

Timelike holographic complexity

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Motivated by the pseudoentropy program, we study timelike subregion complexity within the holographic complexity = volume framework, extending previous spatial constructions to Lorentzian boundary intervals. For hyperbolic timelike regions in pure anti-de Sitter (AdS), we compute the enclosed bulk volume and show that, despite the Lorentzian embedding, the resulting complexity is purely real. We generalize the analysis to AdS black brane geometries, where extremal surfaces may either remain entirely outside the horizon or penetrate it, placing their timelike branch inside the black brane interior. In both configurations, the complexity exhibits the same universal UV divergences as the spacelike case, yet it receives no imaginary contribution—highlighting its causal and geometric origin. This reality stands in sharp contrast to the complex-valued pseudoentropy and indicates that holographic complexity retains a genuinely geometric, real character even under Lorentzian continuation.

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I. INTRODUCTION

Quantum entanglement has long been recognized as a fundamental probe of quantum correlations and spacetime structure in holographic theories [1–3]. Traditionally, entanglement entropy is defined for spatial subregions on a fixed time slice, capturing the pattern of quantum correlations among spatially separated degrees of freedom in the boundary theory. More recently, this notion has been extended beyond equal-time slices, leading to the introduction of a novel quantity known as pseudoentropy [4–6]. This extension allows one to study entanglementlike measures associated with timelike subregions and with transitions between quantum states, thereby enlarging the scope of quantum information probes available in holographic systems.

Pseudoentropy generalizes entanglement entropy to situations involving nontrivial quantum state transitions. Given an initial state $|\psi_i\rangle$ and a final state $|\psi_f\rangle$, one defines a transition matrix

$$\tau = \frac{|\psi_f\rangle\langle\psi_i|}{\langle\psi_i|\psi_f\rangle}, \quad (1)$$

and the pseudoentropy of a subsystem A (for a bipartite system $A \cup B$) as

$$S_A^{\text{pseudo}} = -\text{Tr} \tau_A \log \tau_A, \quad \tau_A = \text{Tr}_B \tau. \quad (2)$$

Unlike conventional entanglement entropy, pseudoentropy is generically complex due to the non-Hermitian nature of the transition matrix τ . Its real part captures a generalized notion of correlation between $|\psi_i\rangle$ and $|\psi_f\rangle$, while its imaginary part encodes phaselike information associated with temporal ordering and causal structure. The emergence of complex values thus signals a richer analytic structure underlying holographic information measures.

Within the AdS/CFT correspondence [7], pseudoentropy naturally motivates the study of timelike extremal surfaces anchored on timelike boundary intervals [8–10]. These surfaces extend the Ryu-Takayanagi (RT) prescription [1,2] into Lorentzian regions of anti-de Sitter (AdS), where the induced metric on the extremal surface may change signature. Such generalizations have attracted increasing attention in recent years (see, e.g., [11–23]). In these constructions, the corresponding holographic entanglement entropy for timelike subregions typically acquires an imaginary part, providing a bulk dual interpretation of complex-valued pseudoentropy.

This naturally raises the question of whether other quantum information measures—most notably quantum complexity—admit a consistent timelike generalization, and if so, what boundary quantity they compute.

Holographic complexity has been proposed as a measure of the computational cost required to prepare a given boundary state from a simple reference state using a minimal set of quantum operations. Two prominent proposals are the “complexity = volume” (CV) conjecture [24,25], which was further extended to spatial subregion

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complexity in [26] (see also [27,28]), and the “complexity = action” (CA) conjecture [29,30], whose subregion version has been explored in [31–33]. These frameworks have substantially deepened our understanding of information storage, state preparation, and dynamical complexity growth in holographic field theories, primarily for spatially defined subsystems and global states.

Despite this progress, the extension of holographic complexity to timelike subregions has remained largely unexplored. Motivated by the pseudoentropy program, in this work we introduce and analyze a timelike version of subregion complexity within the CV proposal. This involves evaluating the bulk volume enclosed by timelike extremal surfaces, thereby extending the original construction of [26] into Lorentzian regions of AdS. Conceptually, this provides a natural holographic candidate for a measure of temporal circuit depth, or equivalently, the complexity associated with time evolution between quantum configurations in a given boundary subregion.

A crucial feature of this construction is that, although it relies on timelike extremal surfaces, the resulting bulk region is everywhere real and Lorentzian. Consequently, the CV functional yields a strictly real quantity. This sharply distinguishes timelike subregion complexity from pseudoentropy, and suggests that holographic complexity retains a purely geometric interpretation even in Lorentzian settings. In particular, the emergence of linear growth in the gravitational computation provides strong evidence that the dual boundary quantity is naturally interpreted as a notion of quantum complexity, since linear growth is widely regarded as a defining dynamical feature of complexity in chaotic quantum systems.

We first focus on timelike hyperbolic subregions in AdS_{d+2} , building on the explicit extremal surface solutions constructed in [8,9]. We compute the associated bulk volumes and analyze their divergence structure in detail. Remarkably, we find that timelike subregion complexity remains purely real, reinforcing its interpretation as a geometric observable and indicating that it probes aspects of spacetime geometry inaccessible to entanglement-based measures.

We then extend our analysis to thermal states dual to AdS black brane geometries, where the presence of an event horizon introduces qualitatively new features. In this setting, timelike extremal surfaces may either remain entirely in the exterior region or penetrate into the black hole interior, depending on the boundary time interval. We show that the resulting complexity exhibits universal UV divergences together with finite, horizon-sensitive contributions, while remaining real in all regimes. This allows us to identify a dimension-dependent bound on the maximal interior penetration of extremal surfaces, providing a sharp geometric constraint on timelike subregion complexity.

The remainder of this paper is organized as follows. In Sec. II, we compute timelike subregion complexity for

hyperbolic regions in pure AdS and discuss its general properties. In Sec. III, we generalize the construction to AdS black brane backgrounds and analyze the behavior of extremal surfaces and complexity near and beyond the horizon. We conclude in Sec. IV with a summary of our results and a discussion of future directions.

II. TIMELIKE SUBREGION AND HOLOGRAPHIC VOLUME

In standard holographic constructions, a subregion is defined as a spatial domain on a constant-time slice. In contrast, here we consider timelike subregions, namely codimension-one regions that extend along the time direction while being localized in space. Such regions have appeared in the study of timelike entanglement entropy and pseudoentropy. These objects should not be interpreted as subsystems in the usual quantum information sense, since they do not admit a tensor factorization of the Hilbert space. Rather, they encode information about the evolution of a spatial region over a finite time interval.

Following [8,9], we consider a timelike boundary subregion in an AdS_{d+2} spacetime with $d \geq 1$. The bulk geometry is described by the metric

$$ds^2 = \frac{R^2}{r^2} (dr^2 + dy^2 - d\xi^2 + \xi^2 dX_{d-1}^2), \quad (3)$$

where y is a spectator coordinate, X_{d-1} denotes the $(d-1)$ -dimensional hyperbolic space (in pure AdS), and R is the AdS radius. The boundary subsystem is chosen to be a timelike hyperbolic region defined by $\xi^2 \leq \frac{T^2}{4}$, with T the boundary time interval.

