

Delineating neutral and charged mesons in magnetic fields

Toru Kojo^{*} and Sakura Itatani[†]

*Theory Center, IPNS, High Energy Accelerator Research Organization
(KEK), 1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan*

and Graduate Institute for Advanced Studies, SOKENDAI, 1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan



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We investigate the properties of neutral and charged mesons in magnetic fields, from weak-field to strong-field regimes. To develop analytic insights, we employ a nonrelativistic quark model with a confining potential of the harmonic oscillator type. Short-range correlations, such as Coulomb and color-magnetic interactions, are treated as perturbations. In particular, we focus on the magnetic field dependence of the relative and the center-of-mass motions. The qualitative trends differ significantly between neutral and charged mesons; for neutral mesons, the transverse momenta are conserved and continuous, while charged mesons exhibit quantized transverse dynamics. The Zeeman effects, arising from intrinsic spins and orbital angular momenta, are carefully examined. In particular, for charged mesons with spins $s \geq 1$, we discuss how the zero-point energy in the internal quark motion cancels the Zeeman energy from the orbital angular momentum, ensuring the energetic stability of mesons with high spins. The effectively reduced dimensionality of these mesons in the strong-field limit is also discussed.

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I. INTRODUCTION

QCD in strong magnetic field regimes has attracted significant attention because of its importance in heavy-ion collisions [1–4] and its potential applications to neutron star physics [5–10], as well as its role as a critical test bed for theoretical concepts (for reviews, see, e.g., Refs. [11–16]). Strong magnetic fields couple to quarks and quantize their transverse dynamics, which in turn affects the vacuum structure and the properties of hadrons [17–50]. These phenomena have been extensively studied in lattice Monte Carlo simulations [51–66], covering aspects such as confinement/deconfinement transitions, chiral restoration, hadron spectra, and thermodynamics. However, the physical interpretation of these lattice results is often not straightforward, triggering renewed interest in the fundamental aspects of QCD dynamics in extreme environments.

In this work we study mesonic states in magnetic fields, from weak ($eB \ll \Lambda_{\text{QCD}}^2$) to strong field regimes ($eB \gg \Lambda_{\text{QCD}}^2$) with $\Lambda_{\text{QCD}} \simeq 0.2\text{--}0.3$ GeV being a typical non-perturbative scale in QCD. Both neutral and charged mesons are analyzed. Our primary interest lies in how

internal quark dynamics affects the overall behavior of mesons, which are frequently treated as pointlike particles in effective models. By isolating the relevant effects one by one, we discuss how the parameters in hadronic effective field theories (EFT) should be modified in the presence of magnetic fields. Specifically, in this paper we focus on the magnetic-field dependence of the meson masses and their transverse kinetic terms. For charged mesons, especially for states with spin and/or orbital angular momenta, the Zeeman terms tend to be canceled by the zero-point energies associated with internal quark motion; this cancellation is one of the key composite-particle effects examined in this work. The analysis of hadronic coupling constants will be presented in a subsequent paper.

Our primary goal is to develop analytic insights and establish a simple, intuitive physical picture. (See Refs. [67–71] for comprehensive studies based on Nambu-Jona-Lasinio or quark-meson models.) To this end, we adopt a simple nonrelativistic quark model [72–75], which enables many quantities to be computed analytically [43,76]. For applications to heavy quarkonia systems, see Refs. [16,27]. In this work, we stick to the nonrelativistic framework for several reasons. While it is possible to extend this framework to a relativistic version [77,78], such an extension introduces technical complexities that might distract us from developing a simple and intuitive picture. Furthermore, for ground-state mesons, the large transverse kinetic energies at high B (which typically challenge the validity of the nonrelativistic framework) are largely canceled by Zeeman effects, as seen in relativistic frameworks

^{*}Contact author: torukojo@post.kek.jp

[†]Contact author: itasaku@post.kek.jp

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(see, e.g., Ref. [22]).¹ The remaining B -dependence is relatively weak, suggesting that a nonrelativistic framework can still effectively capture the key properties of these low-lying states, to a similar extent as nonrelativistic models explain low-lying states at $B = 0$.

This work builds upon and extends our previous investigations into this topic [24,25]. In this paper, we present a more comprehensive and conceptually refined analysis compared to our prior work, laying the groundwork for forthcoming studies that will encompass multi-quark systems and hadronic media. Specifically, we have significantly improved our treatment of the pseudomomentum and guiding center coordinates, given more systematic and structured descriptions of charged mesons, and developed more physically motivated treatments of short-range correlations which require special care in the limit of large magnetic fields.

This paper is structured as follows. In Sec. II we review the kinetic motion of charged particles in magnetic fields, and in Sec. III summarize our modeling of long- and short-range interactions, and adjust the parameters to reproduce meson spectra at $B = 0$. In Sec. IV we study neutral meson spectra at finite B . In Sec. V we analyze charged mesons. Section VI is devoted to summary.

II. PREPARATION: KINETIC MOTIONS OF CHARGED PARTICLES

In this section we summarize our setup, reviewing the basics of the transverse motion in magnetic fields. Throughout this paper, the quark mass refers to the constituent quark mass, and we assume it to be B -independent. This assumption is supported by the lattice results for the chiral condensate at $B \gg \Lambda_{\text{QCD}}^2$, $\langle \bar{q}q \rangle \sim |eB| \langle \bar{q}q \rangle_{2D} \sim |eB| \Lambda_{\text{QCD}} [59]$, where the overall factor $|eB|$ comes from the Landau degeneracy and $\langle \bar{q}q \rangle_{2D}$ is computed from quark propagators with the dimensional reduction. If the dynamically generated quark masses were sensitive to the magnetic field, it would be very difficult to explain the linear scaling with respect to B [21].

A. Single charged particle

First we consider a single charged particle. We consider a nonrelativistic Hamiltonian,

$$H_{\text{single}} = m + \frac{\mathbf{\Pi}^2}{2m} - \boldsymbol{\mu} \cdot \mathbf{B}, \quad (1)$$

¹For example, for a free spin-1/2 elementary particle with charge $q > 0$, the relativistic Landau energy may be written as $E^2 = m^2 + p_z^2 + qB(2n + 1 - \sigma_z)$. For the lowest Landau level, the nonrelativistic expansion gives $E = m + \frac{p_z^2 + qB(1 - \sigma_z)}{2m} + \dots$. Thus, the apparently large $O(B/m)$ contribution disappears for $\sigma_z = 1$.

where B is constant and applied in the z -direction. The kinetic momentum and magnetic moment are

$$\mathbf{\Pi} = \mathbf{p} - e\mathbf{A}, \quad \boldsymbol{\mu} = \frac{e}{2m}\boldsymbol{\sigma}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad (2)$$

respectively. (We take the Landé g -factor to be $g = 2$, treating the constituent quark as an elementary Dirac fermion.)

We define a pseudomomentum [25,27],

$$\mathcal{K} \equiv \mathbf{\Pi} + e\mathbf{B} \times \mathbf{r} \equiv e\mathbf{B} \times \mathbf{X}. \quad (3)$$

This can be related to the *guiding center* coordinate,

$$\mathbf{X} = -\frac{\mathbf{B} \times \mathcal{K}}{eB^2}, \quad B = |\mathbf{B}|, \quad (4)$$

which characterizes the center of the cyclotron motion.

The commutation rules among $\mathbf{\Pi}$ and $\mathcal{K}$ are

$$[\Pi^x, \Pi^y] = ieB_z = -[\mathcal{K}^x, \mathcal{K}^y], \quad (5)$$

and all the other commutators are zero; in particular, $[\Pi^\alpha, \mathcal{K}^\beta] = 0 = [\Pi^\alpha, X^\beta]$ for any sets of (α, β) .

We note that Π^x and Π^y do not commute with $\mathbf{\Pi}^2$ and hence cannot be used as conserved quantities; indeed, the Lorentz force keeps changing the direction of kinetic momentum $\mathbf{\Pi} \sim m\mathbf{v}$. Meanwhile, $\mathcal{K}$, or the corresponding guiding center $\mathbf{X}$, commutes with $\mathbf{\Pi}^2$, as the location of the center does not affect the energy spectrum.

For explicit computations it is convenient to choose a specific gauge. Here we take the symmetric gauge

$$\mathbf{A} = \frac{\mathbf{B} \times \mathbf{r}}{2}, \quad (6)$$

with which

$$\mathbf{\Pi} = \mathbf{p} - \frac{e\mathbf{B} \times \mathbf{r}}{2}, \quad \mathcal{K} = \mathbf{p} + \frac{e\mathbf{B} \times \mathbf{r}}{2}. \quad (7)$$

Writing $\mathbf{\Pi}^2 = p_z^2 + \mathbf{\Pi}_\perp^2$, we note

$$\mathbf{\Pi}_\perp^2 = p_\perp^2 + \frac{(eB)^2}{4} r_\perp^2 - eBL_z. \quad (8)$$

The first two terms correspond to a usual harmonic oscillator in two dimensions. We first define the creation and annihilation operators for the x component through

$$x = \frac{1}{\sqrt{|eB|}}(a_x + a_x^\dagger), \quad p_x = -i\sqrt{|eB|}/2(a_x - a_x^\dagger), \quad (9)$$

and also for the y component in the same way. Then we construct the creation and annihilation operators for $+$ and $-$ circular modes,

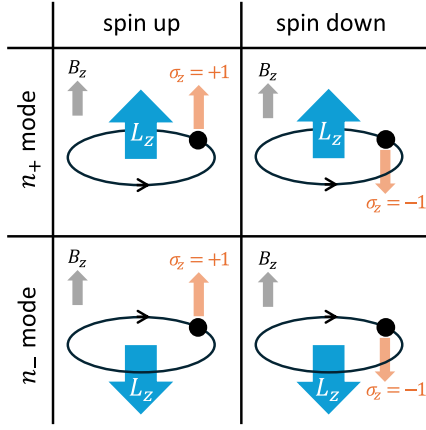


FIG. 1. Schematic illustration of the Zeeman splitting for orbital and spin angular momenta of a positively charged particle. The energy is lowered (raised) when the orbital angular momentum L_z or the spin σ_z is parallel (antiparallel) to the external magnetic field B_z . In the symmetric gauge, the n_+ mode corresponds to the orbital motion aligned with the magnetic field.

$$a_{\pm}^{\dagger} = \frac{1}{\sqrt{2}}(a_x^{\dagger} \pm i a_y^{\dagger}), \quad (10)$$

with which the angular momentum can be written as

$$L_z = a_+^{\dagger} a_+ - a_-^{\dagger} a_-. \quad (11)$$

In our definition, the orbital magnetic moment points to the z direction for the $+$ mode; see Fig. 1. Now, for later purposes we generalize the Hamiltonian for the transverse dynamics as

$$H_{\perp} = \frac{p_{\perp}^2}{2m} + \frac{m\omega_0^2}{2} r_{\perp}^2 - \Omega_0 L_z. \quad (12)$$

For charged particles free from potentials, $\omega_0 = |eB|/2m$ and $\Omega_0 = eB/2m$. The spectrum for $H_{\perp}$ is

$$E_{n_r, \ell_z}^{\perp} = \omega_0(2n_r + 1) + \omega_0 \left(|\ell_z| - \frac{\Omega_0}{\omega_0} \ell_z \right), \quad (13)$$

or equivalently,

$$E_{n_+, n_-}^{\perp} = (\omega_0 - \Omega_0)n_+ + (\omega_0 + \Omega_0)n_- + \omega_0, \quad (14)$$

where $2n_{\pm} = 2n_r + |\ell_z| \pm \ell_z$ with $n_r, n_{\pm}$ being non-negative integers and ℓ_z an integer. The Zeeman energy associated with the orbital angular motion reduces the energy when its direction is aligned with that of magnetic fields, while it costs more energy when the direction is opposite. In the former case, the increase of ℓ_z costs more energy in the kinetic motion, but the Zeeman coupling tends to cancel such energy cost.

In particular, for charged particles in the absence of external potential, we find $\omega_0 = \Omega_0$, so that such

cancellations are exact; this degeneracy for $\ell_z \geq 0$ is nothing but the usual Landau degeneracy for a given Landau level. For $\omega_0 = \Omega_0$, another way to understand this degeneracy is to interpret n_+ as a quantum number for the guiding center coordinates; they should not affect the spectra because of the translational invariance. When a charged particle is within an external potential or interacts with another particle, this degeneracy is lifted, as we discuss in the following sections.

B. Two charged particles

For two-body systems, the structure of our Hamiltonian is (i, j are used as a particle label)

$$H = \sum_{j=1,2} \left[m_j + \frac{\Pi_j^2}{2m_j} - \boldsymbol{\mu}_j \cdot \mathbf{B} \right] + V(r_{12}). \quad (15)$$

Here we included a potential which depends on the distance $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$.

