

Light-cone sum rules analysis of the semileptonic $D_s^+ \rightarrow f_0(980)$ $(\rightarrow \pi^+ \pi^-) e^+ \nu_e$ decay incorporating the $f_0(980)$ mixing state twist-2 distribution amplitudes

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The isospin-singlet scalar meson $f_0(980)$ is hypothesized to consist of two energy eigenstates, forming a mixture of $\frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d)$ and $\bar{s}s$. Building on this framework, we apply the quantum chromodynamics (QCD) sum rules approach in the background field theory to compute the $f_0(980)$ decay constant, yielding $f_{f_0}(\mu_0 = 1 \text{ GeV}) = 0.386 \pm 0.009 \text{ GeV}$. Subsequently, we derive the first two ξ -moments of the leading-twist light-cone distribution amplitude ϕ_{2,f_0} . Using QCD light-cone sum rules, we then calculate the $D_s^+ \rightarrow f_0(980)$ transition form factor (TFF) $f_+(q^2)$, obtaining $f_+(0) = 0.516_{-0.024}^{+0.027}$ at the large-recoil region. We extend $f_+(q^2)$ to the full physical region via a simplified series expansion parametrization, enabling calculations of the differential decay widths and the branching fraction $\mathcal{B}(D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+ \pi^-) e^+ \nu_e) = (1.783_{-0.189}^{+0.227}) \times 10^{-3}$. Our theoretical predictions align well with the latest BESIII Collaboration measurements within reasonable errors.

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I. INTRODUCTION

Light scalar mesons present a longstanding classification challenge in particle physics. The understanding of the structure of scalar mesons with masses around 1 GeV is one of the prominent topics in modern hadronic physics. Originally interpreted as orbitally excited quark-antiquark ($q\bar{q}$) states, these particles have sparked debates over tetraquark ($q\bar{q}q\bar{q}$) or molecular configurations, particularly due to their strong coupling to $\pi\pi$ and KK thresholds which complicate traditional spectroscopic analysis. The ambiguity arises from the interplay between confinement and chiral symmetry breaking, driving investigations into non-perturbative quantum chromodynamics (QCD) effects,

multi-quark dynamics, and hadronic molecules [1]. The unique properties of light scalar mesons such as $f_0(500)$, $f_0(980)$ and $a_0(980)$ have been a central focus in hadron physics research for decades. Unraveling their internal quark-gluon substructure is a cornerstone of investigating QCD in the nonperturbative regime, as these systems provide critical insights into the dynamical mechanisms underlying chiral symmetry breaking—a phenomenon fundamental to understanding quark confinement and hadron mass generation in the low-energy sector of QCD.

The weak interaction between leptons and hadrons in the final state renders semileptonic decays of charm mesons an ideal experimental framework for probing the quark components in the wave functions of light scalar mesons [2]. This “clean” decay system offers dual advantages: first, the spectator light quark mechanism dominates scalar state formation, enabling precise analysis of their quark flavor composition through comparisons of Cabibbo-favored and Cabibbo-suppressed processes; second, extracting the hadronic form factors that characterize strong interactions between final-state quarks provides critical

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experimental benchmarks for testing nonperturbative theoretical approaches such as the lattice QCD and the QCD sum rules (QCD SR) [3]. Notably, the form factors and branching fractions of these decays are highly sensitive to the internal structure of light scalar states, making investigations of their dynamics a core experimental tool for unraveling the intrinsic nature of these particles. This endeavor bridges the gap between precision measurements of weak decays and strong interaction theories [4].

In this paper, we focus on the light scalar meson $f_0(980)$. Its mass and width are measured as 990 ± 20 MeV and 10–100 MeV, according to Ref. [5]. Given its mass below 1 GeV, competing interpretations of its internal quark structure have emerged. Analyses of the $f_0(980)$ -meson as a quarkonium ($q\bar{q}$) state were performed in Refs. [6–10]. Different scenarios for the admixture of nonstrange and strange $s\bar{s}$ components have been developed, which range from a pure or dominant $s\bar{s}$ state to mixtures with $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$. Alternative models propose composite structures, such as diquark-diantiquark ($q^2\bar{q}^2$) [11–13], compact tetraquarks ($q\bar{q}q\bar{q}$) [14–16] or meson-meson bound states [17–19]. Within this mass range, the possibility of a scalar glueball bound state has also been considered [20–22]. Resolving the fundamental properties of this intriguing hadron state therefore requires more precise experimental measurements, advanced nonperturbative QCD techniques, and continued theoretical investigations.

From the experimental perspective, the *BABAR* collaboration identified evidence for the $f_0(980)$ resonance in semileptonic decays by analyzing the $f_0(980) \rightarrow K^+K^-$ channel, where interference between an S -wave and the dominant P -wave ϕ decay was observed [23]. The CLEO collaboration pioneered the first measurement of the product branching fraction for $D_s^+ \rightarrow f_0(980)$ with the cascade decay $f_0(980) \rightarrow \pi^+\pi^-$ [24]. In 2009, they proposed that the $f_0(980)$ wave function combines strange and nonstrange quark-antiquark components. Analyzing $D_s^+ \rightarrow f_0(980)$ decay branching fractions, they determined a mixing angle consistent with $|\bar{s}s\rangle$ dominance, providing evidence for why $f_0(980)$ decays predominantly to $\pi^+\pi^-$ rather than K^+K^- [25]. The BESIII collaboration first measured $D_s^+ \rightarrow f_0(980)$ in 2019 and observed the S -wave contribution, setting an upper limit of $\mathcal{B}(D_s^+ \rightarrow f_0(980)e^+\nu_e, f_0(980) \rightarrow \pi^+\pi^-) < 2.8 \times 10^{-5}$ [26]. Recently, BESIII collaboration updated these measurements with improved precision, reporting $\mathcal{B}(D_s^+ \rightarrow f_0(980)e^+\nu_e, f_0(980) \rightarrow \pi^0\pi^0) = (0.79 \pm 0.14 \pm 0.03) \times 10^{-3}$ for the neutral channel [27] and $\mathcal{B}(D_s^+ \rightarrow f_0(980)e^+\nu_e, f_0(980) \rightarrow \pi^+\pi^-) = (1.72 \pm 0.13 \pm 0.10) \times 10^{-3}$ for the charged one [28]. More notably, the BESIII collaboration extracted the $D_s^+ \rightarrow f_0(980)$ transition form factor (TFF) under the Flatté resonant model with an