The area functional for a codimension-two extremal surface anchored on this region is

$$A = H_{d-1} R^d \int dr \frac{\xi^{d-1} \sqrt{1 - \xi'^2}}{r^d}, \quad (4)$$

where $H_{d-1} = \text{Vol}(X_{d-1})$ is the transverse volume. Extremizing this functional yields two distinct branches of solutions [8,9],

$$\xi^2 = r^2 + \frac{T^2}{4}, \quad \xi^2 = r^2 - \frac{T^2}{4}, \quad (5)$$

corresponding to spacelike and timelike extremal surfaces, respectively. The timelike branch exists only for $r \geq T/2$, thereby introducing a causal cutoff in the bulk geometry.

Together, these two branches form the complete extremal surface configuration associated with the timelike subregion. As shown in [8,9], while the area of the spacelike branch remains real, the timelike branch contributes an imaginary term to the holographic entanglement entropy. This feature motivates examining the corresponding construction in holographic complexity, where one might

expect qualitatively different behavior due to the volume-based nature of the proposal.

In this work we propose a geometric quantity associated with a timelike boundary region and conjecture that it captures a notion of complexity in the dual field theory. Since the boundary region extends along a timelike direction, it does not define a subsystem in the usual sense and therefore does not admit a standard reduced density matrix description. Instead, we adopt an interpretation inspired by pseudoentropy and timelike entanglement entropy. Consider a spatial subregion A of the boundary theory and two Cauchy slices at times t_1 and t_2 . One may associate to these slices the reduced states $\rho_A(t_1)$ and $\rho_A(t_2)$, or more generally a transition matrix between them. We propose that the bulk quantity defined in this paper provides a measure of the complexity associated with transforming the configuration of the system in region A at time t_1 to that at time t_2 . In this sense, it may be viewed as a candidate for a transition complexity associated with a timelike interval.

A precise definition of the boundary state associated with a timelike region requires specifying the algebra of observables localized on such a region. This is subtle, as timelike-separated regions do not define commuting operator algebras. In analogy with pseudoentropy, one may instead consider transition amplitudes or generalized density matrices associated with pairs of states. In this work, we do not attempt to construct this object explicitly. Rather, we take a pragmatic approach and propose that the bulk geometric quantity captures a complexitylike measure associated with the transition between configurations of the system in a fixed spatial region at different times. At present, this identification should be regarded as a conjecture. A first-principles derivation from the boundary theory, including a precise definition of the relevant operator algebra associated with a timelike region, remains an open problem. Nevertheless, the geometric construction we propose satisfies several properties expected of a complexity measure, such as positivity, real-valuedness, and a natural dependence on the length of the timelike interval.

It is natural to expect that such a transition complexity may be related to existing field-theoretic notions of quantum complexity, such as circuit complexity defined via unitary gates, path-integral optimization, or Lorentzian tensor network constructions. In these approaches, one attempts to quantify the minimal computational cost required to transform one quantum state into another under suitable locality and causality constraints. While a precise mapping between our geometric proposal and these frameworks remains to be established, the qualitative features we observe—such as positivity, real-valuedness, and linear growth at late times—are consistent with expectations from these definitions. We leave a more direct field-theoretic formulation of timelike subregion complexity for future work.

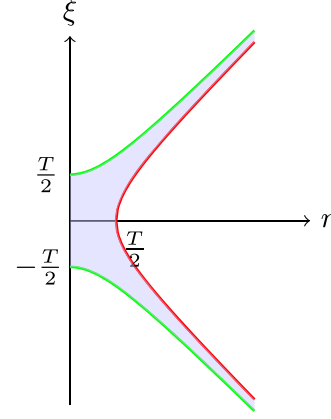


FIG. 1. The colored region between the spacelike (green) and timelike (red) extremal surfaces represents the bulk volume associated with the timelike subregion complexity.

Within the subregion CV prescription [26], holographic complexity is proportional to the bulk volume enclosed by the relevant extremal surfaces,

$$V = H_{d-1} R^{d+1} \int_{\xi \leq f(r)} dr d\xi \frac{\xi^{d-1}}{r^{d+1}}, \quad (6)$$

where $f(r)$ denotes the extremal profiles given in Eq. (5). The bulk region entering our definition is constructed geometrically as the domain bounded by the boundary timelike region and the associated extremal surfaces, as illustrated in Fig. 1. Since the extremal surfaces are defined covariantly, the resulting construction is independent of coordinate choices. In all examples considered in this work, the symmetry of the setup uniquely determines the bulk region.

Performing the ξ integration yields the volume difference between the two branches,

$$V = \frac{2H_{d-1}R^{d+1}}{d} \left[\int_{\epsilon}^{\infty} dr \frac{(r^2 + \frac{T^2}{4})^{\frac{d}{2}}}{r^{d+1}} - \int_{\frac{T}{2}}^{\infty} dr \frac{(r^2 - \frac{T^2}{4})^{\frac{d}{2}}}{r^{d+1}} \right], \quad (7)$$

where ϵ is a UV cutoff near the AdS boundary at $r = 0$. The lower limit of the second integral reflects the fact that the timelike branch exists only for $r \geq T/2$, while the overall factor of 2 arises from symmetry of the configuration.

Following [26], the timelike subregion complexity is defined as

$$\mathcal{C}_T = \frac{V}{GR}, \quad (8)$$

where G is the $(d+1)$ -dimensional Newton constant. We regard this construction as a minimal covariant extension of subregion CV complexity to timelike domains,

defined by extremal surfaces compatible with boundary causal structure.

It is important to emphasize that although neither branch alone defines a conventional homology region, the pair together admits a natural covariant interpretation: they bound the maximal real Lorentzian bulk domain that can be reached by spacelike surfaces anchored to the timelike boundary interval. The bulk volume entering the CV prescription is therefore not chosen *ad hoc*, but is fixed uniquely by causal and geometric considerations.

Although each integral in Eq. (7) separately exhibits an infrared divergence as $r \rightarrow \infty$, these divergences cancel in the difference, leaving a finite IR contribution. The remaining UV divergences are regulated by the cutoff ϵ .

