

Infrared spectra of some strongly-coupled chiral gauge theories

Stefano Bolognesi^{*,†}, Kenichi Konishi[‡], and Matteo Orso[‡]

Department of Physics “E. Fermi”, University of Pisa, Largo Pontecorvo, 3, Ed. C, 56127 Pisa, Italy
and INFN, Sezione di Pisa, Largo Pontecorvo, 3, Ed. C, 56127 Pisa, Italy



(Received 19 May 2026; accepted 25 June 2026; published 5 August 2026)

Several simple asymptotically free chiral gauge theories are studied. The only “free parameters” of our models are the choice of the gauge group, the matter Weyl fermion representations, and the relative magnitudes of the renormalization-group-invariant scales Λ_i associated with each gauge group. None of our models has nontrivial, nonabelian global symmetries (“family-like” fermion representations). We rely on some recent theoretical developments on the dynamics of strongly coupled chiral gauge theories based on the generalized symmetries and associated new types of anomaly matching consideration, but also on the solid knowledge on vector-like gauge theories such as QCD and supersymmetric Yang-Mills theories. The structures of the infrared effective theories, RG flows, and light spectra found in these models are surprisingly rich and intriguing.

DOI: [10.1103/4xhl-2fnw](https://doi.org/10.1103/4xhl-2fnw)

I. INTRODUCTION

Our understanding of the dynamics of strongly coupled gauge theories in four dimensions is still quite preliminary. Exceptions are the vector-like theories such as QCD and $\mathcal{N}=2$ supersymmetric gauge theories, where many detailed results are known either by numerical (the lattice) or exact analytical methods. Unfortunately, in the case of chiral gauge theories, which could possibly be relevant to model building beyond the Standard Model, our quantitative grip of their properties is still quite poor. One feels that it is necessary to make our theoretical tools more powerful before we will be able to take concrete, realistic steps in building the correct theory of the fundamental interactions, which reduces at low energies to the Standard $SU(3) \times SU(2)_L \times U(1)_Y$ model with the known quarks and leptons and their properties, and accounting for the Higgs sector in a totally natural fashion. The aim of the present work is to take a few modest steps beyond the earlier [1–16] and more recent series of investigations [17–33] on the dynamics of chiral gauge theories.

The models considered here contain a set of nonabelian gauge fields and fermions in some simple representations. The gauge interactions are all asymptotically free (AF). The

only freedom we have, apart from the choice of the gauge groups and of the matter fermions, is the relative ordering of the renormalization group (RG) invariant mass scales of each gauge group, $\{\Lambda_i\}$, $i = 1, 2, \dots$

Our models do not possess any global symmetry groups, except perhaps some $U(1)$ groups, which are possible Adler–Bell–Jackiw (ABJ) anomaly free combinations of the classical fermion–number groups, associated with each type of fermion. This occurs in the first model considered below. *A priori*, these $U(1)$ symmetries could have ’t Hooft anomalies, but actually this turns out not to be the case. The conventional ’t Hooft anomaly matching consideration does not play a central role in this work, in finding out the dynamics of the systems as they flow towards the infrared (IR).

We wish to find out, given the model, which type of low-energy effective theories can result dynamically and which kind of light spectrum and small mass parameters emerge, depending on the relative values of $\{\Lambda_i\}$. Whichever low-energy effective theory and particle spectra we find in each case will be totally natural, in the sense that all radiative corrections and renormalizations eventually (in principle) lead back in the ultraviolet (UV), to the original, simple asymptotically free (hence, free) theory.

This paper is organized as follows. In Sec. II, we consider an $SU(N) - SU(N+4)$ model which is a sort of combination of the so-called Bars–Yankielowicz and Georgi–Glashow models. In Sec. III, quiver $SU(N)^n$ models are studied. Section IV examines an $SU(N) - Sp(6) - Sp(6)$ model, which is a generalization of the Pati–Salam model. In Sec. V, we analyze an $SU(10)$ gauge theory with a single Weyl fermion in the fifth-rank antisymmetric (self-adjoint) representation. Section VI summarizes what is found in these models.

*Contact author: stefano.bolognesi@unipi.it

†Contact author: kenichi.konishi@unipi.it

‡Contact author: matteo.orso@phd.unipi.it

II. THE $SU(N) - SU(N+4)$ MODEL

The first model we consider here is an $SU(N) \times SU(N+4)$ gauge theory with left-handed fermions shown in Table I. The gauge interactions become stronger towards the infrared. The infrared spectra of the model will be different, depending on which of the $SU(N+4)$ or $SU(N)$ gauge interactions is strongest first along the RG flow.

When $g_{SU(N+4)} = 0$, the model is the well-known “ $\psi\eta$ ” (Bars–Yankielowicz) model investigated in [3,7,8,17,21–24,26–28,31], with free χ fermions that act as spectator fermions to make the global $SU(N+4)$ anomaly free. If instead $g_{SU(N)} = 0$, the model is the “ $\chi\eta$ ” (Georgi–Glashow) model studied in [8,17,22–24,26–28,31], with extra free spectator fermions ψ .

The global $U(1)$ symmetry is free of ABJ anomalies with respect to $SU(N+4)$ or to $SU(N)$. Also, the $[U(1)]^3$ as well as $[U(1)] - [\text{gravity}]^2$ anomalies turn out to cancel, and hence do not introduce any significant anomaly matching constraint on the possible infrared dynamics.

A. $SU(N)$ interactions getting strong first

Let us first consider the case where the $SU(N)$ interactions become strong at some scale Λ_1 . We assume that at that mass scale $SU(N+4)$ interactions are weak and can be treated perturbatively. In the limit $g_{SU(N+4)} \rightarrow 0$, the model reduces to the Bars–Yankielowicz model, with a free extra fermions χ . In other words,

$$g_1(\equiv g_{SU(N)}) \gg g_2(\equiv g_{SU(N+4)}), \quad (2.1)$$

in the UV. Recent observations based on a $\mathbb{Z}_2$ anomaly [21,22,27,28] (see more below) suggest that when the $\psi\eta$ system becomes strongly coupled at the mass scale $\sim \Lambda_1$, it goes into dynamical Higgs phase with “color–flavor locked” bifermion condensate formation [1,3,7,8],

$$\langle \psi^{ij} \eta_{ja} \rangle = \delta_a^i (\Lambda_1)^2, \quad i, j = 1, 2, \dots, N, \quad (2.2)$$

$$a = 1, 2, \dots, N+4.$$

The fermions in the UV and in the IR are shown in Table II, taken from [21]. The low-energy, massless degrees of freedom are $\frac{N^2+7N}{2}$ massless baryons,

TABLE I. The gauge groups and matter (Weyl) fermions. $U(1)$ is the unique, anomaly free combination of the global fermion number symmetries.

Fields \ groups	$SU(N)$	$SU(N+4)$	$U(1)$
ψ	$\square \square$	$\underline{1}$	$N+4$
η	$\square$	$\square$	$-(N+2)$
χ	$\underline{1}$	$\square$	N

TABLE II. Color–flavor locked phase in the $\psi\eta$ model. a_1 or b_1 stand for $1, 2, \dots, N$, and a_2 or b_2 for the rest of the flavor indices, $N+1, \dots, N+4$. The fermion parity $(\mathbb{Z}_2)_F$ acts as $\psi \rightarrow -\psi$, $\eta \rightarrow -\eta$.

Fields \ groups	$SU(N)'$	$SU(4)$	$U(1)'$	$(\mathbb{Z}_2)_F$
UV				
ψ	$\square \square$	$\frac{N(N+1)}{2} \cdot \underline{1}$	$N+4$	1
η_{a_1}	$\square \square \oplus \square$	$N^2 \cdot \underline{1}$	$-(N+4)$	-1
η_{a_2}	$4 \cdot \square$	$N \cdot \square$	$-\frac{N+4}{2}$	-1
IR				
$\mathcal{B}_{[a_1 b_1]}$	$\square$	$\frac{N(N-1)}{2} \cdot \underline{1}$	$-(N+4)$	-1
$\mathcal{B}_{[a_1 b_2]}$	$4 \cdot \square$	$N \cdot \square$	$-\frac{N+4}{2}$	-1

$$\mathcal{B}_{[ab]} = \psi^{ij} \eta_{ia} \eta_{jb}, \quad a, b = 1, 2, \dots, N+4, \quad (2.3)$$

together with $8N+1$ Nambu–Goldstone (NG) bosons.

Let us recall briefly the role the discrete $(\mathbb{Z}_2)_F$ symmetry of Table II played in constraining the infrared dynamics of these models. As shown in [21,22,27,28], the fermion $(\mathbb{Z}_2)_F$ symmetry suffers from various mixed anomalies such as $[(\mathbb{Z}_2)_F] - [\mathbb{Z}_N]^2$ and $[(\mathbb{Z}_2)_F] - [\mathbb{Z}_{N+4}]^2$, in $N = \text{even}$ models, where $\mathbb{Z}_N$ and $\mathbb{Z}_{N+4}$ are 1-form (“generalized”) symmetries [34–44]. These anomalies fail to be matched by massless “baryons” in a hypothetical confining phase with no global symmetry breaking, sometimes discussed in the literature. Such a phase therefore cannot be dynamically realized in the IR. No problem arises in the dynamical Higgs phase assumed here: the condensates $\langle \psi\eta \rangle$ (2.2) indeed break spontaneously both the global 0-form $U(1)$ and 1-form $\mathbb{Z}_N$ color center (or the flavor $\mathbb{Z}_{N+4}$ center) symmetries, thus “matching” the impossibility of gauging these center symmetries together with $(\mathbb{Z}_2)_F$.

By taking into account the χ fields, decomposed as a direct sum over irreducible representations (irreps) of $SU(N)' \times SU(4)$, and renaming the baryons as

$$\mathcal{B}_{[a_1 b_1]} \rightarrow \lambda_1, \quad \mathcal{B}_{[a_1 b_2]} \rightarrow \lambda_2, \quad (2.4)$$

one has the low-energy gauge groups and the fermions of this model in Table III. Note that both $SU(N)'$ and $SU(4)$ are anomaly free and $U(1)'$ is free of the ABJ and $[U(1)']^3$, $[U(1)'] - [\text{gravity}]^2$ anomalies.

$U(1)'$ is an anomaly free, unbroken combination of a broken subgroup of $SU(N+4)$, generated by

$$\eta \rightarrow e^{iQ\beta} \eta, \quad Q = \begin{pmatrix} 4\mathbb{1}_N & 0 \\ 0 & -N\mathbb{1}_4 \end{pmatrix} \quad (2.5)$$

and $U(1)$ of Table I,

TABLE III. The spectrum of the massless fermions in the model of Table I at mass scale below Λ_1 , where the $SU(N)$ gauge interactions are assumed to have become strongly coupled.