integrated luminosity of 7.33fb^{-1} , yielding $f_+(0) = 0.518 \pm 0.018 \pm 0.036$ at the full recoiled point, including statistical and systematic uncertainties [28]. Leveraging the Flatté formula, BESIII's latest measurements further constrain the $f_0(980)$ -meson's properties. Using the relation between the branching fraction (BF) and the mixing angle θ in the $q\bar{q}$ mixture model $\sin\theta|\frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d)\rangle + \cos\theta|\bar{s}s\rangle$ [29,30], they confirmed the dominance of the $s\bar{s}$ component [28].

Theoretical predictions for the hadronic TFFs rely on assumptions about the structure of $f_0(980)$, and calculations of the BFs for the semileptonic decay $D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e$ are highly sensitive to its internal composition. Thus, measurements for these quantities can help unravel its nature. Currently, diverse theoretical groups have predicted the TFFs and the BFs of $D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e$, including the light-front quark model (LFQM) [10], chiral unitarity approach (χ UA) [3], covariant confined quark model (CCQM) [4], covariant light-front dynamics (CLFD) [31], three-point sum rules (3PSR) [32–34] and light-cone sum rules (LCSR) [29,30,35]. Existing theoretical predictions for TFFs and BFs exhibit significant discrepancies with experimental measurements. To systematically address this gap, we will employ the QCD sum rule (QCD SR) approach to develop a comprehensive theoretical description of the $D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e$ decay process. This description incorporates the $f_0(980)$ decay constant, twist-2 light-cone distribution amplitude (LCDA) under the LCSR framework, the $D_s^+ \rightarrow f_0(980)$ TFF and the related BF, aiming to reconcile theoretical predictions with experimental data.

The LCSR approach [36–39] is a powerful tool for determining nonperturbative parameters of hadronic states. In this framework, the amplitude of the process is factorized into a perturbatively calculable short-distance component and a nonperturbative component encoded in LCDAs. Applying LCSR involves performing an operator product expansion (OPE) in the light-cone vicinity ($x^2 \approx 0$), after which the resulting nonperturbative hadronic matrix elements are then parametrized using LCDAs of increasing twists. The higher twist terms are generally power suppressed, which may also be effectively suppressed within proper Borel windows. In LCSR calculations for the TFFs and BFs of $D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e$, the decay constant and twist-2 LCDA of the $f_0(980)$ meson serve as critical input parameters. Despite their fundamental role, theoretical investigations of these observables remain limited. While Refs. [40,41] pioneered the determination of $f_0(980)$ decay constants and twist-2 LCDAs using QCD SR approach, this analysis relied on the simplifying assumption of a pure strange-antistrange $\bar{s}s$ configuration for the $f_0(980)$. This approximation overlooks potential contributions from nonvalence components or mixing effects with other scalar meson states, which may introduce

systematic uncertainties in subsequent phenomenological analyses. In this paper, we model the $f_0(980)$ as a mixed state comprising $\frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d)$ and $\bar{s}s$ quark components. Using QCD sum rules within the background field theory framework (BFTSR), we systematically compute its decay constant and twist-2 LCDA.

The BFTSR approach is to derive the well-known Shifman-Vainshtein-Zakharov (SVZ) sum rules [42,43] in the framework of background field theory (BFT) [44–46]. The SVZ sum rules takes the correlation function constructed by inserting currents representing hadrons between two vacuum states as the starting point. In deep Euclidean region, the operator product expansion is performed on the correlation function, while in the physical region, the complete set of hadron states is inserted to obtain its hadron representation. Then, the dispersion relation is used to match the two, and the Borel transform is performed to suppress the contribution of continuous states. Finally, a sum rule containing the physical quantity we want to study is obtained. In the OPE framework, the short- and long- distance quark-gluon interactions are separated. The former are calculated using QCD perturbation theory, while the latter are parametrized as the universal nonperturbative parameters—vacuum condensations. To compare with the traditional procedures of SVZ sum rules, the BFTSR provides a unified description based on the field theory [44,45]. The BFT is a phenomenological theory, whose basic idea is to rewrite the fields as the sum of the classical background fields and the quantum fields. The former describe nonperturbative effects, while the latter describe quantum fluctuations on these basis, namely perturbative effects [46]. The introduction of BFT provides a clear physical picture for the SVZ sum rules, which also has advantages in the deriving the formulas for the propagators, the vacuum matrix elements and etc. [47]. Thus the BFT provides a self-consistent description for the vacuum condensates and provides a systematic way to derive the SVZ sum rules. At present, the BFTSR approach has been successfully applied to calculate LCDAs of multiple mesons, enhancing the utility of QCD sum rules in hadron physics. This approach not only yields more reliable theoretical predictions for experimental observations but also provides a powerful tool for probing the nonperturbative mechanisms and hadron structure in QCD.