While the integrals in Eq. (7) can be expressed in terms of hypergeometric functions for arbitrary d , it is more illuminating to present explicit results in lower dimensions. For $d = 1, 2, 3, 4$, we find

$$\text{AdS}_3: \quad C_T = \frac{R}{G} \frac{T}{2\epsilon}, \quad (9)$$

$$\text{AdS}_4: \quad C_T = \frac{V_1 R^2}{2G} \left(\frac{T^2}{8\epsilon^2} + \log \frac{T}{2\epsilon} + \frac{1}{2} \right), \quad (10)$$

$$\text{AdS}_5: \quad C_T = \frac{V_2 R^3}{3G} \left(\frac{T^3}{24\epsilon^3} + \frac{3T}{4\epsilon} \right), \quad (11)$$

$$\text{AdS}_6: \quad C_T = \frac{V_3 R^4}{4G} \left(\frac{T^4}{64\epsilon^4} + \frac{T^2}{4\epsilon^2} + \log \frac{T}{2\epsilon} + \frac{3}{4} \right). \quad (12)$$

The divergence structure closely parallels that of spacelike subregion complexity: power-law UV divergences associated with short-distance correlations, and universal logarithmic terms appearing in even dimensions. The leading divergence scales with the volume of the boundary subregion, confirming that the dominant contribution to holographic complexity arises from the near-boundary geometry.

A key distinction from timelike entanglement entropy now becomes apparent. The expressions in Eq. (9) are manifestly real. While the timelike branch of the extremal surface contributes an imaginary term to entanglement entropy—due to analytic continuation across the light cone—the complexity computation integrates over a strictly real Lorentzian bulk region with $r \geq T/2$. As a result, the CV prescription yields a real-valued geometric quantity.

The fact that the proposed quantity is real-valued follows directly from the choice of the Lorentzian volume element $\sqrt{-\det g_{\text{ind}}}$, which is the natural generalization of the standard CV prescription to timelike hypersurfaces. This choice ensures that the resulting volume is real and positive, as expected for a complexity measure. We emphasize that this prescription is a direct Lorentzian continuation of the standard CV proposal for spacelike hypersurfaces, in which the volume element is defined

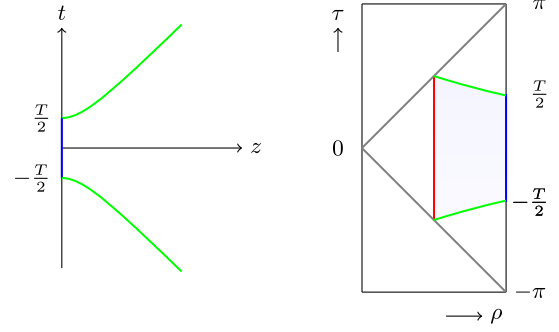


FIG. 2. Extremal surfaces for a timelike entangling region (blue interval) in Poincaré (left) and global (right) coordinates. In the global case, in addition to the spacelike extremal surface (green), there exists a timelike one (red). The timelike subregion complexity corresponds to the bulk volume enclosed between these two surfaces, illustrated by the shaded (colored) region. This figure essentially reproduces Fig. 3 of [8] for clarity and comparison. Here, (r, z) and (τ, ρ) denote the Poincaré and global coordinate systems, respectively.

using the induced metric on the bulk slice. The use of $\sqrt{-\det g_{\text{ind}}}$ is therefore not arbitrary, but follows from requiring covariance and consistency with the Euclidean/spacelike formulation. Alternative prescriptions would either violate reparametrization invariance or fail to reproduce the standard CV result in the appropriate limit. In contrast, the complex nature of timelike holographic entanglement entropy arises from the use of length functionals that do not incorporate this sign adjustment. From this perspective, the reality of our result is not a dynamical feature but rather a consequence of adopting a physically appropriate definition of volume in Lorentzian signature.

To build further intuition, consider the three-dimensional case, where the extremal surfaces can be visualized explicitly in both Poincaré and global coordinates, as shown in Fig. 2 (see also [8]). In the entanglement entropy computation, the timelike branch becomes evident upon transforming to global coordinates, giving rise to the imaginary contribution characteristic of pseudoentropy. For complexity, however, the relevant bulk volume is not that enclosed by the spacelike surface alone, but rather the region bounded jointly by the spacelike and timelike branches, illustrated by the shaded domain in Fig. 2.

One might wonder whether the coexistence of spacelike and timelike branches implies a hidden causal connection that could, in principle, generate an imaginary contribution to the complexity. At present, however, it remains unclear whether such a causal volume genuinely exists.¹ Instead, the explicit calculation shows that any potential phase factors cancel, leaving a purely real result. Indeed, from a technical standpoint, for odd d both endpoints of the

¹I would like to thank T. Takayanagi for a comment on this point.

timelike integral acquire phase factors that precisely cancel. Timelike subregion complexity therefore measures only the real, causally accessible Lorentzian bulk volume, while pseudoentropy captures additional phase information associated with temporal correlations.

Comparing with the spacelike subregion results of [26], we find that although the divergence structure of timelike complexity closely mirrors the spacelike case, the two are not related by a simple Wick rotation. Both exhibit identical hierarchies of UV divergences, but their finite terms differ both quantitatively and conceptually. In particular, for odd-dimensional AdS backgrounds, timelike complexity lacks the universal constant term present in the spacelike case. This absence suggests that timelike complexity has a genuinely distinct geometric origin, rooted in the Lorentzian causal structure of the extremal hypersurface.

This distinction can be further clarified by reconsidering the extremization problem for a spacelike spherical subregion at the boundary. The corresponding area functional admits two solutions,

$$\xi(r) = \sqrt{\ell^2 - r^2}, \quad \xi(r) = i\sqrt{\ell^2 + r^2}, \quad (13)$$

where ℓ is the radius of the boundary region. The first solution is the familiar Ryu-Takayanagi surface [1], which underlies standard computations of entanglement entropy and spacelike subregion complexity [26]. The second solution is purely imaginary and is usually discarded as unphysical, since it does not correspond to a real Lorentzian surface in the bulk. From the present perspective, however, this “discarded” branch plays a crucial role. Upon analytic continuation, it reproduces precisely the timelike extremal branch encountered in timelike entanglement entropy and timelike subregion complexity. The imaginary embedding $\xi = i\sqrt{\ell^2 + r^2}$ represents a continuation across the light cone, encoding the causal extension of the boundary region into the timelike domain. Its contribution is responsible for the imaginary part of pseudoentropy, reflecting phase information associated with nonunitary or time-reflected evolutions of the boundary state.

In this sense, the real and imaginary branches are not independent, but rather analytic continuations of a single underlying geometric structure. The real branch probes spatial entanglement, while the imaginary branch captures temporal and causal correlations. This unified picture naturally bridges spacelike and timelike holographic observables.²

Within the CV framework, however, only the real Lorentzian bulk volume contributes directly, yielding a

²It is worth noting that [34,35] propose a systematic holographic framework for timelike entanglement entropy based on complexified bulk geometries. I thank M. Heller for bringing these insightful works to my attention.

strictly real measure of complexity. Although the extremal configuration for timelike subregions includes a timelike branch, the resulting complexity remains real, underscoring the fundamentally geometric—as opposed to entropic—nature of holographic complexity.

