

Unified study of scalar, vector, and tensor two-meson form factors in $U(3)$ resonance chiral theory

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We perform a systematic study of two-meson form factors of the scalar, vector, and antisymmetric tensor types within the framework of the $U(3)$ resonance chiral theory. The complete perturbative form factors in both the strangeness-conserving and strangeness-changing channels are calculated by incorporating one-loop light-flavor pseudoscalar meson contributions and tree-level resonance exchanges. With these newly calculated chiral results, we construct the corresponding unitarized form factors by incorporating meson-meson final-state interactions. The parameter values obtained in previous meson-meson scattering studies are then exploited to predict the corresponding form factors. Different types of form factors are found to exhibit rather distinct resonance structures across channels.

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I. INTRODUCTION

Light-flavor pseudoscalar meson form factors (FFs), namely, the meson matrix elements of the quark currents, are critical objects in particle physics. Not only can they provide a bridge to probe the properties of electromagnetic and weak currents in hadron physics, but they are also ideal objects to investigate the nonperturbative QCD dynamics. In this work, we will focus on evaluating the two-meson FFs. The pioneer study of the latter quantities [1] in chiral perturbation theory (χ PT) dates back almost to the same time as the establishment of χ PT [2–4]. The calculations of the scalar and vector two-meson FFs have been pursued up to the two-loop order in the $SU(2)$ [5,6] and $SU(3)$ [7,8] χ PT frameworks. Higher-order χ PT calculation is of great use to precisely pin down the low-energy behavior of the FFs, especially their values and slopes at zero recoiled energy. However, in many phenomenological studies, such as semileptonic and weak decays of the charmed/bottomed hadrons, the hadronic decays of the τ lepton, the

contributions from the hadronic vacuum polarization, and the light-by-light processes to the muon $g-2$, and so on, one would also need to explicitly incorporate the hadron resonance contributions in the intermediate energy ranges, where χ PT alone becomes insufficient, since it is based on a low-energy expansion.

In this regard, resonance chiral theory ($R\chi T$) [9], which constructs the interactions between hadronic resonances and light pseudoscalar mesons respecting chiral symmetry, is a valuable approach to extend the χ PT FFs to higher-energy regions. Furthermore, the synthesis of chiral amplitudes, unitarity, and analyticity offers a remarkable framework to systematically investigate light-flavor chiral dynamics of the nonperturbative strong interactions both in the low- and intermediate-energy regions, especially when hadron resonances emerge in the system under study [10–13]. For the elastic two-meson FF, the Omnès function [14] gives a strict solution that fully respects unitarity, analyticity, and Watson’s final-state theorem [15]. Nevertheless, when extending to the coupled-channel case, it is challenging to set up a strictly rigorous formalism for multichannel FFs; see, e.g., recent progress in Refs. [16–19]. In this work, rather than pursuing a purely dispersive analysis, we employ the chiral unitarized approach [20,21] to carry out a comprehensive study of the two-meson FFs with scalar, vector, and antisymmetric tensor (hereafter simply referred to as tensor) structures within the framework of $U(3)$ resonance chiral theory. Compared to dispersive methods, the chiral unitarized approach offers several important advantages. In particular,

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since it incorporates the quark-mass dependence dictated by chiral symmetry, it provides a natural and reliable framework for performing chiral extrapolations of lattice QCD results, which also allows one to make correlated predictions for related channels within the same theoretical setup. Furthermore, within the chiral unitarized approach, the nonperturbative final-state strong interactions of the meson pairs in the FFs are governed by the same objects that also appear in the meson-meson scattering amplitudes. As a consequence, the unitarized FFs can be pure predictions in the chiral unitary approach, since almost all the free parameters in the FFs can be fixed by fitting the scattering amplitudes to the experimental scattering data and relevant event distributions [22,23].

The vector FFs of pions and kaons are calculated in Ref. [24] within the unitarized $SU(3)$ $R\chi T$ by including the octet pseudoscalar π, K, η_8 and explicit vector resonance states. The calculation of the pion vector FF is further pursued by including the full $1/N_C$ corrections in $R\chi T$ in Refs. [25,26]. The strangeness-changing scalar FFs of $K\pi$, $K\eta$, and $K\eta'$ are computed in Ref. [27] by combining the tree-level $R\chi T$ amplitudes and dispersion relations. In addition, the unitarized FFs are also constructed by taking the $SU(2)$ [28] and $SU(3)$ [16] χPT amplitudes with local higher-order terms accompanied by low energy constants (LECs). The $U(3)$ chiral framework explicitly incorporates the QCD $U_A(1)$ anomaly effect, which is the main source of the large mass of the singlet η_0 . It has been exploited to calculate the strangeness-conserving scalar FFs in Refs. [22,23] by including the light pseudoscalar meson one-loop amplitudes within $R\chi T$. In this work, we push forward the computation of both the strangeness-conserving and -changing FFs of the scalar and vector types by including the full light pseudoscalar meson one-loop amplitudes within the framework of $R\chi T$. Meanwhile, the tensor FFs [29] of the light pseudoscalar mesons are found to play vital roles in searching for phenomena beyond the Standard Model [30–35]. We will calculate these interesting objects by completing two-meson tensor FFs up to the one-loop level in $U(3)$ $R\chi T$ and also performing the unitarization.

This work is organized as follows. In Sec. II, we first introduce the relevant chiral Lagrangians within the framework of $U(3)$ $R\chi T$. Then, by taking into account the light-flavor pseudoscalar meson one-loop contributions and the tree-level resonance-exchange contributions, we calculate the complete perturbative amplitudes of the scalar, vector, and tensor form factors for two-meson systems and further construct their unitarized counterparts by incorporating the nonperturbative meson-meson final-state interactions. In Sec. III, we discuss the phenomenological results of the various scalar, vector, and tensor FFs, with particular attention paid to the resonance structures in different channels. Finally, our conclusions and a brief outlook are given in Sec. IV.

II. THEORETICAL FRAMEWORK

A. Relevant chiral Lagrangian

We aim at the computation of the full sets of the two-meson FFs with the scalar, vector, and tensor types within the framework of $R\chi T$ developed in Ref. [9]. Comparing with conventional χPT [4], whose dynamical fields are the pseudo-Nambu-Goldstone bosons (pNGBs) π, K, η_8 , the heavier degrees of freedom, i.e., the bare/preexisting resonance states, are explicitly incorporated in a chiral covariant way within the framework of $R\chi T$. One could predict the higher-order χPT LECs in terms of the resonance couplings and masses once the preexisting resonance states are integrated out. The predictions of the $O(p^4)$ LECs from $R\chi T$ are found to be in nice agreement with their phenomenological values [9]. This resonance-saturation strategy is particularly advantageous in the $U(3)$ framework: extending the pure-pNGB chiral Lagrangian to higher orders introduces a large number of *a priori* undetermined LECs [36], whose determination would go beyond the scope of the present work. By retaining the lightest resonances of the spectrum as explicit degrees of freedom, we instead saturate the dominant higher-order LECs in terms of a small set of resonance couplings and masses. On the one hand, this indicates that any amplitude calculated within the $R\chi T$ framework will automatically respect the chiral symmetry in the low-energy region. On the other hand, the explicit introduction of the heavier preexisting resonance states also provides an opportunity to extend the application region of χPT with pure pNGBs to higher-energy ranges, especially when the resonances start to contribute. Additionally, it is often necessary to combine $R\chi T$ with a unitarization approach. This is a must in order to fit data to reaction probabilities (e.g., cross sections or event distributions) as a function of energy. Furthermore, for some channels, the unitarity corrections to the χPT series can be even more important than those stemming from the exchange of preexisting resonances. In particular, they may become so large that they can generate resonances dynamically, like the lightest scalar resonances [37,38] or the $\Lambda(1405)$ [39]. Indeed unitarized $R\chi T$ has been extensively applied to a wide range of phenomenological processes, such as the meson-meson scattering [38,40–43], two-photon annihilation into two mesons [44,45], hadronic τ decays [46–57], electron-positron annihilation into multi-meson states [58–60], etc.

In this work, not only the preexisting resonance states will be explicitly included, but we also take into account the dynamical field of the singlet η_0 state, which mixes with the octet η_8 to give the physical states η and η' . The large mass of the singlet η_0 component—which dominantly constitutes the physical η' and lies around 1 GeV—is widely understood as a consequence of the strong $U_A(1)$ anomaly in QCD. Therefore, the inclusion of the dynamical η_0 allows us to simultaneously study the processes involving $\pi, K, \eta,$

and η' . The QCD $U_A(1)$ anomaly effect can be systematically incorporated in the chiral framework via the $U(3)$ chiral theory (also dubbed as large- N_C chiral theory) [36,61], which needs to implement the δ power counting scheme by treating the $1/N_C$ expansion in the same way as the momentum squared and light-flavor quark masses, namely, $O(\delta) \sim O(p^2) \sim O(m_q) \sim O(1/N_C)$. The reason behind this is that, in the large- N_C limit, the QCD $U_A(1)$ anomaly would disappear and the singlet η_0 would become the ninth pNGB in the chiral limit. As a result, the octet π, K, η_8 and the singlet η_0 would form the pNGB nonet.

The leading order (LO) Lagrangian in $U(3)$ chiral theory is given by [62–65]

$$\mathcal{L}_0 = \frac{F^2}{4} \langle u_\mu u^\mu \rangle + \frac{F^2}{4} \langle \chi_+ \rangle + \frac{F^2}{3} M_0^2 \ln^2 \det u, \quad (1)$$

where the chiral tensors are

$$\begin{aligned} U &= u^2 = e^{i\frac{\sqrt{2}\Phi}{F}}, & \chi &= 2B(s + ip), \\ \chi_\pm &= u^\dagger \chi u^\dagger \pm u \chi^\dagger u, & u_\mu &= iu^\dagger D_\mu U u^\dagger, \\ D_\mu U &= \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu), \end{aligned} \quad (2)$$

and the matrix of the nonet pNGB fields reads

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 & \pi^+ & K^+ \\ \pi^- & \frac{-1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 & K^0 \\ K^- & \bar{K}^0 & \frac{-2}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 \end{pmatrix}. \quad (3)$$

The kinetic Lagrangian of the vector resonance nonet reads [9]

$$\mathcal{L}_{\text{kin}}^V = -\frac{1}{2} \left\langle \nabla^\lambda V_{\lambda\mu} \nabla_\nu V^{\nu\mu} - \frac{1}{2} M_V^2 V_{\mu\nu} V^{\mu\nu} \right\rangle, \quad (4)$$

where the antisymmetric tensor is adopted to describe the vector resonances. For the scalar (S) and pseudoscalar (P) resonances, their kinetic Lagrangians are given by

$$\mathcal{L}_{\text{kin}}^R = \frac{1}{2} \langle \nabla^\mu R_8 \nabla_\mu R_8 - M_{R_8}^2 R_8^2 \rangle + \frac{1}{2} \langle \partial^\mu R_1 \partial_\mu R_1 - M_{R_1}^2 R_1^2 \rangle, \quad (5)$$

where R_8 and R_1 with $R = (S, P)$ denote the $SU(3)$ octet and singlet, respectively. The minimal Lagrangian describing the interaction of vector resonances consists of two operators [9]

$$\mathcal{L}_{2,V} = \frac{F_V}{2\sqrt{2}} \langle V_{\mu\nu} f^{\mu\nu}_+ \rangle + \frac{iG_V}{\sqrt{2}} \langle V_{\mu\nu} u^\mu u^\nu \rangle. \quad (6)$$

For the scalar and pseudoscalar resonances, we follow Refs. [22,23] to separately treat the interaction operators involving the octet and singlet states

$$\begin{aligned} \mathcal{L}_S &= c_d \langle S_8 u_\mu u^\mu \rangle + c_m \langle S_8 \chi_+ \rangle + \tilde{c}_d S_1 \langle u_\mu u^\mu \rangle \\ &\quad + \tilde{c}_m S_1 \langle \chi_+ \rangle, \end{aligned} \quad (7)$$

$$\mathcal{L}_P = id_m \langle P_8 \chi_- \rangle + i\tilde{d}_m P_1 \langle \chi_- \rangle. \quad (8)$$

The operators in Eq. (8) will cause mixing terms between the pseudoscalar resonances and pNGBs, which can be

eliminated by performing proper field redefinitions. See Ref. [23] for details about this procedure, as we do not repeat it here.

Compared to the vector and scalar sectors, the phenomenology of pseudoscalar resonances remains considerably less understood. This is mainly due to the scarcity and limited precision of experimental data, the large widths and strong overlapping among states, and the complications associated with their mixing with the pNGBs and with the nontrivial effects of the $U_A(1)$ anomaly. Consequently, the pseudoscalar resonance parameters are affected by sizable uncertainties. To partially account for these uncertainties, an L_8 -like operator is introduced in Ref. [22],

$$\frac{\delta L_8}{2} \langle \chi_+ \chi_+ + \chi_- \chi_- \rangle. \quad (9)$$

It is noted that δL_8 is different from L_8 in χ PT and their relation can be written as $L_8^{\chi\text{PT}} = L_8^{\text{Res}} + \delta L_8$, where L_8^{Res} corresponds to the resonance contributions at leading N_C and δL_8 can be interpreted as the remnant part subleading in $1/N_C$. Finally, two additional terms in the $U(3)$ Lagrangian are also considered

$$\mathcal{L}_\Lambda = \Lambda_1 \frac{F^2}{12} D_\mu \psi D^\mu \psi - i\Lambda_2 \frac{F^2}{12} \langle U^\dagger \chi - \chi^\dagger U \rangle \psi, \quad (10)$$

with

$$\psi = -i \ln \det U, \quad D_\mu \psi = \partial_\mu \psi - 2\langle a_\mu \rangle, \quad a_\mu = (r_\mu - l_\mu)/2. \quad (11)$$

The two operators in Eq. (10) are subleading in $1/N_C$ and cannot be generated by the resonance Lagrangians given above.

To calculate the tensor FF in chiral effective field theory, it is convenient to introduce the antisymmetric tensor source $\tilde{\mu}^{\mu\nu}$ in the QCD Lagrangian $\bar{q}\sigma_{\mu\nu}\tilde{\mu}^{\mu\nu}q$. The corresponding formalism in χ PT was first studied in Ref. [29]. The LO χ PT Lagrangian coupled to the tensor external source scales as $\mathcal{O}(p^4)$ and the relevant operator to our study is given by

$$\mathcal{L}_T^{(4)} = -i\Lambda_2^T \langle t_+^{\mu\nu} u_\mu u_\nu \rangle, \quad (12)$$

with

$$\begin{aligned} t_\pm^{\mu\nu} &= u^\dagger t^{\mu\nu} u^\dagger \pm u t^{\mu\nu} u, \\ t^{\mu\nu} &= \frac{1}{4}(g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho} - i\epsilon^{\mu\nu\lambda\rho})\tilde{t}_{\lambda\rho}. \end{aligned} \quad (13)$$

For the Levi-Civita tensor $\epsilon^{\mu\nu\lambda\rho}$, we use the convention of $\epsilon^{0123} = 1$ throughout. In addition, we also include the resonance contributions to the tensor FFs. The corresponding resonance Lagrangian, involving a single resonance field and the lowest-order couplings to the external tensor source, reads [31]

$$\mathcal{L}_{T,V} = F_V^T M_V \langle V_{\mu\nu} t_+^{\mu\nu} \rangle, \quad (14)$$

where the tensor coupling F_V^T has mass dimension. Possible terms from the axial-vector resonances are expected to be tiny and hence are neglected.

B. Calculation of the scalar form factors

By including the tree-level resonance exchange together with the pNGB one-loop diagrams, the strangeness-conserving two-meson scalar FFs were computed in Refs. [22,23]. As one of the main novelties of the present work, we extend this analysis to the strangeness-changing channels within the same framework. In addition, our calculation improves upon Ref. [27], which only accounted for tree-level resonance exchanges, by incorporating the full set of pNGB one-loop contributions.

The two-meson scalar FF corresponds to the matrix element of the scalar quark current, i.e.,

$$F_{S,PQ}^{mn}(s) = \frac{1}{B} \langle 0 | \bar{q}^m q^n | PQ \rangle, \quad (15)$$

where m and n denote the quark-flavor indices, S in the subscript refers to the scalar FF, P and Q stand for the two mesons in the final states, and s is the energy squared of the PQ system. The results will be presented using the quark-flavor basis rather than the singlet-octet basis for the quark currents. In the quark-flavor basis, the combinations $(\bar{u}u +$

TABLE I. Relevant two-meson channels coupled to the various scalar currents.