Subsequently, we employ the LCSR method to compute the $D_s^+ \rightarrow f_0(980)$ TFFs. By incorporating the decay width formula recently reported by the BESIII Collaboration, we derive the corresponding differential decay widths and BFs. Our results substantially reduce the discrepancy between theoretical predictions and experimental measurements, offering new theoretical constraints for unraveling the structure of light scalar mesons.

This paper is structured as follows. In Sec. II, we first briefly introduce the decay width for the semileptonic

$D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e$. We then outline the principles of the BFTSR framework for calculating the moments of the $f_0(980)$ meson's twist-2 LCDA, as well as the LCSR framework for calculating the TFF associated with this semileptonic decay. In Sec. III, we present our numerical results and conduct a detailed comparison with other experimental and theoretical predictions. Section IV is dedicated to a summary.

II. CALCULATION TECHNOLOGY

The decay diagram at hadronic level for the process $D_s^+ \rightarrow f_0(980)(\rightarrow P^+P^-)e^+\nu_e$ is shown in Fig. 1, where P represents π or K , respectively. After emitting the off-shell W^+ -boson, the hadronic sector is the $\bar{s}s$ which will couple to the isosinglet component $f_0(980)$. At the energy scale near 1 GeV, the $f_0(980)$ -meson, due to its inherent instability, exhibits significant coupling with the $\pi^+\pi^-$ through strong interaction [25]. Then the decay amplitudes for the $D_s^+ \rightarrow f_0(980)e^+\nu_e \equiv \pi^+\pi^-e^+\nu_e$ is given as [48]

$$\mathcal{A}(D_s^+ \rightarrow \pi^+\pi^-e^+\nu) = \hat{A} \left(\frac{i}{D_{f_0}} \times ig_{\pi^+\pi^-} \right), \quad (1)$$

where D_{f_0} is the $f_0(980)$ -meson denominators of the resummed propagators, encompasses the self-energy correction, the width effect, and the coupling information of the $f_0(980)$. It is typically expressed in the form

$$D_{f_0}(s) = s - m_{f_0}^2 + \Pi_{f_0}(s), \quad (2)$$

where $\Pi_{f_0}(s)$ is self-energy function. The real part of the self-energy function causes a shift in the propagator pole, reflecting the correction of the $f_0(980)$ -meson mass from the bare mass m_{f_0} to the physical mass s . The imaginary part is directly related to the decay width Γ_{f_0} , embodying the probability of $f_0(980)$ decaying into $\pi^+\pi^-$ or K^+K^- via strong interaction. Consequently, the expression for D_{f_0} can be further articulated as [49,50]

$$D_{f_0}(s) = s - m_{f_0}^2 + i\sqrt{s}[\Gamma_{\pi^+\pi^-}^{f_0}(s) + \Gamma_{K^+K^-}^{f_0}(s)]. \quad (3)$$

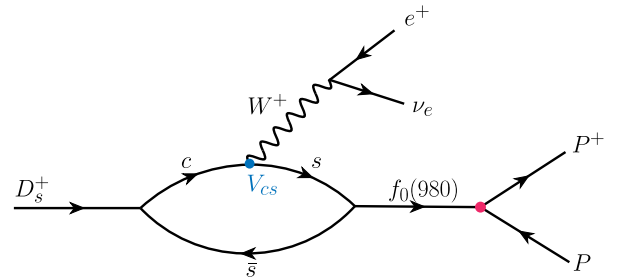


FIG. 1. The $D_s^+ \rightarrow f_0(980)(\rightarrow P^+P^-)e^+\nu_e$ decay diagram, where P represents π or K , respectively.

$\Gamma_{\pi^+\pi^-}^{f_0}(s)$ and $\Gamma_{K^+K^-}^{f_0}(s)$ are nonzero mass-dependent decay width, which the specific expressions are, respectively

$$\begin{aligned}\Gamma_{\pi^+\pi^-}^{f_0}(s) &= \frac{1}{16\pi\sqrt{s}} |g_{\pi^+\pi^-}|^2 \rho_{\pi\pi}(s), \\ \Gamma_{K^+K^-}^{f_0}(s) &= \frac{1}{16\pi\sqrt{s}} |g_{K^+K^-}|^2 \rho_{K\bar{K}}(s),\end{aligned}\quad (4)$$

where $g_{\pi^+\pi^-} = \sqrt{16\pi g_1}$ and $g_{K^+K^-} = \sqrt{16\pi g_2}$, with the constants g_1 and g_2 are the $f_0(980)$ couplings to $\pi^+\pi^-$ and K^+K^- final states, respectively. The factors $\rho_{\pi\pi}$ and $\rho_{K\bar{K}}$ are given by Lorentz invariant phase space [51]

$$\rho_{\pi\pi}(s) = \sqrt{\left[1 - \frac{(m_\pi - m_\pi)^2}{s}\right] \left[1 - \frac{(m_\pi + m_\pi)^2}{s}\right]}, \quad (5)$$

$$\rho_{K\bar{K}}(s) = \sqrt{\left[1 - \frac{(m_K - m_{\bar{K}})^2}{s}\right] \left[1 - \frac{(m_K + m_{\bar{K}})^2}{s}\right]}. \quad (6)$$