III. TIMELIKE SUBREGION COMPLEXITY FOR ADS BLACK BRANES

It is natural to extend our discussion of timelike subregion complexity to thermal states, which are holographically dual to AdS black hole or black brane geometries. In this context, the thermal nature of the boundary state is encoded in the presence of an event horizon, which introduces an additional geometric scale and modifies the causal structure relevant for extremal surfaces anchored to timelike subregions. For concreteness, we consider a $(d+2)$ -dimensional AdS black hole spacetime with metric

$$ds^2 = \frac{R^2}{r^2} \left(-f(r)dt^2 + \frac{dr^2}{f(r)} + dx^2 + dY_{d-1}^2 \right), \quad (14)$$

where

$$f(r) = 1 - \frac{r^{d+1}}{r_h^{d+1}}, \quad (15)$$

and r_h denotes the horizon radius. The conformal boundary is located at $r = 0$. Here, R is the AdS radius, and dY_{d-1}^2 denotes the metric on the transverse $(d-1)$ -dimensional space. The temperature of the dual thermal state is given by the Hawking temperature $T_H = (d+1)/(4\pi r_h)$.

We consider a timelike subregion on the boundary defined by a finite time interval $t \in [-T/2, T/2]$ at fixed spatial coordinate x , while extending uniformly along the transverse y_i directions. Although this configuration lacks the manifest symmetry of the hyperbolic subregions discussed earlier, the extremization procedure can still be carried out by considering timelike embeddings of the form $t = t(r)$, following the general framework for timelike holographic complexity [9]. The induced metric on such a hypersurface is

$$ds_{\text{ind}}^2 = \frac{R^2}{r^2} \left[\left(\frac{1}{f(r)} - f(r)t'(r)^2 \right) dr^2 + dY_{d-1}^2 \right], \quad (16)$$

where $t'(r) = dt/dr$. The corresponding area functional reads

$$A = L^{d-1} R^d \int dr \frac{\sqrt{\frac{1}{f(r)} - f(r)t'(r)^2}}{r^d}, \quad (17)$$

with $L^{d-1} = \text{Vol}(Y_{d-1})$. Extremizing this functional yields a conserved quantity,

$$\frac{-f(r)t'(r)}{r^d \sqrt{\frac{1}{f(r)} - f(r)t'(r)^2}} = \text{constant}, \quad (18)$$

which can be interpreted as an effective “energy” associated with the surface profile $t(r)$. This conservation law follows from the fact that the Lagrangian does not depend explicitly on t , making it analogous to energy conservation in classical mechanics. For real-valued $t'(r)$, this constant must itself be real; however, for timelike subregions this requirement is subtle. This subtlety results in a possibility that the extremal surface extends beyond the horizon, as we will demonstrate explicitly.

To systematically capture the different possible extremal surface configurations anchored to timelike intervals, it is convenient to square the conservation equation and introduce a constant $\pm 1/b^2$,

$$\frac{f^2(r)t'^2(r)}{r^{2d}(\frac{1}{f(r)} - f(r)t'^2(r))} = \pm \frac{1}{b^2}, \quad (19)$$

where b is a positive constant with dimensions of $[\text{length}]^d$, to be fixed by boundary conditions. The choice of sign distinguishes two qualitatively different classes of extremal surface solutions. Solving algebraically for $t'(r)$ gives the first-order differential equation (see also [36])

$$t'(r) = \pm \frac{r^d}{f(r)\sqrt{r^{2d} \pm b^2 f(r)}}, \quad (20)$$

to be integrated subject to the boundary condition $t(\epsilon) = \pm T/2$, where ϵ is a UV cutoff. The overall $\pm$ sign corresponds to the two time-reflection-symmetric branches of the surface (one for increasing t , one for decreasing t), while the $\pm$ inside the square root differentiates the two solution classes: the minus sign corresponds to the exterior solution, and the plus sign to the interior solution.

From the expression for $t'(r)$, a turning point r_0 occurs when $t'(r_0) \rightarrow \infty$. This divergence can arise in two ways: first, when $f(r_0) = 0$, corresponding to the horizon at $r_0 = r_h$; and second, when the argument of the square root vanishes, namely $r_0^{2d} \pm b^2 f(r_0) = 0$. Depending on the sign choice, this yields turning points either outside or inside the horizon:

- (i) Exterior solution (minus sign): The condition $r_0^{2d} - b^2 f(r_0) = 0$ with $f(r_0) > 0$ gives a turning point $r_0 < r_h$ outside the horizon.
- (ii) Interior solution (plus sign): The condition $r_0^{2d} + b^2 f(r_0) = 0$ with $f(r_0) < 0$ requires $r_0 > r_h$, placing the turning point inside the horizon.

In both cases, the full extremal surface consists of two branches that meet at the horizon $r = r_h$. Both classes share a common structure: the first (spacelike) branch extends from the boundary cutoff $r = \epsilon$ to the horizon $r = r_h$, while the second (timelike) branch connects the horizon to the

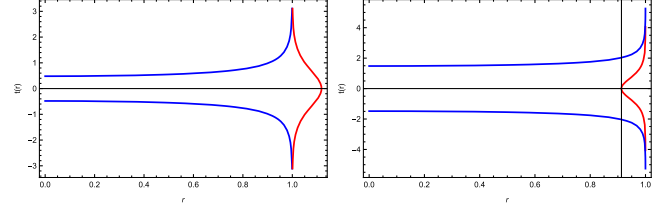


FIG. 3. Two examples of extremal surfaces. The left panel corresponds to the case where the turning point lies behind the horizon, while the right panel shows the configuration with two distinct branches. The blue and red curves represent spacelike and timelike extremal surfaces, respectively. The horizon is at $r_h = 1$.

turning point r_0 . The distinction lies in the origin of the second branch: for exterior solutions it originates from $r_0 < r_h$, while for interior solutions it comes from $r_0 > r_h$. See Fig. 3.

This interior possibility parallels extremal surfaces in time-dependent backgrounds such as Vaidya geometries [37], where horizon crossing is essential for capturing causal correlations. Although the bulk geometry here is static, the anchoring of extremal surfaces on a nonequal-time boundary interval forces the embedding functions to depend nontrivially on the radial direction. In this sense, the extremization problem inherits features typically associated with dynamical backgrounds, even in the absence of explicit bulk time dependence. The timelike nature of the boundary subregion effectively introduces a notion of time evolution, thereby justifying the existence of interior branches that cross the horizon. This analogy suggests that our construction captures dynamical aspects of complexity beyond what equal-time surfaces can probe.

Using the turning point condition, we can determine b in terms of r_0

$$b_1 = \frac{r_0^d}{\sqrt{f(r_0)}}(\text{exterior}), \quad b_2 = \frac{r_0^d}{\sqrt{-f(r_0)}}(\text{interior}). \quad (21)$$

The turning point r_0 is then fixed by imposing the total boundary time separation T via

$$T = -2 \int_0^{r_0} t'(r) dr. \quad (22)$$

Before proceeding, we clarify the interpretation of T and the structure of solutions. In the present setup, T denotes the length of a timelike interval on the boundary, not a dynamical evolution time in a time-dependent background. Accordingly, we refer to different regimes as short and long timelike intervals rather than early and late times.

