Exact Schur index in closed form for non-Lagrangian theories

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The Schur index is a powerful tool to probe the spectrum and dualities of four-dimensional $\mathcal{N}=2$ superconformal field theories, deeply related to two-dimensional vertex operator algebras (VOAs). In this paper, we compute the Schur index in closed form for two series of non-Lagrangian theories. We explore and classify the Argyres-Douglas (AD) theories $D_p^b(\mathfrak{sl}_N,[Y])$ realized as the $SU(2)$ gauging of two AD matter theories, where we identify several infinite families with interesting central charge relations analogous to the $a_{4d}=c_{4d}$ of $\mathcal{N}=4$ theories. We focus on $D_{N-4}(\mathfrak{sl}(N),[N-4,4])$ and $D_{N-2}(\mathfrak{sl}(N),[N-3,3])$ and compute their flavored and unflavored Schur and Wilson line indices in compact form. We also explore their large-N behavior and show that they arise as special limits of the $SU(2)$ SQCD flavored index, also analogous to the relation among the $a_{4d}=c_{4d}$ theories. We also generalize the elliptic function integration formula in the presence of higher-order poles to compute in closed form the partially flavored indices of the Minahan-Nemeschansky E_6 and E_7 theories. Our results point to a universal structure underlying the residues of elliptic integrands, Wilson loop indices, and nonvacuum modules of the corresponding VOAs.

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I. INTRODUCTION

The superconformal index is a fundamental observable to analyze and characterize four-dimensional (4D) $\mathcal{N}=2$ superconformal field theories (SCFTs). It encodes the representation-theoretic structure of the protected local operator spectrum. The index admits various simplifying limits, among which is the Schur limit [1]. The Schur index is particularly interesting because it has been shown to participate prominently in the 4D/two-dimensional (2D) or the SCFT/vertex-operator-algebra (VOA) correspondence, where every 4D $\mathcal{N}=2$ SCFT is associated with a 2D VOA, and the Schur index is precisely identified with the vacuum character of the algebra [2].

There are several ways of computing the Schur index. For Lagrangian theories, the index is written as a contour integral of a multivariate elliptic function, where the integrand represents the Schur index in the zero-coupling limit, and the contour integral implements the gauging by projecting onto the gauge-invariant states. The integrand can be expanded in q-series, and then the contour integral can be performed termwise, resulting in a q-series expansion of the Schur index.

For class-$\mathcal{S}$ theories defined with respect to a puncture Riemann surface, including families of non-Lagrangian $T_{N\geq 3}$ theories and Argyres-Douglas theories, the Schur index can be computed exploiting the topological quantum field theory (TQFT) nature of the index [1,3–10]. In this approach, the index is typically expressed as a sum over irreducible representations of the gauge group, with each term in the sum being a product of contributions from the punctures and the genus of the Riemann surface.

Some non-Lagrangian 4D $\mathcal{N}=2$ SCFTs can also be obtained through supersymmetry-enhancing renormalization group flow starting from 4D $\mathcal{N}=1$ Lagrangian theories [11–18]. In these cases, the Schur index can be computed using the $\mathcal{N}=1$ index of the UV theory with appropriate modifications to account for decoupled operators along the flow. The usual output of this approach is either the contour integral that comes from the $\mathcal{N}=1$ Lagrangian theory or the q-series expansion.

Compared with the q-series expression or infinite sum over representations, the Schur index in some closed form can be more useful in some circumstances, such as in studying the modular property of the index, representation-theoretic problems of the associated VOA, and the large-N behavior/holography interpretation. There had been series of efforts to perform exact computation of the Schur index. The Fermi-gas formalism was first introduced to compute

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the unflavored $\mathcal{N}=4$ $SU(N)$ Schur index in Ref. [19], then applied to circular quiver theories [20], and more recently generalized to the flavored index and $\mathcal{N}=4$ theories with other gauge groups [21,22], and with insertion of line operators. In Ref. [23], integration formula of elliptic functions and Eisenstein series were proposed to compute various types of Lagrangian Schur index, and subsequently to include of surface and line defects, where the results are written in terms of quasi-modular or quasi-Jacobi forms [24–27].

There are mainly two known series of closed-form Schur index for non-Lagrangian theories. First is the class of Argyres-Douglas theories whose associated VOAs are given by affine Kac-Moody algebras at boundary admissible levels (and a few nonadmissible levels) and W-algebras obtained from quantum Drinfeld-Sokolov reduction of these affine Kac-Moody algebras [28–30]. The second class is the $\mathcal{T}_{p,N}$ theories whose Schur index is directly related to that of the 4D $\mathcal{N}=4$ theories [31–33].

In this paper, we apply the integration formula to more non-Lagrangian theories and compute the Schur index in closed form. We focus on two classes of theories. The first is the Argyres-Douglas theories $D_p^b(\mathfrak{sl}(N),[Y])$, which has been a subject of intensive study [28–30,34–44]. Here, p and N are not coprime. These theories can be constructed by going to special points in the Coulomb branch [45], through geometric engineering on singular Calabi-Yau threefolds [46,47], or in class-$\mathcal{S}$ construction with irregular punctures [34,39,48,49]. As shown in Refs. [37,50–52], such theories can be written in a duality frame conformal gauging two $D_{p_i}(\mathfrak{sl}(N_i),[Y_i])$ by an $SU(N)$ vector multiplet, and possibly with some additional fundamental hypermultiplets. In the simplest situation, each pair p_i, N_i are, respectively, coprime, and the two Argyres-Douglas matter theories do not have exactly marginal deformations. In this case, the Schur index of each component theory is known in closed form in terms of plethystic exponential [5,28,30], and the gauging gives rise to a multivariate integral of elliptic function. This allows us to analytically compute the Schur index using the integration formula [23]. For simplicity, we focus on the cases where the gauge node is an $SU(2)$ gauge node, whose Schur index involves only a single contour integral with integrand $\mathcal{Z}$. Cases with higher-rank gauge node, or where the component theories are not in the coprime class, can be treated similarly, though the computation will be more involved, which we leave for future work.

The requirement of having only one $SU(2)$ gauge node and coprime pairs (p_i,N_i) put significant constraints on the choice of b, p, N, Y. This leads to a partial classification for the type-I cases, i.e., when $b=N$. Even with this restriction, there are still infinitely many possibilities. Among them, we identify several interesting infinite families, whose central charges satisfy simple relations

$$a_{4\mathrm{d}}-c_{4\mathrm{d}}=\Delta,\quad \Delta=-\frac{1}{12},-\frac{1}{12}+\frac{1}{4(N-4)},-\frac{1}{8},-\frac{5}{24}, \tag{1}$$

reminiscent of the $\mathcal{N}=4$ and the $\mathcal{T}_{p,N}$ theories. In particular, we study systematically the $D_{N-4}(\mathfrak{sl}(N),[N-4,4])$, $D_{N-2}(\mathfrak{sl}(N),[N-3,3])$ series. In fact, the difference $a_{4d}-c_{4d}$ is directly related to the 't Hooft anomaly for the $U(1)_r$ symmetry

$$\mathrm{Tr}U(1)_r=48(a_{4\mathrm{d}}-c_{4\mathrm{d}}). \tag{2}$$

This relation can also be employed to analyze central charges by interpreting the closure of a regular puncture as a renormalization group (RG) flow. In this picture, the 't Hooft anomaly matching condition holds between the UV and IR theories, up to possible additional contributions from Goldstone multiplets associated with flavor symmetry breaking [51]. We compute the Schur index for these two families in closed form. Curiously, the Schur index of both series are just special limit of the flavored index of the $SU(2)$ theory with four flavors, in complete analogy to the relation between the $\mathcal{T}_{p,N}$ theories and the $\mathcal{N}=4$ $SU(N)$ theories [31–33,53]. We also explore the large-N behavior of the Schur/Wilson line index for these two families.

Bogomol'nyi-Prasad-Sommerfield (BPS) nonlocal operators preserving the supercharges defining the Schur index are believed to give rise to modules of the associated VOA [54–56]. For the two $SU(2)$ Lagrangian theories, the $\mathcal{N}=4$ $SU(2)$ and the $SU(2)$ SQCD theory, the Wilson line index can be written schematically as

$$\mathcal{I}_L(b,q)=\sum_i R_i(b,q)ch_i(b,q)=R_0'\mathcal{I}+\sum_j R_j'\mathrm{Res}_j, \tag{3}$$

where R_i,R_j' are some rational functions of $q^{1/2}$ and the flavor fugacities b, $ch_i(b,q),\mathrm{Res}_j$ denote the irreducible characters of the associated VOA and the residues of $\mathcal{Z}$ [24,57–59]. In the two examples, all the characters ch_i and residues Res_j solve the same set of flavored modular linear differential equations (MLDEs) [25,57,60–66], and the characters ch_i are given as integral linear combination of the residues (and the Schur index $\mathcal{I}$) [64,67].

The presence of a gauge group allows us to couple the non-Lagrangian theory to BPS nonlocal operators. We consider the simplest possibility by introducing the Wilson line and compute the line index [24,68,69]. Although the full analysis of flavor modular differential equations is currently lacking, we find in various examples combinations of residues of $\mathcal{Z}$ that admit unflavoring limit and solve the same unflavored MLDE as the unflavored Schur index. Said differently, the specific combination of residues is a linear combination of characters.

We also explore the closed-form Schur index of the Minahan-Nemeschansky E_6 and E_7 theory. Recently, the

unflavored Schur index of these theories has been computed using the generalized partition function method [36], where the index is written as a contour integral of an elliptic function, even though the theory has no Lagrangian description. Nonetheless, the integration formula can be applied. Explicitly, we find that the generalized partition function method can be slightly generalized to include one flavor fugacity, allowing us to extract the partially flavored Schur index in closed form. In this computation, due to the presence of higher-order poles, we develop new integration formula that incorporates generalized residues.

Although the contour integral formula does not follow from a Lagrangian description, computationally we can always compute a "Wilson line index" by inserting into the integral an $SU(2)$ character and summing over the generalized residues. *A priori*, one would not have expected these residues or line index to have any relation with the $(E_6)_{-3}$, $(E_7)_{-4}$ modules. Surprisingly, we do find that the leading partially flavored residues solve the expected partially flavored MLDEs coming from the null states of the VOAs, and hence they are linear combinations of VOA characters. All of these results [together with those from Argyres-Douglas (AD) theories] suggest a universal relation between the residues of an elliptic integrand $\mathcal{Z}$ that integrates to the index, the Wilson line index, and the VOA representation theory.

The rest of the paper is organized as follows. In Sec. II, we review the landscape of type-A Argyres-Douglas theories and summarize the classification of those admitting an $SU(2)$ gauge theory description. In Sec. III, we compute the Schur index in closed form for several simple examples of $D_p^b(\mathfrak{sl}_N,[Y])$, including both type I and type II. In Secs. IV and V, we focus on two infinite families $D_{N-4}(\mathfrak{sl}(N),[N-4,4])$, $D_{N-2}(\mathfrak{sl}(N),[N-3,3])$. We show that their flavored Schur index is a simple specialization of the $SU(2)$ supersymmetric quantum chromodynamics (SQCD) Schur index. We compute both the flavored and unflavored Schur/Wilson line index in compact closed form, construct a nonvacuum module character from residues, and study the large-N behavior of these indices. In Sec. VI, we compute the partially flavored Schur index of the Minahan-Nemeschansky E_6 and E_7 theories and show that the first order generalized residues are linear combinations of $(E_6)_{-3}$, $(E_7)_{-4}$ characters.

II. LANDSCAPE OF TYPE-*A* ARGYRES-DOUGLAS THEORIES

A. Generalities on Argyres-Douglas theories

The AD theory was first discovered as the IR fixed point at a special locus on the Coulomb branch of pure $SU(3)$ gauge theory [45]. They are intrinsically non-Lagrangian in the $\mathcal{N}=2$ sense, which can be seen from the presence of local operators with fractional scaling dimension. Alternatively, an AD theory can be constructed by compactifying the six-dimensional type-$J=A,\ D,\ E$ $\mathcal{N}=(2,0)$ theory on a Riemann surface with one irregular puncture and (optionally) one regular puncture [49].

The Coulomb branch of a 4D $\mathcal{N}=2$ superconformal field theory on $\mathbb{R}^3\times S^1$ can be described by a Hitchin system [70] and hence is characterized by a Higgs field $\Phi(z)$, where z denotes the coordinate on the Riemann sphere. The irregular puncture takes the general form

$$\Phi(z)=\frac{T_k}{z^{2+\frac{k}{n}}}\mathrm{d}z+\sum_{l=-n}^{k-1}\frac{T_l}{z^{2+\frac{l}{n}}}\mathrm{d}z+\cdots, \tag{4}$$

where the leading matrix T_k is a semisimple element of A_{N-1}. A necessary condition for the theory to be superconformal is that the Levi subalgebras for each T_i satisfy [37]

$$\mathfrak{l}_k=\mathfrak{l}_{k-1}=\cdots=\mathfrak{l}_{-n+1}\supset\mathfrak{l}_{-n}. \tag{5}$$

The classification of irregular punctures relies on the single-valued-ness of the Higgs field under the transformation $z\to e^{2\pi i}z$. The allowed leading matrix then takes the form [37,71]

$$T_k=\begin{pmatrix} a_1^{(k)}\Sigma_n & & & & \\ & a_2^{(k)}\Sigma_n & & & \\ & & \ddots & & \\ & & & a_m^{(k)}\Sigma_n & \\ & & & & 0_{N-mn=N-b} \end{pmatrix}, \tag{6}$$

where

$$\Sigma_n:=\begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{n-1} \end{pmatrix},\qquad \omega=e^{2\pi i/n}. \tag{7}$$

The regular punctures for type-*A* theories were classified in Ref. [48] and take the general form

$$\Phi(z)=\frac{\Lambda}{z}+\cdots, \tag{8}$$

where Λ is a nilpotent element of A_{N-1}. Thus, the classification of regular punctures corresponds to the classification of nilpotent orbits, and they can be labeled by Young tableaux $[Y]$.

In this paper, we require the eigenvalues of each T_i to be distinct. In summary, an AD theory can be labeled as

$$D_p^b(\mathfrak{sl}_N,[Y]),\qquad p=k+b. \tag{9}$$

For certain specific values of b and $[Y]$, we follow the convention of Refs. [43,44] and simplify the notation as

$$\begin{aligned} D_p^N(\mathfrak{sl}_N,[Y]) &=: D_p(\mathfrak{sl}_N,[Y]),\\ D_p^b(\mathfrak{sl}_N,[1^N]) &=: D_p^b(\mathfrak{sl}_N),\\ D_p^N(\mathfrak{sl}_N,[1^N]) &=: D_p(\mathfrak{sl}_N). \end{aligned} \tag{10}$$

We also refer to the cases where $b=N$ and $b=N-1$ as type I and type II theories, respectively.

Class-$\mathcal{S}$ theories with regular punctures admit different pants decompositions related by S-duality. Similarly, the landscape of AD theories also enjoys various dualities that help simplify the overall picture, expressing an AD theory in terms of gauging of other AD matter theories [37,38,50–52].

In this paper, we make use of the duality discussed in Refs. [51,52] and later generalized in Ref. [50], where a generic AD theory $D_p^b(\mathfrak{sl}_N,[Y])$ is expressed in terms of a type-I theory and some additional free hypermultiplets,

$$\begin{aligned} D_p^b(\mathfrak{sl}_N,[Y]) \cong\ & D_p(\mathfrak{sl}_{(N-b)p+b},[(p-1)^{N-b},Y])\\ & \otimes H_{\text{free}}\ \text{free hypermultiplets}, \end{aligned} \tag{11}$$

where

$$H_{\text{free}} = (N-b)\sum_{i=k+b}^{N}(i-k-b+1)l_i \tag{12}$$

and l_i denotes the multiplicity of the part i in the Young diagram $[Y]=[N^{l_N},\ldots,1^{l_1}]$. The additional hypermultiplets transform in the fundamental representation of each simple flavor factor arising from the regular punctures.

Therefore, the study of generic $D_p^b(\mathfrak{sl}_N,[Y])$ reduces to that of the type-I theories. A type-I theory with exactly marginal deformation can be further expressed as a gauge theory with simpler AD matters. The regular puncture is labeled by the partition $[Y]$,

$$[Y]=\left[N^{l_N},\ldots,2^{l_2},1^{l_1}\right],\qquad L=\sum_{i=1}^{N}l_i, \tag{13}$$

one can define the "conjugate" partition (here the $q\in\mathbb{N}$, not to be confused with the $q=e^{2\pi i\tau}$ in the Jacobi theta functions)

$$[\tilde{Y}]:=[(q-1)^{l_1},(q-2)^{l_2},\ldots,(q-N)^{l_N}],\quad q:=\frac{p}{m}. \tag{14}$$

The entries in the partition $\widetilde{[Y]}$ sum to $qL-N$. We separate the entries into three subpartitions based on the sign: positive, negative, and zero. Let $[\tilde{Y}_+]$ be the subpartition of positive entries and $[\tilde{Y}_-]$ be the subpartition of the absolute values of the negative entries. The sums of these two subpartitions are denoted $\tilde{N}_+$ and $\tilde{N}_-$, respectively, and the number of zero entries is denoted $\tilde{N}_0$.

Then, the general type I theory[1] can be expressed as as a gauge theory with AD matters,

$$\begin{array}{c} D_p\left(\mathfrak{sl}_N,[Y]\right)\cong D_q\left(\mathfrak{sl}_{qL-\frac{N}{m}},\left[\widetilde{Y}_+,1^{N-\frac{N}{m}-\widetilde{N}_-}\right]\right)\\ \uparrow\\ \mathfrak{su}\left(N-\frac{N}{m}-\widetilde{N}_-\right)\text{——}\boxed{\widetilde{N}_0}\\ \downarrow\\ D_{(m-1)q}\left(\mathfrak{sl}_{N-\frac{N}{m}},\left[\widetilde{Y}_-,1^{N-\frac{N}{m}-\widetilde{N}_-}\right]\right) \end{array} \tag{15}$$

In the gauge theory description, the first AD matter contains no exactly marginal deformation since $\gcd((m-1)q,N-\frac{N}{m})=1$. In the second AD matter, the number of exactly marginal deformations is

$$\gcd\left((m-1)q,N-\frac{N}{m}\right)-1=m-2, \tag{16}$$

making the total number of exactly marginal deformations of $D_p(\mathfrak{sl}_N,[Y])$ equal to $m-1=\gcd(N,p)-1$. This decomposition can be applied recursively to decompose any $D_p^b(\mathfrak{sl}_N,[Y])$ into a generalized quiver gauge theory with basic AD matter without exactly marginal deformations.

B. Argyres-Douglas theories as $SU(2)$ gauge theories

In this paper, we are interested in computing the exact Schur index of the simplest class of AD theories, which can be realized as $SU(2)$ gauge theories coupled to two AD matters without exactly marginal deformations. Schematically, they look like

$$\begin{array}{c} D_{p_1}(\mathfrak{sl}_{N_1},[Y_1])\longleftarrow\mathfrak{su}(2)\longrightarrow D_{p_2}(\mathfrak{sl}_{N_2},[Y_2]),\\ \boxed{\tilde{N}_0} \end{array} \tag{17}$$

where $gcd(p_1,N_1)=gcd(p_2,N_2)=1$. Recall that the number of exactly marginal operators in $D_p(\mathfrak{sl}_N,[Y])$ with $p=N+k$ is given by the formula[2]

$$\begin{aligned} &\gcd(N,k)-1 \quad \text{for } k\neq mN,\\ &\gcd(N,k)-2 \quad \text{for } k=mN. \end{aligned} \tag{18}$$

[1]We consider only the case where k is not an integer multiple of N.

[2]In this paper, we focus on the former situation.

The coprime condition on p_i, N_i implies that both AD matters have no exactly marginal deformations. Imposing that the full theory $D_p(\mathfrak{sl}_N,[Y])$ has only one exactly marginal coupling and the gauge group to be $SU(2)$, the duality (15) puts a set of constraints on the parameters.

The requirement that the full theory has only one exactly marginal deformation and with only one $SU(2)$ gauge node implies

$$m=\gcd(N,k)=2,\ N-\frac{N}{2}-\tilde{N}_-=2,\ p=k+N=mq=2q. \tag{19}$$

Substituting the definition of $\tilde{N}_-$, we have the following constraints:

$$\begin{aligned} l_{q+1}+2l_{q+2}+\cdots+(N-q)l_N&=\frac{N}{2}-2,\\ l_1+2l_2+\cdots Nl_N&=N. \end{aligned} \tag{20}$$

The condition $\gcd(N,k)=2$ requires both N, k to be even. Furthermore, if $N=2n$, $k=2n'$, then we must have $\gcd(n,n')=1$:

(i) If $N=2$, then $N-\frac{N}{2}-\tilde{N}_-=2$ cannot be satisfied. Hence, $N\geq 4$.

(ii) If $N=4$, then $k=2 \bmod 4 \geq -2$. Note that $\frac{N}{2}-2=0$ and hence $l_{q+1}=l_{q+2}=\cdots=l_N=0$. This places no further constraint on k but allows only a subset of the partitions $[Y]$.

(iii) If $N>4$ such that $\frac{N}{2}-2>0$, then q must be strictly less than N; otherwise, the left-hand side of the first equation in (20) would be zero. This in turns places strong constraints on k on which we now elaborate.

1. Theories with $k>0$

First, we denote $N=2n>4$ and consider $k=2n'>0$. The first equation in (20) can be rearranged,

$$\begin{aligned} &l_1+2l_2+\cdots+Nl_N-(l_1+2l_2+\cdots+ql_q)\\ &\quad-q(l_{q+1}+l_{q+2}+\cdots+l_N)=\frac{N}{2}-2, \end{aligned} \tag{21}$$

giving

$$l_1+2l_2+\cdots+ql_q+q(l_{q+1}+l_{q+2}+\cdots+l_N)=\frac{N}{2}+2. \tag{22}$$

Recall that $q=n+n'>n=N/2$. The left-hand side of (22) as a function of integer partition Y assumes its minimal at $[Y]=[N]$: as $[Y]$ varies away from $[Y]=[N]$, $l_1,\ldots,l_q$ starts picking up nonzero values, while $l_{q+1}+\cdots+l_N$ remains 1 for a while. In this process, the left-hand side of (22) increases, until $l_{q+1}=l_{q+2}=\cdots=l_N=0$, forcing the expression to simply equal N; this is the maximal value, since the moment $l_{q+1}+\cdots+l_N$ drops to zero $l_1+2l_2+\cdots+ql_q$ increases at least by $q+1$. To summarize,

$$\begin{aligned} n+n'&\leq l_1+2l_2+\cdots+ql_q\\ &\quad+q(l_{q+1}+l_{q+2}+\cdots+l_N)=n+2. \end{aligned} \tag{23}$$

Hence,

$$n'\leq 2,\quad k=2,4. \tag{24}$$

When $n'=2$, the only partition allowed by the constraint (22) is $[Y]=[N]$. When $n'=1$, there is only $Y=[N-1,1]$. We summarize the allowed configurations for $k>0$ below.

Even N	k	$[Y]$	Comments
$N(\neq 4)$	4	$[N]$	$\cong D_{N+4}(\mathfrak{sl}_4,[4])$ $\cong D_{N+4}(\mathfrak{sl}_{N+8},[N+4,4])$
$N(\neq 4)$	2	$[N-1,1]$	$\cong D_{N+2}(\mathfrak{sl}_{N+4},[N+1,3])$
4	$4n'+2\geq 6$	$[4]$ $[3,1],[2,2],[1^4]$	$\cong$ first row
4	2	$[2,2],[3,1],[2,1^2],[1^4]$	

In this table, we have made some identification of entries utilizing the dualities among type-I Argyres-Douglas theories as discussed in Ref. [50]. Recall that the duality between general type-I Argyres-Douglas theories is encoded in the gauging description [50],

$$D_p(\mathfrak{sl}_N,[Y])\equiv D_p(\mathfrak{sl}_{pl+N},[p^l,Y]) \tag{25}$$

and

$$D_p(\mathfrak{sl}_N,[Y]) \cong D_p(\mathfrak{sl}_{pL-N},[Y^c]), \tag{26}$$

where $[Y^c]=[\,(p-1)^{l_1},(p-2)^{l_2},\ldots,(p-N)^{l_N}\,]$. For instance, one finds the duality

$$\begin{aligned} D_{4+N}(\mathfrak{sl}_N,[N]) &\cong D_{4+N}(\mathfrak{sl}_4,[4]) \\ &\cong D_{N+4}(\mathfrak{sl}_{N+8},[N+4,4]) \\ D_{N+2}(\mathfrak{sl}_N,[N-1,1]) &\cong D_{2+N}(\mathfrak{sl}_{N+4},[N+1,3]). \end{aligned} \tag{27}$$

This gives the "comments" column in the table.

2. Theories with $k<0$

For $2-N\le k=2n'<0$, the analysis is similar, where the left-hand side of (22) assumes its maximum $N=2n$ and minimum $q=n+n'$ at $[Y]=[1^N]$ and $[Y]=[N]$, respectively. Given that $n+n'\le n+2\le N=2n$, and the fact that the left-hand side of (22) can take all integer values between $n+n'$ and N, there are always some partitions $[Y]$ satisfying (22).

However, we should remove unphysical solutions where the partition $[Y]=[r^{l_r>0},Y']$ with $r>p$. From the gauge theory description (15), when $r>p=2q$, the conjugate partition $[\tilde{Y}]$ must contain at least one negative entry, $\tilde{Y}=[\ldots,(q-r)^{l_r},\ldots]$. Hence, $\tilde{Y}_-=[\ldots,(r-q)^{l_r},\ldots]$, while $(r-q)>(m-1)q=q$, making the second AD matter $D_{(m-1)q}(\mathfrak{sl}_{N-\frac{N}{m}},[\tilde{Y}_-,1^{N-\frac{N}{m}-\tilde{N}_-}])$ in (15) unphysical. We also identify and omit cases that have been listed in the previous table for $k>0$ using the dualities (25) and (26). Let us spell out the physical solutions in detail.

Consider $n'=-1$, then $q=n-1$. The sum $l_{q+1}+\cdots+l_N\in\{0,1,2\}$, where 2 is assumed when $l_{q+1}=1, l_{i\neq q+1}=0$. It is then straightforward to see that the only possibility of satisfying (22) is $Y=[N-3,$ partitions of 3], except for the special case $n=4$; in this case, $2q=2n-2=n+2$, and there is additional solution $Y=[4^2]$.

Consider $n'=-2$, then $q=n-2$, $p=2n-4$. The sum $l_{q+1}+\cdots+l_N\in\{0,1,2\}$. When the sum equals 0, the left-hand side of (22) gives $N=2n>n+2$. When the sum equals 2, $l_{q+1}+l_{q+2}=2$, and the left-hand side of (22) gives three possible values,

$$2q+0=2n-4,\quad 2q+1=2n-3,\quad 2q+2=2n-2. \tag{28}$$

Only $n=5$ is allowed such that $2n-3=n+2$, since $\gcd(n,n')=1$. Finally, when the sum equals 1, since $n+2=q+4$, only the partitions $Y=[N-4,$ partitions of 4] satisfy the constraint (22). However, these are dual to the cases with $n'=2$ in the previous table, and hence are omitted.