Isospin, strangeness	Scalar currents	Two-meson channels
$I = 0, S = 0$	$(\bar{u}u + \bar{d}d)/\sqrt{2}$ $\bar{s}s$	$\pi\pi, K\bar{K}, \eta\eta, \eta\eta', \eta'\eta'$ $\pi\pi, K\bar{K}, \eta\eta, \eta\eta', \eta'\eta'$
$I = 1, S = 0$	$\bar{u}d$	$\pi\eta, K\bar{K}, \pi\eta'$
$I = \frac{1}{2}, S = -1$	$\bar{u}s$	$\bar{K}\pi, \bar{K}\eta, \bar{K}\eta'$

$\bar{d}d)/\sqrt{2}$ and $\bar{s}s$ constitute the isoscalar sector, while the bilinears $(\bar{u}u - \bar{d}d)/\sqrt{2}$, $\bar{u}d$, and $\bar{d}u$ form an isovector triplet. For strangeness-changing transitions, the relevant operators arrange into two isospin-1/2 doublets, namely, $(\bar{d}s, \bar{u}s)$ and $(\bar{s}u, \bar{s}d)$. Since CP -violating effects are not considered in this work, these two doublets are related by charge conjugation and therefore governed by identical strong interaction dynamics. Furthermore, we work in the isospin (I) symmetric limit throughout, implying that there are four independent scalar quark currents, which are chosen as $(\bar{u}u + \bar{d}d)/\sqrt{2} (I = 0)$, $\bar{s}s (I = 0)$, $\bar{u}d (I = 1)$, and $\bar{u}s (I = \frac{1}{2})$. In Table I, we summarize the two-meson channels coupled to the various scalar currents and find that there are 16 independent two-meson scalar FFs for the four types of currents. In order to construct the two-meson FFs in the isospin basis, we follow the same conventions as those given in Eq. (19) of Ref. [41], which were employed in the $U(3)$ meson-meson scattering case. For the charged kaon states, however, we use the phase convention

$$|K^+\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle, \quad |K^-\rangle = -\left| \frac{1}{2}, -\frac{1}{2} \right\rangle \quad (16)$$

in the subsequent analysis of the FFs, in order to be consistent with Refs. [16,66]. This only amounts to a different phase convention for the charged kaon isospin components and does not affect the remaining results of Refs. [22,23,41]. The isoscalar two-meson states are given by

$$\begin{aligned} |\pi\pi\rangle_{I=0} &= -\frac{|\pi^+\pi^-\rangle + |\pi^-\pi^+\rangle + |\pi^0\pi^0\rangle}{\sqrt{6}}, \\ |K\bar{K}\rangle_{I=0} &= -\frac{|K^+K^-\rangle + |K^0\bar{K}^0\rangle}{\sqrt{2}}, \\ |\eta\eta\rangle_{I=0} &= \frac{|\eta\eta\rangle}{\sqrt{2}}, \\ |\eta\eta'\rangle_{I=0} &= |\eta\eta'\rangle, \\ |\eta'\eta'\rangle_{I=0} &= \frac{|\eta'\eta'\rangle}{\sqrt{2}}, \end{aligned} \quad (17)$$

and the isovector two-meson states are

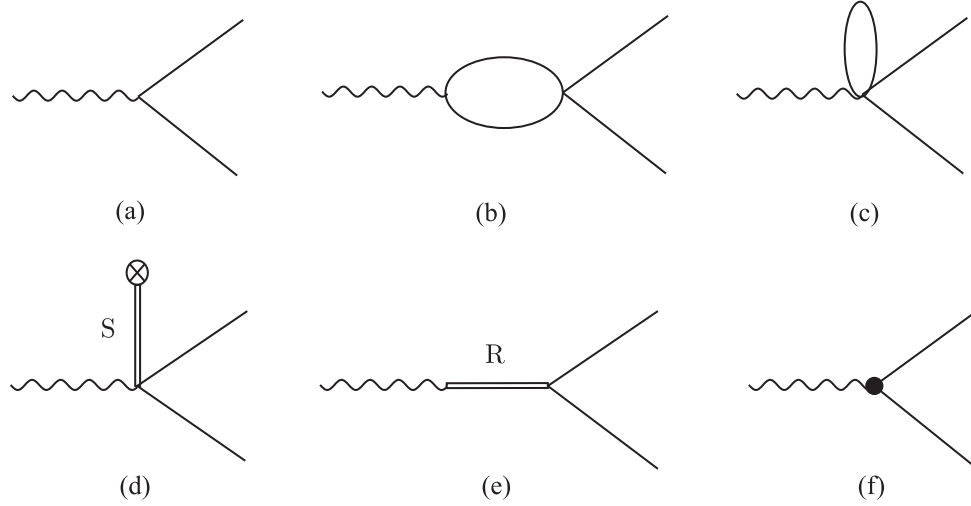


FIG. 1. Feynman diagrams for the two-meson FFs. The wiggly lines denote the scalar/vector/tensor external sources for the scalar/vector/tensor FFs. The double lines denote the proper resonance fields. Panels (a), (b), and (c) represent, respectively, the leading-order contact contribution, the two-meson loop contribution, and the tadpole-loop contribution. (d) Only appears in the scalar and vector FFs, and the circle cross symbol connected to the scalar resonance stands for the vacuum, which is caused by the tadpole contribution from the c_m and $\tilde{c}_m$ terms in Eq. (7). The type of the resonance R in (e) depends on the different FFs, $R = S$ for scalar and $R = V$ for vector and tensor. (f) Arises from the two local next-to-leading-order (NLO) operators in Eq. (10), the pseudoscalar resonance operator in Eq. (8) after performing the appropriate field redefinitions, and the additional L_8 -like operator in Eq. (9), which only enters in the scalar FFs.

$$\begin{aligned}
 |\pi\pi\rangle_{I=1} &= \frac{|\pi^0\pi^- \rangle - |\pi^-\pi^0 \rangle}{2}, \\
 |K\bar{K}\rangle_{I=1} &= -|K^0K^- \rangle, \\
 |\pi\eta\rangle_{I=1} &= |\pi^-\eta \rangle, \\
 |\pi\eta'\rangle_{I=1} &= |\pi^-\eta' \rangle,
 \end{aligned} \tag{18}$$

where the symmetry factor of $\sqrt{2}$ is introduced for the identical particle pairs, such as the $\pi\pi$, $\eta\eta$, and $\eta'\eta'$, consistent with the convention of the partial-wave meson-meson amplitudes in Refs. [22,23,41]. The two-meson states in the isospin-1/2 case are defined as

$$\begin{aligned}
 |\bar{K}\pi\rangle_{I=\frac{1}{2}} &= -\frac{\sqrt{2}|\bar{K}^0\pi^- \rangle + |K^-\pi^0 \rangle}{\sqrt{3}}, \\
 |\bar{K}\eta\rangle_{I=\frac{1}{2}} &= -|K^-\eta \rangle, \\
 |\bar{K}\eta'\rangle_{I=\frac{1}{2}} &= -|K^-\eta' \rangle.
 \end{aligned} \tag{19}$$

The relevant Feynman diagrams to the FFs calculated in this work are illustrated in Fig. 1, which include both the

tree-level resonance exchanges and the pNGB one-loop effects. It should also be noted that the wave function renormalizations of the pNGB fields, as shown in Fig. 2, are also taken into account. For clarity, let us indicate that similar diagrams hold for the perturbative calculations of the meson-meson scattering amplitudes up to one loop in $U(3)$ R χ T, but then the resonance exchanges and pNGB loops include both the direct s channel as well as the crossed ones, namely, in the t and u channels [41].

The LO expressions of the various FFs defined by the $\bar{u}s$ current, corresponding to Fig. 1(a), are

$$\begin{aligned}
 F_{S,K^-\pi^0}^{\bar{u}s,\text{LO}} &= \frac{1}{\sqrt{2}}, & F_{S,K^-\eta}^{\bar{u}s,\text{LO}} &= -\frac{\sqrt{6}c_\theta + 4\sqrt{3}s_\theta}{6}, \\
 F_{S,K^-\eta'}^{\bar{u}s,\text{LO}} &= \frac{4\sqrt{3}c_\theta - \sqrt{6}s_\theta}{6},
 \end{aligned} \tag{20}$$

where $c_\theta = \cos \theta$ and $s_\theta = \sin \theta$, and θ denotes the LO η - η' mixing angle

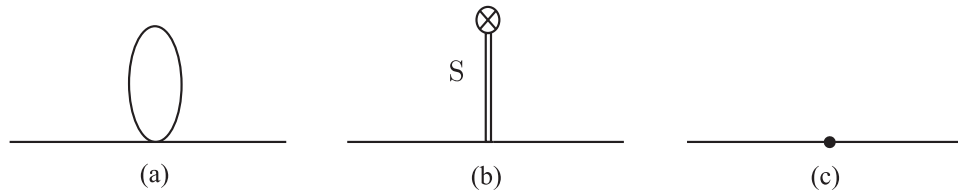


FIG. 2. Feynman diagrams contributing to the pNGB self-energy. Panels (a), (b), and (c) represent, respectively, the pNGB tadpole-loop contribution, the scalar-resonance tadpole contribution, and the local counterterm contribution. The meanings of the different symbols are illustrated in Fig. 1.

$$\sin \theta = - \left(\sqrt{1 + \frac{(3M_0^2 - 2\Delta^2 + \sqrt{9M_0^4 - 12M_0^2\Delta^2 + 36\Delta^4})^2}{32\Delta^4}} \right)^{-1}, \quad (21)$$

with $\Delta^2 = m_{\bar{K}}^2 - m_{\pi}^2$, and m_{π} and $m_{\bar{K}}$ denoting the LO masses of the pion and kaon, respectively. Although the LO expressions of the scalar FFs with isoscalar and isovector quark currents have been given in Ref. [23], we present the results here for the sake of completeness. The LO results for the isoscalar currents are

$$\begin{aligned} F_{S,\pi^0\pi^0}^{\frac{\bar{u}u+\bar{d}d}{\sqrt{2}},\text{LO}} &= \sqrt{2}, & F_{S,\pi^0\pi^0}^{\bar{s}s,\text{LO}} &= 0, \\ F_{S,K^0\bar{K}^0}^{\frac{\bar{u}u+\bar{d}d}{\sqrt{2}},\text{LO}} &= \frac{1}{\sqrt{2}}, & F_{S,K^0\bar{K}^0}^{\bar{s}s,\text{LO}} &= 1, \\ F_{S,\eta\eta}^{\frac{\bar{u}u+\bar{d}d}{\sqrt{2}},\text{LO}} &= -\frac{\sqrt{2}}{3} (c_\theta^2 + 2\sqrt{2}c_\theta s_\theta - 2), & F_{S,\eta\eta}^{\bar{s}s,\text{LO}} &= \frac{2}{3} (c_\theta^2 + 2\sqrt{2}c_\theta s_\theta + 1), \\ F_{S,\eta'\eta'}^{\frac{\bar{u}u+\bar{d}d}{\sqrt{2}},\text{LO}} &= \frac{\sqrt{2}}{3} (\sqrt{2}c_\theta^2 - c_\theta s_\theta - \sqrt{2}s_\theta^2), & F_{S,\eta'\eta'}^{\bar{s}s,\text{LO}} &= -\frac{2}{3} (\sqrt{2}c_\theta^2 - c_\theta s_\theta - \sqrt{2}s_\theta^2), \\ F_{S,\eta'\eta'}^{\frac{\bar{u}u+\bar{d}d}{\sqrt{2}},\text{LO}} &= \frac{\sqrt{2}}{3} (-s_\theta^2 + 2\sqrt{2}c_\theta s_\theta + 2), & F_{S,\eta'\eta'}^{\bar{s}s,\text{LO}} &= \frac{2}{3} (s_\theta^2 - 2\sqrt{2}c_\theta s_\theta + 1), \end{aligned} \quad (22)$$

and the scalar FFs of the isovector current are given by

$$F_{S,\pi^-\eta}^{\bar{u}d,\text{LO}} = \frac{\sqrt{2}c_\theta - 2s_\theta}{\sqrt{3}}, \quad F_{S,K^0K^-}^{\bar{u}d,\text{LO}} = 1, \quad F_{S,\pi^-\eta'}^{\bar{u}d,\text{LO}} = \frac{2c_\theta + \sqrt{2}s_\theta}{\sqrt{3}}. \quad (23)$$

The complete expressions for the strangeness-changing ($I = \frac{1}{2}$) scalar FFs are given in Eqs. (A1)–(A3) of the Appendix. The explicit expressions for the remaining scalar FFs with isoscalar and isovector quark currents have been provided in Ref. [23].

For the scalar FFs, when restricting the intermediate states to two-meson channels, unitarity implies the following relation above the corresponding thresholds:

$$\text{Im } F_{S,j}^I(s) = \sum_k T_{jk}^{IJ}(s)^* \rho_k(s) F_{S,k}^I(s). \quad (24)$$

Here $T_{jk}^{IJ}(s)$ denotes the meson-meson scattering partial-wave amplitude with isospin I and angular momentum J (with $J = 0$ for scalar FFs), while j and k label the coupled two-meson channels and $\rho_k(s)$ denotes the kinematical factor

$$\rho_k(s) = \frac{\sqrt{[s - (m_a + m_b)^2][s - (m_a - m_b)^2]}}{16\pi s} \equiv \frac{q_k(s)}{8\pi\sqrt{s}}, \quad (25)$$

where m_a and m_b are the masses of the two mesons in the channel k , and $q_k(s)$ is the three momentum in the center-of-mass frame. However, the perturbative FFs and scattering amplitudes calculated within chiral effective field theory do not fulfill the unitarity relation in Eq. (24)

exactly. Instead, unitarity is satisfied only perturbatively, namely, order by order in the χ PT expansion.

In order to reliably describe the resonance dynamics, one needs to restore the unitarity relations. For the elastic case when there is only a single channel, one can strictly solve Eq. (24) and write the FF in terms of the Omnés function [14]. However, its rigorous extension to coupled channels is considerably more involved, since it requires as input the coupled-channel scattering amplitudes over a broad energy region, where sizable uncertainties are typically present. In addition, the Omnés solution requires knowledge of the phase of the FF up to asymptotically high energies, which may introduce further important uncertainties. In this work, instead of relying on the dispersive method, we will follow Refs. [22,23] to implement the unitarization of the scalar FFs using the following formula:

$$F_S^I(s) = [1 + N^{IJ}(s) \cdot g^{IJ}(s)]^{-1} \cdot R_S^I(s), \quad (26)$$

with

$$N^{IJ}(s) = T^{IJ}(s)^{(2)+\text{Res}+\text{Loop}} + T^{IJ}(s)^{(2)} \cdot g^{IJ}(s) \cdot T^{IJ}(s)^{(2)}, \quad (27)$$

$$R_S^I(s) = F_S^I(s)^{(2)+\text{Res}+\text{Loop}} + N^{IJ}(s)^{(2)} \cdot g^{IJ}(s) \cdot F_S^I(s)^{(2)}, \quad (28)$$

where $J = 0$ should be taken for scalar FFs. Here, $T^{IJ}(s)^{(2),\text{Res,Loop}}$ and $F_S^I(s)^{(2),\text{Res,Loop}}$ denote, respectively, the $\mathcal{O}(p^2)$, tree-level resonance-exchange, and pNGB loop contributions to the perturbative meson-meson scattering amplitudes and two-meson scalar FFs calculated within R χ T. Furthermore, $g^{IJ}(s)$ is an $n \times n$ diagonal matrix whose k th nonvanishing diagonal entry is given by

$$16\pi^2 g_k^{IJ}(s) = a_{SL}(\mu) + \log \frac{m_b^2}{\mu^2} - x_+ \log \frac{x_+ - 1}{x_+} - x_- \log \frac{x_- - 1}{x_-}, \quad (29)$$

$$x_{\pm} = \frac{s + m_a^2 - m_b^2}{2s} \pm \frac{1}{2s} \sqrt{-4s(m_a^2 - i0^+) + (s + m_a^2 - m_b^2)^2}, \quad (30)$$

corresponding to the two-point one-loop function with the subtraction constant a_{SL} . The imaginary part of $g_k^{IJ}(s)$ turns out to be $\text{Im } g_k^{IJ}(s) = -\rho_k(s)$. It is then straightforward to demonstrate that $N^{IJ}(s)$ and $R_S^I(s)$ do not contain right-hand cut (RHC). This follows because the RHC generated by chiral loops in the first term of Eqs. (27) and (28) is exactly canceled by the RHC from the $g^{IJ}(s)$ function in the second term; the full RHC of the unitarized FF is thus resumed through the $[1 + N^{IJ}(s) \cdot g^{IJ}(s)]^{-1}$ factor in Eq. (26). See Refs. [22,23,41] for details. The unitarized meson-meson scattering can be similarly constructed as

$$T^{IJ}(s) = [1 + N^{IJ}(s) \cdot g^{IJ}(s)]^{-1} \cdot N^{IJ}(s). \quad (31)$$

We stress that by construction the unitarized FFs and scattering amplitudes in Eqs. (26) and (31) are exactly reproduced up to the order to which they are perturbatively calculated. This is guaranteed by the matching procedure used to calculate $N^{IJ}(s)$ in Eq. (27) and $R_S^I(s)$ in Eq. (28). These equations just originate by expanding up to $g^{IJ}(s)$, the one loop calculated at $\mathcal{O}(p^4)$, the nonperturbative formulas in Eqs. (26) and (31), and then solving for the functions $N^{IJ}(s)$ and $R_S^I(s)$, such that the perturbative scattering amplitudes $T^{IJ}(s)^{(2)+\text{Res}+\text{Loop}}$ and the form factor $F_S^I(s)^{(2)+\text{Res}+\text{Loop}}$ are exactly reproduced up to the order they are calculated. This approach allows proceeding systematically in the matching procedure with even higher orders of the strong amplitudes and form factors calculated in chiral theory.

