

Radiative lepton model in a noninvertible fusion rule

Takaaki Nomura^{1,*}, Hiroshi Okada^{2,†} and Yoshihiro Shigekami^{2,‡}

¹*College of Physics, Sichuan University, Chengdu 610065, China*

²*Department of Physics, Henan Normal University, Xinxiang 453007, China*



(Received 8 November 2025; accepted 21 May 2026; published 15 June 2026)

We propose a radiatively induced lepton mass model introducing a Z_2 gauging Z_5 fusion rule. In our framework, the charged-lepton mass matrix is generated at one-loop level via dynamical breaking of the fusion rule. On the other hand, the neutrino mass matrix is induced at the one-loop level without breaking the fusion rule. As a direct consequence of the loop induced charged lepton masses, we can also consider lepton flavor violations, electron and muon $g-2$, and charged-lepton electric dipole moments that come into our valid phenomenological discussion. Then, we perform numerical analysis and show some interesting tendency on the Dirac CP phase, two Majorana phases, charged-lepton electric dipole moments and the neutrinoless double beta decay, all of which depends on their arguments where we fix the absolute values of our free parameters in order to satisfy experimental data of the lepton masses and mixing angles.

DOI: [10.1103/4g19-wmcs](https://doi.org/10.1103/4g19-wmcs)

I. INTRODUCTION

The radiative seesaw model is one of the attractive scenarios to understand a mechanism of minuscule masses such as active neutrinos and electrons at low energy scale beyond the standard model (BSM). In addition, we can potentially discuss plenty of phenomenology such as lepton flavor violations (LFVs), electron/muon anomalous magnetic dipole moments (electron/muon $g-2$), charged-lepton electric dipole moments (EDMs), and dark matter (DM), all of which can be connected to such tiny mass particles inside the loop. To obtain relevant terms in order to induce loop diagrams as leading contributions, noninvertible fusion rules are recently applied to these kinds of models. These selection rules possess some intriguing natures. The first remarkable nature is that the rules can forbid undesired terms at tree level but violate at loop level in a dynamical manner. The second one is that these rules provide an exact symmetry such as Z_2 that is never broken at any orders and plays a role in stabilizing the DM candidate. The third one is that they give some mass textures to reduce mass parameters in quark and lepton sectors, and possibly lead to predictions.¹ Note here that the

second and third nature can be replaced by the other symmetries such as discrete flavor symmetries [23–26]. However, the first one is totally unique for the case of the noninvertible fusion rule. These natures pave the way for constructing new models and it is worth exploring applications of the rules.

In this work, we apply Z_2 gauging Z_5 fusion rule $\mathbf{Z}_5^{\text{NI}}$, which consists of three elements $\{\mathbb{I}, a, b\}$, where all are supposed to be real and commutable each other; backgrounds and rules for Z_2 gauging Z_N symmetry can be referred to Refs. [9,10]. For these elements, we have

$$\begin{aligned} a \otimes a &= \mathbb{I} \oplus b, & a \otimes b &= a \oplus b, \\ b \otimes b &= \mathbb{I} \oplus a, & \mathbb{I} \otimes (\mathbb{I}, a, b) &= (\mathbb{I}, a, b). \end{aligned} \quad (1)$$

Here, it is worthwhile to comment on the mathematical nature of our noninvertible symmetry. Since all the generators $\mathbb{I}, a, b$, mutually commute, one might naively expect that $\mathbf{Z}_5^{\text{NI}}$ shares the characteristics of Abelian group symmetries such as Z_N or $U(1)$. However, the multiplication rules given in Eq. (1) exhibit a structure reminiscent of non-Abelian groups. Nevertheless, the algebraic structure of $\mathbf{Z}_5^{\text{NI}}$ differs fundamentally from that of non-Abelian groups. In an ordinary Abelian group, the product of two arbitrary generators always results in a single generator, i.e., $a \otimes b = c$. Therefore, the fusion rule in Eq. (1) cannot be realized within the framework of conventional Abelian groups. Consequently, $\mathbf{Z}_5^{\text{NI}}$ provides a genuinely new structure that is not reducible to any standard group symmetry.²

²These features can be generalized to other noninvertible symmetries.

*Contact author: nomura@scu.edu.cn

†Contact author: hiroshi3okada@htu.edu.cn

‡Contact author: shigekami@htu.edu.cn

¹See Refs. [1–22], which proposed several ways to apply these rules for phenomenology.

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

Owing to these distinctive features, it opens up the possibility of constructing novel scenarios that cannot be achieved using conventional group-theoretic symmetries. In the main text, we explicitly demonstrate these differences in detail. For a radiative lepton mass model, we introduce left- and right-handed Majorana fermions, isospin doublet inert boson η , singly charged-boson S^- .³ We make use of the first and second nature in the $\mathbf{Z}_5^{\text{NI}}$ fusion rules. Therefore, the charged-lepton mass matrix can be induced at one loop via dynamical breaking of $\mathbf{Z}_5^{\text{NI}}$. On the other hand, the neutrino mass matrix is generated at the one-loop level where $\mathbf{Z}_5^{\text{NI}}$ provides an accidental Z_2 symmetry that never can be broken at any order. This is the same as the case of Ma model [27,28]. In addition, through the charged-lepton sector, we can also discuss LFVs, electron/muon $g-2$ and charged-lepton EDMs.⁴

This paper is organized as follows. In Sec. II, we review our lepton seesaw model and explain how to induce the small charged-lepton masses as well as the neutrino masses. We also discuss several phenomenology related to the charged leptons. In Sec. III, we demonstrate several numerical χ^2 results satisfying neutrino observables, LFVs, electron/muon $g-2$, and charged-lepton EDMs in cases of normal and inverted hierarchies. In Sec. IV, we summarize and conclude.

II. MODEL SETUP

Here, we review our model setup in which we introduce Z_2 gauging Z_5 fusion rule $\mathbf{Z}_5^{\text{NI}}$. First, the lepton masses at tree level are forbidden by assigning $\mathbf{Z}_5^{\text{NI}}$ charge to lepton particles. Then, in order to generate the lepton mass matrix at the one-loop level, we introduce three families of left/right neutral Majorana fermions $N_{L/R}$, an isospin doublet inert boson $\eta \equiv [\eta^+, (\eta_R + i\eta_I)/\sqrt{2}]^T$ and an isospin singlet singly-charged-boson S^- in addition to the SM particles. Moreover, η and S^- do not acquire nonzero vacuum expectation values (VEVs). Note here that the charged-lepton mass matrix is generated by dynamically breaking of this symmetry,⁵ and LFVs, $g-2$, and EDMs have the same origin as the charged-lepton mass

³We can also realize the same interactions by applying Fibonacci fusion rules that is the minimum fusion ones. However, this scenario induces terms that are forbidden at tree level, by radiative corrections. Some of these radiative corrections include infinity that never be able to be canceled out. We thus do not adopt the minimal Fibonacci fusion rules to keep predictability of the model. We briefly show some more detail in the model part.

⁴Even though we can discuss the nature of DM candidate, we focus on the other phenomenology such as EDMs in this paper.

⁵It means noninvertible symmetry is broken by radiative corrections where charged lepton mass is forbidden at tree level by the symmetry. Note that effective operator inducing neutrino mass, $LLHH$, is allowed by the symmetry. These features can be explicitly observed from diagrams for the radiative lepton masses in Figs. 1 and 2.

TABLE I. Charge assignments of relevant fermions and bosons under $SU(2)_L \otimes U(1)_Y \otimes \mathbf{Z}_5^{\text{NI}}$. All fermions have to have three generations, and all new particles are singlet under $SU(3)_C$.

	L_L	ℓ_R	N_L	N_R	H	η	S^-
$SU(2)_L$	2	1	1	1	2	2	1
$U(1)_Y$	$-\frac{1}{2}$	-1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	-1
$\mathbf{Z}_5^{\text{NI}}$	b	a	a	a	$\mathbb{I}$	b	b

matrix [29–31]. On the other hand the neutrino mass matrix is obtained without violating the fusion rule, as we will show later. The particle contents and their charge assignments are summarized in Table I.