The general cases with $1-n\le n'<-2$ are a lot trickier. However, we can show that for given n' and relative large N such that $n>|3n'|$ there are no new solutions. The large-N condition implies the following useful inequalities:

$$\begin{aligned} &3q=3n+3n'>2n, \qquad 2-n'<n+n'=q, \\ &\qquad 2n-2|n'|>n+2. \end{aligned} \tag{29}$$

Because of the first inequality, the sum $l_{q+1}+\cdots+l_N\in\{0,1,2\}$. When the sum equals 0, the left-hand side of (22) gives $N=2n>n+2$. When the sum equals 2, the multiplicities must have $l_{q+1}+l_{q+2}+\cdots+l_{q+|n'|}=2$ and $l_{i>q+|n'|}=0$. The left-hand side of (22) gives a set of possible values,

$$\begin{aligned} &=2q+(2n-(2q+i+j)) \\ &=2n-(i+j)\ge 2n-2|n'|, \quad i,j\in\{1,2,\ldots,|n'|\}. \end{aligned} \tag{30}$$

But $2n-2|n'|>n+2$, hence no solution. The only remaining choice is to have the sum equals 1. Since $n+2=q\cdot 1+(2-n')$, only the partitions $Y=[N-(2-n')$ and partitions of$(2-n')]$ satisfy the constraint (22). However, these are unphysical since $n'<-2\Rightarrow N+n'-2>p$. For $n<|3n'|$, there are solutions which do not exhibit illuminating patterns, some of which are listed in Table I. Note that the three infinite families at the bottom of the table can be identified as type-II Argyres-Douglas theories with $b=2$ through the duality (11), up to some free hypermultiplets (as indicated by the $\sim$ sign).

In the above, we have identified a few infinite series of AD theories that can be realized as $SU(2)$ gauge theories coupled to two AD matters without exactly marginal deformations. Their central charges satisfy interesting relations, analogous to the relation $a_{4d}-c_{4d}=0$ of the $\mathcal{N}=4$ theories and $\mathcal{T}_{p,N}$ theories [32,33,53].

TABLE I. Cases with $k<0$ and $N\le 10$.

Even N	k	$[Y]$	
4	−2	$[1^4]$	
6	−2	$[3,2,1],[3,1^3]$	
8	−2	$[4,4],[5,3],[5,2,1],[5,1^3]$	
10	−4	$[5,4,1]$	
10	−6	$[3^3,1]$	
14	−6	$[7,6,1]$	
16	−6	$[9,7],[8,8]$	
$N\ge 10$	−2	$[N-3,1^3]$	$\cong D^{b=2}_{N-2}(\mathfrak{sl}_3,[1^3])$
		$[N-3,3]$	$\cong D^{b=2}_{N-2}(\mathfrak{sl}_3,[3])$
		$[N-3,2,1]$	$\sim D^{b=2}_{N-2}(\mathfrak{sl}_3,[2,1])$

Theories	$a_{4d}-c_{4d}$
$D_{N-4}(\mathfrak{sl}_N,[N-4,4])$	$-\frac{1}{12}+\frac{1}{4(N-4)}$
$D_{N-2}(\mathfrak{sl}_N,[N-3,3])$	$-\frac{1}{12}$
$D_{N-2}(\mathfrak{sl}_N,[N-3,2,1])$	$-\frac{1}{8}$
$D_{N-2}(\mathfrak{sl}_N,[N-3,1,1,1])$	$-\frac{5}{24}$

In the following discussions, we will compute the large-N limit of the Schur index and Wilson line index of the first two infinite families. It would be very interesting to explore the holography interpretation.

III. SCHUR INDEX AND NONVACUUM CHARACTERS

The Schur index can be defined for any 4D $\mathcal{N}=2$ SCFT, given by the trace formula

$$\mathcal{I}(q,b)=q^{\frac{c_{4d}}{2}}\mathrm{tr}_{\mathcal{H}}q^{E-R}b^{f}. \tag{31}$$

Here, E denotes the scaling dimension, R denotes the $SU(2)_{\mathcal{R}}$ charge, and b denotes the flavor fugacity associated to the flavor charge f. It is also the Schur limit $t\to q$ of the full $\mathcal{N}=2$ superconformal index $\mathcal{I}(p,q,t;b)$. The index can be viewed as an $S^3\times S^1$ partition function with suitable background $U(1)_r, SU(2)_{\mathcal{R}}$ gauge field [72,73]. In the 4D/2D correspondence [2], the Schur index is identified with the vacuum character of the associated VOA and also viewed as a T^2 partition function where the complex moduli of T^2 is denoted by τ, with $q=e^{2\pi i\tau}$ in the Schur index. In this section, we explicitly compute a few examples of the Schur index for the type-A Argyres-Douglas theories using the integration formula proposed in Ref. [23].

A. Index of $D_p(\mathfrak{sl}_N,[Y])$ with $\gcd(p,N)=1$

From the duality (15), the basic building blocks in the decomposition of type-I AD theories are those with $\gcd(p,N)=\gcd(k,N)=1$. These theories have been studied extensively [5,28–30]. When $[Y]=[1^N]$ is the maximal regular puncture, the associated VOA is given by the affine Kac-Moody algebra $\widehat{\mathfrak{su}}(N)_{k_{2d}}$, where the level

$$k_{2d}=-N+\frac{N}{p} \tag{32}$$

is boundary admissible. For general regular puncture $[Y]$, the associated VOA is given by the simple $\mathcal{W}$-algebra $W_{k_{2d}}(\mathfrak{su}(N),Y)$ obtained from the quantum Drinfeld-Sokolov reduction of $\widehat{\mathfrak{su}}(N)_{k_{2d}}$ specified by $[Y]$. The Schur index of $D_p(\mathfrak{sl}_N,[Y])$ or the vacuum character of the associated VOA is well known [30,74] and can be written in the form of plethystic exponential.

For a generic regular puncture labeled by $[Y]$, the corresponding flavor symmetry is given by the commutant G_F of the $SU(2)$ embedding specified by the partition $[Y]$. Explicitly, writing $[Y]=[1^{l_1},\ldots,N^{l_N}]$, the flavor symmetry is given by $G_F=(\prod_{i=1}^{N}U(l_i))/U(1)$. The adjoint representation of $SU(N)$ decomposes under $SU(2)\times G_F$ as

$$\mathfrak{g}=\sum_j V_j\otimes R_j, \tag{33}$$

where V_j denotes the spin-j representation of $SU(2)$ and R_j is the corresponding representation of G_F. Note that $j\in\mathbb{Z}$. The Schur index of $D_p(\mathfrak{sl}_N,[Y])$ then takes the form [30]

$$\mathrm{PE}\left[\frac{\sum_j q^{1+j}\chi_{R_j}(z)-q^{N+k}\sum_j q^{-j}\chi_{R_j}(z)}{(1-q)(1-q^{N+k})}\right]. \tag{34}$$

As an example, we consider the theory $D_p(\mathfrak{sl}_N,[1^N])$.

The associated flavor symmetry $G_F=SU(N)$, and the decomposition is trivial. The only nontrivial contribution corresponds to $j=0$ with $\chi_{R_j}=\chi_{\mathrm{adj}}$, leading to the Schur index

$$\mathrm{PE}\left[\frac{q-q^{N+k}}{(1-q)(1-q^{N+k})}\chi_{\mathrm{adj}}(z)\right]. \tag{35}$$

Example: $D_p(\mathfrak{sl}_N,[r^m,1^{N-rm}])$

The associated flavor symmetry $G_F=SU(N-rm)\times SU(m)\times U(1)$. The decomposition of the adjoint representation of $\mathfrak{sl}_N$ under this symmetry reads

$$\begin{aligned}\chi^{\mathfrak{sl}_N}_{\mathrm{adj}}=&\chi^{SU(N-rm)}_{\mathrm{adj}}+\sum_{j=0}^{r-1}\chi_{V_j}\left(\chi^{SU(m)}_{\mathrm{adj}}+1\right)\\&+\left(a\chi^{SU(m)}_{\square}\chi^{SU(N-rm)}_{\overline{\square}}+a^{-1}\chi^{SU(m)}_{\overline{\square}}\chi^{SU(N-rm)}_{\square}\right)\chi_{V_{\frac{r-1}{2}}},\end{aligned} \tag{36}$$

where a is the fugacity associated with the $U(1)$ symmetry and $\square,\bar{s}quare$ denote, respectively, the fundamental and antifundamental representations of the corresponding SU group. The Schur index takes the form

$$\begin{aligned}\mathrm{PE}\Bigg[\frac{1}{(1-q)(1-q^{N+k})}\Bigg((&q-q^{N+k})\chi^{SU(N-rm)}_{\mathrm{adj}}\\&+(\chi^{SU(m)}_{\mathrm{adj}}+1)\sum_{i=1}^{r}(q^i-q^{N+k-i+1})\\&+\left(q^{\frac{r+1}{2}}-q^{N+k-\frac{r-1}{2}}\right)\left(a\chi^{SU(m)}_{\square}\chi^{SU(N-rm)}_{\overline{\square}}\right.\\&\left.+a^{-1}\chi^{SU(m)}_{\overline{\square}}\chi^{SU(N-rm)}_{\square}\right)\Bigg)\Bigg].\end{aligned} \tag{37}$$

1. Vector and hypermultiplet contributions

In addition to the above building blocks, we must also include the contributions from vector multiplets and hypermultiplets.

The Schur index of a hypermultiplet in the representation R of the flavor symmetry includes contributions from a pair of half-hypermultiplets transforming in R and its conjugate $\bar{R}$,

$$\mathrm{PE}\left[\frac{q^{\frac{1}{2}}}{(1-q)}\left(a\chi_R(\mathbf{x})+a^{-1}\chi_{\bar{R}}(\mathbf{x})\right)\right], \tag{38}$$

where a denotes the $U(1)$ fugacity which rotates the scalars $q,\bar{q}$ oppositely and $\chi_R(\mathbf{x})$ is the character of R under the flavor symmetry. The Schur index of a vector multiplet is

$$\mathrm{PE}\left[-\frac{2q}{1-q}\chi_{\mathrm{adj}}(\mathbf{z})\right]. \tag{39}$$

2. Unphysical theories from the Schur index

There is a physical consistency requirement arising from the structure of the Schur index. Recall that the Schur index involves the plethystic exponential of

$$\frac{\sum_j q^{1+j}\chi_{R_j}(z)-q^{h^\vee+k}\sum_j q^{-j}\chi_{R_j}(z)}{(1-q)(1-q^{h^\vee+k})}. \tag{40}$$

The sum over spin j can be split into two parts,

$$\sum_j \frac{q^{1+j}-q^{p-j}}{1-q}\chi_{\mathcal{R}_j}(z)=\sum_{j\geq p}\frac{q^{1+j}-q^{p-j}}{1-q}\chi_{\mathcal{R}_j}(z)+\sum_{j<p}\frac{q^{1+j}-q^{p-j}}{1-q}\chi_{\mathcal{R}_j}(z). \tag{41}$$

The latter sum can be expanded as

$$=\sum_{j<p}(q^{1+j}+q^{2+j}+\cdots+q^{p-2j-2})\chi_{\mathcal{R}_j}(z), \tag{42}$$

while the former

$$=-\sum_{j\geq p}q^{p-j}(q^0+q^1+\cdots+q^{2j-p})\chi_{\mathcal{R}_j}(z). \tag{43}$$

Each integer spin $j\geq p$ contributes one $\frac{-1}{1-q^p}$ in the plethystic exponential, leading to a factor

$$\mathrm{PE}\left[\sum_{\substack{j\in\mathbb{N}\\ j\geq p}}^{n}\frac{-1}{1-q^p}\right]=(1;q^p)^{\#\text{integer spin}j\geq p}=0, \tag{44}$$

which is unphysical.

Now, consider a general regular puncture specified by $Y=[r^m,h^n,\ldots]$, with $r\geq h\geq\cdots$. The fundamental representation of $SU(N)$ then decomposes as

$$\chi_{\square}^{SU(N)}=\chi_{\frac{r-1}{2}}^{SU(2)}\chi_{\square}^{SU(m)}+\cdots. \tag{45}$$

Consequently, the adjoint representation necessarily contains a factor with maximal spin j:

$$\chi_{\frac{r-1}{2}}^{SU(2)}\chi_{\frac{r-1}{2}}^{SU(2)}=\chi_{r-1}^{SU(2)}+\cdots. \tag{46}$$

Therefore, the consistency requirement becomes

$$p>r-1. \tag{47}$$

B. $D_6(\mathfrak{sl}_4,[2,2])$

We consider the $D_6(\mathfrak{sl}_4,[2,2])$ theory as our first nontrivial example of a type-I AD theory with a duality frame as an $SU(2)$ gauge theory. In this and the following simple examples, we will compute analytically the flavored Schur index using the integration formula [23],

$$\oint\frac{da}{2\pi i a}\mathcal{Z}(\mathfrak{a})=\mathcal{Z}(\mathfrak{a}_0)+\sum_j R_j E_1\begin{bmatrix}-1\\ a_j/a_0q^{\pm\frac{1}{2}}\end{bmatrix}(\tilde{\tau}). \tag{48}$$

Here, we follow the convention $a=e^{2\pi i\mathfrak{a}}$, $q=e^{2\pi i\tau}$. The integrand $\mathcal{Z}(\mathfrak{a})$ is an elliptic function with double periodicity $\mathcal{Z}(\mathfrak{a})=\mathcal{Z}(\mathfrak{a}+1)=\mathcal{Z}(\mathfrak{a}+\tilde{\tau})$; note that $\tilde{\tau}$ is not necessarily the same τ that defines the Schur index. For the formula to work, $\mathcal{Z}(\mathfrak{a})$ has only simple poles $\mathfrak{a}_j$ within the fundamental parallelogram spanned by 1 and $\tilde{\tau}$. The residues R_j are given by

$$R_j=2\pi i\,\mathrm{Res}_{\mathfrak{a}\to\mathfrak{a}_j}\mathcal{Z}(\mathfrak{a})=\mathrm{Res}_{a\to a_j}\frac{1}{a}\mathcal{Z}(\mathfrak{a}_j). \tag{49}$$

Finally, poles $\mathfrak{a}_j=\text{real}+\lambda\tau$ with a positive λ is called an imaginary pole and a real pole otherwise. The factor $q^{\pm\frac{1}{2}}=q^{\frac{1}{2}}$ or $q^{-\frac{1}{2}}$ when $\mathfrak{a}_j$ is real or imaginary, respectively.

Once the flavored index is obtained, we take the unflavored limit to obtain the unflavored Schur index in closed form and find the unflavored modular differential equation that it satisfies. Although the residues R_j that appear in the computation individually do not have unflavoring limit, we will show in all examples that there exist nontrivial combinations that admit unflavoring limit, which give rise to additional solutions to the unflavored modular differential equation. This phenomenon suggest close relation between the residues and the module character of the associated VOA, which we leave for future investigation.

The gauge theory description of $D_6(\mathfrak{sl}_4,[2,2])$ is given by

$$D_3(\mathfrak{sl}_2,[1,1]) \longleftarrow \mathfrak{su}(2) \longrightarrow D_3(\mathfrak{sl}_4,[1,1,1,1]) \qquad \boxed{0} \tag{50}$$

Concretely, we read off the following data:

$$p=6,\quad N=4,\quad m=2,\quad k=3,\quad q=\frac{p}{m}=3,\quad Y=[2,2],\quad L=2 \tag{51}$$

$$\tilde{Y}=[1,1],\qquad \tilde{Y}_+=[1,1],\qquad \tilde{Y}=\varnothing, \tag{52}$$

$$N-\frac{N}{m}-\tilde{N}_-=2,\qquad qL-\frac{N}{m}=4,\qquad N-\frac{N}{m}=2. \tag{53}$$

The building blocks are $D_3(\mathfrak{sl}_2,[1^2])$ and $D_3(\mathfrak{sl}_4,[1^4])$, with the Schur index given by

$$\mathcal{I}_{D_3(\mathfrak{sl}_2,[1^2])}=\mathrm{PE}\left[\frac{q-q^{2+1}}{(1-q)(1-q^{2+1})}(a^2+a^{-2}+1)\right], \tag{54}$$

$$\mathcal{I}_{D_3(\mathfrak{sl}_4,[1^4])}=\mathrm{PE}\left[\frac{q-q^{4+(-1)}}{(1-q)(1-q^{4+(-1)})}\left(-1+\sum_{A,B=1}^{4}\frac{a_A}{a_B}\right)\right]_{a_1a_2a_3a_4=1}. \tag{55}$$

Upon gauging the $SU(2)$ subgroup, we identify the fugacities

$$a_1=ab_1,\qquad a_2=a^{-1}b_1,\qquad a_3=b_1b_2,\qquad a_4=b_1^{-3}b_2^{-1}. \tag{56}$$

The Schur index of $D_6(\mathfrak{sl}_4,[2,2])$ is given by the contour integral

$$\oint\frac{da}{2\pi ia}\mathcal{I}_{D_3(\mathfrak{sl}_2,[1^2])}(\mathfrak{a})\mathcal{I}_{D_3(\mathfrak{sl}_4,[1^4])}(\mathfrak{a},\mathfrak{b})\mathcal{I}_{\mathrm{VM}}(\mathfrak{a})=\oint\frac{da}{2\pi ia}\mathcal{Z}(\mathfrak{a}), \tag{57}$$

where the integrand can be rewritten in terms of ϑ_1 functions,

$$\mathcal{Z}(\mathfrak{a})=\frac{\eta(3\tau)^{15}}{q^{\frac{15}{4}}\eta(\tau)^3}\frac{\vartheta_1(2\mathfrak{a})^2\vartheta_1(4\mathfrak{b}_1+2\mathfrak{b}_2|3\tau)}{\prod_{\alpha=\pm}\vartheta_1(2\alpha\mathfrak{a}+\tau|3\tau)^2}\frac{1}{\prod_{\alpha,\beta=\pm}\vartheta_1(\alpha\tau+\beta\mathfrak{a}+\mathfrak{b}_2|3\tau)\vartheta_1(\alpha\tau+\beta\mathfrak{a}+4\mathfrak{b}_1+\mathfrak{b}_2|3\tau)}. \tag{58}$$

The integrand contains apparent double poles. However, using the identity

$$\prod_{\alpha=-1,0,1}\vartheta_1(\alpha\tau+\mathfrak{z}|3\tau)=\frac{q^{-1/3}\eta(3\tau)^3}{\eta(\tau)}\vartheta_1(\mathfrak{z}), \tag{59}$$

the integrand can be further simplified to

$$\mathcal{Z}(\mathfrak{a})=-\frac{1}{2}\frac{\eta(\tau)^3}{q^{7/4}\eta(3\tau)^3}\frac{\vartheta_1(4\mathfrak{b}_1+2\mathfrak{b}_2|3\tau)}{\vartheta_1(4\mathfrak{b}_1+2\mathfrak{b}_2)}\prod_{\alpha=\pm}\frac{\vartheta_1(2\alpha\mathfrak{a}|3\tau)\vartheta_1(\alpha\mathfrak{a}+\mathfrak{b}_2|3\tau)\vartheta_1(\alpha\mathfrak{a}+4\mathfrak{b}_1+\mathfrak{b}_2|3\tau)}{\vartheta_1(\alpha\mathfrak{a}+\mathfrak{b}_2)\vartheta_1(\alpha\mathfrak{a}+4\mathfrak{b}_1+\mathfrak{b}_2)}. \tag{60}$$

After the rewriting, the double poles are removed by canceling with zeroes. This integrand is elliptic with respect to $\mathfrak{a}$ with period 3τ,

$$\mathcal{Z}(\mathfrak{a}+3\tau)=\mathcal{Z}(\mathfrak{a}+1)=\mathcal{Z}(\mathfrak{a}). \tag{61}$$

To apply the integration formula, we identify the relevant poles,

$$\pm\mathfrak{b}_2+\ell\tau,\qquad \pm(4\mathfrak{b}_1+\mathfrak{b}_2)+\ell\tau,\qquad \ell=0,1,2. \tag{62}$$

Note that $\pm\mathfrak{b}_1$ and $\pm(4\mathfrak{b}_1+\mathfrak{b}_2)$ are canceled by a zero in the numerator, and hence there are only eight nonzero residues, which sum to zero since the integrand is elliptic. The integration formula leads to the closed form,

$$\begin{aligned}\mathcal{I}_{D_6(\mathfrak{sl}_4,[2,2])}(q,b)=&-\frac{\eta(3\tau)^6}{q^{23/12}\eta(\tau)^2\vartheta_1(4\mathfrak{b}_1|3\tau)\vartheta_1(4\mathfrak{b}_1+2\mathfrak{b}_2)}\\&\times\sum_{\pm}\mp\left(\frac{\vartheta_1(\pm\tau+2\mathfrak{b}_2|3\tau)}{\vartheta_1(\mp\tau+4\mathfrak{b}_1|3\tau)\vartheta_1(\pm\tau+4\mathfrak{b}_1+2\mathfrak{b}_2|3\tau)\vartheta_1(2\mathfrak{b}_2|3\tau)}E_1\begin{bmatrix}-1\\ b_2q^{\pm\frac{1}{2}}\end{bmatrix}(3\tau)\right.\\&\left.+\frac{\vartheta_1(\pm\tau+8\mathfrak{b}_1+2\mathfrak{b}_2|3\tau)}{\vartheta_1(\pm\tau+4\mathfrak{b}_1|3\tau)\vartheta_1(\pm\tau+4\mathfrak{b}_1+2\mathfrak{b}_2|3\tau)\vartheta_1(8\mathfrak{b}_1+2\mathfrak{b}_2|3\tau)}E_1\begin{bmatrix}-1\\ b_1^4b_2q^{\pm\frac{1}{2}}\end{bmatrix}(3\tau)\right).\end{aligned} \tag{63}$$

The expression is a sum of four terms, where the coefficients of the Eisenstein series are the residues from the poles. Let us denote these residues as R_i,

$$R_1=\Lambda\frac{\vartheta_1(2\mathfrak{b}_2+\tau|3\tau)}{\vartheta_1(2\mathfrak{b}_2|3\tau)\vartheta_1(4\mathfrak{b}_1-\tau|3\tau)\vartheta_1(4\mathfrak{b}_1+2\mathfrak{b}_2+\tau|3\tau)}, \tag{64}$$

$$R_2=-\Lambda\frac{\vartheta_1(2\mathfrak{b}_2-\tau|3\tau)}{\vartheta_1(2\mathfrak{b}_2|3\tau)\vartheta_1(4\mathfrak{b}_1+\tau|3\tau)\vartheta_1(4\mathfrak{b}_1+2\mathfrak{b}_2-\tau|3\tau)}, \tag{65}$$

$$R_3=\Lambda\frac{\vartheta_1(8\mathfrak{b}_1+2\mathfrak{b}_2+\tau|3\tau)}{\vartheta_1(8\mathfrak{b}_1+2\mathfrak{b}_2|3\tau)\vartheta_1(4\mathfrak{b}_1+\tau|3\tau)\vartheta_1(4\mathfrak{b}_1+2\mathfrak{b}_2+\tau|3\tau)}, \tag{66}$$

$$R_4=-\Lambda\frac{\vartheta_1(8\mathfrak{b}_1+2\mathfrak{b}_2-\tau|3\tau)}{\vartheta_1(8\mathfrak{b}_1+2\mathfrak{b}_2|3\tau)\vartheta_1(4\mathfrak{b}_1-\tau|3\tau)\vartheta_1(4\mathfrak{b}_1+2\mathfrak{b}_2-\tau|3\tau)}, \tag{67}$$

where

$$\Lambda=\frac{\eta(3\tau)^6}{q^{23/12}\eta(\tau)^2\vartheta_1(4\mathfrak{b}_1|3\tau)\vartheta_1(4\mathfrak{b}_1+2\mathfrak{b}_2)}. \tag{68}$$

As residues of an elliptic function with only simple poles, these four residues satisfy

$$R_1-R_2-R_3+R_4=0. \tag{69}$$

It is straightforward to take the unflavored limit given in a closed form,

$$\begin{aligned}\mathcal{I}_{D_6(\mathfrak{sl}_4,[2,2])}(q)=&-\frac{i\vartheta_1^{(3)}(0|3\tau)\vartheta_1^{(1)}(\tau|3\tau)^2}{48\pi^5\eta(3\tau)\vartheta_1(\tau|3\tau)^3}+\frac{i\vartheta_1^{(3)}(0|3\tau)\vartheta_1^{(2)}(\tau|3\tau)}{48\pi^5\eta(3\tau)\vartheta_1(\tau|3\tau)^2}+\frac{3i\vartheta_1^{(1)}(\tau|3\tau)^4}{16\pi^4\vartheta_1(\tau|3\tau)^5}\\&-\frac{i\vartheta_1^{(2)}(\tau|3\tau)\vartheta_1^{(1)}(\tau|3\tau)^2}{4\pi^4\vartheta_1(\tau|3\tau)^4}+\frac{3\vartheta_1^{(2)}(\tau|3\tau)\vartheta_1^{(1)}(\tau|3\tau)}{8\pi^3\vartheta_1(\tau|3\tau)^3}+\frac{i\vartheta_1^{(3)}(\tau|3\tau)\vartheta_1^{(1)}(\tau|3\tau)}{6\pi^4\vartheta_1(\tau|3\tau)^3}\\&-\frac{\vartheta_1^{(3)}(\tau|3\tau)}{8\pi^3(\vartheta_1(\tau|3\tau))^2}-\frac{3i\vartheta_1^{(2)}(\tau|3\tau)^2}{32\pi^4\vartheta_1(\tau|3\tau)^3}-\frac{\vartheta_1^{(1)}(\tau|3\tau)^3}{4\pi^3\vartheta_1(\tau|3\tau)^4}-\frac{i\vartheta_1^{(4)}(\tau|3\tau)}{96\pi^4\vartheta_1(\tau|3\tau)^2}.\end{aligned} \tag{70}$$

Expanded in q-series, the Schur index reads

$$\begin{aligned}\mathcal{I}_{D_6(\mathfrak{sl}_4,[2,2])}(q) = q^{\frac{7}{4}}(1 &+ 4q + 20q^2 + 76q^3 + 263q^4 + 816q^5 + 2381q^6 + 6482q^7 + 16841q^8 + 41776q^9 + 99863q^{10} \\ &+ 230608q^{11} + 517077q^{12} + 1128130q^{13} + 2402649q^{14} + 5004024q^{15} + 10213532q^{16} \\ &+ 20459252q^{17} + 40283830q^{18} + 78055088q^{19} + 149007235q^{20} + 280514896q^{21} + 521243939q^{22} \\ &+ 956743326q^{23} + 1735941489q^{24} + 3115550718q^{25} + \cdots).\end{aligned} \tag{71}$$

The unflavored index satisfies a 12th-order unflavored monic MLDE,

$$\begin{aligned}0 = \Big[&D_q^{(12)} - 1490E_4D_q^{(10)} - 68460E_6D_q^{(9)} - 644625E_4^2D_q^{(8)} - 8022000E_6E_4D_q^{(7)} + 56202500E_4^3D_q^{(6)} \\ &- 212170000E_6^2D_q^{(6)} - 954765000E_6E_4^2D_q^{(5)} - 9022740625E_4^4D_q^{(4)} - 17200960000E_6^2E_4D_q^{(4)} \\ &- 441015750000E_6E_4^3D_q^{(3)} - 174724200000E_6^3D_q^{(3)} - 1100917031250E_4^5D_q^{(2)} - 9954509250000E_6^2E_4^2D_q^{(2)} \\ &- 55495407187500E_6E_4^4D_q^{(1)} - 75831126000000E_6^3E_4D_q^{(1)} - 49750337109375E_4^6 - 587402077500000E_6^2E_4^3 \\ &- 137044278000000E_6^4\Big]\mathcal{I}_{D_6(\mathfrak{sl}_4,[2,2])}(q) = 0.\end{aligned} \tag{72}$$