Fields \ groups	$SU(N)'$	$SU(4)$	$U(1)'$
λ_1	$\overline{\square}$	$\underline{1}$	$-(N+4)$
λ_2	$\square$	$\square$	$-\frac{N+4}{2}$
χ_1	$\overline{\square}$	$\underline{1}$	$N+4$
χ_2	$\square$	$\overline{\square}$	$\frac{N+4}{2}$
χ_3	$\underline{1}$	$\square$	0

$$U(1): \psi \rightarrow e^{i(N+4)\alpha} \psi, \quad \eta \rightarrow e^{-i(N+2)\alpha} \eta, \quad \chi \rightarrow e^{iN\alpha} \chi. \quad (2.6)$$

Requiring that the vacuum expectation value (VEV) (2.2) leave it unbroken gives the condition

$$U(1)': \quad 2\alpha + 4\beta = 0, \quad \beta = -\frac{\alpha}{2}. \quad (2.7)$$

The color-flavor diagonal $SU(N)'$, which is a global symmetry in the limit $g_2 \equiv g_{SU(N+4)} \rightarrow 0$, is actually a local gauge symmetry for $g_2 \neq 0$. Its coupling constant $g'_{SU(N)}$ is a combination,

$$g' \equiv \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}}, \quad g_1 \equiv g_{SU(N)}, \quad g_2 \equiv g_{SU(N+4)}, \quad (2.8)$$

(see Appendix A). At that mass scale Λ_1 , $g_2 \equiv g_{SU(N+4)}$ is assumed to be weakly coupled,

$$g_2(\Lambda_1) \ll g_1(\Lambda_1). \quad (2.9)$$

We see that just below the scale Λ_1 , the surviving $SU(N)'$ gauge interactions are actually weakly coupled, with

$$g' \equiv g'_{SU(N)}(\Lambda_1) \simeq g_2, \quad (2.10)$$

due to the hierarchy (2.8), (2.9). Thus, the part of $SU(N+4)$ that becomes strong and is locked to $SU(N)$ survives, but in a weak form.

The (first term of the) beta functions of $SU(N)'$ and $SU(4)$ below the mass scale Λ_1 are:

$$\begin{aligned} \frac{dg'}{d \log \mu} &\equiv \beta(g') = -\frac{9N-4}{3} \frac{(g')^3}{16\pi^2} + o((g')^5) \\ &\equiv b'_0 \frac{(g')^3}{16\pi^2} + o((g')^5), \\ \frac{dg_2}{d \log \mu} &\equiv \beta(g_2) = -\frac{42-2N}{3} \frac{g_2^3}{16\pi^2} + o(g_2^5) \\ &\equiv b_0^{(2)} \frac{g_2^3}{16\pi^2} + o(g_2^5). \end{aligned} \quad (2.11)$$

Thus $SU(N)'$ and $SU(4)$ are both asymptotically free as long as $N < 21$, which we assume to be the case below.

The system at the scale below Λ_1 also contains $8N+1$ pseudo-NG bosons in the case $SU(N+4)$ is a global symmetry. The $8N$ (would-be) NG bosons, which are not absorbed by the $SU(N)$ gauge bosons, correspond to the nondiagonal generators of $SU(N+4)$

$$T^{ik}, \quad T^{ki}, \quad i = 1, 2, \dots, N, \quad k = N+1, \dots, N+4, \quad (2.12)$$

i.e., in $(\underline{N}, \underline{4})$ of $SU(N)' \times SU(4)$, and absorbed by the associated $SU(N+4)$ gauge bosons. The last NG boson corresponds to the $SU(N+4)$ generator (2.5) and is a singlet $(\underline{1}, \underline{1})$ of $SU(N)' \times SU(4)$. This NG boson gets absorbed by the corresponding broken $SU(N+4)$ gauge boson.

Below Λ_1 , both $SU(N)'$ and $SU(4)$ gauge interactions grow stronger towards the IR. Which of them becomes strongly coupled first depends on the value of N , as their values just below Λ_1 are close to each other (see (2.10)). The RG flow of the coupling constants in the $SU(N) - SU(N+4)$ model of this section is illustrated in Fig. 1 for two possible cases, discussed in Secs. II A 1 and II A 2.

1. Large N

Taking the value of b_0 (2.11) as a rough indication, one may conclude that for a relatively large N ($N \geq 5$) $SU(N)'$, couplings become stronger first, flowing towards the IR at some mass scale Λ_2 ,

$$(\Lambda_2)^2 = (\Lambda_1)^2 e^{16\pi^2/b'_0 g'(\Lambda_1)^2} \ll (\Lambda_1)^2. \quad (2.13)$$

The $SU(N)'$ theory is a QCD, with a Dirac pair (λ_1, χ_1) in the anti-antisymmetric tensor representation and $N_f = 4$ flavors of the fundamental “quarks,” (λ_2, χ_2) , which are all expected to condense in pairs and get mass. The global $U(1)'$ symmetry is unbroken, and thus no NG bosons appear and the χ_3 has no charge with respect to it.

At mass scales below Λ_2 , there remains an unbroken $SU(4)$ gauge system with a self-adjoint fermion χ_3 : it will evolve further towards the IR and is expected to become strongly coupled at mass scale Λ_3 . $\Lambda_3 \ll \Lambda_2 \ll \Lambda_1$ goes into confinement phase, forming the condensate

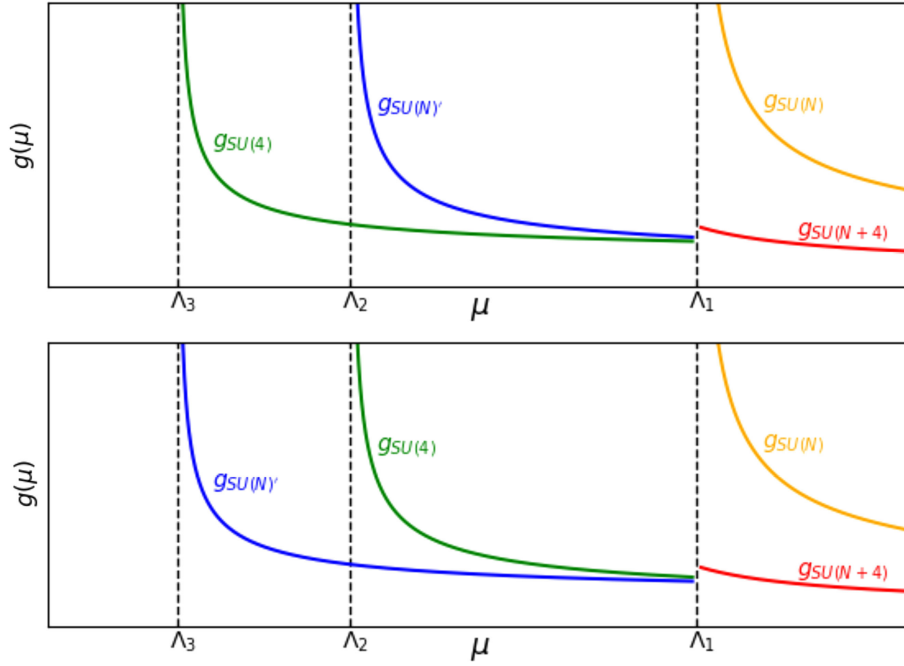


FIG. 1. The RG flow of the coupling constants in the $SU(N) - SU(N+4)$ model of Table I. The top picture corresponds to the theory with $N \geq 5$, and the lower corresponds to $N \leq 4$.

$$\langle \chi_3 \chi_3 \rangle, \quad (2.14)$$

and producing mesons $\chi_3 \chi_3$ of mass of the order of Λ_3 ,

$$\Lambda_3 \ll \Lambda_2 \ll \Lambda_1. \quad (2.15)$$

2. Small N

If $SU(4)$ gets stronger first towards the IR ($N \leq 4$), the system will go into the $SU(4)$ confinement phase at some scale $\Lambda_2 \ll \Lambda_1$, with condensates

$$\langle \lambda_2 \chi_2 \rangle, \quad \langle \chi_3 \chi_3 \rangle \quad (2.16)$$

of the order of $(\Lambda_2)^3$ forming. These fermions get dynamical masses of the order of Λ_2 and decouple from the system at lower energies. At mass scales below Λ_2 , the system is an $SU(N)'$ gauge theory which looks like a one-family (antisymmetric) tensor QCD with fermions λ_1 , χ_1 (see Table III). It will get strongly coupled at Λ_3 ($\Lambda_3 \ll \Lambda_2 \ll \Lambda_1$), confines, and produces mesons and baryons.

B. $SU(N+4)$ getting strong first

Let us consider now the case in which $SU(N+4)$ coupling gets strong first, towards the IR at some scale Λ , in our $SU(N) - SU(N+4)$ model of Table I. The $SU(N)$ coupling constant $g_1 \equiv g_{SU(N)}$ is now assumed to be small at Λ . The dynamics of the system will then basically be the same as in what was known as the Georgi-Glashow model (the “ $\chi\eta$ model”), studied in

[8,17,22–24,26–28,31]. A color-flavor locked bifermion condensate

$$\langle \chi^{[ab]} \eta_{ib} \rangle = \text{const}(\Lambda_2)^3 \delta_i^a, \quad i, a, b = 1, 2, \dots, N \quad (2.17)$$

is expected to form, based on considerations of mixed anomalies, as in the case of the Bars-Yankielowicz models. Such a VEV leaves a diagonal $SU(N)' \subset SU(N) \times SU(N+4)$ as well as $SU(4) \subset SU(N+4)$ as the unbroken gauge groups. The massless fermions surviving the condensate formation (2.17) have been listed in Appendix D of [24], see Table IV below.¹

The massless baryons

$$B_{\{ij\}} = \chi^{[ab]} \eta_{ai} \eta_{bj}, \quad i, j = 1, 2, \dots, N, \quad (2.18)$$

are believed to form and saturate all the anomalies associated with $SU(N)' \times U(1)'$. This is manifestly seen in Table IV; it is an example of the “natural anomaly matching” [31].²

In the present model, the “color-flavor locked” $SU(N)'$ gauge group is actually a local gauge group, as the original

¹Below, in adapting the results found in [24] to present model of Table I, we make a shift $N \rightarrow N+4$, and take the complex conjugation of all the representations. Lastly, the indices are appropriately renamed so as to agree with those used in previous sections in the present paper.

²After the fermions participating in the condensate pairwise get Dirac masses and decouple, the remaining massless fermions in the UV and in the IR are identical, with respect to the unbroken groups.

TABLE IV. Color-flavor locking in the “ $\chi\eta$ ” (generalized Georgi-Glashow) model. The color indices a_1 , b_1 , and i run up to N and the rest, a_2 or b_2 , covers $N+1, \dots, N+4$.

Fields \ groups		$SU(N)'$	$U(1)'$	$SU(4)$
UV	$\chi^{a_1 b_1}$	$\square$	N	$\underline{1}$
	$\chi^{a_1 a_2}$	$\square$	$\frac{N}{2}$	$\square$
	$\chi^{a_2 b_2}$	$\underline{1}$	0	$\square$
	$\eta_{a_1, i}$	$\square \oplus \bar{\square}$	$-N$	$\underline{1}$
	$\eta_{a_2, i}$	$\bar{\square}$	$-\frac{N}{2}$	$\bar{\square}$
IR	$B_{\{ij\}}$	$\square$	$-N$	$\underline{1}$
	χ	$\underline{1}$	0	$\square$

$SU(N)$ group is local, though weakly coupled. By taking into account the ψ field, and renaming

$$B_{\{ij\}} \rightarrow \kappa, \quad (2.19)$$

the infrared spectrum of the system below the mass scale Λ is summarized in Table V. It is just a one-flavor (symmetric) tensorial QCD, together with a decoupled $SU(4)$ sector with a single fermion χ in the self-adjoint antisymmetric tensor representation ($\chi^{a_2 b_2}$ of Table IV). We assume that $SU(4)$ confines and the condensate

$$\langle \chi\chi \rangle \neq 0, \quad (2.20)$$

forms and χ acquires mass dynamically. So do the tensor “quarks” ψ, κ , confined by the $SU(N)'$ forces. The RG flow of the coupling constants in this case is illustrated in Fig. 2.