Unlike the single particle case, the potential commutes with neither $\mathcal{K}_1$ nor $\mathcal{K}_2$, but commutes with the total pseudomomentum $\mathcal{K}_1 + \mathcal{K}_2 \equiv \mathcal{K}_R$. But in general we can label eigenstates of the Hamiltonian only by either $\mathcal{K}_R^x$ or $\mathcal{K}_R^y$, since $\mathcal{K}_R^x$ and $\mathcal{K}_R^y$ do not commute in general,

$$[\mathcal{K}_R^x, \mathcal{K}_R^y] = -ie_R B_z, \quad (e_R \equiv e_1 + e_2). \quad (16)$$

Clearly, the exception is the charge neutral case, $e_R = 0$, for which $\mathcal{K}_R^x$ and $\mathcal{K}_R^y$ can be used simultaneously to label the eigenstates of the Hamiltonian.

III. LONG- AND SHORT-RANGE INTERACTIONS

For two-body forces we separately treat the long-range and short-range interactions. Following the standard approach, we treat the long-range confining potential at leading order and the short-range part as perturbations. In vacuum, it is essential to have the confining potential in our unperturbed Hamiltonian; without it quarks would fly away and the perturbative framework never works.

A. Confining potential

For a confining potential, we use the harmonic oscillator which is standard in nonrelativistic quark models,

$$V_{\text{conf}}(r) = \lambda r^2, \quad \lambda \sim \Lambda_{\text{QCD}}^3. \quad (17)$$

Our motivation for using the harmonic oscillator comes simply from its tractability in analytic computations. More realistically, the confining potential should be linear at long distance which is necessary to derive the Regge trajectories. But in this work we restrict ourselves to low-lying states so that the impact of this simplification should not be significant.

B. Short-range potential

Next we turn to short-range interactions. As in usual quark models, we focus on the color-electric (Coulomb) and color-magnetic interaction.

1. Color-electric interaction

The color-electric interaction is

$$V_E(r) = -\frac{4\alpha_s}{3r}. \quad (18)$$

At strong magnetic fields, $eB \gg \Lambda_{\text{QCD}}^2$, actually the details of short-range interactions become more important than the long-range interactions. In fact, the model dependence largely appears from the treatment of the short-range correlations.

As we discuss later, we evaluate the expectation values of the following type,

$$\langle 1/r \rangle \sim \Lambda_z \mathcal{B} \int \frac{dz d^2 \mathbf{r}_\perp}{\sqrt{z^2 + (\mathbf{r}_\perp - \mathbf{r}_\perp^0)^2}} e^{-\frac{\Lambda_z^2}{2} z^2} e^{-\frac{\mathcal{B}}{2} r_\perp^2}, \quad (19)$$

where we assume $\Lambda_z \sim \Lambda_{\text{QCD}}$ and $\mathcal{B}^2 \sim B^2 + \Lambda_{\text{QCD}}^4$. The factor $\Lambda_{\text{QCD}} \mathcal{B}$ and the Gaussian functions for z and $r_\perp$ come from the wave functions. At large B , the range of $r_\perp$ is strongly limited to $\sim B^{-1/2}$. Here we consider the case $\mathbf{r}_\perp^0 = 0$. If we neglect the small $r_\perp$ in the denominator, we would encounter the integral

$$\mathcal{B} \int_0^{\sim \Lambda_z^{-1}} dz \int_0^{\sim B^{-1/2}} \frac{d^2 \mathbf{r}_\perp}{z} \sim \int_0^{\sim \Lambda_z^{-1}} \frac{dz}{z}, \quad (20)$$

which diverges logarithmically because of the short distance singularity around $z \sim 0$. For such small z , the small $1/B$ term in r must be kept to cut off the UV contributions, leaving

$$\langle 1/r \rangle \sim \Lambda_z \int_{\sim B^{-1/2}}^{\sim \Lambda_z^{-1}} \frac{dz}{z} \sim \Lambda_z \ln \frac{\mathcal{B}}{\Lambda_z^2}. \quad (21)$$

This perturbation grows logarithmically with $\mathcal{B}$; at very large $\mathcal{B}$ it would induce too much attraction between a quark and an antiquark, driving the meson unstable. Such instability, however, has not been found in lattice simulations.

Although this observation is based on a perturbative evaluation, it is unlikely that nonperturbative treatment of $1/r$ forces cures the inconsistency with the lattice results; with such improvement the wave function should have more concentration for smaller $r_\perp$ and the disagreement would become even worse. Thus, a remedy for the instability, if it exists, should be found already at the evaluation in perturbative treatments.

A plausible scenario is that the running of coupling constants tempers the growth of the attraction [43,76]. At

short distance of $\sim B^{-1/2}$, the typical momentum transfer is large, and the corresponding value of α_s is small. At large B , we use the one-loop running coupling ($N_f = 3$),

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln \frac{Q^2}{\Lambda_{\text{QCD}}^2}}, \quad \beta_0 = 11 - 2N_f/3, \quad (22)$$

taking $Q^2 \sim \mathcal{B}$. The logarithm in the denominator behaves as $\sim \ln(\mathcal{B}/\Lambda_{\text{QCD}}^2)$, which largely cancels the logarithm appearing in $\langle 1/r \rangle$. Accordingly the $\langle 1/r \rangle$ does not grow indefinitely for $\mathcal{B} \gg \Lambda_{\text{QCD}}^2$.

In this paper we parametrize the effective momentum transfer as ($\Lambda_z^4 = 8\mu\lambda$)

$$Q_B^2 \equiv m_1^2 + m_2^2 + \Lambda_z^2 + c_B eB, \quad (23)$$

and use the coupling $\alpha_s(Q_B^2)$ to evaluate the short-range interactions. We try to cover the following situations: (i) for larger quark masses, the wave function tends to be more compact so that Q^2 should be larger; (ii) for $B \rightarrow 0$ and small $m_{1,2}$, the Q^2 should be characterized by the confining scale Λ_z ; (iii) for $B \gg \Lambda_{\text{QCD}}^2$, the momentum transfer should be large, $Q^2 \sim B$.

Here, the parameter c_B is introduced because our results for neutral mesons, which can have very compact structures in the transverse direction, are sensitive to which Q^2 are used. In our numerical estimates we vary c_B and show how the results are affected.

2. Color-magnetic interaction

The traditional form of the color-magnetic interaction takes the form,

$$V_M^{\text{trad}}(r) = \frac{32\pi\alpha_s}{9m_1 m_2} \tilde{\delta}(\mathbf{r}) \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2, \quad (24)$$

where the $\tilde{\delta}(\mathbf{r})$ can be the usual δ -function or a strongly localized function such as $\sim e^{-\Lambda_{\text{UV}}^2 r^2}$. In momentum space, its Fourier transform leads to $\sim e^{-q^2/\Lambda_{\text{UV}}^2}$, cutting off the momenta beyond Λ_{UV} .

Since the expression (24) is arranged for the low energy regime with momentum transfer up to $\sim \Lambda_{\text{UV}}$, it is not clear whether the above parametrized form is reasonable also for the regime of $B \gg \Lambda_{\text{UV}}$; here the momentum transfer in the transverse direction can be as large as B , exceeding the cutoff scale. Also there is a significant asymmetry between the z - and transverse directions.

For this reason we examine the original relativistic expression which eventually leads to Eq. (24) in a certain limit. We start from the spin dependent term in momentum space,

$$V_M^{\text{rela}} = \frac{16\pi\alpha_s}{3E_1(\mathbf{p}_1)E_2(\mathbf{p}_2)} \mathcal{P}_s(\mathbf{q}) \quad (25)$$

with

$$\mathcal{P}_s(\mathbf{q}) \equiv \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 - (\mathbf{q} \cdot \boldsymbol{\sigma}_1)(\mathbf{q} \cdot \boldsymbol{\sigma}_2)/q^2. \quad (26)$$

Here the first factor arises from the vertices for a particle 1 and 2, $\mathbf{q}$ the momentum transfer, and $\mathbf{p}_{1,2}$ the momenta of the particle 1 and 2. If $\mathbf{p}_{1,2}$ are small, one can make a replacement, $E_{1,2} \rightarrow m_{1,2}$, and take the spatial average $q_i q_j \rightarrow \delta_{ij} q^2/3$, with which a constant in momentum space yields the δ -function in coordinate space. Meanwhile, in the presence of large B , the momentum transfer and typical particle momenta are large, $\mathbf{p}_{1,2}^2 \sim \mathbf{q}^2 \gg m_{1,2}^2$.

In order to capture the trend of large momentum transfer while keeping the expression as simple as possible, we consider the following compromised form,

$$V_M(\mathbf{q}) \equiv \frac{16\pi\alpha_s}{3} \mathcal{F}_s(m_1, m_2) \mathcal{P}_s(\mathbf{q}), \quad (27)$$

where we introduce a form factor ($\mu_{\text{IR}}^{-2} \equiv m_1^{-2} + m_2^{-2}$)

$$\mathcal{F}_s(m_1, m_2) \equiv C_M \left[\frac{e^{-q^2/\mu_{\text{IR}}^2}}{m_1 m_2} + \frac{1 - e^{-q^2/\mu_{\text{IR}}^2}}{q^2 + c_m m_1 m_2} \right]. \quad (28)$$

The factor C_M and c_m are some factors of $O(1)$. This form factor behaves as $\mathcal{F}_s \rightarrow 1/m_1 m_2$ for $\mathbf{q} \rightarrow 0$ and $\mathcal{F}_s \rightarrow 1/q^2$ for $\mathbf{q} \rightarrow \infty$, as requested. The parameter c_m is introduced just for the second term to vanish for $\mathbf{q} \rightarrow 0$.

In practice, we need to compute the following expectation value,

$$\langle V_M(r) \rangle = \int_r |\psi(\mathbf{r})|^2 \int_q e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}_\perp^0)} V_M(\mathbf{q}). \quad (29)$$

For the ground state, the wave function is $|\psi(\mathbf{r})|^2 \sim \Lambda_z \mathcal{B} e^{-\frac{\Lambda_z^2}{2} z^2} e^{-\frac{\mathbf{B}_\perp^2}{2} r_\perp^2}$, and we find

$$\begin{aligned} \langle V_M(r) \rangle &\sim \int_q V_M(\mathbf{q}) e^{-\frac{q_z^2}{2\Lambda_z^2} - \frac{q_\perp^2}{2\mathbf{B}_\perp^2} - i\mathbf{q}_\perp \cdot \mathbf{r}_\perp^0} \\ &= \langle V_1^s \rangle \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \langle V_{\text{II}}^s \rangle \sigma_1^z \sigma_2^z, \end{aligned} \quad (30)$$

where ($J = \text{I, II}$)

$$\langle V_J^s \rangle = \frac{16\pi\alpha_s}{3} \int_q e^{-\frac{q_z^2}{2\Lambda_z^2} - \frac{q_\perp^2}{2\mathbf{B}_\perp^2} - i\mathbf{q}_\perp \cdot \mathbf{r}_\perp^0} \mathcal{F}_s \mathcal{P}_J, \quad (31)$$

with

$$\mathcal{P}_\text{I} = \frac{1}{2} \left(1 + \frac{q_z^2}{q^2} \right), \quad \mathcal{P}_\text{II} = \frac{1}{2} \left(1 - \frac{3q_z^2}{q^2} \right). \quad (32)$$

The spin terms were split because the spatial wave functions are anisotropic. At $B = 0$ and $\mathbf{r}_\perp^0 = 0$, one recovers the usual expression with $\mathcal{P}_\text{I} \rightarrow 2/3$ and $\mathcal{P}_\text{II} \rightarrow 0$.

The major modification in the present treatment is the behavior at large B which are improved by taking into account the scaling $E_{1,2} \sim |\mathbf{q}|$ at large $\mathbf{q}$. The integration over q_z yields a factor $\sim \Lambda_{\text{QCD}}$. The integration for $\mathbf{q}_\perp$ is divided to $|\mathbf{q}_\perp| < m_{1,2}$ and $m_{1,2} < |\mathbf{q}_\perp| < \sqrt{B}$. They yield

$$\langle V_J^s \rangle \sim c_1 \Lambda_{\text{QCD}} + c_2 \Lambda_{\text{QCD}} \ln \frac{B}{m_{1,2}^2}, \quad (33)$$

where $c_{1,2} \sim O(1)$. As in the color-electric case, we find the logarithmic enhancement for a large B , which is largely canceled by using the running coupling $\alpha_s(B)$. Thanks to this, the perturbative corrections remain $\sim \Lambda_{\text{QCD}}$.