Here, E_n denotes $E_n(\tau)$ for brevity. The equation has ten rational indicial roots and two irrational indicial roots,

$$\alpha = -\frac{1}{4}, \quad \left[\frac{1}{12}\right]_2, \quad \left[\frac{5}{12}\right]_2, \quad \left[\frac{3}{4}\right]_2, \quad \frac{13}{12}, \quad \frac{17}{12}, \frac{7}{4}, \frac{1}{12}(27 \pm 2\sqrt{5}), \tag{73}$$

where the subscripts denote the multiplicity. The presence of irrational indicial roots suggests the existence of another unflavored nonmonic MLDE, which is of 10th order but at modular weight 32,

$$\begin{aligned}0 = \Big[&(20E_4^3 + 147E_6^2)D_q^{(10)} - 3360E_6E_4^2D_q^{(9)} - (19500E_4^4 + 84525E_6^2E_4)D_q^{(8)} + (1932000E_4^3E_6 - 6379800E_6^3)D_q^{(7)} \\ &- (6195000E_4^5 - 12090750E_4^2E_6^2)D_q^{(6)} + (508620000E_4^4E_6 - 1289427000E_4E_6^3)D_q^{(5)} \\ &+ (75625000E_4^6 - 1073896250E_6^2E_4^3 + 172872000E_6^4)D_q^{(4)} - (6457500000E_6E_4^5 + 10109925000E_6^3E_4^2)D_q^{(3)} \\ &- (36360937500E_4^7 - 154931109375E_6^2E_4^4 + 817900650000E_6^4E_4)D_q^{(2)} \\ &+ 525000E_6(2267000E_4^6 - 15161335E_6^2E_4^3 + 2247336E_6^4)D_q^{(1)} \\ &- (1009687500000E_4^8 - 6180571289062.5E_6^2E_4^5 + 51391350000000E_6^4E_4^2)\Big]\mathcal{I}_{D_6(\mathfrak{sl}_4,[2,2])}.\end{aligned} \tag{74}$$

The indicial roots are precisely the previous 10 rational indicial roots. The largest root $\frac{7}{4}$ corresponds to the Schur index, while the minimal root $\alpha_{\min}$ is related to the 4D central charges,

$$6 = -24\alpha_{\min} = c_{\text{eff}} = 48(c_{4\text{d}} - a_{4\text{d}}), \qquad c_{4\text{d}} = \frac{11}{4}, \qquad a_{4\text{d}} = \frac{21}{8}. \tag{75}$$

The four residues R_i have precisely one linear combination that admits smooth limit $b_i \to 1$, which leads to an additional solution to both of the unflavored equations:

$$\begin{aligned}&\frac{1}{2}(R_1 + R_2 + R_3 + R_4)|_{\mathfrak{b}_i \to 0} \\ &\quad = -\frac{2}{\eta(\tau)^6}\left(E_1\begin{bmatrix}1\\q\end{bmatrix}(3\tau)^3 + 3E_1\begin{bmatrix}1\\q\end{bmatrix}(3\tau)E_2\begin{bmatrix}1\\q\end{bmatrix}(3\tau) + 3E_3\begin{bmatrix}1\\q\end{bmatrix}(3\tau)\right) \\ &\quad = q^{\frac{7}{4}}\left(\frac{1}{q} + 9 + 54q + 246q^2 + 954q^3 + 3267q^4 + 10215q^5 + 29637q^6 + \cdots\right).\end{aligned} \tag{76}$$

C. $D_6(\mathfrak{sl}_4,[3,1])$

According to the duality (15), the theory can be decomposed as

$$D_3(\mathfrak{sl}_4,[2,1,1]) \longleftarrow \mathfrak{su}(2) \longrightarrow D_3(\mathfrak{sl}_2,[1,1]) \qquad \mathfrak{su}(2)\ \text{with}\ \boxed{1} \tag{77}$$

The Schur index of $D_3(\mathfrak{sl}_2,[1,1])$ is given by

$$\mathrm{PE}\left[\frac{q-q^3}{(1-q)(1-q^3)}\chi_{\mathrm{adj}}(z)\right], \tag{78}$$

and the schur index of $D_3(\mathfrak{sl}_4,[2,1,1])$ takes the form

$$\mathrm{PE}\left[\frac{1}{(1-q)(1-q^3)}\left((q-q^3)\chi^{SU(2)}_{\mathrm{adj}}+\sum_{i=1}^{r}(q^i-q^{4-i})+\left(q^{\frac{3}{2}}-q^{\frac{5}{2}}\right)\left(a\chi^{SU(2)}_{\square}+a^{-1}\chi^{SU(2)}_{\square}\right)\right)\right]. \tag{79}$$

Collecting all building blocks, the Schur index of $D_6(\mathfrak{sl}_4,[3,1])$ is given by the contour integral

$$\mathcal{I}_{D_6(\mathfrak{sl}_4,[3,1])}=\oint\frac{da}{2\pi ia}\frac{\eta(\tau)\eta(3\tau)^3\vartheta_1(2\mathfrak{a}|3\tau)^2}{2q^{\frac{5}{6}}\prod_{\pm}\vartheta_4(\pm\mathfrak{a}+\mathfrak{b}_1|3\tau)\vartheta_4(\pm\mathfrak{a}+\mathfrak{b}_2)}=\oint\frac{da}{2\pi ia}\mathcal{Z}(\mathfrak{a}). \tag{80}$$

The integrand $\mathcal{Z}(\mathfrak{a})$ is elliptic with period 3τ, and hence there are eight imaginary poles within the fundamental parallelogram,

$$\pm\mathfrak{b}_1+\frac{3\tau}{2},\qquad \pm\mathfrak{b}_2+\frac{\tau}{2},\qquad \pm\mathfrak{b}_2+\frac{3\tau}{2},\qquad \pm\mathfrak{b}_2+\frac{5\tau}{2}. \tag{81}$$

The sum of residues of these poles vanish since the integrand is elliptic. Using the integration formula, the Schur index reads

$$\begin{aligned}\mathcal{I}_{D_6(\mathfrak{sl}_4,[3,1])}=&\frac{i\eta(\tau)\vartheta_1(2\mathfrak{b}_1|3\tau)}{2q^{\frac{5}{6}}\prod_{\pm}\vartheta_1(\mathfrak{b}_1\pm\mathfrak{b}_2)}\sum_{\alpha=\pm}\alpha E_1\begin{bmatrix}-1\\ b_1^\alpha\end{bmatrix}(3\tau)-\frac{i\eta(3\tau)^3\vartheta_1(2\mathfrak{b}_2|3\tau)^2}{2q^{\frac{5}{6}}\eta(\tau)^2\vartheta_1(2\mathfrak{b}_2)\prod_{\pm}\vartheta_1(\mathfrak{b}_1\pm\mathfrak{b}_2)}\sum_{\alpha=\pm}\alpha E_1\begin{bmatrix}-1\\ b_2^\alpha\end{bmatrix}(3\tau)\\ &-\frac{iq^{\frac{1}{6}}\eta(3\tau)^3}{2\vartheta_1(2\mathfrak{b}_2)\eta(\tau)^2}\sum_{\alpha=\pm}\frac{\alpha\vartheta_1(-2\tau+2\alpha\mathfrak{b}_2|3\tau)^2}{b_2^{2\alpha}\prod_{\beta=\pm}\vartheta_1(\beta\tau+\alpha\mathfrak{b}_1-\beta\alpha\mathfrak{b}_2|3\tau)}E_1\begin{bmatrix}-1\\ b_2^\alpha/q\end{bmatrix}(3\tau)\\ &-\frac{i\eta(3\tau)^3}{2q^{\frac{11}{6}}\vartheta_1(2\mathfrak{b}_2)\eta(\tau)^2}\sum_{\alpha=\pm}\frac{\alpha\vartheta_1(+\tau+2\alpha\mathfrak{b}_2|3\tau)^2}{\prod_{\beta=\pm}\vartheta_1(2\beta\tau+\alpha\mathfrak{b}_1+\alpha\beta\mathfrak{b}_2|3\tau)}E_1\begin{bmatrix}-1\\ b_2^\alpha/q\end{bmatrix}(3\tau).\end{aligned} \tag{82}$$

The six coefficients in front of the Eisenstein series are again residues of the simple poles of the integrand $\mathcal{Z}(\mathfrak{a})$. For later convenience, let us denote these coefficients by R_i,

$$R_1=\frac{i\eta(\tau)\vartheta_1(2\mathfrak{b}_1,q^3)}{2q^{5/6}\vartheta_1(\mathfrak{b}_1-\mathfrak{b}_2|\tau)\vartheta_1(\mathfrak{b}_1+\mathfrak{b}_2|\tau)}, \tag{83}$$

$$R_2=\frac{i\eta(3\tau)^3\vartheta_1(2\mathfrak{b}_2,q^3)^2}{2q^{5/6}\eta(\tau)^2\vartheta_1(2\mathfrak{b}_2|\tau)\vartheta_1(\mathfrak{b}_1-\mathfrak{b}_2|3\tau)\vartheta_1(\mathfrak{b}_1+\mathfrak{b}_2|3\tau)}, \tag{84}$$

$$R_3=-\frac{ib_2^2q^{\frac{1}{6}}\eta(3\tau)^3\vartheta_1(-2\mathfrak{b}_2-2\tau|3\tau)^2}{2\eta(\tau)^2\vartheta_1(2\mathfrak{b}_2|\tau)\vartheta_1(-\mathfrak{b}_1-\mathfrak{b}_2-\tau|3\tau)\vartheta_1(-\mathfrak{b}_1+\mathfrak{b}_2+\tau|3\tau)}, \tag{85}$$

$$R_4=\frac{ib_2^{-2}q^{\frac{1}{6}}\eta(3\tau)^3\vartheta_1(2\mathfrak{b}_2-2\tau|3\tau)^2}{2\eta(\tau)^2\vartheta_1(2\mathfrak{b}_2|\tau)\vartheta_1(\mathfrak{b}_1-\mathfrak{b}_2+\tau|3\tau)\vartheta_1(\mathfrak{b}_1+\mathfrak{b}_2-\tau|3\tau)}, \tag{86}$$

$$R_5 = -\frac{i\eta(3\tau)^3\vartheta_1(\tau-2\mathfrak{b}_2|3\tau)^2}{2q^{\frac{11}{6}}\eta(\tau)^2\vartheta_1(2\mathfrak{b}_2|\tau)\vartheta_1(-\mathfrak{b}_1-\mathfrak{b}_2+2\tau|3\tau)\vartheta_1(-\mathfrak{b}_1+\mathfrak{b}_2-2\tau|3\tau)}, \tag{87}$$

$$R_6 = \frac{i\eta(3\tau)^3\vartheta_1(2\mathfrak{b}_2+\tau|3\tau)^2}{2q^{\frac{11}{6}}\eta(\tau)^2\vartheta_1(2\mathfrak{b}_2|\tau)\vartheta_1(\mathfrak{b}_1-\mathfrak{b}_2-2\tau|3\tau)\vartheta_1(\mathfrak{b}_1+\mathfrak{b}_2+2\tau|3\tau)}. \tag{88}$$

By inspection, $R_3+R_5=R_4+R_6=0$, and hence only four residues R_1, R_2, R_3, R_4 are independent.

The unflavored limit of the Schur index is given by the closed form

$$\begin{aligned}\mathcal{I}_{D_6(\mathfrak{sl}_4,[3,1])}(q) = \frac{\eta(3\tau)^3}{\eta(\tau)^5}\Bigg(&+E_1\begin{bmatrix}-1\\ q\end{bmatrix}(3\tau)+3E_2(3\tau)+E_1\begin{bmatrix}-1\\ q\end{bmatrix}(3\tau)^2\\ &+6E_1\begin{bmatrix}-1\\ q\end{bmatrix}(3\tau)E_1\begin{bmatrix}1\\ q\end{bmatrix}(3\tau)+4E_2\begin{bmatrix}-1\\ 1\end{bmatrix}(3\tau)+2E_2\begin{bmatrix}-1\\ q\end{bmatrix}(3\tau)\Bigg).\end{aligned} \tag{89}$$

It satisfies a seventh-order unflavored monic MLDE,

$$\begin{aligned}0 = \Big[&D_q^{(7)}-200E_4D_q^{(5)}-7000E_6D_q^{(4)}-62000E_4D_q^{(3)}-924000E_4E_6D_q^{(2)}\\ &-30000(80E_4^3+147E_6^2)D_q^{(1)}-29400000E_4^2E_6\Big]\mathcal{I}_{D_6(\mathfrak{sl}_4,[3,1])}(q).\end{aligned} \tag{90}$$

The equation has seven rational indicial roots,

$$\alpha = -\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{5}{6},\frac{5}{6},\frac{7}{6}. \tag{91}$$

The $\alpha=\frac{7}{6}$ solution corresponds to the Schur index. The minimal indicial root $\alpha_{\min}$ is related to the 4D central charges

$$-24\alpha_{\min} = 4 = c_{\text{eff}} = 48(c_{4\text{d}}-a_{4\text{d}}), \qquad c_{4\text{d}}=\frac{5}{3}, \qquad a_{4\text{d}}=\frac{19}{12}. \tag{92}$$

The four independent residues R_i have one unique linear combination that admits smooth limit $b_i\to 1$, which leads to an additional solution to the unflavored equation:

$$\begin{aligned}-2(R_3+R_4)|_{b_i\to 1} &= \frac{2\eta(3\tau)^3}{\eta(\tau)^5}\left(1+3E_1\begin{bmatrix}+1\\ q\end{bmatrix}(3\tau)\right)\\ &= q^{\frac{1}{6}}+11q^{\frac{7}{6}}+50q^{\frac{13}{6}}+188q^{\frac{19}{6}}+583q^{\frac{25}{6}}+1646q^{\frac{31}{6}}+\cdots.\end{aligned} \tag{93}$$

D. $D_{10}(\mathfrak{sl}_4,[4])$

The theory $D_{10}(\mathfrak{sl}_4,[4])$ is also known as (A_3,A_5) in the notation of Ref. [46]. From the duality (15), the theory has the following duality frame [51]:

$$\begin{array}{ccccc} D_5(\mathfrak{sl}_3,[1^3]) & \longleftarrow & \mathfrak{su}(2) & \longrightarrow & D_5(\mathfrak{sl}_2,[1^2]).\\ & & \boxed{0} & & \end{array} \tag{94}$$

We can also read off the following data:

$$p=10, \qquad N=4, \qquad m=2, \qquad k=3, \qquad q=\frac{p}{m}=5, \qquad Y=[4], \qquad L=1 \tag{95}$$

$$\tilde{Y}=[1], \qquad \tilde{Y}_+=[1], \qquad \tilde{Y}=\varnothing, \tag{96}$$

$$N-\frac{N}{m}-\tilde{N}_-=2,\quad qL-\frac{N}{m}=3,\quad N-\frac{N}{m}=2. \tag{97}$$

The basic building blocks of the Schur index for this theory are given by

$$\mathcal{I}_{D_5(\mathfrak{sl}_3,[1,1,1])}=\mathrm{PE}\left[\frac{q-q^{3+3}}{(1-q)(1-q^{3+3})}\left(-1+\sum_{A,B=1}^{3}\frac{a_A}{a_B}\right)\right], \tag{98}$$

$$\mathcal{I}_{D_5(\mathfrak{sl}_2,[1,1])}=\mathrm{PE}\left[\frac{q-q^{2+3}}{(1-q)(1-q^{2+3})}(a^2+a^{-2}+1)\right]. \tag{99}$$

We gauge the diagonal of the flavor subgroup $SU(2)\subset SU(3)$ from $D_5(\mathfrak{sl}_3,[1,1,1])$ and the $SU(2)$ flavor symmetry of $D_5(\mathfrak{sl}_2,[1,1])$. The fugacities are identified by

$$a_1=ab,\qquad a_2=\frac{b}{a},\qquad a_3=\frac{1}{b^2}, \tag{100}$$

which leads to the Schur index

$$\mathcal{I}_{D_{10}(\mathfrak{sl}_4,[4])}=\oint\frac{da}{2\pi ia}\mathcal{I}_{D_5(\mathfrak{sl}_3,[1,1,1])}(\mathfrak{a},\mathfrak{b})\mathcal{I}_{D_5(\mathfrak{sl}_2,[1,1])}(\mathfrak{a})\mathcal{I}_{\mathrm{VM}}(\mathfrak{a})=\oint\frac{da}{2\pi ia}\mathcal{Z}(a), \tag{101}$$

where

$$\mathcal{Z}(\mathfrak{a})=\frac{a^{12}\eta(5\tau)^{19}\vartheta_1(2\mathfrak{a})^2}{2q^{\frac{169}{12}}\eta(\tau)^3}\frac{1}{\prod_\pm\prod_{\ell=1}^4\vartheta_1(\pm3\mathfrak{b}-\ell\tau+\mathfrak{a}|5\tau)}\frac{1}{\prod_{\ell=1}^4\vartheta_1(2\mathfrak{a}-\ell\tau|5\tau)^2}.$$

The apparent double poles can be removed by using the identity

$$\prod_{\ell=0}^{4}\vartheta_1(2\mathfrak{a}-\ell\tau|5\tau)=a^4q^{-3}\vartheta_1(2\mathfrak{a})\frac{\eta(5\tau)^5}{\eta(\tau)}, \tag{102}$$

which implies

$$\mathcal{Z}(\mathfrak{a})=\frac{\eta(\tau)}{2q^{\frac{25}{12}}\eta(5\tau)}\frac{\vartheta_1(2\mathfrak{a}|5\tau)^2\vartheta_1(\pm3\mathfrak{b}+\mathfrak{a}|5\tau)}{\vartheta_1(\pm3\mathfrak{b}+\mathfrak{a})}. \tag{103}$$

The integrand $\mathcal{Z}(\mathfrak{a})$ is elliptic with period 5τ, and hence the relevant poles are

$$\pm3\mathfrak{b}+\ell\tau,\qquad \ell=0,\ldots,4. \tag{104}$$

The corresponding residues are

$$0,\quad \frac{ib^6\vartheta_1(\tau|5\tau)\vartheta_1(6\mathfrak{b}+\tau|5\tau)\vartheta_1(6\mathfrak{b}+2\tau|5\tau)^2}{2q^{13/12}\eta(\tau)^2\eta(5\tau)\vartheta_1(6\mathfrak{b})}, \tag{105}$$

$$\frac{i\vartheta_1(2\tau|5\tau)\vartheta_1(6\mathfrak{b}-\tau|5\tau)^2\vartheta_1(6\mathfrak{b}+2\tau|5\tau)}{2q^{13/12}\eta(\tau)^2\eta(5\tau)\vartheta_1(6\mathfrak{b})}, \tag{106}$$

$$-\frac{i\vartheta_1(2\tau|5\tau)\vartheta_1(6\mathfrak{b}-2\tau|5\tau)\vartheta_1(6\mathfrak{b}+\tau|5\tau)^2}{2q^{13/12}\eta(\tau)^2\eta(5\tau)\vartheta_1(6\mathfrak{b})}, \tag{107}$$

$$-\frac{i\vartheta_1(\tau|5\tau)(\vartheta_1(6\mathfrak{b}-2\tau|5\tau))^2\vartheta_1(6\mathfrak{b}-\tau|5\tau)}{2b^6q^{13/12}\eta(\tau)^2\eta(5\tau)\vartheta_1(6\mathfrak{b})}, \tag{108}$$

$$0,\quad \frac{i\vartheta_1(\tau|5\tau)(\vartheta_1(6\mathfrak{b}-2\tau|5\tau))^2\vartheta_1(6\mathfrak{b}-\tau|5\tau)}{2b^6q^{13/12}\eta(\tau)^2\eta(5\tau)\vartheta_1(6\mathfrak{b})}, \tag{109}$$

$$\frac{i\vartheta_1(2\tau|5\tau)\vartheta_1(6\mathfrak{b}-2\tau|5\tau)\vartheta_1(6\mathfrak{b}+\tau|5\tau)^2}{2q^{13/12}\eta(\tau)^2\eta(5\tau)\vartheta_1(6\mathfrak{b})}, \tag{110}$$

$$-\frac{i\vartheta_1(2\tau|5\tau)\vartheta_1(6\mathfrak{b}-\tau|5\tau)^2\vartheta_1(6\mathfrak{b}+2\tau|5\tau)}{2q^{13/12}\eta(\tau)^2\eta(5\tau)\vartheta_1(6\mathfrak{b})}, \tag{111}$$

$$-\frac{ib^6\vartheta_1(\tau|5\tau)\vartheta_1(6\mathfrak{b}+\tau|5\tau)\vartheta_1(6\mathfrak{b}+2\tau|5\tau)^2}{2q^{13/12}\eta(\tau)^2\eta(5\tau)\vartheta_1(6\mathfrak{b})}. \tag{112}$$

The Schur index in closed form reads

$$\mathcal{I} = \frac{i}{q^{13/12}\eta(\tau)^2\eta(5\tau)\vartheta_1(6\mathfrak{b})}\Bigg(\tag{113}$$

$$\vartheta_1(2\tau|5\tau)\sum_{\alpha=\pm}\alpha\vartheta_1(-6\alpha\mathfrak{b}+2\tau|5\tau)\vartheta_1(6\alpha\mathfrak{b}|5\tau)^2 E_1\begin{bmatrix}-1\\ b^{3\alpha}q^{\frac{1}{2}}\end{bmatrix}(5\tau) \tag{114}$$

$$-\vartheta_1(\tau|5\tau)\sum_{\alpha=\pm}\alpha b^{6\alpha}\vartheta_1(6\alpha\mathfrak{b}+\tau|5\tau)\vartheta_1(6\alpha\mathfrak{b}+2\tau|5\tau)^2 E_1\begin{bmatrix}-1\\ b^{-3\alpha}q^{\frac{3}{2}}\end{bmatrix}(5\tau)\Bigg). \tag{115}$$

The coefficients of the Eisenstein series come from the residues of integrand $\mathcal{Z}(\mathfrak{a})$, where only three are actually linear independent. The unflavored limit gives

$$\begin{aligned}\mathcal{I}_{D_{10}(\mathfrak{sl}_4,[4])}(q) = -\frac{\eta(5\tau)}{\eta(\tau)^3}\Bigg(&1+E_1\begin{bmatrix}+1\\ q\end{bmatrix}(5\tau)+3E_1\begin{bmatrix}+1\\ q^2\end{bmatrix}(5\tau)-2E_2(5\tau)\\ &+E_1\begin{bmatrix}+1\\ q\end{bmatrix}(5\tau)^2+E_1\begin{bmatrix}+1\\ q^2\end{bmatrix}(5\tau)^2-2E_2\begin{bmatrix}+1\\ q\end{bmatrix}(5\tau)-2E_2\begin{bmatrix}1\\ q^2\end{bmatrix}(5\tau)\Bigg).\end{aligned} \tag{116}$$

The unflavored Schur index satisfies a 16th-order unflavored monic MLDE,

$$\begin{aligned}0 = \Bigg[&D_q^{(16)} - \frac{11171050208504}{2189278165}E_4D_q^{(14)} - \frac{464197483598576}{2189278165}E_6D_q^{(13)}\\
&+\frac{9860330680149964}{2189278165}E_4^2D_q^{(12)} + \frac{779235372995814112}{2189278165}E_6E_4D_q^{(11)}\\
&+\left(\frac{32456611659695917304}{54731954125}E_4^3 + \frac{200995901927582792800}{54731954125}E_6^2\right)D_q^{(10)} - \frac{1187733051446491930128}{54731954125}E_6E_4^2D_q^{(9)}\\
&-\left(\frac{148712814577023697558778}{273659770625}E_4^4 - \frac{155652147396466305175360}{273659770625}E_6^2E_4\right)D_q^{(8)}\\
&-\left(\frac{32836741812014425024826 24}{273659770625}E_6E_4^3 - \frac{57878392079849495808 0000}{273659770625}E_6^3\right)D_q^{(7)}\\
&+\left(\frac{43891294827249381536313 8136}{273659770625}E_4^5 - \frac{13270267216435353195205 1648}{273659770625}E_6^2E_4^2\right)D_q^{(6)}\\
&+\left(\frac{65290881415274231513044 2976}{273659770625}E_6E_4^4 - \frac{11668212890233264022464 37888}{273659770625}E_6^3E_4\right)D_q^{(5)}\\
&-\left(\frac{16835730318835206451559968884}{1368298853125}E_4^6 + \frac{16890224717433561754340326400}{1368298853125}E_6^2E_4^3\right.\\
&\left.+\frac{29935674500039554254749440000}{1368298853125}E_6^4\right)D_q^{(4)}\\
&-\left(\frac{44389198978248691680429 7504032}{1368298853125}E_6E_4^5 + \frac{85848527366234686773233 272576}{1368298853125}E_6^3E_4^2\right)D_q^{(3)}\\
&-\left(\frac{38442665063764147273325 9675640}{1368298853125}E_4^7 + \frac{13182902853285529377609 168626592}{1368298853125}E_6^2E_4^4\right.\\
&\left.+\frac{73321538761297845436720 65337600}{1368298853125}E_6^4E_4\right)D_q^{(2)}\end{aligned}$$

$$
\begin{aligned}
&- \Bigg(\frac{3673534225094072904022225764800}{1368298853125} E_6 E_4^6 + \frac{153792329386800366550880928720384}{1368298853125} E_6^3 E_4^3 \\
&\quad + \frac{260990084723734593774015846400000}{1368298853125} E_6^5 \Bigg) D_q^{(1)} \\
&- \Bigg(\frac{17740578315181492500104332125}{1368298853125} E_4^8 + \frac{4899941128665400768454044221440}{1368298853125} E_6^2 E_4^5 \\
&\quad + \frac{60815743075714497663999142935398 4}{1368298853125} E_6^4 E_4^2 \Bigg) \Bigg] \mathcal{I}_{D_{10}(\mathfrak{sl}_4,[4])}(q). \qquad (117)
\end{aligned}
$$

The equation has 12 rational indicial roots

$$
-\frac{7}{60}, \qquad \left[\frac{1}{12}\right]_2, \qquad \left[\frac{17}{60}\right]_2, \qquad \frac{29}{60}, \qquad \frac{41}{60}, \qquad \left[\frac{53}{60}\right]_2, \qquad \frac{13}{12}, \qquad \frac{77}{60}, \qquad \frac{25}{12}, \qquad (118)
$$

and four additional irrational ones. The largest indicial root $\alpha = \frac{25}{12}$ corresponds to the Schur index, while the minimal indicial root $\alpha_{\min}$ is related to the 4D central charges

$$
-24\alpha_{\min} = \frac{14}{5} = c_{\text{eff}} = 48(c_{4\text{d}} - a_{4\text{d}}), \qquad c_{4\text{d}} = \frac{25}{6}, \qquad a_{4\text{d}} = \frac{493}{120}. \qquad (119)
$$

Among the eight nonzero residues listed earlier, there are only three independent ones, from which we can construct two combinations $ch_{1,2}$ that admit the smooth $b \to 1$ limit,

$$
ch_1 = \frac{i\vartheta_1(6\mathfrak{b} + \tau|5\tau)}{2q^{13/12}\eta(\tau)^2\eta(5\tau)\vartheta_1(6\mathfrak{b}|\tau)} (b^6 \vartheta_1(\tau|5\tau)\vartheta_1(6\mathfrak{b} + 2\tau|5\tau)^2 + \vartheta_1(2\tau|5\tau)\vartheta_1(6\mathfrak{b} + \tau|5\tau)\vartheta_1(6\mathfrak{b} - 2\tau|5\tau)), \qquad (120)
$$

$$
ch_2 = \frac{i\vartheta_1(2\tau|5\tau)}{2q^{13/12}\eta(\tau)^2\eta(5\tau)\vartheta_1(6\mathfrak{b}|\tau)} (\vartheta_1(6\mathfrak{b} + 2\tau|5\tau)\vartheta_1(6\mathfrak{b} - \tau|5\tau)^2 + \vartheta_1(6\mathfrak{b} + \tau|5\tau)^2\vartheta_1(6\mathfrak{b} - 2\tau|5\tau)). \qquad (121)
$$

Sending $b \to 1$, the two expressions reduce to

$$
\begin{aligned}
ch_1 &= \frac{\eta(5\tau)}{\eta(\tau)^3} \left(1 - E_1 \begin{bmatrix} +1 \\ q \end{bmatrix} (5\tau) + 3E_1 \begin{bmatrix} +1 \\ q^2 \end{bmatrix} \right) \\
&= q^{\frac{25}{12}} \left(\frac{1}{q} + 1 + 7q + 13q^2 + 35q^3 + 71q^4 + 155q^5 + 293q^6 + \cdots \right), \qquad (122)
\end{aligned}
$$

$$
\begin{aligned}
ch_2 &= -\frac{2\eta(5\tau)}{\eta(\tau)^3} \left(2E_1 \begin{bmatrix} +1 \\ q \end{bmatrix} (5\tau) - E_1 \begin{bmatrix} 1 \\ q^2 \end{bmatrix} (5\tau) \right) \\
&= q^{\frac{25}{12}} \left(\frac{1}{q^2} + \frac{7}{q} + 23 + 70q + 173q^2 + 407q^3 + 882q^4 + 1830q^5 + 3610q^6 + \cdots \right). \qquad (123)
\end{aligned}
$$

Both of these expressions solves the same 16th-order unflavored MLDE as the unflavored Schur index, corresponding to the indicial roots $\frac{13}{12}, \frac{1}{12}$.

