P_T^2 \cos 2\phi)^2 \cos^2 \theta + P_T^4 \sin^2 2\phi}. \quad (35)$$

A necessary condition for the existence of maximal entanglement ($\mathcal{C} = 1$) is that the system is in a pure state. When $P_L = 1$ or $P_T = 1$, $\rho_{Y\bar{Y}}$ is in a pure state and Eq. (33) holds as equality. As can be seen, hyperon polarization remains nonzero ($\|\mathbf{B}_L^\pm\| \neq 0$) across full scattering angles. Therefore maximal entanglement is unachievable since $\mathcal{C}[\rho_{Y\bar{Y}}^{P_L}] < 1$ as constrained by Eq. (32). Even so, the system accesses the largest possible entanglement. On the other hand, tuning the transverse polarization degree to maximum degree makes it possible to realize vanishing polarization of hyperons by properly choosing θ and ϕ as shown by Eq. (35) and discussed in detail in Sec. IV.

Taking the Λ channel as an example, Fig. 11 shows the evolution of the upper bounds in Eqs. (31) and (33) as the longitudinal or transverse beam polarization degree increases in comparison of the CHSH parameters $\mathcal{B}[\rho_{\Lambda\bar{\Lambda}}]$. The upper bound in Eq. (31) is more stringent than that in Eq. (33). When P_L or P_T approaches the maximum degree, the two upper bounds become close to each other and ultimately equal at any scattering angle when the system is in a pure state. It can be also seen in Fig. 11 that if $P_T = 1$, maximal entanglement is attained across the full θ range at particular angles $\phi = k\pi$, ($k = 0, 1, 2$).

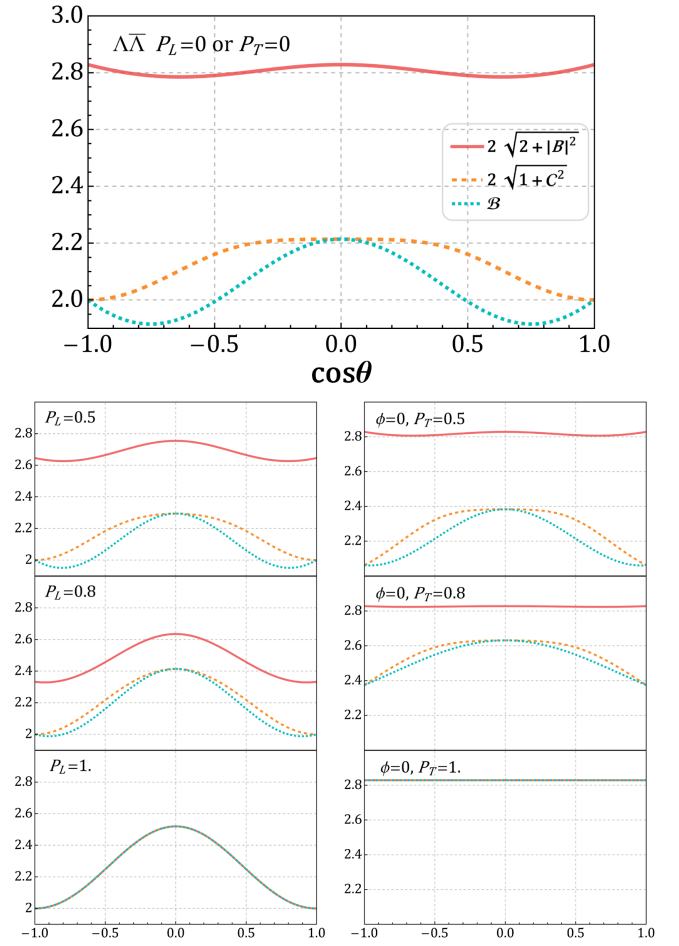


FIG. 11. Upper bounds in Eq. (31) (orange dashed lines) and Eq. (33) (red solid lines) for the Λ channel are compared to the CHSH parameter $\mathcal{B}[\rho]$ (green dotted lines) as a function of $\cos \theta$. Upper panel: unpolarized lepton beams. Lower left panels: longitudinal polarized electron beam. Lower right panels: transversely polarized lepton beams with $\phi = 0$. The polarization degrees P_L or P_T are indicated in each panel.