TABLE V. The low-energy spectrum of the $SU(N) - SU(N+4)$ model of Table I, below the scale Λ_2 at which the $SU(N+4)$ gauge interactions get strong.

Fields \ groups		$SU(N)'$	$U(1)'$	$SU(4)$
κ		$\square$	$-N$	$\underline{1}$
		$\bar{\square}$		
ψ		$\square$	N	$\underline{1}$
χ		$\underline{1}$	0	$\square$

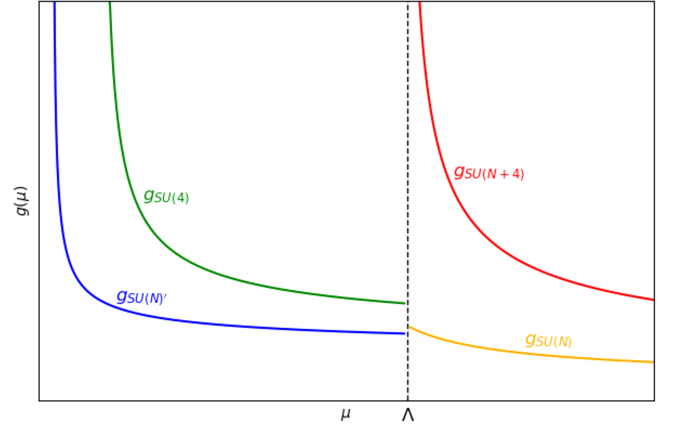


FIG. 2. The RG flow of the system in Table I, where the $SU(N+4)$ interactions become strong first towards the IR at the mass scale Λ at which the other, $SU(N)$ gauge interactions are assumed to be weak, $g_{SU(N)} \ll 1$. Just below Λ , both the diagonal $SU(N)' \subset SU(N) \times SU(N+4)$ and $SU(4) \subset SU(N+4)$ are weakly coupled. They are both asymptotically free and evolve independently.

III. THE QUIVER MODEL WITH MULTIPLE $SU(N)$ s

A. $SU(N)^3$ quiver model

The second model we consider is a quiver $SU(N)^3$ model, an $SU(N)_1 \times SU(N)_2 \times SU(N)_3$ gauge theory with fermions in the following quiver-type bifundamental representations (Table VI).

Let us assume that one of the $SU(N)$ factors gets strongly coupled first. As the three $SU(N)$ s appear symmetrically, we may assume without loss of generality that

$$\Lambda_1 \gg \Lambda_2, \Lambda_3. \quad (3.1)$$

That is, we assume that $SU(N)_2$ and $SU(N)_3$ are weakly coupled on the scale Λ_1 where $SU(N)_1$ is strongly coupled. That is,

$$g_1(\mu) \gg g_2(\mu), g_3(\mu) \quad (3.2)$$

near $\mu \simeq \Lambda_1$, where $g_1(\mu) \geq 1$.

TABLE VI. The quiver $SU(N)^3$ model with fermions ψ, κ, λ . There are no anomaly free combinations of classical $U(1)$ symmetries.

Fields \ gauge group		$SU(N)_1$	$SU(N)_2$	$SU(N)_3$
ψ		$\square$	$\bar{\square}$	$\underline{1}$
		$\bar{\square}$		
κ		$\bar{\square}$	$\underline{1}$	$\square$
λ		$\underline{1}$	$\square$	$\bar{\square}$

Neglecting first the weak $SU(N)_2$ and $SU(N)_3$ interactions, i.e., by setting $g_2 = g_3 = 0$, the system is just a $N_f = N$ flavored $SU(N)$ QCD, together with free (spectator) fermions λ that take care of the $SU(N)_2$ and $SU(N)_3$ anomalies. We know that a “quark condensate”

$$\langle \psi_a^i \kappa_i^k \rangle = \text{const} \delta_a^k (\Lambda_1)^3, \quad i, a, k = 1, 2, \dots, N \quad (3.3)$$

forms, and the $SU(N)_2 \times SU(N)_3$ flavor symmetry is broken to the diagonal

$$SU(N)_0 \subset SU(N)_2 \times SU(N)_3. \quad (3.4)$$

The fermions ψ and κ become massive, acquiring dynamical masses of the order of Λ_1 . They form $SU(N)_1$ -singlet mesons and baryons of masses of the order of Λ_1 .

The orthogonal, axial $SU(N)_A \subset SU(N)_2 \times SU(N)_3$ is broken spontaneously and produces massless $N^2 - 1$ NG bosons. In the presence of the weak $SU(N)_2 \times SU(N)_3$ gauge interactions, these $N^2 - 1$ massless “pions” get absorbed by the axial $SU(N)_A$ gauge bosons, giving them masses of the order of $g' \Lambda_1$, $g' \equiv \sqrt{g_2^2 + g_3^2}$. The decomposition and gauge-field mixing is discussed in Appendix B.

At mass scales much lower than $g' \Lambda_1 (\ll \Lambda_1)$, the massless degrees of freedom are the $SU(N)_0$ gauge bosons and the fermions λ in the adjoint representation of $SU(N)_0$. Note that in the decomposition of λ in $SU(N)_0 \subset SU(N)_2 \times SU(N)_3$,

$$\underline{N} \otimes \underline{N}^* = \underline{N^2 - 1} \oplus \underline{1}, \quad (3.5)$$

the singlet part decouples from the $SU(N)_0$ gauge bosons, i.e. it is a free, massless fermion, λ_0 .

At mass scales below $g' \Lambda_1$ the massless fermions λ are seen to be coupled as

$$\mathcal{D}_\mu \lambda = \partial_\mu \lambda - ig_0 [A_\mu^{(0)}, \lambda], \quad (3.6)$$

to the gauge bosons $A_\mu^{(0)}$ of the unbroken gauge group $SU(N)_0$, where

$$g_0 \equiv \frac{g_2 g_3}{g'} = \frac{g_2 g_3}{\sqrt{g_2^2 + g_3^2}}. \quad (3.7)$$

Note that it is smaller than both g_2 and g_3 . This formula also shows that the evolution towards the IR of the system below the mass scale $g_{2,3} \Lambda_1$ is essentially independent of the detailed relation between Λ_2 and Λ_3 , or the relative strengths between the small coupling constants $g_2(\mu)$ and $g_3(\mu)$ near $\mu \sim \Lambda_1$.

The system is now $\mathcal{N} = 1$ supersymmetric pure Yang-Mills (SYM) theory, thoroughly studied earlier in [45–52]. The fermion λ plays the role of the gauge fermion (the “gluino”). What is the RG-invariant mass scale Λ of this

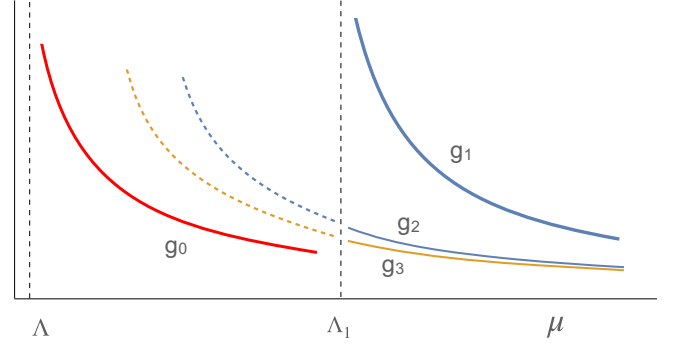


FIG. 3. A schematic view of the RG flow of the coupling constants in the quiver $SU(N)^3$ model. $g_{1,2,3}(\mu)$ represent the coupling constants of $SU(N)_1$, $SU(N)_2$, and $SU(N)_3$, respectively, as functions of the mass scale μ . At $\mu \simeq \Lambda_1$, the $SU(N)_1$ interactions become strong, and are assumed to confine and form the $\psi\kappa$ condensate, (3.3). The axial combination of $SU(N)_2$ and $SU(N)_3$ gets spontaneously broken and the corresponding gauge bosons acquire masses $\sim g_{2,3} \Lambda_1$. Below $g' \Lambda_1$, the system becomes effectively an $\mathcal{N} = 1$ supersymmetric $SU(N)$ Yang-Mills theory.

emergent $SU(N)$ SYM? In terms of the $SU(N)_0$ coupling constant (3.7) below the scale Λ_1 , it is given by

$$\Lambda^2 = (\Lambda_1)^2 e^{16\pi^2/b_0 g_0(\Lambda_1)^2} \ll (\Lambda_1)^2, \quad (3.8)$$

where

$$b_0 = -3N \quad (3.9)$$

is the well-known (first coefficient of the) beta function of the $SU(N)$ SYM theory. The RG flow of this model is sketched in Fig. 3.

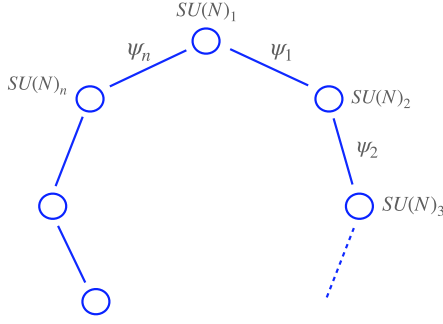
There are N vacua, in which the gluino condensate takes the known exact values [45–52]

$$\left\langle \frac{\text{Tr} \lambda \lambda}{16\pi^2} \right\rangle = \Lambda^3 e^{2\pi i k/N}, \quad k = 1, 2, \dots, N. \quad (3.10)$$

In each vacuum, at mass scales much lower than $\sim \Lambda$, the world consists of just the free fermion(s), λ_0 .

B. $SU(N)^n$ quiver model

A question comes into one’s mind naturally: What happens in the $SU(N)^n \equiv SU(N)_1 \times SU(N)_2 \times \dots \times SU(N)_n$ model with fermions in a bifundamental representation of the two adjacent $SU(N)$ gauge group factors (see Fig. 4)? By repeating the argument above for the $SU(N)^3$ quiver model, it is easy to convince oneself that when one of the $SU(N)$ factors gets strongly coupled and goes into confinement phase, the two “adjacent” $SU(N)$ factors combine into “axial,” broken $SU(N)$, and unbroken diagonal $SU(N)$ gauge groups. The net effect is the chain evolution towards the IR,

FIG. 4. A $SU(N)^n$ quiver model.

$$SU(N)^n \rightarrow SU(N)^{n-2} \rightarrow SU(N)^{n-4} \rightarrow \dots \quad (3.11)$$

where at each step the model is a quiver gauge theory with each fermion in the bifundamental representation with respect to the two adjacent gauge groups. When n is odd, the system eventually ends up with the $SU(N)^3$ quiver model discussed above, and in the extreme infrared it evolves as a pure $\mathcal{N} = 1$ supersymmetric Yang-Mills system, plus a decoupled free massless fermion.