E. $D_8(\mathfrak{sl}_6, [5,1])$

For a last stand-alone example, we consider the $D_8(\mathfrak{sl}_6, [5,1])$ theory. The theory admits the gauge theory description

$$
\begin{array}{ccccc}
D_4(\mathfrak{sl}_5, [3,1,1]) & \longleftarrow & \mathfrak{su}(2) & \longrightarrow & D_4(\mathfrak{sl}_3, [1,1,1]) \\
 & & \boxed{0} & &
\end{array} \qquad (124)
$$

The theory has central charges

$$a_{4d} = \frac{49}{12}, \qquad c_{4d} = \frac{25}{6}, \tag{125}$$

and it belongs to the infinite series $D_{N+2}(\mathfrak{sl}(N), [N-1,1])$, which is also dual to $D_{N+2}(\mathfrak{sl}(N+4), [N+1,3])$. We will discuss this series later.

The $SU(2) \subset SU(3)$ flavor subgroup from $D_4(\mathfrak{sl}(3), [1,1,1])$ is gauged with the $SU(2)$ flavor symmetry from $D_4(\mathfrak{sl}_5), [3,1,1]$. The leftover $U(1)^2$ flavor fugacities are labeled by $b_{1,2}$. The full Schur index is given by the contour integral

$$\mathcal{I}_{D_8(\mathfrak{sl}_6,[5,1])} = -\oint \frac{da}{2\pi i a} \frac{\eta(4\tau)^2}{2q^{17/6}} \frac{\vartheta_1(2\mathfrak{a}|4\tau)^2 \prod_{\pm} \vartheta_1(\mathfrak{b}_1 \pm \mathfrak{b}_2|4\tau)}{\prod_{\pm} \vartheta_1(\mathfrak{a} \pm \mathfrak{b}_1|\tau)\vartheta_1(-2\tau + \mathfrak{a}_1 \pm \mathfrak{b}_2|4\tau)} = \oint \frac{da}{2\pi i a} \mathcal{Z}(\mathfrak{a}).$$

The integrand $\mathcal{Z}(\mathfrak{a})$ is periodic with respect to 4τ,

$$\mathcal{Z}(\mathfrak{a} + 4\tau) = \mathcal{Z}(\mathfrak{a} + 1) = \mathcal{Z}(\mathfrak{a}). \tag{126}$$

There are 10 simple imaginary and real poles $\pm\mathfrak{b}_2 + 2\tau$ and $\pm\mathfrak{b}_1 + \ell\tau|_{\ell=0,1,2,3}$. We then compute the index using the integration formula, which gives

$$\begin{aligned}
&= -\frac{\eta(4\tau)^6 \vartheta_1(\tau|4\tau)\vartheta_1(2\mathfrak{b}_1 - 2\tau|4\tau)}{2q^{\frac{19}{12}}\eta(\tau)^4 \vartheta_1(2\mathfrak{b}_1|4\tau)\vartheta_1(2\mathfrak{b}_1 - \tau|4\tau)\vartheta_1(\mathfrak{b}_1 - \mathfrak{b}_2 - \tau|4\tau)\vartheta_1(\mathfrak{b}_1 + \mathfrak{b}_2 - \tau|4\tau)} E_1\begin{bmatrix} -1 \\ b_1/q \end{bmatrix} \\
&\quad + \frac{i\eta(4\tau)^2 \vartheta_1(2\mathfrak{b}_1|4\tau)^2 \vartheta_1(2\tau|4\tau)\vartheta_1(2\mathfrak{b}_1 - 2\tau|4\tau)}{b_1 q^{\frac{5}{6}}\eta(\tau)^3 \vartheta_1(2\mathfrak{b}_1|\tau)\vartheta_1(\mathfrak{b}_1 - \mathfrak{b}_2|4\tau)\vartheta_1(\mathfrak{b}_1 + \mathfrak{b}_2|4\tau)} E_1\begin{bmatrix} -1 \\ b_1 \end{bmatrix}(4\tau) \\
&\quad + \frac{i\eta(4\tau)^2 \vartheta_1(\tau|4\tau)\vartheta_1(2\mathfrak{b}_1 - 2\tau|4\tau)^2 \vartheta_1(2\mathfrak{b}_1 - \tau|4\tau)}{b_1^3 q^{\frac{5}{6}}\eta(\tau)^3 \vartheta_1(2\mathfrak{b}_1|\tau)\vartheta_1(\mathfrak{b}_1 - \mathfrak{b}_2 + \tau|4\tau)\vartheta_1(\mathfrak{b}_1 + \mathfrak{b}_2 + \tau|4\tau)} E_1\begin{bmatrix} -1 \\ b_1 q \end{bmatrix}(4\tau) \\
&\quad - \frac{i\eta(4\tau)^2 \vartheta_1(\tau|4\tau)\vartheta_1(2\mathfrak{b}_1 - 2\tau|4\tau)^2 \vartheta_1(2\mathfrak{b}_1 + \tau|4\tau)}{2b_1 q^{\frac{5}{6}}\eta(\tau)^3 \vartheta_1(2\mathfrak{b}_1|\tau)\vartheta_1(\mathfrak{b}_1 - \mathfrak{b}_2 - \tau|4\tau)\vartheta_1(\mathfrak{b}_1 + \mathfrak{b}_2 - \tau|4\tau)} E_1\begin{bmatrix} -1 \\ b_1 q^{-1} \end{bmatrix}(4\tau) \\
&\quad + \frac{i\vartheta_1(2\mathfrak{b}_2|4\tau)\vartheta_1(\mathfrak{b}_1 - \mathfrak{b}_2 - 2\tau|4\tau)\vartheta_1(\mathfrak{b}_1 + \mathfrak{b}_2 - 2\tau|4\tau)}{b_1 q^{\frac{5}{6}}\eta(4\tau)\vartheta_1(\mathfrak{b}_1 - \mathfrak{b}_2|\tau)\vartheta_1(\mathfrak{b}_1 + \mathfrak{b}_2|\tau)} E_1\begin{bmatrix} -1 \\ b_2 \end{bmatrix}(4\tau).
\end{aligned} \tag{127}$$

The coefficients of the Eisenstein series are simply the residues of $\mathcal{Z}(\mathfrak{a})$.

The unflavoring limit is given by the simple formula

$$\begin{aligned}
\mathcal{I}_{D_8(\mathfrak{sl}_{10},[7,3])}(q) &= \frac{\eta(4\tau)^4}{\eta(\tau)^2\vartheta_1(\tau|4\tau)} \left(3E_2(4\tau) + 4E_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}(4\tau) + 2E_2\begin{bmatrix} -1 \\ q \end{bmatrix}(4\tau) \right. \\
&\quad \left. - E_1\begin{bmatrix} -1 \\ q \end{bmatrix}(4\tau)^2 + 2E_1\begin{bmatrix} -1 \\ q \end{bmatrix}(4\tau) E_1\begin{bmatrix} +1 \\ q \end{bmatrix}(4\tau) \right) \\
&= 1 + 2q + 6q^2 + 20q^3 + 50q^4 + 120q^5 + 280q^6 + 616q^7 + 1303q^8 + \cdots.
\end{aligned} \tag{128}$$

It is annihilated by a 14th-order monic modular differential operator,

$$
\begin{aligned}
&D_q^{(14)} - \frac{41356535E_4}{13471}D_q^{(12)} - \frac{1651331220E_6}{13471}D_q^{(11)} + \frac{11772659875E_4^2}{13471}D_q^{(10)} + \frac{1018752772625E_6E_4}{13471}D_q^{(9)} \\
&\quad + \frac{625(8603926907E_4^3 + 11155004154E_6^2)}{13471}D_q^{(8)} + \frac{100111961381250E_6E_4^2}{13471}D_q^{(7)} \\
&\quad - \frac{1250(1174872118745E_4^3 - 4752943356784E_6^2)E_4}{13471}D_q^{(6)} - \frac{13125E_6(125333104655E_4^3 - 2372060191536E_6^2)}{13471}D_q^{(5)} \\
&\quad + \frac{62500(1660587751085E_4^3 - 6515126570848E_6^2)E_4^2}{13471}D_q^{(4)} \\
&\quad - \frac{109375E_6(18868960484060E_4^3 + 63405268916423E_6^2)E_4}{13471}D_q^{(3)} \\
&\quad - \frac{15625(815283970908600E_4^6 + 5873692279036750E_6^2E_4^3 + 2077585779126753(E_6)^4)}{13471}D_q^{(2)} \\
&\quad - \frac{1640625E_6(283955673730300E_4^3 + 654661271404749E_6^2)E_4^2}{13471}D_q^{(1)} \\
&\quad - \frac{7109375\mathrm{cch}E_4(47764261202400E_4^6 + 726944797649700E_6^2E_4^3 + 416991195316829E_6^4)}{13471}. \qquad (129)
\end{aligned}
$$

The unflavored MLDE has 11 rational indicial roots,

$$
\alpha = -\frac{1}{6}, \qquad \left[\frac{1}{12}\right]_2, \qquad \left[\frac{1}{3}\right]_2, \qquad \frac{7}{12}, \qquad \left[\frac{5}{6}\right]_2, \qquad \left[\frac{13}{12}\right]_2, \qquad \frac{25}{12}, \qquad (130)
$$

where the subscripts denote the multiplicity of the roots, with an additional three irrational roots. The 11 rational roots implies the existence of an 11th-order unflavored MLDE at higher modular weight.

As we saw earlier, the integrand $\mathcal{Z}(\mathfrak{a})$ has several residues R_j, but only two of them are independent. They form one combination which admits smooth unflavoring limit, giving

$$
\begin{aligned}
ch_1 &= -\frac{\eta(4\tau)^4}{\sqrt{q}\eta(\tau)^2\vartheta_1(\tau|4\tau)^4}\left(1 + 4E_1\begin{bmatrix}1\\ q\end{bmatrix}(4\tau)\right) \\
&= q^{\frac{1}{12}} + 10q^{\frac{13}{12}} + 51q^{\frac{25}{12}} + 190q^{\frac{37}{12}} + 601q^{\frac{49}{12}} + 1702q^{\frac{61}{12}} + 4422q^{\frac{73}{12}} + \cdots. \qquad (131)
\end{aligned}
$$

Again, this is another solution to the unflavored MLDE corresponding to the second smallest indicial root $\frac{1}{12}$.

F. Exact formula for the $D_p^b(\mathfrak{sl}_N,[Y])$ theory

In this subsection, we briefly discuss the Schur index of the more general $D_p^b(\mathfrak{sl}_N,[Y])$ theory, where $[Y]$ is a generic regular puncture. Recall from (11) that a generic $D_p^b(\mathfrak{sl}_N,[Y])$ theory is dual to a type-I Argyres-Douglas theory together with a set of free hypermultiplets. For later convenience, we record here the duality relation explicitly:

$$
D_p^b(\mathfrak{sl}_N,[Y]) \cong D_p(\mathfrak{sl}_{(N-b)p+b},[(p-1)^{N-b},Y]) \otimes (N-b)\sum_{i=p}^{N}(i-p+1)\ell_i \text{ free hypermultiplets.} \qquad (132)
$$

The right-hand side includes free hypermultiplets originating from the breaking of flavor symmetry after choosing the general regular puncture $[Y]$. In the special case of the full puncture $[Y]=[1^N]$, the duality simplifies to

$$
D_p^b(\mathfrak{sl}_N) \cong D_p(\mathfrak{sl}_{(N-b)p+b},[(p-1)^{N-b},1^N]). \qquad (133)
$$

Here, the right-hand side is a type-I Argyres-Douglas theory $D_{p'}(\mathfrak{sl}(N'),[r^m,1^{N'-rm}])$, where the relevant data are given by

$$
b' = N' = (N-b)p + b, \qquad k' = p - b, \qquad r = (p-1), \qquad m = N - b. \qquad (134)
$$

We would like to compute the Schur index of $D_p^b(\mathfrak{sl}_N,[Y])$. A convenient approach exploits the TQFT nature of the Schur index [1,6,75,76]. From the TQFT perspective, the Schur index is expressed as a 2D q-deformed Yang-Mills correlator

$$\mathcal{I}_{D_p^b(\mathfrak{sl}_N,[Y])}(q,z)=\sum_\lambda \psi_\lambda^{p,b,N}(q)\psi_\lambda^{Y}(z), \tag{135}$$

where $\psi_\lambda^{p,b,N}(q)$ denotes the wave function of the irregular puncture and the wave function associated with a generic regular puncture is given by

$$\psi_\lambda^{Y}(\boldsymbol{a})=\mathrm{PE}\left[\sum_j \frac{q^{j+1}}{1-q}\mathrm{Tr}_{R_j}(\boldsymbol{a})\right]\chi_\lambda(\boldsymbol{a}q^{\Lambda_Y}). \tag{136}$$

Specializing to $[Y]=[1^N]$, using (133), the left-hand side of (135) is known by analyzing the type-I Schur index on the right, and one can solve the wave function $\psi_\lambda^{p,b,N}(q)$ of the irregular puncture. Then, we can combine $\psi_\lambda^{p,b,N}(q)$ with the known wave function of the chosen regular puncture to deduce the Schur index of $D_p^b(\mathfrak{sl}_N,[Y])$.

For simplicity, we focus on the case with $\gcd(p,b)=1$, although the idea certainly extends beyond this case. In this setting, the theory $D_p(\mathfrak{sl}_{(N-b)p+b},[(p-1)^{N-b},1^N])$ contains no mass parameters in the irregular puncture and is completely determined by the formula (37). The general Schur index of $D_p^b(\mathfrak{sl}_N)$ takes the form

$$\mathcal{I}_{D_p^b(\mathfrak{sl}_N)}(q,z)=\sum_\lambda \psi_\lambda^{\mathrm{irregular}}(q)\psi_\lambda^{\mathrm{full}}(z), \tag{137}$$

where

$$\psi_\lambda^{\mathrm{full}}(z)=\mathrm{PE}\left[\frac{q}{1-q}\chi_{\mathrm{adj}}(z)\right]\chi_\lambda(z). \tag{138}$$

Denote the Schur index of $D_p(\mathfrak{sl}_{(N-b)p+b},[(p-1)^{N-b},1^N])$ by $\mathcal{I}(q,z)$. Then, the irregular puncture contributes

$$\psi_\lambda^{p,b,N}(q)=\oint[dz]\,\mathrm{PE}\left[-\frac{q}{1-q}\chi_{\mathrm{adj}}(z)\right]\mathcal{I}(q,z)\chi_\lambda(z). \tag{139}$$

Substituting $\psi_\lambda^{p,b,N}(q)$ into (137), and using the identity

$$\sum_\lambda \chi_\lambda(z)\chi_\lambda(z')=\frac{1}{\Delta_J(z)}\delta(z-z'), \tag{140}$$

with respect to the Haar measure $\Delta_J(z)$, we obtain the wave function associated with a generic regular puncture specified by the Young diagram Y:

$$\psi_\lambda^{Y}(\boldsymbol{a})=\mathrm{PE}\left[\sum_j \frac{q^{j+1}}{1-q}\mathrm{Tr}_{R_j}(\boldsymbol{a})\right]\chi_\lambda(\boldsymbol{a}q^{\Lambda_Y}). \tag{141}$$

We find the $\mathcal{I}_{D_p^b(\mathfrak{sl}_N,[Y])}$ to be simply

$$\mathcal{I}_{D_p^b(\mathfrak{sl}_N,[Y])}=\mathrm{PE}\left[\sum_j \frac{q^{j+1}}{1-q}\mathrm{Tr}_{R_j}(\boldsymbol{a})\right]\,\mathrm{PE}\left[-\frac{q}{1-q}\chi_{\mathrm{adj}}(\boldsymbol{a}q^{\Lambda_Y})\right]\mathcal{I}(q,\boldsymbol{a}q^{\Lambda_Y}). \tag{142}$$

Recalling $\boldsymbol{a}q^{\Lambda_Y}$ solves

$$\chi_{\mathrm{adj}}(\boldsymbol{a}q^{\Lambda_Y})=\sum_j \chi_{\mathcal{R}_j}(\mathbf{a})\chi_{V_j}^{\mathfrak{su}(2)}, \tag{143}$$

and the Schur index simplifies to

$$\mathrm{PE}\left[\sum_j \frac{q^{j+1}}{1-q}\mathrm{Tr}_{R_j}(\boldsymbol{a})\right]\,\mathrm{PE}\left[-\sum_j \frac{q}{1-q}\mathrm{Tr}_{R_j}(\boldsymbol{a})\sum_{m=-j}^{j}q^m\right]\mathcal{I}(q,\boldsymbol{a}q^{\Lambda_Y})=\mathrm{PE}\left[-\sum_j \frac{q}{1-q}\mathrm{Tr}_{R_j}(\boldsymbol{a})\sum_{m=-j}^{j-1}q^m\right]\mathcal{I}(q,\boldsymbol{a}q^{\Lambda_Y}). \tag{144}$$

For an explicit example, let us consider the theory $D_3^5(\mathfrak{sl}_6,[3,3])$, with $k=-2$, $b=5$, $p=3$, $N=6$. To begin, we first study the theory $D_3^5(\mathfrak{sl}_6)$ with a full puncture. The duality relation implies

$$D_3^5(\mathfrak{sl}_6)\cong D_3(\mathfrak{sl}_8,[2,1^6]), \tag{145}$$

from which we deduce that the Schur index of $D_3^5(\mathfrak{sl}_6)$ is given by

$$\mathrm{PE}\left[\frac{(q^{3/2}-q^{5/2})(a^{-1}\chi_{\square}^{SU(6)}+a\chi_{\overline{\square}}^{SU(6)})+(q-q^3)\chi_{\mathrm{adj}}^{SU(6)}-q^3+q}{(1-q)(1-q^3)}\right]. \tag{146}$$

Now, we partially close the full puncture to $Y=[3,3]$, corresponding to the flavor symmetry $SU(2)$, by working out the representation R_j in the decomposition of adjoint of $\mathfrak{sl}_6$. From the character decomposition, we know the fugacity should take the form

$$\{aq,a,aq^{-1},bq,b,bq^{-1}\},\quad ab=1, \tag{147}$$

for the Young diagram $Y=[3,3]$. The character decomposes as

$$\chi_1(a)+\chi_1(q)(1+\chi_1^f(a))+\chi_2(q)(1+\chi_1^f(a)), \tag{148}$$

and we denote the character of the spin-j representation of the flavor symmetry $SU(2)$ by χ_i^f. We read off

$$\mathrm{tr}_{\mathcal{R}_{j=1,2}}(\boldsymbol{a})=(1+\chi_1^f(a)). \tag{149}$$

Hence, we can conclude that the prefactor is

$$\mathrm{PE}\left[-\frac{q}{1-q}(1+\chi_1(a))\sum_{j=1}^{2}\sum_{m=-j}^{j-1}q^m\right]. \tag{150}$$

Combining these two contributions, we obtain the Schur index of $D_3^5(\mathfrak{sl}_6,[3,3])$.

Now, we proceed to check the consistency of our formula (144). Recall that the general duality involves $(N-b)\sum_{i=p}^{N}(i-p+1)l_i$ free hypermultiplets. If we choose the regular puncture Y such that the only nonzero l_i are those with $i<p$, then the duality directly implies the equivalence

$$D_p^b(\mathfrak{sl}_N,[Y])\cong D_p(\mathfrak{sl}_{(N-b)p+b},[(p-1)^{N-b},Y]), \tag{151}$$

which allows us to test the consistency of the formula.

As an example, let us again start from (145), but now consider the regular puncture

$$[Y]=[2^3]. \tag{152}$$

In this case, the duality predicts

$$D_3^5(\mathfrak{sl}_6,[2^3])\cong D_3(\mathfrak{sl}_8,[2^4]). \tag{153}$$

On the right-hand side, the regular puncture $[2^4]$ induces the following decomposition under $\mathfrak{su}(2)\times\mathfrak{su}(4)$:

$$\chi_{\mathrm{adj}}^{SU(4)}+\chi_1\left(1+\chi_{\mathrm{adj}}^{SU(4)}\right). \tag{154}$$

Hence, the Schur index of $D_3(\mathfrak{sl}_8,[2^4])$ takes the form

$$\mathrm{PE}\left[\frac{q(1+q)\chi_{\mathrm{adj}}^{SU(4)}}{1-q^3}\right]. \tag{155}$$

On the left-hand side, our formula prescribes the substitution of variables as

$$aq^{1/2},\ aq^{-1/2},\ bq^{1/2},\ bq^{-1/2},\ cq^{1/2},\ cq^{-1/2},\ abc=1. \tag{156}$$

The decomposition under $\mathfrak{su}(2)\times\mathfrak{su}(3)$ is then

$$\chi_{\mathrm{adj}}^{SU(3)}+\chi_1(1+\chi_{\mathrm{adj}}^{SU(3)}). \tag{157}$$

The prefactor contributes

$$\mathrm{PE}\left[-\frac{q}{1-q}(1+\chi_{\mathrm{adj}}^{SU(3)})(1+q^{-1})\right]. \tag{158}$$

Collecting all contributions, the Schur index of $D_3^5(\mathfrak{sl}_6,[2^3])$ becomes

$$\mathrm{PE}\left[\frac{q(q+1)(abd^2(a^2b^2+a+b)+d(a^2b^2+a+b)(ab(a+b)+1)+ab(ab(a+b)+1))}{a^2b^2d(1-q^3)}\right]. \tag{159}$$

After a redefinition of the variables a, b, d, this expression reproduces the Schur index of $D_3(\mathfrak{sl}_8,[2^4])$.

IV. $D_{N-4}(\mathfrak{sl}(N),[N-4,4])$

With the simplest examples worked out in the previous section, we now turn to the series $D_{N-4}(\mathfrak{sl}(N),[N-4,4])$ where $N=4n+2=2(2n+1)\geq 18$. This series admit the following gauge theory description:

$$D_{\frac{N}{2}-2}(\mathfrak{sl}(\tfrac{N}{2}-4),[\tfrac{N}{2}-6,1,1]) \longleftarrow \underset{\boxed{0}}{\mathfrak{su}(2)} \longrightarrow D_{\frac{N}{2}-2}(\mathfrak{sl}(\tfrac{N}{2}),[\tfrac{N}{2}-2,1,1]) \tag{160}$$

As discussed in Refs. [40,50,51], this theory is dual to the $D_{N-4}(\mathfrak{sl}_4,[4])$, which includes the example $D_{10}(\mathfrak{sl}_4,[4])$ discussed above when we specialize $N=14$. In the notation of Ref. [46], the theory is is also known as (A_3,A_{N-9}). The theory receives $U(1)$ flavor symmmetry from the irregular puncture. The associated VOA is given by the logarithmic $\mathcal{B}(\frac{N-4}{4})_{\mathfrak{sl}_4}$ [77]. In the following, we stick to the notation $D_{N-4}(\mathfrak{sl}(N),[N-4,4])$.

When $N=4n+2$, the central charges of $D_{N-4}(\mathfrak{sl}(N),[N-4,4])$ satisfy

$$a_{4\mathrm{d}}-c_{4\mathrm{d}}=-\frac{1}{12}+\frac{1}{4(N-4)},\qquad N=4n+2\geq 18, \tag{161}$$

which is analogous to the well-known $a_{4\mathrm{d}}=c_{4\mathrm{d}}$ relation. As a side note, for $N=4n$, there is a simpler relation:

$$a_{4\mathrm{d}}-c_{4\mathrm{d}}=\frac{1}{8},\qquad N=4n\geq 20. \tag{162}$$

This $N=4n$ series admit an $SU(3)$ gauge description, whose Schur index can also be computed using our integration formula. However, for computational simplicity, we will only focus on the $N=4n+2$ series in this paper.