Under the symmetry, the renormalizable Lagrangian is given by

$$\begin{aligned}
 -\mathcal{L}_\ell = & y_{ii}^\ell \overline{L_{Li}} \eta \ell_{Ri} + f_{ia}^R \overline{L_{Li}} \tilde{\eta} N_{Ra} + f_{ia}^L \overline{L_{Li}} \tilde{\eta} N_{La}^C \\
 & + g_{ai}^R \overline{N_{Ra}}^C \ell_{Ri} S^+ + g_{ai}^L \overline{N_{La}}^C \ell_{Ri} S^+ + (M_D)_{aa} \overline{N_{La}} N_{Ra} \\
 & + (M_R)_{ab} \overline{N_{Ra}}^C N_{Rb} + (M_L)_{ab} \overline{N_{La}}^C N_{Lb} + \text{H.c.}, \quad (2)
 \end{aligned}$$

where $\tilde{\eta} \equiv i\tau_2 \eta^*$ being τ_2 the second Pauli matrix. y^ℓ and M_D can be diagonal matrices without loss of generality. The Higgs potential is given by

$$\begin{aligned}
 \mathcal{V} = & -\mu_H^2 |H|^2 + \mu_\eta^2 |\eta|^2 + \mu_S^2 |S^-|^2 + \mu_0 (H^T (i\tau_2) \eta S^- + \text{H.c.}) \\
 & + \lambda_H |H|^4 + \lambda_\eta |\eta|^4 + \lambda_S |S^-|^4 + \lambda_{H\eta} |H|^2 |\eta|^2 \\
 & + \lambda'_{H\eta} |H^\dagger \eta|^2 + \lambda''_{H\eta} ((H^\dagger \eta)^2 + \text{H.c.}) \\
 & + \lambda_{HS} |H|^2 |S^-|^2 + \lambda_{\eta S} |\eta|^2 |S^-|^2. \quad (3)
 \end{aligned}$$

Especially, the $\lambda''_{H\eta}$ term contributes to generating the neutrino mass matrix at one-loop level, and the μ_0 term contributes to the radiative charged-lepton mass matrix.

Before moving to see the radiative mass generation for each fermion sector, we comment on the physical phases of the model. In the Higgs potential in Eq. (3), there are two possible complex parameters, μ_0 and $\lambda''_{H\eta}$. These phases can be absorbed into the two phases of H , η , and $S^\pm$. Since other parameters in the Higgs potential are real by definition, there is no physical CP source from the potential. In the Lagrangian in Eq. (2), on the other hand, we have lots of complex parameters. It is clear that all these phases cannot be removed by considering phase redefinition for remaining fields. As a result, we truly have physical CP sources which result in nonzero (and nontrivial) predictions for charged-lepton EDMs. For example, following phases are invariant under any rephasing for fields:

$$\theta_{\text{phys}} = \arg \left[\frac{f_{ia}^R g_{ia}^L}{f_{ia}^L g_{ia}^R} \right], \quad \arg \left[\frac{(M_D)_{aa}^2}{(M_R)_{aa} (M_L)_{aa}} \right], \dots, \quad (4)$$

with no summation for i, a .

Furthermore, we emphasize the role of noninvertible symmetry. If we use ordinary global symmetry, like Z_M or global $U(1)$ symmetry, it is not possible to realize the scenario as explained below. Let us consider Z_4 symmetry is used. First, we need to assign Z_4 charge “-1” for the inert scalar doublet η in any setup, in order to have the $(H^\dagger \eta)^2$ term,⁶ which is necessary for the neutrino mass generation (SM Higgs should not have charge to generate quark mass). Then S^- should also have Z_4 charge -1 for the $H\eta S^-$ term. In this case, we can write terms $\{\overline{L}_L \eta \ell_R, \overline{L}_L \tilde{\eta} N_R, \overline{N}_R^C \ell_R S^+\}$ by assigning charge “ i ” for $\{\overline{L}_L, N_{L(R)}, \ell_R\}$. However, we cannot write Majorana mass terms of $N_{L(R)}$, and our one-loop diagrams generating lepton masses cannot be written. As an another option, we can assign -1 charge for $\overline{L}_L$ as well as η and $S^\pm$, while ℓ_R and $N_{L(R)}$ do not have charge. In this case, we cannot write the $\overline{N}_R^C \ell_R S^+$ term, and charged lepton masses are not generated. These difficulties cannot be improved even if we use a simple Z_2 , larger Z_M , or global $U(1)$ symmetry, and therefore, such a usual symmetry cannot be used for our scenario. By using the noninvertible fusion rule, on the other hand, we can realize it in use of the rules in Eq. (1). Therefore our model is a good example to realize the scenario that cannot be obtained from the usual symmetry. Indeed, we have many free parameters, since we need several Yukawa couplings to induce charged lepton and neutrino masses at loop level. However, our main point in this paper is to construct a model in which all lepton masses are generated by radiative corrections rather than getting specific predictions.

Notice that if we use the Fibonacci rules, $1 \otimes \tau = \tau$ and $\tau \otimes \tau = 1 \oplus \tau$, assigning τ to η and S^- , we can write the term $(H^\dagger \eta)(\eta^\dagger \eta)$. This term can induce terms such as $(H^\dagger \eta)(H^\dagger H)$ and $\eta^\dagger H$ through radiative correction. These loop corrections generate divergences that cannot be absorbed by renormalization, since the corresponding tree-level counterterms are absent. Consequently, the divergences cannot be eliminated unless a finite cutoff scale is introduced. In addition, $\eta^\dagger H$ term spoils inert nature of η and the mechanism of the model does not work. Therefore we need $\mathbf{Z}_S^{\mathbf{N1}}$ as a relevant minimal selection rule where $(H^\dagger \eta)(\eta^\dagger \eta)$ is forbidden as $b \otimes b \otimes b$ does not contain $\mathbb{I}$.

A. Neutral fermion mass matrix

The mass matrix of the neutral fermion is induced via the second line of Lagrangian in Eq. (2), and its form is given by

$$(\overline{N}_R^C \quad \overline{N}_L) \begin{pmatrix} M_R & M_D \\ M_D & M_L \end{pmatrix} \begin{pmatrix} N_R \\ N_L^C \end{pmatrix} \equiv (\overline{N}_R^C \quad \overline{N}_L) M_N \begin{pmatrix} N_R \\ N_L^C \end{pmatrix}, \quad (5)$$

⁶For this term, it is clear that we cannot use any discrete Z_M symmetry with odd number of M .

where we omit generation indices, and each block is understood as a 3×3 matrix. M_N is then understood as a 6×6 symmetric matrix. Note that due to our assumption about absence of VEVs for $\{\eta, S^-\}$, neutrinos do not mix with $N_{L,R}$. In order to diagonalize the above mass matrix, we suppose

$$M_R, M_L \ll M_D, \quad (6)$$

which is an appropriate hypothesis when we consider the ‘t Hooft sense. Then, we first transform M_N as follows:

$$\begin{aligned} M'_N &\equiv V_N^{(1)T} M_N V_N^{(1)} \\ &\equiv \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} M_R & M_D \\ M_D & M_L \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} M_D + \frac{M_L + M_R}{2} & \frac{M_L - M_R}{2} \\ \frac{M_L - M_R}{2} & -M_D + \frac{M_L + M_R}{2} \end{pmatrix} \\ &\equiv \begin{pmatrix} M_D + m_+ & m_- \\ m_- & -M_D + m_+ \end{pmatrix}, \end{aligned} \quad (7)$$

where $m_- \lesssim m_+ \ll M_D$. M'_N is then block diagonalized as follows:

$$(\Omega P)^T M'_N (\Omega P) = \begin{pmatrix} M_D - m_+ & \mathbf{0} \\ \mathbf{0} & M_D + m_+ \end{pmatrix} \equiv M''_N, \quad (8)$$

$$\Omega P U_N = \begin{pmatrix} \mathbf{1} & -\frac{1}{2} M_D^{-1} m_- \\ \frac{1}{2} m_-^\dagger M_D^{-1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ i \cdot \mathbf{1} & \mathbf{0} \end{pmatrix}. \quad (9)$$