When n is even, the system ends up with an $SU(N)_1 - SU(N)_2$ quiver model in the infrared limit. When one of the $SU(N)$ factors, say $SU(N)_1$, becomes strongly coupled at the mass scale Λ , it confines and all fermions get a dynamical mass of the order of Λ , producing massive mesons and baryons of mass $\sim \Lambda$. Depending on the coupling $g_{SU(N)_2}$ near Λ , there may appear somewhat lighter broken gauge bosons, of masses of the order of

$$g_{SU(N)_2} \Lambda. \quad (3.12)$$

IV. THE $SU(N) - Sp(6) - Sp(6)$ MODEL

The third model we consider is an $SU(N) \times Sp(6)_L \times Sp(6)_R$ gauge theory³ with fermions shown in Table VII. $N = 2n$, $n \in \mathbb{N}$ is taken to be even to ensure the absence of the Witten anomaly [55]. Both $Sp(6)_{L,R}$ are asymptotically free for $N < 44$:

$$b_0[Sp(6)_{L,R}] = -\frac{11 \cdot 4 - N}{3} < 0, \quad N < 44, \quad (4.1)$$

whereas $SU(N)$ is AF for any $N \geq 2$:

³We use the terminology $Sp(6)_L$ and $Sp(6)_R$, having in mind an analogy of this system with an $SU(N)N_f = 6$ QCD, with $\psi = \psi_L$ and $\tilde{\psi} \equiv \psi_R^c$. The choice of this model was inspired by attempts to construct a model of color-electroweak-family unification [53,54], based on the $SU(4) \times Sp(6)_L \times Sp(6)_R$ group, even though in the present work we make no attempt to construct a realistic model of the fundamental interactions.

TABLE VII. The matter Weyl fermions in the $SU(N) \times Sp(6)_L \times Sp(6)_R$ model.

Fields \ gauge group	$SU(N)$	$Sp(6)_L$	$Sp(6)_R$
ψ	$\square$	$\underline{6}$	$\underline{1}$
$\tilde{\psi}$	$\bar{\square}$	$\underline{1}$	$\underline{6}$

$$b_0[SU(N)] = -\frac{11 \cdot N - 2 \cdot 6}{3} < 0. \quad (4.2)$$

Again, the infrared spectra and the structure of the low-energy effective theories are considerably different, depending on which gauge interactions get stronger first, flowing towards the IR.

A. $SU(N)$ more strongly coupled than $Sp(6)$

Let us first consider the case in which the $SU(N)$ forces become strong at some scale Λ , assuming that at that scale $Sp(6)_{L,R}$ interactions are weakly coupled,

$$g_{Sp_L} \sim g_{Sp_R} \ll g_{SU(N)}. \quad (4.3)$$

Below, we shall sometimes denote g_{Sp_L}, g_{Sp_R} generically as g_{weak} .

In the limit

$$g_{Sp_L} = g_{Sp_R} = 0, \quad (4.4)$$

i.e., $g_{\text{weak}} = 0$, the system is just an $SU(N)$ QCD with six flavors $N_F = 6$ of quarks. The global symmetry of the system would be

$$G_F = SU(6)_L \times SU(6)_R \times U(1)_V, \quad (4.5)$$

and one expects that the quark condensate

$$\langle \psi_a^i \tilde{\psi}_i^b \rangle = \Lambda^3 \delta_a^b, \quad (4.6)$$

forms and breaks the global symmetry as

$$G_F \rightarrow SU(6)_V \times U(1)_V, \quad (4.7)$$

generating 35 NG bosons of the broken $SU(6)_A$ symmetry.

In our system, the subgroup $Sp(6)_L \times Sp(6)_R \subset SU(6)_L \times SU(6)_R$ is weakly gauged. This appears to present a typical but subtle vacuum alignment problem of strong interaction dynamics versus global symmetry and weak gauging. The situation actually looks more straightforward once we assume that the effect of the strong $SU(N)$ dynamics, (4.6), is not affected substantially by the weak gauging of the subgroup $Sp(6)_L \times Sp(6)_R \subset SU(6)_L \times SU(6)_R$.

The condensate (4.6) reads

$$\begin{aligned}\langle\psi_1\tilde{\psi}^1\rangle &= \langle\psi_2\tilde{\psi}^2\rangle = \langle\psi_3\tilde{\psi}^3\rangle = \langle\psi_4\tilde{\psi}^4\rangle \\ &= \langle\psi_5\tilde{\psi}^5\rangle = \langle\psi_6\tilde{\psi}^6\rangle = \Lambda^3.\end{aligned}\quad (4.8)$$

By appropriately identifying the six components of $\tilde{\psi}$ as

$$\tilde{\psi} = \Omega\psi', \quad \Omega = \begin{pmatrix} \mathbf{0}_3 & \mathbb{1}_3 \\ -\mathbb{1}_3 & \mathbf{0}_3 \end{pmatrix}, \quad \psi' \sim \underline{6} \quad (4.9)$$

the condensation (4.6), (4.8) can be rewritten as

$$\langle\psi\Omega\psi'\rangle \propto \mathbb{1}, \quad (4.10)$$

i.e., in the singlet of the decomposition in $Sp(6)$,

$$\underline{6} \otimes \underline{6} = \underline{1} \oplus \underline{14} \oplus \underline{21}. \quad (4.11)$$

This means the weak gauge symmetry is broken as

$$Sp(6)_L \times Sp(6)_R \rightarrow Sp(6)_{\text{diag}}. \quad (4.12)$$

This implies that, among the 35 would-be NG bosons of the $N_f = 6$ QCD, 21 corresponding to the broken axial $Sp(6)_A$ symmetry are absorbed by the corresponding gauge bosons giving them mass, $\sim g_{\text{weak}}\Lambda$. This is a dynamical Higgs phenomenon.

The rest, 14 of them, can naturally be assigned to the multiplet $\underline{14}$ of the unbroken $Sp(6)_{\text{diag}}$ symmetry. They will acquire a mass of the order of $g_{\text{weak}}\Lambda$. Note that our system has no exact global symmetry, but only an approximate global symmetry (4.5), which is broken *explicitly* by the $Sp(6)_L \times Sp(6)_R$ gauging. Hence, exactly massless physical NG bosons cannot appear. The expected mass $g_{\text{weak}}\Lambda$ of these scalar particles in $\underline{14}$ is due to the standard quadratic divergence diagram, but here the (physical) cutoff is provided by the fact that these are composite particles, pointlike only at the mass scale $\mu \ll \Lambda$.⁴

At the same time, $SU(N)$ interactions confine all fermions participating in the bifermion condensate (4.6): all ψ and $\tilde{\psi}$ (ψ') are confined and get Dirac (constituent) mass of the order of Λ . Mesons and baryons,

$$\psi\tilde{\psi}, \quad \psi\psi\ldots\psi, \quad \tilde{\psi}\tilde{\psi}\ldots\tilde{\psi} \quad (4.13)$$

⁴This might be considered as a typical situation where the Coleman-Weinberg potential [56] comes into play. The massless particles in $\underline{14}$ here, however, are would-be NG bosons. A vacuum instability (a nonvanishing VEV of these particles) would imply a modification of the symmetry breaking pattern (4.6), (4.10), and (4.12), which cannot occur for arbitrarily weak $Sp(6)_L \times Sp(6)_R$ couplings. A study of the Coleman-Weinberg potential in a similar situation [57] appears to be consistent with this consideration.

of mass of the order of Λ will appear, but they are now in the irreps of $Sp(6)_{\text{diag}}$, rather than of $SU(6)_V$. For weak $Sp(6)_L \times Sp(6)_R$ gauging, however, they will remain in approximate multiplets of $SU(6)_V$. For instance, the “ ρ ” mesons in $\underline{35}$

$$\underline{6} \otimes \underline{6}^* = \underline{1} \oplus \underline{35} \quad (4.14)$$

will split as $\underline{14} \oplus \underline{21}$ of $Sp(6)_{\text{diag}}$, see (4.11), by small mass differences of the order of $g_{\text{weak}}\Lambda$. These massive mesons and baryons carry charges with respect to $Sp(6)_{\text{diag}}$ and are coupled to the $Sp(6)_{\text{diag}}$ gauge bosons, accordingly.

Summarizing, the spectrum of the $SU(N) \times Sp(6)_L \times Sp(6)_R$ theory of Table VII is as follows.

- (i) Massive mesons and baryons of mass, $\sim\Lambda$, in various irreps of $Sp(6)_{\text{diag}} \subset Sp(6)_L \times Sp(6)_R$. They are also in approximate multiplets of $SU(6)_V$.
- (ii) The fermions $\psi, \tilde{\psi}$ are all confined and get dynamical, constituent masses of the order of Λ .
- (iii) Massive gauge bosons of broken $Sp(6)_A$ group, of masses of the order of $g_{\text{weak}}\Lambda$.
- (iv) Massless gauge bosons of unbroken $Sp(6)_{\text{diag}}$.
- (v) Light pseudo NG bosons of masses $\sim g_{\text{weak}}\Lambda$, in $\underline{14}$ of $Sp(6)_{\text{diag}}$.

Symmetries (approximate, weakly gauged, spontaneously broken or unbroken) acting on the fermions in this model are illustrated in Fig. 5. The coupling constant flow looks like Fig. 6.

B. An $Sp(6)$ getting stronger first

If one of the $Sp(6)$ factors, e.g., $Sp(6)_L$, gets strongly coupled instead, when $SU(N)$ and $Sp(6)_R$ are still weakly coupled, it is most likely that a condensate

$$\langle\psi\psi\rangle \quad (4.15)$$

forms which is in the singlet of $Sp(6)_L$. As it is an antisymmetric product with the metric Ω , as in (4.9) and (4.10), the VEV (4.15) must also be antisymmetric in $SU(N)$ indices: it is in the antisymmetric tensor representation

$$\langle\psi\psi\rangle \sim \begin{array}{|c|} \hline \square \\ \hline \end{array} \quad (4.16)$$

of $SU(N)$. The VEV (4.16) spontaneously breaks $SU(N)$. Following [58–61], we assume that the symmetry-breaking pattern is

$$SU(N) \rightarrow Sp(N), \quad N = 2n. \quad (4.17)$$

An alternative symmetry-breaking pattern $SU(N) \rightarrow SU(N-2) \times SU(2)$ induced by a scalar VEV in the two-index antisymmetric representation has been considered in

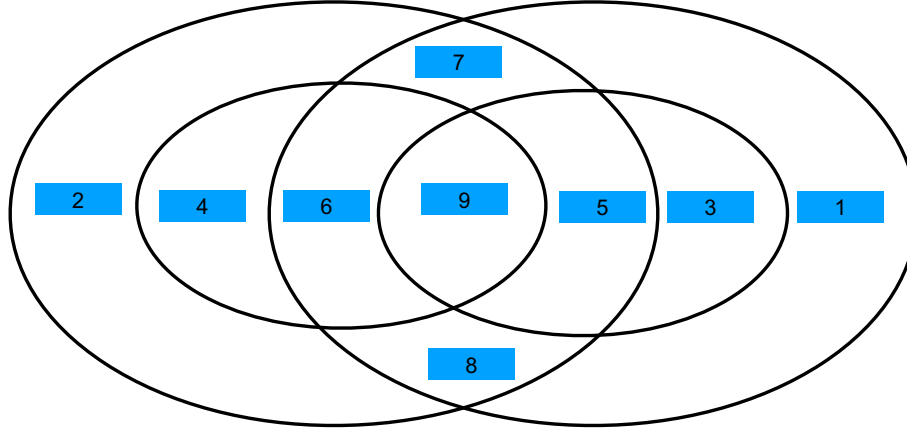


FIG. 5. Symmetries acting on the fermions. The two large ellipses including the sections (2467895) and (1367895) represent $SU(6)_L$ and $SU(6)_R$ transformations, respectively. The two smaller ellipses containing the sections (469) and (359) represent the subgroups $Sp(6)_L \subset SU(6)_L$ and $Sp(6)_R \subset SU(6)_R$, respectively. The combined section (56789) represents the intersection $SU(6)_V = SU(6)_L \cap SU(6)_R$, while the section (9) gives $Sp(6)_{\text{diag}} = Sp(6)_L \cap Sp(6)_R$. The section (5,6) represent some generators of $SU(6)_V$, which is weakly gauged but not part of $Sp(6)_{\text{diag}}$. The section (7,8) corresponds to the $SU(6)_V$ transformations which are broken by the weak gauging. The discussion in the text suggests that the sections (5) and (6) are actually absent.

the literature. In our context, such a breaking pattern seems to be less likely. When the $Sp(6)_L$ interactions become strongly coupled, it will generate the VEV, (4.16), but also confine. In the decomposition of the ψ field under $SU(N-2) \times SU(2)$, $(N-2, 1) \oplus (1, 2)$, only the second component directly participates in the condensate, but the first component will also be confined to form a massive $Sp(6)_L$ singlet mesons. At this point, the resulting $SU(N-2)$ gauge symmetry would become anomalous, having only massless fermions in $(N-2)^*$ from $\tilde{\psi}$.