In this unitarized formalism, the unitarity relation in Eq. (24) is satisfied as long as $N_{ij}^{IJ}(s)$ and $R_{S,j}^I(s)$ are real matrices above the upper threshold of each ij channel coupled. This is always the case if $\sqrt{s}$ is greater than the highest threshold of the coupled channels. Below it, there could be a violation of exact unitarity that, with the final set of free parameters adopted, is typically visible

(appreciable) only in a very narrow range of s , because some matrix elements of $N^{IJ}(s)$ and/or $R^I(s)$ turn complex there due to crossed-channel exchanges. This deficiency would be improved if these functions were calculated at higher order in the chiral expansion, and they could be avoided if other unitarization techniques were implemented, like the once-iterated N/D method. However, there is a price to pay because of a typically large increase in complexity and in the number of free parameters associated with subtractions that should be taken in order to end with finite dispersion relations. We consider that our approach is well settled, being very predictive and phenomenologically successful, as it has been probed in many instances in meson-meson, meson-baryon, and baryon-baryon scattering. See Refs. [22,23,41,67] for the successful reproduction of experimental and lattice data in connection with the topics discussed in this work. Let us also stress that the algebraic unitarization employed here in Eq. (26) is also consistent with the asymptotic $1/s$ falloff dictated by perturbative QCD in the chiral and large- N_C limits. See the detailed discussions in Sec. VI C of Ref. [23].

It should be noted that, compared to the unitarized scattering amplitude $T^{IJ}(s)$, no additional unknown parameters enter in the unitarized FF $F_S^I(s)$. In other words, once the unitarized scattering amplitudes are fixed by the relevant meson-meson scattering data, we can give pure predictions for the unitarized scalar FFs. Both the theoretical formulas and phenomenological results of the strangeness-conserving scalar FFs with $I = 0$ have been given in Refs. [22,23]. Here, our focus is primarily on evaluating the strangeness-changing scalar FFs. In addition, we also update the strangeness-conserving scalar FFs with $I = 1$ by taking the revised parameters from Ref. [67], which are obtained by fitting lattice finite-volume energy levels of the $\pi\eta$, $K\bar{K}$, and $\pi\eta'$ systems.

For the scalar FFs of the physical states with isoscalar quark currents, their relations can be easily obtained as

$$F_{S,\pi^0\pi^0}^{\bar{u}u+\bar{d}d} = F_{S,\pi^+\pi^-}^{\bar{u}u+\bar{d}d} = F_{S,\pi^-\pi^+}^{\bar{u}u+\bar{d}d}, \quad F_{S,K^+K^-}^{\bar{u}u+\bar{d}d} = F_{S,K^0\bar{K}^0}^{\bar{u}u+\bar{d}d}, \quad (32)$$

and

$$F_{S,\pi^0\pi^0}^{\bar{s}s} = F_{S,\pi^+\pi^-}^{\bar{s}s} = F_{S,\pi^-\pi^+}^{\bar{s}s}, \quad F_{S,K^+K^-}^{\bar{s}s} = F_{S,K^0\bar{K}^0}^{\bar{s}s}. \quad (33)$$

For the scalar FFs with isovector currents, we have the following relations:

$$F_{S,\pi^0\eta^{(\prime)}}^{\bar{u}u-\bar{d}d} = F_{S,\pi^-\eta^{(\prime)}}^{\bar{u}d} = F_{S,\pi^+\eta^{(\prime)}}^{\bar{d}u}, \quad F_{S,K^+K^-}^{\bar{u}u-\bar{d}d} = -F_{S,K^0\bar{K}^0}^{\bar{u}u-\bar{d}d} = \frac{1}{\sqrt{2}} F_{S,K^0K^-}^{\bar{u}d} = \frac{1}{\sqrt{2}} F_{S,K^+\bar{K}^0}^{\bar{d}u}. \quad (34)$$

For the scalar FFs with isospin-1/2 quark currents, their relations are given by

$$\begin{aligned} F_{S,K^0\pi^+}^{\bar{s}u} &= F_{S,\bar{K}^0\pi^-}^{\bar{u}s} = F_{S,K^+\pi^-}^{\bar{s}d} = F_{S,K^-\pi^+}^{\bar{d}s} = \sqrt{2}F_{S,K^+\pi^0}^{\bar{s}u} \\ &= \sqrt{2}F_{S,K^-\pi^0}^{\bar{u}s} = -\sqrt{2}F_{S,K^0\pi^0}^{\bar{s}d} = -\sqrt{2}F_{S,\bar{K}^0\pi^0}^{\bar{d}s}, \\ F_{S,K^+\eta^{(\prime)}}^{\bar{s}u} &= F_{S,K^-\eta^{(\prime)}}^{\bar{u}s} = F_{S,K^0\eta^{(\prime)}}^{\bar{s}d} = F_{S,\bar{K}^0\eta^{(\prime)}}^{\bar{d}s}. \end{aligned} \quad (35)$$

C. Calculation of the vector form factors

In this part, we calculate the vector FF, and it can be extracted from the matrix element of the vector current via

$$\begin{aligned} \langle 0 | \bar{q}^m \gamma^\mu q^n | P(p_1) Q(p_2) \rangle &\equiv (p_2 - p_1)^\mu F_{V+,PQ}^{mn}(s) \\ &+ (p_2 + p_1)^\mu F_{V-,PQ}^{mn}(s), \end{aligned} \quad (36)$$

where $F_{V+,PQ}^{mn}(s)$ corresponds to the vector FF [hereinafter abbreviated as $F_{+,PQ}^{mn}(s)$ in this work] and $F_{V-,PQ}^{mn}(s)$ is the scalar component of matrix element. The scalar FFs discussed previously can be related to the quantities of $F_{V+,PQ}^{mn}(s)$ and $F_{V-,PQ}^{mn}(s)$ [1]. After taking into account the charge-conjugate and isospin symmetries, there are seven types of independent two-meson vector FFs, as illustrated in Table II. The perturbative vector FFs in R χ T can be obtained by calculating the corresponding Feynman diagrams in Fig. 1, where wavy lines should be taken as the vector external sources. The explicit LO expressions of the seven independent vector FFs read

$$\begin{aligned} F_{+,K^+K^-}^{\bar{u}u+\bar{d}d,\text{LO}} &= -\frac{1}{\sqrt{2}}, & F_{+,K^+K^-}^{\bar{s}s,\text{LO}} &= 1, \\ F_{+,\pi^+\pi^0}^{\bar{u}d,\text{LO}} &= -\sqrt{2}, & F_{+,\bar{K}^0K^-}^{\bar{u}d,\text{LO}} &= -1, & F_{+,K^-\pi^0}^{\bar{u}s,\text{LO}} &= -\frac{1}{\sqrt{2}}, \\ F_{+,K^-\eta}^{\bar{u}s,\text{LO}} &= -\frac{\sqrt{6}}{2}c_\theta, & F_{+,K^-\eta'}^{\bar{u}s,\text{LO}} &= -\frac{\sqrt{6}}{2}s_\theta. \end{aligned} \quad (37)$$

Because of the lengthy formulas, the complete expressions for these seven perturbative vector FFs are not displayed in

TABLE II. Relevant two-meson channels coupled to vector currents.

Isospin, strangeness	Vector currents	Two-meson channels
$I = 0, S = 0$	$\frac{\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d}{\sqrt{2}}$	$K\bar{K}$
	$\bar{s}\gamma^\mu s$	$K\bar{K}$
$I = 1, S = 0$	$\bar{u}\gamma^\mu d$	$\pi\pi, K\bar{K}$
$I = \frac{1}{2}, S = -1$	$\bar{u}\gamma^\mu s$	$\bar{K}\pi, \bar{K}\eta, \bar{K}\eta'$

the main text, but, instead, are presented in Eqs. (A4)–(A10) of the Appendix.

Next, we perform the unitarization of the vector FF $F_+^I(s)$ with definite isospin I , whose unitarity relation can be expressed in the matrix form as [24]

$$\text{Im } F_+^I(s) = \tilde{Q}(s)^{-1} \cdot T^{IJ}(s)^* \cdot \rho(s) \cdot \tilde{Q}(s) \cdot F_+^I(s), \quad (38)$$

where the diagonal matrix $\tilde{Q}$ is defined by $\tilde{Q}_{ij}(s) = q_j(s)\delta_{ij}$ and $J = 1$ should be taken for vector FFs. Following the unitarization procedure presented in the previous subsection, we build the unitarized vector FFs

$$F_+^I(s) = \tilde{Q}(s)^{-1} \cdot [1 + N^{IJ}(s) \cdot g^{IJ}(s)]^{-1} \cdot \tilde{Q}(s) \cdot R_+^I(s), \quad (39)$$

$$\begin{aligned} R_+^I(s) &= F_+^I(s)^{(2)+\text{Res}+\text{Loop}} + \tilde{Q}(s)^{-1} \cdot N^{IJ}(s)^{(2)} \\ &\cdot g^{IJ}(s) \cdot \tilde{Q}(s) \cdot F_+^I(s)^{(2)}, \end{aligned} \quad (40)$$

where N^{IJ} has been given in Eq. (27) and the superscripts (2), Res, and Loop stand for the contributions from the LO, tree-level resonance exchange, and pNGB one-loop diagrams, respectively, to the vector FFs in R χ T. Notice that we have factorized out the matrix of momenta in Eq. (39) because the interactions of the meson-meson channels with the vector sources always involve it, which is actually a basic step to end with the standard definition of form factors. It is straightforward to verify that the unitarized vector FF of Eq. (39) and the unitarized meson-meson scattering amplitude of Eq. (31) satisfy the unitarity relation shown in Eq. (38).

By using the seven independent types of unitarized vector FFs in the isospin bases as given in Table II, one can straightforwardly compute the two-meson vector FFs of different physical states. The relations of the vector FFs with isoscalar quark currents for different physical states are

$$F_{+,K^+K^-}^{\bar{u}u+\bar{d}d} = F_{+,K^0\bar{K}^0}^{\bar{u}u+\bar{d}d}, \quad F_{+,K^+K^-}^{\bar{s}s} = F_{+,K^0\bar{K}^0}^{\bar{s}s}. \quad (41)$$

The vector FFs associated with isovector quark currents for different physical states satisfy the following relations:

$$\begin{aligned} F_{+,\pi^+\pi^-}^{\bar{u}u-\bar{d}d} &= F_{+,\pi^-\pi^0}^{\bar{u}d} = -F_{+,\pi^+\pi^0}^{\bar{d}u}, \\ F_{+,K^0\bar{K}^0}^{\bar{u}u-\bar{d}d} &= -F_{+,K^+K^-}^{\bar{u}u-\bar{d}d} = -\frac{1}{\sqrt{2}}F_{+,\bar{K}^0K^-}^{\bar{u}d} = -\frac{1}{\sqrt{2}}F_{+,K^+\bar{K}^0}^{\bar{d}u}. \end{aligned} \quad (42)$$

And for the vector quark currents with $I = 1/2$, we have the following relationships:

$$\begin{aligned}
 F_{+,K^+\pi^0}^{\bar{s}u} &= \frac{1}{\sqrt{2}} F_{+,K^0\pi^+}^{\bar{s}u} = -F_{+,K^-\pi^0}^{\bar{u}s} = -\frac{1}{\sqrt{2}} F_{+,\bar{K}^0\pi^-}^{\bar{u}s} \\
 &= -F_{+,K^0\pi^0}^{\bar{s}d} = \frac{1}{\sqrt{2}} F_{+,K^+\pi^-}^{\bar{s}d} = F_{+,\bar{K}^0\pi^0}^{\bar{d}s} \\
 &= -\frac{1}{\sqrt{2}} F_{+,K^-\pi^+}^{\bar{d}s}, \\
 F_{+,K^+\eta'}^{\bar{s}u} &= -F_{+,K^-\eta'}^{\bar{u}s} = F_{+,K^0\eta'}^{\bar{s}d} = -F_{+,\bar{K}^0\eta'}^{\bar{d}s}. \quad (43)
 \end{aligned}$$

D. Calculation of the tensor form factors

The tensor FFs are defined by the following matrix elements of the tensor currents:

$$\langle 0 | \bar{q}^m \sigma^{\mu\nu} q^n | P(p_1) Q(p_2) \rangle \equiv i \frac{\Lambda_2^T}{F_\pi^2} (p_1^\mu p_2^\nu - p_1^\nu p_2^\mu) F_{T,PQ}^{mn}(s), \quad (44)$$

where $F_{T,PQ}^{mn}(s)$ stands for the tensor FFs. After calculating the Feynman diagrams in Fig. 1 by taking the external sources as the tensor ones, one can obtain the perturbative results of the various tensor FFs. The relevant two-meson channels coupled to the various tensor quark currents are the same as the vector cases in Table II. The LO results for the seven independent tensor FFs are

$$\begin{aligned}
 F_{T,K^+K^-}^{\frac{\bar{u}u+\bar{d}d}{\sqrt{2}},\text{LO}} &= -\frac{1}{\sqrt{2}}, & F_{T,K^+K^-}^{\bar{s}s,\text{LO}} &= 1, \\
 F_{T,\pi^-\pi^0}^{\bar{u}d,\text{LO}} &= -\sqrt{2}, & F_{T,K^0K^-}^{\bar{u}d,\text{LO}} &= -1, \\
 F_{T,K^-\pi^0}^{\bar{u}s,\text{LO}} &= -\frac{1}{\sqrt{2}}, & F_{T,K^-\eta}^{\bar{u}s,\text{LO}} &= -\frac{\sqrt{6}}{2} c_\theta, \\
 F_{T,K^-\eta'}^{\bar{u}s,\text{LO}} &= -\frac{\sqrt{6}}{2} s_\theta. \quad (45)
 \end{aligned}$$

The full expressions of these seven independent perturbative tensor FFs can be found in Eqs. (A11)–(A17) in the Appendix.