Here, bold number $\mathbf{0}(\mathbf{1})$ represents 3×3 zero (unit) matrix. Finally, the above mass matrix is diagonalized by

$$\begin{aligned} U_N^T M''_N U_N &= \begin{pmatrix} U_R^T & \mathbf{0} \\ \mathbf{0} & U_L^T \end{pmatrix} \begin{pmatrix} M_D - m_+ & \mathbf{0} \\ \mathbf{0} & M_D + m_+ \end{pmatrix} \\ &\times \begin{pmatrix} U_R & \mathbf{0} \\ \mathbf{0} & U_L \end{pmatrix} \\ &= \begin{pmatrix} D_R & \mathbf{0} \\ \mathbf{0} & D_L^C \end{pmatrix}. \end{aligned} \quad (10)$$

M_N is approximately diagonalized via $V_N \equiv V_N^{(1)} \Omega P U_N$ as $V_M^T M_N V_N \sim \text{diag}(D_R, D_L^C)$. Note that we have used an assumption that M_D should be proportional to the 3×3 unit matrix in order to achieve our diagonalization [32,33];

$$M_D \approx M_0 \times \mathbf{1}. \quad (11)$$

Then, $M_D \pm m_+$ is given by

$$M_D \pm m_+ \simeq \begin{pmatrix} M_0 \pm \delta_{11} & \pm\delta_{12} & \pm\delta_{13} \\ \pm\delta_{12} & M_0 \pm \delta_{22} & \pm\delta_{23} \\ \pm\delta_{13} & \pm\delta_{23} & M_0 \pm \delta_{33} \end{pmatrix}. \quad (12)$$

Since the diagonal elements are greater than all the off-diagonal elements, $M_D \pm m_+$ is approximately diagonalized by the following mixing matrices:

$$U_{R(L)} \approx \begin{pmatrix} 1 & \epsilon_{12}^{R(L)} & \epsilon_{13}^{R(L)} \\ -\epsilon_{12}^{R(L)} & 1 & \epsilon_{23}^{R(L)} \\ -\epsilon_{13}^{R(L)} & -\epsilon_{23}^{R(L)} & 1 \end{pmatrix}, \quad \epsilon_{ij}^{R(L)} \approx -\frac{\delta_{ij}}{\delta_{ii} - \delta_{jj}}, \quad (i \neq j). \quad (13)$$

Since $\epsilon^{R(L)}$ is expected to be tiny, we suppose $|\delta_{ij}| \ll |\delta_{ii}|$ for $i \neq j$ with $i/j = 1, 2, 3$. As a result, we find mass eigenstates for $N_{R,L}$ as

$$N_{R_a} \equiv \sum_{\alpha=1}^6 (V_N)_{a\alpha} \psi_{R_a}, \quad N_{L_a}^C \equiv \sum_{\alpha=1}^6 (V_N)_{a+3,\alpha} \psi_{R_a}. \quad (14)$$

Hereafter, mass eigenvalues for ψ_{R_a} are indicated by D_a .

B. Charged-lepton mass matrix

The mass matrix of charged-leptons is induced at one-loop level via the following relevant Lagrangian:

$$f_{ia}^R \overline{L_{Li}} \tilde{\eta} N_{R_a} + g_{bj}^L \overline{N_{Lb}} \ell_{R_j} S^+ + f_{ia}^L \overline{L_{Li}} \tilde{\eta} N_{L_a}^C + g_{bj}^R \overline{N_{Rb}^C} \ell_{R_j} S^+ + \text{H.c.},$$

where $\mu_0 H^T(i\tau_2)\eta S^-$ plays a role in connecting $S^\pm$ and $\eta^\pm$ to construct the one-loop diagram.⁷ Then, these terms are rewritten in terms of mass eigenstates of $N_{L,R}$ [ψ_{R_a} in Eq. (14)] as follows (summation for a, b are implicitly understood):

$$\begin{aligned} & \frac{f_{ia}^R (V_N)_{a\alpha}}{\sqrt{2}} \overline{\ell_{Li}} \eta^- \psi_{R_a} + (V_N^T)_{\alpha,b+3} g_{bj}^L \overline{\psi_{R_a}^C} \ell_{R_j} S^+ + \frac{f_{ia}^L (V_N)_{a+3,\alpha}}{\sqrt{2}} \overline{\ell_{Li}} \eta^- \psi_{R_a} + (V_N^T)_{\alpha,b} g_{bj}^R \overline{\psi_{R_a}^C} \ell_{R_j} S^+ \\ & \equiv F_{ia} \overline{\ell_{Li}} \eta^- \psi_{R_a} + G_{aj} \overline{\psi_{R_a}^C} \ell_{R_j} S^+, \end{aligned} \quad (15)$$

where we define $F_{ia} \equiv \sum_{\alpha=1}^3 [f_{ia}^R (V_N)_{a\alpha} + f_{ia}^L (V_N)_{a+3,\alpha}]/\sqrt{2}$ and $G_{aj} \equiv \sum_{b=1}^3 [(V_N^T)_{\alpha,b+3} g_{bj}^L + (V_N^T)_{\alpha,b} g_{bj}^R]$. The resultant mass matrix for charged-lepton shown in Fig. 1 is given by

$$(m_\ell)_{ij} = \frac{v_H \mu'_0}{\sqrt{2}(4\pi)^2 M_0} \frac{\tilde{F}_{ia} \tilde{D}_a \tilde{G}_{aj}}{\tilde{D}_a^2 - \tilde{m}_S^2} \left[\frac{\tilde{D}_a^2}{\tilde{D}_a^2 - \tilde{m}_S^2} \ln\left(\frac{\tilde{D}_a^2}{\tilde{m}_S^2}\right) + \frac{\tilde{D}_a^2}{\tilde{D}_a^2 - \tilde{m}_\eta^2} \ln\left(\frac{\tilde{D}_a^2}{\tilde{m}_\eta^2}\right) \right], \quad (16)$$

where $\mu'_0 \equiv f_{33}^R g_{33}^R \mu_0$, M_0 is a typical scale of M_D [see Eq. (11)], $\tilde{m}_{S,\eta} \equiv m_{S,\eta}/M_0$, $\tilde{D}_a \equiv D_a/M_0$, $\tilde{F} \equiv F/f_{33}^R$, and $\tilde{G} \equiv G/g_{33}^R$ with f_{33}^R, g_{33}^R being, respectively, (3, 3) component of f^R and g^R . We define the diagonalization of m_ℓ as $D_\ell \equiv V_{eL}^\dagger m_\ell V_{eR}$.

C. Active neutrino mass matrix

The active neutrino mass is induced at one-loop level through interaction terms with couplings F and $(H^\dagger \eta)^2$, and given by [28]

⁷Hereafter, we compute the loop diagrams with a method of mass insertion approximation. In particular, we do not perform diagonalization for new scalar particles.