The amplitude $\hat{A}$ can be expressed in terms of the TFFs [52]

$$\begin{aligned}\langle f_0(980)(p) | \bar{s} \gamma_\mu \gamma_5 c | D_s^+(p+q) \rangle \\ = -2if_+(q^2)p_\mu - i[f_+(q^2) + f_-(q^2)]q_\mu,\end{aligned}\quad (7)$$

where $f_\pm(q^2)$ are TFFs. Since this transition is between a pseudoscalar D_s^+ and a scalar $f_0(980)$, only the axial-vector part of the current contribution. The term containing $f_-(q^2)$ gets multiplied by the positron mass squared, and becomes vanishingly small due to chiral suppression [53]. The double differential decay width is then derived as [28,51]

$$\begin{aligned}\frac{d\Gamma(D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e)}{ds dq^2} \\ = \frac{G_F^2 |V_{cs}|^2}{192\pi^3 m_{D_s^+}^3} \lambda^{3/2}(m_{D_s^+}^2, m_{f_0}^2, q^2) |f_+(q^2)|^2 \frac{P(s)}{\pi},\end{aligned}\quad (8)$$

where s is the invariant mass square of the two pseudo-scalars π^+, π^- . For the $f_0(980)$, one may follow the BESIII analysis of the $J/\Psi \rightarrow \phi \pi^+ \pi^-$ [54], $s_{\min} = 0.9 \text{ GeV}/c^2$, $s_{\max} = 1.0 \text{ GeV}/c^2$. Due to the limitation of the physical region q^2 , in the actual calculation process, our value is $s_{\min} \leq s \leq (m_{D_s^+} - m_{f_0})^2$. G_F is the Fermi constant, $|V_{cs}|$ is the Cabibbo-Kobayashi-Maskawa (CKM) matrix element, $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ and $P(s)$ is based on the relativistic Flatté formula due to the open K^+K^- channel as follows:

$$P(s) = \frac{g_1 \rho_{\pi\pi}}{[m_{f_0}^2 - s - i[g_1 \rho_{\pi\pi}(s) + g_2 \rho_{K\bar{K}}(s)]]^2}. \quad (9)$$

Here the $\rho_{\pi\pi}$ and $\rho_{K\bar{K}}$ are individual phase space factors.

The TFF $f_+(q^2)$ can be calculated by using the LCSR method. In order to uniformly express the derivation process of the LCSRs for $D_s^+ \rightarrow f_0(980)$ TFFs, we introduce the following vacuum-to- $f_0(980)$ correlator

$$\begin{aligned}\Pi_\mu(p, q) &= i \int d^4x e^{iq \cdot x} \langle f_0(980)(p) | T \{ \bar{s}(x) \gamma_\mu \gamma_5 c(x) \\ &\quad \times \bar{c}(0) i \gamma_5 s(0) \} | 0 \rangle \\ &= F[q^2, (p+q)^2] p_\mu + \tilde{F}[p^2, (p+q)^2] q_\mu.\end{aligned}\quad (10)$$

We first calculate the correlator (10) in QCD. Under the kinematic conditions $q^2 \ll m_c^2$ and $(p+q)^2 \ll m_c^2$ (where m_c denotes the heavy charm quark mass), the heavy quark propagating within the correlation function exhibits significant virtuality. This regime corresponds to spatial separations approaching the light-cone limit $x^2 \rightarrow 0$ [55], as dictated by the dominance of small momentum transfer scales relative to the heavy quark mass. In this scenario, the OPE framework simplifies through the contraction of heavy quark fields via Wick's theorem, enabling light-cone expansion of the heavy quark propagator. The resulting formalism allows for the factorization of short-distance perturbative effects from long-distance nonperturbative dynamics, characteristic of LCDA analyses. Thus one can contract the heavy quark fields, and the following light-cone expansion of the heavy quark propagator

$$\langle 0 | c_\alpha^i(x) \bar{c}_\beta^j(0) | 0 \rangle = i \int \frac{d^4k}{(2\pi^4)} e^{-ik \cdot x} \left(\delta^{ij} \frac{k + m_c}{k^2 - m_c^2} + \dots \right)_{\alpha\beta}. \quad (11)$$

In this work, only the two-particle part is displayed. The multiparticle contributions are justifiably disregarded due to their minimal impact, representing no more than 1% of the total [56]. Then the vacuum-to- $f_0(980)$ matrix element can be expanded in terms of the $f_0(980)$ -meson LCDA's of increasing twist, i.e.,

$$\begin{aligned}\langle f_0(980)(p) | \bar{s}_\alpha^i(x) s_\beta^j(0) | 0 \rangle \\ = \frac{\delta^{ij}}{12} \tilde{f}_{f_0} \int_0^1 du e^{iup \cdot x} \left[\not{p} \phi_{2;f_0}(u) + m_{f_0} \phi_{3;f_0}^p(u) \right. \\ \left. - \frac{1}{6} m_{f_0} \sigma_{\mu\nu} p^\mu x^\nu \phi_{3;f_0}^\sigma(u) \right]_{\beta\alpha} + \dots\end{aligned}\quad (12)$$

with the $\phi_{2;f_0}(u)$ and $\phi_{3;f_0}^{p,\sigma}(u)$ are the two-particle twist-2, 3 LCDA, respectively. We neglect contributions from twists higher than 3, as it is well established from

a large number of LCSR calculations of semileptonic decay TFFs that higher-twist LCDAs contribute typically less than 5% [57–61]. The investigation of higher twist LCDAs effects are associated with considerable theoretical uncertainties, while their expected contribution is minor, so this study concentrates exclusively to contributions from twist-2 and twist-3 LCDAs. Using the OPE for the invariant amplitudes F and $\tilde{F}$, we obtain

$$F_{\text{QCD}}[q^2, (p+q)^2] = im_c \bar{f}_{f_0} \int_0^1 \frac{du}{m_c^2 - (up+q)^2} \times \left\{ \phi_{2;f_0}(u) - \frac{m_{f_0}}{m_c} u \phi_{3;f_0}^p(u) - \frac{m_{f_0}}{6m_c} \times \left[2 + \frac{m_c^2 + q^2 - u^2 p^2}{m_c^2 - (up+q)^2} \right] \phi_{3;f_0}^\sigma(u) \right\}, \quad (13)$$

$$\tilde{F}_{\text{QCD}}[q^2, (p+q)^2] = im_c \bar{f}_{f_0} \int_0^1 \frac{du}{m_c^2 - (up+q)^2} \times \left\{ -\frac{m_{f_0}}{m_c} \phi_{3;f_0}^\sigma(u) - \frac{m_{f_0}}{6m_c} \times \left[1 - \frac{m_c^2 + q^2 - u^2 p^2}{m_c^2 - (up+q)^2} \right] \frac{\phi_{3;f_0}^\sigma(u)}{u} \right\}. \quad (14)$$