A. Flavored Schur index

The Schur index of the $D_{\frac{N}{2}-2}(\mathfrak{sl}(\frac{N}{2}-4),[\frac{N}{2}-6,1,1])$ and $D_{\frac{N}{2}-2}(\mathfrak{sl}(\frac{N}{2}),[\frac{N}{2}-2,1,1])$ theories can be computed using (37),

$$\begin{aligned}\mathcal{I}_{D_{\frac{N}{2}-2}(\mathfrak{sl}(\frac{N}{2}-4),[\frac{N}{2}-6,1,1])}=&\,\mathrm{PE}\left[\frac{1}{(1-q)(1-q^{\frac{N}{2}-2})}\left((q-q^{\frac{N}{2}-2})\chi^{SU(2)}_{\mathrm{adj}}(a)+\sum_{i=1}^{\frac{N}{2}-6}(q^i-q^{\frac{N}{2}-2-i+1})\right)\right]\\&\times\mathrm{PE}\left[\frac{q^{\frac{1}{2}(\frac{N}{2}-6+1)}-q^{\frac{N}{2}-2-\frac{1}{2}(\frac{N}{2}-6-1)}}{(1-q)(1-q^{\frac{N}{2}-2})}(b_1+b_1^{-1})(a+a^{-1})\right],\end{aligned} \tag{163}$$

and

$$\begin{aligned}\mathcal{I}_{D_{\frac{N}{2}-2}(\mathfrak{sl}(\frac{N}{2}),[\frac{N}{2}-2,1,1])}=&\,\mathrm{PE}\left[\frac{1}{(1-q)(1-q^{\frac{N}{2}-2})}\left((q-q^{\frac{N}{2}-2})\chi^{SU(2)}_{\mathrm{adj}}(a)+\sum_{i=1}^{\frac{N}{2}-2}(q^i-q^{\frac{N}{2}-2-i+1})\right)\right]\\&\times\mathrm{PE}\left[\frac{q^{\frac{1}{2}(\frac{N}{2}-2+1)}-q^{\frac{N}{2}-2-\frac{1}{2}(\frac{N}{2}-2-1)}}{(1-q)(1-q^{\frac{N}{2}-2})}(b_2+b_2^{-1})(a+a^{-1})\right]\\=&\,\mathrm{PE}\left[\frac{(q-q^{\frac{N}{2}-2})}{(1-q)(1-q^{\frac{N}{2}-2})}\chi^{SU(2)}_{\mathrm{adj}}(a)\right]=\mathcal{I}_{D_{\frac{N}{2}-2}(\mathfrak{sl}(2),[1,1])},\end{aligned} \tag{164}$$

where we have identified the two $SU(2)$ flavor fugacities from the two theories. The last equality manifests the duality $D_{N-4}(\mathfrak{sl}(N),[N-4,4])=D_{N-4}(\mathfrak{sl}_4,[4])$. The plethystic exponential can be simplified using

$$\frac{1}{(1-q)(1-q^{\frac{N}{2}-2})}\sum_{i=1}^{\frac{N}{2}-6}(q^i-q^{\frac{N}{2}-2-i+1})=\frac{(1+q+q^2+q^3)}{(1-q^{\frac{N}{2}-2})}\sum_{i=1}^{\frac{N}{2}-6}q^i. \tag{165}$$

Then, the Schur index of the full theory is given by the integral

$$\mathcal{I} = \oint \frac{da}{2\pi i a} \mathcal{I}_{\mathrm{VM}}(\mathfrak{a}) \mathcal{I}_{D_{\frac{N}{2}-2}(\mathfrak{sl}(\frac{N}{2}-4),[\frac{N}{2}-6,1,1])}(\mathfrak{a},\mathfrak{b}_1) \mathcal{I}_{D_{\frac{N}{2}-2}(\mathfrak{sl}(\frac{N}{2}),[\frac{N}{2}-2,1,1])}(\mathfrak{a}), \tag{166}$$

where the integrand reads

$$\mathcal{Z}(\mathfrak{a}) = \frac{q^{-\frac{7}{24}(N-4)}\eta((\frac{N}{2}-2)\tau)^8}{2} \prod_{i=1}^{\frac{N}{2}-6} \prod_{j=0}^{3} \frac{1}{(q^{i+j}, q^{\frac{N}{2}-2})} \frac{\vartheta_1(2\mathfrak{a} + (\frac{N}{2}-2)\tau | (\frac{N}{2}-2)\tau)^2}{\prod_{\alpha=\pm}\prod_{\ell=-1,1,3,5} \vartheta_1(\mathfrak{a} + \alpha\mathfrak{b}_1 + (\ell - \frac{N}{2})\frac{\tau}{2} | (\frac{N}{2}-2)\tau)}. \tag{167}$$

The relevant poles are simple and imaginary, located at

$$\alpha\mathfrak{b}_1 - \left(\ell - \frac{N}{2}\right)\frac{\tau}{2}, \qquad \alpha = \pm, \ell = -1, 1, 3, 5, \tag{168}$$

with residues

$$R_{\alpha,\ell} = \frac{q^{-\frac{7}{24}(N-4)}\eta((\frac{N}{2}-2)\tau)^8}{2} \prod_{i=1}^{\frac{N}{2}-6} \prod_{j=0}^{3} \frac{1}{(q^{i+j}, q^{\frac{N}{2}-2})} \frac{\vartheta_1((N-2-\ell)\tau + 2\alpha\mathfrak{b}_1 | (\frac{N}{2}-2)\tau)^2}{\prod_{\alpha'=\pm}\prod_{\ell'=-1,1,3,5} \vartheta_1((\alpha'+\alpha)\mathfrak{b}_1 + (\ell'-\ell)\frac{\tau}{2} | (\frac{N}{2}-2)\tau)}. \tag{169}$$

As the result of the residue computation, any $\vartheta_1(0|(\frac{N}{2}-2)\tau)$ in the denominator should be replaced by $\frac{1}{2\pi i}\vartheta_1'(0|(\frac{N}{2}-2)\tau) = -i\eta((\frac{N}{2}-2)\tau)^3$. The Schur index is then given by

$$\mathcal{I}_{D_{N-4}(\mathfrak{sl}(N),[N-4,4])} = \sum_{\alpha=\pm}\sum_{\ell=-1,1,3,5} R_{\alpha,\ell} E_1\left[\begin{matrix}-1\\ b_1^{\alpha} q^{\frac{1}{2}(\frac{N}{2}-\ell)} q^{\frac{1}{2}}\end{matrix}\right]\left(\left(\frac{N}{2}-2\right)\tau\right). \tag{170}$$

The integrand $\mathcal{Z}(\mathfrak{a})$ can also be rewritten, using the shift property of the Jacobi theta functions, as

$$\mathcal{Z}(\mathfrak{a}) = \frac{q^{-\frac{7}{24}(N-4)+\frac{1}{4}}}{2\prod_{i=1}^{\frac{N}{2}-6}\prod_{j=1}^{4}(q^{i+j}, q^{\frac{N}{2}-2})} \frac{\eta((\frac{N}{2}-2)\tau)^8 \vartheta_1(2\mathfrak{a}|(\frac{N}{2}-2)\tau)^2}{\prod_{\alpha=\pm}\prod_{\ell=1}^{4}\vartheta_4(\pm\mathfrak{a} + \mathfrak{m}_\ell | (\frac{N}{2}-2)\tau)}, \tag{171}$$

where the parameters $\mathfrak{m}_i$ are defined as

$$\mathfrak{m}_\ell = \mathfrak{b}_1 - \frac{3\tau}{2} + (\ell - 1)\tau, \qquad \ell = 1, \ldots, 4. \tag{172}$$

We notice that the second factor is nothing but the integrand that computes the $SU(2)$ theory with four flavors, whose flavor parameters $\mathfrak{m}_i$ are set to the above special values and the modular parameter τ is rescaled to $\tilde{\tau}$. Concretely,

$$\mathcal{I}_{D_{N-4}(\mathfrak{sl}(N),[N-4,4])}(q, b_1) = \frac{q^{-\frac{7}{24}(N-4)+\frac{1}{4}}}{2\prod_{i=1}^{\frac{N}{2}-6}\prod_{j=1}^{4}(q^{i+j}, q^{\frac{N}{2}-2})} \mathcal{I}_{SU(2)\mathrm{SQCD}}(q^{\frac{N}{2}-2}, m_\ell), \tag{173}$$

where

$$m_\ell = b_1 q^{-\frac{3}{2}} q^{(\ell-1)}. \tag{174}$$

B. Unflavored limit

It is straightforward to compute the unflavored limit. We use the following identity to simplify the expressions:

$$(-i)^n q^{\frac{1}{12}n(n+1)} \prod_{\ell=1}^{\lfloor\frac{2n+1}{2}\rfloor} \vartheta_1(\ell\tau|(2n+1)\tau) = \eta(\tau)\eta((2n+1)\tau)^{n-1}, \quad n = 1, 2, \ldots, \tag{175}$$

$$\prod_{i=1}^{\frac{N}{2}-6}\prod_{\ell=0}^{3}(q^{i+\ell}, q^{\frac{N}{2}-2}) = -\frac{\prod_{\ell=1}^{\lfloor\frac{N}{4}-1\rfloor}\vartheta_1(\ell\tau|(\frac{N}{2}-2)\tau)^{\min(4,\ell)}}{q^{-\frac{1}{4}(2N-29)}q^{-\frac{1}{24}(\frac{N}{2}-2)M}\eta((\frac{N}{2}-2)\tau)^M},$$
$$M \coloneqq \sum_{\ell=1}^{\lfloor\frac{N}{4}-1\rfloor}\min(\ell, 4). \tag{176}$$

To list a few explicit examples with low N, we have

$$\begin{aligned}\mathcal{I}_{D_{14}(\mathfrak{sl}_4,[4])}(q) = \frac{iq^{-\frac{13}{3}}\eta(7\tau)^2}{\eta(\tau)^3\vartheta_1(2\tau|7\tau)}\Bigg(&E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](7\tau) - 2E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](7\tau) + E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](7\tau)\\ &+ 2E_2(7\tau) + 2E_2\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](7\tau) + 2E_2\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](7\tau) + E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](7\tau)^2 - E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](7\tau)^2\\ &+ 4E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](7\tau)E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](7\tau) - 2E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](7\tau)E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](7\tau)\Bigg),\end{aligned} \tag{177}$$

$$\begin{aligned}\mathcal{I}_{D_{18}(\mathfrak{sl}_4,[4])}(q) = \frac{\vartheta_1(\tau|9\tau)\vartheta_1(3\tau|9\tau)}{q^{\frac{35}{6}}\eta(\tau)^4}\Bigg(&-E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](9\tau) + 2E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](9\tau) - E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](9\tau) - 2E_2(9\tau) - 2E_2\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](9\tau)\\ &- 2E_2\left[\begin{smallmatrix}1\\q^4\end{smallmatrix}\right](9\tau) - 3E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](9\tau)^2 - E_1\left[\begin{smallmatrix}1\\q^4\end{smallmatrix}\right](9\tau)^2 + 2E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](9\tau)E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](9\tau)\\ &- 4E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](9\tau)E_1\left[\begin{smallmatrix}1\\q^4\end{smallmatrix}\right](9\tau) + 2E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](9\tau)E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](9\tau) + 2E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](9\tau)E_1\left[\begin{smallmatrix}1\\q^4\end{smallmatrix}\right](9\tau)\Bigg),\end{aligned} \tag{178}$$

$$\begin{aligned}\mathcal{I}_{D_{22}(\mathfrak{sl}_4,[4])}(q) = \frac{\vartheta_1(\tau|11\tau)\vartheta_1(3\tau|11\tau)}{q^{\frac{25}{3}}\eta(\tau)^4}\Bigg(&-E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](11\tau) + 2E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](11\tau) - E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](11\tau) - 2E_2(11\tau) - 2E_2\left[\begin{smallmatrix}1\\q^4\end{smallmatrix}\right](11\tau)\\ &- 2E_2\left[\begin{smallmatrix}1\\q^5\end{smallmatrix}\right](11\tau) - E_1\left[\begin{smallmatrix}1\\q^4\end{smallmatrix}\right](11\tau)^2 - E_1\left[\begin{smallmatrix}1\\q^5\end{smallmatrix}\right](11\tau)^2 + 2E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](11\tau)E_1\left[\begin{smallmatrix}1\\q^4\end{smallmatrix}\right](11\tau)\\ &- 4E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](11\tau)E_1\left[\begin{smallmatrix}1\\q^5\end{smallmatrix}\right](11\tau) + 2E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](11\tau)E_1\left[\begin{smallmatrix}1\\q^4\end{smallmatrix}\right](11\tau) + 2E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](11\tau)E_1\left[\begin{smallmatrix}1\\q^5\end{smallmatrix}\right](11\tau)\\ &- 2E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](11\tau)E_1\left[\begin{smallmatrix}1\\q^4\end{smallmatrix}\right](11\tau)\Bigg).\end{aligned} \tag{179}$$

$$\begin{aligned}\mathcal{I}_{D_{26}(\mathfrak{sl}_4,[4])}(q) = \frac{\vartheta_1(\tau|13\tau)\vartheta_1(3\tau|13\tau)}{q^{\frac{65}{6}}\eta(\tau)^4}\Bigg(&-E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](13\tau) + 2E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](13\tau) - E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](13\tau)\\ &- 2E_2(13\tau) - 2E_2\left[\begin{smallmatrix}1\\q^5\end{smallmatrix}\right](13\tau) - 2E_2\left[\begin{smallmatrix}1\\q^6\end{smallmatrix}\right](13\tau) - E_1\left[\begin{smallmatrix}1\\q^5\end{smallmatrix}\right](13\tau)^2 - E_1\left[\begin{smallmatrix}1\\q^6\end{smallmatrix}\right](13\tau)^2\\ &+ 2E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](13\tau)E_1\left[\begin{smallmatrix}1\\q^5\end{smallmatrix}\right](13\tau) - 4E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](13\tau)E_1\left[\begin{smallmatrix}1\\q^6\end{smallmatrix}\right](13\tau) + 2E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](13\tau)E_1\left[\begin{smallmatrix}1\\q^5\end{smallmatrix}\right](13\tau)\\ &+ 2E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](13\tau)E_1\left[\begin{smallmatrix}1\\q^6\end{smallmatrix}\right](13\tau) - 2E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](13\tau)E_1\left[\begin{smallmatrix}1\\q^5\end{smallmatrix}\right](13\tau)\Bigg),\end{aligned} \tag{180}$$

$$\mathcal{I}_{D_{30}(\mathfrak{sl}_4,[4])}(q)=\frac{\vartheta_1(\tau|15\tau)\vartheta_1(3\tau|15\tau)}{q^{40/3}\eta(\tau)^4}\left(-E_1\left[\begin{matrix}1\\q\end{matrix}\right](15\tau)+2E_1\left[\begin{matrix}1\\q^2\end{matrix}\right](15\tau)-E_1\left[\begin{matrix}1\\q^3\end{matrix}\right](15\tau)-2E_2(15\tau)-2E_2\left[\begin{matrix}1\\q^6\end{matrix}\right](15\tau)\right.$$
$$-2E_2\left[\begin{matrix}1\\q^7\end{matrix}\right](15\tau)-E_1\left[\begin{matrix}1\\q^6\end{matrix}\right](15\tau)^2-E_1\left[\begin{matrix}1\\q^7\end{matrix}\right](15\tau)^2+2E_1\left[\begin{matrix}1\\q\end{matrix}\right](15\tau)E_1\left[\begin{matrix}1\\q^6\end{matrix}\right](15\tau)$$
$$-4E_1\left[\begin{matrix}1\\q\end{matrix}\right](15\tau)E_1\left[\begin{matrix}1\\q^7\end{matrix}\right](15\tau)+2E_1\left[\begin{matrix}1\\q^2\end{matrix}\right](15\tau)E_1\left[\begin{matrix}1\\q^6\end{matrix}\right](15\tau)+2E_1\left[\begin{matrix}1\\q^2\end{matrix}\right](15\tau)E_1\left[\begin{matrix}1\\q^7\end{matrix}\right](15\tau)$$
$$\left.-2E_1\left[\begin{matrix}1\\q^3\end{matrix}\right](15\tau)E_1\left[\begin{matrix}1\\q^6\end{matrix}\right](15\tau)\right). \tag{181}$$

From the above results, the closed form unflavored Schur index can be easily read off, with $\tilde{\tau}\coloneqq\frac{N-4}{2}\tau$,

$$\mathcal{I}_{D_{N-4}(\mathfrak{sl}(N),[N-4,4])}(q)=\mathcal{I}_{D_{N-4}(\mathfrak{sl}_4,[4])}(q)$$
$$=\frac{\vartheta_1(\tau|\frac{N-4}{2}\tau)\vartheta_1(3\tau|\frac{N-4}{2}\tau)}{q^{\frac{10N-190}{24}}\eta(\tau)^4}\left(-E_1\left[\begin{matrix}1\\q\end{matrix}\right](\tilde{\tau})+2E_1\left[\begin{matrix}1\\q^2\end{matrix}\right](\tilde{\tau})-E_1\left[\begin{matrix}1\\q^3\end{matrix}\right](\tilde{\tau})\right.$$
$$-2E_2(\tilde{\tau})-2E_2\left[\begin{matrix}1\\q^{\frac{N-10}{4}}\end{matrix}\right](\tilde{\tau})-2E_2\left[\begin{matrix}1\\q^{\frac{N-6}{4}}\end{matrix}\right](\tilde{\tau})-E_1\left[\begin{matrix}1\\q^{\frac{N-10}{4}}\end{matrix}\right](\tilde{\tau})^2-E_1\left[\begin{matrix}1\\q^{\frac{N-6}{4}}\end{matrix}\right](\tilde{\tau})^2$$
$$+2E_1\left[\begin{matrix}1\\q\end{matrix}\right](\tilde{\tau})E_1\left[\begin{matrix}1\\q^{\frac{N-10}{4}}\end{matrix}\right](\tilde{\tau})-4E_1\left[\begin{matrix}1\\q\end{matrix}\right](\tilde{\tau})E_1\left[\begin{matrix}1\\q^{\frac{N-6}{4}}\end{matrix}\right](\tilde{\tau})+2E_1\left[\begin{matrix}1\\q^2\end{matrix}\right](\tilde{\tau})E_1\left[\begin{matrix}1\\q^{\frac{N-10}{4}}\end{matrix}\right](\tilde{\tau})$$
$$\left.+2E_1\left[\begin{matrix}1\\q^2\end{matrix}\right](\tilde{\tau})E_1\left[\begin{matrix}1\\q^{\frac{N-6}{4}}\end{matrix}\right](\tilde{\tau})-2E_1\left[\begin{matrix}1\\q^3\end{matrix}\right](\tilde{\tau})E_1\left[\begin{matrix}1\\q^{\frac{N-10}{4}}\end{matrix}\right](\tilde{\tau})\right). \tag{182}$$

C. Large-N limit

Let us series expand the Schur index for $N=18, 22, 26, 30, 34$, where we drop the $q^{-\frac{c}{24}}$ factor,

$$\mathcal{I}_{D_{14}(\mathfrak{sl}(18),[14,4])}(b)=1+q+3q^2+6q^3+13q^4+\left(b^2+\frac{1}{b^2}+23\right)q^5$$
$$+\left(2b^2+\frac{2}{b^2}+45\right)q^6+\left(78+6b^2+\frac{6}{b^2}\right)q^7+\left(141+12b^2+\frac{12}{b^2}\right)q^8$$
$$+\left(239+27b^2+\frac{27}{b^2}\right)q^9+\left(409+b^4+52b^2+\frac{52}{b^2}+\frac{1}{b^4}\right)q^{10}+\cdots, \tag{183}$$

$$\mathcal{I}_{D_{18}(\mathfrak{sl}(22),[18,4])}(b)=1+q+3q^2+6q^3+13q^4+23q^5+45q^6+\left(78+b^2+\frac{1}{b^2}\right)q^7+\left(141+2b^2+\frac{2}{b^2}\right)q^8$$
$$+(239+6/b^2+6b^2)q^9+\left(409+12b^2+\frac{12}{b^2}\right)q^{10}+\left(27b^2+\frac{27}{b^2}+674\right)q^{11}+\cdots,$$
$$\mathcal{I}_{D_{22}(\mathfrak{sl}(26),[22,4])}(b)=1+q+3q^2+6q^3+13q^4+23q^5+45q^6+78q^7+141q^8+(239+1/b^2+b^2)q^9$$
$$+\left(409+\frac{2}{b^2}+2b^2\right)q^{10}+\left(674+\frac{6}{b^2}+6b^2\right)q^{11}+\left(12b_1^2+\frac{12}{b_1^2}+1116\right)q^{12}+\cdots. \tag{184}$$

From these results, we observe a clear large-N limit,

$$\lim_{N\to\infty}\mathcal{I}_{D_{N-4}(\mathfrak{sl}(N),[N-4,4])}(b)=1+q+3q^2+6q^3+13q^4+23q^5+45q^6+78q^7$$
$$+141q^8+239q^9+409q^{10}+674q^{11}+1116q^{12}+\cdots, \tag{185}$$

where the explicit q-series arises from the following analytic function,

$$\frac{(1-q)^3(1-q^2)^2(1-q^3)}{(q;q)^4}. \tag{186}$$

To show this large-N behavior, first note that the large-N limit is independent of the flavor fugacity b; whenever a term in the q-series develops independence of b, it stabilizes and assumes the fixed value in the large-N limit. Since b drops out, this limit can be seen by analyzing the closed-form unflavored Schur index (182). First of all, the denominator in (186) corresponds to the $\eta(\tau)^{-4}$ in (182). As $N \to +\infty$, the numerator in (182) contributes only

$$\begin{aligned}\vartheta_1(\tau|\tilde{\tau})\vartheta_1(3\tau|\tilde{\tau}) &\sim q^{2\frac{N-4}{2\times 8}} q^{1/2} q^{3/2}(1-q^{\frac{N-4}{2}})^2(1-qq^{\frac{N-4}{2}})(1-q^{-1})(1-q^3 q^{\frac{N-4}{2}})(1-q^{-3}) \\ &\sim q^{\frac{N}{8}-\frac{5}{2}} - q^{\frac{N}{8}-\frac{3}{2}} - q^{\frac{N}{8}+\frac{1}{2}} + q^{\frac{N}{8}+\frac{3}{2}} + O\left(q^{\frac{5N}{8}}\right).\end{aligned} \tag{187}$$

The combination of Eisenstein is trickier, but by inspection, it approximates to

$$\sim q^{\frac{N-4}{3}-3}(-1+2q+q^2-4q^3+q^4+2q^5-q^6)+O(q^{N-10}). \tag{188}$$

Putting all pieces together including a suitable overall q factors, we see that

$$\mathcal{I}_{D_{N-4}(\mathfrak{sl}(N),[N-4,4])}(q) \overset{N\to+\infty}{\longrightarrow} \frac{(1-q)^3(1-q^2)^2(1-q^3)}{(q;q)^4}. \tag{189}$$

Performing the plethystic logarithm, we obtain the single letter index

$$i_{D_{\infty-4}(\mathfrak{sl}(\infty),[\infty-4,4])}(q) = \frac{q+q^2+q^3+q^4}{1-q}. \tag{190}$$

At closer inspection, we observe that at large finite N

$$\begin{aligned}(q,q)^4\mathcal{I}_{D_{N-4}(\mathfrak{sl}(N),[N-4,4])}(q) &= (1-q)^3(1-q^2)^2(1-q^3) + 2q^{\frac{N}{2}-4}[(1-q)(1-q^2)(1-q^3)]^2(1+q) \\ &\quad + q^{2(\frac{N}{2}-4)}(1-q)^2(1-q^2)^2(1-q^3)(1-q^5)(2-5q+3q^2-5q^3+2q^4) \\ &\quad + q^{3(\frac{N}{2}-4)}(1-q)^3(\cdots).\end{aligned} \tag{191}$$

In terms of ratio,

$$\begin{aligned}\frac{\mathcal{I}_{D_{N-4}(\mathfrak{sl}(N),[N-4,4])}}{\mathcal{I}_{D_{\infty-4}(\mathfrak{sl}(\infty),[\infty-4,4])}} &= 1 + 2q^{\frac{N}{2}-4}(1+q^2)(1+q+q^2) + q^{2(\frac{N}{2}-4)}\left(\sum_{i=1}^{4} q^i\right)(2-5q+3q^2-5q^3+2q^4) \\ &\quad + q^{3(\frac{N}{2}-4)}2\left(\sum_{i=1}^{4} q^i\right)(1-2q+q^2-3q^3+2q^4-3q^5+q^6-2q^7+q^8)+\cdots.\end{aligned} \tag{192}$$

This structure is very much similar to the giant graviton expansion of $\mathcal{N}=4$ $U(N)$ theory discussed in Refs. [78,79], and it would be interesting to explore the holography interpretation of the large-N expansion.

D. Wilson line index

Using the $SU(2)$ gauge theory description of $D_{N-4}(\mathfrak{sl}(N),[N-4,4])$, we include a spin-$j$ Wilson line coupled to the $SU(2)$ gauge group. The Wilson line index is given by the integral

$$\mathcal{I}_j = \oint \frac{da}{2\pi i a}\chi_j(a)\mathcal{Z}(\mathfrak{a}), \tag{193}$$

where $\chi_j(a)=\sum_{m=-j}^{+j}a^{2m}$ is the $SU(2)$ character. When $j\in\mathbb{Z}$, the $m=0$ term integrates to the original index $\mathcal{I}_{D_{N-4}(\mathfrak{sl}(N),[N-4,4])}$.

The flavored Wilson line index can be easily computed using the integration formula,

$$\mathcal{I}_j=\delta_{j\in\mathbb{Z}}\mathcal{I}_{D_{N-4}(\mathfrak{sl}(N),[N-4,4])}-\sum_{\substack{m=-j\\m\neq0}}^{+j}\sum_{\alpha=\pm}\sum_{\ell}R_{\alpha,\ell}\frac{(b^\alpha q^{-\frac{1}{2}(\ell-\frac{\tau}{2})})^{2m}q^{-\frac{2m}{2}(\frac{N}{2}-2)}}{q^{\frac{2m}{2}(\frac{N}{2}-2)}-q^{-\frac{2m}{2}(\frac{N}{2}-2)}},\tag{194}$$

where the residues $R_{\alpha,\ell}$ are given in (169).

It is straightforward to compute the unflavored limit. For example, when $N=14$,

$$\mathcal{I}_j=\delta_{j\in\mathbb{Z}}\mathcal{I}_{D_{10}(\mathfrak{sl}(14),[10,4])}(q)-\frac{\eta(\tilde\tau)^2}{\vartheta_1(\tau|\tilde\tau)^2\vartheta_1(2\tau|\tilde\tau)^2}\frac{\Lambda_{14}}{q^{m(\frac{N}{2}-2)}-q^{-m(\frac{N}{2}-2)}}\sum_{\substack{m=-j\\m\neq0}}^{+j}\left(\frac{2m(-1-q^{2m}q^{4m}-q^{6m})}{2q^{\frac{18m+16}{6}}}\right.$$
$$\left.+\frac{+2(q^{6m}-1)+2(-1+2q^{2m}-2q^{4m}+q^{6m})E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](\tilde\tau)}{2q^{\frac{18m+16}{6}}}\frac{+2(-2-q^{2m}+q^{4m}+2q^{6m})E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](\tilde\tau)}{2q^{\frac{18m+16}{6}}}\right).\tag{195}$$

For $N=18$,

$$\mathcal{I}_j=\delta_{j\in\mathbb{Z}}\mathcal{I}_{D_{10}(\mathfrak{sl}(14),[10,4])}(q)+\frac{\eta(\tilde\tau)^2}{\vartheta_1(\tau|\tilde\tau)^2\vartheta_1(2\tau|\tilde\tau)^2}\frac{\Lambda_{18}}{q^{m(\frac{N}{2}-2)}-q^{-m(\frac{N}{2}-2)}}\sum_{\substack{m=-j\\m\neq0}}^{+j}\left(\frac{-2m(1+q^{2m}+q^{4m}+q^{6m})}{2q^{\frac{18m+23}{6}}}\right.$$
$$\left.+\frac{2(-1+2q^{2m}-2q^{4m}+q^{6m})E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](\tilde\tau)}{2q^{\frac{18m+23}{6}}}+\frac{2(-1+q^{2m})(1+q^{2m})^2E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](\tilde\tau)+2(1-q^{6m})E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](\tilde\tau)}{2q^{\frac{18m+23}{6}}}\right).\tag{196}$$

For general $N=4n+2\geq18$ and j,

$$\mathcal{I}_j=\delta_{j\in\mathbb{Z}}\mathcal{I}_{D_{10}(\mathfrak{sl}(14),[10,4])}(q)+\frac{\eta(\tilde\tau)^2}{\vartheta_1(\tau|\tilde\tau)^2\vartheta_1(2\tau|\tilde\tau)^2}\frac{\Lambda_{N}}{q^{m(\frac{N}{2}-2)}-q^{-m(\frac{N}{2}-2)}}\sum_{\substack{m=-j\\m\neq0}}^{+j}\left(\frac{-2m(1+q^{2m}+q^{4m}+q^{6m})}{2q^{\frac{1}{12}(N+36m+38)}}\right.$$
$$\left.+\frac{2(-1+2q^{2m}-2q^{4m}+q^{6m})E_1\left[\begin{smallmatrix}1\\q\end{smallmatrix}\right](\tilde\tau)}{2q^{\frac{1}{12}(N+36m+38)}}+\frac{2(-1+q^{2m})(1+q^{2m})^2E_1\left[\begin{smallmatrix}1\\q^2\end{smallmatrix}\right](\tilde\tau)+2(1-q^{6m})E_1\left[\begin{smallmatrix}1\\q^3\end{smallmatrix}\right](\tilde\tau)}{2q^{\frac{1}{12}(N+36m+38)}}\right).\tag{197}$$

Now, we can compute the multipoint functions of Wilson line index and work out the large-N limit,

$$\langle W_jW_j\rangle\overset{N\to+\infty}{\longrightarrow}q^{\frac{1}{4}}\frac{(1-q)^3(1-q^2)^2(1-q^3)}{(q,q)^4}\tag{198}$$

$$\langle W_jW_jW_j\rangle|_{j\in\mathbb{Z}+\frac{1}{2}}\overset{N\to+\infty}{\longrightarrow}4q^{\frac{N-9}{4}}\frac{(1-q)^2(1-q^2)^2(1-q^3)(1-q^4)}{(q,q)^4},\tag{199}$$

$$\langle W_jW_jW_j\rangle|_{j\in\mathbb{Z}}\overset{N\to+\infty}{\longrightarrow}q^{\frac{1}{4}}\frac{(1-q)^3(1-q^2)^2(1-q^3)}{(q,q)^4},\tag{200}$$

$$\langle W_j W_j W_j W_j \rangle \overset{N\to+\infty}{\longrightarrow} q^{\frac{1}{4}}(2j+1)\frac{(1-q)^3(1-q^2)^2(1-q^3)}{(q,q)^4} \tag{201}$$

$$\langle W_j W_j W_j W_j W_j \rangle|_{j\in\mathbb{Z}} \overset{N\to+\infty}{\longrightarrow} q^{\frac{N-9}{4}}\frac{(1-q)^2(1-q^2)^2(1-q^3)(1-q^4)}{(q,q)^4} \tag{202}$$

$$\langle W_j W_j W_j W_j W_j \rangle|_{j\in\mathbb{Z}+\frac{1}{2}} \overset{N\to+\infty}{\longrightarrow} q^{\frac{1}{4}}\frac{5j^2+5j+2}{2}\frac{(1-q)^3(1-q^2)^2(1-q^3)}{(q,q)^4}. \tag{203}$$

Note that the $\langle W_j W_j \rangle$, $\langle (W_j)^3|_{j\in\mathbb{Z}} \rangle$, $\langle (W_j)^4 \rangle$, and $\langle (W_j)^5|_{j\in\mathbb{Z}+\frac{1}{2}} \rangle$ are all proportional to the large-N Schur index up to a simple q factor,

$$\mathcal{I}_{D_{\infty-4}(\mathfrak{sl}(\infty),[\infty-4,4])} = \frac{(1-q)^3(1-q^2)^2(1-q^3)}{(q;q)^4}. \tag{204}$$

V. $D_{N-2}(\mathfrak{sl}(N),[N-3,3])$

In this section, we analyze the type-I series $D_{N-2}(\mathfrak{sl}(N),[N-3,3])$ with even $N \geq 12$. Alternatively, this series coincides with the type-II series $D^{b=2}_{N-2}(\mathfrak{sl}_3,[3])$. They admit the following gauge theory description:

$$D_{\frac{N}{2}-1}(\mathfrak{sl}(\tfrac{N}{2}-2),[\tfrac{N}{2}-4,1,1]) \longleftarrow \mathfrak{su}(2) \longrightarrow D_{\frac{N}{2}-1}(\mathfrak{sl}(\tfrac{N}{2}),[\tfrac{N}{2}-2,1,1]). \tag{205}$$

The 4D central charges $a_{4\mathrm{d}}, c_{4\mathrm{d}}$ are given by the formula

$$a_{4d} = \frac{202-101N+12N^2}{12(N-2)}, \qquad c_{4d} = \frac{50-25N+3N^2}{3(N-2)}. \tag{206}$$

Note that these central charges satisfy a property reminiscent to the $a_{4\mathrm{d}} = c_{4\mathrm{d}}$ relation valid for $\mathcal{N}=4$ theories and the special families of $\hat{\Gamma}(G)$ in Ref. [32],

$$a_{4\mathrm{d}} = c_{4\mathrm{d}} - \frac{1}{12}. \tag{207}$$

This central charge relation is also similar to that of the $D_{N-4}(\mathfrak{sl}(N),[N-4,4])$, without the $O(1/N)$ tail.