A related question is the uniqueness of extremal surfaces for a given T . This is governed by the functional dependence $T(r_0)$, which determines the turning point r_0 as a function of the boundary interval. In the regimes we analyze explicitly, near the boundary ($r_0 \ll r_h$) and near

the horizon ($r_0 \simeq r_h$), $T(r_0)$ is monotonic, implying a unique extremal surface for each T . While a rigorous proof of monotonicity across the entire intermediate regime is technically involved and beyond the scope of this work, we find no indication of nonmonotonic behavior in the integrals. The absence of multiple turning points or additional branches in the regimes we control suggests that the solution we consider is the relevant saddle.

Nevertheless, we cannot rigorously exclude the possibility of additional solutions in intermediate regimes. We therefore interpret our results as describing the dominant branch, continuously connected to the near-boundary and near-horizon limits.

Explicitly,

$$\begin{aligned}
 T &= -2 \left(\int_0^{r_h} \frac{r^d dr}{f(r) \sqrt{r^{2d} + b_1^2 f(r)}} \right. \\
 &\quad \left. + \int_{r_h}^{r_0} \frac{r^d dr}{f(r) \sqrt{r^{2d} - b_1^2 f(r)}} \right) (\text{exterior}), \\
 T &= -2 \left(\int_0^{r_h} \frac{r^d dr}{f(r) \sqrt{r^{2d} + b_2^2 f(r)}} \right. \\
 &\quad \left. + \int_{r_h}^{r_0} \frac{r^d dr}{f(r) \sqrt{r^{2d} - b_2^2 f(r)}} \right) (\text{interior}). \quad (23)
 \end{aligned}$$

In both expressions, the first integral represents the contribution from the shared boundary-to-horizon branch. It is important to verify that this integral has no additional poles outside the horizon besides the expected divergence at $r = r_h$. The second integral, whose integrand differs in sign between the two configurations, captures the contribution from the horizon to the turning point.

The timelike subregion complexity, following the CV prescription (8), is computed from the volume of the bulk region enclosed by these extremal surfaces. After performing the integration over the transverse directions and the angular coordinate, one obtains

$$\begin{aligned}
 \mathcal{C}_T^{(\text{ext})} &= \frac{2L^{d-1}R^d}{dG} \left(\frac{T}{2\epsilon^d} - \int_\epsilon^{r_h} \frac{dr}{f(r) \sqrt{r^{2d} + b_1^2 f(r)}} \right. \\
 &\quad \left. - \int_{r_h}^{r_0} \frac{dr}{f(r) \sqrt{r^{2d} - b_1^2 f(r)}} \right), \quad (24)
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{C}_T^{(\text{int})} &= \frac{2L^{d-1}R^d}{dG} \left(\frac{T}{2\epsilon^d} - \int_\epsilon^{r_h} \frac{dr}{f(r) \sqrt{r^{2d} + b_2^2 f(r)}} \right. \\
 &\quad \left. - \int_{r_h}^{r_0} \frac{dr}{f(r) \sqrt{r^{2d} - b_2^2 f(r)}} \right). \quad (25)
 \end{aligned}$$

The factor of two accounts for time-reflection symmetry. The first term, $T/(2\epsilon^d)$, is the universal UV-divergent contribution proportional to the boundary time interval volume. The remaining integrals yield the finite, physically interesting part. Crucially, these expressions remain purely real even when the extremal surface penetrates the black hole interior ($r_0 > r_h$). This reality contrasts sharply with timelike entanglement entropy or pseudoentropy, which acquire imaginary contributions when surfaces extend behind horizons, highlighting a fundamental difference between complexity and entanglement measures.

We do not find evidence for multiple competing extremal surfaces in the regimes analyzed. However, smooth behavior alone does not guarantee uniqueness, and we cannot exclude the possibility of additional branches in other parameter regimes. Establishing global dominance would in principle require a more complete analysis of the solution space, possibly including numerical exploration. We therefore interpret the solutions presented here as the physically relevant saddles within the class of symmetric configurations considered, and treat the identified solution as the dominant saddle based on the available evidence.

We now analyze the behavior of timelike complexity across different regimes, determined by the location of the turning point relative to the horizon: the deep exterior regime ($r_0 \ll r_h$), the near-horizon regime ($r_0 \simeq r_h$), and the interior penetration regime ($r_h < r_0$). The analysis confirms that the timelike complexity, $\mathcal{C}_T$, remains manifestly real in all regimes, which is consistent with its interpretation as a geometric volume.

A. Deep exterior regime

When $r_0 \ll r_h$, the relevant extremal surface lies entirely in the exterior region, and the corresponding conserved quantity behaves as

$$b_1 \sim r_0^d. \quad (26)$$

Although we assume $r_0 \ll r_h$, the radial coordinate in the first integral appearing in the expression for the boundary time interval T ranges over the full interval $r \in [0, r_h]$. The dominant contribution to this integral arises from the near-horizon region. Setting

$$r = r_h(1 - \epsilon), \quad \epsilon \rightarrow 0^+, \quad (27)$$

we have $f(r) \simeq (d+1)\epsilon$. Then, the first integral contributing to T behaves as

$$\int_0^{r_h} \frac{r^d dr}{f(r) \sqrt{r^{2d} + b_1^2 f(r)}} \sim \int \frac{-r_h d\epsilon}{(d+1)\epsilon} \sim -\frac{r_h}{d+1} \ln \epsilon, \quad (28)$$

which is the familiar infrared logarithmic divergence near the horizon.

This divergence is canceled by the near-horizon contribution of the second integral in the expression for T . As a result, the dominant finite contribution to T originates from the turning-point region of the second integral. Indeed, it is worth emphasizing that both the boundary time T and the timelike complexity $\mathcal{C}_T$ are expressed as sums of two radial integrals coming from spacelike and timelike extremal surfaces, split at the horizon. A universal feature of these integrals is that near $r = r_h$, each integral individually develops an IR logarithmic divergence, though these divergences cancel between the two pieces, leading to the fact that the finite, physical contribution arises from the vicinity of the turning point r_0 . Therefore, in all regimes we focus on the near-turning-point region.