On the other hand, the correlation function (10) in the timelike q^2 -region can be treated with the hadronic representation, which is achieved by inserting a complete set of intermediate states with the same quantum numbers in the correlation function. By isolating the pole term of the lowest pseudoscalar D_s^+ -meson, we obtain the following representation of the correlation function from the hadronic side

$$\Pi_\mu(p, q) = \frac{\langle f_0(980)(p) | \bar{s} \gamma_\mu \gamma_5 c | D_s^+(p+q) \rangle \langle D_s^+(p+q) | \bar{c} i \gamma_5 s | 0 \rangle}{m_{D_s^+}^2 - (p+q)^2} + \sum_H \frac{\langle f_0(980)(p) | \bar{s} \gamma_\mu \gamma_5 c | D_s^H(p+q) \rangle \langle D_s^H(p+q) | \bar{c} i \gamma_5 s | 0 \rangle}{m_{D_s^H}^2 - (p+q)^2} \quad (15)$$

where $\langle D_s^+(p+q) | \bar{c} i \gamma_5 s | 0 \rangle = m_{D_s^+}^2 f_{D_s^+} / (m_c + m_s)$ with $f_{D_s^+}$ is D_s^+ -meson decay constant. By substituting the parametrized Eq. (7) (as the TFF) into the above equation, the invariant amplitude in the hadronic representation can be obtained as follows:

$$F_{\text{had}}[p^2, (p+q)^2] = \frac{-2im_{D_s^+}^2 f_{D_s^+} f_+(q^2)}{(m_c + m_s)[m_{D_s^+}^2 - (p+q)^2]} + \dots$$

$$\tilde{F}_{\text{had}}[p^2, (p+q)^2] = \frac{-im_{D_s^+}^2 f_{D_s^+} [f_+(q^2) + f_-(q^2)]}{(m_c + m_s)[m_{D_s^+}^2 - (p+q)^2]} + \dots \quad (16)$$

Equation (8) indicates that the decay width $\Gamma(D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+ \pi^-) e^+ \nu_e)$ primarily depends on the TFF $f_+(q^2)$. Therefore, we will explain its derivation in some more detail in the following. One can derive a general dispersion relation for $F[q^2, (p+q)^2]$ and subsequently apply the usual Borel transformation with respect to the momentum squared $(p+q)^2$ of the heavy meson [62] to suppress the contributions from the unknown continuum states and higher resonances

$$F(q^2, M^2) = \int_{t_{\min}}^\infty \rho(q^2, s) e^{-s/M^2} ds \quad (17)$$

with $t_{\min} = (m_c + m_s)^2$. Here the spectral density is given by

$$\rho(q^2, s) = \frac{1}{\pi} \text{Im} F_{\text{had}}(q^2, s)$$

$$= \delta(s - m_{D_s^+}^2) \frac{-2im_{D_s^+}^2 f_{D_s^+}}{m_c + m_s} f_+(q^2) + \frac{1}{\pi} \text{Im} F_{\text{QCD}}(q^2, s) \theta(s - s_0). \quad (18)$$

In this formulation, the contributions from excited states and continuum states contributions in $F_{\text{had}}(q^2, s)$ are incorporated through the imaginary part of the QCD correlation function $\text{Im} F_{\text{QCD}}(q^2, s)/\pi$, with nonperturbative effects demarcated by an effective continuum threshold parameter s_0 under the quark-hadronic duality approximation [42,43]. The Borel transformation, as implemented in Eq. (17), systematically suppresses potential subtraction arising from dispersion relation while enhancing the convergence of the OPE by exponentially restraining excited states. Substituting Eq. (18) into Eq. (17), one can get

$$F(q^2, M^2) = \frac{-2im_{D_s^+}^2 f_{D_s^+} f_+(q^2)}{m_c + m_s} e^{m_{D_s^+}^2/M^2} + \frac{1}{\pi} \int_{s_0}^\infty F_{\text{had}}(q^2, s) e^{-s/M^2} ds \quad (19)$$

On the other hand, the invariant amplitude after Borel transformation can also be written as

$$F(q^2, M^2) = \frac{1}{\pi} \int_{t_{\min}}^\infty F_{\text{had}}(q^2, s) e^{-s/M^2} ds \quad (20)$$