B. Separable state with transversely polarized beams

In the case of transverse polarized beams, minimum values of $\mathcal{B}[\rho_{Y\bar{Y}}^{P_T}]$ and $\mathcal{C}[\rho_{Y\bar{Y}}^{P_T}]$, labeled as $\min_{\theta, \phi} \mathcal{B}[\rho_{Y\bar{Y}}^{P_T}]$ and $\min_{\theta, \phi} \mathcal{C}[\rho_{Y\bar{Y}}^{P_T}]$, can be found across the full range of solid angles, as shown in Fig. 12. It is observed that, over a substantial P_T range, the system is of vanishing entanglement, and the CHSH inequality is not violated, indicating the separability of the system at a certain azimuthal angle ϕ^* :

- (a) $\phi^* = k\pi$, ($k = 0, 1, 2$) for Σ^+ , Σ^0 channels with $\alpha_\psi < 0$; and
- (b) $\phi^* = \pi/2 + k\pi$, ($k = 0, 1$) for Λ , Ξ^- , and Ξ^0 channels with $\alpha_\psi > 0$.

At these ϕ^* , $\mathbf{B}_T^\pm$ in Eq. (6) has only one nonzero component, and $\rho_{Y\bar{Y}}^{P_T}$ as an X -state with the parameters in Eq. (8) given by

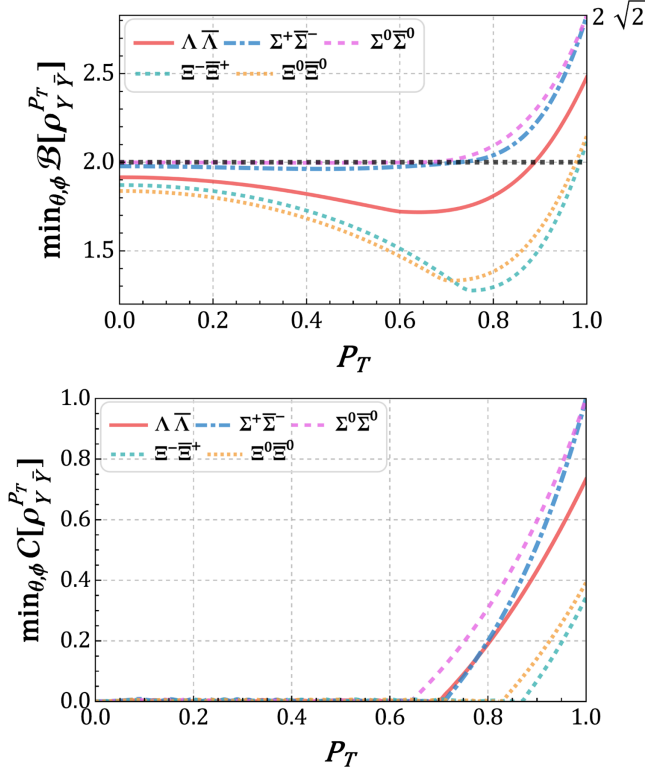


FIG. 12. The $\min_{\theta, \phi} \mathcal{B}[\rho_{Y\bar{Y}}^{P_T}]$ (upper panel) and $\min_{\theta, \phi} \mathcal{C}[\rho_{Y\bar{Y}}^{P_T}]$ (lower panel) as a function of P_T in $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Sigma^0, \Xi^-, \Xi^0$.

$$\begin{aligned}
 B_z &= -\frac{\beta_\psi(1 \mp P_T^2) \sin 2\theta}{2\chi'_T}, \\
 \lambda_1 &= \frac{(1 \pm P_T^2)(1 + \alpha_\psi) + (1 \mp P_T^2)\sqrt{\zeta}}{2\chi'_T}, \\
 \lambda_2 &= \frac{(1 \pm P_T^2)(1 + \alpha_\psi) - (1 \mp P_T^2)\sqrt{\zeta}}{2\chi'_T}, \\
 \lambda_3 &= -\frac{\alpha_\psi \sin^2 \theta \pm P_T^2(1 + \alpha_\psi \cos^2 \theta)}{\chi'_T}, \quad (36)
 \end{aligned}$$

with $\chi'_T = 1 + \alpha_\psi \cos^2 \theta \pm P_T^2 \alpha_\psi \sin^2 \theta$ and $\zeta = (\alpha_\psi + \cos 2\theta)^2 + \gamma_\psi^2 \sin^2 2\theta$. The upper and lower signs of “ $\pm$ ” and “ $\mp$ ” are for case a and case b of ϕ^* , respectively. The $\min_{\phi} \mathcal{B}[\rho_{Y\bar{Y}}^{P_T}]$ is given by