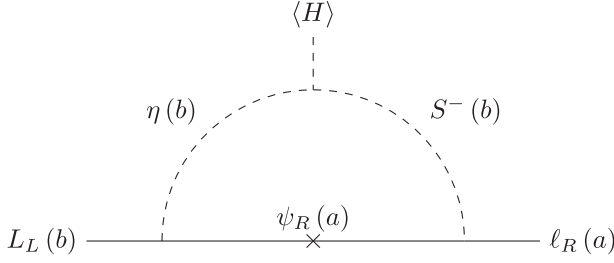


FIG. 1. Charged-lepton masses at one-loop level. a, b in the parentheses are the $\mathbf{Z}_5^{\text{NI}}$ assignment for corresponding fields, $\langle H \rangle$ denotes the VEV of the SM Higgs doublet, and we omit the generation indices for fermions. It is clear that these effective masses have $a \otimes b$ which does not have $\mathbb{I}$, and therefore, our fusion rule is radiatively broken by these charged-lepton masses.

$$(m_\nu)_{ij} = \frac{(f_{33}^R)^2 M_0}{(4\pi)^2} \tilde{F}_{ia} \tilde{D}_a \tilde{F}_{aj}^T \left[\frac{\tilde{m}_R^2}{\tilde{m}_R^2 - \tilde{D}_a^2} \ln \left[\frac{\tilde{m}_R^2}{\tilde{D}_a^2} \right] - \frac{\tilde{m}_I^2}{\tilde{m}_I^2 - \tilde{D}_a^2} \ln \left[\frac{\tilde{m}_I^2}{\tilde{D}_a^2} \right] \right] \quad (17)$$

$$\equiv \kappa_\nu (\tilde{m}_\nu)_{ij}, \quad (18)$$

where $\kappa_\nu \equiv (f_{33}^R)^2 M_0$, $m_R \equiv M_0 \tilde{m}_R$ is the mass of η_R and $m_I \equiv M_0 \tilde{m}_I$ is that of η_I . Corresponding diagram is shown in Fig. 2. We define m_ν is diagonalized by $D_\nu = U_\nu^T m_\nu U_\nu$. Then, the observed mixing matrix $U \equiv V_{eL}^\dagger U_\nu$ and κ_ν can be rewritten in terms of rescaled neutrino mass eigenvalues $\tilde{D}_\nu (\equiv D_\nu / \kappa_\nu)$ and atmospheric neutrino mass-squared difference Δm_{atm}^2 as follows:

$$(\text{NH}): \kappa_\nu^2 = \frac{|\Delta m_{\text{atm}}^2|}{\tilde{D}_{\nu_3}^2 - \tilde{D}_{\nu_1}^2}, \quad (\text{IH}): \kappa_\nu^2 = \frac{|\Delta m_{\text{atm}}^2|}{\tilde{D}_{\nu_2}^2 - \tilde{D}_{\nu_3}^2}, \quad (19)$$

where NH and IH, respectively, stand for the normal and inverted hierarchies, respectively. Subsequently, the solar neutrino mass-squared difference is determined by the relation

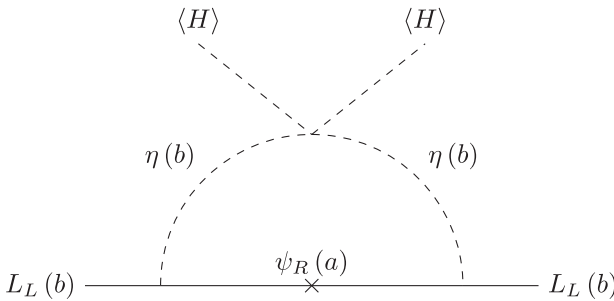


FIG. 2. Neutrino masses at the one-loop level, with the same notation as in Fig. 1. It is clear that these effective masses have $b \otimes b$ which include $\mathbb{I}$, and therefore, these do not break our fusion rule.

$$\Delta m_{\text{sol}}^2 = \kappa_\nu^2 (\tilde{D}_{\nu_2}^2 - \tilde{D}_{\nu_1}^2). \quad (20)$$

The effective mass for neutrinoless double beta decay can be written by

$$\langle m_{ee} \rangle = \kappa_\nu |\tilde{D}_{\nu_1} \cos^2 \theta_{12} \cos^2 \theta_{13} + \tilde{D}_{\nu_2} \sin^2 \theta_{12} \cos^2 \theta_{13} e^{i\alpha_2} + \tilde{D}_{\nu_3} \sin^2 \theta_{13} e^{i(\alpha_3 - 2\delta_{CP})}|, \quad (21)$$

where $\theta_{12,23,13}$ are mixing angles for U in the standard parametrization, δ_{CP} is a Dirac CP phase, and $\alpha_{2,3}$ are Majorana phases, defined in Majorana phase matrix P as $P \equiv \text{diag}(1, e^{i\alpha_2/2}, e^{i\alpha_3/2})$. A predicted value of $\langle m_{ee} \rangle$ is constrained by the current KamLAND-Zen data; $\langle m_{ee} \rangle < (28-122)$ meV at 90% confidence level [34]. Furthermore the sum of neutrino masses is constrained by the minimal standard cosmological model $\Lambda_{\text{CDM}} + \sum D_\nu$ that provides the upper bound $\sum D_\nu \leq 120$ meV [35,36], although it becomes weaker if the data are analyzed in the context of extended cosmological models [37]. Recently, DESI and CMB data combination provides more stringent upper bound on the sum as $\sum D_\nu \leq 72$ meV [38]. Direct search for neutrino mass is done by the Karlsruhe Tritium Neutrino (KATRIN) experiment [39], which is the first sub-eV sensitivity on $m_{\nu_e}^2 = (0.26 \pm 0.34)$ eV² at 90% CL. Here, $m_{\nu_e}^2$ is defined by

$$\kappa_\nu^2 [(\tilde{D}_{\nu_1} \cos \theta_{13} \cos \theta_{12})^2 + (\tilde{D}_{\nu_2} \cos \theta_{13} \sin \theta_{12})^2 + (\tilde{D}_{\nu_3} \sin \theta_{13})^2]. \quad (22)$$

In numerical analysis, we will adopt NuFit 6.0 [40] to experimental ranges of neutrino observables in addition to the above constraints of neutrino masses.

D. LFVs, charged-lepton $g-2$ and EDMs

In this subsection, we will discuss LFVs, charged-lepton $g-2$ and EDMs, in order to compare with the experimental constraints in the numerical analyses. These contributions are arisen from the one-loop diagram adding external photon to charged scalar line in the diagram of Fig. 1. Therefore, these contributions also violate our fusion rule of $\mathbf{Z}_5^{\text{NI}}$. The dominant LFVs and charged-lepton $g-2$ and EDMs are induced via the following terms:

$$F'_{ia} \bar{\ell}_{Li} \psi_{Ra} \eta^- + G'_{aj} \bar{\psi}_{Rj}^C \ell_{Rj} S^+ + \text{H.c.}, \quad (23)$$

where $F' \equiv V_{eL}^\dagger F$, $G' \equiv G V_{eR}$, and all the above particles are supposed to be mass eigenstates. Then, the branching ratios for LFVs, charged-lepton $g-2$ and EDMs are, respectively, given by

$$\text{BR}(\ell_i \rightarrow \ell_j \gamma) \approx \frac{48\pi^3 C_{ij} \alpha_{\text{em}}}{G_F^2 m_{\ell_i}^2} [|a_{Rij}|^2 + |a_{Lij}|^2], \quad (24)$$

$$\Delta a_{\ell_i} \approx -m_{\ell_i} [a_{R_{ii}} + a_{L_{ii}}], \quad d_{\ell_i} = \frac{e}{2} \text{Im} [a_{R_{ii}} - a_{L_{ii}}], \quad (25)$$

where G_F is Fermi constant, $\alpha_{\text{em}} \approx 1/137$ is fine structure constant, and $C_{21} \approx 1$, $C_{31} \approx 0.1784$, $C_{32} \approx 0.1736$, $(\ell_1, \ell_2, \ell_3) \equiv (e, \mu, \tau)$. a_R and a_L in our model are, respectively, computed as follows:

$$a_{R_{ij}} \approx \frac{\sqrt{2} v_H \mu'_0}{(4\pi)^2 M_0^3} \tilde{F}'_{ja} \tilde{D}_\alpha \tilde{G}'_{ai} F_{lfs}^\alpha, \quad a_{L_{ij}} \approx \frac{\sqrt{2} v_H \mu'_0}{(4\pi)^2 M_0^3} \tilde{G}'_{ja} \tilde{D}_\alpha \tilde{F}'_{ai} F_{lfs}^\alpha, \quad (26)$$

$$F_{lfs}^\alpha = \int dx_1 dx_2 dx_3 dx_4 dx_5 \delta(1 - x_1 - x_2 - x_3 - x_4 - x_5) \left(\frac{3}{\Delta_\alpha^2} + \frac{\tilde{m}_S^2 + \tilde{m}_\eta^2}{\Delta_\alpha^3} \right), \quad (27)$$

where $\Delta_\alpha \equiv x_1 \tilde{D}_\alpha^2 + (x_2 + x_4) \tilde{m}_S^2 + (x_3 + x_5) \tilde{m}_\eta^2$ and x_i s are Feynman parameters.