With the symmetry breaking, (4.17), the fermionic spectrum below Λ consists solely of $\tilde{\psi}$ field, as in Table VIII.

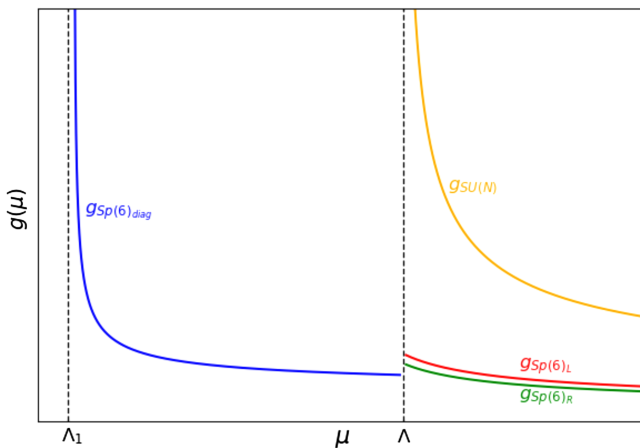


FIG. 6. The coupling constant flow of the $SU(N) - Sp(6) - Sp(6)$ model of Table VII, assuming that $SU(N)$ interactions become strong first at Λ , as the mass scale μ is lowered. Below Λ , the axial combination of $Sp(6)$ is broken, and the system flows as an $Sp(6)_{\text{diag}}$ theory with matter scalars in 14 of mass $\sim g_{\text{weak}} \Lambda$.

Note that this system is free from both gauge and Witten anomalies.

The one-loop coefficients of the beta functions of $Sp(N)$ and $Sp(6)_R$ are respectively ($n = N/2$):

$$b_0[Sp(N)] = -\frac{11n+5}{3}, \quad (4.18)$$

$$b_0[Sp(6)_R] = -\frac{44-2n}{3}. \quad (4.19)$$

In particular, (4.18) is negative for all $n \geq 2$, whereas (4.19) is positive for $n > 22$.

Let us assume that the $Sp(N)$ interactions get strong, at a new mass scale $\Lambda_2 \ll \Lambda_1$. Again, a bifermion condensate,

$$\langle \tilde{\psi} \tilde{\psi} \rangle \propto (\Lambda_2)^3 \neq 0, \quad (4.20)$$

is assumed to form. Due to the most attractive channel (MAC) criterion [1], $\tilde{\psi} \tilde{\psi}$ is a singlet of $Sp(N)$, antisymmetric in the $Sp(N)$ indices. By statistics, (4.20) must also be antisymmetric in the $Sp(6)_R$ indices. The simplest possibility is that it is also an $Sp(6)_R$ singlet. $Sp(6)_R$ remains unbroken. The $\tilde{\psi}$ field is confined and gets a dynamical mass.

Below Λ_2 , the system is described by a pure $Sp(6)_R$ gauge theory. It will become strongly coupled and confine

TABLE VIII. Representation of the fermionic matter content for energies $\mu \ll \Lambda_1$.

Field \ gauge group	$Sp(N)$	$Sp(6)_R$
$\tilde{\psi}$	$\underline{N}$	$\underline{6}$

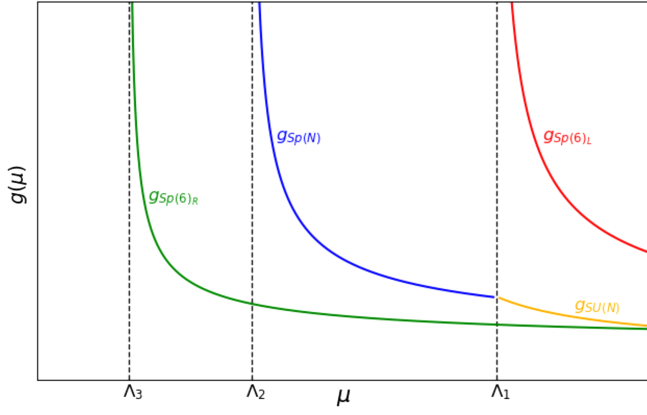


FIG. 7. The coupling constant RG flow of the $SU(N) - Sp(6)_L - Sp(6)_R$ model of Table VII, but this time one of the $Sp(6)$ gauge forces, e.g., $Sp(6)_L$, becomes strong first, at Λ_1 . At that scale a bifermion condensate breaks $SU(N)$ to $Sp(N)$, which then flows towards the infrared, getting strongly coupled at Λ_2 , whereas $Sp(6)_R$, unaffected, flows towards the IR as a pure $Sp(6)_R$ system.

at another mass scale, $\Lambda_3 \ll \Lambda_2 \ll \Lambda_1$. The RG flow in this case is illustrated in Fig. 7.

V. AN AF-IF $SU(10)$ MODEL

The final model we consider is an $SU(10)$ gauge theory with a single Weyl fermion in the self-adjoint (pseudoreal) fifth-rank antisymmetric representation,

$$\psi \sim \underline{252} = \begin{pmatrix} \square \\ \square \\ \square \\ \square \\ \square \end{pmatrix}, \quad (5.1)$$

with the first coefficient of the beta function

$$b_0[SU(10)] = -\frac{11 \times 10 - 70}{3} = -\frac{40}{3} < 0. \quad (5.2)$$

At some scale, Λ_1 $SU(10)$ becomes strongly coupled. The system has been studied recently [19,20], by making advantage of the generalized symmetries and associated new 't Hooft anomaly matching criteria [30,31,34–44]. Here, the relevant symmetries are the standard, nonanomalous $\mathbb{Z}_{70}$ subgroup of $U(1)_\psi$,

$$\mathbb{Z}_{70}: \psi \rightarrow e^{2\pi i k/70} \psi, \quad k = 1, 2, \dots, 70 \quad (5.3)$$

respected by the instantons, and a 1-form $\mathbb{Z}_5^C \subset \mathbb{Z}_{10}^C$ center symmetry,

$$\mathbb{Z}_5^C: e^i \oint^A \rightarrow e^{2\pi i \ell/5} e^i \oint^A, \quad \ell = 1, 2, \dots, 5 \quad (5.4)$$

acting on the Polyakov (or Wilson) loop $e^i \oint^A$, but not acting on ψ . By gauging this 1-form $\mathbb{Z}_5^C$ center symmetry, one finds that (due to the effective fractional instantons) the $\mathbb{Z}_{70}$ symmetry becomes anomalous (a mixed anomaly), and gets broken as

$$\mathbb{Z}_{70} \longrightarrow \mathbb{Z}_{14}. \quad (5.5)$$

Such a reduction of symmetry means that the vacuum cannot be invariant under $\mathbb{Z}_{70}$. The most natural possibility [20] is that the ψ field condenses as $SU(10)$ becomes strong,

$$\langle \psi \psi \rangle \neq 0. \quad (5.6)$$

Such a condensate would break $\mathbb{Z}_{70}$ more strongly, i.e., as $\mathbb{Z}_{70} \rightarrow \mathbb{Z}_2$, than the generalized anomaly suggests, but that is logically consistent. We are indeed familiar with this situation.⁵

Due to statistics, $\psi\psi$ condensate cannot be an $SU(10)$ and Lorentz singlet. Let us assume that $\psi\psi$ is in the adjoint representation of $SU(10)$. As the quadratic Casimir for ψ is quite large, it is possible that the condensation occurs at mass scale where

$$|(C_2(\text{adj}) - C_2(\underline{252}) - C_2(\underline{252}))g^2| = \frac{35}{2}g^2 \sim 1, \quad (5.7)$$

i.e., at a relatively small value of g^2 .

How the $SU(10)$ gauge symmetry is broken by the condensate $\langle \psi\psi \rangle$ in the adjoint representation is not known. The discussion of this section is necessarily of a more speculative nature than that of other sections of this work. For definiteness, we assume here that it breaks the gauge symmetry as

$$SU(10) \rightarrow SU(8) \times SU(2) \times U(1), \quad (5.8)$$

where $U(1)$ is generated by the charge

$$Q = \begin{pmatrix} \mathbb{1}_8 & 0 \\ 0 & -4\mathbb{1}_2 \end{pmatrix}. \quad (5.9)$$

It might be thought that all components of the ψ field acquire Dirac-like mass term and decouple, leaving a pure $SU(8) \times SU(2) \times U(1)$ gauge theory below the mass scale Λ_1 .

Actually, the situation is a little subtler. Decomposition of the ψ field as the sum of the irreducible representations of the unbroken gauge group $SU(8) \times SU(2) \times U(1)$ gives

⁵In the standard QCD with N_f massless flavors, the axial $U(1)_A$ is broken by anomaly and instantons to $\mathbb{Z}_{2N_f}$, but we know that the effect of the quark condensate $\langle \bar{\psi}\psi \rangle \neq 0$ is stronger: $\mathbb{Z}_{2N_f}$ is broken by the vacuum to $\mathbb{Z}_2$.

TABLE IX. Decomposition of the ψ field under the subgroup $SU(8) \times SU(2) \times U(1)$.

Fields \ gauge group	$SU(8)$	$SU(2)$	$U(1)$
ψ_1	$\begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \end{array}$	$\underline{1}$	5
ψ_2	$\begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \end{array}$	$\square$	0
ψ_3	$\begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \end{array}$	$\underline{1}$	-5

Table IX. It can be seen that the pair (ψ_1, ψ_3) can form an $SU(8) \times SU(2) \times U(1)$ invariant Dirac fermion and therefore probably get mass dynamically. In contrast, due to statistics, the ψ_2 field cannot form an $SU(8) \times SU(2) \times U(1)$ and Lorentz invariant pair $\psi_2\psi_2$ (such a composite field would be identically zero; see also (5.14) and (5.15) below). This means that the fermion ψ_2 remains as a massless degree of freedom, below the mass scale, Λ_1 .