$$\min_{\phi} \mathcal{B}[\rho_{Y\bar{Y}}^{P_T}] = 2\sqrt{m'}, \quad (37)$$

where m' is the sum of the two largest eigenvalues of $\{\lambda_1^2, \lambda_2^2, \lambda_3^2\}$ in Eq. (36). From Eqs. (23) and (24), $\min_{\phi} \mathcal{C}[\rho_{Y\bar{Y}}^{P_T}]$ and $\min_{\phi} \mathcal{N}[\rho_{Y\bar{Y}}^{P_T}]$ are given by

TABLE II. Critical P_T values of $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Sigma^0, \Xi^-,$ and Ξ^0 .

$J/\psi \rightarrow$	Bell nonlocality	Nonseparability
$\Lambda\bar{\Lambda}$	0.889	0.699
$\Sigma^+\bar{\Sigma}^-$	0.727	0.711
$\Sigma^0\bar{\Sigma}^0$	0.681	0.609
$\Xi^-\bar{\Xi}^+$	0.985	0.869
$\Xi^0\bar{\Xi}^0$	0.971	0.820

$$\begin{aligned}
 \min_{\phi} \mathcal{C}[\rho_{Y\bar{Y}}^{P_T}] &= \frac{1}{2} \max \left\{ 0, \lambda_1 - \lambda_2 + \lambda_3 - 1, \right. \\
 &\quad \left. \times \lambda_1 + \lambda_2 - \sqrt{(1 + \lambda_3)^2 - 4B_z^2} \right\}, \quad (38)
 \end{aligned}$$

$$\begin{aligned}
 \min_{\phi} \mathcal{N}[\rho_{Y\bar{Y}}^{P_T}] &= \frac{1}{4} \max \left\{ 0, \lambda_1 - \lambda_2 + \lambda_3 - 1, \right. \\
 &\quad \left. \times \sqrt{(\lambda_1 + \lambda_2)^2 + 4B_z^2} - 1 - \lambda_3 \right\}. \quad (39)
 \end{aligned}$$

For each channel, when the polarization degree P_T is increased to a critical value, nonseparability of the system is emergent. If P_T keeps increasing to another critical value, the Bell nonlocality of the system appears with the CHSH inequality violated. This is consistent with the hierarchy of quantum correlations mentioned in Sec. VA. Those critical values are summarized in Table II.

The system is separable with fixed P_T and ϕ^* at scattering angles θ_{Sep} and $\pi - \theta_{\text{Sep}}$, indicating from Eqs. (38) and (39) that

$$\min_{\phi} \mathcal{C}(\theta_{\text{Sep}}) = \min_{\phi} \mathcal{N}(\theta_{\text{Sep}}) = 0. \quad (40)$$

Furthermore, a useful theorem states that if a rank-2 density matrix ρ is separable, its partially transposed density operator ρ^Γ is also of rank-2 [79]. Since $\rho_{Y\bar{Y}}$ is always of rank-2, $\rho_{Y\bar{Y}}^\Gamma$ must also be rank-2 within the separable region, implying the relation

$$\det \rho_{Y\bar{Y}}^\Gamma|_{\theta_{\text{Sep}}} = 0, \quad (41)$$

which implies that the separability criterion in Eq. (20) is exactly saturated. Here $\cos \theta_{\text{Sep}}$ can be plotted as a function of P_T as shown in Fig. 13 at fixed ϕ^* . For unpolarized lepton beams, the system is separable at $\theta_{\text{Sep}} = 0$ and π . It can be seen that increasing the transverse polarization degree moves θ_{Sep} to the direction of $\pi/2$. There are two θ_{Sep} at some P_T range for $\Lambda\bar{\Lambda}$, $\Xi^+\bar{\Xi}^-$, and $\Xi^0\bar{\Xi}^0$ pairs. When P_T is large enough, the separate state vanishes for all channels.