To account for the nonperturbative meson-meson interactions, one can use a similar formalism as the vector case

in Eq. (39) to perform the unitarization for the tensor FFs, i.e.,

$$F_T^I(s) = \tilde{Q}(s)^{-1} \cdot [1 + N^{IJ}(s) \cdot g^{IJ}(s)]^{-1} \cdot \tilde{Q}(s) \cdot R_T^I(s), \quad (46)$$

with

$$\begin{aligned}
 R_T^I(s) &= F_T^I(s)^{(2)+\text{Res}+\text{Loop}} + \tilde{Q}(s)^{-1} \cdot N^{IJ}(s)^{(2)} \\
 &\quad \cdot g^{IJ}(s) \cdot \tilde{Q}(s) \cdot F_T^I(s)^{(2)}. \quad (47)
 \end{aligned}$$

The unitarized tensor FFs in Eq. (46) satisfy the unitarity relations

$$\text{Im } F_T^I(s) = \tilde{Q}(s)^{-1} \cdot T^{IJ}(s)^* \cdot \rho(s) \cdot \tilde{Q}(s) \cdot F_T^I(s). \quad (48)$$

It is noted that $J = 1$ has been taken when building the unitarized tensor FFs. The tensor FFs in the isospin bases used in the unitarization procedure are constructed according to the conventions of Eqs. (17)–(19). It turns out that the relations among the tensor FFs for the physical states are the same as those in the vector case, given in Eqs. (41)–(43); see, for example, the discussion in Ref. [35].

III. PHENOMENOLOGICAL DISCUSSIONS

In this section, we present the phenomenological results for the scalar, vector, and tensor FFs. All the unknown parameters entering the various FFs have already been determined in previous studies of $U(3)$ meson-meson scattering [23,41], except for F_V in Eq. (6), which enters the vector FFs, and the tensor-current couplings Λ_2^T in Eq. (12) and F_V^T in Eq. (14). For the vector sector, we take $F_V = 130$ MeV, which is close to the large- N_C estimate $\sqrt{2}F_\pi$ [9]. For the tensor sector, the couplings are fixed to $\Lambda_2^T = 12$ MeV and $F_V^T = F_V/\sqrt{2}$ in order to facilitate a direct comparison with the results of Ref. [31]. For all remaining parameters, we adopt the results obtained from the best fit in Ref. [23], including both the central values and the associated uncertainties, as our baseline setup. This choice will be referred to as fit I in the following discussions. For comparison purposes, we also consider the parameter set determined in Ref. [41], which we denote as fit II.

To be specific, we explicitly give the values of the different parameters in fit I below,

$$\begin{aligned}
 c_d &= (19.8_{-5.2}^{+2.0}) \text{ MeV}, & c_m &= (41.9_{-9.2}^{+3.9}) \text{ MeV}, & M_{S_8} &= (1397_{-61}^{+73}) \text{ MeV}, \\
 M_{S_1} &= (1100_{-63}^{+30}) \text{ MeV}, & M_\rho &= (801.2_{-6.9}^{+8.2}) \text{ MeV}, & M_{K^*} &= (910.0_{-9.1}^{+7.0}) \text{ MeV}, \\
 G_V &= (62.1_{-2.1}^{+1.9}) \text{ MeV}, & M_0 &= (951_{-50}^{+50}) \text{ MeV}, & a_{SL}^{00,\pi\pi} &= -1.27_{-0.12}^{+0.12}, \\
 a_{SL}^{\frac{1}{2}0,K\pi} &= -1.12_{-0.17}^{+0.12}, & a_{SL}^{\frac{1}{2}0,K\eta'} &= -1.25_{-1.23}^{+1.11}, & a_{SL}^{10,\pi\eta} &= 2.0_{-4.5}^{+3.3}, & a_{SL}^{00,K\bar{K}} &= -0.95_{-0.16}^{+0.33}, \\
 a_{SL}^{\frac{1}{2}0,K\eta} &= -0.08_{-1.04}^{+0.38}, & \delta L_8 &= 0.23_{-0.19}^{+0.29} \times 10^{-3}, & \Lambda_2 &= -0.37_{-0.19}^{+0.19}. \quad (49)
 \end{aligned}$$

See Ref. [23] for details about the way they are determined. It should be noted that we use M_ρ to estimate the numerical value of the parameter M_V . The large- N_C relation is imposed for the scalar resonance couplings in this fit, namely, $\tilde{c}_d = \frac{c_d}{\sqrt{3}}$ and $\tilde{c}_m = \frac{c_m}{\sqrt{3}}$. The parameters related to the pseudoscalar resonances are fixed at $d_m = \sqrt{3}\tilde{d}_m = 30$ MeV and $M_{P_8} = M_{P_1} = 1350$ MeV [23]. The relationships between the remaining subtraction constants used in the former reference read

$$\begin{aligned} a_{SL}^{00,\eta\eta} &= a_{SL}^{00,\eta\eta'} = a_{SL}^{00,\eta'\eta'} = a_{SL}^{00,K\bar{K}}, \\ a_{SL}^{20,\pi\pi} &= a_{SL}^{00,\pi\pi}, \quad a_{SL}^{\bar{3}0,K\pi} = a_{SL}^{\bar{4}0,K\pi}, \\ a_{SL}^{10,\pi\eta'} &= a_{SL}^{10,K\bar{K}} = a_{SL}^{00,K\bar{K}}. \end{aligned} \quad (50)$$

Moreover, all subtraction constants for the vector channels are taken to be equal to $a_{SL}^{00,\pi\pi}$. Reference [23] also includes an additional multiplet of bare scalar resonances above 2 GeV. Although these states primarily serve as a smooth background contribution in the energy region below 1.5 GeV, they are included in our analysis to maintain consistency with the original fitting setup; we refer the reader to Ref. [23] for further details. For fit II, we do not explicitly list the corresponding parameter values here and instead refer to Ref. [41] for the complete setup.

The determinations of the subtraction constants in the isovector scalar scatterings with $\pi\eta$, $K\bar{K}$, and $\pi\eta'$ coupled channels, i.e., $a_{SL}^{10,\pi\eta} = 0.56$ and $a_{SL}^{10,K\bar{K}} = a_{SL}^{10,\pi\eta'} = -1.62$, are later updated in Ref. [67] by introducing in the fit the precise finite-volume energy levels from the lattice QCD simulations [68]. We will denote such a fit as fit III in later discussions.

With the aforementioned parameters from the fits of Refs. [23,41,67], a plethora of resonances have been found in the meson-meson scattering amplitudes, including the isoscalar scalars $f_0(500)$, $f_0(980)$, and $f_0(1370)$, the isospin-1/2 scalars $K_0^*(700)$ and $K_0^*(1430)$, the isovector scalars $a_0(980)$ and $a_0(1450)$, and the vectors $\rho(770)$, $K^*(892)$, and $\phi(1020)$. Interestingly, these resonances manifest themselves in rather different ways for the various two-meson FFs, as we will discuss below.

A. Results of the scalar form factors

For the scalar FFs with the isoscalar quark currents, although the curves have been explicitly provided for the $\pi\pi$ channel in Ref. [23], we repeat the results here for the sake of completeness, and the error bands are further included in Fig. 3. The $f_0(500)$ shows up as the mild bump in $F_{S,\pi\pi}^{(\bar{u}u+\bar{d}d)/\sqrt{2}}$, i.e., the $\pi\pi$ scalar FF of the $(\bar{u}u + \bar{d}d)/\sqrt{2}$ current, while it is barely seen in $F_{S,\pi\pi}^{\bar{s}s}$, viz. the $\pi\pi$ scalar FF of the $\bar{s}s$ current. The $f_0(980)$ shows rather different signals in $F_{S,\pi\pi}^{(\bar{u}u+\bar{d}d)/\sqrt{2}}$ and $F_{S,\pi\pi}^{\bar{s}s}$, where a prominent kink structure emerges in the former case and an obvious narrow peak manifests in the latter. Broad bumps appearing in both of them around 1.4 GeV are caused by the $f_0(1370)$ resonance.

The curves corresponding to the $K\bar{K}$ scalar FFs of the isoscalar quark currents $(\bar{u}u + \bar{d}d)/\sqrt{2}$ and $\bar{s}s$ are given in Fig. 4. Prominent peaks around the $K\bar{K}$ thresholds, where the $f_0(980)$ resonance resides, show up in both FFs; whereas the $f_0(1370)$ signal is not profound in these two FFs. For the $\eta\eta$ scalar FFs, as illustrated in Fig. 5, similar magnitudes are seen for $F_{S,\eta\eta}^{(\bar{u}u+\bar{d}d)/\sqrt{2}}$ and $F_{S,\eta\eta}^{\bar{s}s}$.

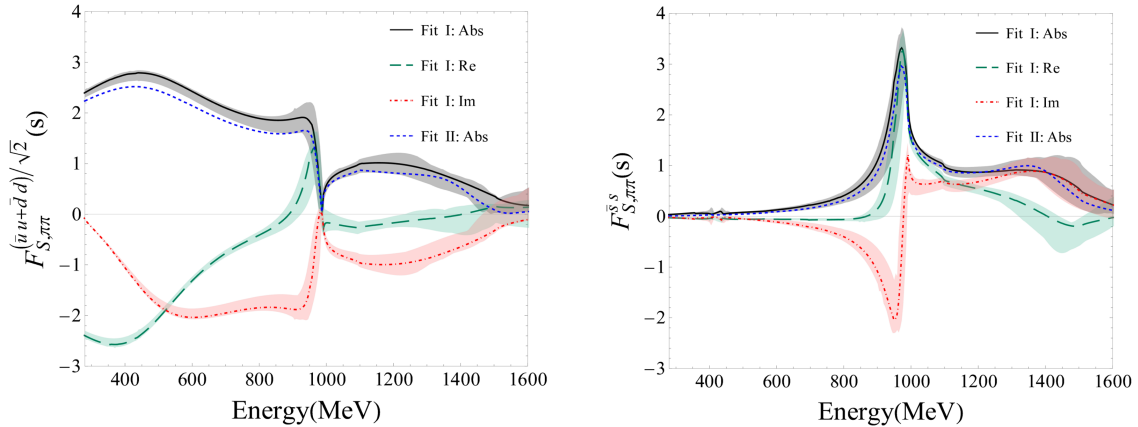


FIG. 3. Scalar $\pi\pi$ FFs of the isoscalar quark currents: $F_{S,\pi\pi}^{(\bar{u}u+\bar{d}d)/\sqrt{2}}$ (left) and $F_{S,\pi\pi}^{\bar{s}s}$ (right). In each panel, the black solid curve shows the modulus as a function of the $\pi\pi$ energy, obtained with the parameter set of fit I. The light-gray shaded areas indicate the corresponding error bands obtained by propagating the parameter uncertainties of fit I. The real and imaginary parts from fit I are denoted by the green long-dashed and red dot-dashed curves, respectively; the light-green and light-pink shaded areas indicate the corresponding error bands, also propagated from the fit I parameter uncertainties. The blue dashed curve in each panel shows the modulus obtained with the parameter set of fit II. In order to avoid the overloading of the figures, only the central curves for the modulus from fit II are provided.

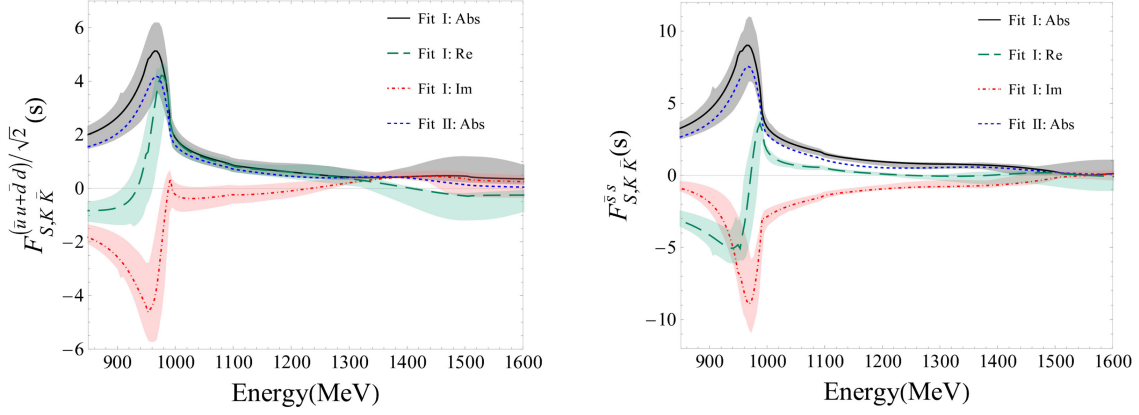


FIG. 4. Scalar $K\bar{K}$ FFs of the isoscalar quark currents: $F_{S,K\bar{K}}^{(\bar{u}u+\bar{d}d)/\sqrt{2}}$ (left) and $F_{S,K\bar{K}}^{\bar{s}s}$ (right). The line styles and shaded bands follow the same conventions as in Fig. 3.

The scalar $\pi\eta$, $K\bar{K}$, and $\pi\eta'$ FFs of the isovector quark current $\bar{u}d$ are illustrated in Fig. 6, where we show the results of fit I and fit III, corresponding to the best fits of Refs. [23,67], respectively. For the isovector scalar $\pi\eta$ and $K\bar{K}$ FFs, the most obvious structures appear around the $K\bar{K}$ threshold, where the $a_0(980)$ resonance resides. The higher $a_0(1450)$ resonance manifests as a clear bump around 1.4 GeV in the $\pi\eta$ channel, while its signal is not very visible in the $K\bar{K}$ case. The $\pi\eta'$ FF is smooth around the $\pi\eta'$ threshold, since the $a_0(980)$ resonance lies below this threshold. The excited $a_0(1450)$ does not behave as a bump in the $\pi\eta'$ FF.

The $\bar{K}\pi$, $\bar{K}\eta$, and $\bar{K}\eta'$ scalar FFs of the isospin-1/2 quark current $\bar{u}s$ are exhibited in Fig. 7. The magnitude trends of the three different FFs with respect to the two-meson energies seem roughly similar; namely, the excited scalar $K_0^*(1430)$ manifests like a shoulder around 1.4 GeV in all of them. The low-lying broad $K_0^*(700)$ scalar resonance is not very apparent in the $\bar{K}\pi$ FF.

B. Results of the vector form factors

In this section, we perform the phenomenological analyses of the various two-meson vector FFs. The plots

corresponding to the $\pi\pi$ and $K\bar{K}$ vector FFs of the isovector quark current $\bar{u}\gamma^\mu d$ are illustrated in Fig. 8. The $\rho(770)$ resonance peak is prominent and also dominates the line shapes of both $F_{+\pi\pi}^{\bar{u}d}$ and $F_{+K\bar{K}}^{\bar{u}d}$ in the focused energy region below 1.2 GeV, though the $\rho(770)$ lies far below the $K\bar{K}$ threshold.

The curves representing the vector $\bar{K}\pi$ and $\bar{K}\eta$ FFs of the isospin-1/2 quark current $\bar{u}\gamma^\mu s$ are given in Fig. 9. Similarly, the $K^*(892)$ resonance peak is noticeable and rules the spectra of both $F_{+\bar{K}\pi}^{\bar{u}s}$ and $F_{+\bar{K}\eta}^{\bar{u}s}$ in the focused energy region below 1.2 GeV, albeit the $K^*(892)$ lies below the $\bar{K}\eta$ threshold.

In Fig. 10, we present the results of the vector $K\bar{K}$ FFs associated with the isoscalar quark currents $(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d)/\sqrt{2}$ and $\bar{s}\gamma^\mu s$. In the case of $F_{+K\bar{K}}^{\bar{s}s}$, a pronounced $\phi(1020)$ symmetric peak appears and provides the dominant contribution in the energy region below 1.2 GeV. By contrast, an asymmetric peak around 1 GeV is observed in $F_{+K\bar{K}}^{(\bar{u}u+\bar{d}d)/\sqrt{2}}$; also, a zero appears just above 1 GeV, a feature not seen in the other vector FFs. This zero originates from the inclusion of the bare ϕ resonance in $N^{IJ}(s)$ and enters this form factor indirectly through a kaon loop, owing to the

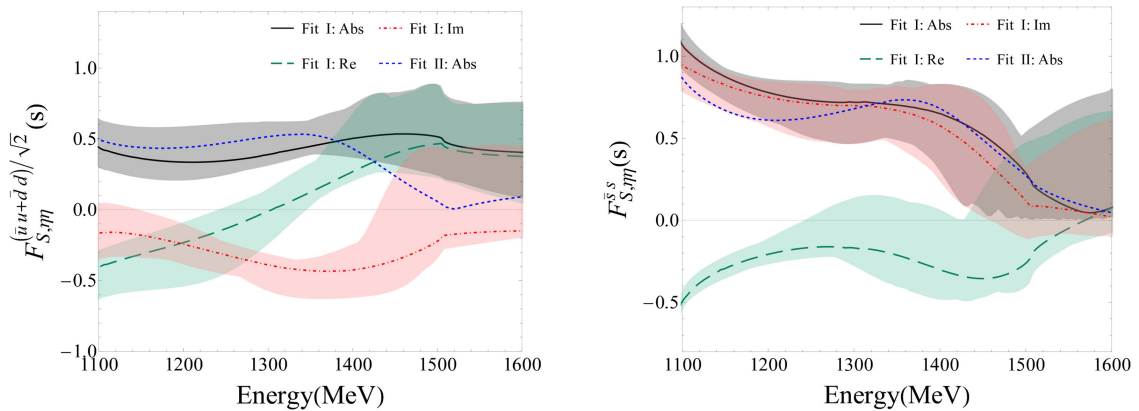


FIG. 5. Scalar $\eta\eta$ FFs: $F_{S,\eta\eta}^{(\bar{u}u+\bar{d}d)/\sqrt{2}}$ (left) and $F_{S,\eta\eta}^{\bar{s}s}$ (right). The line styles and shaded bands follow the same conventions as in Fig. 3.