As the fermion pair $\psi_1 - \psi_3$ formed a massive meson, neutral of the $U(1)$ charge, and the remaining fermion ψ_2 carries no charge with respect to it, the $U(1)$ gauge group becomes a free theory, with its photon decoupled from the rest of the system.

With only ψ_2 as the matter fermion field, $SU(8)$ is still asymptotically free, as

$$b_8 = -\frac{11 \times 8 - 20 \times 2}{3} = -16 < 0, \quad (5.10)$$

whereas $SU(2)$ is infrared-free (IF):

$$b_2 = -\frac{11 \times 2 - 70}{3} = +\frac{68}{3} > 0. \quad (5.11)$$

While the $SU(8)$ effective coupling constant $g_8(\mu)$ grows larger towards the infrared, the $SU(2)$ coupling gets weaker. The RG flow in this model is illustrated in Fig. 8. Even though such a picture is based on the one-loop results, (5.10), (5.11), and g_8 and g_2 evolutions will certainly mix under renormalization beyond the one-loop approximation, it is probably safe to conclude that the $SU(8)$ gauge interactions become strongly coupled at some mass scale

$$(\Lambda_2)^2 = (\Lambda_1)^2 e^{16\pi^2/b_8 g_8(\Lambda_1)^2} \ll (\Lambda_1)^2. \quad (5.12)$$

It is then natural to assume that the ψ_2 -field condensate will form:

$$\langle \psi_2 \psi_2 \rangle \neq 0. \quad (5.13)$$

It might be thought that this will give dynamical mass to the ψ_2 field and wash away any interesting physics below the mass scale Λ_2 . Here again, we find a nontrivial physics. The reason is that the dynamics of the condensate formation (5.13) is now dictated by the strong $SU(8)$ forces, in contrast to what happened at the higher mass scale Λ_1 , (5.6). The $SU(2)$ interactions now play a secondary role. This means that the above condensate (5.13) has the $SU(8)$ indices saturated as

$$\langle \phi^{ij} \rangle, \quad \phi^{ij} \equiv \psi_2^{a_1 a_2 a_3 a_4 i} \psi_2^{a_5 a_6 a_7 a_8 j} \epsilon_{a_1 a_2 \dots a_8}, \quad (5.14)$$

i.e., an $SU(8)$ and Lorentz singlet, but it cannot also be an $SU(2)$ singlet, as

$$\epsilon_{ij} \phi^{ij} \equiv 0. \quad (5.15)$$

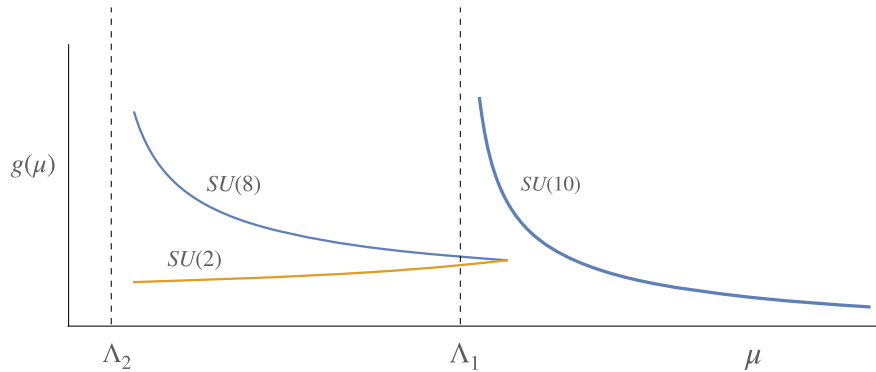


FIG. 8. The coupling constant RG flow of the $SU(10)$ model assuming the breaking (5.8). Due to the large quadratic Casimir, the breaking might happen when the coupling constant is still relatively small. The two unbroken subgroups are one AF and the other IF.

In other words, (5.14) must be an $SU(2)$ triplet. That is, the $\psi_2\psi_2$ condensate (5.13) breaks the $SU(2)$ gauge symmetry as

$$SU(2) \rightarrow U(1). \quad (5.16)$$

If the $SU(2)$ were a global symmetry, this would mean the presence of two exactly massless NG bosons. But as the $SU(2)$ are weakly coupled *gauge* symmetry, we have a dynamical Higgs phenomenon. Two of the $SU(2)$ gauge bosons (“W-bosons”) get massive, with mass of the order of

$$\sim g_2(\Lambda_2)\Lambda_2, \quad (5.17)$$

while the third $SU(2)$ gauge boson remains exactly massless: another “photon.” Note that even if a $U(1) \subset SU(2)$ gauge symmetry remains unbroken, the fermions ψ_2 all get massive, as both $\psi_2^{a_1 \dots a_4 i}$, $i = 1, 2$, participate in the condensate and as a result get a dynamical mass.

At the mass scales much below $g_2\Lambda_2$, therefore, the world is a free $U(1) \times U(1)$ system with no light matter coupled to it. The general state describes an arbitrary excitation and number of two types of photons.

VI. SUMMARY

In this paper, we studied the infrared spectra of several simple asymptotically free chiral gauge theories, with the aim of getting a better grasp on the strong gauge dynamics of these types of systems. We considered models with a few nonabelian gauge group factors and no nonabelian global symmetries. Depending on the gauge groups involved, on the fermion representations, based on the relative strength of their coupling constants at some fixed UV mass scale, varieties of IR effective theories emerge. In the deep infrared, some systems reduce to one with a single kind of free massless fermion (the model of Sec. III), or two kinds of free photons (in the system of Sec. V). In others, the system is gapped. At somewhat higher mass scales, the physics we find varies from an $SU(4)$ theory with a self-adjoint antisymmetric tensor fermion (Sec. II A), an $SU(N)$ QCD with symmetric tensor quarks (Sec. II A 2), an $Sp(6)$ theory with massive scalars in 14 (Sec. IV A), a pure $\mathcal{N} = 1$ Supersymmetric $SU(N)$ Yang–Mills theory (Sec. III), an $SU(8) \times SU(2) \times U(1)$ theory with an asymptotically free $SU(8)$ and infrared-free $SU(2) \times U(1)$ gauge interactions (Sec. V), and so on.

Though these results are of some interest, clearly a more elaborate UV model is needed to produce dynamically an $SU(3) \times SU(2) \times U(1)$ theory, with asymptotically free ($SU(3)$) and infrared-free ($SU(2) \times U(1)$) gauge groups, three families of quarks and leptons, and some light effective Higgs scalar bosons doing their required job (of breaking correctly the electroweak gauge symmetry and giving mass to the fermions) as the low-energy effective theory. In particular, we need a systematic

understanding of how the nontrivial quark CKM matrix (and the analogous lepton mixing matrix) arises, and how the mechanism generating these generally complex, non-vanishing quark and lepton masses leaves at the same time $\theta_{\text{eff}} \simeq 0$ in the QCD sector. We hope to discuss these questions in the near future.

ACKNOWLEDGMENTS

We thank Andrea Luzio for various discussions on previous related projects. This work is supported by the INFN special initiative grant, GAST (“Gauge and String Theories”).

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: THE GAUGE-FIELD MIXING IN THE $SU(N) - SU(N+4)$ MODEL

In the dynamical Higgs phenomenon, an appropriate set of NG bosons are absorbed by the original $SU(N)$ gauge fields, making them massive and leaving orthogonal combinations of $SU(N) \times SU(N+4)$ massless, an exact $SU(N)' \times SU(4)$ gauge theory. Let us work out these things explicitly.

We introduce an effective scalar field representing the composite,

$$\phi_a^i \equiv \psi^{ij} \eta_{ja}, \quad (A1)$$

in

$$\square, \bar{\square} \quad (A2)$$

of $SU(N) \times SU(N+4)$. It acquires a VEV of the form

$$\langle \phi \rangle \sim v(\mathbf{1}_{N \times N} \quad \mathbf{0}_{N \times 4}), \quad (A3)$$

breaking the symmetries as

$$SU(N) \times SU(N+4) \times U(1) \rightarrow SU(N)' \times SU(4) \times U(1)'. \quad (A4)$$

As we noted earlier, in the absence of $SU(N+4)$ gauge interactions ($g_2 = 0$), there would be $8N+1$ physical massless NG bosons. $SU(N)' \times SU(4)$ is an unbroken gauge symmetries, with massless fermions in Table III. The kinetic term of ϕ is

$$|(\partial_\mu - ig_1 T_{\square}^A A_\mu^A - ig_2 t_{\bar{\square}}^A B_\mu^A) \phi|^2, \quad (A5)$$

for the part of ϕ_a^j , $j, a = 1, 2, \dots, N$, and

$$|(\partial_\mu - ig_1 T_{\square}^A A_\mu^A - ig_2 t_{SU(4)}^A B_\mu^A) \phi|^2. \quad (\text{A6})$$

for the rest of ϕ_a^j , $j = 1, 2, \dots, N$, $a = N+1, \dots, N+4$. Under the VEV of the form, (A3), the kinetic term (A5) gives rise to massive $SU(N)$ gauge bosons for a combination of $SU(N)$ and $SU(N+4)$,

$$gG_\mu^A \equiv g_1 A_\mu^A - g_2 B_\mu^A, \quad g \equiv \sqrt{g_1^2 + g_2^2}, \\ A = 1, 2, \dots, N^2 - 1, \quad (\text{A7})$$

of mass, gv . The orthogonal combination,

$$gC_\mu^A \equiv g_2 A_\mu^A + g_1 B_\mu^A, \quad g \equiv \sqrt{g_1^2 + g_2^2}, \\ A = 1, 2, \dots, N^2 - 1, \quad (\text{A8})$$

remain massless and gives rise to low-energy $SU(N)'$ gauge interactions. Inverting these,

$$A_\mu^A \equiv \frac{1}{g}(g_1 G_\mu^A + g_2 C_\mu^A), \quad B_\mu^A \equiv \frac{1}{g}(-g_2 G_\mu^A + g_1 C_\mu^A). \quad (\text{A9})$$

The 4×4 generators of $SU(4) \subset SU(N+4)$, B_μ^B remain massless and describe the low-energy $SU(4)$ gauge interactions.

There are also the gauge bosons B_μ^A of $SU(N+4)$, corresponding to the generators having matrix elements $a = 1, 2, \dots, N$, $b = N+1, N+2, \dots, N+4$, or $a = N+1, N+2, \dots, N+4$, $b = 1, 2, \dots, N$, which become massive absorbing the NG bosons. Finally, the gauge boson B_μ^A of $SU(N+4)$ for the generators

$$\begin{pmatrix} 4\mathbb{1}_N & 0 \\ 0 & -N\mathbb{1}_4 \end{pmatrix} \quad (\text{A10})$$

also gets massive absorbing the last NG boson.

For simplicity, let us write the low-energy effective action for the fermions taking into account the unbroken $SU(N)' \times SU(4)$ gauge interactions only, even though they are in general also coupled to the $N^2 - 1 + 8N + 1 = N^2 + 8N$ massive broken gauge bosons, i.e., an analogue of the real-world “weak interactions.”