Equating Eqs. (19) and (20), the LCSR of TFF $f_+(q^2)$ can be obtained as

$$f_+(q^2) = \frac{(m_c + m_s)\bar{f}_{f_0}}{2m_{D_s}^2 f_{D_s}} e^{m_{D_s}^2/M^2} \left\{ \int_{u_0}^1 du e^{-(m_c^2 - \bar{u}q^2 + u\bar{u}m_{f_0}^2)/(uM^2)} \left[-m_c \frac{\phi_{2;f_0}(u)}{u} \right. \right. \\ \left. \left. + m_{f_0} \left(\frac{2}{u} + \frac{4um_c^2 m_{f_0}^2}{(m_c^2 - q^2 + u^2 m_{f_0}^2)^2} - \frac{m_c^2 + q^2 - u^2 m_{f_0}^2}{m_c^2 - q^2 + u^2 m_{f_0}^2} \frac{d}{du} \right) \frac{\phi_{3;f_0}^\sigma(u)}{6} + m_{f_0} \times \phi_{3;f_0}^p(u) \right] \right. \\ \left. + e^{-(m_c^2 - \bar{u}q^2 + u\bar{u}m_{f_0}^2)/(uM^2)} \frac{m_{f_0}}{6m_c} \frac{m_c^2 + q^2 - u^2 m_{f_0}^2}{m_c^2 - q^2 + u^2 m_{f_0}^2} \phi_{3;f_0}^\sigma(u) \right|_{u \rightarrow 1} \Bigg\}. \quad (21)$$

with $u_0 = \left[\sqrt{(q^2 - s_0 + m_{f_0}^2)^2 + 4m_{f_0}^2(m_c^2 - q^2)} + q^2 - s_0 + m_{f_0}^2 \right] / 2m_{f_0}^2$. Based on the conformal spin invariance, the LCDAs can be expanded in terms of Gegenbauer polynomials $C_n^{3/2}$. For the twists-3 LCDAs, we use the asymptotic form, i.e., the first term of the Gegenbauer expansion [29],

$$\begin{aligned} \phi_{3;f_0}^p(u) &= 1, \\ \phi_{3;f_0}^\sigma(u) &= 6u(1-u). \end{aligned} \quad (22)$$

According to Eq. (8), it can be found that the width of $D_s^+ \rightarrow f_0(980)$ is mainly $\pi^+\pi^-$, which also indicates that $f_0(980)$ is not purely $s\bar{s}$ state, but should contains the composition of $u\bar{u}$ and $d\bar{d}$ pairs. Under $SU_f(3)$ symmetry, the light scalar meson $f_0(980)$ can have two possible quark structures, leading to two possible mixing scenarios. For the $\bar{q}^2 q^2$ structure, the $f_0(980)$ can be expressed as follows [63,64]

$$|f_0(980)\rangle = \left| \frac{1}{\sqrt{2}} (u\bar{u} + d\bar{d})s\bar{s} \right\rangle \cos \theta_I + |u\bar{u}d\bar{d}\rangle \sin \theta_I, \quad (23)$$

It has been shown that $\theta_I = 174.6^\circ$ [63]. It is extremely challenging to calculate $f_0(980)$ as a tetraquark state within the QCD sum rules. For the interpolating operators used, the pole contribution analysis yields very low upper limits for M^2 . There is no OPE convergence in the region allowed by the pole/continuum constraint. This means that for reasonable values of the continuum threshold and Borel mass, it is impossible to simultaneously satisfy both OPE convergence and pole dominance's key validity criteria for reliable QCD sum rules results [65]. For light scalar mesons like $f_0(980)$, since the relative contribution of high-dimensional condensed matter is greater, this makes the truncation of the OPE unreasonable. Therefore, for light scalar mesons including $f_0(980)$, there are no parameter values that can satisfy all the necessary conditions, which weakens the reliability of interpreting $f_0(980)$ as a tetraquark state.

The light scalar meson $f_0(980)$ have another possible $\bar{q}q$ structure [10,29,30,33]

$$|f_0(980)\rangle = |s\bar{s}\rangle \cos \theta + |n\bar{n}\rangle \sin \theta, \quad (24)$$

with θ is mixing angle and $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$. Experimental implications for the mixing angle have been discussed in detail in [12,66,67]. Here, we primarily adopt the results obtained by the BESIII'24 Collaboration, namely: $\theta = (19.7 \pm 12.8)^\circ$ [28]. In general, the $f_0(980)$ -meson twist-2 LCDA $\phi_{2;f_0}(u)$ can be expanded as a Gegenbauer polynomial series [40,68]

$$\phi_{2;f_0}(x, \mu) = \bar{f}_{f_0}(\mu) 6x\bar{x} \left[B_0(\mu) + \sum_{n=1}^{\infty} B_n(\mu) C_n^{3/2}(\xi) \right] \quad (25)$$

where $\bar{x} = (1-x)$, $\xi = (2x-1)$, and B_0, B_n are Gegenbauer moments, $C_n^{3/2}$ are the Gegenbauer polynomials. f_{f_0} is the $f_0(980)$ -meson decay constant and $\bar{f}_{f_0}$ indicates that the decay constant f_{f_0} varies with the energy scale. Because of the charge conjugation invariance, all even Gegenbauer moments of $\phi_{2;f_0}(x)$ vanish, so that $B_{2m} = 0$ for $m = (0, 1, \dots)$ in Eq. (25). As for the odd moments, we will adopt BFTSR-calculated results.

According to the $f_0(980)$ quark components, its twist-2 LCDA can be decomposed into two parts for calculation, namely

$$\phi_{2;f_0}(x, \mu) = \cos \theta \phi_{2;f_0}^{(s\bar{s})}(x, \mu) + \sin \theta \phi_{2;f_0}^{(n\bar{n})}(x, \mu). \quad (26)$$