C. Spectra at $B=0$

Using the parametrization in the previous sections, we adjust the parameters to reproduce pseudoscalar mesons (π, K) and vector mesons (ρ, K^*, ϕ). The list of parameters is summarized in Table. I. For this fitting procedure, only the first five parameters ($m_{u,d}, m_s, \lambda, \Lambda_{\text{QCD}}, C_M$) are important; the parameter c_m has a minor impact while the term with c_B becomes important only at large B . When we optimize the parameters, we first fix $\Lambda_{\text{QCD}} = 0.250$ GeV for α_s , and then consider the range which we believe to be physically reasonable; $m_{u,d} = 0.25\text{--}0.35$ GeV, $m_s = 0.40\text{--}0.55$ GeV, $\lambda^{1/3} = 0.10\text{--}0.25$ GeV, and $C_M = 0.1\text{--}3.0$. Using $\Lambda_{\text{QCD}} = 0.20$ GeV or 0.30 GeV affects details of the optimization but does not change the overall trend.

In Table. II, we examine various contributions to the spectra by sequentially adding interactions; starting with the sum of constituent quark masses, we add confining interactions (+conf), Coulomb (+CE), and color-magnetic (+CM) interactions. It is important to notice that the kinetic energy is large because confinement forces quark wave function to be compact. This kinetic energy is largely canceled by the Coulomb interaction, leaving the spectra

TABLE I. Comparison of calculated and experimental meson masses (in GeV) with model parameters. To reproduce the spectra of (π, K, ρ, K^*, ϕ), the first five parameters are important; meanwhile details of c_m are not very important, and the term with c_B is activated only at finite B .

	Value	Eq.	Comment
$m_{u,d}$	0.264 [GeV]		
m_s	0.425 [GeV]		
$\lambda^{1/3}$	0.170 [GeV]	(17)	Confining force
Λ_{QCD}	0.250 [GeV]	(22)	Running α_s
C_M	1.22	(28)	Color-magnetic force
c_m	0.100	(28)	Color-magnetic force
c_B	0.1–1.0	(23)	Q^2 at finite B

TABLE II. Calculated masses include contributions from constituent masses ($m_1 + m_2$), confinement (+conf), Coulomb (+CE), and color-magnetic (+CM) interactions. Experimental values are taken from the Particle Data Group (PDG). Model results for pseudoscalar meson $\eta_s = s\bar{s}$ are also shown.

	$m_1 + m_2$	+conf	+CE	Total (+CM)	Exp.(PDG)
π	0.528	0.937	0.610	0.147	0.140
ρ	0.528	0.937	0.610	0.764	0.775
K	0.689	1.06	0.805	0.486	0.494
K^*	0.689	1.06	0.805	0.911	0.892
η_s	0.850	1.17	0.946	0.698	—
ϕ	0.850	1.17	0.946	1.03	1.019

close to the sum of constituent quark masses. Finally, the color-magnetic interaction splits the spectra into the low- and high-lying modes. In the following sections, we will delineate how these trends are affected by large magnetic fields.

IV. NEUTRAL MESONS

In this section we discuss the properties of neutral mesons, setting $q \equiv -e_1 = e_2$. As mentioned, the total pseudomomentum is conserved. Moreover $\mathcal{K}_R^x$ and $\mathcal{K}_R^y$ commute with each other. It is instructive to express $\mathcal{K}_R$ in terms of the guiding centers,

$$\mathcal{K}_R = \sum_{j=1,2} e_j \mathbf{B} \times \mathbf{X}_j = -\mathbf{B}_q \times (\mathbf{X}_1 - \mathbf{X}_2), \quad (34)$$

where we introduced a notation $\mathbf{B}_q = q\mathbf{B}$. This means that the distance between two guiding centers, $\mathbf{X}_1 - \mathbf{X}_2$, is conserved, so that we expect the impact of interactions to appear to be $\langle V \rangle \sim V(\mathbf{X}_1 - \mathbf{X}_2)$.

Below we choose coordinates as

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{M}, \quad \boldsymbol{\rho} = \mathbf{r}_2 - \mathbf{r}_1, \quad (35)$$

with $M = m_1 + m_2$. The corresponding momenta are

$$\mathbf{P}_R = \sum_i \mathbf{p}_i, \quad \mathbf{p}_\rho = -\frac{m_2}{M} \mathbf{p}_1 + \frac{m_1}{M} \mathbf{p}_2. \quad (36)$$

We decompose our leading order Hamiltonian into the following pieces,

$$H_0 = M + H_z + H_\perp + V_E + H_{\text{spin}}, \quad (37)$$

where

$$\begin{aligned} H_z &= \frac{P_R^2}{2M} + \frac{(p_\rho^z)^2}{2\mu} + \lambda \rho_z^2, \\ H_\perp &= \sum_{j=1,2} \frac{(\Pi_j^\perp)^2}{2m_j} + \lambda \rho_\perp^2, \\ V_E &= -\frac{4}{3} \frac{\alpha_s}{\sqrt{\rho_z^2 + \rho_\perp^2}}, \\ H_{\text{spin}} &= \frac{B_q}{2} \left(\frac{\sigma_1^z}{m_1} - \frac{\sigma_2^z}{m_2} \right) + V_M. \end{aligned} \quad (38)$$

The most nontrivial part is the treatment of $H_\perp$, and we will rewrite it shortly. After deriving the eigenvalues of H_0 , we construct perturbation theories for short-range interactions.

A. Transverse motions

It is convenient to label eigenstates of the Hamiltonian by $\mathbf{K}_\perp$ such that

$$\mathcal{K}_R \Phi_{\mathbf{K}_\perp} = \mathbf{K}_\perp \Phi_{\mathbf{K}_\perp}. \quad (39)$$

Then, in the symmetric gauge,

$$\mathcal{K}_R = \mathbf{P}_R^\perp + \frac{\mathbf{B}_q}{2} \times \boldsymbol{\rho}. \quad (40)$$

The solution of the eigenvalue equation for $\mathcal{K}_R$ is

$$\begin{aligned} \Phi_{\mathbf{K}_\perp}(\mathbf{R}_\perp, \boldsymbol{\rho}) &= \exp \left[i \mathbf{R}_\perp \cdot \left(\mathbf{K}_\perp - \frac{\mathbf{B}_q}{2} \times \boldsymbol{\rho} \right) \right] \varphi_{\mathbf{K}_\perp}(\boldsymbol{\rho}) \\ &\equiv e^{i\Theta_{\mathbf{K}_\perp}(\mathbf{R}_\perp, \boldsymbol{\rho})} \varphi_{\mathbf{K}_\perp}(\boldsymbol{\rho}), \end{aligned} \quad (41)$$

where the $\mathbf{R}_\perp$ -dependence is factorized. Using this form, our eigenvalue equation for the Hamiltonian [27],

$$H_\perp \Phi_{\mathbf{K}_\perp}(\mathbf{R}_\perp, \boldsymbol{\rho}) = E_{\mathbf{K}_\perp} \Phi_{\mathbf{K}_\perp}(\mathbf{R}_\perp, \boldsymbol{\rho}), \quad (42)$$

can be transformed into

$$H_\perp^{\text{eff,temp}} \varphi_{\mathbf{K}_\perp}(\boldsymbol{\rho}) = E_{\mathbf{K}_\perp} \varphi_{\mathbf{K}_\perp}(\boldsymbol{\rho}), \quad (43)$$

where $H_\perp^{\text{eff,temp}} \equiv e^{-i\Theta_{\mathbf{K}_\perp}} H_\perp e^{i\Theta_{\mathbf{K}_\perp}}$. The explicit form is (reduced mass: $\mu^{-1} = m_1^{-1} + m_2^{-1}$)

$$\begin{aligned} H_\perp^{\text{eff,temp}} &= \frac{(\mathbf{p}_\rho^\perp)^2}{2\mu} + \left(\frac{B_q^2}{8\mu} + \lambda \right) \rho_\perp^2 + \frac{K_\perp^2}{2M} \\ &\quad + \frac{B_q}{2} \left(\frac{1}{m_1} - \frac{1}{m_2} \right) L_\rho^z - \frac{\mathbf{B}_q}{M} \cdot (\boldsymbol{\rho}_\perp \times \mathbf{K}_\perp). \end{aligned} \quad (44)$$

Here we have attached “temp” (temporal) since we will rewrite it shortly for shifted variables.

This expression can be misleading, since it would give impressions that the spectra depend on $\mathbf{K}_\perp$ even in the case of noninteracting limit, $\lambda \rightarrow 0$; for $\lambda = 0$, there should be

only two independent charged particles which yield only quantized spectra. The present confusing expression comes from the failure to eliminate all linear terms in dynamical variables.

In eliminating the linear terms, shifting ρ by a constant is insufficient, since the attempt to eliminate the $\rho \times K_\perp$ term yields another linear term in $p_\rho^\perp$ from L_ρ^z . We need to shift $\rho_\perp$ and $p_\rho^\perp$ at the same time, $\rho_\perp \rightarrow \rho_\perp + \rho_\perp^0$ and $p_\rho^\perp \rightarrow p_\rho^\perp + p_0^\perp$. This shift with constant terms does not change the commutation relations for $p_\rho^\perp$ and $\rho_\perp$. Explicitly, the proper shifts turn out to be

$$\begin{aligned}\rho_\perp^0 &= \frac{\mathbf{K} \times \mathbf{B}_q}{B_q^2 + 2M\lambda}, \\ p_0^\perp &= \frac{m_1 - m_2}{2M} \frac{B_q^2 \mathbf{K}_\perp}{B_q^2 + 2M\lambda},\end{aligned}\quad (45)$$

which generate terms that cancel the $K_\perp^2/2M$ term. One should also notice that these shift also affect the coordinates in V_E and V_M ; at large $K_\perp$, the $\rho_\perp^0$ serves as a cutoff for short-range interactions.

Now, with new variables the previous $H_\perp^{\text{eff,temp}}$ is replaced with

$$\begin{aligned}H_\perp^{\text{eff}} &= \frac{\lambda K_\perp^2}{B_q^2 + 2M\lambda} + \frac{(p_\rho^\perp)^2}{2\mu} + \frac{B_q^2}{8\mu} \rho_\perp^2 \\ &+ \frac{\mathbf{B}_q \cdot \mathbf{L}_\rho}{2} \left(\frac{1}{m_1} - \frac{1}{m_2} \right),\end{aligned}\quad (46)$$

where we introduced a notation

$$B_q^2 = B_q^2 + 8\mu\lambda. \quad (47)$$

In the present expression, it is clear that $K_\perp^2$ term vanishes for $\lambda = 0$; in this case, only discrete spectra appear, as they should.

The qualitative meaning of the $K_\perp^2$ term becomes clearer if we replace $K_\perp^2$ with the guiding center coordinates,

$$\frac{\lambda K_\perp^2}{B_q^2 + 2M\lambda} = \frac{\lambda(\mathbf{X}_1 - \mathbf{X}_2)^2}{1 + 2M\lambda/B_q^2}. \quad (48)$$

At large B , this is nothing but the energy from confining potential whose separation is specified by the conserved distance between the guiding centers for two particles. Note also that the limit $B_q \rightarrow 0$ reproduces the standard $K_\perp^2/2M$ term.

The spectra take the form

$$\begin{aligned}E_{n_r, \ell_z}^\perp &= \frac{\lambda K_\perp^2}{B_q^2 + 2M\lambda} + \frac{B_q}{2\mu} (2n_r + 1) \\ &+ \frac{B_q}{2} \left[\frac{|\ell_z|}{\mu} + \ell_z \frac{B_q}{B_q} \left(\frac{1}{m_1} - \frac{1}{m_2} \right) \right],\end{aligned}\quad (49)$$

or equivalently,

$$\begin{aligned}E_{K_\perp}^{n_+, n_-} &= \frac{\lambda K_\perp^2}{B_q^2 + 2M\lambda} + \frac{B_q}{2\mu} + \frac{1}{2} \left[\frac{B_q}{\mu} + B_q \left(\frac{1}{m_1} - \frac{1}{m_2} \right) \right] n_+ \\ &+ \frac{1}{2} \left[\frac{B_q}{\mu} + B_q \left(\frac{1}{m_2} - \frac{1}{m_1} \right) \right] n_-.\end{aligned}\quad (50)$$

From this expression, it is clear that $\lambda \rightarrow 0$ yields the spectra of two independent charged particles. Note that the coefficients of n_+ and n_- are manifestly positive definite, and any internal excitations cost the energy of $\sim B$. Finally we note that there is a zero-point energy $B_q/2\mu$ which, in the ground state, is largely canceled by the Zeeman energy from the intrinsic spins.