The kinetic terms of the fermions are

$$D_\mu \psi = (\partial_\mu - ig_1 T_{\square}^A A_\mu^A) \psi, \\ D_\mu \eta = (\partial_\mu - ig_1 T_{\square}^A A_\mu^A - ig_2 t_{\square}^A B_\mu^A) \eta, \quad (\text{A11}) \\ D_\mu \chi = (\partial_\mu - ig_2 t_{\square}^A B_\mu^A) \chi.$$

Decomposing the fermions with respect the unbroken symmetry groups, $SU(N)' \times SU(4)$ as in Table III, their kinetic terms become

$$D_\mu \lambda_1 = (\partial_\mu - ig_2 t_{\square}^A B_\mu^A) \lambda_1 = (\partial_\mu - ig' t_{\square}^A C_\mu^A) \lambda_1, \quad t^A \in \mathfrak{su}(N)', \\ D_\mu \lambda_2 = (\partial_\mu - ig' t_{\square}^A C_\mu^A - ig_2 t_{\square}^B B_\mu^B) \lambda_2, \quad t^B \in \mathfrak{su}(4), \\ D_\mu \chi_1 = (\partial_\mu - ig' t_{\square}^A C_\mu^A) \chi_1, \\ D_\mu \chi_2 = (\partial_\mu - ig' t_{\square}^A C_\mu^A - ig_2 t_{\square}^B B_\mu^B) \chi_2, \\ D_\mu \chi_3 = (\partial_\mu - ig_2 t_{\square}^B B_\mu^B) \chi_3, \quad (\text{A12})$$

where

$$g' \equiv \frac{g_1 g_2}{g} \equiv \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} \simeq g_2 \ll g_1. \quad (\text{A13})$$

APPENDIX B: THE GAUGE-FIELD MIXING IN THE $SU(N)^3$ QUIVER MODEL

The effective scalar which acts as the Higgs field is the composite

$$\phi_a^k \sim \psi_a^i \kappa_i^k. \quad (\text{B1})$$

It transforms under $SU(N)_2 \times SU(N)_3$ as (by using a matrix notation),

$$\phi \rightarrow U_3 \phi U_2^\dagger, \quad U_2 \in SU(N)_2, \quad U_3 \in SU(N)_3. \quad (\text{B2})$$

In the kinematic term for ϕ , $\partial_\mu \phi$ transforms as

$$\begin{aligned}
\partial_\mu \phi &\rightarrow \partial_\mu (U_3 \phi U_2^\dagger) \\
&= U_3 (\partial_\mu \phi) U_2^\dagger + U_3 \tilde{\partial}_\mu \phi U_2^\dagger + U_3 \phi \partial_\mu U_2^\dagger \\
&= U_3 (\partial_\mu \phi) U_2^\dagger - U_3 (\partial_\mu U_3^\dagger) U_3 \phi U_2^\dagger + U_3 \phi U_2^\dagger U_2 \partial_\mu U_2^\dagger.
\end{aligned} \tag{B3}$$

In order to have a locally $SU(N)_2 \times SU(N)_3$ invariant kinematic term, one must write the covariant derivative as

$$\mathcal{D}_\mu \phi = \partial_\mu \phi - ig_3 A_\mu^{(3)} \phi + ig_2 \phi A_\mu^{(2)}, \tag{B4}$$

with the gauge-field transformations,

$$\begin{aligned}
A_\mu^{(2)} &\rightarrow U_2 \left(A_\mu^{(2)} + \frac{i}{g_2} \partial_\mu \right) U_2^\dagger, \\
A_\mu^{(3)} &\rightarrow U_3 \left(A_\mu^{(3)} + \frac{i}{g_3} \partial_\mu \right) U_3^\dagger.
\end{aligned} \tag{B5}$$

The first term in the kinetic term transforms as in (B3). The second and third terms in (B4) get transformed as

$$\begin{aligned}
-ig_3 A_\mu^{(3)} \phi &\rightarrow -ig_3 U_3 \left(A_\mu^{(3)} + \frac{i}{g_3} \partial_\mu \right) U_3^\dagger U_3 \phi U_2^\dagger \\
&= U_3 (-ig_3 A_\mu^{(3)} + (\partial_\mu U_3^\dagger) U_3) \phi U_2^\dagger,
\end{aligned} \tag{B6}$$

and

$$\begin{aligned}
ig_2 \phi A_\mu^{(2)} &\rightarrow ig_2 U_3 \phi U_2^\dagger U_2 \left(A_\mu^{(2)} + \frac{i}{g_2} \partial_\mu \right) U_2^\dagger \\
&= U_3 (ig_2 \phi A_\mu^{(2)} U_2^\dagger - \phi U_2^\dagger U_2 (\partial_\mu U_2^\dagger)).
\end{aligned} \tag{B7}$$

We note that the second terms in (B6) and in (B7) cancel the second and the third terms of (B3), respectively. Thus, the kinetic term of ϕ , (B4), transforms covariantly as

$$\mathcal{D}_\mu \phi \rightarrow U_3 (\mathcal{D}_\mu \phi) U_2^\dagger, \tag{B8}$$

as desired.

The $SU(N)_2 \times SU(N)_3$ gauge fields mix as follows:

$$g' B_\mu \equiv g_2 A_\mu^{(2)} - g_3 A_\mu^{(3)}, \quad g' A_\mu^{(0)} \equiv g_3 A_\mu^{(2)} + g_2 A_\mu^{(3)}, \tag{B9}$$

or inverting them,

$$A_\mu^{(2)} \equiv \frac{1}{g'} (g_2 B_\mu + g_3 A_\mu^{(0)}), \quad A_\mu^{(3)} \equiv \frac{1}{g'} (-g_3 B_\mu + g_2 A_\mu^{(0)}), \tag{B10}$$

where

$$g' \equiv \sqrt{g_2^2 + g_3^2}. \tag{B11}$$

Indeed, in the Higgs vacuum, (3.3),

$$\langle \phi \rangle \propto v \mathbb{1}, \tag{B12}$$

we see that the B_μ field gets mass,

$$g' v, \tag{B13}$$

by the Higgs mechanism. The other field $A_\mu^{(0)}$ remains massless and describes the unbroken $SU(N)_0$ gauge theory. The coupling of the λ field with the $SU(N)_0$ gauge field can be found as follows. λ field transforms as

$$\lambda \rightarrow U_2 \lambda U_3^\dagger \tag{B14}$$

(see Table VI). So its kinetic terms are given by (in the matrix notation)

$$\mathcal{D}_\mu \lambda = \partial_\mu \lambda - ig_2 A_\mu^{(2)} \lambda + ig_3 \lambda A_\mu^{(3)}, \tag{B15}$$

and by using the mixing relation (B10), we see that the new kinetic term in the theory below Λ_1 is given, neglecting the coupling to the heavy, broken gauge field B_μ , simply by

$$\mathcal{D}_\mu \lambda = \partial_\mu \lambda - ig_0 [A_\mu^{(0)}, \lambda], \tag{B16}$$

where

$$g_0 \equiv \frac{g_2 g_3}{g'} \equiv \frac{g_2 g_3}{\sqrt{g_2^2 + g_3^2}}. \tag{B17}$$

Note that is smaller than either g_2 or g_3 .

APPENDIX C: A FEW GROUP-THEORY FORMULAE FOR $SU(N)$

For convenience of the reader, we report here a few group-theory formulae relevant to an $SU(N)$ gauge theory, which were already presented as an Appendix in [20].

The Dynkin index T_R is defined by

$$\text{tr}(t_R^a t_R^b) \equiv T_R \delta^{ab}, \tag{C1}$$

where t_R^a are the generators of $SU(N)$ in the representation R . Summing over $a = b$, one gets

$$d(R) C_2(R) = T_R (N^2 - 1), \quad \sum_a t_R^a t_R^a \equiv C_2(R) \mathbb{1}_{d(R)}, \tag{C2}$$

where $d(R)$ is the dimension of the representation and $C_2(R)$ is the quadratic Casimir. For the fundamental representation, one has

$$C_2(F) = \frac{N^2 - 1}{2N}, \quad d(F) = N, \quad T_F = \frac{1}{2}, \tag{C3}$$

and for the adjoint,

$$C_2(\text{adj}) = N, \quad d(\text{adj}) = N^2 - 1, \quad T_{\text{adj}} = N, \quad (\text{C4})$$

these two are quite familiar. For a rectangular Young tableau, the quadratic Casimir is⁶

$$C_2(\overbrace{f, \dots, f}^k, 0, \dots) = \frac{kf(N+f)(N-k)}{2N}, \quad (\text{C5})$$

where f is the number of the boxes in a row and k the number of rows.

For the order n -antisymmetric representation, $f = 1$, $k = n$, it is

$$C_2(R) = \frac{n(N-n)(N+1)}{2N}. \quad (\text{C6})$$

Taking into account the multiplicity

$$d(R) = \frac{N(N-1) \cdots (N-n+1)}{n!} = \frac{N!}{n!(N-n)!}, \quad (\text{C7})$$

the Dynkin index of totally antisymmetric single column representation of height n is given by

$$T_R = \frac{(N-2)(N-3) \cdots (N-n)}{2(n-1)!}. \quad (\text{C8})$$

For the special cases with $N = 2n$ even, we have

$$C_2(R) = \frac{N(N+1)}{8}, \quad d(R) = \binom{N}{N/2} \quad (\text{C9})$$

and

$$2T_R = \binom{N-2}{N/2-1} = \binom{2n-2}{n-1}. \quad (\text{C10})$$

By using this expression, it is easy to see that for $N = 4\ell$, $n = 2\ell$, $2T_R$ is a multiple of ℓ , whereas for $N = 4\ell + 2$, $2T_R$ contains $2\ell + 1$ as a divisor. To prove it, note that the general combinatoric number

$$\binom{m}{r} = \frac{m(m-1) \cdots (m-r+1)}{r!} = \frac{m!}{r!(m-r)!}, \quad (\text{C11})$$

is always an integer. But

⁶See, for example, the book of Barut and Raczka [62], p. 259 (apart from a factor $1/2$ which is included, so that $C_2(F) = (N^2 - 1)/2N$ for the fundamental). See also [63] for reference.

$$\binom{m}{r} = \binom{m}{r-1} \cdot \frac{m-r+1}{r} \quad (\text{C12})$$

and both $\binom{m}{r}$ and $\binom{m}{r-1}$ are integers. Therefore, $m - r + 1$ is a divisor of $\binom{m}{r}$. Applying this for $N = 4\ell$, one finds that

$$2T_R = \binom{4\ell-2}{2\ell-1} \quad (\text{C13})$$

has a divisor 2ℓ , hence ℓ . For $N = 4\ell + 2$,

$$2T_R = \binom{4\ell}{2\ell} \quad (\text{C14})$$

has a divisor, $2\ell + 1$.

For the symmetric representation of rank 2, $f = 2$, $k = 1$, so

$$C_2(R) = \frac{(N+2)(N-1)}{N}. \quad (\text{C15})$$

By taking into account the multiplicity,

$$d(R) = \frac{N(N+1)}{2}, \quad (\text{C16})$$

one finds

$$2T_R = 2 \frac{d(R)C_2(R)}{N^2 - 1} = N + 2. \quad (\text{C17})$$

For the symmetric representation of rank m , $f = m$, $k = 1$, so

$$C_2(R) = \frac{m(N+m)(N-1)}{2N}. \quad (\text{C18})$$

By taking into account the multiplicity,

$$d(R) = \frac{(N+m-1)!}{m!(N-1)!}, \quad (\text{C19})$$

one finds

$$2T_R = \frac{d(R)C_2(R)}{N^2 - 1} = \frac{(N+m)!}{(N+1)!(m-1)!}. \quad (\text{C20})$$

$2T_R$ is a multiple of m , as can be shown following a similar consideration as (C11)–(C14).