Experimental upper bounds and future prospects for the branching ratios of the LFV decays are given by [41–44]

$$\text{Current: } \text{BR}(\mu \rightarrow e\gamma) \lesssim 3.1 \times 10^{-13}, \quad \text{BR}(\tau \rightarrow e\gamma) \lesssim 3.3 \times 10^{-8}, \quad \text{BR}(\tau \rightarrow \mu\gamma) \lesssim 4.4 \times 10^{-8}, \quad (28)$$

$$\text{Future: } \text{BR}(\mu \rightarrow e\gamma) \lesssim 6.0 \times 10^{-14}, \quad \text{BR}(\tau \rightarrow e\gamma) \lesssim 9.0 \times 10^{-9}, \quad \text{BR}(\tau \rightarrow \mu\gamma) \lesssim 6.9 \times 10^{-9}. \quad (29)$$

Here, we estimate our input parameters to satisfy the LFVs to prepare the range of numerical analysis in the next section, focusing on the most stringent bound on $\mu \rightarrow e\gamma$ for safety. When we simply fix the following values $\tilde{F}' = \tilde{G}' = 0.01$, $\tilde{D}_\alpha F_{lfs}^\alpha = 1$, $\mu'_0 = 1$ GeV and $M_0 = 1000$ GeV, $\text{BR}(\mu \rightarrow e\gamma) \approx 6.95 \times 10^{-13}$ that is larger by factor of ≈ 2.24 than the experimental upper bound. However, as one can understand from Eq. (26), the branching ratio hardly depends on the value of M_0 , $\text{BR}(\mu \rightarrow e\gamma) \propto 1/M_0^6$. Using this fact, $\text{BR}(\mu \rightarrow e\gamma) \lesssim 3.1 \times 10^{-13}$ can be easily obtained by setting $M_0 \gtrsim 1144$ GeV, and thus, it is easy to satisfy the LFVs as well as other bounds. Experimental results for charged-lepton $g-2$ at 1σ are given by [45–52]⁸

$$\text{Current: } \Delta a_e \simeq (3.41 \pm 1.64) \times 10^{-13}, \quad \Delta a_\mu \simeq (38 \pm 63) \times 10^{-11}, \quad (30)$$

$$\text{Past: } \Delta a_e \simeq (4.8 \pm 3.0) \times 10^{-13}, \quad \Delta a_\mu \simeq (25.1 \pm 5.9) \times 10^{-10} \quad \text{or} \quad (24.9 \pm 4.8) \times 10^{-10}. \quad (31)$$

Experimental upper bounds for EDMs are, respectively, given by [54]

$$|d_e| < 4.1 \times 10^{-30} \text{ e cm (JILA)}, \quad |d_\mu| < 1.9 \times 10^{-19} \text{ e cm (Muon } g-2), \quad (32)$$

$$|\text{Re}(d_\tau)| < 4.5 \times 10^{-17} \text{ e cm (Belle)}, \quad |\text{Im}(d_\tau)| < 2.5 \times 10^{-17} \text{ e cm (Belle)}. \quad (33)$$

Furthermore, indirect bounds for the muon and tau EDMs provide more stringent constraints than the direct experimental ones, which are obtained by evaluating the muon-loop induced light-by-light CP odd amplitude [55,56]:

$$|d_\mu| < 6.3 \times 10^{-21} \text{ e cm}, \quad |d_\tau| < 4.1 \times 10^{-19} \text{ e cm}, \quad (34)$$

by using the latest electron EDM bound. Note that these bounds are obtained by the current electron EDM bound, and therefore, an improvement of the electron EDM bound will also push down these upper bounds.

III. NUMERICAL RESULTS

Here, we perform $\Delta\chi^2$ numerical analysis and investigate our allowed regions to satisfy whole the experimental results discussed above. Here, we assume all the relevant observables to be the Gaussian distribution under the $\Delta\chi^2$ analysis, and hence,

$$\Delta\chi^2 = \sum_i \left(\frac{O_i^{\text{obs}} - O_i^{\text{th}}}{\delta O_i^{\text{exp}}} \right)^2, \quad (35)$$

where $O_i^{\text{obs(th)}}$ is observed (theoretically obtained) value of corresponding observables and δO_i^{exp} indicates the experimental error at 1σ . We adopt four reliable observables; $\{\sin^2 \theta_{12}, \sin^2 \theta_{13}, \Delta m_{\text{atm}}^2, \Delta m_{\text{sol}}^2\}$, to estimate the $\Delta\chi^2$ analysis and take our

⁸For Δa_e , there is another result by using the fine-structure constant α determined by ^{133}Cs , whose result has opposite sign, $\Delta a_e = (-10.1 \pm 2.7) \times 10^{-13}$ [53].

TABLE II. Best-fit experimental values in the NH case, corresponding to $\Delta\chi_{\min} \approx 1.06$. Here we display these values that only depend on absolute values.

Parameter	BF	Parameter	BF	Parameter	BF
s_{12}	0.553	s_{23}	0.763	s_{13}	0.147
Δm_{sol}^2	$7.32 \times 10^{-23} \text{ GeV}^2$	Δm_{atm}^2	$2.54 \times 10^{-21} \text{ GeV}^2$	D_{ν_1}	26.5 meV
m_{ν_e}	27.9 meV	$\sum D_\nu$	111 meV	m_τ	1.777 GeV
m_e	0.000511 GeV	m_μ	0.106 GeV		
Δa_e	8.74×10^{-17}	Δa_μ	2.52×10^{-12}		
$\text{BR}(\mu \rightarrow e\gamma)$	3.54×10^{-24}	$\text{BR}(\tau \rightarrow e\gamma)$	2.35×10^{-27}	$\text{BR}(\tau \rightarrow \mu\gamma)$	4.08×10^{-21}

allowed regions up to 5σ .⁹ On the other hand, we independently request $\sin^2 \theta_{23}$ to be within 3σ level of Nufit 6.0, since the experimental error is deviated from Gaussian distribution. In our parameter space, we fix $f^R = f^L$ and $g^R = g^L$ for simplicity.¹⁰ Then, we randomly select our input parameters as follows:

$$10^{-3} \leq \frac{|\delta_{11,22,33}|}{\text{GeV}} \leq 0.1, \quad 10^{-5} \leq \frac{|\delta_{12,13,23}|}{\text{GeV}} \leq 10^{-3},$$

$$10^{-5} \leq \frac{|m_-|}{\text{GeV}} \leq 0.1, \quad (36)$$

$$10^{-7} \leq \delta\tilde{m} \leq 10^{-5}, \quad 10^{-5} \leq |f_R| \leq 1,$$

$$10^{-5} \leq |g_R| \leq 0.1, \quad (37)$$

$$100 \leq \frac{M_0}{\text{GeV}} \leq 10^5, \quad 0.1 \leq \frac{\mu'_0}{\text{GeV}} \leq 10,$$

$$0.1 \leq \tilde{m}_I \leq 10, \quad 10^{-6} \leq \tilde{m}_S \leq 10^{-4}, \quad (38)$$

where δ_{ij} is the component of m_+ , $\tilde{m}_R \equiv \tilde{m}_I + \delta\tilde{m}$, $\tilde{m}_\eta \equiv \tilde{m}_I$. Arguments run over $[-\pi, +\pi]$.¹¹ Note that arbitrary three components of g^R can directly be determined by charged-lepton mass matrix putting experimental results for charged-lepton masses. Here, we select $g_{11,22,32}^R$. Starting at the above regions in our input parameters, we specify the parameter space of their absolute values to fit the neutrino mass-square differences, three mixing angles, LFVs, and lepton $g-2$.¹² But, we keep the wide ranges of their arguments that play a crucial role in determining the Dirac CP phase δ_{CP} , two Majorana phases $\alpha_{2,3}$ and EDMs $d_{e,\mu,\tau}$.