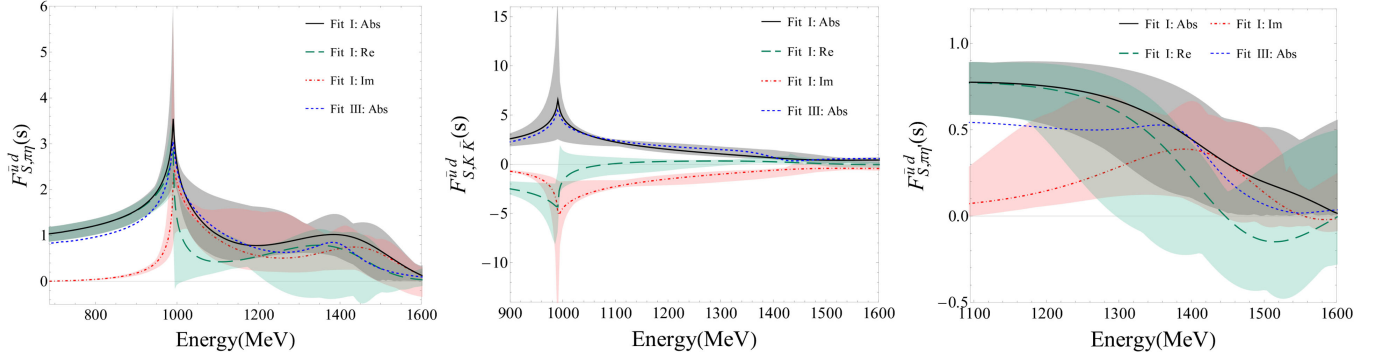


FIG. 6. Scalar $\pi\eta$ (left), $K\bar{K}$ (middle), and $\pi\eta'$ (right) FFs of the isovector quark current $\bar{u}d$: $F_{S,\pi\eta}^{\bar{u}d}$, $F_{S,K\bar{K}}^{\bar{u}d}$, and $F_{S,\pi\eta'}^{\bar{u}d}$. The blue dashed curves show the modulus using the parameter set of the NLO fit obtained in Ref. [67]. All other line styles and shaded bands follow the same conventions as in Fig. 3.

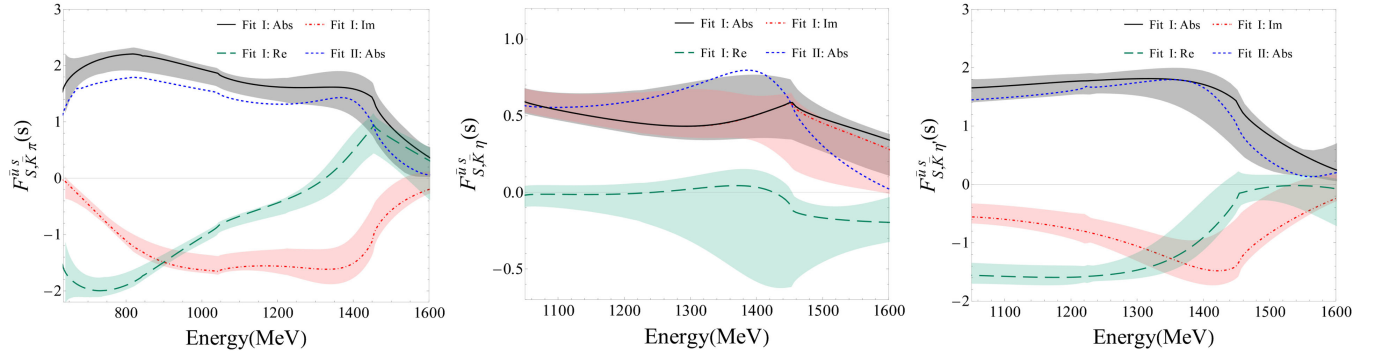


FIG. 7. Scalar $\bar{K}\pi$ (left), $\bar{K}\eta$ (middle), and $\bar{K}\eta'$ (right) FFs of the isospin-1/2 quark current $\bar{u}s$: $F_{S,\bar{K}\pi}^{\bar{u}s}$, $F_{S,\bar{K}\eta}^{\bar{u}s}$, and $F_{S,\bar{K}\eta'}^{\bar{u}s}$. The line styles and shaded bands follow the same conventions as in Fig. 3.

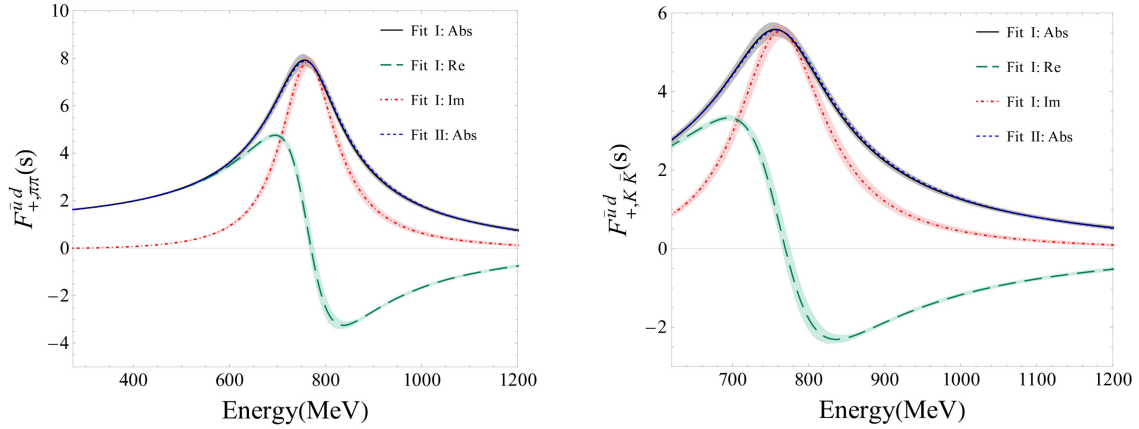


FIG. 8. Vector $\pi\pi$ (left) and $K\bar{K}$ (right) FFs of the isovector quark current $\bar{u}\gamma^\mu d$: $F_{+,\pi\pi}^{\bar{u}d}$ and $F_{+,K\bar{K}}^{\bar{u}d}$. The line styles and shaded bands follow the same conventions as in Fig. 3.

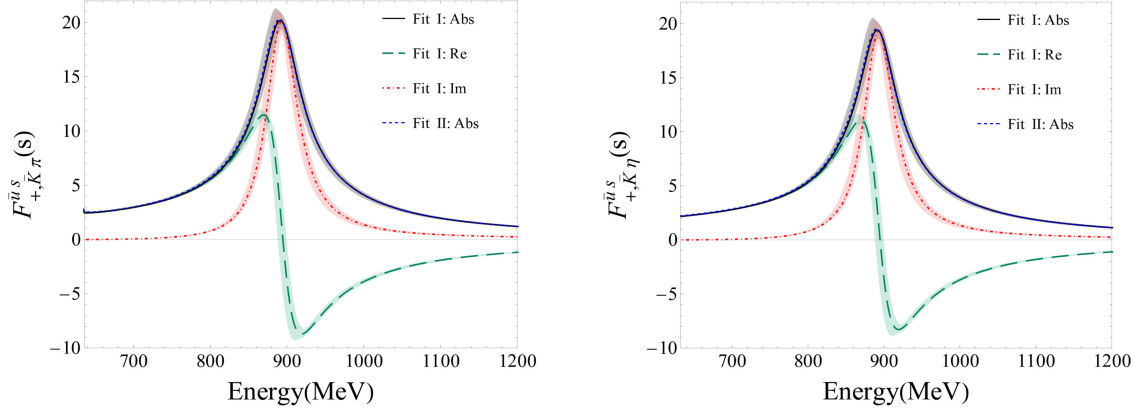


FIG. 9. Vector $\bar{K}\pi$ (left) and $\bar{K}\eta$ (right) FFs of the isospin-1/2 quark current $\bar{u}\gamma^\mu s$: $F_{+,K\pi}^{\bar{u}s}$ and $F_{+,K\eta}^{\bar{u}s}$. The line styles and shaded bands follow the same conventions as in Fig. 3.

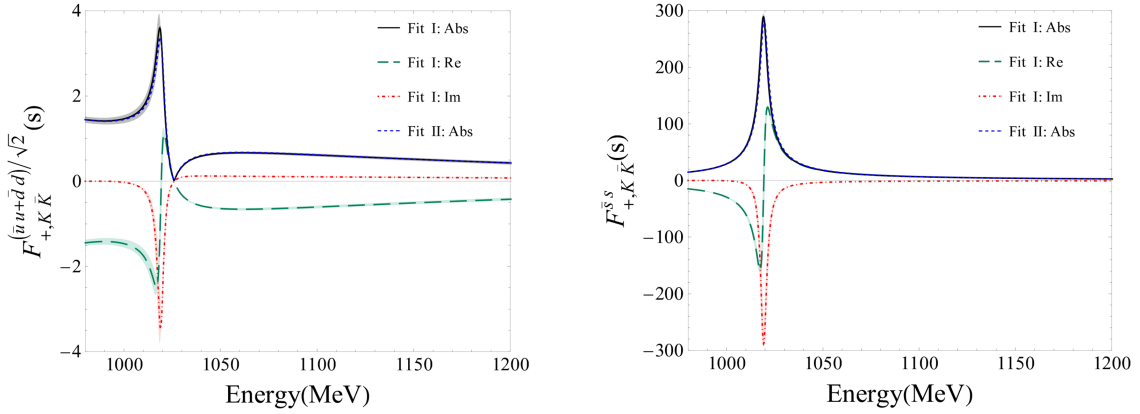


FIG. 10. Vector $K\bar{K}$ FFs of the isoscalar quark currents $(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d)/\sqrt{2}$ (left) and $\bar{s}\gamma^\mu s$ (right): $F_{+,K\bar{K}}^{(\bar{u}u+\bar{d}d)/\sqrt{2}}$ and $F_{+,K\bar{K}}^{\bar{s}s}$. The line styles and shaded bands follow the same conventions as in Fig. 3.

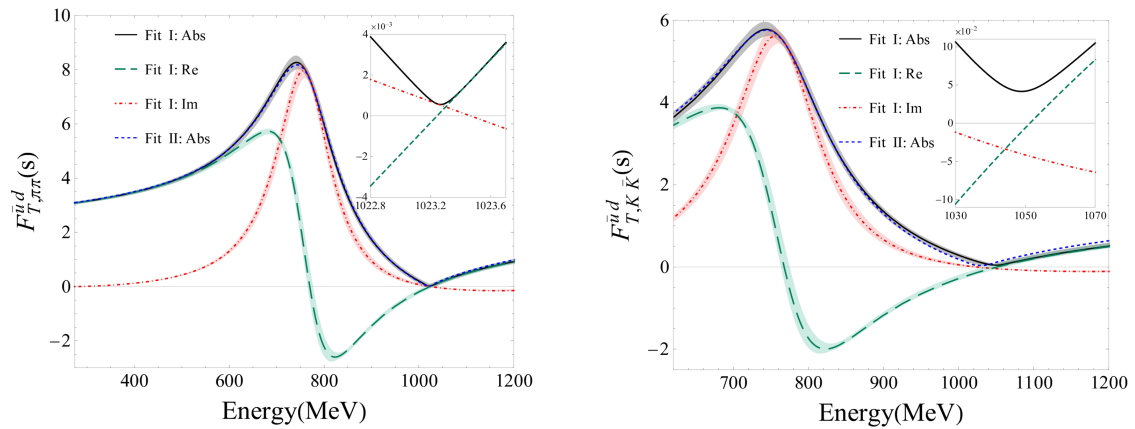


FIG. 11. Tensor $\pi\pi$ (left) and $K\bar{K}$ (right) FFs of the isovector quark current $\bar{u}\sigma^{\mu\nu}d$: $F_{T,\pi\pi}^{\bar{u}d}$ and $F_{T,K\bar{K}}^{\bar{u}d}$. The insets in the upper-right corners show magnified views in the energy regions where the FFs approach zero. The line styles and shaded bands follow the same conventions as in Fig. 3.

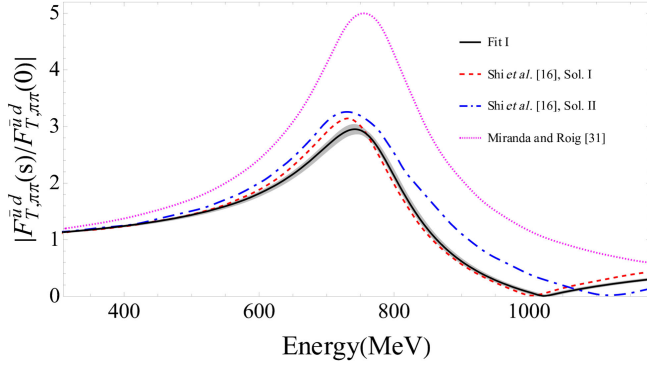


FIG. 12. Comparison of the normalized $|F_{T,\pi\pi}^{ud}(s)/F_{T,\pi\pi}^{ud}(0)|$ obtained using different approaches. The black solid line shows the result of fit I from the present work, while the light-gray shaded band indicates the uncertainty range obtained from the parameter errors of fit I. The red dashed line and the blue dash-dotted line represent the curves obtained using the parameter values from original fit and fit 2 in Ref. [16], respectively, and are labeled as Sol. I and Sol. II in the figure. The magenta dotted line corresponds to the result of Ref. [31], obtained using dispersive representation.

product $N^{IJ}(s)g^{IJ}(s)$ appearing in the denominator of Eq. (39). At higher orders in the approach, one expects this zero to be replaced by a minimum once the perturbative calculation of the form factor is extended to include the full one-loop contribution with bare resonances in the vertices.

C. Results of the tensor form factors

In this subsection, we analyze the phenomenological consequences of the two-meson tensor FFs. In the energy region considered here, below approximately 1.2 GeV, the dominant underlying dynamics governing the various tensor FFs are similar to the vector ones, since they are largely ruled by the ground-state vector resonances. However, for the isovector and isospin- $\frac{1}{2}$ channels, some obvious differences are evident between

the vector and tensor FFs, being the presence of the approximate zeros in the latter case above 1 GeV. Likewise, for the isoscalar channel of the $(\bar{u}u + \bar{d}d)/\sqrt{2}$ quark current, the tensor FF develops an additional zero around 1.1 GeV compared with the vector one. For the tensor FFs, we verify that the perturbative expressions give zeros in the region around 1.0–1.1 GeV. Such perturbative zero points are only marginally affected by the unitarization procedure.