A. Flavored Schur index

The Schur index for the left and right nodes in the gauge theory description reads

$$\mathcal{I}_{D_{\frac{N}{2}-1}(\mathfrak{sl}(\frac{N}{2}-2),[\frac{N}{2}-4,1,1])} \tag{208}$$

$$= \mathrm{PE}\left[\frac{1}{(1-q)(1-q^{\frac{N}{2}-1})}\left((q-q^{\frac{N}{2}-1})\chi^{SU(2)}_{\mathrm{adj}}(a) + \sum_{i=1}^{\frac{N}{2}-4}(q^i - q^{\frac{N}{2}-1-i+1})\right)\right]$$
$$\times \mathrm{PE}\left[\frac{(q^{\frac{1}{2}(\frac{N}{2}-4+1)} - q^{\frac{N}{2}-1-\frac{1}{2}(\frac{N}{2}-4-1)})}{(1-q)(1-q^{\frac{N}{2}-1})}(b_1+b_1^{-1})(a+a^{-1})\right] \tag{209}$$

$$\mathcal{I}_{D_{\frac{N}{2}-1}(\mathfrak{sl}(\frac{N}{2}),[\frac{N}{2}-2,1,1])} = \mathrm{PE}\left[\frac{1}{(1-q)(1-q^{\frac{N}{2}-1})}\left((q-q^{\frac{N}{2}-1})\chi^{SU(2)}_{\mathrm{adj}}(a)+\sum_{i=1}^{\frac{N}{2}-2}(q^i-q^{\frac{N}{2}-1-i+1})\right)\right] \times \mathrm{PE}\left[\frac{(q^{\frac{1}{2}(\frac{N}{2}-2+1)}-q^{\frac{N}{2}-1-\frac{1}{2}(\frac{N}{2}-2-1)})}{(1-q)(1-q^{\frac{N}{2}-1})}(b_2+b_2^{-1})(a+a^{-1})\right]. \tag{210}$$

After carefully simplifying the plethystic exponential, we find that the full Schur index of $D_{\frac{N}{2}-1}(\mathfrak{sl}(N),[N-3,3])$ is given by the integral expression

$$\mathcal{I}_{D_{\frac{N}{2}-1}(\mathfrak{sl}(N),[N-3,3])} = \oint \frac{da}{2\pi i a}\frac{q^{-\frac{1}{6}(N-2)(N-3)}\eta(\frac{N}{2}-1)^{10}\prod_{i=1}^{\frac{N}{2}-2}(q^i,q^{\frac{N}{2}-1})}{2\eta(\tau)^2\prod_{i=0}^{\frac{N}{2}-5}\prod_{j=0}^{2}(q^{i+j+1},q^{\frac{N}{2}-1})\prod_{i=0}^{\frac{N}{2}-3}(q^{1+i},q^{\frac{N}{2}-1})}\frac{\vartheta_1(2\mathfrak{a}|(\frac{N}{2}-1)\tau)}{\prod_{\alpha=\pm}\vartheta_4(\alpha\mathfrak{a}+\mathfrak{m}_\ell|(\frac{N}{2}-1)\tau)}, \tag{211}$$

where $\mathfrak{m}_\ell$ are given by

$$\mathfrak{m}_\ell = \mathfrak{b}_{1,2}, \qquad \pm\mathfrak{b}_1 \pm \tau. \tag{212}$$

For brevity, we often denote the overall factor of q-Pochhammer symbols

$$\Lambda_N = \prod_{i=1}^{\frac{N}{2}-2}(q^i,q^{\frac{N}{2}-1})\prod_{i=0}^{\frac{N}{2}-5}\prod_{j=0}^{2}\frac{1}{(q^{i+j+1},q^{\frac{N}{2}-1})}\prod_{i=0}^{\frac{N}{2}-3}\frac{1}{(q^{1+i},q^{\frac{N}{2}-1})} = \frac{1}{\prod_{\ell=2}^{\frac{N}{2}-3}(q^\ell,q^{\frac{N}{2}-1})\prod_{\ell=3}^{\frac{N}{2}-4}(q^\ell,q^{\frac{N}{2}-1})}. \tag{213}$$

Similar to the theories $D_{N-4}(\mathfrak{sl}(N),[N-4,4])$, the Schur index is also closely related to that of the $SU(2)$ SQCD with four flavors,

$$\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-2,2])}(q,b) = \frac{q^{-\frac{1}{6}(N-2)(N-3)}\eta(\frac{N}{2}-1)^2\prod_{i=1}^{\frac{N}{2}-2}(q^i,q^{\frac{N}{2}-1})}{\eta(\tau)^2\prod_{i=0}^{\frac{N}{2}-5}\prod_{j=0}^{2}(q^{i+j+1},q^{\frac{N}{2}-1})\prod_{i=0}^{\frac{N}{2}-3}(q^{1+i},q^{\frac{N}{2}-1})}\mathcal{I}_{SU(2)\mathrm{SQCD}}(q^{\frac{N}{2}-1},m_\ell), \tag{214}$$

where the four flavor fugacities on the right take the special values

$$m_\ell = b_1, \quad b_2, \quad b_1q^{-1}, \quad b_1q. \tag{215}$$

The integrand is elliptic with respect to $(\frac{N}{2}-1)\tau$. It has eight imaginary simple poles,

$$\pm\mathfrak{b}_1+\ell\tau+\frac{1}{2}\left(\frac{N}{2}-1\right)\tau, \qquad \ell=-1,0,1, \tag{216}$$

$$\pm\mathfrak{b}_2+\frac{1}{2}\left(\frac{N}{2}-1\right)\tau. \tag{217}$$

The residues are given by, with $\tilde{\tau}=(\frac{N}{2}-1)\tau$,

$$R_{\alpha\mathfrak{b}_1+\ell\tau+\frac{1}{2}(\frac{N}{2}-1)\tau} = \Lambda_N\frac{iq^{-\frac{N-3}{3}}\eta(\tilde{\tau})^7}{2\eta(\tau)^2\vartheta_1(\tau|\tilde{\tau})}\frac{\alpha(-1)^{\ell-1}\vartheta_1(2\ell\tau+2\mathfrak{b}_1|\tilde{\tau})}{\vartheta_1((|\ell|+1)\tau|\tilde{\tau})\prod_\pm\vartheta_1(\alpha\ell\tau+\mathfrak{b}_1\pm\mathfrak{b}_2)\prod_{\substack{\ell'\neq-\ell\\ \in\{0,-1,1\}}}\vartheta_1(2\mathfrak{b}_1+\alpha\ell'\tau)}, \tag{218}$$

$$R_{\alpha\mathfrak{b}_2+\frac{1}{2}(\frac{N}{2}-1)\tau} = -\Lambda_N\alpha\frac{iq^{-\frac{N-3}{3}}\eta(\tilde{\tau})^{10}\vartheta_1(2\mathfrak{b}_2|\tilde{\tau})}{2\eta(\tau)^2\prod_{\ell=-1}^{1}\prod_\pm\vartheta_1(\mathfrak{b}_1\pm\mathfrak{b}_2+\ell\tau|\tilde{\tau})}. \tag{219}$$

Note that these eight residues sum to zero, and only four are linear independent.

Using these residues, the Schur index then reads

$$\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-2,2])}(q,b)=\sum_{\ell=-1,0,1}\sum_{\alpha=\pm}R_{\alpha\mathfrak{b}_1+\ell\tilde{\tau}+(\frac{N}{2}-1)\tau}E_1\left[\begin{matrix}-1\\ b_1^{\alpha}q^{\ell}+q^{\frac{1}{2}(\frac{N}{2}-1)}q^{-\frac{1}{2}}\end{matrix}\right](\tilde{\tau})+\sum_{\alpha=\pm}R_{\alpha\mathfrak{b}_2+\frac{1}{2}(\frac{N}{2}-1)\tau}E_1\left[\begin{matrix}-1\\ b_2^{\alpha}q^{\frac{1}{2}(\frac{N}{2}-1)}q^{-\frac{1}{2}}\end{matrix}\right](\tilde{\tau}). \tag{220}$$

B. Large-N unflavored Schur index

To analyze the unflavored Schur index, we consider two distinct subcases, $N=4n$ and $N=4n+2$, $n=3,4,\ldots$, which have different unflavored Schur index closed-form formula but share the same large-N behavior. First, we list a few flavored Schur indices:

$$\begin{aligned}\mathcal{I}_{D_{10}(\mathfrak{sl}(12),[9,3])}&=1+2q+6q^2+14q^3+(32+b_1^2+b_1b_2+b_1b_2^{-1}+b_i\leftrightarrow b_i^{-1})q^4\\&\quad+(66+3b_1^2+3b_1b_2+3b_1b_2^{-1}+b_i\leftrightarrow b_i^{-1})q^5+\cdots\end{aligned} \tag{221}$$

$$\begin{aligned}\mathcal{I}_{D_{14}(\mathfrak{sl}(16),[13,3])}&=1+2q+6q^2+14q^3+32q^4+66q^5+(136+b_1^2+b_1b_2+b_1b_2^{-1}+b_i\leftrightarrow b_i^{-1})q^6\\&\quad+(262+3b_1^2+3b_1b_2+3b_1b_2^{-1}+b_i\leftrightarrow b_i^{-1})q^7+\cdots\end{aligned} \tag{222}$$

$$\begin{aligned}\mathcal{I}_{D_{22}(\mathfrak{sl}(24),[21,3])}&=1+2q+6q^2+14q^3+32q^4+66q^5+136q^6+262q^7+499q^8+916q^9\\&\quad+(1654+b_1^2+b_1b_2+b_1b_2^{-1}+b_i\leftrightarrow b_i^{-1})q^{10}+(2912+3b_1^2+3b_1b_2+3b_1b_2^{-1}+b_i\leftrightarrow b_i^{-1})q^{11}\end{aligned} \tag{223}$$

$$\begin{aligned}\mathcal{I}_{D_{10}(\mathfrak{sl}(12),[9,3])}&=1+q+6q^2+14q^3+\left(32+b_1^2+b_1b_2+\frac{b_1}{b_2}+b_i\leftrightarrow b_i^{-1}\right)q^4\\&\quad+(66+3b_1^2+3b_1b_2+3b_1b_2^{-1}+b_i\leftrightarrow b_i^{-1})q^5+\cdots,\end{aligned} \tag{224}$$

$$\begin{aligned}\mathcal{I}_{D_{14}(\mathfrak{sl}(16),[13,3])}&=1+2q+6q^2+14q^3+32q^4+66q^5+(136+b_1^2+b_1b_2+b_1b_2^{-1}+b_i\leftrightarrow b_i^{-1})q^6\\&\quad+(262+3b_1^2+3b_1b_2+3b_1b_2^{-1}+b_i\leftrightarrow b_i^{-1})q^7+\cdots,\end{aligned} \tag{225}$$

$$\begin{aligned}\mathcal{I}_{D_{18}(\mathfrak{sl}(20),[17,3])}&=1+2q+6q^2+14q^3+32q^4+66q^5+136q^6+262q^7+(499+b_1^2+b_1b_2+b_1b_2^{-1}+b_i\leftrightarrow b_i^{-1})q^8\\&\quad+(916+3b_1^2+3b_1b_2+3b_1b_2^{-1}+b_i\leftrightarrow b_i^{-1})q^9+\cdots.\end{aligned} \tag{226}$$

We first consider the unflavored Schur index for $N=4n$. The above explicit q-series suggest the index to have a large-N limit that is independent of the flavor fugacities b_i. Let us now work out what the limiting q-series is by computing the unflavored Schur index.

Direct computation shows that the unflavored Schur index for $N=4n$ takes the form with $\tilde{\tau}\coloneqq(\frac{N}{2}-1)\tau$, where the q factor normalizes the series to start with 1,

$$\begin{aligned}\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-3,3])}(q)=-\Lambda_N\frac{q^{-\frac{N}{3}+\frac{3}{4}}\eta(\tilde{\tau})^4}{2\eta(\tau)^2\vartheta_1(\tau|\tilde{\tau})}\Bigg(&+1+4E_1\left[\begin{matrix}1\\ q\end{matrix}\right](\tilde{\tau})-2E_1\left[\begin{matrix}1\\ q^2\end{matrix}\right](\tilde{\tau})+4E_1\left[\begin{matrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-1)}\end{matrix}\right](\tilde{\tau})\\&-6E_2(\tilde{\tau})-4E_2\left[\begin{matrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-3)}\end{matrix}\right](\tilde{\tau})-8E_2\left[\begin{matrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-1)}\end{matrix}\right](\tilde{\tau})-2E_1\left[\begin{matrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-3)}\end{matrix}\right](\tilde{\tau})^2\\&+8E_1\left[\begin{matrix}1\\ q\end{matrix}\right](\tilde{\tau})E_1\left[\begin{matrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-3)}\end{matrix}\right](\tilde{\tau})-4E_1\left[\begin{matrix}1\\ q^2\end{matrix}\right](\tilde{\tau})E_1\left[\begin{matrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-3)}\end{matrix}\right](\tilde{\tau})\Bigg),\end{aligned} \tag{227}$$

where Λ_N is define above,

$$\Lambda_N = \frac{1}{\prod_{\ell=2}^{\frac{N}{2}-3}(q^\ell, q^{\frac{N}{2}-1})\prod_{\ell=3}^{\frac{N}{2}-4}(q^\ell, q^{\frac{N}{2}-1})}. \tag{228}$$

In deriving this closed form, we also make use of the identity

$$\prod_{\ell=1}^{n}\vartheta_1(\ell\tau|2n\tau) = i^n q^{-\frac{1}{24}(n^2+4n+1)}\eta(2n\tau)^{n-1}(q^n, q^{2n}), \tag{229}$$

$$(q^n, q^{2n})^2 = \frac{-iq^{\frac{n}{3}}\vartheta_1(n\tau|2n\tau)}{\eta(2n\tau)}. \tag{230}$$

Without the Λ_N factor, we observe that the remaining pieces of $\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-3,3])}(q)$ have a large-N limit simply given by

$$\Lambda_N^{-1}\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-3,3])}(q) \overset{N\to+\infty}{\longrightarrow} \frac{1}{(q,q)^2} = 1 + 2q + 5q^2 + 10q^3 + 20q^4 + \cdots. \tag{231}$$

For finite N, the correction sets in at the $(\frac{N}{2}-3)$th order,

$$\Lambda_N^{-1}\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-3,3])}(q) = \frac{1}{(q,q)^2} + \mathcal{O}(q^{\frac{N}{2}-3}). \tag{232}$$

The Λ_N factor itself admits a simple large-N limit,

$$\Lambda_N^{-1} = \frac{(1-q)^2(1-q^2)}{(q;q)^2} + O(q^{\frac{N}{2}-3}), \tag{233}$$

which gives rise to the large-N limit

$$\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-3,3])}(q) = \frac{(1-q)^2(1-q^2)}{(q;q)^4} + q^{\frac{N}{2}-3}\mathcal{I}_1^{(N)}(q). \tag{234}$$

In fact, at large finite $N = 4n$, the correction term itself is also a nested sum of stabilizing series,

$$\mathcal{I}_1^{(N)}(q) = 1 + 10q + 29q^2 + 74q^3 + 157q^4 + \cdots + q^{\frac{N}{2}-3}\mathcal{I}_2^{(N)}(q) \tag{235}$$

$$\mathcal{I}_n^{(N)}(q) = \mathcal{I}_n^{(\infty)}(q) + q^{\frac{N}{2}-3}\mathcal{I}_{n+1}^{(N)}(q), \quad n = 2, 3, \ldots. \tag{236}$$

This is similar to the discussions in the previous section and the giant graviton expansion [78,79].

Next, we comment on the case with $N = 4n+2$. The unflavored Schur index takes the following form, with $\tilde{\tau} := (\frac{N}{2}-1)\tau$:

$$\begin{aligned}\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-3,3])}(q) = \Lambda_N \frac{q^{-\frac{N}{3}+\frac{3}{4}}\eta(\tilde{\tau})^4}{4\eta(\tau)^2\vartheta_1(\tau,q^{14})^4}\Bigg(&-1 - 8E_1\begin{bmatrix}1\\ q\end{bmatrix}(\tilde{\tau}) + 4E_1\begin{bmatrix}1\\ q^2\end{bmatrix}(\tilde{\tau}) + 2E_1\begin{bmatrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-3)}\end{bmatrix}(\tilde{\tau}) - 6E_1\begin{bmatrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-1)}\end{bmatrix}(\tilde{\tau})\\ &+ 12E_2(\tilde{\tau}) + 8E_2\begin{bmatrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-3)}\end{bmatrix}(\tilde{\tau}) + 16E_2\begin{bmatrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-1)}\end{bmatrix}(\tilde{\tau}) + 4E_1\begin{bmatrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-3)}\end{bmatrix}(\tilde{\tau})^2\\ &- 16E_1\begin{bmatrix}1\\ q\end{bmatrix}(\tilde{\tau})E_1\begin{bmatrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-3)}\end{bmatrix}(\tilde{\tau}) + 8E_1\begin{bmatrix}1\\ q^2\end{bmatrix}(\tilde{\tau})E_1\begin{bmatrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-3)}\end{bmatrix}(\tilde{\tau})\\ &+ 4E_1\begin{bmatrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-3)}\end{bmatrix}(\tilde{\tau})E_1\begin{bmatrix}1\\ q^{\frac{1}{2}(\frac{N}{2}-1)}\end{bmatrix}(\tilde{\tau})\Bigg).\end{aligned} \tag{237}$$

Here, the overall q factor normalizes the series to start with 1.

Although this $\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-3,3])}$ looks rather different from that of the $N = 4n$ case, the large-N behavior is actually the same. Plugging in $N = 4n+2$ into (227), the resulting q-series differs from the correct answer using (237) precisely starting from order $q^{\frac{N}{2}-3}$.

To summarize, for both $N = 4n, 4n+2$, we have found the large-N behavior

$$\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-3,3])}(q) = \frac{(1-q)^2(1-q^2)}{(q;q)^4} + q^{\frac{N}{2}-3}(1+\cdots). \tag{238}$$

Its plethystic logarithm gives a single particle index

$$i_{D_{\infty-2}(\mathfrak{sl}(\infty),[\infty-3,3])}(q) = \frac{2q+q^2+q^3}{1-q}. \tag{239}$$

C. Wilson line index

Using the $SU(2)$ gauge theory description, we may include a spin-j Wilson line coupled to the $SU(2)$ gauge group. The Wilson line index is given by the integral

$$\mathcal{I}_j = \oint \frac{da}{2\pi i a}\chi_j(a)\mathcal{Z}(\mathfrak{a}), \tag{240}$$

where $\chi_j(a) = \sum_{m=-j}^{+j} a^{2m}$ is the $SU(2)$ character. When $j \in \mathbb{Z}$, the $m = 0$ term integrates to the original index $\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-3,3])}$.

The Wilson line index $\mathcal{I}_j$ can be easily computed using the integration formula,

$$\begin{aligned}\mathcal{I}_j = {} & \delta_{j\in\mathbb{Z}}\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-3,3])}(q,b) + \sum_{\substack{m=-j\\ m\neq 0}}^{+j}\sum_{\alpha}\sum_{\ell} R_{\alpha\mathfrak{b}_1+\ell\tau+\frac{1}{2}(\frac{N}{2}-1)\tau}\frac{(b_1^\alpha q^\ell q^{\frac{1}{2}(\frac{N}{2}-1)})^{2m}\tilde{q}^{-m}}{\tilde{q}^m-\tilde{q}^{-m}} \\ & + \sum_{\alpha=\pm} R_{\alpha\mathfrak{b}_2+\frac{1}{2}(\frac{N}{2}-1)\tau}\frac{(b_2^\alpha q^{\frac{1}{2}(\frac{N}{2}-1)})^{2m}\tilde{q}^{-m}}{q^m-q^{-m}},\end{aligned} \tag{241}$$

where the residues are computed in (218) and (219).

The unflavoring limit of the Wilson line index is straightforward. For example, when $j = \frac{1}{2}$, the unflavored Schur index is, with $\tilde{\tau} = (\frac{N}{2}-1)\tau$,

$$\mathcal{I}_{\frac{1}{2}} = -\Lambda_N \frac{\eta(\tilde{\tau})^4}{\eta(\tau)^2\vartheta_1(\tau|\tilde{\tau})}\frac{1}{q^{\frac{N}{2}}}\frac{1}{q^{\frac{1}{2}(\frac{N}{2}-1)}-q^{-\frac{1}{2}(\frac{N}{2}-1)}}\left(-1-4q-q^2-4E_1\begin{bmatrix}1\\ q\end{bmatrix}(\tilde{\tau})+4q^2E_1\begin{bmatrix}1\\ q\end{bmatrix}(\tilde{\tau})+2E_1\begin{bmatrix}1\\ q^2\end{bmatrix}(\tilde{\tau})-2q^2E_1\begin{bmatrix}1\\ q^2\end{bmatrix}(\tilde{\tau})\right), \tag{242}$$

and when $j = 1$,

$$\begin{aligned}\mathcal{I}_1 = {} & \mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-3,3])}(q) + 2\Lambda_N\frac{\eta(\tilde{\tau})^4}{\eta(\tau)^2\vartheta_1(\tau|\tilde{\tau})^4}\frac{1}{q^{\frac{N}{2}+1}}\frac{1}{q^{\frac{N}{2}-1}-q^{-\frac{N}{2}+1}} \\ & \times\left(1+4q^2+q^4+2E_1\begin{bmatrix}1\\ q\end{bmatrix}(\tilde{\tau})-2q^4E_1\begin{bmatrix}1\\ q\end{bmatrix}(\tilde{\tau})-E_1\begin{bmatrix}1\\ q^2\end{bmatrix}(\tilde{\tau})+q^4E_1\begin{bmatrix}1\\ q^2\end{bmatrix}(\tilde{\tau})\right).\end{aligned} \tag{243}$$

For general j and N, the unflavored Wilson line index is given by

$$\begin{aligned}\mathcal{I}_j = {} & \delta_{j\in\mathbb{Z}}\mathcal{I}_{D_{N-2}(\mathfrak{sl}(N),[N-3,3])}(q) \\ & -\sum_{\substack{m=-j\\ m\neq 0}}^{+j}\frac{\Lambda_N\eta(\tilde{\tau})^4}{\eta(\tau)^2\vartheta_1(\tau|\tilde{\tau})^4}\frac{m(1+4q^{2m}+q^{4m})-(q^{4m}-1)(2E_1\left[\begin{smallmatrix}1\\ q\end{smallmatrix}\right](\tilde{\tau})-E_1\left[\begin{smallmatrix}1\\ q^2\end{smallmatrix}\right](\tilde{\tau}))}{q^{\frac{N}{2}+2m-1}(q^{m(\frac{N}{2}-1)}-q^{-m(\frac{N}{2}-1)})}.\end{aligned} \tag{244}$$

Let us consider the basic building block of these Wilson line indices (without Λ_N),

$$\mathcal{I}(m) := \frac{\eta(\tilde{\tau})^4}{\eta(\tau)^2\vartheta_1(\tau|\tilde{\tau})^4}\frac{m(1+4q^{2m}+q^{4m})-(q^{4m}-1)(2E_1\left[\begin{smallmatrix}1\\ q\end{smallmatrix}\right](\tilde{\tau})-E_1\left[\begin{smallmatrix}1\\ q^2\end{smallmatrix}\right](\tilde{\tau}))}{q^{\frac{N}{2}+2m-1}(q^{m(\frac{N}{2}-1)}-q^{-m(\frac{N}{2}-1)})}. \tag{245}$$

Using the plethystic logarithm, we make the following observations of the large-N behavior:

$$\mathcal{I}\left(m=\frac{1}{2}\right)\overset{N\to+\infty}{\longrightarrow} q^{\frac{N-6}{4}}\frac{2(1+q)^2}{(q,q)^2}, \qquad \mathcal{I}(1)\overset{N\to+\infty}{\longrightarrow}\frac{1}{(q,q)^2}. \tag{246}$$

Similarly, we can compute the multipoint functions of the spin-j Wilson lines and their large-N limits,

$$\mathcal{I}_{\frac{1}{2}} \xrightarrow{N\to+\infty} \frac{1}{(q,q)^2}, \qquad \langle W_j W_j\rangle = \sum_{\ell=0}^{2j} \mathcal{I}_\ell \xrightarrow{N\to+\infty} \frac{1}{(q,q)^2}, \tag{247}$$

$$\langle W_j W_j W_j\rangle|_{j\in\mathbb{Z}} \xrightarrow{N\to+\infty} \frac{1}{(q,q)^2},$$
$$\langle W_j W_j W_j\rangle|_{j\in\mathbb{Z}+\frac{1}{2}} \xrightarrow{N\to+\infty} q^{\frac{N-6}{4}} \frac{4(1+q)^2}{(q,q)^2}, \tag{248}$$

where the large-N limit of the two-point function is independent of j.