⁹ 3σ with four degrees of freedom corresponds to $\Delta\chi^2 \approx 34.555$ [57].

¹⁰Note that, this simplification does not eliminate all physical CP phases in our model: As mentioned in Eq. (4), we still have second example of the physical CP phase, as well as other ones which coming from, e.g., off-diagonal elements of $f^{R(L)}$, $g^{R(L)}$, and M_{RL} . Therefore, we can discuss predictions for charged-lepton EDMs in our models.

¹¹If there are no symbols of absolute values $|\dots|$ in Eqs. (36)–(38), such parameters are considered as real parameters.

¹²This is because there are too many parameters and too much time to find out all the allowed regions.

A. NH

At first, we show our benchmark point that does not depend on phases in Table II. The best fit (BF) value corresponds to $\Delta\chi_{\min} \approx 1.06$. It is clear that the branching ratios for LFVs are enough small to satisfy current bounds. Moreover, the prediction of Δa_μ is within 1σ of current results, while that of Δa_e is within 3σ .

Under Table II, we figure out the neutrinoless double beta decay in terms of phases δ_{CP} , α_2 , and α_3 in Fig. 3, as blue, red, and green points, respectively. The maximum value of neutrinoless double beta decay just reaches the upper limit of 28 meV. We find the following allowed region-tendencies of phases; $30^\circ \lesssim \delta_{CP} \lesssim 60^\circ$, $200^\circ \lesssim \delta_{CP} \lesssim 240^\circ$, $-10^\circ \lesssim \alpha_2 \lesssim 0^\circ$ and $-10^\circ \lesssim \alpha_3 \lesssim 20^\circ$.

Next, we show our EDM predictions of $|d_e|$ (left), $|d_\mu|$ (center) and $|d_\tau|$ (right) for the case of NH in Fig. 4, in term of three phases, where each of color plots are the same as the one in Fig. 3. These figures suggest that $|d_e|$ reaches the experimental upper bound $4.1 \times 10^{-30} e \text{ cm}$, while typical values of muon and tau EDM are $|d_\mu| \simeq (1-4) \times 10^{-25} e \text{ cm}$ and $|d_\tau| \simeq (2-6) \times 10^{-24} e \text{ cm}$. Interestingly, the constraint $|d_e| < 4.1 \times 10^{-30} e \text{ cm}$ results in $\langle m_{ee} \rangle \lesssim 28 \text{ meV}$,

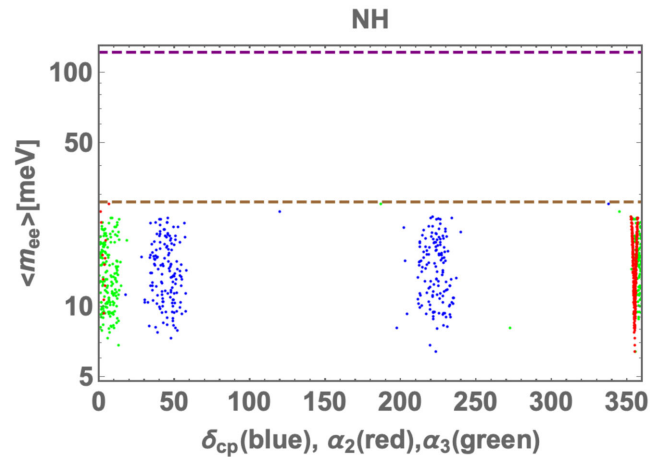


FIG. 3. Allowed regions for $\langle m_{ee} \rangle$ in terms of phases δ_{CP} (blue), α_2 (red), and α_3 (green) in case of NH. Brown (lower) and purple (upper) dashed lines correspond to $\langle m_{ee} \rangle = 28$ and 122 meV, respectively.

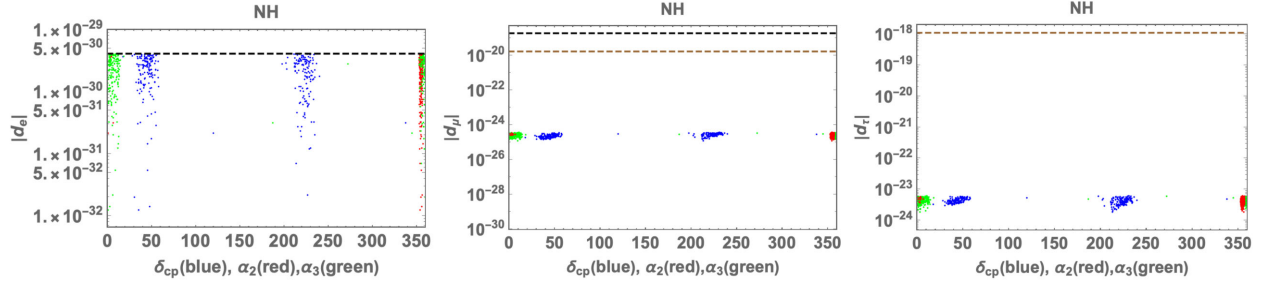


FIG. 4. Allowed regions for EDMs in terms of phases δ_{CP} , α_2 , and α_3 in case of NH, with same color manner as in Fig. 3. The black dashed line is the direct experimental upper bound, while brown corresponds to the indirect upper bound.

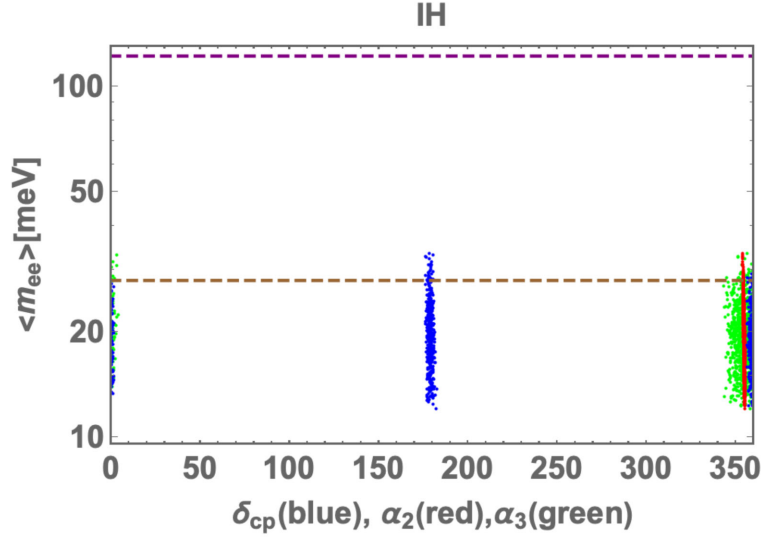


FIG. 5. Allowed regions for $\langle m_{ee} \rangle$ in terms of phases δ_{CP} (blue), α_2 (red), and α_3 (green) in case of IH. The meanings of dashed lines are the same as in Fig. 3.

which is more stringent upper limit of the neutrinoless double beta decay. It is also emphasized that the predictions of $|d_\mu|$ and $|d_\tau|$ are several orders of magnitude larger than those from the minimal-flavor violation relation:

$|d_\mu| \simeq |d_e|(m_\mu/m_e) \lesssim 8.5 \times 10^{-28} \text{ e cm}$ and $|d_\tau| \simeq |d_e|(m_\tau/m_e) \lesssim 1.4 \times 10^{-26} \text{ e cm}$. This is because origins of physical CP phases for $d_{e,\mu,\tau}$ are different from each other. Another interesting point is mostly fixed predictions

TABLE III. Best-fit experimental values in the IH case, corresponding to $\Delta\chi_{\min} \approx 0.539$. Here we select the above values that dominantly depend on absolute values.