In Fig. 11, we show the plots of the $\pi\pi$ and $K\bar{K}$ tensor FFs for the isovector quark current $\bar{u}\sigma^{\mu\nu}d$, both of which are dominated by the $\rho(770)$ for $E < 1.2$ GeV. It is noted that the two FFs exhibit near-zero structures in the energy region around 1 GeV. Magnified views of these near-zero regions are shown in the insets of Fig. 11. In Fig. 12, we compare the normalized quantity $|F_{T,\pi\pi}^{ud}(s)/F_{T,\pi\pi}^{ud}(0)|$ from our study with those in Refs. [16,31]. The magnitude obtained in Ref. [31] is clearly larger than others. The results of Ref. [16] from the $SU(3)$ inverse amplitude method (IAM), by taking different parameters, look roughly similar to ours from the unitarized $R_\chi T$ amplitude. However, differences are also seen: (1) the magnitudes of the FF in Ref. [16] are a bit larger than ours; (2) slight variations are observed for the positions of the near-zero points in $|F_{T,\pi\pi}^{ud}(s)|$. Regarding the quantity of $F_{T,\pi\pi}^{ud}(0)$, somewhat different values arise from different approaches; e.g., our prediction of $F_{T,\pi\pi}^{ud}(0) = 2.9 \pm 0.1$ is larger than the value of 1.8 from the IAM calculation in Ref. [16] and the LO value of 1.4 used by the dispersive method in Ref. [31]. Nevertheless, it is pointed out that the result of $F_{T,\pi\pi}^{ud}(0)$ in the IAM approach [16] depends on the poorly determined $O(p^6)$ LECs, which, in principle, bear rather large uncertainties.

In Fig. 13, we illustrate the results of the $\bar{K}\pi$ and $\bar{K}\eta$ tensor FFs for the isospin-1/2 quark current $\bar{u}\sigma^{\mu\nu}s$, both of which are mostly ruled by the $K^*(892)$ resonance for $E < 1.2$ GeV. Similar to Fig. 11, the

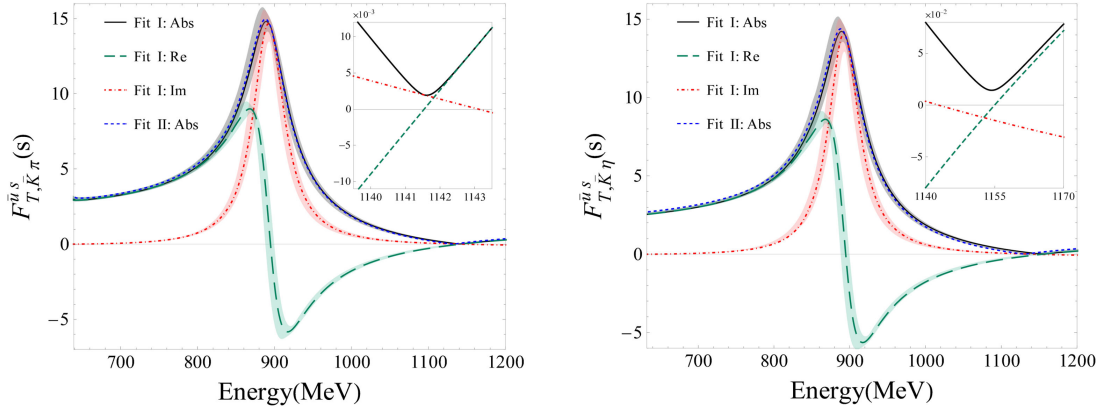


FIG. 13. Tensor $\bar{K}\pi$ (left) and $\bar{K}\eta$ (right) FFs of the isospin-1/2 quark current $\bar{u}\sigma^{\mu\nu}s$: $F_{T,K\pi}^{us}$ and $F_{T,K\eta}^{us}$. The insets in the upper-right corners show magnified views in the energy regions where the FFs approach zero. The line styles and shaded bands follow the same conventions as in Fig. 3.

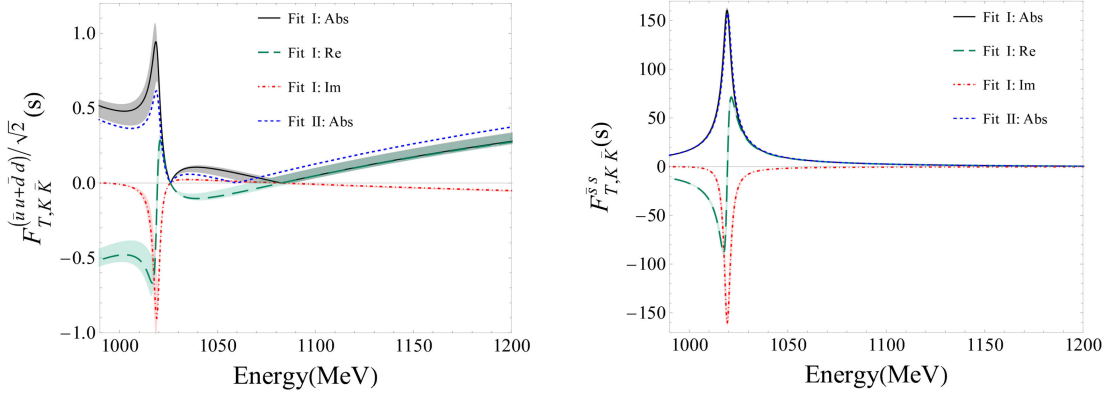


FIG. 14. Tensor $K\bar{K}$ FFs of the isoscalar quark currents $(\bar{u}\sigma^{\mu\nu}u + \bar{d}\sigma^{\mu\nu}d)/\sqrt{2}$ (left) and $\bar{s}\sigma^{\mu\nu}s$ (right): $F_{T,K\bar{K}}^{(\bar{u}u+\bar{d}d)/\sqrt{2}}$ and $F_{T,K\bar{K}}^{\bar{s}s}$. The line styles and shaded bands follow the same conventions as in Fig. 3.

two FFs exhibit near-zero structures around 1.1 GeV. Magnified views of these near-zero regions are shown in the insets of Fig. 13.

Figure 14 displays the tensor FFs for the $K\bar{K}$ channel induced by the isoscalar quark currents $(\bar{u}\sigma^{\mu\nu}u + \bar{d}\sigma^{\mu\nu}d)/\sqrt{2}$ and $\bar{s}\sigma^{\mu\nu}s$. A clear symmetric resonance peak from the $\phi(1020)$ is found in $F_{T,K\bar{K}}^{\bar{s}s}$, which dominates the behavior of this FF throughout the region below 1.2 GeV. On the other hand, an asymmetric peak structure is seen in $F_{T,K\bar{K}}^{(\bar{u}u+\bar{d}d)/\sqrt{2}}$ around 1 GeV. This latter tensor FF develops two zeros in the energy range between 1.0 and 1.1 GeV. The lighter one originates because of the same mechanism as explained with respect to the analogous vector FF in Fig. 10. The higher zero has its origin due to the zero in the perturbative calculation of the FF.

IV. SUMMARY AND CONCLUSIONS

In this work, we have performed a thorough study of the two-meson FFs of light pseudoscalar mesons within the framework of $U(3)$ resonance chiral theory. At the perturbative level, we have calculated the complete two-meson scalar, vector, and tensor FFs for both the strangeness-conserving and strangeness-changing channels by taking into account both the tree-level resonance-exchange contributions and the pNGB one-loop contributions.

Next, to account for the meson-meson final-state interactions, we have further constructed the unitarized scalar, vector, and tensor FFs by matching the unitarized meson-meson scattering amplitudes to the perturbative results. An important feature of this approach is that the same dynamical ingredients governing meson-meson scattering also play a crucial role in determining the final-state interactions in the FFs. Therefore, once the scattering amplitudes are constrained by the experimental data and relevant event distributions, most of the parameters entering the FFs can be fixed accordingly, so that many of the FFs

are already strongly constrained and can, in many cases, be predicted.

In the numerical calculations, we have mainly employed parameter sets from previous $U(3)$ meson-meson scattering studies, using the best-fit values from Ref. [23] as the baseline input. Based on this fit, all the FFs have been evaluated, and the corresponding uncertainty bands have been provided. For comparison, we have also evaluated the FFs using the parameter values given in Refs. [41,67]. Phenomenologically, we have discussed the main features of various scalar, vector, and tensor FFs in the isoscalar, isovector, and isospin-1/2 channels. In the scalar sector, the scalar resonances $f_0(500)$, $f_0(980)$, $f_0(1370)$, $a_0(980)$, $a_0(1450)$, $K_0^*(700)$, and $K_0^*(1430)$ manifest themselves rather differently depending on the quark current and final-state channel. In the vector sector, the low-energy behavior below 1.2 GeV is largely governed by the ground-state vector resonances, with the $\rho(770)$ dominating the isovector $\pi\pi$ and $K\bar{K}$ FFs, the $K^*(892)$ controlling the $K\pi$ and $K\eta$ channels, and the $\phi(1020)$ producing a pronounced enhancement in the vector FFs of the isoscalar $K\bar{K}$ channel. The tensor FFs exhibit a resonance pattern qualitatively similar to the vector case, again being mainly controlled by the $\rho(770)$, $K^*(892)$, and $\phi(1020)$ in the low-energy region. We have further compared our $\pi\pi$ tensor FF $F_{T,\pi\pi}^{\bar{u}d}(s)$ with previous results obtained using a dispersive approach and the inverse amplitude method, finding overall agreement with the latter. However, small differences remain in both the overall magnitude and the positions of the corresponding near-zero structures.

Our results for the various FFs are expected to provide useful theoretical input for future studies of the hadronic τ decays and the semileptonic D meson decays.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: FULL EXPRESSIONS FOR FORM FACTORS IN $U(3)$ $R_{\chi T}$

In this appendix, we present the complete expressions for the perturbative FFs, including the strangeness-changing scalar FFs as well as all the vector and tensor FFs. The explicit expressions for the isoscalar and isovector scalar FFs can be found in Ref. [23]. It should be particularly noted that, although the original expressions of the FFs are written in terms of the bare decay constant F , the final results are reexpressed in terms of the physical pion decay constant F_π . The relation between the decay constant F in the chiral limit and F_π has been explicitly given in Eq. (C2) of Ref. [41]. The higher-order η - η' mixing matrix elements, δ_k and $s_\delta = \sin \theta_\delta$, which characterize the mixing between the η and η' states beyond LO, are explicitly given in Refs. [23,41] and will not be discussed further here. The explicit expressions of the one-loop functions $A_0(m^2)$ and $B_0(s, m_a^2, m_b^2)$ can also be found in Ref. [41].

The expressions of the strangeness-changing scalar FFs read

$$\begin{aligned}
 F_{S, K^- \pi^0}^{\bar{u}s} &= \frac{F_{S, K^- \pi^0}^{\bar{u}s}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left\{ 1 + \frac{4c_m[c_m(m_K^2 + m_\pi^2) - c_d(m_K^2 + m_\pi^2 - s)]}{F_\pi^2(M_{S_8}^2 - s)} \right. \\
 &\quad - \frac{8(\tilde{c}_d - \tilde{c}_m)\tilde{c}_m(2m_K^2 + m_\pi^2)}{F_\pi^2 M_{S_1}^2} + \frac{4(c_d - c_m)c_m(m_K^2 - m_\pi^2)}{3F_\pi^2 M_{S_8}^2} - \frac{8d_m^2(m_K^2 + m_\pi^2)}{F_\pi^2 M_{F_8}^2} \\
 &\quad + \frac{16(m_K^2 + m_\pi^2)\delta L_8}{F_\pi^2} \left. \right\} + \frac{1}{\sqrt{2}} \left\{ \frac{(m_K^2 - m_\pi^2)A_0(m_K^2)}{64F_\pi^2 \pi^2 s} - \frac{(3m_K^2 - 3m_\pi^2 + 2s)A_0(m_\pi^2)}{128F_\pi^2 \pi^2 s} \right. \\
 &\quad - \frac{[c_\theta^2(-9m_K^2 + 9m_\pi^2 + 6s) + 6\sqrt{2}c_\theta(-3m_K^2 + 3m_\pi^2 + s)s_\theta - 24s s_\theta^2]A_0(m_\eta^2)}{1152F_\pi^2 \pi^2 s} \\
 &\quad + \frac{[8c_\theta^2 s + 2\sqrt{2}c_\theta(-3m_K^2 + 3m_\pi^2 + s)s_\theta + (3m_K^2 - 3m_\pi^2 - 2s)s_\theta^2]A_0(m_{\eta'}^2)}{384F_\pi^2 \pi^2 s} - \frac{B_0(s, m_{\eta'}^2, m_K^2)}{1152F_\pi^2 \pi^2 s} \\
 &\quad \times \left\{ c_\theta^2[-9m_K^4 + (7m_\pi^2 - 9s)s + 3m_{\eta'}^2(3m_K^2 - 3m_\pi^2 + s) + m_K^2(9m_\pi^2 + 2s)] - 16(2m_K^2 + m_\pi^2)ss_\theta^2 \right. \\
 &\quad + 2\sqrt{2}c_\theta[-9m_K^4 + m_K^2(9m_\pi^2 - 2s) + (5m_\pi^2 - 9s)s + 3m_{\eta'}^2(3m_K^2 - 3m_\pi^2 + s)]s_\theta \left. \right\} \\
 &\quad + \frac{B_0(s, m_{\eta'}^2, m_K^2)}{1152F_\pi^2 \pi^2 s} \{ 16c_\theta^2(2m_K^2 + m_\pi^2)s + 2\sqrt{2}c_\theta[-9m_K^4 + m_K^2(9m_\pi^2 - 2s) \\
 &\quad + (5m_\pi^2 - 9s)s + 3m_{\eta'}^2(3m_K^2 - 3m_\pi^2 + s)]s_\theta + [9m_K^4 - (7m_\pi^2 - 9s)s - 3m_{\eta'}^2(3m_K^2 - 3m_\pi^2 + s) \\
 &\quad - m_K^2(9m_\pi^2 + 2s)]s_\theta^2 \} - \frac{[3m_K^4 + 3m_\pi^4 + 2m_\pi^2 s - 5s^2 + m_K^2(-6m_\pi^2 + 2s)]B_0(s, m_K^2, m_\pi^2)}{128F_\pi^2 \pi^2 s} \left. \right\}, \quad (A1)
 \end{aligned}$$