B. Perturbative corrections and spin dependence

Using the eigenfunctions for $H_z + H_\perp$ with the quantum numbers $\mathbf{n} = (n_z, n_r, \ell_z)$, we evaluate the short-range interactions as the first order of perturbation,

$$\langle V_E \rangle_{\mathbf{n}}, \quad \langle V_M \rangle_{\mathbf{n}} = \langle V_I^s \rangle_{\mathbf{n}} (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) + \langle V_{II}^s \rangle_{\mathbf{n}} \sigma_1^z \sigma_2^z. \quad (51)$$

These expectation values are evaluated numerically. The electric potential may be simply added to the eigenenergy from H_z and $H_\perp$. The magnetic part appears in the spin dependent part of the Hamiltonian,

$$\begin{aligned}\langle H_{\text{spin}} \rangle_{\mathbf{n}} &= \frac{B_q}{2} \left(\frac{\sigma_1^z}{m_1} - \frac{\sigma_2^z}{m_2} \right) \\ &+ \langle V_I^s \rangle_{\mathbf{n}} (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) + \langle V_{II}^s \rangle_{\mathbf{n}} \sigma_1^z \sigma_2^z.\end{aligned}\quad (52)$$

Here $(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$ can be diagonalized by a state with the total spin S , and the corresponding eigenvalue is $2[S(S+1) - 3/2]$. Meanwhile, the Zeeman energy is diagonal only for $|S=1, S_z=\pm 1\rangle$; the Zeeman term converts $|S=0, S_z=0\rangle$ into $|S=1, S_z=0\rangle$ and vice versa.

For $|S=1, S_z=\pm 1\rangle$, the spectra are

$$(E_{S=1}^{S_z=\pm 1})_{\mathbf{n}} = \pm \frac{B_q}{2} \left(\frac{1}{m_1} - \frac{1}{m_2} \right) + \langle V_I^s + V_{II}^s \rangle_{\mathbf{n}}. \quad (53)$$

For $S_z = 0$ states, we diagonalize the matrix

$$\begin{bmatrix} -3\langle V_I^s \rangle_{\mathbf{n}} - \langle V_{II}^s \rangle_{\mathbf{n}} & \frac{B_q}{2\mu} \\ \frac{B_q}{2\mu} & \langle V_I^s \rangle_{\mathbf{n}} - \langle V_{II}^s \rangle_{\mathbf{n}} \end{bmatrix} \quad (54)$$

and find the spectra $(\langle V_{I+II}^s \rangle = \langle V_I^s + V_{II}^s \rangle)$,

$$(E_{S=\pm 1}^{S_z=0})_{\mathbf{n}} = -\langle V_{I+II}^s \rangle_{\mathbf{n}} \pm \sqrt{4\langle V_I^s \rangle_{\mathbf{n}}^2 + \left(\frac{B_q}{2\mu} \right)^2}. \quad (55)$$

Among all these spectra, the $S_z = 0$ state with the energy $(E_{-}^{S_z=0})_n$ leads to the smallest energy.

C. Spatial wave functions

Short-range interactions are numerically evaluated using the wave functions of the unperturbed Hamiltonian. The spatial wave functions take the forms,

$$\Phi_n^{\text{nt}}(z, \rho_\perp) = \psi_{n_z}(z) \phi_{n_\perp}^{\text{nt}}(\rho_\perp), \quad (56)$$

where each wave function is normalized to one. The wave function in the z -direction is

$$\psi_{n_z}(z) = \mathcal{N}_{n_z} H_{n_z}(z_\Lambda) e^{-\frac{\Lambda^2}{4} z^2}, \quad (57)$$

with $z_\Lambda = (2\mu\lambda)^{1/4} z \equiv \Lambda z / \sqrt{2}$, a normalization constant $\mathcal{N}_{n_z} = (\Lambda/2^{n_z} n_z! \sqrt{2\pi})^{1/2}$, and H_{n_z} being the Hermite polynomials. For the relative motion in the transverse direction, the unperturbed wave functions for neutral mesons take the form

$$\phi_{n_\perp}^{\text{nt}}(\rho_\perp) = \mathcal{N}_{n_\perp}^{\text{nt}} e^{i\ell_z \theta} \rho_B^{|\ell_z|} L_{n_\perp}^{|\ell_z|}(\rho_B^2) e^{-\frac{B_q}{4} \rho_\perp^2}, \quad (58)$$

where $L_{n_\perp}^{|\ell_z|}$ is the associated Laguerre polynomials and the normalization constant is $\mathcal{N}_{n_\perp}^{\text{nt}} = [\mathcal{B}_q n_\perp! / 2\pi(n_\perp + |\ell_z|)!]^{1/2}$. The effective transverse coordinate is $\rho_B = \rho_\perp (\mathcal{B}_q/2)^{1/2}$ with the parameter $\mathcal{B}_q$ which was given in Eq. (47).

We are interested in the modes with $n_z = n_\perp = l_z = 0$ for which $H_{n_z=0} = 1$ and $L_{n_\perp=0}^{|\ell_z|} = 1$. We use the following wave function,

$$\Phi_{\text{gs}}^{\text{nt}}(z, \rho_\perp) = \mathcal{N}_{n_z} \mathcal{N}_{n_\perp}^{\text{nt}} e^{-\frac{\Lambda^2}{4} z^2} e^{-\frac{B_q}{4} \rho_\perp^2}, \quad (59)$$

to evaluate the color-electric and magnetic interactions. The analytic structure was examined in Sec. III B.

Now we summarize the integral to be evaluated. The expressions suitable for numerical evaluations are summarized in the Appendix. B. Including the shift $\rho_\perp^0 \sim O(K_\perp/B)$ in Eq. (45), the color-electric energy is ($n_z = n_\perp = 0$)

$$\langle V_E \rangle = -\frac{4\alpha_s}{3} |\mathcal{N}_{n_z}|^2 |\mathcal{N}_{n_\perp}^{\text{nt}}|^2 \int_r \frac{e^{-\frac{\Lambda^2}{2} z^2 - \frac{B_q}{2} \rho_\perp^2}}{\sqrt{\rho_z^2 + (\rho_\perp + \rho_\perp^0)^2}}, \quad (60)$$

whose qualitative trend was discussed in Sec. III B 1.

The color-magnetic energy is

$$\langle V_M \rangle = \langle V_I^s \rangle \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \langle V_{\text{II}}^s \rangle \sigma_1^z \sigma_2^z, \quad (61)$$

with ($J = \text{I, II}$)

$$\langle V_J^s \rangle = \frac{16\pi\alpha_s}{3} \int_q e^{-\frac{q_z^2}{2\Lambda^2} - \frac{q_\perp^2}{2B_q} - i q_\perp \cdot r_\perp^0} \mathcal{F}_s \mathcal{P}_J, \quad (62)$$

where $\mathcal{F}_s$ and $\mathcal{P}_J$ are given in Eqs. (28) and (32), respectively.

D. Spectra

Collecting the energies evaluated so far, the spectra for neutral mesons can be written as

$$E_n^{S, S_z}(\mathbf{K}) = M + E_{K_z}^{n_z} + E_{\mathbf{K}_\perp}^{n_\perp} + \langle V_E \rangle_n + (E_S^z)_n. \quad (63)$$

We are particularly interested in the ground state energy given by the state $\mathbf{n} = 0$, $S = -$, and $S_z = 0$,

$$E_{\text{gs}}(\mathbf{K}) = M + \langle V_E \rangle + \frac{K_z^2}{2M} + \frac{\lambda K_\perp^2}{B_q^2 + 2M\lambda} + \frac{1}{2} \sqrt{\frac{2\lambda}{\mu}} + \frac{\mathcal{B}_q}{2\mu} - \langle V_{\text{I+II}}^s \rangle - \sqrt{4\langle V_I^s \rangle^2 + \left(\frac{B_q}{2\mu}\right)^2}. \quad (64)$$

The numerical evaluation is shown shortly.

We examine how the spectrum changes from weak to strong magnetic fields. We set $\mathbf{K} = 0$. At $B = 0$,

$$E_{\text{gs}}^{B=0}(0) = M + \langle V_E \rangle^{B=0} + \frac{3}{2} \sqrt{\frac{2\lambda}{\mu}} - 3\langle V_I \rangle^{B=0}. \quad (65)$$

At $B \gg \Lambda_{\text{QCD}}^2$,

$$E_{\text{gs}}^{B \gg \Lambda_{\text{QCD}}^2}(0) \simeq M + \langle V_E \rangle + \frac{1}{2} \sqrt{\frac{2\lambda}{\mu}} - \langle V_{\text{I+II}}^s \rangle, \quad (66)$$

where $O(1/B)$ terms are dropped.

The ground state spectra of $u\bar{u}$, $d\bar{s}$, and $s\bar{s}$ mesons for $\mathbf{K} = 0$ are shown in Fig. 2. Except for a small enhancement at small B , the ground state energies substantially decrease from their vacuum values.

To understand this behavior, it is instructive to switch off the short-range interactions by setting $\alpha_s \rightarrow 0$ (Fig. 2, dashed lines). With increasing B , the zero-point energy of the transverse motion and the Zeeman energy largely cancel, so the energy at large B reduces as

$$\frac{3}{2} \sqrt{\frac{2\lambda}{\mu}} \rightarrow \frac{1}{2} \sqrt{\frac{2\lambda}{\mu}} + \frac{\mathcal{B}_q - B_q}{2\mu}, \quad (67)$$

where $\mathcal{B}_q - B_q \sim \mu\lambda/B_q$ is negligible at large B . Hence the reduction at $\alpha_s = 0$ may be understood as the effective disappearance of the transverse kinetic energy. For instance, the energy reduction in the large B limit is $\sqrt{2\lambda/\mu} \simeq 0.272(0.215)$ GeV for the $u\bar{u}$ ($s\bar{s}$) meson.

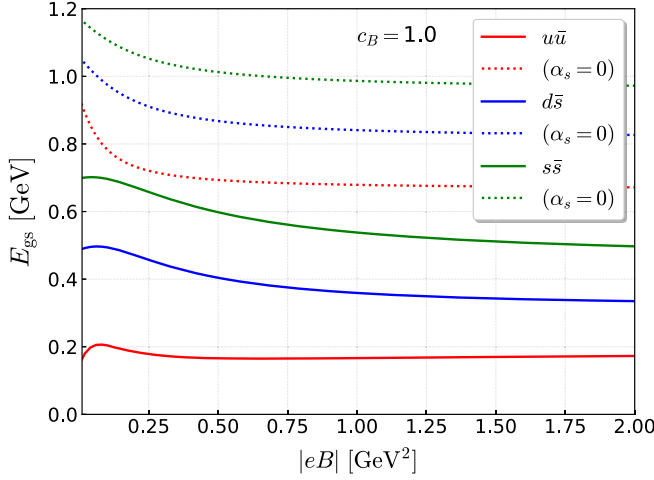


FIG. 2. Magnetic field dependence of the ground state energies for $u\bar{u}$, $d\bar{s}$, and $s\bar{s}$ mesons for $\mathbf{K} = 0$. The solid and dotted lines represent calculations with and without short-range interactions [$\alpha_s(Q_B) \neq 0$ and $\alpha_s = 0$]. For Q_B , we use the expression with $c_B = 1.0$.

Now we switch on the short-range interactions, see Figs. 2 and 3 for $u\bar{u}$, $d\bar{s}$, and $s\bar{s}$ spectra. Their impact sensitively depend on our choice of Q^2 in the running coupling $\alpha_s(Q^2)$, as we have emphasized in Sec. III B. If we neglect the running of α_s at a small transverse distance of $\sim B^{-1/2}$, the energy reduction caused by the short-range interactions becomes too large (see the dotted lines in Fig. 3); the spectrum in the $u\bar{u}$ channel becomes even

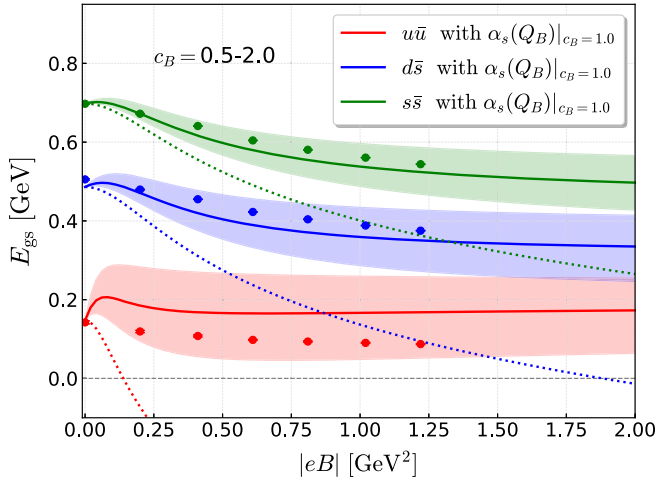


FIG. 3. Ground-state energy levels of $u\bar{u}$, $d\bar{s}$, and $s\bar{s}$ mesons as functions of $|eB|$, with the running coupling $\alpha_s(Q_B)$ and $\mathbf{K} = 0$. The colored bands represent the uncertainty originating from the scale parameter $c_B \in [0.5, 2.0]$, where the solid lines denote the central value, $c_B = 1.0$. A smaller c_B yields more mass reduction. For comparison, the results for a constant coupling (corresponding to $c_B = 0$) are indicated with the dotted curves. The data points are taken from the lattice QCD results reported in Ref. [51].