In the present cases in the turning-point region, we may approximate $f(r) \simeq 1$, yielding

$$\int_{r_h}^{r_0} \frac{r^d dr}{f(r) \sqrt{r^{2d} - b_1^2 f(r)}} \approx \int_{r_h}^{r_0} \frac{r^d dr}{\sqrt{r^{2d} - r_0^{2d}}}. \quad (29)$$

Since $r_0 \ll r_h$, the dominant contribution comes from $r \approx r_0$. Changing variables to $u = r/r_0$, we find

$$T \approx 2 \int_{r_0}^{r_h} \frac{r^d dr}{\sqrt{r^{2d} - r_0^{2d}}} \approx 2r_0 \int_1^\infty \frac{u^d du}{\sqrt{u^{2d} - 1}} \quad (30)$$

$$= 2 \frac{\sqrt{\pi} \Gamma(\frac{d+1}{2d})}{2\Gamma(\frac{1}{2d})} r_0 \quad (31)$$

Similarly, the dominant contribution to the complexity comes from the near-turning point region

$$\begin{aligned} \int_{r_h}^{r_0} \frac{dr}{f(r) \sqrt{r^{2d} - b_1^2 f(r)}} &\approx \int_{r_h}^{r_0} \frac{dr}{\sqrt{r^{2d} - r_0^{2d}}} \\ &\approx \frac{\sqrt{\pi} \Gamma(\frac{d+1}{2d})}{d\Gamma(\frac{1}{2d})} r_0^{1-d}. \end{aligned} \quad (32)$$

Thus, for $r_0 \ll r_h$ (short timelike interval), the finite part of the complexity scales as

$$\mathcal{C}_T^{\text{finite}} \sim r_0^{1-d} \sim T^{1-d}. \quad (33)$$

This indicates that at short times the black hole has little effect on the boundary complexity, and the system behaves similarly to pure AdS space, as expected.

We emphasize that this power-law behavior for short timelike intervals in the black brane geometry is not directly comparable to the pure AdS hyperbolic results of Eq. (9). Instead, it should be compared with the pure AdS strip entangling region in Poincaré coordinates, for which the timelike subregion complexity exhibits the same T^{1-d} scaling in the finite part. This agreement is expected

because at short intervals the extremal surface remains far from the horizon, and the geometry is well approximated by pure AdS.

B. Near-horizon regime

Let us consider the case where $r_0 \sim r_h$. We note that in the near-horizon regime, the turning point could approach the horizon from either side. We now consider the case where the turning point approaches the horizon from the exterior,

$$r_0 = r_h(1 - \delta), \quad \delta \rightarrow 0^+. \quad (34)$$

In this limit,

$$b_1^2 = \frac{r_0^{2d}}{f(r_0)} \simeq \frac{r_h^{2d}}{(d+1)\delta}. \quad (35)$$

As before, both integrals contributing to T and $\mathcal{C}_T$ develop logarithmic divergences near the horizon, which cancel between spacelike and timelike contributions. The dominant finite behavior again comes from the turning-point region.

Introducing

$$r = r_0(1 + \epsilon) = r_h(1 - (\delta - \epsilon)), \quad 0 < \epsilon < \delta, \quad (36)$$

we have $f(r) \simeq (d+1)(\delta - \epsilon)$. Thus, the turning-point contribution to T becomes

$$\begin{aligned} \int_{r_h}^{r_0} \frac{r^d dr}{f(r) \sqrt{r^{2d} - b_1^2 f(r)}} &\sim \int_\delta^0 \frac{-r_h d\epsilon}{(d+1)(\delta - \epsilon) \sqrt{1 - \frac{\delta - \epsilon}{\delta}}} \\ &\sim -\frac{r_h}{d+1} \ln \frac{1}{\delta}, \end{aligned} \quad (37)$$

where the last equality is obtained by evaluating the integral at upper limit $\epsilon \rightarrow 0$. Thus one gets

$$T \sim \frac{2r_h}{d+1} \ln \frac{1}{\delta}. \quad (38)$$

Inverting this relation yields

$$\delta \sim \exp\left(-\frac{d+1}{2r_h} T\right) = \exp(-2\pi T_H T), \quad (39)$$

where $T_H = \frac{d+1}{4\pi r_h}$ is the Hawking temperature. This expression reveals an exponential decay of the turning point's distance from the horizon as the boundary time increases.

The dominant contribution to the complexity comes from the region near $r \sim r_0$, where the extremal surface is nearly null. Using the same notation as above for the second integral of $\mathcal{C}_T^{(\text{ext})}$ in (24) one finds

$$\int_{r_h}^{r_0} \frac{dr}{f(r)\sqrt{r^{2d} - b_1^2 f(r)}} \sim \int_{\delta}^0 \frac{-r_h d\epsilon}{(d+1)(\delta - \epsilon)r_h^d \sqrt{1 - \frac{\delta - \epsilon}{\delta}}} - \frac{1}{r_h^d} \frac{r_h}{d+1} \ln \frac{1}{\delta}. \quad (40)$$

Thus

$$\mathcal{C}_T^{\text{finite}} \simeq \frac{L^{d-1} R^d}{d G r_h^d} \frac{2r_h}{d+1} \ln \frac{1}{\delta} + \mathcal{O}(e^{-2\pi T_H T}). \quad (41)$$

Using (38), we obtain the linear growth

$$\mathcal{C}_T^{\text{finite}} \simeq \frac{4S}{dL} T + \mathcal{O}(e^{-2\pi T_H T}), \quad (42)$$

where $S = \frac{R^d L^d}{4G r_h^d}$ is the total Bekenstein-Hawking entropy of the black brane (for transverse volume L^d). The same linear growth holds for the interior near-horizon limit [the near-turning-point integrals are identical after accounting for the sign flip in $f(r)$]. This should be understood as the celebrated late times linear growth of complexity whose slope of the linear growth phase is related to the mass of the black brane.

We now turn to the case where the turning point approaches the horizon from the interior,

$$r_0 = r_h(1 + \delta), \quad \delta \rightarrow 0^+. \quad (43)$$

The conserved quantity becomes

$$b_2^2 = -\frac{r_0^{2d}}{f(r_0)} \simeq -\frac{r_h^{2d}}{(d+1)\delta}. \quad (44)$$

The infrared divergences from either side of the horizon again cancel, leaving the turning-point contribution dominant. Setting

$$r = r_0(1 - \epsilon) = r_h(1 + \delta - \epsilon), \quad 0 < \epsilon < \delta, \quad (45)$$

with $f(r) \simeq -(d+1)(\delta - \epsilon)$, one finds

$$\int_{r_h}^{r_0} \frac{r^d dr}{f(r)\sqrt{r^{2d} + b_2^2 f(r)}} \sim \int_{\delta}^0 \frac{r_h d\epsilon}{(d+1)(\delta - \epsilon)\sqrt{1 - \frac{\delta - \epsilon}{\delta}}} \sim \frac{r_h}{d+1} \ln \frac{1}{\delta}. \quad (46)$$

It is also straightforward to evaluate the near-turning-point contribution to the complexity

$$\int_{r_h}^{r_0} \frac{dr}{f(r)\sqrt{r^{2d} + b_2^2 f(r)}} \sim \int_{\delta}^0 \frac{r_h d\epsilon}{(d+1)(\delta - \epsilon)r_h^d \sqrt{1 - \frac{\delta - \epsilon}{\delta}}} \sim \frac{1}{r_h^d} \frac{r_h}{d+1} \ln \frac{1}{\delta}. \quad (47)$$

As a result, one finds that the complexity exhibits the same linear growth

$$\mathcal{C}_T^{\text{finite}} \simeq \frac{4S}{dL} T + \mathcal{O}(e^{-2\pi T_H T}). \quad (48)$$

Thus, for both exterior and interior approaches to the horizon, the finite part of the timelike complexity grows linearly with boundary time.