Let us first introduce the calculation procedures for the ξ -moments of $\phi_{2;f_0}^{(s\bar{s})}(x, \mu)$ for the first part. The twist-2 LCDA $\phi_{2;f_0}^{(s\bar{s})}(x, \mu)$ and the twist-3 LCDA $\phi_{3;f_0}^{p(s\bar{s})}(x, \mu)$ for the scalar meson $f_0(980)$ made of the $s\bar{s}$ component are given by [40]

$$\begin{aligned} \langle 0 | \bar{s}(z) \gamma_\mu s(-z) | f_0(980) \rangle &= p_\mu \bar{f}_{f_0}^{(s\bar{s})} \int_0^1 dx e^{i\xi(p \cdot x)} \phi_{2;f_0}^{(s\bar{s})}(x, \mu) \\ \langle 0 | \bar{s}(z) s(-z) | f_0(980) \rangle &= m_{f_0} \bar{f}_{f_0}^{(s\bar{s})} \int_0^1 dx e^{i\xi(p \cdot x)} \phi_{3;f_0}^{p(s\bar{s})}(x, \mu) \end{aligned} \quad (27)$$

where $z^2 = 0$ and m_{f_0} is $f_0(980)$ -meson mass. Expanding both sides of Eq. (27) into z -series, one gets

$$\begin{aligned}
\langle 0 | \bar{s}(0) \not{z} (i\vec{z} \cdot \vec{D})^n s(0) | f_0(980) \rangle &= \bar{f}_{f_0}^{(s\bar{s})} (z \cdot p)^{n+1} \langle \xi_{2,f_0}^{(s\bar{s}),n} \rangle \\
\langle 0 | \bar{s}(0) s(0) | f_0(980) \rangle &= m_{f_0} \bar{f}_{f_0}^{(s\bar{s})} \langle \xi_{3,f_0}^{p(s\bar{s}),0} \rangle
\end{aligned} \quad (28)$$

The covariant derivative satisfies the relation $(i\vec{z} \cdot \vec{D})^n = (i\vec{z} \cdot \vec{D} - i\vec{z} \cdot \vec{D})^n$. Based on the basic procedure of QCD SR, one can adopt the following two-point correlator to derive the sum rules for $f_0(980)$ -meson twist-2 LCDA moments $\langle \xi_{2,f_0}^{(s\bar{s}),n} \rangle|_\mu$, which can be read off

$$\begin{aligned}
\Pi_{2,f_0}^{(s\bar{s})}(z, q) &= i \int d^4x e^{iq \cdot x} \langle 0 | T \{ J_n^{(s\bar{s})}(x), \hat{J}_0^{(s\bar{s}),\dagger}(0) \} | 0 \rangle \\
&= (z \cdot q)^{n+1} I_{2,f_0}^{(s\bar{s})}(q^2)
\end{aligned} \quad (29)$$

where $J_n^{(s\bar{s})}(x) = \bar{s}(x) \not{z} (i\vec{z} \cdot \vec{D})^n s(x)$ and $\hat{J}_0^{(s\bar{s}),\dagger}(0) = \bar{s}(0) s(0)$ are interpolating currents. $\langle \xi_{2,f_0}^{(s\bar{s}),n} \rangle|_\mu = \int_0^1 dx (2x - 1)^n \phi_{2,f_0}^{(s\bar{s})}(x, \mu)$ is the n_{th} ξ -moment. Because of the G-parity, only odd moments of $\langle \xi_{2,f_0}^{(s\bar{s}),n} \rangle|_\mu$ are nonzero, i.e., $n = (1, 3, \dots)$. Within the framework of BFTSR, the application of Feynman diagrammatic rules enables a systematic OPE treatment of the correlation function (29) in deep Euclidean region $q^2 \ll 0$. This expansion decomposes the correlation function into three distinct components: (1) the s -quark propagators $S_F^s(0, x)$ and $S_F^s(x, 0)$, representing quark propagation from space-time points 0 to x and x to 0, respectively; (2) vertex operators $(i\vec{z} \cdot \vec{D})^n$ are covariant derivative interactions; (3) a series of local gauge-invariant operators of increasing dimensionality. For the Lorentz invariant function $\Pi_{2,f_0}^{(s\bar{s})}(z, q)$, the OPE generates an asymptotic series organized by operator dimension, with the leading contributions arising from low-dimensional condensates while higher-dimensional terms are suppressed by powers of the renormalization scale. The expectation values of these operators in the nonperturbative (physical) vacuum are known as vacuum condensates, whose detailed expression can be found in Refs. [69,70]. By inserting a complete set of hadronic states into correlator (29) in physical region, whose hadronic representation can be read as

$$\begin{aligned}
\text{Im } I_{2,f_0}^{\text{had}}(s) &= \pi m_{f_0} \delta(s - m_{f_0}^2) \bar{f}_{f_0}^{(s\bar{s})2} \langle \xi_{2,f_0}^{(s\bar{s}),n} \rangle|_\mu \langle \xi_{3,f_0}^{p(s\bar{s}),0} \rangle|_\mu \\
&\quad + \text{Im } I_{2,f_0}^{\text{pert}}(s) \theta(s - s_{f_0})
\end{aligned} \quad (30)$$

where s_{f_0} is the continuum threshold, $\langle \xi_{3,f_0}^{p(s\bar{s}),0} \rangle|_\mu$ is the O_{th} ξ -moment of $f_0(980)$ two-particle twist-3 LCDA. Substituting the resulted OPE and hadronic representation of correlator Eq. (29), into the following dispersion relation after Borel transformation,

$$\frac{1}{\pi} \frac{1}{M^2} \int_{4m_s^2}^{s_{f_0}} ds e^{-s/M^2} \text{Im } I_{2,f_0}^{\text{had}}(s) = L_{M^2} I_{2,f_0}^{\text{QCD}}(q^2) \quad (31)$$