Parameter	BF	Parameter	BF	Parameter	BF
s_{12}	0.557	s_{23}	0.695	s_{13}	0.150
Δm_{sol}^2	$7.60 \times 10^{-23} \text{ GeV}^2$	Δm_{atm}^2	$2.47 \times 10^{-21} \text{ GeV}^2$	D_{ν_3}	0.0166 meV
m_{ν_e}	48.6 meV	$\sum D_\nu$	98.6 meV	m_τ	1.777 GeV
m_e	0.000511 GeV	m_μ	0.106 GeV		
Δa_e	7.35×10^{-16}	Δa_μ	2.95×10^{-11}		
$\text{BR}(\mu \rightarrow e\gamma)$	3.93×10^{-20}	$\text{BR}(\tau \rightarrow e\gamma)$	1.78×10^{-23}	$\text{BR}(\tau \rightarrow \mu\gamma)$	2.64×10^{-17}

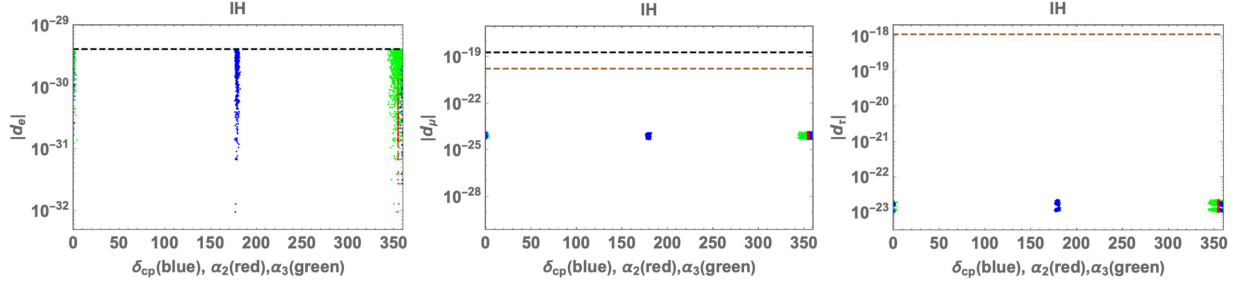


FIG. 6. Same plot as Fig. 4, but for the case of IH.

of $|d_\mu|$ and $|d_\tau|$ are obtained by the focused benchmark point. This is because they tend to be determined by the arguments whose allowed region is narrow for our input parameters, while $|d_e|$ tends to be predicted by wide ones.¹³

B. IH

Next, we show our benchmark point that does not depend on phases in Table III. The BF value results in $\Delta\chi_{\min} \approx 0.539$. Compared with the case of NH, we obtain about one order of magnitude larger Δa_e and Δa_μ . On the other hand, we found that the IH case predicts about 4 orders of magnitude larger LFVs than those in the NH case, which still satisfy the current bounds.

Under Table III, we figure out the neutrinoless double beta decay in terms of phases δ_{CP} , α_2 , and α_3 in Fig. 5, with same color manner as in Fig. 3. The several points exceed the upper limit of 28 meV, and the maximum value reaches 33.5 meV. We find δ_{CP} is localized at nearby 0° and 180° , while Majorana phases are $\alpha_2 \sim 355^\circ$ and $-20^\circ \lesssim \alpha_3 \lesssim 0^\circ$.

In Fig. 6, we show EDMs $|d_e|$ (left), $|d_\mu|$ (center), and $|d_\tau|$ (right), in term of three phases, where each of color plots are the same as the one in Fig. 3. These figures suggest that $|d_e|$ can reach the experimental upper bound 4.1×10^{-30} e cm, but typical values of $|d_{\mu,\tau}|$ are, respectively, 10^{-24} and 10^{-23} e cm. Interestingly, the constraint $|d_e| < 4.1 \times 10^{-30}$ e cm leads us to the similar result of the NH case; $\langle m_{ee} \rangle \lesssim 33.5$ meV which is just above the stringent upper limit of the neutrinoless double beta decay. The EDM dependencies are similar to those in the NH case.

IV. SUMMARY AND DISCUSSION

We have proposed a radiatively induced lepton mass model by introducing a Z_2 gauging Z_5 fusion rule to forbid tree-level lepton masses while keeping relevant interactions to induce them via radiative corrections. The

charged-lepton mass matrix is generated at one loop level via dynamical breaking of the fusion rule. On the other hand, the neutrino mass matrix is induced at the one-loop level without breaking the fusion rule. These facts lead us to discuss LFVs, electron and muon $g - 2$, and charged-lepton EDMs, where these phenomenology originate from the charged-lepton sector.

Due to too many free parameters, we have specified the absolute values of parameter set in order to fit the neutrino oscillation data and the charged-lepton mass observables. In addition, we have fixed the set of best fit values in cases of NH and IH. Then, once we have randomly selected the arguments of our free parameters and scanned whole their ranges, we have obtained our observables such as the Dirac CP phase, two Majorana phases, charged-lepton EDMs and the neutrinoless double beta decay, all of which depends on their arguments. As results of our numerical analyses, we have found several interesting tendencies; $30^\circ \lesssim \delta_{CP} \lesssim 60^\circ$, $200^\circ \lesssim \delta_{CP} \lesssim 240^\circ$, $-10^\circ \lesssim \alpha_2 \lesssim 0^\circ$ and $-10^\circ \lesssim \alpha_3 \lesssim 20^\circ$ in NH, and $\delta_{CP} \sim 0^\circ, 180^\circ$, $\alpha_2 \sim 355^\circ$ and $-20^\circ \lesssim \alpha_3 \lesssim 0^\circ$ in IH. We have also discovered that the constraint $|d_e| < 4.1 \times 10^{-30}$ e cm has led us to the maximum prediction of $\langle m_{ee} \rangle$ to be 28 meV for NH and 33.5 meV for IH, both of which are around more stringent upper limit of the neutrinoless double beta decay. Furthermore, the predictions of the muon and tau EDMs are several orders of magnitude larger than those obtained by the rough minimal-flavor violation relation, due to different origin of the CP phase.

We would like to mention the application of our mechanism to the quark sector. Although we did not focus on the quark sector in this work, we can construct the model for radiative quark masses by assigning the proper non-invertible fusion rule to quarks. The simplest model is to assign b to Q_L and a to d_R so that the tree-level down-type Yukawa couplings are forbidden. Then, one would obtain the same structure as in the case of our charged-lepton sector. Note that the up-type quark sector is not favored to apply this mechanism, since top-Yukawa coupling is order one,¹⁴ and its radiative generation will cause the issues, e.g.,

¹³In fact, we found that some arguments are required to be narrow range for fitting the experimental observables, while some of them are not so strictly constrained. Although it might be interesting to explore its dependence, we will not show them in detail, since we have a lot of parameters. Note here that absolute values of d_μ and d_τ are much larger than those of d_e .

¹⁴The radiative masses for up and charm quarks are acceptable, due to small Yukawa couplings.

requiring too large couplings which is not allowed in the sense of perturbativity. Therefore, down-type quark masses are generated via one-loop level after radiative breaking of the fusion rule. In this case, $b \rightarrow s\ell\bar{\ell}$ anomalies can be taken into account through η , as well as several constraints such as neutral meson mixings and $b \rightarrow s\gamma$. Detailed analysis is beyond our scope, but it is also worth constructing a radiative quark mass model in future.

Before closing our paper, we comment on a possible dark matter candidate in the model. Potentially, we have two dark matter candidates; the lightest neutral fermion ψ_{R1} or the lightest neutral component of η which are similar to the case of the Ma model. However, our allowed mass region through the numerical analysis suggests that these masses are $\mathcal{O}(1)$ PeV scale in case of $m_S \sim 100$ GeV, and DMs decay so quickly that they cannot be DM candidates. In case of the bosonic one, it decays into, e.g., pairs of $\bar{\ell}\ell$ through y^ℓ . In case of the fermionic one, it decays into $\ell^- W^+ \nu$ through $g^{L/R}$ and kinetic term of η due to mixing

between S^+ and η^+ . Thus, we concentrate on the other phenomenology in the current paper, instead of dark matter phenomenology. In a different parameter set, however, it would be a promising dark matter candidate which would be discussed elsewhere.