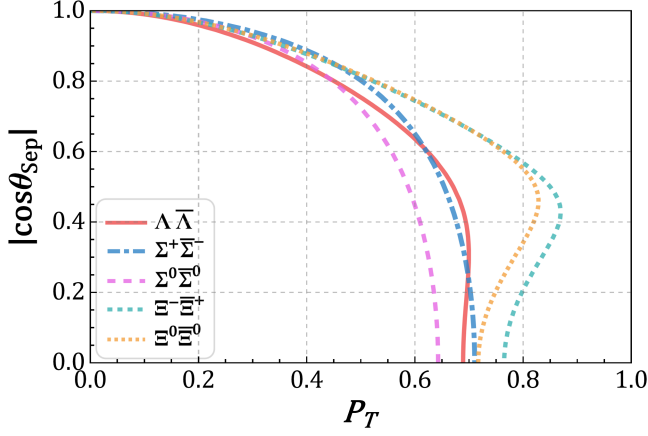


FIG. 13. The $\cos \theta_{\text{Sep}}$ as a function of the beam transverse polarization degree P_T in $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Sigma^0, \Xi^-,$ and Ξ^0 .

VI. CONCLUSION

This work systematically investigates the influence of longitudinal and transverse beam polarization on quantum entanglement and Bell nonlocality of hyperon-antihyperon pairs produced in electron-positron annihilation. Based on the explicit spin density matrix, the CHSH parameter, concurrence, and negativity of the $Y\bar{Y}$ system are quantitatively evaluated. It is found that both polarization configurations can enhance quantum correlations, though the manifestations differ remarkably. Longitudinal polarization of beams broadens the angular range of Bell nonlocality and increases monotonically the entanglement of the system, but fails to attain maximal entanglement even at maximum polarization degree. In contrast, transverse polarization enables the system to be maximally entangled across the full range of scattering angles if choosing optimally the azimuthal angle. The underlying reason is ascribed to different reactions of hyperon polarization to beam polarization modes. The hierarchy of quantum correlations under two polarization configurations of beams and separable states in the case of transversely polarized beams is thoroughly discussed as well.

Compared to the $\Lambda\bar{\Lambda}$, $\Xi^-\bar{\Xi}^-$, and $\Xi^0\bar{\Xi}^0$ channels, the $J/\psi \rightarrow \Sigma^+\bar{\Sigma}^+$ and $\Sigma^0\bar{\Sigma}^0$ channels not only possess negative decay parameters, but also exhibit much smaller relative phases between electric and magnetic couplings. As a result, Bell nonlocality and quantum entanglement in the respective $Y\bar{Y}$ systems exhibit distinctly different sensitivities to the manipulation of lepton beam polarization. As a byproduct, this provides a novel perspective on the anomalous decay parameters of the Σ channels, revealing that these anomalies

physically manifest as distinct responses in quantum correlations.

Transversely polarized beams are already available at BESIII as discussed in literature [48], and longitudinally polarized beams would be feasible at the future Super τ -Charm facility [88]. The findings in this paper highlight the potential of exploiting polarized beams as a tool to study quantum correlations of $Y\bar{Y}$ systems, and also as a means to explore theoretically the quantum information in other high-energy reactions. Furthermore, whether the manipulation of beam polarization can be utilized to mitigate fundamental loopholes (such as the detection or locality loopholes using only classical observables) in Bell tests remains an intriguing open question for particle physics.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX: SUPPLEMENTARY FIGURES FOR J/ψ AND $\psi(3686) \rightarrow Y\bar{Y}$

In Fig. 14, the CHSH parameters and concurrence for $J/\psi \rightarrow \Sigma^0\bar{\Sigma}^0$ are shown as a complementary figure in the main text. In Fig. 15, the negativity of $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Sigma^0, \Xi^-,$ and Ξ^0 is shown under longitudinal and transverse beam polarization configurations. For the transverse beam polarization, the dashed curve $\mathcal{N}(\theta = 0, \pi) = \mathcal{N}(\theta = \pi/2)$ in the lower panels of Fig. 15 divides each panel into two regions. For the region above the dashed curve, $\max_{\theta} \mathcal{N}$ locates at $\theta = 0, \pi$. For the region below the dashed curve, $\max_{\theta} \mathcal{N}$ locates at $\theta = \pi/2$.