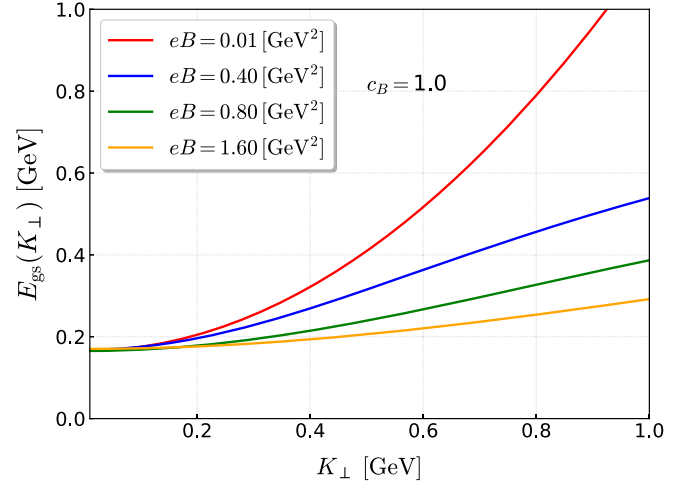


FIG. 4. The ground-state energies for the $u\bar{u}$ meson as functions of transverse momentum $K_\perp$, for various $|eB|$. At larger B , the spectra become more insensitive to the value of $K_\perp$. $c_B = 1.0$ is chosen.

unstable. This too large reduction in the mass is tempered by using α_s with $Q^2 \sim B$. While it is sensible to set $Q^2 \sim B$, we do not know precisely the numerical coefficient in front of B . For this reason we vary the parameter c_B for Q_B^2 in Eq. (23) for the range to be compatible with the lattice results. In Fig. 3, the band for the range $c_B \in [0.5, 2.0]$ is shown to display the impact of our choice on Q_B . Lighter mesons are more sensitive to Q_B .

Finally we take a look at the ground-state energy for the $u\bar{u}$ meson at finite transverse momentum $\mathbf{K}_\perp$ which is related to the distance between the two guiding centers as $|\mathbf{X}| \sim |\mathbf{K}_\perp|/B_q$. The energy shift at finite $\rho_\perp$ originates from two competing factors: (i) a reduction in the transverse center-of-mass kinetic energy [Eq. (48)], and (ii) an energy penalty arising from the weakened short-range attraction at finite $\rho_\perp$. The results are shown in Fig. 4. As the magnetic field B increases, the energy dependence on $K_\perp$ becomes less pronounced. In the limit of $B \rightarrow \infty$, the energy becomes independent of $K_\perp$, reflecting the effective dimensional reduction in the energy spectra.

V. CHARGED MESONS

For charged mesons, the analyses of the spectra are much more involved. $\mathcal{K}_R$ is conserved as in the neutral meson case, but the x - and y -components do not commute, $[\mathcal{K}_R^x, \mathcal{K}_R^y] \neq 0$. Below we choose the eigenvalue of $\mathcal{K}_R^x$ to label the eigenstates of the Hamiltonian. We look for the eigenstates of $\mathcal{K}_R^x$,

$$\mathcal{K}_R^x \Phi_{K_x} = K_x \Phi_{K_x}. \quad (68)$$

We expect a charged meson to circulate in a closed orbit with which the spectra should be quantized. Meanwhile, the eigenvalue equation for $\mathcal{K}_R^x$ seems to yield a continuous

eigenvalue K_x . Combining these observations, we expect that the spectra become independent of K_x , making the spectra discrete. We confirm this expectation shortly.

A. Transverse motions

For charged systems in coordinates $\mathbf{R}$ and $\mathbf{r}$, the pseudomomentum takes the form (in the symmetric gauge),

$$\mathcal{K}_R = \mathbf{P}_R + \frac{B}{2} \times (e_R \mathbf{R} + e_\rho \boldsymbol{\rho}), \quad (69)$$

where we write

$$e_\rho = \frac{e_2 m_1 - e_1 m_2}{M}. \quad (70)$$

We look for the eigenfunctions of $\mathcal{K}_R^x$. The equation to be solved is

$$\left[-i \frac{\partial}{\partial R_x} - \frac{B}{2} (e_R R_y + e_\rho \rho_y) \right] \Phi_{K_x} = K_x \Phi_{K_x}. \quad (71)$$

The solution takes the form

$$\begin{aligned} \Phi_{K_x}(\mathbf{R}, \boldsymbol{\rho}) &= \exp \left[i R_x \left(K_x + \frac{B}{2} (e_R R_y + e_\rho \rho_y) \right) \right] \varphi_{K_x}(R_y, \boldsymbol{\rho}) \\ &\equiv e^{i\Theta_c} \varphi_{K_x}(R_y, \boldsymbol{\rho}). \end{aligned} \quad (72)$$

As we have done for the neutral meson case, we convert

$$H_{c\perp} \Phi_{K_x}(R_y, \boldsymbol{\rho}) = E_{K_x} \Phi_{K_x}(R_y, \boldsymbol{\rho}), \quad (73)$$

into

$$\begin{aligned} H_{c\perp}^{\text{eff}} &= \frac{(P_R^y)^2}{2M} + \left[\frac{(Be_R)^2}{2M} + \frac{(Be_\rho)^2}{8\mu} \right] R_y^2 + \frac{(\mathbf{p}_\rho^\perp)^2}{2\mu} + \left[\frac{\mu(Be_R)^2}{8M^2} + \lambda + \frac{(Be_\rho)^2}{8\mu} - \frac{\mu B^2 e_\rho e_R}{4M} \left(\frac{1}{m_1} - \frac{1}{m_2} \right) \right] \rho_\perp^2 \\ &+ \left[-\frac{Be_R}{2M} + \frac{Be_\rho}{2} \left(\frac{1}{m_1} - \frac{1}{m_2} \right) \right] L_\rho^z - \frac{3(Be_\rho)^2}{8M} \rho_x^2 + \left[\frac{5B^2 e_R e_\rho}{4M} - \frac{(Be_\rho)^2}{4} \left(\frac{1}{m_1} - \frac{1}{m_2} \right) \right] \rho_y R_y + \frac{Be_\rho}{2\mu} R_y p_\rho^x - \frac{Be_\rho}{2M} \rho_x P_R^y. \end{aligned} \quad (78)$$

The terms in the first two lines are axially symmetric and can be analytically diagonalized. The complication comes from the third line which is not axially symmetric. The full Hamiltonian contains anisotropic terms and the coupling between the center-of-mass and internal motion. In principle one can diagonalize analytically but the expression is too complicated. We first consider the $e_\rho = 0$ case for which we can derive the spectra analytically. Then, we derive the full spectra numerically.

$$H_{c\perp}^{\text{eff,temp}} \varphi_{K_x}(R_y, \boldsymbol{\rho}) = E_{K_x} \varphi_{K_x}(R_y, \boldsymbol{\rho}), \quad (74)$$

where $H_{c\perp}^{\text{eff,temp}} \equiv e^{-i\Theta_{K_x}} H_{c\perp} e^{i\Theta_{K_x}}$.

In the Hamiltonian, the K_x comes out when we act P_R^x on the phase Θ_c . We look at Π_j^x ($j = 1, 2$)

$$\begin{aligned} e^{-i\Theta_c} \Pi_j^x e^{i\Theta_c} &= \frac{m_j}{M} [K_x + Be_R R_y + P_R^x] + \frac{B}{2} e_\rho \rho_y \\ &- \eta_j \left[p_\rho^x + \frac{B}{2} e_\rho R_y + \frac{B\mu}{2M} e_R \rho_y \right], \end{aligned} \quad (75)$$

where $\eta_1 = -\eta_2 = 1$. Now we consider a constant shift of the operators R_y and p_ρ^x as

$$R_y \rightarrow R_y - \frac{K_x}{Be_R}, \quad p_\rho^x \rightarrow p_\rho^x + \frac{e_\rho}{2e_R} K_x, \quad (76)$$

with which the K_x term disappears

$$\begin{aligned} e^{-i\Theta_c} \Pi_j^x e^{i\Theta_c} &\rightarrow \frac{m_j}{M} [Be_R R_y + P_R^x] + \frac{B}{2} e_\rho \rho_y \\ &- \eta_j \left[p_\rho^x + \frac{B}{2} e_\rho R_y + \frac{B\mu}{2M} e_R \rho_y \right]. \end{aligned} \quad (77)$$

The other terms in the Hamiltonian do not contain K_x terms and are unaffected by the above constant shifts. Hence, we verified that the spectra are independent of the value of K_x . For this reason, below we no longer attach the subscript K_x to the energy and states.

After some calculations including the above-mentioned shifts, we arrive at

B. Analytic insights: $e_\rho = 0$ case

The $e_\rho = 0$ case corresponds to $m_1 = m_2 = M/2$ and $e_1 = e_2 = e_R/2$, which are not realized in physical mesonic systems. (For diquarks made of identical quarks, this condition can be satisfied [79].) But it is still useful to study this case as a baseline to diagnose a more general $e_\rho \neq 0$ case.

The energy spectra take the form

$$E_{n_R, \tilde{n}_c}^{S, S_z}(K_z) = M + E_{K_z}^{n_z} + E_{n_R}^{n_1} + \langle V_E \rangle_{n_c} + (E_S^{S_z})_{n_c}. \quad (79)$$

The short-range potentials are evaluated using the eigenfunctions of $H_0^\perp$. The Hamiltonian at $e_\rho = 0$ is

$$H_0^\perp = \frac{(P_R^y)^2}{2M} + \frac{(Be_R)^2}{2M} R_y^2 + \frac{2(\mathbf{p}_\rho^\perp)^2}{M} + \left[\frac{(Be_R)^2}{32M} + \lambda \right] \rho_\perp^2 - \frac{Be_R}{2M} L_\rho^z, \quad (80)$$

which yields the following spectrum,

$$(E_{n_R}^\perp)_0 = \frac{Be_R}{2M} (2n_R + 1) + \frac{\mathcal{B}_R}{2M} (2n_r + |\ell_z| + 1) - \frac{Be_R}{2M} \ell_z, \quad (81)$$

or equivalently,

$$(E_{n_R}^\perp)_0 = \frac{Be_R}{2M} (2n_R + 1) + \frac{\mathcal{B}_R}{2M} (n_+ + n_- + 1) - \frac{Be_R}{2M} (n_+ - n_-), \quad (82)$$

where

$$\mathcal{B}_R = \sqrt{(Be_R)^2 + 32M\lambda}. \quad (83)$$

The angular momentum in a meson tends to cancel the zero-point energy from the kinetic motion. At large B , states with $\ell_z \geq 0$ are almost degenerate with a small splitting due to $\mathcal{B}_R - Be_R \sim \lambda M/B$. Thus, at large B , there are many low energy excitations with $\ell_z \geq 0$.

There are also contributions from the spin-dependent term $(E_S^{S_z})_{n_c}$. The spin term is diagonalized by (S, S_z) ,

$$(E_S^{S_z})_{n_c,0} = -\frac{Be_R}{M} S_z + \langle V_I^S \rangle_{n_c} [2S(S+1) - 3] + 4\langle V_{II}^S \rangle_{n_c} S_1^z S_2^z. \quad (84)$$

At large B , the ground state is given by the $S = S_z = 1$ state in which the Zeeman energy exactly cancels the zero-point energy from the center-of-mass motion and almost completely cancels that from the relative motion.

The low-lying spectra which are insensitive to B are given by $n_R = n_r = 0$ modes with $\ell_z \geq 0$,

$$E_{\ell_z \geq 0}^{n_R=n_r=0} = M + E_{K_z=0}^{n_z=0} + \langle V_E + V_{I+\Pi}^S \rangle_{n_r=0, \ell_z} + \frac{\mathcal{B}_R - Be_R}{2M} (1 + |\ell_z|). \quad (85)$$

The main findings from this special setup for large B are: (i) for $\ell_z > 0$, the orbital kinetic energy and Zeeman energy tends to cancel, leaving small excitation energies of $\sim \lambda/B$ which vanish for noninteracting limit, $\lambda \rightarrow 0$; (ii) the zero-point energy of the center-of-mass motion cancels exactly

with the half of the Zeeman energy; (iii) the zero-point energy of the relative-motion largely cancels with the half of the Zeeman energy; (iv) in the limit of $\lambda \rightarrow 0$, the energy levels of $\ell_z \geq 0$ are all degenerate.