ACKNOWLEDGMENTS

T. N. is supported by the Fundamental Research Funds for the Central Universities. H. O. is supported by Zhongyuan Talent (Talent Recruitment Series) Foreign Experts Project. Y. S. is supported by Natural Science Foundation of China under Grant No. W2433006.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

-
- [1] Z. Jiang, B.-Y. Qu, and G.-J. Ding, *Phys. Rev. D* **112**, 115029 (2025).
 - [2] T. Kobayashi, H. Otsuka, M. Tanimoto, and T. T. Yanagida, *arXiv:2510.01680*.
 - [3] S. Jangid and H. Okada, *arXiv:2508.16174*.
 - [4] H. Okada and Y. Shigekami, *arXiv:2507.16198*.
 - [5] Y. Choi, H. T. Lam, and S.-H. Shao, *Phys. Rev. Lett.* **129**, 161601 (2022).
 - [6] C. Cordova, S. Hong, S. Koren, and K. Ohmori, *Phys. Rev. X* **14**, 031033 (2024).
 - [7] C. Cordova and K. Ohmori, *Phys. Rev. X* **13**, 011034 (2023).
 - [8] C. Cordova, S. Hong, and S. Koren, *Phys. Rev. X* **15**, 031011 (2025).
 - [9] T. Kobayashi, H. Otsuka, and M. Tanimoto, *J. High Energy Phys.* **12** (2024) 117.
 - [10] T. Kobayashi and H. Otsuka, *J. High Energy Phys.* **11** (2024) 120.
 - [11] T. Kobayashi, Y. Nishioka, H. Otsuka, and M. Tanimoto, *J. High Energy Phys.* **05** (2025) 177.
 - [12] M. Suzuki and L.-X. Xu, *arXiv:2503.19964*.
 - [13] Q. Liang and T. T. Yanagida, *Phys. Lett. B* **868**, 139706 (2025).
 - [14] T. Kobayashi, H. Otsuka, M. Tanimoto, and H. Uchida, *J. High Energy Phys.* **08** (2025) 189.
 - [15] T. Kobayashi, H. Mita, H. Otsuka, and R. Sakuma, *arXiv:2506.10241*.
 - [16] T. Kobayashi, H. Okada, and H. Otsuka, *J. High Energy Phys.* **12** (2025) 111.
 - [17] T. Nomura and H. Okada, *arXiv:2506.16706*.
 - [18] J. Dong, T. Jeric, T. Kobayashi, R. Nishida, and H. Otsuka, *Phys. Rev. D* **113**, 056028 (2026).
 - [19] T. Nomura and O. Popov, *arXiv:2507.10299*.
 - [20] J. Chen, C.-Q. Geng, H. Okada, and J.-J. Wu, *Nucl. Phys. B* **1025**, 117391 (2026).
 - [21] T. Kobayashi, H. Otsuka, and T. T. Yanagida, *Phys. Rev. D* **113**, 055016 (2026).
 - [22] M. Suzuki, L.-X. Xu, and H. Y. Zhang, *arXiv:2508.14970*.
 - [23] G. Altarelli and F. Feruglio, *Rev. Mod. Phys.* **82**, 2701 (2010).
 - [24] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, *Prog. Theor. Phys. Suppl.* **183**, 1 (2010).
 - [25] T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu, and M. Tanimoto, *An Introduction to Non-Abelian Discrete Symmetries for Particle Physicists* (Springer Berlin Heidelberg, 2022), 10.1007/978-3-662-64679-3.
 - [26] H. Ishimori, T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu, and M. Tanimoto, *An Introduction to Non-Abelian Discrete Symmetries for Particle Physicists* (Springer Berlin Heidelberg, 2012), Vol. 858, 10.1007/978-3-642-30805-5.
 - [27] Z.-j. Tao, *Phys. Rev. D* **54**, 5693 (1996).
 - [28] E. Ma, *Phys. Rev. D* **73**, 077301 (2006).
 - [29] K. S. Khaw, Y. Nakai, R. Sato, Y. Shigekami, and Z. Zhang, *J. High Energy Phys.* **02** (2023) 234.
 - [30] H. Okada and K. Yagyu, *Phys. Rev. D* **89**, 053008 (2014).
 - [31] H. Okada and K. Yagyu, *Phys. Rev. D* **90**, 035019 (2014).
 - [32] M. E. Catano, R. Martinez, and F. Ochoa, *Phys. Rev. D* **86**, 073015 (2012).
 - [33] Y. Kajiyama, H. Okada, and T. Toma, *Eur. Phys. J. C* **73**, 2381 (2013).
 - [34] S. Abe *et al.* (KamLAND-Zen Collaboration), *Phys. Rev. Lett.* **135**, 262501 (2025).

- [35] S. Vagnozzi, E. Giusarma, O. Mena, K. Freese, M. Gerbino, S. Ho, and M. Lattanzi, *Phys. Rev. D* **96**, 123503 (2017).
- [36] N. Aghanim *et al.* (Planck Collaboration), *Astron. Astrophys.* **641**, A6 (2020); **652**, C4(E) (2021).
- [37] K. A. Olive (Particle Data Group), *Chin. Phys. C* **38**, 090001 (2014).
- [38] A. G. Adame *et al.* (DESI Collaboration), *J. Cosmol. Astropart. Phys.* **02** (2025) 021.
- [39] M. Aker *et al.* (KATRIN Collaboration), *Nat. Phys.* **18**, 160 (2022).
- [40] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler, J. a. P. Pinheiro, and T. Schwetz, *J. High Energy Phys.* **12** (2024) 216.
- [41] A. M. Baldini *et al.* (MEG Collaboration), *Eur. Phys. J. C* **76**, 434 (2016).
- [42] B. Aubert *et al.* (BABAR Collaboration), *Phys. Rev. Lett.* **104**, 021802 (2010).
- [43] F. Renga (MEG Collaboration), *Hyperfine Interact.* **239**, 58 (2018).
- [44] K. Afanaciev *et al.* (MEG II Collaboration), *Eur. Phys. J. C* **84**, 216 (2024); **84**, 1042(E) (2024).
- [45] X. Fan, Ph.D. thesis, Harvard University, 2022.
- [46] X. Fan, T. G. Myers, B. A. D. Sukra, and G. Gabrielse, *Phys. Rev. Lett.* **130**, 071801 (2023).
- [47] S. Navas *et al.* (Particle Data Group), *Phys. Rev. D* **110**, 030001 (2024).
- [48] R. Aliberti *et al.*, *Phys. Rep.* **1143**, 1 (2025).
- [49] D. P. Aguillard *et al.* (Muon $g - 2$ Collaboration), *Phys. Rev. Lett.* **135**, 101802 (2025).
- [50] L. Morel, Z. Yao, P. Cladé, and S. Guellati-Khélifa, *Nature (London)* **588**, 61 (2020).
- [51] B. Abi *et al.* (Muon $g - 2$ Collaboration), *Phys. Rev. Lett.* **126**, 141801 (2021).
- [52] D. P. Aguillard *et al.* (Muon $g - 2$ Collaboration), *Phys. Rev. Lett.* **131**, 161802 (2023).
- [53] R. H. Parker, C. Yu, W. Zhong, B. Estey, and H. Müller, *Science* **360**, 191 (2018).
- [54] A. Abada and T. Toma, *J. High Energy Phys.* **08** (2024) 128.
- [55] Y. Ema, T. Gao, and M. Pospelov, *Phys. Rev. Lett.* **128**, 131803 (2022).
- [56] Y. Ema, T. Gao, and M. Pospelov, *Phys. Lett. B* **835**, 137496 (2022).
- [57] H. Okada and Y. Orikasa, *arXiv:1908.08409*.