The decay parameters of $\psi(3686) \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Sigma^0, \Xi^-,$ and Ξ^0 are collected in Table III. The CHSH parameter $\mathcal{B}[\rho_{Y\bar{Y}}^{P_L}]$, negativity $\mathcal{N}[\rho_{Y\bar{Y}}^{P_L}]$, and concurrence $\mathcal{C}[\rho_{Y\bar{Y}}^{P_L}]$ are shown in Figs. 16 and 17 for $\psi(3686) \rightarrow Y\bar{Y}$

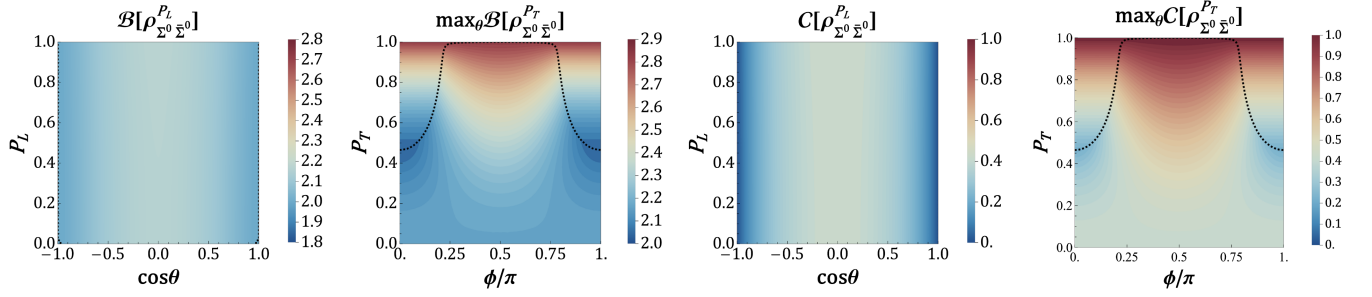


FIG. 14. CHSH parameters and concurrence for $J/\psi \rightarrow \Sigma^0 \bar{\Sigma}^0$. The dashed curves are for $\mathcal{B}(\theta = 0, \pi) = \mathcal{B}(\theta = \pi/2)$ and $\mathcal{C}(\theta = 0, \pi) = \mathcal{C}(\theta = \pi/2)$, respectively.

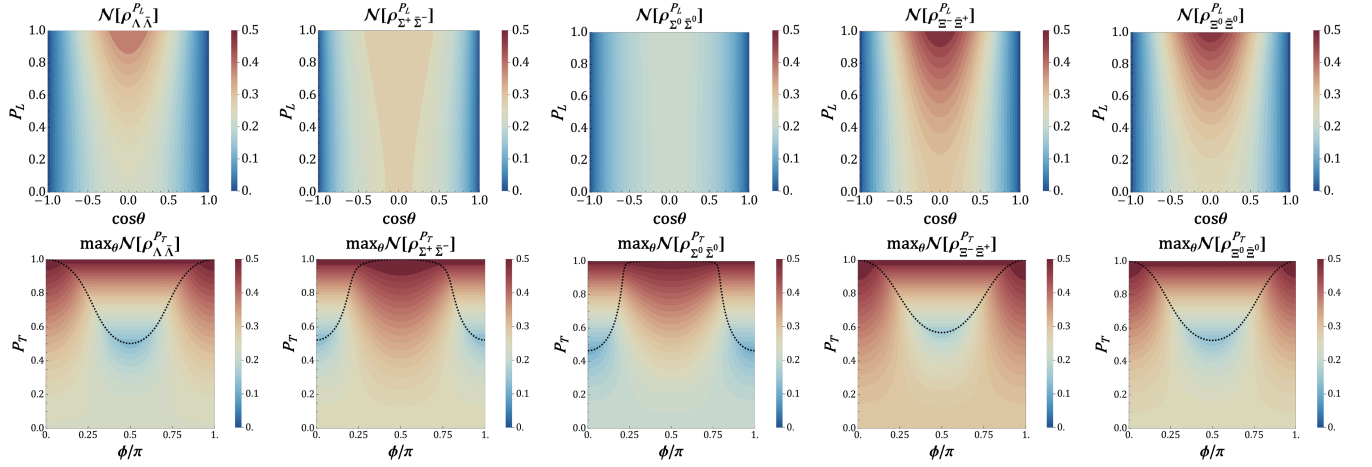


FIG. 15. Negativity $\mathcal{N}[\rho_{YY}^{P_L}]$ as a function of P_L and θ (upper panels), and $\max_{\theta} \mathcal{N}[\rho_{YY}^{P_T}]$ as a function of P_T and ϕ (lower panels) in $J/\psi \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Sigma^0, \Xi^-, \text{ and } \Xi^0$. The dashed curve is $\mathcal{N}(\theta = 0, \pi) = \mathcal{N}(\theta = \pi/2)$.