C. Perturbative expansion with respect to e_ρ

For general cases of $e_1 \neq e_2$ and $m_1 \neq m_2$, the expressions become complicated. Still, if $\delta e = e_1 - e_2$ and $\delta m = m_1 - m_2$ can be regarded as small, one can analytically compute corrections to the $e_\rho = 0$ results. The e_ρ may be expressed as

$$e_\rho = \frac{e_R \delta m}{2M} - \frac{\delta e}{2}, \quad (86)$$

which is the sum of δe and δm corrections. In the light quark sector, the former is a good approximation, and the latter may be regarded as the $1/N_c$ corrections, since $e_{u,d} = (\pm 1 + 1/N_c)/2$; for charged combinations $(u, \bar{d})$ and $(d, \bar{u})$, the charge difference is $\delta e = e/N_c$.

Computing to the second order of these perturbations, we find the correction to the ground-state energy at large B , with the quanta $n_R = n_r = \ell_z = 0$ and $S = S_z = 1$, is (for details, see Appendix. A)

$$\delta E_{\ell_z=0}^{n_R=n_r=0} = -\frac{(Be_\rho)^2}{2M} \left[\frac{(\frac{\xi}{2} - \frac{7}{2\xi})^2}{\mathcal{B}_R + Be_R} + \frac{(\frac{\xi}{2} + \frac{3}{2\xi})^2}{\mathcal{B}_R + 3Be_R} \right] + \frac{Be_\rho^2}{4Me_R} \frac{\xi^2 + 10}{\xi^2} + \frac{8\delta m^2 \lambda}{M^2 Be_R \xi^2}, \quad (87)$$

where $\xi = [1 + 32M\lambda/(Be_R)^2]^{1/4}$. In the limit of $B \rightarrow \infty$, the scaling of parameters is $\mathcal{B}_R \rightarrow Be_R$, $\xi \rightarrow 1$. In this limit the terms of $O(B)$ cancel, leaving

$$\delta E_{\ell_z=0}^{n_R=n_r=0} \sim O(\lambda/B) \quad (B \rightarrow \infty). \quad (88)$$

Hence the insensitivity of the ground-state energy to B , as seen for the $e_\rho = 0$ case, remains valid to $O(\delta e^2, \delta m^2)$. This conclusion holds also for $\ell_z > 0$ with $n_R = n_r = 0$. This result indicates that for general cases with $e_\rho \neq 0$, the overall trend of the spectra remains similar as the $e_\rho = 0$ cases.

D. General spectra for $e_\rho \neq 0$

The full Hamiltonian for the transverse kinetic term takes a complicated form and we derive the spectra numerically. What we need is the eigenfrequencies and the corresponding wave functions which are utilized to compute the expectation values of the short-range interactions.

1. Eigenfrequencies

First we define $X = (\rho_x, \rho_y, R_y, p_\rho^x, p_\rho^y, P_R^y)$ and write the Hamiltonian as

$$H_{\perp}^{\text{eff}} = \frac{1}{2} X_i A_{ij} X_j, \quad A_{ij} = A_{ji}. \quad (89)$$

The commutation relations for X 's are

$$[X_i, X_j] = iJ_{ij}, \quad J = \begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix}, \quad (90)$$

where J is the fundamental symplectic matrix. Our goal is to convert the Hamiltonian into the form

$$H_{\perp}^{\text{eff}} = \sum_{\alpha=1}^3 \omega_{\alpha} \left(b_{\alpha}^{\dagger} b_{\alpha} + \frac{1}{2} \right), \quad (91)$$

with the constraint $[b_{\alpha}, b_{\beta}^{\dagger}] = \delta_{\alpha\beta}$, and find the relation between X and $(b, b^{\dagger})$,

$$X_i = T_{i\alpha} \tilde{b}_{\alpha}, \quad \tilde{b} = (b_1, b_2, b_3, b_1^{\dagger}, b_2^{\dagger}, b_3^{\dagger}). \quad (92)$$

We can also express the commutation relation as $[\tilde{b}_{\alpha}, \tilde{b}_{\beta}] = J_{\alpha\beta}$. This imposes the condition $iJ = TJT^T$ which is satisfied in our ordering of operators in $\tilde{b}$. The rest is the normalization of overall coefficients.

To find the spectra, we examine the Heisenberg equation,

$$\dot{X}_k = i[H_{\perp}^{\text{eff}}, X_k] = (JA)_{kj} X_j. \quad (93)$$

We consider a linear transformation $X_k = V_{k\alpha} \tilde{X}_{\alpha}$ with $V^{-1}(JA)V = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_6)$,

$$\dot{\tilde{X}}_{\alpha} = \lambda_{\alpha} \tilde{X}_{\alpha} \rightarrow \tilde{X}_{\alpha}(t) = e^{\lambda_{\alpha} t} \tilde{X}_{\alpha}(0). \quad (94)$$

We note that, if some component has the eigenvalue λ_{α} , there is also a component having the eigenvalue $-\lambda_{\alpha}$. We further note that $\tilde{X}_{\alpha}^{\dagger}$ has the eigenvalue λ_{α}^* . These conditions are fulfilled for $\lambda_{\alpha} = i\omega_{\alpha}$ so that $\lambda_{\alpha}^* = -\lambda_{\alpha}$. Now we can set

$$\tilde{X}_{1,2,3} = (b_1, b_2, b_3), \quad \tilde{X}_{4,5,6} = (b_1^{\dagger}, b_2^{\dagger}, b_3^{\dagger}), \quad (95)$$

where $\tilde{X}_{1-3}$ and $\tilde{X}_{4-6}$ have the eigenvalues $-i\omega_{1-3}$ and $i\omega_{1-3}$, respectively, and the matrix V turns out to be the matrix T in Eq. (92). These frequencies are nothing but those in Eq. (91). Now Eq. (91) satisfies the time evolution,

$$\begin{aligned} b_{\alpha}(t) &= e^{-i\omega_{\alpha} t} b_{\alpha} = e^{iH_{\perp}^{\text{eff}} t} b_{\alpha} e^{-iH_{\perp}^{\text{eff}} t}, \\ b_{\alpha}^{\dagger}(t) &= e^{i\omega_{\alpha} t} b_{\alpha}^{\dagger} = e^{iH_{\perp}^{\text{eff}} t} b_{\alpha}^{\dagger} e^{-iH_{\perp}^{\text{eff}} t}, \end{aligned} \quad (96)$$

as requested in Eq. (94).

2. Wave functions

To find the ground state wave functions for the transverse Hamiltonian, we look for the state such that

$$b_{\alpha}|0_{\perp}\rangle = T_{\alpha i}^{-1} X_i |0_{\perp}\rangle = 0, \quad (\alpha = 1, 2, 3). \quad (97)$$

Writing $x_{\alpha} = (\boldsymbol{\rho}_{\perp}, R_y)$ and $p_{\alpha} = (\mathbf{p}_{\rho}, P_R^y)$, this takes the form

$$\sum_{\beta=1}^3 (f_{\alpha\beta} x_{\beta} + g_{\alpha\beta} p_{\beta}) |0_{\perp}\rangle = 0, \quad (98)$$

where $f_{\alpha\beta} = T_{\alpha\beta}^{-1}$ and $g_{\alpha\beta} = T_{\alpha, \beta+3}^{-1}$ are 3×3 matrices. The form of the solution is

$$\langle \mathbf{x} | 0_{\perp} \rangle = \mathcal{N}_{\perp} \exp \left[-\frac{1}{2} x_{\alpha} M_{\alpha\beta} x_{\beta} \right], \quad (99)$$

where the matrix $M (= M^T)$ can be computed as

$$M_{\alpha\beta} = i[g^{-1}f]_{\alpha\beta}, \quad (100)$$

with the normalization constant $|\mathcal{N}_{\perp}|^2 = \sqrt{\det \mathcal{M} / \pi^3}$, where $\mathcal{M} \equiv \text{Re } M$. The wave functions for excited states can be created by applying creation operators $b_{\alpha}^{\dagger}$.

3. Matrix elements

In our computations of the matrix elements, we evaluate the expectation values of functions independent of R_y . The electric potential is $(\mathcal{N}_{\perp}^{\perp} = \mathcal{N}_{n_z=0} \mathcal{N}_{\perp})$

$$\begin{aligned} \langle V_E \rangle_{\text{gs}} &= |\mathcal{N}_{\perp}^{\perp}|^2 \int_{R_y, \mathbf{r}} V_E(\mathbf{r}) e^{-x_{\alpha} \mathcal{M}_{\alpha\beta} x_{\beta} - \frac{\Lambda^2}{2} z^2} \\ &= -\frac{4\alpha_s}{3} |\mathcal{N}_{\perp}^{\perp}|^2 \sqrt{\frac{\pi}{\mathcal{M}_{RR}}} \int_r \frac{e^{-\rho_{\perp}^2 \Sigma_{ab} \rho_{\perp}^2 - \frac{\Lambda^2}{2} z^2}}{r}, \end{aligned} \quad (101)$$

where integrating R_y renormalizes the matrix for $\boldsymbol{\rho}_{\perp}$,

$$\Sigma_{ab} = \mathcal{M}_{ab} - \frac{\mathcal{M}_{Ra} \mathcal{M}_{Rb}}{\mathcal{M}_{RR}}, \quad (a, b = x, y). \quad (102)$$

For the magnetic potential, we find (for $J = \text{I, II}$)

$$\begin{aligned} \langle V_J^s \rangle_{\text{gs}} &= \int_{\mathbf{q}} V_J^s(\mathbf{q}) e^{-\frac{1}{4} q_a^{\perp} (\Sigma^{-1})_{ab} q_b^{\perp} - \frac{q_z^2}{2\Lambda^2}} \\ &= \frac{16\pi\alpha_s}{3} \int_{\mathbf{q}} e^{-q_a^{\perp} \tilde{\Sigma}_{ab} q_b^{\perp}} \mathcal{F}_s \mathcal{P}_J, \end{aligned} \quad (103)$$

where $\mathcal{F}_s$ and $\mathcal{P}_J$ are given in Eqs. (28) and (32), respectively, and $\tilde{\Sigma} \equiv \Sigma^{-1}/4 + I_{zz}/2\Lambda^2$. Useful formulas to reduce the three-dimensional integral over (z, ρ_x, ρ_y) to one-dimensional integral is given in the Appendix. B.

4. Spectra

Now we examine numerical results for the ground state spectra for $u\bar{d}$ and $u\bar{s}$ mesons.

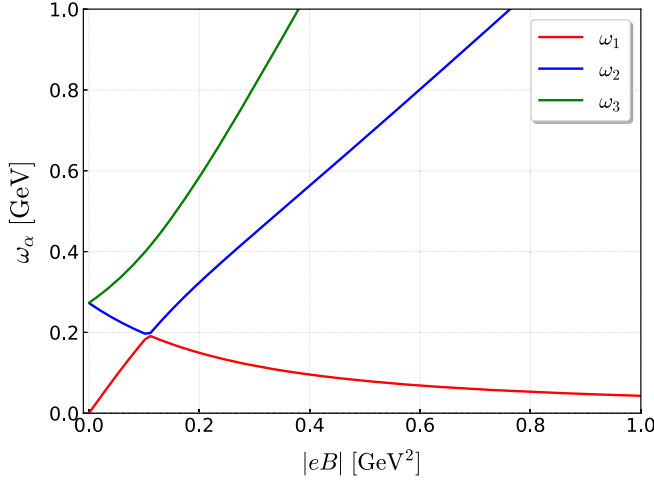


FIG. 5. Eigenfrequencies ω_α ($\alpha = 1, 2, 3$) for the transverse Hamiltonian $H_\perp^{\text{eff}}$, see Eq. (91). The $u\bar{d}$ meson is considered. These eigenmodes are the mixtures of the center-of-mass, +, and – modes. The level repulsion occurs around $|eB| \simeq 0.12 \text{ GeV}^2$.

We begin with the eigenfrequencies ω_α ($\alpha = 1, 2, 3$) of the transverse Hamiltonian $H_\perp^{\text{eff}}$, see Eq. (91) and Fig. 5. In the limit of $e_\rho = 0$ (see Sec. VB), we have already seen the center-of-mass motion with the energy cost $\sim |B|/M$, and two orbital angular motions with the magnetic moments parallel and antiparallel to the magnetic field, with the energy cost of $\sim |B|/M$ and $\sim \lambda/|B|$, respectively. The small energy in the latter is the consequence of the orbital Zeeman energy that tends to cancel the zero-point energy of the orbital motion.