The growth rate $\frac{4S}{dL}$ is now expressed directly in terms of the black-hole entropy S and the transverse size L . Any remaining temperature dependence enters only through $S \propto T_H^d$ [with $T_H = (d+1)/(4\pi r_h)$] and the subleading exponential corrections. This form facilitates a direct comparison with Lloyd's bound $\frac{dC}{dt} \leq \frac{2E}{\pi}$ for the full system, since the Arnowitt-Deser-Misner energy satisfies $E = \frac{d}{d+1} S T_H$. The factor $1/(dL)$ reflects the subregion nature of our observable (the timelike interval is localized at fixed x). A more precise field-theoretic identification of the dual boundary quantity may restore the familiar saturation of Lloyd's bound in the appropriate limit; we leave this for future work.

C. Interior penetration and maximum depth

Let us consider the case where the timelike branch of the extremal surfaces lies behind the horizon which in turn requires $r_0 > r_h$. An immediate question one may ask is how far the timelike complexity can probe inside the horizon. To address this question we note that for interior turning points, the turning-point condition is

$$V(r) \equiv r^{2d} + b_2^2 f(r) = 0. \quad (49)$$

On the other hand a valid turning point requires:

$$V(r_0) = 0, \quad V'(r_0) < 0, \quad (50)$$

so that the square root changes sign correctly. Therefore the limiting value $r_{\text{max}}(d)$, if it exists, is determined by the marginal condition

$$V(r_{\text{max}}) = 0, \quad V'(r_{\text{max}}) = 0. \quad (51)$$

Using

$$f'(r) = -\frac{d+1}{r_h^{d+1}} r^d, \quad (52)$$

one finds

$$r_{\max} = r_h \left(\frac{2d}{d-1} \right)^{\frac{1}{d+1}}, \quad (d > 1). \quad (53)$$

Beyond $r_{\max}$, the effective potential $V(r)$ no longer admits a physical turning point. Consequently, no extremal surface anchored to a finite boundary time interval exists, and the timelike complexity $\mathcal{C}_T$ ceases to be well-defined for deeper penetration. In this sense, the black brane interior acts as an effective potential barrier for timelike extremal surfaces. We also note that as $d \rightarrow \infty$, the bound (53) tends to r_h^+ ; the accessible interior region collapses toward the horizon, reflecting the increasingly steep gravitational potential of higher-dimensional black branes.

The case $d = 1$ (Bañados–Teitelboim–Zanelli (BTZ) black hole) requires separate treatment: the double-root condition becomes ill defined as $d \rightarrow 1$, reflecting qualitative differences in the extremal surface equations. The BTZ interior, being locally AdS_3 everywhere, lacks a geometric cutoff on turning point depth, yet complexity growth remains bounded due to near-horizon redshift effects.

Note that in the allowed interior range

$$r_h < r_0 < r_{\max}(d), \quad (54)$$

there is no horizon divergence and no exponential redshift. Dimensional analysis therefore implies

$$T \sim r_0, \quad \mathcal{C}_T \sim r_0^{1-d}, \quad (55)$$

leading to a power-law relation $\mathcal{C}_T \sim T^{1-d}$.

The scale $r_{\max}$ corresponds to the maximal radial depth reached by real extremal surfaces. Beyond this point, the surfaces cease to exist as real solutions. This suggests that $r_{\max}$ plays the role of an effective cutoff for the region probed by the timelike interval. From a boundary perspective, this limitation may be interpreted as a finite temporal resolution effect: a boundary observer with access only to a finite time interval cannot probe arbitrarily deep regions of the bulk. The scale $r_{\max}(d)$ thus quantifies a precise relation between the duration of the boundary process and the maximal depth of spacetime that can be accessed through complexity. A precise boundary interpretation of this geometric bound remains to be understood, but it is tempting to view it as reflecting a limitation in resolving deep infrared physics from timelike probes.

Taken together, our results suggest a refined picture of holographic complexity. Timelike subregion complexity does not probe the entire spacetime indiscriminately; instead, it selectively captures regions that are both causally accessible and computationally relevant to a boundary observer with finite temporal resolution. The black hole interior is consequently stratified into layers of decreasing informational accessibility, with only the region up to

$r_{\max}(d)$ contributing meaningfully to the complexity of the mixed thermal state.

This behavior is physically intuitive: the deeper interior layers are causally disconnected on timescales accessible to a finite-time boundary observer. The dimension-dependent bound $r_{\max}(d)$ quantifies this causal-computational trade-off. In higher dimensions, the gravitational potential becomes steeper, and the accessible interior region shrinks toward the horizon, reflecting the increasing difficulty of probing deep interior degrees of freedom.

Our analysis shows that timelike subregion complexity behaves distinctly from entanglement-based measures such as pseudoentropy. While the latter acquire imaginary parts when extremal surfaces cross horizons—reflecting subtle analytic continuation issues—complexity remains manifestly real in all regimes. This reality underscores its purely geometric origin as a volume, rather than its interpretation as an analytically continued quantum informational measure. Consequently, $\mathcal{C}_T$ offers a robust, well-defined holographic observable for studying quantum processes in thermal states, complementing entanglement-based probes while avoiding their interpretive subtleties.

The dynamical profile of $\mathcal{C}_T$ —short-interval power-law growth $\mathcal{C}_T \sim T^{1-d}$, followed by linear late-time growth $\mathcal{C}_T \sim T$ —qualitatively matches expectations for quantum complexity in chaotic systems [24,25]. The linear growth rate $\frac{d\mathcal{C}_T}{dT} \approx \frac{4S}{dL}$ resembles Lloyd’s bound [38]. $1/(dL)$.

The interior region $r_h < r_0 < r_{\max}(d)$ functions as a “complexity reservoir”: as boundary time increases, extremal surfaces penetrate deeper, harvesting complexity from a growing volume. Yet this exploration is strictly bounded by $r_{\max}(d)$, creating a “complexity horizon” beyond which no finite-time surfaces exist. This finite penetration depth contrasts with eternal black holes, where extremal surfaces can in principle reach arbitrarily deep, highlighting the physical difference between finite-time subregion complexity and late-time eternal black hole complexity.

In summary, timelike subregion complexity provides a sharp geometric tool for probing the interior structure of black holes and the thermalization dynamics of holographic systems. Its well-defined, real-valued behavior across all regimes, combined with its physically intuitive scaling with time and dimension, makes it a promising candidate for a holographic complexity measure that is both computationally meaningful and observationally robust.

IV. CONCLUSION AND OUTLOOK

In this work, we have systematically investigated timelike subregion complexity within the holographic CV framework. Motivated by the pseudoentropy program, which extends entanglement-based quantities to timelike subregions, we studied the geometric volume enclosed by timelike extremal surfaces anchored to finite boundary time intervals.