under the longitudinal beam polarization configuration, and those under the transverse beam polarization configuration are shown in Figs. 18 and 19. Under the transverse beam polarization, the dashed curve

$\mathcal{B}, \mathcal{N}, \mathcal{C}(\theta = 0, \pi) = \mathcal{B}, \mathcal{N}, \mathcal{C}(\theta = \pi/2)$ in Fig. 18 divides each panel into two regions. For the region above the dashed curve, $\max_{\theta} \mathcal{B}, \mathcal{N}, \mathcal{C}$ locates at $\theta = 0, \pi$. For the region below the dashed curve, $\max_{\theta} \mathcal{B}, \mathcal{N}, \mathcal{C}$ locates at $\theta = \pi/2$.

TABLE III. Decay parameters of $\psi(3686) \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Sigma^0, \Xi^-, \text{ and } \Xi^0$.

Decay channels	α_{ψ}	$\Delta\Phi/\text{rad}$	Ref.
$\psi(3686) \rightarrow \Lambda\bar{\Lambda}$	$0.69 \pm 0.07 \pm 0.02$	$0.40 \pm 0.14 \pm 0.03$	[66,89]
$\psi(3686) \rightarrow \Sigma^+\bar{\Sigma}^-$	$0.682 \pm 0.030 \pm 0.011$	$0.397 \pm 0.07 \pm 0.014$	[62,66]
$\psi(3686) \rightarrow \Sigma^0\bar{\Sigma}^0$	$0.814 \pm 0.028 \pm 0.028$	$0.512 \pm 0.085 \pm 0.034$	[63,66]
$\psi(3686) \rightarrow \Xi^-\bar{\Xi}^+$	$0.693 \pm 0.048 \pm 0.049$	$0.667 \pm 0.111 \pm 0.058$	[66,90]
$\psi(3686) \rightarrow \Xi^0\bar{\Xi}^0$	$0.665 \pm 0.086 \pm 0.081$	$-0.050 \pm 0.150 \pm 0.020$	[66,68]