$$\begin{aligned}
 F_{S, K^- \eta}^{\bar{u}s} &= \sqrt{\frac{2}{3}} \left\{ -\left(\frac{c_\theta}{2} + \sqrt{2}s_\theta\right) + \frac{4(\tilde{c}_d - \tilde{c}_m)\tilde{c}_m(2m_K^2 + m_\pi^2)(c_\theta + 2\sqrt{2}s_\theta)}{F_\pi^2 M_{S_1}^2} \right. \\
 &\quad + \frac{2c_m[c_d(m_\eta^2 + m_K^2 - s)(c_\theta + 2\sqrt{2}s_\theta) + c_m(-5c_\theta m_K^2 + 3c_\theta m_\pi^2 - 4\sqrt{2}m_K^2 s_\theta)]}{F_\pi^2(M_{S_8}^2 - s)} \left. \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{2c_m(m_K^2 - m_\pi^2)[-c_m(11c_\theta + 4\sqrt{2}s_\theta) + c_d(c_\theta + 2c_\theta^3 + 2\sqrt{2}s_\theta + 8\sqrt{2}c_\theta^2s_\theta + 16c_\theta s_\theta^2)]}{3F_\pi^2 M_{S_8}^2} \\
& + \frac{1}{4}(s_\theta - 2\sqrt{2}c_\theta)(\delta_k + 2s_\delta) + \frac{1}{4}s_\theta^2(c_\theta + 2\sqrt{2}s_\theta)\Lambda_1 - \sqrt{2}s_\theta\Lambda_2 \\
& + \frac{4d_m^2(5c_\theta m_K^2 - 3c_\theta m_\pi^2 + 4\sqrt{2}m_K^2 s_\theta) - \frac{8(5c_\theta m_K^2 - 3c_\theta m_\pi^2 + 4\sqrt{2}m_K^2 s_\theta)\delta L_8}{F_\pi^2}}{F_\pi^2 M_{P_8}^2} \Bigg\} \\
& + \frac{1}{\sqrt{6}} \left\{ \frac{[c_\theta^3(-9m_\eta^2 + 9m_K^2 - 10s) - 6\sqrt{2}c_\theta^2(3m_\eta^2 - 3m_K^2 + 4s)s_\theta - 24c_\theta s s_\theta^2 - 16\sqrt{2}s s_\theta^3]}{384F_\pi^2 \pi^2 s} A_0(m_\eta^2) \right. \\
& + \frac{[-8c_\theta^3 s + 6\sqrt{2}c_\theta^2(3m_\eta^2 - 3m_K^2 + s)s_\theta + 3c_\theta(-3m_\eta^2 + 3m_K^2 + 2s)s_\theta^2 - 2\sqrt{2}s s_\theta^3]}{384F_\pi^2 \pi^2 s} A_0(m_{\eta'}^2) \\
& + \frac{1}{128F_\pi^2 \pi^2 s} \{3c_\theta^3(m_\eta^2 - m_K^2 + s) + c_\theta[3m_\eta^2(-3 + s_\theta^2) - 3m_K^2(-3 + s_\theta^2) - s(3 + s_\theta^2)] \\
& - 12\sqrt{2}s s_\theta + 8\sqrt{2}c_\theta^2 s s_\theta\} A_0(m_K^2) + \frac{(9c_\theta m_\eta^2 - 9c_\theta m_K^2 + 6c_\theta s - 6\sqrt{2}s s_\theta)}{128F_\pi^2 \pi^2 s} A_0(m_\pi^2) \\
& + \frac{1}{384F_\pi^2 \pi^2 s} \{c_\theta^3[9m_\eta^4 + 9m_K^4 - 18m_K^2 s + 4m_\pi^2 s + 9s^2 - 6m_\eta^2(3m_K^2 + s)] \\
& + 2\sqrt{2}c_\theta^2[9m_\eta^4 + 9m_K^4 - 26m_K^2 s + 8m_\pi^2 s + 9s^2 - 6m_\eta^2(3m_K^2 + s)]s_\theta + 16c_\theta(-5m_K^2 + 2m_\pi^2)s s_\theta^2 \\
& - 32\sqrt{2}m_K^2 s s_\theta^3\} B_0(s, m_\eta^2, m_K^2) + \frac{1}{384F_\pi^2 \pi^2 s} \{16c_\theta^3(-2m_K^2 + m_\pi^2)s \\
& + 6\sqrt{2}c_\theta^2[-3m_K^4 + 2m_K^2 s - 2m_\pi^2 s - 3s^2 + m_{\eta'}^2(3m_K^2 + s) + m_\eta^2(-3m_{\eta'}^2 + 3m_K^2 + s)]s_\theta \\
& + 3c_\theta[3m_K^4 + m_\eta^2(3m_{\eta'}^2 - 3m_K^2 - s) + 10m_K^2 s - 4m_\pi^2 s + 3s^2 - m_{\eta'}^2(3m_K^2 + s)]s_\theta^2 \\
& + 4\sqrt{2}(-2m_K^2 + m_\pi^2)s s_\theta^3\} B_0(s, m_{\eta'}^2, m_K^2) + \frac{1}{128F_\pi^2 \pi^2 s} \{-4\sqrt{2}(2m_K^2 + m_\pi^2)s s_\theta \\
& + c_\theta[-9m_K^4 + (7m_\pi^2 - 9s)s + 3m_\eta^2(3m_K^2 - 3m_\pi^2 + s) + m_K^2(9m_\pi^2 + 2s)]\} B_0(s, m_K^2, m_\pi^2) \Bigg\}, \tag{A2}
\end{aligned}$$

$$\begin{aligned}
F_{S,K-\eta'}^{\bar{u}s} &= \sqrt{\frac{2}{3}} \left\{ \sqrt{2}c_\theta - \frac{s_\theta}{2} - \frac{4(\tilde{c}_d - \tilde{c}_m)\tilde{c}_m(2m_K^2 + m_\pi^2)(2\sqrt{2}c_\theta - s_\theta)}{F^2 M_{S_1}^2} + \frac{1}{4}(\delta_k - 2s_\delta)(c_\theta + 2\sqrt{2}s_\theta) \right. \\
& + \frac{2c_m[c_m(4\sqrt{2}c_\theta m_K^2 - 5m_K^2 s_\theta + 3m_\pi^2 s_\theta) - c_d(m_{\eta'}^2 + m_K^2 - s)(2\sqrt{2}c_\theta - s_\theta)]}{F^2(M_{S_8}^2 - s)} \\
& + \frac{2c_m(m_K^2 - m_\pi^2)[c_m(4\sqrt{2}c_\theta - 11s_\theta) + c_d(s_\theta + 16c_\theta^2 s_\theta + 2s_\theta^3 - 2\sqrt{2}c_\theta - 8\sqrt{2}c_\theta s_\theta^2)]}{3F^2 M_{S_8}^2} \\
& + \frac{8(4\sqrt{2}c_\theta m_K^2 - 5m_K^2 s_\theta + 3m_\pi^2 s_\theta)\delta L_8}{F^2} - \frac{4d_m^2(4\sqrt{2}c_\theta m_K^2 - 5m_K^2 s_\theta + 3m_\pi^2 s_\theta)}{F^2 M_{P_8}^2} \\
& + \frac{c_\theta^2(-2\sqrt{2}c_\theta + s_\theta)\Lambda_1}{4} + \sqrt{2}c_\theta\Lambda_2 \Bigg\} + \frac{1}{\sqrt{6}} \left\{ \frac{A_0(m_\eta^2)}{384F^2 \pi^2 s} [2\sqrt{2}c_\theta^3 s - 6\sqrt{2}c_\theta(3m_\eta^2 - 3m_K^2 + s)s_\theta^2 \right. \\
& - 8s s_\theta^3 + c_\theta^2(-9m_\eta^2 s_\theta + 9m_K^2 s_\theta + 6s s_\theta)] + \frac{1}{384F^2 \pi^2 s} [16\sqrt{2}c_\theta^3 s - 24c_\theta^2 s s_\theta \\
& + 6\sqrt{2}c_\theta(3m_{\eta'}^2 - 3m_K^2 + 4s)s_\theta^2 + (-9m_{\eta'}^2 + 9m_K^2 - 10s)s_\theta^3] A_0(m_{\eta'}^2) + \frac{A_0(m_K^2)}{128F^2 \pi^2 s}
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ c_\theta^2(3m_\eta^2 - 3m_K^2 - s)s_\theta + 4\sqrt{2}c_\theta s(3 - 2s_\theta^2) + 3s_\theta \left[m_\eta^2(-3 + s_\theta^2) - m_K^2(-3 + s_\theta^2) - c_\theta^2 s \right] \right\} \\
& + \frac{(6\sqrt{2}c_\theta s + 9m_\eta^2 s_\theta - 9m_K^2 s_\theta + 6ss_\theta)A_0(m_\pi^2)}{128F_\pi^2 \pi^2 s} - \frac{1}{384F_\pi^2 \pi^2 s} \{ 4\sqrt{2}c_\theta^3(-2m_K^2 + m_\pi^2)s \\
& + 3c_\theta^2[-3m_K^4 - 10m_K^2 s + 4m_\pi^2 s - 3s^2 + m_\eta^2(3m_K^2 + s) + m_\eta^2(-3m_\eta^2 + 3m_K^2 + s)]s_\theta \\
& + 6\sqrt{2}c_\theta[-3m_K^4 + 2m_K^2 s - 2m_\pi^2 s - 3s^2 + m_\eta^2(3m_K^2 + s) + m_\eta^2(-3m_\eta^2 + 3m_K^2 + s)]s_\theta^2 \\
& + 16(2m_K^2 - m_\pi^2)ss_\theta^3 \} B_0(s, m_\eta^2, m_K^2) + \frac{B_0(s, m_\eta^2, m_K^2)}{384F_\pi^2 \pi^2 s} \{ 16c_\theta^2(-5m_K^2 + 2m_\pi^2)ss_\theta \\
& - 2\sqrt{2}c_\theta[9m_\eta^4 + 9m_K^4 - 26m_K^2 s + 8m_\pi^2 s + 9s^2 - 6m_\eta^2(3m_K^2 + s)]s_\theta^2 + 32\sqrt{2}c_\theta^3 m_K^2 s \\
& + [9m_\eta^4 + 9m_K^4 - 18m_K^2 s + 4m_\pi^2 s + 9s^2 - 6m_\eta^2(3m_K^2 + s)]s_\theta^3 \} + \frac{B_0(s, m_K^2, m_\pi^2)}{128F_\pi^2 \pi^2 s} \\
& \times \left\{ 4\sqrt{2}c_\theta(2m_K^2 + m_\pi^2)s + [(7m_\pi^2 - 9s)s - 9m_K^4 + 3m_\eta^2(3m_K^2 - 3m_\pi^2 + s) + m_K^2(9m_\pi^2 + 2s)]s_\theta \right\} \}. \quad (A3)
\end{aligned}$$

The explicit expressions of the various vector FFs are

$$F_{+,K^+K^-}^{\bar{u}u+\bar{d}d} = -\frac{1}{\sqrt{2}} \left\{ 1 + \frac{F_V G_V s}{F_\pi^2(M_\omega^2 - s)} + \frac{s - 6m_K^2}{96F_\pi^2 \pi^2} + \frac{A_0(m_K^2)}{16F_\pi^2 \pi^2} + \frac{(s - 4m_K^2)B_0(s, m_K^2, m_K^2)}{64F_\pi^2 \pi^2} \right\}, \quad (A4)$$

$$F_{+,K^+K^-}^{\bar{s}s} = 1 + \frac{F_V G_V s}{F_\pi^2(M_\phi^2 - s)} + \frac{s - 6m_K^2}{96F_\pi^2 \pi^2} + \frac{A_0(m_K^2)}{16F_\pi^2 \pi^2} - \frac{(4m_K^2 - s)B_0(s, m_K^2, m_K^2)}{64F_\pi^2 \pi^2}, \quad (A5)$$

$$\begin{aligned}
F_{+,\pi^-\pi^0}^{\bar{u}d} = & -\sqrt{2} \left\{ 1 + \frac{F_V G_V s}{F_\pi^2(M_\rho^2 - s)} + \frac{s - 2m_K^2 - 4m_\pi^2}{96F_\pi^2 \pi^2} + \frac{A_0(m_K^2)}{48F_\pi^2 \pi^2} \right. \\
& \left. + \frac{A_0(m_\pi^2)}{24F_\pi^2 \pi^2} + \frac{(s - 4m_K^2)B_0(s, m_K^2, m_K^2)}{192F_\pi^2 \pi^2} + \frac{(s - 4m_\pi^2)B_0(s, m_\pi^2, m_\pi^2)}{96F_\pi^2 \pi^2} \right\}, \quad (A6)
\end{aligned}$$

$$\begin{aligned}
F_{+,K^0K^-}^{\bar{u}d} = & -\left\{ 1 + \frac{F_V G_V s}{F_\pi^2(M_\rho^2 - s)} + \frac{s - 2m_K^2 - 4m_\pi^2}{96F_\pi^2 \pi^2} + \frac{A_0(m_K^2)}{48F_\pi^2 \pi^2} \right. \\
& \left. + \frac{A_0(m_\pi^2)}{24F_\pi^2 \pi^2} + \frac{(s - 4m_K^2)B_0(s, m_K^2, m_K^2)}{192F_\pi^2 \pi^2} + \frac{(s - 4m_\pi^2)B_0(s, m_\pi^2, m_\pi^2)}{96F_\pi^2 \pi^2} \right\}, \quad (A7)
\end{aligned}$$

$$\begin{aligned}
F_{+,K^-\pi^0}^{\bar{u}s} = & \frac{1}{\sqrt{2}} F_{+,\bar{K}^0\pi^-}^{\bar{u}s} = -\frac{1}{\sqrt{2}} \left\{ 1 + \frac{F_V G_V s}{F_\pi^2(M_{K^*}^2 - s)} \right\} + \frac{1}{\sqrt{2}} \left\{ \frac{3m_K^2 + 3m_\pi^2 - s + c_\theta^2(3m_\eta^2 + 3m_K^2 - s)}{192F_\pi^2 \pi^2} \right. \\
& + \frac{(3m_\eta^2 + 3m_K^2 - s)s_\theta^2}{192F_\pi^2 \pi^2} + \frac{c_\theta^2(m_\eta^2 - m_K^2 - 2s)A_0(m_\eta^2)}{128F_\pi^2 \pi^2 s} + \frac{(m_\eta^2 - m_K^2 - 2s)s_\theta^2 A_0(m_\eta^2)}{128F_\pi^2 \pi^2 s} \\
& + \left[\frac{m_K^2 - m_\pi^2 - 5s + c_\theta^2(-m_\eta^2 + m_K^2 + s) + (-m_\eta^2 + m_K^2 + s)s_\theta^2}{128F_\pi^2 \pi^2 s} \right] A_0(m_K^2) \\
& - \frac{(m_K^2 - m_\pi^2 + 2s)A_0(m_\pi^2)}{128F_\pi^2 \pi^2 s} - \frac{c_\theta^2[m_\eta^4 + (m_K^2 - s)^2 - 2m_\eta^2(m_K^2 + s)]B_0(s, m_\eta^2, m_K^2)}{128F_\pi^2 \pi^2 s} \\
& - \frac{s_\theta^2[m_\eta^4 + (m_K^2 - s)^2 - 2m_\eta^2(m_K^2 + s)]B_0(s, m_\eta^2, m_K^2)}{128F_\pi^2 \pi^2 s} \\
& \left. - \frac{[m_K^4 + (m_\pi^2 - s)^2 - 2m_K^2(m_\pi^2 + s)]B_0(s, m_K^2, m_\pi^2)}{128F_\pi^2 \pi^2 s} \right\}, \quad (A8)
\end{aligned}$$

$$\begin{aligned}
F_{+,K^-\eta}^{\bar{u}s} = & -\sqrt{\frac{3}{2}}c_\theta \left\{ 1 + \frac{F_V G_V s}{F_\pi^2(M_{K^*}^2 - s)} \right\} + \frac{4c_d c_m (m_K^2 - m_\pi^2)(-\sqrt{2}c_\theta + \sqrt{2}c_\theta^3 - 2s_\theta + 4c_\theta^2 s_\theta)}{\sqrt{3}F_\pi^2 M_{S_8}^2} \\
& + \frac{1}{2}\sqrt{\frac{3}{2}}\{s_\theta(2s_\delta + \delta_k) + c_\theta s_\theta^2 \Lambda_1\} + \sqrt{\frac{3}{2}} \left\{ \frac{c_\theta[3m_K^2 + 3m_\pi^2 + c_\theta^2(3m_\eta^2 + 3m_K^2 - s) - s]}{192F_\pi^2 \pi^2} \right. \\
& + \frac{c_\theta(3m_{\eta'}^2 + 3m_K^2 - s)s_\theta^2}{192F_\pi^2 \pi^2} + \frac{c_\theta^3(m_\eta^2 - m_K^2 - 2s)A_0(m_\eta^2)}{128F_\pi^2 \pi^2 s} + \frac{c_\theta(m_{\eta'}^2 - m_K^2 - 2s)s_\theta^2 A_0(m_{\eta'}^2)}{128F_\pi^2 \pi^2 s} \\
& + \frac{c_\theta[m_K^2 - m_\pi^2 - 9s + c_\theta^2(-m_\eta^2 + m_K^2 + 5s)] + c_\theta(-m_{\eta'}^2 + m_K^2 + s)s_\theta^2}{128F_\pi^2 \pi^2 s} A_0(m_K^2) \\
& - \frac{c_\theta(m_K^2 - m_\pi^2 + 2s)A_0(m_\pi^2)}{128F_\pi^2 \pi^2 s} - \frac{c_\theta^3[m_\eta^4 + (m_K^2 - s)^2 - 2m_\eta^2(m_K^2 + s)]B_0(s, m_\eta^2, m_K^2)}{128F_\pi^2 \pi^2 s} \\
& - \frac{c_\theta s_\theta^2[m_{\eta'}^4 + (m_K^2 - s)^2 - 2m_{\eta'}^2(m_K^2 + s)]B_0(s, m_{\eta'}^2, m_K^2)}{128F_\pi^2 \pi^2 s} \\
& \left. - \frac{c_\theta[m_K^4 + (m_\pi^2 - s)^2 - 2m_K^2(m_\pi^2 + s)]B_0(s, m_K^2, m_\pi^2)}{128F_\pi^2 \pi^2 s} \right\}, \tag{A9}
\end{aligned}$$