- [1] S. Raby, S. Dimopoulos, and L. Susskind, Tumbling gauge theories, *Nucl. Phys.* **B169**, 373 (1980).
- [2] S. Dimopoulos, S. Raby, and L. Susskind, Light composite fermion, *Nucl. Phys.* **B173**, 208 (1980).
- [3] I. Bars and S. Yankielowicz, Composite quarks and leptons as solutions of anomaly constraints, *Phys. Lett.* **101B**, 159 (1981).
- [4] E. Eichten and J. Preskill, Chiral gauge theories on the lattice, *Nucl. Phys.* **B268**, 179 (1986).
- [5] J. Goity, R. D. Peccei, and D. Zeppenfeld, Tumbling and complementarity in a chiral gauge theory, *Nucl. Phys.* **B262**, 95 (1985).
- [6] E. Eichten, R. D. Peccei, J. Preskill, and D. Zeppenfeld, Chiral gauge theories in the $1/n$ expansion, *Nucl. Phys.* **B268**, 161 (1986).
- [7] T. Appelquist, A. G. Cohen, M. Schmaltz, and R. Shrock, New constraints on chiral gauge theories, *Phys. Lett. B* **459**, 235 (1999).
- [8] T. Appelquist, Z. y. Duan, and F. Sannino, Phases of chiral gauge theories, *Phys. Rev. D* **61**, 125009 (2000).
- [9] E. Poppitz and Y. Shang, Chiral lattice gauge theories via mirror-fermion decoupling: A mission (im)possible?, *Int. J. Mod. Phys. A* **25**, 2761 (2010).
- [10] A. Armoni and M. Shifman, A chiral $SU(N)$ gauge theory planar equivalent to Super-Yang-Mills, *Phys. Rev. D* **85**, 105003 (2012).
- [11] T. Appelquist and R. Shrock, Ultraviolet to infrared evolution of chiral gauge theories, *Phys. Rev. D* **88**, 105012 (2013).
- [12] Y. L. Shi and R. Shrock, $A_k \bar{F}$ chiral gauge theories, *Phys. Rev. D* **92**, 105032 (2015).
- [13] Y. L. Shi and R. Shrock, Renormalization-group evolution and nonperturbative behavior of chiral gauge theories with fermions in higher-dimensional representations, *Phys. Rev. D* **92**, 125009 (2015).
- [14] T. A. Ryttov and R. Shrock, β'_{IR} at an infrared fixed point in chiral gauge theories, *Phys. Rev. D* **97**, 016020 (2018).
- [15] G. 't Hooft, Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking, *NATO Sci. Ser. B* **59**, 135 (1980).
- [16] C. Csáki and H. Murayama, Discrete anomaly matching, *Nucl. Phys.* **B515**, 114 (1998).
- [17] S. Bolognesi, K. Konishi, and M. Shifman, Patterns of symmetry breaking in chiral QCD, *Phys. Rev. D* **97**, 094007 (2018).
- [18] S. Bolognesi and K. Konishi, Dynamics and symmetries in chiral $SU(N)$ gauge theories, *Phys. Rev. D* **100**, 114008 (2019).
- [19] S. Yamaguchi, 't Hooft anomaly matching condition and chiral symmetry breaking without bilinear condensate, *J. High Energy Phys.* **01** (2019) 014.
- [20] S. Bolognesi, K. Konishi, and A. Luzio, Gauging 1-form center symmetries in simple $SU(N)$ gauge theories, *J. High Energy Phys.* **01** (2020) 048.
- [21] S. Bolognesi, K. Konishi, and A. Luzio, Dynamics from symmetries in chiral $SU(N)$ gauge theories, *J. High Energy Phys.* **09** (2020) 001.
- [22] S. Bolognesi, K. Konishi, and A. Luzio, Probing the dynamics of chiral $SU(N)$ gauge theories via generalized anomalies, *Phys. Rev. D* **103**, 094016 (2021).
- [23] S. Bolognesi, K. Konishi, and A. Luzio, Strong anomaly and phases of chiral gauge theories, *J. High Energy Phys.* **08** (2021) 028.
- [24] S. Bolognesi, K. Konishi, and A. Luzio, Anomalies and phases of strongly-coupled chiral gauge theories: Recent developments, *Int. J. Mod. Phys. A* **37**, 2230014 (2022).
- [25] S. Bolognesi, K. Konishi, and A. Luzio, Dynamical Abelianization and anomalies in chiral gauge theories, *J. High Energy Phys.* **12** (2022) 110.
- [26] S. Bolognesi, K. Konishi, and A. Luzio, Dynamics of strongly-coupled chiral gauge theories, *J. Phys. Conf. Ser.* **2531**, 012006 (2023).
- [27] S. Bolognesi, K. Konishi, and A. Luzio, The $\mathbb{Z}_2$ anomaly in some chiral gauge theories, *J. High Energy Phys.* **08** (2023) 125.
- [28] K. Konishi, S. Bolognesi, and A. Luzio, Anomalies and dynamics in strongly-coupled gauge theories. New criteria for different phases and lessons from supersymmetric theories, *Proc. Sci., CORFU2023* (2024) 126 [arXiv: 2403.15775].
- [29] M. M. Anber, S. Hong, and M. Son, New anomalies, TQFTs, and confinement in bosonic chiral gauge theories, *J. High Energy Phys.* **02** (2022) 062.
- [30] M. M. Anber and S. Y. L. Chan, 2-index chiral gauge theories, *J. High Energy Phys.* **10** (2023) 025.
- [31] S. Bolognesi, K. Konishi, A. Luzio, and M. Orso, Natural anomaly matching, *Phys. Rev. D* **110**, 114037 (2024).
- [32] S. Bolognesi, K. Konishi, A. Luzio, and M. Orso, Chiral gauge theories, generalized anomalies and breakdown of the color-flavor-locked center symmetry, *J. High Energy Phys.* **05** (2025) 062.
- [33] H. L. Li, Á. Pastor-Gutiérrez, S. Vatan, and L. X. Xu, Dynamical symmetry breaking in Georgi-Glashow chiral-gauge theories, *J. High Energy Phys.* **12** (2025) 020.
- [34] N. Seiberg, Modifying the sum over topological sectors and constraints on supergravity, *J. High Energy Phys.* **07** (2010) 070.
- [35] A. Kapustin and N. Seiberg, Coupling a QFT to a TQFT and duality, *J. High Energy Phys.* **04** (2014) 001.
- [36] O. Aharony, N. Seiberg, and Y. Tachikawa, Reading between the lines of four-dimensional gauge theories, *J. High Energy Phys.* **08** (2013) 115.
- [37] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, Generalized global symmetries, *J. High Energy Phys.* **02** (2015) 172.
- [38] D. Gaiotto, A. Kapustin, Z. Komargodski, and N. Seiberg, Theta, time reversal, and temperature, *J. High Energy Phys.* **05** (2017) 091.
- [39] H. Shimizu and K. Yonekura, Anomaly constraints on deconfinement and chiral phase transition, *Phys. Rev. D* **97**, 105011 (2018).
- [40] Y. Tanizaki, Y. Kikuchi, T. Misumi, and N. Sakai, Anomaly matching for the phase diagram of massless $\mathbb{Z}_N$ -QCD, *Phys. Rev. D* **97**, 054012 (2018).
- [41] Z. Komargodski, T. Sulejmanpasic, and M. Ünsal, Walls, anomalies, and deconfinement in quantum antiferromagnets, *Phys. Rev. B* **97**, 054418 (2018).
- [42] M. M. Anber and E. Poppitz, Two-flavor adjoint QCD, *Phys. Rev. D* **98**, 034026 (2018).

- [43] M. M. Anber and E. Poppitz, Anomaly matching, (axial) Schwinger models, and high-T super Yang-Mills domain walls, *J. High Energy Phys.* **09** (2018) 076.
- [44] Y. Tanizaki, Anomaly constraint on massless QCD and the role of skyrmions in chiral symmetry breaking, *J. High Energy Phys.* **08** (2018) 171.
- [45] D. Amati, G. C. Rossi, and G. Veneziano, Instanton effects in supersymmetric gauge theories, *Nucl. Phys.* **B249**, 1 (1985).
- [46] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Instantons in supersymmetric theories, *Nucl. Phys.* **B223**, 445 (1983).
- [47] I. Affleck, M. Dine, and N. Seiberg, Dynamical supersymmetry breaking in supersymmetric QCD, *Nucl. Phys.* **B241**, 493 (1984).
- [48] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Supersymmetric instanton calculus (gauge theories with matter), *Nucl. Phys.* **B260**, 157 (1985).
- [49] D. Amati, K. Konishi, Y. Meurice, G. C. Rossi, and G. Veneziano, Nonperturbative aspects in supersymmetric gauge theories, *Phys. Rep.* **162**, 169 (1988).
- [50] N. M. Davies, T. J. Hollowood, V. V. Khoze, and M. P. Mattis, Gluino condensate and magnetic monopoles in supersymmetric gluodynamics, *Nucl. Phys.* **B559**, 123 (1999).
- [51] D. Finnell and P. Pouliot, Instanton calculations versus exact results in four-dimensional SUSY gauge theories, *Nucl. Phys.* **B453**, 225 (1995).
- [52] K. Konishi and A. Ricco, Calculating gluino condensates in $N = 1$ SYM from Seiberg-Witten curves, *Phys. Lett. B* **570**, 118 (2003).
- [53] T. K. Kuo and N. Nakagawa, The generation problem and the symplectic group, *Nucl. Phys.* **B250**, 641 (1985).
- [54] J. Davighi and J. Tooby-Smith, Electroweak flavour unification, *J. High Energy Phys.* **09** (2022) 193.
- [55] E. Witten, An $SU(2)$ anomaly, *Phys. Lett.* **117B**, 324 (1982).
- [56] S. R. Coleman and E. J. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking, *Phys. Rev. D* **7**, 1888 (1973).
- [57] M. Piai, A. Pierce, and J. G. Wacker, Composite vector mesons from QCD to the little Higgs, *arXiv:hep-ph/0405242*.
- [58] L. F. Li, Group theory of the spontaneously broken gauge symmetries, *Phys. Rev. D* **9**, 1723 (1974).
- [59] D. d. Wu, The symmetry breaking pattern of scalars in low dimension representations, *Nucl. Phys.* **B199**, 523 (1982); *Nucl. Phys.* **B213**, 545(E) (1983).
- [60] J. E. Kim, Possibility of $SU(n)$ antisymmetric tensor Higgs fields, *Phys. Rev. D* **26**, 2009 (1982).
- [61] P. Jetzer, J. M. Gerard, and D. Wyler, Possible symmetry breaking patterns with totally symmetric and antisymmetric representations, *Nucl. Phys.* **B241**, 204 (1984).
- [62] A. O. Barut and R. Raczka, *Theory of Group Representations and Applications* (PWN—Polish Scientific Publishers, Warszawa, 1980).
- [63] D. D. Dietrich and F. Sannino, Conformal window of $SU(N)$ gauge theories with fermions in higher dimensional representations, *Phys. Rev. D* **75**, 085018 (2007).