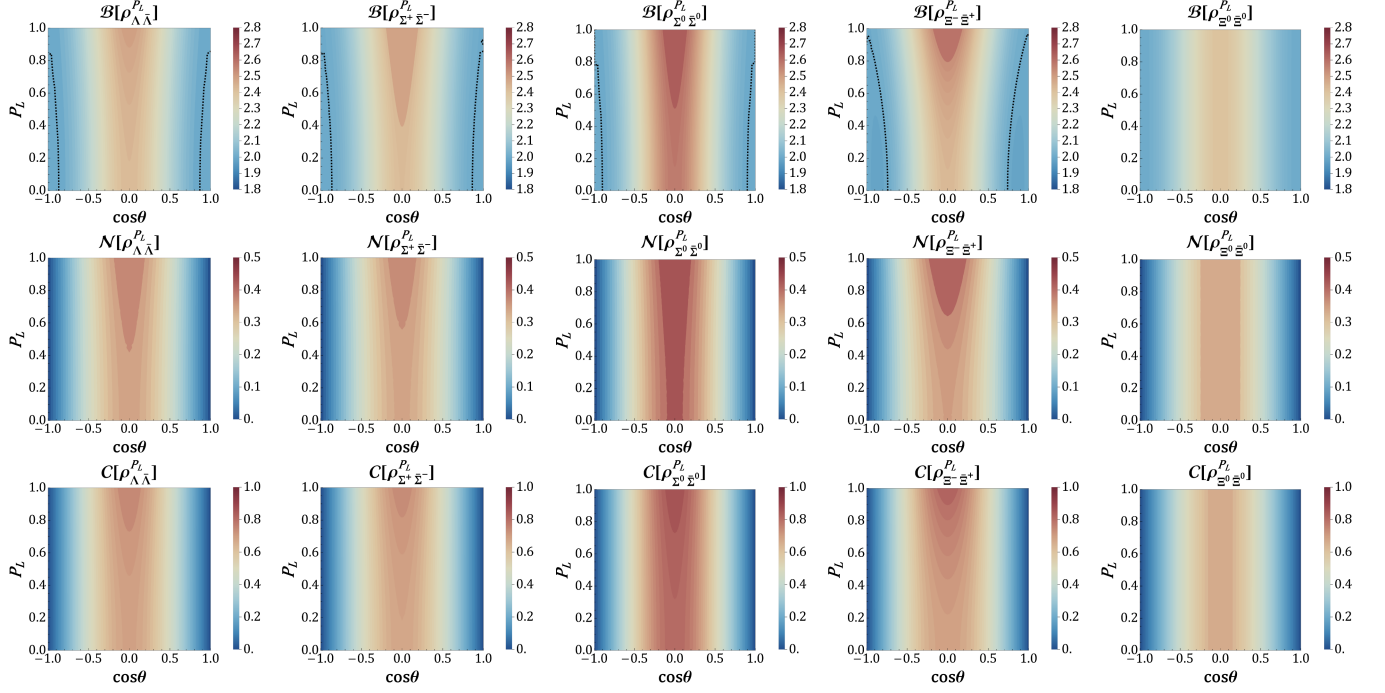


FIG. 16. CHSH parameter $\mathcal{B}[\rho_{Y\bar{Y}}^{P_L}]$ (upper panels), negativity $\mathcal{N}[\rho_{Y\bar{Y}}^{P_L}]$ (middle panels), and concurrence $\mathcal{C}[\rho_{Y\bar{Y}}^{P_L}]$ (lower panels) as a function of P_L and θ in $\psi(3686) \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Sigma^0, \Xi^-,$ and Ξ^0 . The dashed line in each upper panel is $\mathcal{B} = 2$.

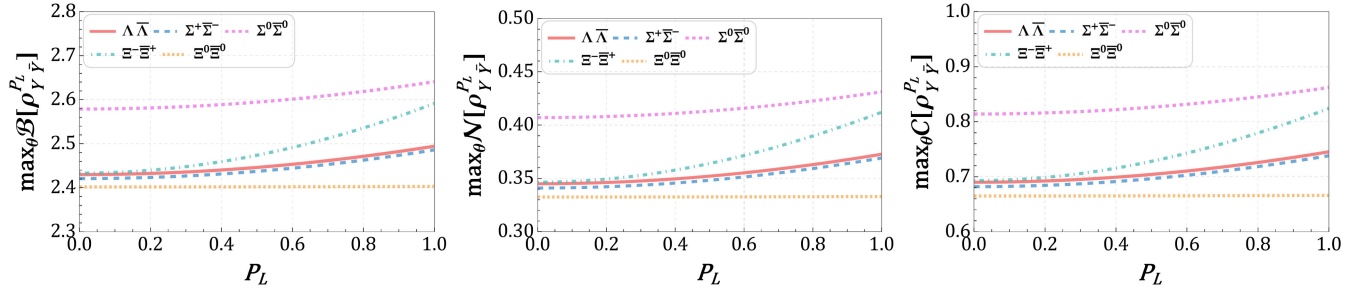


FIG. 17. The $\max_{\theta} \mathcal{B}[\rho_L]$ (left), negativity $\max_{\theta} \mathcal{N}[\rho_L]$ (middle), and concurrence $\max_{\theta} \mathcal{C}[\rho_L]$ (right) as a function P_L and θ in $\psi(3686) \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Sigma^0, \Xi^-,$ and Ξ^0 .