$$\begin{aligned}
F_{+,K^-\eta'}^{\bar{u}s} = & -\sqrt{\frac{3}{2}}s_\theta \left\{ 1 + \frac{F_V G_V s}{F_\pi^2(M_{K^*}^2 - s)} \right\} + \frac{4c_d c_m (m_K^2 - m_\pi^2)[c_\theta(2 - 4s_\theta^2) + \sqrt{2}s_\theta(-1 + s_\theta^2)]}{\sqrt{3}F_\pi^2 M_{S_8}^2} \\
& + \frac{1}{2}\sqrt{\frac{3}{2}}\{c_\theta(-2s_\delta + \delta_k) + c_\theta^2 s_\theta \Lambda_1\} + \frac{1}{\sqrt{2}} \left\{ \frac{[3m_K^2 + 3m_\pi^2 + c_\theta^2(3m_\eta^2 + 3m_K^2 - s) - s]s_\theta}{64\sqrt{3}F_\pi^2 \pi^2} \right. \\
& + \frac{(3m_{\eta'}^2 + 3m_K^2 - s)s_\theta^3}{64\sqrt{3}F_\pi^2 \pi^2} + \frac{\sqrt{3}c_\theta^2(m_\eta^2 - m_K^2 - 2s)s_\theta A_0(m_\eta^2)}{128F_\pi^2 \pi^2 s} + \frac{\sqrt{3}(m_{\eta'}^2 - m_K^2 - 2s)s_\theta^3 A_0(m_{\eta'}^2)}{128F_\pi^2 \pi^2 s} \\
& + \left\{ \frac{\sqrt{3}[m_K^2 - m_\pi^2 - 9s + c_\theta^2(-m_\eta^2 + m_K^2 + s)]s_\theta}{128F_\pi^2 \pi^2 s} + \frac{\sqrt{3}(-m_{\eta'}^2 + m_K^2 + 5s)s_\theta^3}{128F_\pi^2 \pi^2 s} \right\} A_0(m_K^2) \\
& - \frac{\sqrt{3}(m_K^2 - m_\pi^2 + 2s)s_\theta A_0(m_\pi^2)}{128F_\pi^2 \pi^2 s} - \frac{\sqrt{3}c_\theta^2[m_\eta^4 + (m_K^2 - s)^2 - 2m_\eta^2(m_K^2 + s)]s_\theta B_0(s, m_\eta^2, m_K^2)}{128F_\pi^2 \pi^2 s} \\
& - \frac{\sqrt{3}[m_{\eta'}^4 + (m_K^2 - s)^2 - 2m_{\eta'}^2(m_K^2 + s)]s_\theta^3 B_0(s, m_{\eta'}^2, m_K^2)}{128F_\pi^2 \pi^2 s} \\
& \left. - \frac{\sqrt{3}[m_K^4 + (m_\pi^2 - s)^2 - 2m_K^2(m_\pi^2 + s)]s_\theta B_0(s, m_K^2, m_\pi^2)}{128F_\pi^2 \pi^2 s} \right\}. \tag{A10}
\end{aligned}$$

The explicit expressions of the tensor FFs are given by

$$\begin{aligned}
F_{T,K^+K^-}^{\frac{\bar{u}u+\bar{d}d}{\sqrt{2}}} = & -\frac{1}{\sqrt{2}} \left\{ 1 + \frac{\sqrt{2}F_V^T G_V M_V}{\Lambda_2^T(M_\omega^2 - s)} + \frac{8c_d c_m (m_\pi^2 - m_K^2)}{F_\pi^2 M_{S_8}^2} + \frac{s - 6m_K^2}{96F_\pi^2 \pi^2} + \frac{(c_\theta^2 - 2\sqrt{2}c_\theta s_\theta + 2s_\theta^2)A_0(m_\eta^2)}{96F_\pi^2 \pi^2} \right. \\
& \left. + \frac{(2c_\theta^2 + 2\sqrt{2}c_\theta s_\theta + s_\theta^2)A_0(m_{\eta'}^2)}{96F_\pi^2 \pi^2} + \frac{7A_0(m_\pi^2)}{32F_\pi^2 \pi^2} + \frac{(s - 4m_K^2)B_0(s, m_K^2, m_\pi^2)}{64F_\pi^2 \pi^2} \right\}, \tag{A11}
\end{aligned}$$

$$\begin{aligned}
F_{T,K^+K^-}^{\bar{s}s} = 1 + & \frac{\sqrt{2}F_V^T G_V M_V}{\Lambda_2^T(M_\phi^2 - s)} - \frac{8c_d c_m(m_K^2 - m_\pi^2)}{F_\pi^2 M_{S_8}^2} - \left\{ \frac{6m_K^2 - s}{96F_\pi^2 \pi^2} \right. \\
& - \frac{(2c_\theta^2 + 2\sqrt{2}c_\theta s_\theta + s_\theta^2)A_0(m_\eta^2)}{48F_\pi^2 \pi^2} - \frac{(c_\theta^2 - 2\sqrt{2}c_\theta s_\theta + 2s_\theta^2)A_0(m_{\eta'}^2)}{48F_\pi^2 \pi^2} \\
& \left. - \frac{A_0(m_K^2)}{16F_\pi^2 \pi^2} - \frac{A_0(m_\pi^2)}{8F_\pi^2 \pi^2} + \frac{(4m_K^2 - s)B_0(s, m_K^2, m_K^2)}{64F_\pi^2 \pi^2} \right\}, \tag{A12}
\end{aligned}$$

$$\begin{aligned}
F_{T,\pi^-\pi^0}^{\bar{u}d} = -\sqrt{2} \left\{ 1 + \frac{\sqrt{2}F_V^T G_V M_V}{\Lambda_2^T(M_\rho^2 - s)} - \frac{2m_K^2 + 4m_\pi^2 - s}{96F_\pi^2 \pi^2} + \frac{(c_\theta^2 - 2\sqrt{2}c_\theta s_\theta + 2s_\theta^2)A_0(m_\eta^2)}{96F_\pi^2 \pi^2} \right. \\
+ \frac{(2c_\theta^2 + 2\sqrt{2}c_\theta s_\theta + s_\theta^2)A_0(m_{\eta'}^2)}{96F_\pi^2 \pi^2} + \frac{A_0(m_K^2)}{12F_\pi^2 \pi^2} + \frac{13A_0(m_\pi^2)}{96F_\pi^2 \pi^2} \\
\left. + \frac{(s - 4m_K^2)B_0(s, m_K^2, m_K^2)}{192F_\pi^2 \pi^2} + \frac{(s - 4m_\pi^2)B_0(s, m_\pi^2, m_\pi^2)}{96F_\pi^2 \pi^2} \right\}, \tag{A13}
\end{aligned}$$

$$\begin{aligned}
F_{T,K^0K^-}^{\bar{u}d} = - \left\{ 1 + \frac{\sqrt{2}F_V^T G_V M_V}{\Lambda_2^T(M_\rho^2 - s)} - \frac{8c_d c_m(m_K^2 - m_\pi^2)}{F_\pi^2 M_{S_8}^2} + \frac{s - 2m_K^2 - 4m_\pi^2}{96F_\pi^2 \pi^2} \right. \\
+ \frac{(\sqrt{2}c_\theta^2 - 4c_\theta s_\theta + 2\sqrt{2}s_\theta^2)A_0(m_\eta^2)}{96\sqrt{2}F_\pi^2 \pi^2} + \frac{(2\sqrt{2}c_\theta^2 + 4c_\theta s_\theta + \sqrt{2}s_\theta^2)A_0(m_{\eta'}^2)}{96\sqrt{2}F_\pi^2 \pi^2} + \frac{A_0(m_K^2)}{12F_\pi^2 \pi^2} \\
\left. + \frac{13A_0(m_\pi^2)}{96F_\pi^2 \pi^2} + \frac{(-4m_K^2 + s)B_0(s, m_K^2, m_K^2)}{192F_\pi^2 \pi^2} + \frac{(-4m_\pi^2 + s)B_0(s, m_\pi^2, m_\pi^2)}{96F_\pi^2 \pi^2} \right\}, \tag{A14}
\end{aligned}$$

$$\begin{aligned}
F_{T,K^-\pi^0}^{\bar{u}s} = \frac{1}{\sqrt{2}} F_{T,\bar{K}^0\pi^-}^{\bar{u}s} = -\frac{1}{\sqrt{2}} \left\{ 1 + \frac{\sqrt{2}F_V^T G_V M_V}{\Lambda_2^T(M_{K^*}^2 - s)} - \frac{4c_d c_m(m_K^2 - m_\pi^2)}{F_\pi^2 M_{S_8}^2} \right\} \\
- \frac{-3m_\pi^2 + s + c_\theta^2(-3m_\eta^2 - 3m_K^2 + s) - 3m_{\eta'}^2 s_\theta^2 + s s_\theta^2 - 3m_K^2(1 + s_\theta^2)}{192\sqrt{2}F_\pi^2 \pi^2} \\
+ \frac{1}{\sqrt{2}} \left\{ -\frac{[c_\theta^2(-3m_\eta^2 + 3m_K^2 - 2s) + 4\sqrt{2}c_\theta s s_\theta + 8s s_\theta^2]A_0(m_\eta^2)}{384F_\pi^2 \pi^2 s} \right. \\
+ \frac{[-8c_\theta^2 s + 4\sqrt{2}c_\theta s s_\theta + (3m_{\eta'}^2 - 3m_K^2 + 2s)s_\theta^2]A_0(m_{\eta'}^2)}{384F_\pi^2 \pi^2 s} \\
+ \frac{[-m_\pi^2 - 13s + c_\theta^2(-m_\eta^2 + m_K^2 + s) - m_{\eta'}^2 s_\theta^2 + s s_\theta^2 + m_K^2(1 + s_\theta^2)]A_0(m_K^2)}{128F_\pi^2 \pi^2 s} \\
- \frac{3(m_K^2 - m_\pi^2 + 18s)A_0(m_\pi^2)}{384F_\pi^2 \pi^2 s} - \frac{c_\theta^2[m_\eta^4 + (m_K^2 - s)^2 - 2m_\eta^2(m_K^2 + s)]B_0(s, m_\eta^2, m_K^2)}{128F_\pi^2 \pi^2 s} \\
- \frac{[m_{\eta'}^4 + (m_K^2 - s)^2 - 2m_{\eta'}^2(m_K^2 + s)]s_\theta^2 B_0(s, m_{\eta'}^2, m_K^2)}{128F_\pi^2 \pi^2 s} \\
\left. - \frac{[m_K^4 + (m_\pi^2 - s)^2 - 2m_K^2(m_\pi^2 + s)]B_0(s, m_K^2, m_\pi^2)}{128F_\pi^2 \pi^2 s} \right\}, \tag{A15}
\end{aligned}$$

$$\begin{aligned}
F_{T,K-\eta}^{\bar{u}s} = & -\sqrt{\frac{3}{2}}c_\theta \left\{ 1 + \frac{\sqrt{2}F_V^T G_V M_V}{\Lambda_2^T (M_{K^*}^2 - s)} - \frac{4c_d c_m (m_K^2 - m_\pi^2)(5 + 2c_\theta^2 + 4\sqrt{2}c_\theta s_\theta)}{3F_\pi^2 M_{S_8}^2} \right\} \\
& + \frac{1}{2}\sqrt{\frac{3}{2}} \{ s_\theta(\delta_k + 2s_\delta) + c_\theta s_\theta^2 \Lambda_1 \} \\
& + \sqrt{\frac{3}{2}} \left\{ -\frac{c_\theta [c_\theta^2(3m_K^2 - 3m_\eta^2 - 2s) + 4\sqrt{2}c_\theta s_\theta s + 8s_\theta^2 s] A_0(m_\eta^2)}{384F_\pi^2 \pi^2 s} \right. \\
& - \frac{c_\theta [8c_\theta^2 s - 4\sqrt{2}c_\theta s s_\theta + (-3m_{\eta'}^2 + 3m_K^2 - 2s)s_\theta^2] A_0(m_{\eta'}^2)}{384F_\pi^2 \pi^2 s} - \frac{c_\theta (m_K^2 - m_\pi^2 + 18s) A_0(m_\pi^2)}{128F_\pi^2 \pi^2 s} \\
& + \frac{c_\theta [-m_\pi^2 - 17s + c_\theta^2(-m_\eta^2 + m_K^2 + 5s) - m_{\eta'}^2 s_\theta^2 + s s_\theta^2 + m_K^2(1 + s_\theta^2)] A_0(m_K^2)}{128F_\pi^2 \pi^2 s} \\
& - \frac{c_\theta [-3m_\pi^2 + s + c_\theta^2(-3m_\eta^2 - 3m_K^2 + s) - 3m_{\eta'}^2 s_\theta^2 + s s_\theta^2 - 3m_K^2(1 + s_\theta^2)]}{192F_\pi^2 \pi^2} \\
& - \frac{c_\theta^3 [m_\eta^4 + (m_K^2 - s)^2 - 2m_\eta^2(m_K^2 + s)] B_0(s, m_\eta^2, m_K^2)}{128F_\pi^2 \pi^2 s} \\
& - \frac{c_\theta [m_{\eta'}^4 + (m_K^2 - s)^2 - 2m_{\eta'}^2(m_K^2 + s)] s_\theta^2 B_0(s, m_{\eta'}^2, m_K^2)}{128F_\pi^2 \pi^2 s} \\
& \left. - \frac{c_\theta [m_K^4 + (m_\pi^2 - s)^2 - 2m_K^2(m_\pi^2 + s)] B_0(s, m_K^2, m_\pi^2)}{128F_\pi^2 \pi^2 s} \right\}, \tag{A16}
\end{aligned}$$

$$\begin{aligned}
F_{T,K-\eta'}^{\bar{u}s} = & -\sqrt{\frac{3}{2}}s_\theta \left\{ 1 + \frac{\sqrt{2}F_V^T G_V M_V}{\Lambda_2^T (M_{K^*}^2 - s)} - \frac{4c_d c_m (m_K^2 - m_\pi^2)(5 - 4\sqrt{2}c_\theta s_\theta + 2s_\theta^2)}{3F_\pi^2 M_{S_8}^2} \right\} \\
& + \frac{1}{2}\sqrt{\frac{3}{2}} \{ c_\theta(\delta_k - 2s_\delta) + c_\theta^2 s_\theta \Lambda_1 \} \\
& + \sqrt{\frac{3}{2}} \left\{ -\frac{s_\theta [c_\theta^2(-3m_\eta^2 + 3m_K^2 - 2s) + 4\sqrt{2}c_\theta s_\theta s + 8s s_\theta^2] A_0(m_\eta^2)}{384F_\pi^2 \pi^2 s} \right. \\
& + \frac{s_\theta [-8c_\theta^2 s + 4\sqrt{2}c_\theta s_\theta s + (3m_{\eta'}^2 - 3m_K^2 + 2s)s_\theta^2] A_0(m_{\eta'}^2)}{384F_\pi^2 \pi^2 s} - \frac{s_\theta (m_K^2 - m_\pi^2 + 18s) A_0(m_\pi^2)}{128F_\pi^2 \pi^2 s} \\
& + \frac{s_\theta [-m_\pi^2 - 17s + c_\theta^2(-m_\eta^2 + m_K^2 + s) - m_{\eta'}^2 s_\theta^2 + 5s s_\theta^2 + m_K^2(1 + s_\theta^2)] A_0(m_K^2)}{128F_\pi^2 \pi^2 s} \\
& - \frac{2s s_\theta [-3m_\pi^2 + s + c_\theta^2(-3m_\eta^2 - 3m_K^2 + s) - 3m_{\eta'}^2 s_\theta^2 + s s_\theta^2 - 3m_K^2(1 + s_\theta^2)]}{384F_\pi^2 \pi^2 s} \\
& - \frac{c_\theta^2 [m_\eta^4 + (m_K^2 - s)^2 - 2m_\eta^2(m_K^2 + s)] s_\theta B_0(s, m_\eta^2, m_K^2)}{128F_\pi^2 \pi^2 s} \\
& - \frac{[m_{\eta'}^4 + (m_K^2 - s)^2 - 2m_{\eta'}^2(m_K^2 + s)] s_\theta^3 B_0(s, m_{\eta'}^2, m_K^2)}{128F_\pi^2 \pi^2 s} \\
& \left. - \frac{[m_K^4 + (m_\pi^2 - s)^2 - 2m_K^2(m_\pi^2 + s)] s_\theta B_0(s, m_K^2, m_\pi^2)}{128F_\pi^2 \pi^2 s} \right\}. \tag{A17}
\end{aligned}$$

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