At $e_\rho \neq 0$, all these modes can mix, but the large B behaviors are similar to the trend found in the $e_\rho = 0$ case; we have two energetic modes (ω_2 and ω_3 modes) with the linear dependence on B , and one low-energy mode (ω_1 mode) which decreases with B . Extrapolating the assignment of quantum numbers for the $e_\rho = 0$ case, one can regard the ω_1 mode to be dominated by the $\ell_z = +1$ excitation. Meanwhile, for small B , the lowest energy mode is dominated by the center-of-mass motion but it is eventually dominated by one of the orbital modes. When these modes (ω_1 and ω_2) approach, the level repulsion occurs, see ω_1 around $|eB| \sim 0.12 \text{ GeV}^2$.

Shown in Fig. 6 is the magnetic field dependence of the ground state energies for $u\bar{d}$ and $u\bar{s}$ mesons, with and without short-range interactions [$\alpha_s(Q_B) \neq 0$ and $\alpha_s = 0$]. For Q_B , we chose $c_B = 1.0$. For $\alpha_s = 0$, as in the neutral meson cases, the mass reduction occurs due to the effective disappearance of the transverse zero-point energy. With the short-range interactions added in, this trend does not change much.

In Fig. 7, we show the ground state energies for $u\bar{d}$ and $u\bar{s}$ mesons calculated with the running $\alpha_s(Q_B)$. The colored bands represent the uncertainty originating from the scale parameter $c_B \in [0.5, 2.0]$, where the solid lines denote the

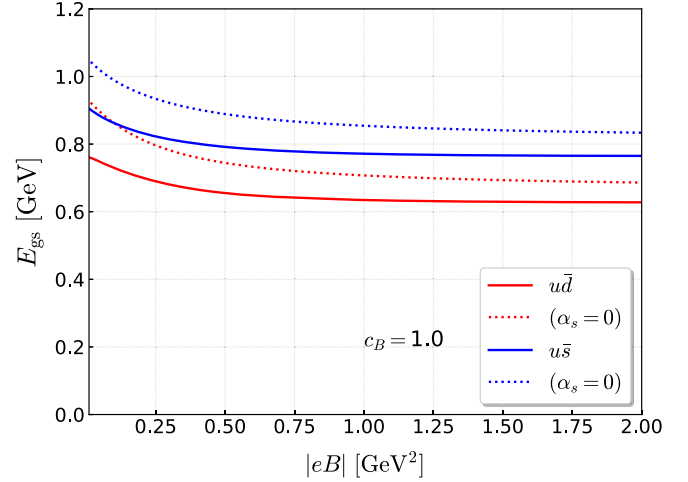


FIG. 6. Magnetic field dependence of the ground state energies for $u\bar{d}$ and $u\bar{s}$ mesons. The solid and dotted lines represent calculations with and without short-range interactions ($\alpha_s(Q_B) \neq 0$ and $\alpha_s = 0$). For Q_B , we use the expression with $c_B = 1.0$.

results for the intermediate value, $c_B = 1.0$. The results for a constant α_s are also shown as dotted lines. For comparison with lattice results, we plot the vacuum masses multiplied by the ratio $C_{\text{ratio}}^{\text{lat}} = m_{\rho_\pi}(B)/m_\rho(0)$ measured on the lattice for $m_\pi = 0.415 \text{ GeV}$ [56]. Since previous lattice analyses have shown that the pion mass dependence is mild between $m_\pi \simeq 0.42 \text{ GeV}$ and 0.81 GeV , we applied the same ratio to the vacuum masses of m_ρ and m_{K^*} .

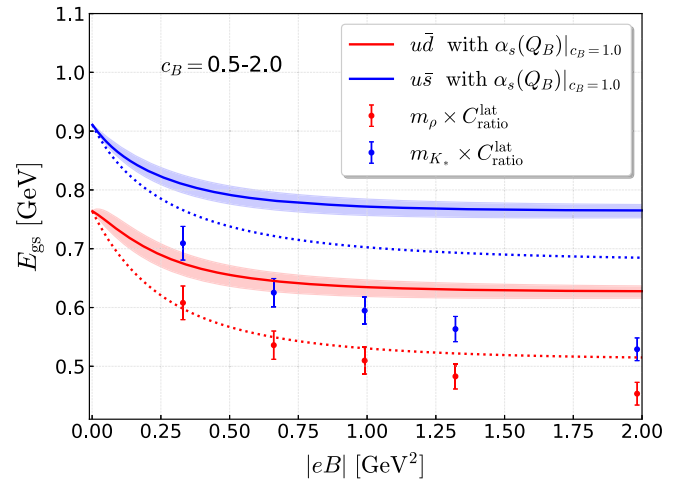


FIG. 7. Ground-state energy levels of $u\bar{d}$ and $u\bar{s}$ mesons as functions of $|eB|$, calculated with the running coupling $\alpha_s(Q_B)$. The colored bands represent the uncertainty originating from the scale parameter $c_B \in [0.5, 2.0]$, where the solid lines denote the results for the central value, $c_B = 1.0$. For comparison, the results for a constant coupling (corresponding to $c_B = 0$) are indicated with the dotted curves. We also plot the vacuum meson multiplied by the ratio $m_\rho(B)/m_\rho(0)$ measured on the lattice for $m_\pi = 0.415 \text{ GeV}$ [56].

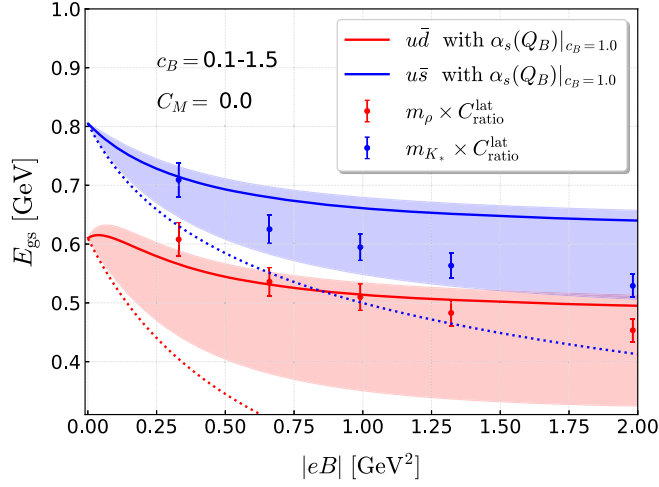


FIG. 8. Same as Fig. 7, but the color-magnetic interaction is switched off ($C_M = 0$). The range of c_B for the band is shifted to the lower values ($c_B = [0.1, 1.5]$) to explore the range for a better fit.

Compared with the lattice data, our model calculations seem to underestimate the mass reduction. Increasing the value of α_s , which affects both the color-electric (attractive) and magnetic (repulsive) interactions, does not resolve the discrepancy. One possible resolution is to assume that the color-magnetic interaction is weaker for ρ and K_{*+} at the relatively large current quark masses used in the lattice calculations [56]. Indeed, if we switch off the color-magnetic interaction in our quark model (by setting $C_M = 0$), the loss of the repulsive term leads to a significant mass reduction, bringing the resulting mass ratio into better agreement with that obtained in the lattice calculation, as shown in Fig. 8. At present, however, we have not yet identified a definitive physical mechanism to explain the significant mass reduction observed on the lattice. Further studies are called for.

VI. SUMMARY

We studied the properties of neutral and charged mesons within a simple constituent quark model. We first investigated how the confining potential affects the dynamics of charged particles in magnetic fields, and then delineated the impact of the short-range interactions. The qualitative trends in the mass reduction of neutral and charged mesons, as found in the lattice simulations, are reproduced. The agreement with the lattice results can be improved to a semiquantitative level if a proper scale for the running coupling is chosen for the short-range interactions.

In addition to the ground-state energies, we also argue that the transverse kinetic terms of neutral mesons have nontrivial behavior at finite B which are substantially different from those obtained by extrapolating from a hadronic effective Lagrangian at $B = 0$ [20,24,25]. The impact of transverse momenta is suppressed by a factor of

$\sim 1/B$, and hence at large B there are many low-energy states. This may be interpreted as an effective dimensional reduction of the meson dynamics. Similar trends can be found for charged mesons with excitations of $\ell_z > 0$. In a hadron resonance gas with these low-lying mesons, thermal contributions at low temperature are significantly enhanced, and this trend is consistent with the findings on the lattice [57].

In this study we found that the details of the running of α_s are crucial for quantitative description of the hadronic spectra within the present framework, especially for states with the spatially compact wave functions. Unfortunately our choice of Q^2 remains largely heuristic and should be improved by more rigorous arguments [80]. Also, quantitatively, our description of charged mesons are not satisfactory. Considering the sensitivity of hadronic spectra to the choice of α_s , the B -dependence of the hadronic spectra should allow us to gain more insights on the running of α_s at the hadronic scale.

As mentioned in the Introduction, this work is aimed to establish a reasonable (qualitative) baseline to study multi-quark systems and hadronic matter in magnetic fields. In subsequent papers, we discuss hadronic scattering at large B and how such scattering processes can be used to describe multi-quark systems. These results provide a useful framework for understanding hadronic dynamics in strong magnetic fields.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The lattice QCD data compared with our model results are available in Ref. [51] for Fig. 3, and in Ref. [56] for Figs. 7 and 8. The data from our model calculations are available from the authors upon request.

APPENDIX A: PERTURBATIVE EVALUATION

To compute the corrections to the analytic expression in the $e_\rho = 0$ (see Sec. VC), we split the Hamiltonian by classifying terms by the order of e_ρ ,

$$H_{\perp}^{\text{eff}} = H_0^{\perp} + H_1^{\perp} + H_2^{\perp} + \dots \quad (\text{A1})$$

Below the reduced mass is expanded as $\mu^{-1} = 4M^{-1} + 4\delta m^2 M^{-3} + \dots$, and $m_1^{-1} - m_2^{-1} = -4\delta m/M^2 + \dots$. Explicitly,

$$H_0^\perp = \frac{(P_R^y)^2}{2M} + \frac{(Be_R)^2}{2M} R_y^2 + \frac{2(\mathbf{p}_\rho^\perp)^2}{M} + \left[\frac{(Be_R)^2}{32M} + \lambda \right] \rho_\perp^2 - \frac{Be_R}{2M} L_\rho^z, \quad (\text{A2})$$

is $O(e_\rho^0)$, and

$$H_1^\perp = \frac{2Be_\rho}{M} R_y p_\rho^x + \frac{5B^2 e_R e_\rho}{4M} \rho_y R_y - \frac{Be_\rho}{2M} \rho_x p_R^y, \quad (\text{A3})$$

is $O(e_\rho^1)$, and

$$H_2^\perp = \frac{(Be_\rho)^2}{2M} R_y^2 + \frac{2\delta m^2}{M^3} (\mathbf{p}_\rho^\perp)^2 + \left[\frac{(Be_\rho)^2}{2M} + \frac{B^2 e_R e_\rho \delta m}{4M^2} - \frac{(Be_R)^2 \delta m^2}{32M^3} \right] \rho_\perp^2 - \frac{3(Be_\rho)^2}{8M} \rho_x^2 - \frac{2Be_\rho \delta m}{M^2} L_\rho^z, \quad (\text{A4})$$

is regarded as $O(e_\rho^2)$. Finally, we also split the Zeeman term as

$$H_{Zm} = -\frac{e_1 B}{2m_1} \sigma_1^z - \frac{e_2 B}{2m_2} \sigma_2^z = -\frac{Be_R}{2M} (\sigma_1^z + \sigma_2^z) + \frac{Be_\rho}{M} (\sigma_1^z - \sigma_2^z) - \frac{Be_\rho \delta m}{M^2} (\sigma_1^z + \sigma_2^z) + O(\delta m^4), \quad (\text{A5})$$

where the first, second, and third terms correspond to $O(e_\rho^0)$, $O(e_\rho^1)$, and $O(e_\rho^2)$, respectively.

For the transverse Hamiltonian of $O(e_\rho)$, the first-order perturbation is

$$\langle H_1^\perp \rangle = 0, \quad (\text{A6})$$

so that the corrections start from $O(e_\rho^2)$. The $O(e_\rho^2)$ correction to the n th level is as

$$(E_{n_R}^\perp)_2 = \sum_{n' \neq n} \frac{\langle n | H_1^\perp | n' \rangle \langle n' | H_1^\perp | n \rangle}{(E_{n_R}^\perp)_0 - (E_{n_R'}^\perp)_0} + \langle n | H_2^\perp | n \rangle = \delta_1 E_{n_R}^\perp + \delta_2 E_{n_R}^\perp + \delta_{2,Zm} E_{n_R}^\perp. \quad (\text{A7})$$

To evaluate $\rho_x p_R^y$, $R_y p_\rho^x$ and $\rho_y R_y$ in $H_1^\perp$, it is convenient to use the creation and annihilation operators

$$R_y = \frac{1}{\sqrt{2Be_R}} (b + b^\dagger), \quad p_R^y = -i\sqrt{\frac{Be_R}{2}} (b - b^\dagger), \quad (\text{A8})$$

and

$$\rho_i = \frac{1}{\sqrt{2\kappa}} (a_i + a_i^\dagger), \quad p_\rho^i = -i\sqrt{\frac{\kappa}{2}} (a_i - a_i^\dagger), \quad (\text{A9})$$

where $i = x, y$ and $\kappa = Be_R \mu / 2M$.