For timelike hyperbolic regions in pure AdS_{d+2} , we showed that the bulk volume bounded by spacelike and timelike extremal surfaces remains strictly real. This behavior stands in sharp contrast with pseudoentropy, which generally acquires imaginary contributions through analytic continuation. The reality of $\mathcal{C}_T$ highlights its fundamentally geometric character: timelike complexity measures a genuine Lorentzian bulk volume rather than an analytically continued entropic quantity.

For thermal states dual to AdS black branes, timelike extremal surfaces may either remain entirely outside the horizon or penetrate into the black hole interior. In both regimes, the resulting timelike complexity $\mathcal{C}_T$ is manifestly real, reinforcing its interpretation as a measure of causally accessible bulk volume constrained by finite boundary time intervals, rather than an intrinsically complex observable.

The appearance of two classes of extremal surface solutions—exterior and interior—naturally invites comparison with the behavior of spacelike entanglement entropy in eternal black holes [39]. In that context, two distinct and complete spacelike extremal surfaces exist for the same boundary subregion: a “short” surface remaining outside the horizon and a “long” surface passing through it. These represent genuine competing saddles, and beyond a critical boundary separation the long surface becomes dominant, producing a sharp phase transition.

The structure of timelike subregion complexity is fundamentally different. Although we also find exterior ($r_0 < r_h$) and interior ($r_0 > r_h$) extremal surfaces, they are not independent alternatives for a fixed boundary time interval T . Instead, the equations of motion uniquely determine a single turning point r_0 for each T , which varies continuously as the boundary time increases. The transition from an exterior to an interior turning point is therefore smooth and does not correspond to a competition between saddles. Consequently, the timelike complexity is always computed from a single extremal surface, without any minimization over multiple complete geometries.

A central geometric outcome of our analysis is the existence of a dimension-dependent upper bound on how far timelike extremal surfaces can penetrate into the black hole interior,

$$\frac{r_{\max}(d)}{r_h} = \left(\frac{2d}{d-1} \right)^{\frac{1}{d-1}}, \quad (d > 1). \quad (56)$$

This maximal depth arises when the extremal surface equation develops a double root, signaling the disappearance of real solutions. Beyond $r_{\max}(d)$, no real timelike extremal surfaces anchored to finite boundary time intervals exist. Importantly, this bound reflects a purely classical geometric obstruction rather than a breakdown of the complexity prescription, and may be interpreted as an intrinsic complexity horizon inside the black hole.

The timelike complexity exhibits three qualitatively distinct regimes. At short times, when the turning point lies well outside the horizon, the finite part of the complexity scales as $\mathcal{C}_T^{\text{finite}} \sim T^{1-d}$, matching the behavior in pure AdS and indicating negligible sensitivity to the black hole. In the near-horizon regime, $\mathcal{C}_T^{\text{finite}}$ grows linearly with time, $\mathcal{C}_T^{\text{finite}} \sim ET$, reflecting the dominance of horizon physics. Remarkably, this linear growth persists throughout the interior window $r_h < r_0 < r_{\max}(d)$ as extremal surfaces probe deeper behind the horizon. The region between r_h and $r_{\max}(d)$ thus acts as a finite “complexity reservoir,” with $r_{\max}(d)$ serving as a complexity horizon beyond which further interior access is forbidden.

The persistence of linear growth is particularly significant from an information-theoretic perspective. A defining feature of circuit complexity in generic chaotic quantum systems is its linear increase under time evolution until saturation. The emergence of robust linear growth from our purely gravitational computation therefore provides non-trivial evidence that the dual boundary quantity captured by $\mathcal{C}_T$ should indeed be interpreted as a form of quantum complexity rather than an entropic observable, which typically saturates at finite times. The existence of a maximal penetration depth $r_{\max}(d)$ then naturally corresponds to a finite complexity budget imposed by causal and temporal constraints.

These observations suggest a natural boundary interpretation. For a boundary subregion A and a finite time interval T , timelike subregion complexity may be viewed as the minimal circuit depth required to transform an initial reduced density matrix $\rho_A(t_i)$ into a final reduced density matrix $\rho_A(t_f)$, using unitary operations localized to A and constrained to act within time $T = t_f - t_i$. In this sense, $\mathcal{C}_T$ measures a process-dependent or relative notion of complexity, quantifying the intrinsic computational cost of implementing a causal transformation between quantum configurations under both spatial locality and temporal restrictions. While pseudoentropy captures phase information associated with transition amplitudes, timelike complexity instead quantifies the geometric “depth” of the corresponding process.

Our results also complement existing insights into bulk interior reconstruction. Much like entanglement wedge reconstruction, access to the black hole interior via complexity is limited, but here the limitation is explicitly quantified by the dimension-dependent scale $r_{\max}(d)$. The resulting complexity horizon is distinct from entanglement wedge boundaries, indicating that complexity probes aspects of bulk geometry inaccessible to entanglement-based observables. The breakdown of extremal surfaces beyond $r_{\max}(d)$ —rather than the emergence of complex-valued quantities—further emphasizes the classical geometric origin of this constraint.

Several important extensions remain open. Developing a precise field-theoretic formulation of timelike subregion

complexity—possibly using circuit complexity, path-integral optimization, or Lorentzian tensor networks—would sharpen its boundary interpretation. It would also be interesting to investigate whether a similar construction can be implemented within the CA proposal, and in particular whether the resulting quantity remains real for timelike subregions. Such a comparison could help clarify which features of timelike complexity are universal and which are specific to the CV framework. Time-dependent backgrounds, such as Vaidya geometries or evaporating black holes, could elucidate how the penetration bound evolves dynamically. Finally, incorporating charge, rotation, or quantum extremal surfaces would test the robustness of the complexity horizon and its relation to quantum gravitational effects.

In summary, timelike complexity provides a sharp and well-defined geometric probe of black hole interiors and thermalization dynamics. Its real-valued nature, robust linear growth, and explicit dimension-dependent penetration bound make it a compelling holographic quantity that complements entanglement-based measures while avoiding their analytic continuation subtleties. The bound $r_{\max}(d)$ offers a concrete geometric target that any consistent boundary definition of timelike complexity must reproduce, paving the way for a deeper understanding of quantum complexity in holography.

Timelike complexity differs from standard spacelike subregion complexity in that it probes the evolution of a system rather than its static structure. While spacelike complexity measures the cost of preparing a state on a given spatial region, the present construction is sensitive to how that state changes over a finite time interval. In this sense, timelike complexity may provide a complementary

probe to timelike entanglement entropy, capturing dynamical aspects of quantum information not accessible through entanglement measures alone.

Recent work on spatial subregion complexity [40,41] has revealed striking dynamical transitions depending on subsystem size and temperature. While their setup focuses on spatial subregions evolving in time, the sharp transitions they observe may have analogs in our timelike framework—potentially corresponding to transitions between different extremal surface branches. Exploring these connections could unify our understanding of how complexity dynamics depends on causal structure and system size.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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