VI. MINAHAN-NEMESCHANSKY THEORIES

A. Integration formula

In this section, we will compute the partially flavored Schur index for the Minahan-Nemeschansky (MN) theories with E_6, E_7 flavor symmetry, exploiting the generalized partition formula proposed in Ref. [36]. The formula inevitably involves contour integrals of elliptic functions with higher-order poles, which is beyond the scope of the integration formula in Ref. [23]. We will first generalize the integration formula to higher-order poles. In this subsection, we first generalized the formula to accommodate higher-order poles and then apply it to compute the Schur index of the MN theories in the next subsection.

Consider an elliptic function of $\mathfrak{a}$ with double periodicity,

$$\mathcal{Z}(\mathfrak{a}+n\tau) = \mathcal{Z}(\mathfrak{a}+1) = \mathcal{Z}(\mathfrak{a}). \tag{249}$$

Here, for generality, we assume a periodicity $\tilde{\tau} \coloneqq n\tau$ in the variable $\mathfrak{a}$ with some positive integer n instead of τ; we also denote $\tilde{q} \coloneqq q^n$. Moreover, the integrand may have n_jth-order poles $\mathfrak{a}_j$ inside the fundamental parallelogram bounded by the vertices $0, \tilde{\tau}, 1, \tilde{\tau}+1$. Following the notation in Ref. [23], we call $\mathfrak{a}_j$ a real pole when $\mathfrak{a}_j \in [0,1)$ and imaginary if otherwise.

The integrand $\mathcal{Z}(\mathfrak{a})$ can be expanded using Weierstrass ζ functions,

$$\mathcal{Z}(\mathfrak{a}) = C(\tilde{\tau}) + \frac{1}{2\pi i}\sum_j \sum_{m=1}^{+\infty} R_{j,m}\zeta^{(m-1)}(\mathfrak{a}-\mathfrak{a}_j|\tilde{\tau}), \tag{250}$$

where $C(\tilde{\tau})$ is a function independent of the variable $\mathfrak{a}$, and we define the generalized residues $R_{j,m}$,

$$R_{j,m} \coloneqq \mathrm{Res}_{\mathfrak{a}=\mathfrak{a}_j}\left[2\pi i \frac{(-1)^{m-1}}{(m-1)!}(\mathfrak{a}-\mathfrak{a}_j)^{m-1}\mathcal{Z}(\mathfrak{a})\right]. \tag{251}$$

Note that $R_{j,m}=0$ when m is larger than the order n_j of the pole $\mathfrak{a}_j$, and hence the above is a finite sum over ζ functions. Also note that

$$\sum_j R_{j,m} = 0 \quad \text{when } m \text{ is odd.} \tag{252}$$

The Weierstrass functions $\zeta(\mathfrak{a}|\tilde{\tau})$ can be expanded in Fourier series,

$$\zeta(\mathfrak{a}-\mathfrak{a}_j|\tilde{\tau}) = -4\pi^2(\mathfrak{a}-\mathfrak{a}_j)E_2(\tilde{\tau}) \mp \pi i + \pi {\sum_n}' \frac{1}{\sin n\pi\tilde{\tau}} \tilde{q}^{\mp\frac{n}{2}} e^{2\pi i n(\mathfrak{a}-\mathfrak{a}_j)} \tag{253}$$

$$\zeta^{(1)}(\mathfrak{a}-\mathfrak{a}_j|\tilde{\tau}) = -4\pi^2 E_2(\tilde{\tau}) + \pi {\sum_n}' \frac{2\pi i n}{\sin n\pi\tilde{\tau}} \tilde{q}^{\mp\frac{n}{2}} e^{2\pi i n(\mathfrak{a}-\mathfrak{a}_j)} \tag{254}$$

$$\zeta^{(m-1)}(\mathfrak{a}-\mathfrak{a}_j|\tilde{\tau}) = +\pi {\sum_n}' \frac{(2\pi i n)^{m-1}}{\sin n\pi\tilde{\tau}} \tilde{q}^{\mp\frac{n}{2}} e^{2\pi i n(\mathfrak{a}-\mathfrak{a}_j)}, \quad m\geq 3, \tag{255}$$

where the $\mp$ corresponds to real/imaginary poles. The Fourier series can then be used to derive the integration formula, giving

$$\oint \frac{da}{2\pi i a}\mathcal{Z}(\mathfrak{a}) = C(\tilde{\tau}) + \sum_{\text{real/img }\mathfrak{a}_j} R_{j,1}\left(\mp\frac{1}{2} - 2\pi i \mathfrak{a}_j E_2(\tilde{\tau})\right) + 2\pi i E_2(\tilde{\tau}) \sum_{\text{real/img }\mathfrak{a}_j} R_{j,2}.$$

We can compute $C(\tilde{\tau})$ from any reference point $\mathfrak{a}_0$,

$$C(\tilde{\tau}) = \mathcal{Z}(\mathfrak{a}_0) - \frac{1}{2\pi i}\sum_j \sum_{m\geq 1} R_{j,m}\zeta^{(m-1)}(\mathfrak{a}-\mathfrak{a}_j|\tilde{\tau})|_{\mathfrak{a}=\mathfrak{a}_0}. \tag{256}$$

It is often convenient to choose $\mathfrak{a}_0$ such that $\mathcal{Z}(\mathfrak{a}_0)=0$ to get rid of the first term.

Analogous to Ref. [23], we may rewrite the results in terms of Eisenstein series. The Weierstrass ζ function is related to the Eisenstein series by

$$\zeta(\mathfrak{z}|\tilde{\tau}) \pm \pi i + 4\pi^2 \mathfrak{z} E_2(\tilde{\tau}) = 2\pi i E_1\begin{bmatrix} -1 \\ z q^{\mp 1/2}\end{bmatrix}(\tilde{\tau}), \tag{257}$$

which helps simplify the final result,

$$\oint \frac{da}{2\pi i a}\mathcal{Z}(\mathfrak{a}) = \mathcal{Z}(\mathfrak{a}_0) + \sum_j R_{j,1} E_1\begin{bmatrix} -1 \\ a_j/a_0 q^{\pm\frac{1}{2}} \end{bmatrix}(\tilde{\tau}) - \frac{1}{2\pi i}\sum_j R_{j,2}(\zeta'(\mathfrak{a}_0-\mathfrak{a}_j|\tilde{\tau}) + 4\pi^2 E_2(\tilde{\tau})) - \frac{1}{2\pi i}\sum_j \sum_{m>2} R_{j,m}\zeta^{(m-1)}(\mathfrak{a}_0-\mathfrak{a}_j|\tilde{\tau}). \tag{258}$$

The $\zeta'(\mathfrak{z}|\tau)$ in the second line can be rewritten in Eisenstein series,

$$\zeta'(\mathfrak{z}|\tau) = 4\pi^2\left(E_1\begin{bmatrix}1\\ z\end{bmatrix}(\tau)^2 + 2E_2\begin{bmatrix}1\\ z\end{bmatrix}(\tau)\right). \tag{259}$$

The subsequent derivatives $\zeta^{(m-1)}(\mathfrak{z}|\tau)$ can also be reorganized in terms of Eisenstein series, but we shall not spell that out here.

Alternatively, one can stick to the Weierstrass ζ function expression and write

$$\oint \frac{da}{2\pi i a}\mathcal{Z}(\mathfrak{a}) = \mathcal{Z}(\mathfrak{a}_0) - \frac{1}{2\pi i}\sum_j\sum_{m=1}^{n_j} R_{j,m}\zeta^{(m-1)}(\mathfrak{a}_0-\mathfrak{a}_j|\tilde{\tau}) + \frac{1}{2\pi i}\sum_j R_{j,1}(\pm\pi i) - 2\pi i E_2(\tilde{\tau})\sum_j R_{j,1}\mathfrak{a}_j + 2\pi i E_2(\tilde{\tau})\sum_j R_{j,2}. \tag{260}$$

Using the definition of standard residue, the generalized residue can be written as

$$R_{j,m} = 2\pi i\frac{(-1)^{m-1}}{(m-1)!}\frac{1}{(n_j-m)!}\frac{d^{n_j-m}}{da^{n_j-m}}\bigg|_{\mathfrak{a}=\mathfrak{a}_j}[(\mathfrak{a}-\mathfrak{a}_j)^{n_j}\mathcal{Z}(\mathfrak{a})]. \tag{261}$$

Also note that

$$\zeta^{(m-1)}(\mathfrak{a}_0-\mathfrak{a}_j|\tilde{\tau}) = (-1)^{m-1}\frac{d^{m-1}}{d\mathfrak{a}^{m-1}}\zeta(\mathfrak{a}_0-\mathfrak{a}|\tilde{\tau}). \tag{262}$$

Putting these ingredients together, the integral can be written simply as combinations of derivatives of the integrand $\mathcal{Z}(\mathfrak{a})$,

$$\oint \frac{da}{2\pi i a}\mathcal{Z}(\mathfrak{a}) = \mathcal{Z}(\mathfrak{a}_0) - \sum_j \frac{1}{(n_j-1)!}\frac{d^{n_j-1}}{d\mathfrak{a}^{n_j-1}}\bigg|_{\mathfrak{a}=\mathfrak{a}_j}[(\mathfrak{a}-\mathfrak{a}_j)^{n_j}\mathcal{Z}(\mathfrak{a})\zeta(\mathfrak{a}_0-\mathfrak{a}|\tilde{\tau})] - \frac{1}{2}\sum_j(\pm)R_{j,1} - 2\pi i E_2(\tilde{\tau})\sum_j R_{j,1}\mathfrak{a}_j + 2\pi i E_2(\tilde{\tau})\sum_j R_{j,2}. \tag{263}$$

Here, $\pm\pi i = +\pi i$ for a real pole and $-\pi i$ for an imaginary pole.

Finally, in the presence of a monomial, the integration of Fourier series is fairly simple, giving

$$\oint \frac{da}{2\pi i a}a^k f(\mathfrak{a}) = \sum_j\sum_{m\geq 1} R_{j,m}\frac{1}{\tilde{q}^{k/2}-\tilde{q}^{-k/2}}(2\pi i k)^{m-1}\tilde{q}^{\pm\frac{k}{2}}a_j^k. \tag{264}$$

This formula can be used to compute the Wilson line index.

B. Partially flavored E_6, E_7 index

Minahan-Nemeschansky theories are a class of $\mathcal{N}=2$ SCFTs with exceptional flavor symmetry $E_{6,7,8}$ [80], which can be realized as class-$\mathcal{S}$ theories with regular punctures. The associated VOAs are given by the $\hat{\mathfrak{e}}_6, \hat{\mathfrak{e}}_7, \hat{\mathfrak{e}}_8$ entries of Deligne-Cvitanović exceptional series at nonadmissible level $k = -\frac{h^\vee}{6} - 1$ [2,81], where $h^\vee$ is the dual Coxeter number of the Lie algebra. The Schur index of these theories has been extensively studied in the physics

literature [82–84]. For example, in Ref. [84], it is shown that the E_6 Schur index can be written as a sum of the $SU(3)$ SQCD Schur index, exploiting the S-duality relation and inversion formula [82,85]. In Ref. [23], the flavored Schur index of the $SU(3)$ SQCD is computed using the integration formula, which produces the E_6 Schur index in closed form. The unflavored Schur index of $E_{6,7,8}$ satisfies second-order unflavored modular differential equations, whose solution can be explicitly written in terms of quasimodular forms [86,87].

The associated VOAs are nonadmissible quasi-lisse, whose irreducible highest weight modules were known in Ref. [87]. Alternatively, the spectrum can be derived using flavor modular differential equations and quasimodular bootstrap approach [64,65]. In this approach, the fully flavored characters are completely constrained by a set of flavored modular differential equations coming from special null states in the VOA. Conversely, once the flavored MLDEs are known, any solution must be some linear combination of the irreducible characters. Although the vacuum character of E_6 theory is known in closed form, the closed form of the nonvacuum characters are less studied. In Ref. [88], a combination of $(E_6)_{-3}$ flavored characters is proposed in closed form using the pure spinor formalism.

In this section, we continue to explore the flavored Schur index of the $E_{6,7}$ theory and nonvacuum characters of the associated VOAs by generalizing the result in Ref. [36]. The unflavored Schur index of $E_{6,7,7+\frac{1}{2},8}$ theory can be computed by the contour integral (up to an overall numerical factor)

$$\mathcal{I}_\alpha = \frac{\eta(\tau)^{-2(\alpha-1)}}{B(2\alpha-1,\alpha)} \oint \frac{da}{2\pi i a} \mathcal{Z}(\mathfrak{a},\mathfrak{b})^\alpha|_{\mathfrak{b}_i=0}, \tag{265}$$

where

$$\mathcal{Z}(\mathfrak{a},\mathfrak{b}) = +\frac{1}{2}\vartheta_1(2\mathfrak{a})^2 \prod_{i=1}^{4} \frac{\eta(\tau)^2}{\vartheta_4(\mathfrak{a}+\mathfrak{b}_i)\vartheta_4(-\mathfrak{a}+\mathfrak{b}_i)} \tag{266}$$

and $B(\alpha,\beta) \coloneqq \frac{\alpha!}{\beta!(\alpha-\beta)!}$ denotes the Binomial coefficient. The dictionary is as follows:

α	2	3	4	5
Theory	E_6	E_7	$E_{7+\frac{1}{2}}$	E_8

We begin with the closed-form unflavored index. Setting all the flavor fugacities $\mathfrak{b}_i = 0$, the integrand reads

$$\mathcal{Z}(\mathfrak{a}) = \frac{1}{2}\eta(\tau)^{8\alpha}\frac{\vartheta_1(2\mathfrak{a})^{2\alpha}}{\vartheta_4(\mathfrak{a})^{8\alpha}}, \tag{267}$$

which is elliptic with periodicity τ and has only one $n = 8\alpha$-order imaginary pole $\mathfrak{a}_1 = \frac{\tau}{2}$. The closed form of the Schur index is given in terms of the residues

$$R_{1,m} = 2\pi i \frac{(-1)^{m-1}}{(m-1)!}\frac{1}{(n-m)!}\frac{d^{n-m}}{d\mathfrak{a}^{n-m}}\bigg|_{\mathfrak{a}=\frac{\tau}{2}}\left[\left(\mathfrak{a}-\frac{\tau}{2}\right)^n \mathcal{Z}(\mathfrak{a})\right]. \tag{268}$$

Since there is only one pole, the residue $R_{1,m=1}$ must vanish. With this simplification, we can write down the unflavored Schur index compactly as

$$\mathcal{I}_\alpha(q) = \frac{\eta(\tau)^{2-2\alpha}}{B(2\alpha-1,\alpha)}\left[\frac{1}{(n-1)!}\frac{d^{n-1}}{d\mathfrak{a}^{n-1}}\bigg|_{\mathfrak{a}=\frac{\tau}{2}}\left[\left(\mathfrak{a}-\frac{\tau}{2}\right)^n \mathcal{Z}(\mathfrak{a})\zeta(\mathfrak{a}|\tau)\right] - 2\pi i E_2(\tau)R_{1,2}\right]. \tag{269}$$

Now, we turn on one flavor fugacity in the above computation, obtaining the partially flavored Schur index using the generalized partition function. The starting point is the fully flavored partition function

$$\mathcal{I}_\alpha(\mathfrak{b}) = \eta(\tau)^{2(\alpha-1)} \oint \frac{da}{2\pi i a} \mathcal{Z}(\mathfrak{a},\mathfrak{b})^\alpha. \tag{270}$$

Consider the simplest case of $\alpha = 2$, corresponding to the E_6 case. The integrand is periodic with respect to τ, but it now has eight imaginary double poles,

$$\mathfrak{a} = \pm\mathfrak{b}_i + \frac{\tau}{2}, \qquad i = 1,2,3,4. \tag{271}$$

The presence of double poles requires the use of the integration formula (258). The relevant residues are

$$\mathrm{Res}_{\pm\mathfrak{b}_i+\frac{\tau}{2},1} = \frac{\pm\eta(\tau)^8\vartheta_1(2\mathfrak{b}_1)^2}{\prod_{j\neq i}\vartheta_1(\mathfrak{b}_i-\mathfrak{b}_j)^2\vartheta_1(\mathfrak{b}_i+\mathfrak{b}_j)^2}\left(-E_1\begin{bmatrix}1\\ b_i^2\end{bmatrix} + \frac{1}{3}\sum_{\alpha=\pm}\sum_{j\neq i}E_1\begin{bmatrix}1\\ b_i b_j^\alpha\end{bmatrix}\right), \tag{272}$$

and

$$\mathrm{Res}_{\pm\mathfrak{b}_i+\frac{\tau}{2},2}=\frac{\pm\eta(\tau)^8\vartheta_1(2\mathfrak{b}_1)^2}{12\pi i\prod_{j\neq i}\vartheta_1(\mathfrak{b}_i-\mathfrak{b}_j)^2\vartheta_1(\mathfrak{b}_i+\mathfrak{b}_j)^2}. \tag{273}$$

From this, we have

$$\mathcal{I}_{\alpha=2}(\mathfrak{b})=\sum_{i=1}^{4}\sum_{\pm}\left[R_{\pm\mathfrak{b}_i+\frac{\tau}{2},1}E_1\begin{bmatrix}-1\\ b_i^{\pm}\end{bmatrix}-\frac{1}{2\pi i}R_{\pm\mathfrak{b}_i+\frac{\tau}{2},2}\left(\zeta'\left(\mp\mathfrak{b}_i-\frac{\tau}{2}\right)+4\pi^2E_2(\tau)\right)\right]. \tag{274}$$

Now, we take the special limit

$$\mathfrak{b}_1\to 3\mathfrak{b},\qquad \mathfrak{b}_{2,3,4}\to\mathfrak{b}, \tag{275}$$

and we find that the result can be series expanded as

$$\begin{aligned}&=q^{13/12}+\frac{(8b^8+16b^6+30b^4+16b^2+8)q^{25/12}}{b^4}\\&\quad+\frac{(35b^{16}+112b^{14}+310b^{12}+480b^{10}+635b^8+480b^6+310b^4+112b^2+35)q^{37/12}}{b^8}\\&\quad+\frac{2(56b^{24}+224b^{22}+749b^{20}+1632b^{18}+3030b^{16}+4184b^{14}+4885b^{12}+4184b^{10})q^{49/12}}{b^{12}}+\cdots.\end{aligned} \tag{276}$$

Each coefficient of $q^{\#}$ is a palindromic polynomial of b with integer coefficients and is in fact a character of E_6 representation where the fugacities are set to a special limit. For example,

$$30+8b^{-4}+16b^{-2}+16b^2+8b^4=\chi^{E_6}_{\mathbf{78}}(b)|_{b_{1,2,4}=b,b_3=b^2,b_5,b_6=1}, \tag{277}$$

$$\begin{aligned}&635+35b^{-8}+112b^{-6}+310b^{-4}+480b^{-2}+480b^2+310b^4+112b^6+35b^8\\&\quad=\chi^{E_6}_{\mathbf{2430}}(b)+\chi^{E_6}_{\mathbf{78}}(b)+\chi^{E_6}_{\mathbf{1}}(b)|_{b_{1,2,4}=b,b_3=b^2,b_5,b_6=1}.\end{aligned} \tag{278}$$

The integral $\mathcal{I}_{\alpha=2}(\mathfrak{b}_1\to 3\mathfrak{b},\mathfrak{b}_{2,3,4}\to\mathfrak{b})$ thus gives the partially flavored Schur index of E_6 theory,

$$\mathcal{I}_{\alpha=2}(\mathfrak{b}_1\to 3\mathfrak{b},\mathfrak{b}_{2,3,4}\to\mathfrak{b})=\mathcal{I}^{E_6}(b)|_{b_{1,2,4}=b,b_3=b^2,b_5,b_6=1}. \tag{279}$$

Further sending $b\to 1$ gives the Schur index in terms of Eisenstein series,

$$\begin{aligned}\mathcal{I}^{E_6}(q)=\frac{32}{77\eta(\tau)^{22}}\Bigg(&760E_4(\tau)E_2\begin{bmatrix}-1\\1\end{bmatrix}^4+1232E_6(\tau)E_2\begin{bmatrix}-1\\1\end{bmatrix}^3-1725E_4(\tau)^2E_2\begin{bmatrix}-1\\1\end{bmatrix}^2-5320E_4(\tau)E_6(\tau)E_2\begin{bmatrix}-1\\1\end{bmatrix}\\&-80E_2\begin{bmatrix}-1\\1\end{bmatrix}^6+1125E_4(\tau)^3-1890E_2(\tau)E_4(\tau)E_6(\tau)\Bigg).\end{aligned} \tag{280}$$

The partially flavored Schur index satisfies the partially flavored modular differential equations,

$$\left[D_q^{(1)}-\frac{1}{48}D_b^2+\left(-\frac{1}{6}E_1\begin{bmatrix}1\\b^2\end{bmatrix}-\frac{2}{3}E_1\begin{bmatrix}1\\b^4\end{bmatrix}\right)D_b+\left(4E_2\begin{bmatrix}1\\b^2\end{bmatrix}+8E_2\begin{bmatrix}1\\b^4\end{bmatrix}\right)\right]\mathcal{I}=0, \tag{281}$$

$$\left[D_q^{(2)}+\left(4E_3\begin{bmatrix}1\\b^2\end{bmatrix}+4E_3\begin{bmatrix}1\\b^4\end{bmatrix}\right)D_b-\left(283E_4(\tau)+288E_4\begin{bmatrix}1\\b^2\end{bmatrix}+144E_4\begin{bmatrix}1\\b^4\end{bmatrix}\right)\right]\mathcal{I}=0, \tag{282}$$

resulting from two null states in the chiral algebra $(\hat{E}_6)_{-3}$ by applying Zhu's recursion formula [61–64],

$$|\mathcal{N}_{\text{Sug}}\rangle = \left(L_{-2} - \frac{1}{2(k+h^\vee)}\sum_{a,b} K_{ab} J^a_{-1} J^b_{-1}\right)|0\rangle, \tag{283}$$

$$|\mathcal{N}_T\rangle = \left(L^2_{-2} - \frac{8+c}{6(k+h^\vee)} K_{ab} J^a_{-3} J^b_{-1} + \frac{2+c}{8(k+h^\vee)} K_{ab} J^a_{-2} J^b_{-2}\right)|0\rangle. \tag{284}$$

When $\alpha = 1$, the full flavored integral gives the Schur index $\mathcal{I}$ of $SU(2)$ SQCD. The associated VOA of the theory is $\widehat{\mathfrak{so}}(8)_{-2}$, whose irreducible highest weight modules are classified in Ref. [87], with the finite highest weight given by $0, -2\omega_1, -\omega_2, -2\omega_3, -2\omega_4$, where 0 corresponds to the vacuum module. The flavored characters of these modules are integral linear combination of the residue $R_{j,1}$ at the simple poles $\mathfrak{a}_j = \mathfrak{b}_j + \frac{\tau}{2}$ of $\mathcal{Z}(\mathfrak{a}, \mathfrak{b})$ [67],

$$ch_{-2\omega_1} = \mathcal{I} - 2R_{1,1}, \tag{285}$$

$$ch_{-\omega_2} = -2\mathcal{I} + 2R_{1,1} + 2R_{2,1}, \tag{286}$$

$$ch_{-2\omega_3} = \mathcal{I} - R_{1,1} - R_{2,1} - R_{3,1} - R_{4,1}, \tag{287}$$

$$ch_{-2\omega_3} = \mathcal{I} - R_{1,1} - R_{2,1} - R_{3,1} + R_{4,1}. \tag{288}$$

All these characters satisfy the same set of flavored modular differential equations derived from the null states $|\mathcal{N}\rangle_{\text{Sug}}, |\mathcal{N}_T\rangle$ in the VOA $\widehat{SO}(8)_{-2}$. From the physics point of view, this is the manifestation of the relation between Wilson line operators in four dimensions and nonvacuum modules in the VOA.

For $\alpha = 2$, *a priori*, we would not expect that relation between the residue of $\mathcal{Z}(\mathfrak{a}, \mathfrak{b})^\alpha$ and the characters of the VOA $(E_6)_{-3}$, since the integral (270) does not follow from a Lagrangian description, and there is no natural way to couple the non-Lagrangian theory to a Wilson line operator. However, it turns out that we can extract additional solution to the above partially flavored MLDEs. This solution is given precisely by the residue

$$\begin{aligned} R_{3\mathfrak{b}+\frac{\tau}{2},m=1}\mathcal{Z}(\mathfrak{a},\mathfrak{b})|_{\mathfrak{b}_1\to 3\mathfrak{b},\mathfrak{b}_{2,3,4}\to\mathfrak{b}} &= \frac{\eta(\tau)^8\vartheta_1(6\mathfrak{b})^2}{\vartheta_1(2\mathfrak{b})^6\vartheta_1(4\mathfrak{b})^2}\left(E_1\begin{bmatrix}1\\ b^2\end{bmatrix} + E_1\begin{bmatrix}1\\ b^4\end{bmatrix} - E_1\begin{bmatrix}1\\ b^6\end{bmatrix}\right) \\ &= -\frac{b^{12}(b^4+b^2+1)(b^8+3b^6+6b^4+3b^2+1)}{2(b^2-1)^{11}(b^2+1)^7} q^{-\frac{11}{12}} \\ &\quad + \frac{b^{10}(4b^{16}+27b^{14}+60b^{12}+90b^{10}+100b^8+90b^6+60b^4+27b^2+4)}{(b^2-1)^{11}(b^2+1)^7} q^{\frac{1}{12}}. \end{aligned} \tag{289}$$

Note that at the limit $\mathfrak{b}_1 \to 3\mathfrak{b}, \mathfrak{b}_{2,3,4} \to \mathfrak{b}$ there is only one independent residue given by the above expression. The higher residues $R_{j,m>1}$ do not give rise to further solutions to the modular differential equations, nor do their linear combinations.