For perturbation treatments for $(\rho_\perp, \mathbf{p}_\rho^\perp)$, it is useful to label the unperturbed states by (n_+, n_-) which is related to (n_r, ℓ_z) by $n_\pm = n_r + (|\ell_z| \pm \ell_z)/2$. The first order Hamiltonian is $(\eta_1 = -\eta_2 = 1)$

$$H_1^\perp = i \frac{Be_\rho}{2M} \sum_{j=\pm} \left[\frac{\xi}{2} - \frac{5\eta_j}{2\xi} - \frac{1}{\xi} \right] a_j^\dagger b^\dagger - i \frac{Be_\rho}{2M} \sum_{j=\pm} \left[\frac{\xi}{2} - \frac{5\eta_j}{2\xi} + \frac{1}{\xi} \right] a_j b^\dagger + (\text{H.c.}), \quad (\text{A10})$$

where $\xi = [1 + 32M\lambda/(Be_R)^2]^{1/4}$.

For the ground state, only $a_j^\dagger b^\dagger$ type terms yield non-zero contributions. The $a_+^\dagger b^\dagger$ increases the quanta $(n_+, n_-, n_R) \rightarrow (n_+ + 1, n_-, n_R + 1)$ with the excitation energy,

$$E_{\Delta n_+=1}^{\Delta n_R=1} = \frac{Be_R}{M} + \frac{\mathcal{B}_R - Be_R}{2M}, \quad (\text{A11})$$

while the $a_-^\dagger b^\dagger$ increases the quanta $(n_+, n_-, n_R) \rightarrow (n_+, n_- + 1, n_R + 1)$ with the energy

$$E_{\Delta n_-=1}^{\Delta n_R=1} = \frac{Be_R}{M} + \frac{\mathcal{B}_R + Be_R}{2M}, \quad (\text{A12})$$

Hence, the energy shift $\delta_1 E$ for the ground state is

$$\delta_1 E_{\text{gs}} = -\frac{(Be_\rho)^2}{2M} \left[\frac{(\frac{\xi}{2} - \frac{7}{2\xi})^2}{\mathcal{B}_R + Be_R} + \frac{(\frac{\xi}{2} + \frac{3}{2\xi})^2}{\mathcal{B}_R + 3Be_R} \right]. \quad (\text{A13})$$

Next we evaluate $\delta_2 E$. The straightforward calculations yield

$$\begin{aligned} \langle n_\perp | \rho_\perp^2 | n_\perp \rangle &= \frac{8}{Be_R \xi^2} (n_+ + n_- + 1), \\ \langle n_\perp | (\mathbf{p}_\rho^\perp)^2 | n_\perp \rangle &= \frac{Be_R \xi^2}{8} (n_+ + n_- + 1), \\ \langle n_R | R_y^2 | n_R \rangle &= \frac{1}{2Be_R} (2n_R + 1). \end{aligned} \quad (\text{A14})$$

The corresponding energy shift is

$$\delta_2 E_{\text{gs}} = \frac{Be_\rho^2}{4Me_R} \frac{\xi^2 + 10}{\xi^2} + \frac{2Be_\rho \delta m}{M^2 \xi^2} + \frac{8\delta m^2 \lambda}{M^2 Be_R \xi^2}, \quad (\text{A15})$$

where we used $\xi^4 - 1 = 32M\lambda/(Be_R)^2$. Including the corrections to the Zeeman energy, the results are summarized as

$$\begin{aligned}
\delta_1 E_{\text{gs}} &= -\frac{(Be_\rho)^2}{2M} \left[\frac{(\frac{\xi}{2} - \frac{7}{2\xi})^2}{\mathcal{B}_R + Be_R} + \frac{(\frac{\xi}{2} + \frac{3}{2\xi})^2}{\mathcal{B}_R + 3Be_R} \right], \\
\delta_2 E_{\text{gs}} &= \frac{Be_\rho^2}{4Me_R} \frac{\xi^2 + 10}{\xi^2} + \frac{2Be_\rho \delta m}{M^2 \xi^2} + \frac{8\delta m^2 \lambda}{M^2 Be_R \xi^2}, \\
\delta_{2,\text{Zm}} E_{\text{gs}} &= -\frac{2Be_\rho \delta m}{M^2 \xi^2},
\end{aligned} \tag{A16}$$

where $\xi = [1 + 32M\lambda/(Be_R)^2]^{1/4}$ and we have used $\xi^4 - 1 = 32M\lambda/(Be_R)^2$ in computations of $\delta_2 E_{\text{gs}}$.

In the limit of $B \rightarrow \infty$, the scaling of parameters is $\mathcal{B}_R \rightarrow Be_R$, $\xi \rightarrow 1$. The perturbative corrections of $O(Be_\rho^2)$ are assembled to cancel,

$$\begin{aligned}
\delta_1 E_{\text{gs}} &\rightarrow -\frac{11Be_\rho^2}{4Me_R}, \\
\delta_2 E_{\text{gs}} &\rightarrow \frac{11Be_\rho^2}{4Me_R} + \frac{2Be_\rho \delta m}{M^2}, \\
\delta_{2,\text{Zm}} E_{\text{gs}} &\rightarrow -\frac{2Be_\rho \delta m}{M^2},
\end{aligned} \tag{A17}$$

and hence the B -dependence appears only through $O(M\lambda/B^2)$.

APPENDIX B: FORMULAS FOR MATRIX ELEMENTS

For perturbative evaluation of short-range potentials for neutral and charged mesons, here we summarize some useful formulas. We begin with

$$\frac{1}{\sqrt{\xi^2 + a^2}} = \int_0^\infty ds \frac{e^{-s(\xi^2 + a^2)}}{\sqrt{\pi s}}. \tag{B1}$$

In computations of V_E , the three-dimensional integral is reduced to the one-dimensional integral,

$$\begin{aligned}
I_E[\Sigma] &= \int_r \frac{e^{-x^T \Sigma x}}{|x - x_0|} \\
&= \int_0^\infty \frac{ds}{\sqrt{\pi s}} \int_r e^{-x^T \Sigma x - s(x - x_0)^2} \\
&= \int_0^\infty ds \frac{\pi e^{s^2 x_0^T (\Sigma + sI)^{-1} x_0 - s x_0^2}}{\sqrt{s \det(\Sigma + sI)}}.
\end{aligned} \tag{B2}$$

For the magnetic potential, we encounter the following types of integrals,

$$I_0^M[\tilde{\Sigma}] = \int_q e^{-q \tilde{\Sigma} q - i q_\perp \cdot r_\perp^0} = \frac{1}{8\pi^{3/2}} \frac{w(r_\perp^0)}{\sqrt{\det \tilde{\Sigma}}}, \tag{B3}$$

$$\begin{aligned}
\tilde{I}_0^M[A] &= \int_q \frac{e^{-q_a \tilde{\Sigma}_{ab} q_b - i q_\perp \cdot r_\perp^0}}{q^2 + m^2} \\
&= \frac{1}{8\pi^{3/2}} \int_0^\infty ds \frac{e^{-sm^2} w(r_\perp^0)}{\sqrt{\det(\tilde{\Sigma} + sI)}},
\end{aligned} \tag{B4}$$

where

$$w(r_\perp^0) = e^{-\frac{1}{4}(r_\perp^0)^T \tilde{\Sigma}^{-1} r_\perp^0}. \tag{B5}$$

We also compute the integral

$$\begin{aligned}
I_j^M[\tilde{\Sigma}] &= \int_q e^{-q^T \tilde{\Sigma} q - i q_\perp \cdot r_\perp^0} \frac{q_j^2}{q^2} \\
&= -\frac{\partial}{\partial \tilde{\Sigma}_{jj}} \int_0^\infty ds \int_q e^{-q^T \tilde{\Sigma} q - s q^2 - i q_\perp \cdot r_\perp^0} \\
&= -\frac{\partial}{\partial \tilde{\Sigma}_{jj}} \frac{1}{8\pi^{3/2}} \int_0^\infty ds \frac{w(r_\perp^0)}{\sqrt{\det(\tilde{\Sigma} + sI)}}
\end{aligned} \tag{B6}$$

and

$$\begin{aligned}
\tilde{I}_j^M[\tilde{\Sigma}] &= \int_q e^{-q^T \tilde{\Sigma} q - i q_\perp \cdot r_\perp^0} \frac{1}{q^2 + m^2} \frac{q_j^2}{q^2} \\
&= -\frac{\partial}{\partial A_{jj}} \int_0^\infty du \int_0^\infty dt \int_q e^{-q^T \tilde{\Sigma} q - t(q^2 + m^2) - u q^2 - i q_\perp \cdot r_\perp^0} \\
&= -\frac{\partial}{\partial A_{jj}} \int_0^\infty ds \int_0^s dt \int_q e^{-q^T [\tilde{\Sigma} + sI] q - t m^2 - i q_\perp \cdot r_\perp^0} \\
&= -\frac{\partial}{\partial A_{jj}} \frac{1}{8\pi^{3/2}} \int_0^\infty ds \frac{(1 - e^{-sm^2}) w(r_\perp^0)}{m^2 \sqrt{\det(\tilde{\Sigma} + sI)}}.
\end{aligned} \tag{B7}$$

In particular, when A is diagonal,

$$I_j^M[\tilde{\Sigma}] = \frac{w(r_\perp^0)}{16\pi^{3/2}} \int_0^\infty ds \frac{1}{\tilde{\Sigma}_{jj} + s} \prod_k \frac{1}{\sqrt{\tilde{\Sigma}_{kk} + s}}, \tag{B8}$$

$$\tilde{I}_j^M[\tilde{\Sigma}] = \frac{w(r_\perp^0)}{16\pi^{3/2}} \int_0^\infty ds \frac{1 - e^{-sm^2}}{m^2 (\tilde{\Sigma}_{jj} + s)} \prod_k \frac{1}{\sqrt{\tilde{\Sigma}_{kk} + s}}. \tag{B9}$$

With these formulas and setting $\tilde{\Sigma}' \equiv \tilde{\Sigma} + \mu_{\text{IR}}^{-2} I$, we compute the integrals appearing in the magnetic energy as

$$\begin{aligned}
\mathcal{G}_0 &\equiv \int_q e^{-q^T \tilde{\Sigma} q - i q_\perp \cdot r_\perp^0} \mathcal{F}_s \\
&= \frac{I_0^M[\tilde{\Sigma}']}{m_1 m_2} + \tilde{I}_0^M[\tilde{\Sigma}] - \tilde{I}_0^M[\tilde{\Sigma}'],
\end{aligned} \tag{B10}$$

$$\begin{aligned}\mathcal{G}_j &\equiv \int_{\mathbf{q}} e^{-\mathbf{q}^T \tilde{\Sigma} \mathbf{q} - i \mathbf{q}_\perp \cdot \mathbf{r}_\perp^0} \mathcal{F}_s \frac{q_j^2}{q^2} \\ &= \frac{I_j^M[\tilde{\Sigma}']}{m_1 m_2} + \tilde{I}_j^M[\tilde{\Sigma}] - \tilde{I}_j^M[\tilde{\Sigma}']. \end{aligned} \quad (\text{B11})$$

The magnetic energy can be written as

$$\langle V_I^s \rangle = \frac{16\pi\alpha_s}{3} \int_{\mathbf{q}} e^{-\mathbf{q}^T \tilde{\Sigma} \mathbf{q} - i \mathbf{q}_\perp \cdot \mathbf{r}_\perp^0} \frac{\mathcal{F}_s}{2} \left(1 + \frac{q_z^2}{q^2} \right) = \frac{8\pi\alpha_s}{3} (\mathcal{G}_0 + \mathcal{G}_z), \quad (\text{B12})$$

$$\langle V_{II}^s \rangle = \frac{16\pi\alpha_s}{3} \int_{\mathbf{q}} e^{-\mathbf{q}^T \tilde{\Sigma} \mathbf{q} - i \mathbf{q}_\perp \cdot \mathbf{r}_\perp^0} \frac{\mathcal{F}_s}{2} \left(1 - \frac{3q_z^2}{q^2} \right) = \frac{8\pi\alpha_s}{3} (\mathcal{G}_0 - 3\mathcal{G}_z). \quad (\text{B13})$$

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