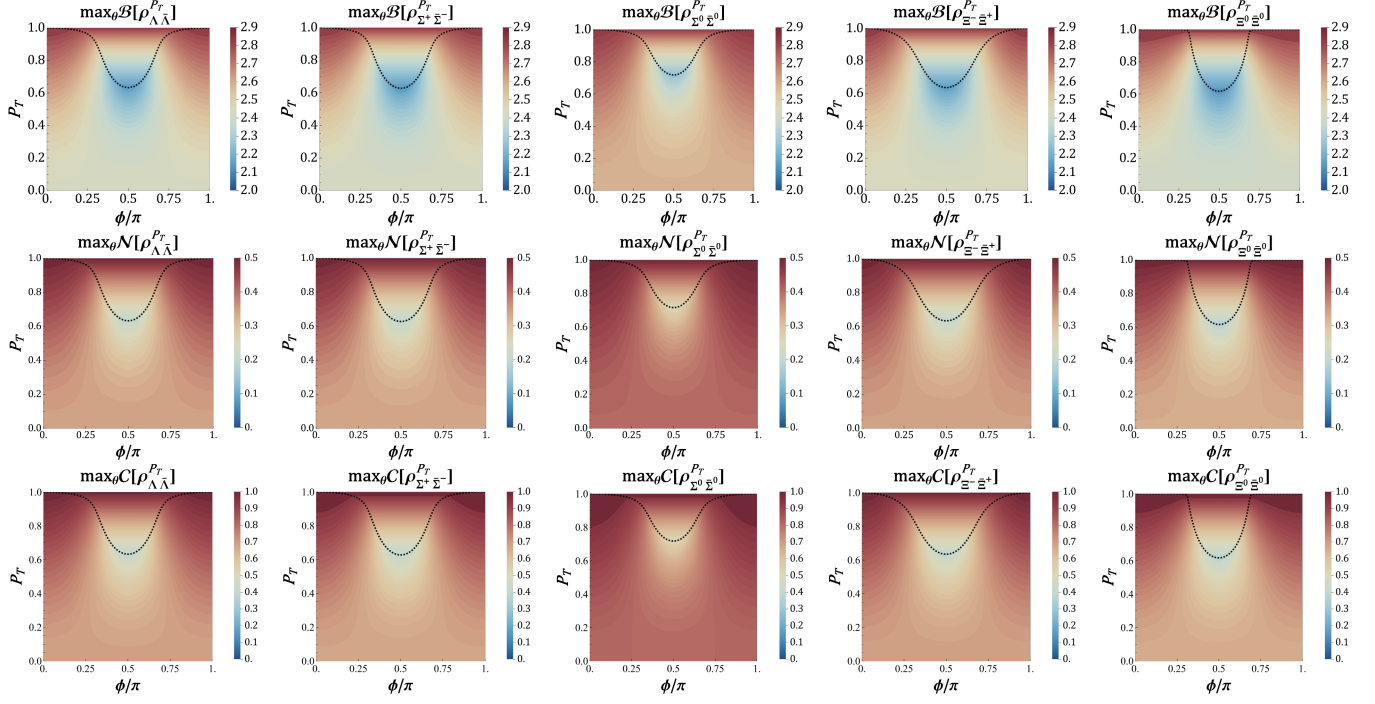


FIG. 18. The $\max_{\theta} \mathcal{B}[\rho_T]$ (upper panels), $\max_{\theta} \mathcal{N}[\rho_T]$ (middle panels), and $\max_{\theta} \mathcal{C}[\rho_T]$ (lower panels) as a function of azimuthal angle ϕ and transverse polarization degree P_T in $\psi(3686) \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Sigma^0, \Xi^-, \text{ and } \Xi^0$. The dashed curve is $\mathcal{B}, \mathcal{N}, \mathcal{C}(\theta = 0, \pi) = \mathcal{B}, \mathcal{N}, \mathcal{C}(\theta = \pi/2)$ in each upper, middle, and lower panel.

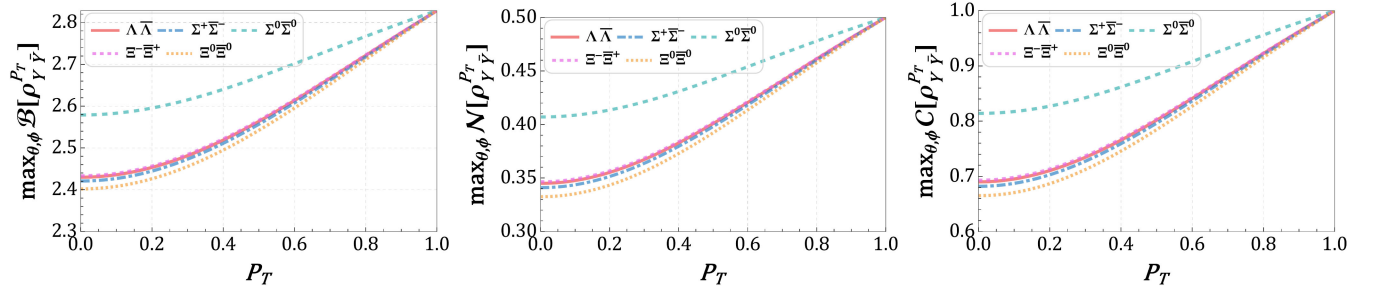


FIG. 19. The $\max_{\theta,\phi} \mathcal{B}[\rho_T]$ (left), $\max_{\theta,\phi} \mathcal{N}[\rho_T]$ (middle), and $\max_{\theta,\phi} \mathcal{C}[\rho_T]$ (right) as a function P_T in $\psi(3686) \rightarrow Y\bar{Y}$ for $Y = \Lambda, \Sigma^+, \Sigma^0, \Xi^-, \text{ and } \Xi^0$.

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