Next, we turn to $\alpha = 3$, corresponding to the E_7 case. In this case, we consider the following contour integral

$$\mathcal{I}_{\alpha=3}(\mathfrak{b}) = \eta(\tau)^{-4}\oint \frac{da}{2\pi i a}\mathcal{Z}(\mathfrak{a},\mathfrak{b})^3|_{\mathfrak{b}_i=\mathfrak{b}}. \tag{290}$$

In the integral, we have taken the required partially flavored limit. The integrand therefore has two 12-order poles

$$\mathfrak{a} = \pm\mathfrak{b} + \frac{\tau}{2}. \tag{291}$$

It is straightforward but tedious to compute the generalized residues $R_{\pm\mathfrak{b}+\frac{\tau}{2},m}$ with $m = 1, 2, \ldots, 12$. Here, we list a few of the simplest results,

$$\text{Res}_{\mathfrak{b}+\frac{\tau}{2},12} = -\frac{1}{40874803200\pi^{11}\eta(\tau)^{36}\vartheta_1(2\mathfrak{b})^6}, \tag{292}$$

$$\text{Res}_{\mathfrak{b}+\frac{\tau}{2},10} = -\frac{i}{\pi^9\eta(\tau)^{16}\vartheta_1(2\mathfrak{b}_1)^6}\left(\frac{1}{30965760}E_1\begin{bmatrix}1\\ b^2\end{bmatrix}^2 + \frac{1}{15482880}E_2\begin{bmatrix}1\\ b^2\end{bmatrix}\right), \tag{293}$$

$$\mathrm{Res}_{\mathfrak{b}+\frac{\tau}{2},9}=\frac{1}{860160\pi^8\eta(\tau)^{16}\vartheta_1(2\mathfrak{b})^6}\left(E_1\begin{bmatrix}1\\b^2\end{bmatrix}^3+3E_1\begin{bmatrix}1\\b^2\end{bmatrix}E_2\begin{bmatrix}1\\b^2\end{bmatrix}+3E_3\begin{bmatrix}1\\b^2\end{bmatrix}\right) \tag{294}$$

$$\begin{aligned}\mathrm{Res}_{\mathfrak{b}+\frac{\tau}{2},8}=\frac{1}{215040\pi^7\eta(\tau)^{16}\vartheta_1(2\mathfrak{b})^6}&\left(6E_4(\tau)+E_1\begin{bmatrix}1\\b^2\end{bmatrix}^4+4E_1\begin{bmatrix}1\\b^2\end{bmatrix}^2E_2\begin{bmatrix}1\\b^2\end{bmatrix}\right.\\&\left.-10E_2\begin{bmatrix}1\\b^2\end{bmatrix}^2+28E_1\begin{bmatrix}1\\b^2\end{bmatrix}E_3\begin{bmatrix}1\\b^2\end{bmatrix}+28E_4\begin{bmatrix}1\\b^2\end{bmatrix}\right),\end{aligned} \tag{295}$$

$$\begin{aligned}\mathrm{Res}_{\mathfrak{b}+\frac{\tau}{2},7}=\frac{1}{1280\pi^6\eta(\tau)^{16}\vartheta_1(2\mathfrak{b})^6}&\left(E_1\begin{bmatrix}1\\b^2\end{bmatrix}^5+5E_1\begin{bmatrix}1\\b^2\end{bmatrix}^3E_2\begin{bmatrix}1\\b^2\end{bmatrix}+7E_1\begin{bmatrix}1\\b^2\end{bmatrix}E_2\begin{bmatrix}1\\b^2\end{bmatrix}^2\right.\\&\left.+E_1\begin{bmatrix}1\\b^2\end{bmatrix}^2E_3\begin{bmatrix}1\\b^2\end{bmatrix}+7E_2\begin{bmatrix}1\\b^2\end{bmatrix}E_3\begin{bmatrix}1\\b^2\end{bmatrix}-5E_1\begin{bmatrix}1\\b^2\end{bmatrix}E_4\begin{bmatrix}1\\b^2\end{bmatrix}-5E_5\begin{bmatrix}1\\b^2\end{bmatrix}\right).\end{aligned}$$

In the end, we obtain a long expression for the partially flavored Schur index of E_7 theory, which series expands as

$$\mathcal{I}_{\alpha=3}(\mathfrak{b})=q^{19/12}+(79+27b^{-2}+27b^2)q^{31/12}+(3239+351b^4+351b^{-4}+1782b^2+1782b^{-2})q^{43/12}+\cdots. \tag{296}$$

The coefficients

$$79+27b^{-2}+27b^2=\chi^{E_7}_{\mathbf{133}}(b)|_{b\to b^*}, \tag{297}$$

$$3239+351b^4+351b^{-4}+1782b^2+1782b^{-2}=\chi^{E_7}_{\mathbf{7371}}(b)+\chi^{E_7}_{\mathbf{133}}(b)+\chi^{E_7}_{\mathbf{1}}|_{b\to b^*}, \tag{298}$$

where $b\to b^*$ represents the special limit

$$b_1\to b^2, b_2\to b^2, b_3\to b^2, b_4\to b, b_5\to 1, b_6\to b, b_7\to b. \tag{299}$$

Accordingly, we have computed the partially flavored Schur index of E_7 theory in terms of the generalized partition function,

$$\mathcal{I}_{\alpha=3}(\mathfrak{b})=\mathcal{I}^{E_7}(b)|_{b\to b^*}. \tag{300}$$

The partially flavored Schur index of E_7 theory also satisfies the following partially flavored modular differential equations,

$$0=\left[D_q^{(1)}-\frac{1}{24}D_b^2-\frac{3}{2}E_1\begin{bmatrix}1\\b^2\end{bmatrix}D_b+\left(E_2+18E_2\begin{bmatrix}1\\b^2\end{bmatrix}\right)\right]\mathcal{I}^{E_7}_{b\to b^*} \tag{301}$$

$$0=\left[D_q^{(2)}+18E_3\begin{bmatrix}1\\b^2\end{bmatrix}D_b-\left(967E_4(\tau)+648E_4\begin{bmatrix}1\\b^2\end{bmatrix}\right)\right]\mathcal{I}^{E_7}_{b\to b^*}. \tag{302}$$

Similar to the $\alpha=2$ case, there is another solution to the above modular differential equations, which is given precisely by the residue

$$\mathrm{Res}_{\mathfrak{b}+\frac{\tau}{2},1}\mathcal{Z}(\mathfrak{a},\mathfrak{b})^3|_{\mathfrak{b}_i=\mathfrak{b}}. \tag{303}$$

The explicit expression of this residue is straightforward to compute, although it is too long to present here. We record its series expansion here,

$$\begin{aligned}&\mathrm{Res}_{\mathfrak{a}\mathfrak{b}+\frac{\tau}{2},1}\eta(\tau)^{-4}\mathcal{Z}(\mathfrak{a},\mathfrak{b})^3\\&=\frac{b^{12}(1+b^2)(1+9b^2+19b^4+9b^6+b^8)}{(1-b^2)^{17}}q^{-\frac{17}{12}}+\frac{79b^{12}+439b^{14}+808b^{16}+808b^{18}+439b^{20}+79b^{22}}{(1-b^2)^{17}}q^{-\frac{5}{12}}\\&\quad+\frac{+27b^{24}+2780b^{22}+8333b^{20}+13391b^{18}+13391b^{16}+8333b^{14}+2780b^{12}+27b^{10}}{(1-b^2)^{17}}q^{\frac{7}{12}}+\cdots.\end{aligned} \tag{304}$$

VII. CONCLUSIONS

In this paper, we focus on two series of non-Lagrangian theories and analyze their flavored Schur index in detail. Our main results revolve around the Argyres-Douglas theories of type A which admit $SU(2)$ gauge theory descriptions with two basic Argyres-Douglas building blocks. We provide a classification of these theories, which contains several infinite series exihibiting interesting a–c relations. It would be interesting to further classify these theories based on the ultimate quiver structure, for example, those admitting $SU(N)$ descriptions, or $SU(2)$ linear quiver descriptions with Argyres-Douglas building blocks.

We compute analytically the Schur index of these theories using the $SU(2)$ gauge theory description. Interestingly, the infinite series (with a parameter N) have a rather simple and elegant closed-form Schur index, which facilitates large N analysis. The Wilson line index and the large N behavior for these theories are also computed. The Schur index of these theories seems to have expansions that resemble those of the giant graviton expansion, and it would be very interesting to explore further the holographic implications.

We also make some observations concerning the representation theory of the associated chiral algebras by computing the Wilson line index, whose unflavored limit solves the relevant MLDE. This suggests that the Wilson line index leads to non-vacuum characters of the associated chiral algebras. This phenomenon partially carries over to the other series of non-Lagrangian theories we investigated at the end of the discussions, where the relevant contour integral descriptions do not have $\mathcal{N}=2$ SCFT origin. These computations strongly suggest deep relations betweeen the elliptic function integral and the underlying chiral algebra representation theory. More generally, one may consider coupling surface defects to these non-Lagrangian theories, and through defect index computation, probe the full space of characters and the modularity of the associated chiral algebras.

Another possible future direction is to extend our analysis to other classes of AD theories, including D- and E-type AD theories, as well as AD theories with nontrivial outer automorphism twists. This requires a deeper understanding of the relevant duality relations and may ultimately lead to a systematic construction of dualities for all AD theories within the framework of class-S theory. The representation theory of the corresponding VOA also deserves further investigation.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: SPECIAL FUNCTIONS AND NOTATION

In the following and in the main text, we use the normal and fraktur font to denote fugacities in characters and Schur indices,

$$a=e^{2\pi i\mathfrak{a}},\quad b=e^{2\pi i\mathfrak{b}},\quad \tilde{b}=e^{2\pi i\tilde{\mathfrak{b}}},\ \ldots,\ z=e^{2\pi i\mathfrak{z}}. \tag{A1}$$

The only exception is $q=e^{2\pi i\tau}$, which is the standard notation.

1. Dedekind eta function

The Dedekind eta function $\eta(\tau)$ is defined as

$$\eta(\tau)\coloneqq q^{\frac{1}{24}}\prod_{n=1}^{+\infty}(1-q^n). \tag{A2}$$

Under modular transformations,

$$\eta(\tau+1)=e^{\frac{\pi i}{12}}\eta(\tau),\qquad \eta\left(-\frac{1}{\tau}\right)=\sqrt{-i\tau}\eta(\tau). \tag{A3}$$

2. Weierstrass ζ function

The Weierstrass ζ function is defined as

$$\zeta(\mathfrak{z}|\tau)=\frac{1}{\mathfrak{z}}+\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}}{}'\left[\frac{1}{\mathfrak{z}-m-n\tau}+\frac{1}{m+n\tau}+\frac{\mathfrak{z}}{2}(m+n\tau)\right]. \tag{A4}$$

The ζ function is an odd function, $\zeta(\mathfrak{z}|\tau)+\zeta(-\mathfrak{z}|\tau)=0$. It is nonelliptic but satisfies the quasiperiodic property

$$\begin{aligned}\zeta(\mathfrak{z}+1|\tau)&=\zeta(\mathfrak{z}|\tau)+2\eta_1(\tau),\\ \zeta(\mathfrak{z}+\tau|\tau)&=\zeta(\mathfrak{z}|\tau)+2\eta_2(\tau),\end{aligned} \tag{A5}$$

where

$$\eta_1(\tau)=-2\pi^2E_2(\tau),\qquad \eta_2(\tau)=\tau\eta_1(\tau)-\pi i. \tag{A6}$$

For our purpose, $\zeta(\mathfrak{z}|\tau)$ and its derivatives form a basis of elliptic functions, which allows contour integration to be carried out analytically.

3. Jacobi theta functions

The standard Jacobi theta functions are defined using the q-Pochhammer symbol $(z;q)\coloneqq\prod_{k=0}^{+\infty}(1-zq)$,

$$\vartheta_1(\mathfrak{z}|\tau) = iq^{\frac{1}{8}}z^{-\frac{1}{2}}(q;q)(z;q)(z^{-1}q;q) = -iq^{\frac{1}{8}}z^{\frac{1}{2}}(q;q)(zq;q)(z^{-1};q), \tag{A7}$$

$$\vartheta_2(\mathfrak{z}|\tau) = q^{\frac{1}{8}}z^{-\frac{1}{2}}(q;q)(-z;q)(-z^{-1}q;q) = q^{\frac{1}{8}}z^{\frac{1}{2}}(q;q)(-zq;q)(-z^{-1};q), \tag{A8}$$

$$\vartheta_3(\mathfrak{z}|\tau) = (q;q)(-zq^{1/2};q)(-z^{-1}q^{1/2};q), \tag{A9}$$

$$\vartheta_4(\mathfrak{z}|\tau) = (q;q)(zq^{1/2};q)(z^{-1}q^{1/2};q). \tag{A10}$$

We will often omit the τ from the notation, and in particular,

$$\vartheta_i(0) = \vartheta_i(\mathfrak{z}=0|\tau), \qquad \vartheta_i^{(k)}(0) = \partial_{\mathfrak{z}}^n|_{\mathfrak{z}=0}\vartheta_i(\mathfrak{z}|\tau). \tag{A11}$$

These functions can be rewritten in infinite sum,

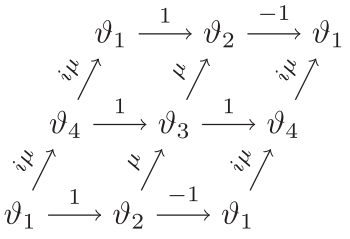
$$\begin{aligned}\vartheta_1(\mathfrak{z}|\tau) &\coloneqq -i\sum_{r\in\mathbb{Z}+\frac{1}{2}}(-1)^{r-\frac{1}{2}}e^{2\pi i r\mathfrak{z}}q^{\frac{r^2}{2}},\\ \vartheta_2(\mathfrak{z}|\tau) &\coloneqq \sum_{r\in\mathbb{Z}+\frac{1}{2}}e^{2\pi i r\mathfrak{z}}q^{\frac{r^2}{2}},\end{aligned} \tag{A12}$$

$$\begin{aligned}\vartheta_3(\mathfrak{z}|\tau) &\coloneqq \sum_{n\in\mathbb{Z}}e^{2\pi i n\mathfrak{z}}q^{\frac{n^2}{2}},\\ \vartheta_4(\mathfrak{z}|\tau) &\coloneqq \sum_{n\in\mathbb{Z}}(-1)^n e^{2\pi i n\mathfrak{z}}q^{\frac{n^2}{2}}.\end{aligned} \tag{A13}$$

The Jacobi theta functions are quasiperiodic functions of $\mathfrak{z}$, for example,

$$\vartheta_1(\mathfrak{z}+m\tau+n) = (-1)^{m+n}e^{-2\pi i m\mathfrak{z}}q^{-\frac{1}{2}m^2}\vartheta_1(\mathfrak{z}). \tag{A14}$$

When $\mathfrak{z}$ is shifted by half-periods, the four Jacobi theta functions transform into each other. We summarize the shift properties of the ϑ functions in the following diagram, where $\mu = e^{-\pi i\mathfrak{z}}e^{-\frac{\pi i}{4}}$, and $f \xrightarrow{a} g$ means



$$\text{either} \qquad f\left(\mathfrak{z}+\frac{1}{2}\right) = ag(\mathfrak{z}) \qquad \text{or} \qquad f\left(\mathfrak{z}+\frac{\tau}{2}\right) = ag(\mathfrak{z})\ , \tag{A15}$$

depending on whether the arrow is horizontal or (slanted) vertical, respectively.

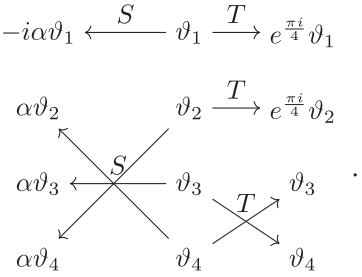
In the main text, we often encounter products of theta functions with fractional shifts, which can be simplified using the identities

$$\prod_{\ell=0}^{n-1}\vartheta_1(\mathfrak{z}-\ell\tau|n\tau) = (-iz^{1/2})^{n-1}q^{-\frac{(n-1)(2n-1)}{12}}\vartheta_1(\mathfrak{z})\frac{\eta(n\tau)^n}{\eta(\tau)}, \tag{A16}$$

$$\prod_{\ell=1}^{n}\vartheta_1(\ell\tau|2n\tau) = i^n q^{-\frac{1}{24}(2n^2+4n+1)}\eta(2n)^{n-1}\eta(\tau)(q^n,q^{2n}), \tag{A17}$$

$$\prod_{\ell=1}^{n}\vartheta_1(\ell\tau|(2n+1)\tau) = i^n\eta(\tau)\eta((2n+1)\tau)^{n-1}. \tag{A18}$$

The modularity of $\vartheta_i(\mathfrak{z}|\tau)$ is well known. Under the S and T transformations, which act on the nome and flavor fugacity as $(\frac{\mathfrak{z}}{\tau},-\frac{1}{\tau}) \xleftarrow{S} (\mathfrak{z},\tau) \xrightarrow{T} (\mathfrak{z},\tau+1)$,



Here, we define $\alpha \coloneqq \sqrt{-i\tau}e^{\frac{\pi i}{\tau}\mathfrak{z}^2}$. These transformation can be used to deduce the modular transformation of the Eisenstein series that we will now review.

4. Eisenstein series

The Eisenstein series[3] depend on two characteristics ϕ,θ, and is defined by the infinite sum,

$$\begin{aligned}E_{k\geq 1}\begin{bmatrix}\phi\\ \theta\end{bmatrix} \coloneqq &-\frac{B_k(\lambda)}{k!}+\frac{1}{(k-1)!}{\sum_{r\geq 01}}'\frac{(r+\lambda)^{k-1}\theta^{-1}q^{r+\lambda}}{1-\theta^{-1}q^{r+\lambda}}\\ &+\frac{(-1)^k}{(k-1)!}\sum_{r\geq 1}\frac{(r-\lambda)^{k-1}\theta q^{r-\lambda}}{1-\theta q^{r-\lambda}},\end{aligned} \tag{A19}$$

where $\phi \coloneqq e^{2\pi i\lambda}$ with $0\leq\lambda<1$, $B_k(x)$ denotes the kth Bernoulli polynomial, and the prime in the sum indicates that the $r=0$ should be omitted when $\phi=\theta=1$. To make certain formula more compact, we also define

[3]In the literature, these functions are often called *twisted* Eisenstein series. In this paper, we will suppress the word twisted.

$$E_0\begin{bmatrix}\phi\\ \theta\end{bmatrix}=-1. \tag{A20}$$

We have suppressed the τ-dependence in the notation. However, when the argument is rescaled to $n\tau$, we will write it explicitly, like

$$E_2\begin{bmatrix}-1\\ z\end{bmatrix}(3\tau),\qquad E_3\begin{bmatrix}1\\ q^{1/4}b\end{bmatrix}(4\tau),\qquad \text{etc.} \tag{A21}$$

Eisenstein series with even $k=2n$ is related to the usual Eisenstein series $E_{2n}(\tau)$ by sending the $\theta,\phi\to 1$. When k is odd, $\theta=\phi=1$ is a vanishing limit, except for the case $k=1$ where it is singular,

$$E_{2n}\begin{bmatrix}+1\\ +1\end{bmatrix}=E_{2n},\ E_1\begin{bmatrix}+1\\ z\end{bmatrix}=\frac{1}{2\pi i}\frac{\vartheta_1'(\mathfrak{z})}{\vartheta_1(\mathfrak{z})},\ E_{2n+1\geq 3}\begin{bmatrix}+1\\ +1\end{bmatrix}=0. \tag{A22}$$

In fact, all $E_k\left[\begin{smallmatrix}\pm 1\\ z\end{smallmatrix}\right]$ are regular as $z\to 1$, except for $E_1\left[\begin{smallmatrix}1\\ z\end{smallmatrix}\right]$, which has a simple pole. We often omit (τ) in $E_n(\tau)$, unless the argument is $n\tau$, where we will write $E_k(n\tau)$.

A closely related property is the symmetry of the Eisenstein series

$$E_k\begin{bmatrix}\pm 1\\ z^{-1}\end{bmatrix}=(-1)^k E_k\begin{bmatrix}\pm 1\\ z\end{bmatrix}. \tag{A23}$$

Often, it is useful to rewrite the Eisenstein series in terms of the Jacobi theta functions,

$$E_k\begin{bmatrix}+1\\ z\end{bmatrix}=\sum_{\ell=0}^{\lfloor k/2\rfloor}\frac{(-1)^{k+1}}{(k-2\ell)!}\left(\frac{1}{2\pi i}\right)^{k-2\ell}\mathbb{E}_{2\ell}\frac{\vartheta_1^{(k-2\ell)}(\mathfrak{z})}{\vartheta_1(\mathfrak{z})}. \tag{A24}$$

Here, $\vartheta_i^{(k)}(\mathfrak{z})$ is the kth derivative in $\mathfrak{z}$. The conversion from $E_k\left[\begin{smallmatrix}-1\\ \pm z\end{smallmatrix}\right]$ can be obtained by replacing ϑ_1 with $\vartheta_{2,3,4}$ appropriately. An immediate consequence of this relation is that the Eisenstein series satisfies the following shift property:

$$E_k\begin{bmatrix}\pm 1\\ zq^{\frac{n}{2}}\end{bmatrix}=\sum_{\ell=0}^{k}\left(\frac{n}{2}\right)^{\ell}\frac{1}{\ell!}E_{k-\ell}\begin{bmatrix}(-1)^n(\pm 1)\\ z\end{bmatrix}. \tag{A25}$$

Using the known modular property of ϑ_i, one can deduce that under the $S:(\tau,\mathfrak{z})\to(-\frac{1}{\tau},\frac{\mathfrak{z}}{\tau})$, the Eisenstein series transform according to the following set of formulas:

$$E_n\begin{bmatrix}+1\\ +z\end{bmatrix}\xrightarrow{S}E_n\begin{bmatrix}+1\\ e^{\frac{2\pi i\mathfrak{z}}{\tau}}\end{bmatrix}\left(-\frac{1}{\tau}\right)=\left(\frac{1}{2\pi i}\right)^n\sum_{\ell=0}^{k}\frac{(-\log z)^{n-\ell}(\log q)^{\ell}}{(n-\ell)!}E_{\ell}\begin{bmatrix}+1\\ z\end{bmatrix}, \tag{A26}$$

$$E_n\begin{bmatrix}-1\\ +z\end{bmatrix}\xrightarrow{S}E_n\begin{bmatrix}-1\\ e^{\frac{2\pi i\mathfrak{z}}{\tau}}\end{bmatrix}\left(-\frac{1}{\tau}\right)=\left(\frac{1}{2\pi i}\right)^n\sum_{\ell=0}^{k}\frac{(-\log z)^{n-\ell}(\log q)^{\ell}}{(n-\ell)!}E_{\ell}\begin{bmatrix}+1\\ -z\end{bmatrix}, \tag{A27}$$

$$E_n\begin{bmatrix}1\\ -z\end{bmatrix}\xrightarrow{S}E_n\begin{bmatrix}1\\ -e^{\frac{2\pi i\mathfrak{z}}{\tau}}\end{bmatrix}\left(-\frac{1}{\tau}\right)=\left(\frac{1}{2\pi i}\right)^n\sum_{\ell=0}^{k}\frac{(-\log z)^{n-\ell}(\log q)^{\ell}}{(n-\ell)!}E_{\ell}\begin{bmatrix}-1\\ z\end{bmatrix}, \tag{A28}$$

$$E_n\begin{bmatrix}-1\\ -z\end{bmatrix}\xrightarrow{S}E_n\begin{bmatrix}-1\\ -e^{\frac{2\pi i\mathfrak{z}}{\tau}}\end{bmatrix}\left(-\frac{1}{\tau}\right)=\left(\frac{1}{2\pi i}\right)^n\sum_{\ell=0}^{k}\frac{(-\log z)^{n-\ell}(\log q)^{\ell}}{(n-\ell)!}E_{\ell}\begin{bmatrix}-1\\ -z\end{bmatrix}. \tag{A29}$$

Under the T-transformation, the Eisenstein series transform according to the following rules:

$$E_n\begin{bmatrix}+1\\+z\end{bmatrix}\xrightarrow{T}E_n\begin{bmatrix}+1\\+z\end{bmatrix},\qquad E_n\begin{bmatrix}-1\\+z\end{bmatrix}\xrightarrow{T}E_n\begin{bmatrix}-1\\-z\end{bmatrix}\tag{A30}$$

$$E_n\begin{bmatrix}+1\\-z\end{bmatrix}\xrightarrow{T}E_n\begin{bmatrix}+1\\-z\end{bmatrix},\qquad E_n\begin{bmatrix}-1\\-z\end{bmatrix}\xrightarrow{T}E_n\begin{bmatrix}-1\\+z\end{bmatrix}.\tag{A31}$$

For reference, we give a few explicit examples of modular transformation of Eisenstein series. Consider the special case $E_2(\tau)=E_2\left[{}^{+1}_{+1}\right]=\lim_{z\to1}E_2\left[{}^{+1}_{z}\right]$. Under the S-transformation $(\tau,\mathfrak{z})\to(-\frac{1}{\tau},\frac{\mathfrak{z}}{\tau})$,

$$E_2\begin{bmatrix}1\\e^{2\pi i\mathfrak{z}}\end{bmatrix}(\tau)\to-\frac{\mathfrak{z}^2}{2}-\mathfrak{z}\tau E_1\begin{bmatrix}1\\e^{2\pi i\mathfrak{z}}\end{bmatrix}+\tau^2E_2\begin{bmatrix}1\\e^{2\pi i\mathfrak{z}}\end{bmatrix}.\tag{A32}$$

When taking the limit $\mathfrak{z}\to0$, the second term does not vanish due the fact that $E_1\left[{}^{1}_{z}\right]$ has a simple pole at $z=1$,

$$\lim_{\mathfrak{z}\to0}\mathfrak{z}E_1\begin{bmatrix}1\\e^{2\pi i\mathfrak{z}}\end{bmatrix}=-\frac{i}{2\pi}.\tag{A33}$$

As a result, we recover the standard modular transformation of $E_2(\tau)$,

$$E_2(\tau)\xrightarrow{S}\tau^2E_2(\tau)+\frac{i}{2\pi}\tau.\tag{A34}$$

Besides $E_2\left[{}^{1}_{1}\right]$, no other $E_{k>2}(\tau)$ has this extra τ-term, since $\lim_{\mathfrak{z}\to0}\mathfrak{z}^{k-1}E_1\left[{}^{1}_{z}\right]=0$.

As a second example, we consider two different ways of computing the modular transformation of $E_2\left[{}^{+1}_{-1}\right]$. First, according to the above rules,

$$E_2\begin{bmatrix}+1\\-1\end{bmatrix}(\tau)\xrightarrow{S}\left(\frac{1}{2\pi i}\right)^2\sum_{\ell=0}^{k}\frac{(-\log z)^{2-\ell}(\log q)^\ell}{(2-\ell)!}E_\ell\begin{bmatrix}-1\\z\end{bmatrix}(\tau)|_{z\to1}=\tau^2E_2\begin{bmatrix}-1\\+1\end{bmatrix}.\tag{A35}$$

Alternatively, we can treat $-1=e^{\frac{2\pi i\mathfrak{z}}{\tau}}$ with $\mathfrak{z}:=\frac{\tau}{2}$, $z:=e^{2\pi i\mathfrak{z}}=q^{\frac{1}{2}}$,

$$\begin{aligned}E_2\begin{bmatrix}+1\\-1\end{bmatrix}(\tau)\xrightarrow{S}E_2\begin{bmatrix}+1\\e^{\frac{\pi i\tau}{\tau}}\end{bmatrix}\left(-\frac{1}{\tau}\right)&=\left(\frac{1}{2\pi i}\right)^2\sum_{\ell=0}^{2}\frac{(-\pi i\tau)^{2-\ell}(2\pi i\tau)^\ell}{(2-\ell)!}E_\ell\begin{bmatrix}-1\\q^{+\frac{1}{2}}\end{bmatrix}(\tau)\\&=\tau^2E_2\begin{bmatrix}-1\\+1\end{bmatrix}(\tau).\end{aligned}\tag{A36}$$

In going to the second line, we make use of the shift property (A25). Combining the S, T transformation, we obtain

$$E_n\begin{bmatrix}-1\\z\end{bmatrix}\xrightarrow{STS}\left(\frac{1}{2\pi i}\right)^n\left[\left(\sum_{k\geq0}\frac{1}{k!}(-\log z)^k y^k\right)\left(\sum_{\ell\geq0}(\log q-2\pi i)^\ell y^\ell E_\ell\begin{bmatrix}-1\\+z\end{bmatrix}\right)\right]_n.$$

There are some useful identities involving Eisenstein series and their products. For example, using the following elegant identity, we can trade the argument $p\tau$ with τ:

$$E_n\begin{bmatrix}\pm1\\b^p\end{bmatrix}(p\tau)=\frac{1}{p}\sum_{\ell=0}^{p-1}E_n\begin{bmatrix}\pm1\\be^{2\pi i\ell/p}\end{bmatrix}(\tau),\qquad p=1,2,3,\ldots.\tag{A37}$$

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