And the final sum rules for the moment of $f_0(980)$ -meson twist-2 LCDA is given by

$$\begin{aligned}
\frac{m_{f_0} \bar{f}_{f_0}^{2(s\bar{s})}}{M^2 e^{m_{f_0}^2/M^2}} \langle \xi_{3,f_0}^{p(s\bar{s}),0} \rangle \langle \xi_{2,f_0}^{(s\bar{s}),n} \rangle &= [1 - (-1)^n] \left\{ -\frac{1}{\pi M^2} \int_{4m_s^2}^{s_{f_0}} ds e^{-s/M^2} \frac{3m_s}{16\pi(n+1)(n+2)} \right. \\
&\quad \times [2n + 3 + (-1)^n] + \left(\frac{1}{M^2} + \frac{nm_s^2}{M^4} \right) \langle \bar{s}s \rangle + \left(-\frac{n}{2M^4} + \frac{5nm_s^2}{18M^6} \right) \langle g_s \bar{s} \sigma T G s \rangle \\
&\quad \left. + \frac{2m_s}{27M^6} \langle g_s \bar{s}s \rangle^2 + L_{M^2} I_{\langle G^2 \rangle} + L_{M^2} I_{\langle G^3 \rangle} + L_{M^2} I_{\langle q^4 \rangle} \right\},
\end{aligned} \quad (32)$$

$$\begin{aligned}
L_{M^2} I_{\langle G^2 \rangle} &= -m_s \frac{\langle \alpha_s G^2 \rangle}{48\pi M^4} \left\{ 4 \left[\ln \left(\frac{M^2}{\mu^2} \right) + 1 - \gamma_E \right] + 2 \frac{n+2}{n+1} - 4(\psi^{(0)}(n+1) + \gamma_E) \right. \\
&\quad + \theta(n-2) \left[(6n+4)\psi_3(n) + \frac{4n^3 + 10n^2 - 6n - 8}{n(n+1)} \right] + \theta(n-1) \left[-\frac{2}{n+1} [3n^2 \right. \\
&\quad \left. \left. + 5n + 3 + (n+1)\psi_2(n)] - 8(n+1) \left[\psi^{(0)}(n+1) + 2\gamma_E - \ln \left(\frac{M^2}{\mu^2} \right) - 1 \right] \right] \right\},
\end{aligned} \quad (33)$$

$$\begin{aligned}
L_{M^2 I_{\langle G^3 \rangle}} = m_s \frac{\langle g_s^3 f G^3 \rangle}{23040 \pi^2 M^6} & \left\{ (1584 - 7776n) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] + [72(108n - 22) \right. \\
& - 96(81n - 5)](\psi^{(0)}(n+1) + \gamma_E) - \frac{12(-1458n^2 - 1471n - 121)}{n+1} + \theta(n-2) \\
& \times \left[72(n-1) \frac{-51n^3 - 35n^2 + 80n + 52}{n(n+1)} + 144(9n+13)\psi_3(n) + 864n(n-1) \right. \\
& \times \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] + \theta(n-1) \left[288(5n+8) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] \right. \\
& - (1440n + 1824)(\psi^{(0)}(n+1) + \gamma_E) - \frac{12}{n+1} (431n^2 - 1281n - 1030) - 96\psi_2(n) \Big] \\
& + \theta(n-3) \left[\frac{54}{n(n^2-1)} (-77n^5 - 249n^4 + 67n^3 - 95n^2 - 294n - 72) - 216(20n^2 + 16n + 9)\psi_1(n) \right. \\
& \left. \left. - 648(n^2 + 5n + 6) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] + 648(n^2 + 5n + 6)(\psi^{(0)}(n+1) + \gamma_E) \right] \right\}, \quad (34)
\end{aligned}$$

$$\begin{aligned}
L_{M^2 I_{\langle q^4 \rangle}} = m_s \frac{(2+\kappa^2)\langle g_s^2 q \bar{q} \rangle^2}{233280 \pi^2 M^6} & \left\{ 648(66n-23) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] + 14904 \times (\psi^{(0)}(n+1) + \gamma_E) \right. \\
& + \frac{216(27n^2 + 40n + 82)}{n+1} + \theta(n-3) \left[3564(n^2 + 5n + 6) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] \right. \\
& + 1188(20n^2 + 16n + 9)\psi_1(n) - 3564(n^2 + 5n + 6)(\psi^{(0)}(n+1) + \gamma_E) \\
& + \frac{54}{n(n^2-1)} (581n^5 + 2355n^4 + 1013n^3 - 991n^2 + 606n + 396) \Big] \\
& + \theta(n-2) \left[-4752n(n-1) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] + 72(46n-45)\psi_3(n) - 108(n-1) \frac{-82n^3 - 35n^2 + 83n + 60}{n^2(n+1)} \right. \\
& + \theta(n-1) \left\{ 414\psi_2(n) - 648(3n-10)[(\psi^{(0)}(n+1) + \gamma_E)] + 648(3n-10) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] \right. \\
& \left. \left. - \frac{18}{n+1} (1555n^2 + 2283n + 368) \right\} \right\}. \quad (35)
\end{aligned}$$

In particular, the sum rule (32) is regarded as that for $\langle \xi_{3;f_0}^{p(s\bar{s}),0} \rangle \langle \xi_{2;f_0}^{(s\bar{s}),n} \rangle$ instead of $\langle \xi_{2;f_0}^{(s\bar{s}),n} \rangle$ in this work due to dependence of $\langle \xi_{3;f_0}^{p(s\bar{s}),n} \rangle$ on the Borel parameter as suggested. This assumption can be confirmed by the sum rule of $\langle \xi_{3;f_0}^{p(s\bar{s}),n} \rangle$ derived from the correlator $i \int d^4x e^{iq \cdot x} \langle 0 | \hat{J}_0^{(s\bar{s})}(0) \hat{J}_0^{(s\bar{s})\dagger}(0) | 0 \rangle$. Following the above sum rule calculation procedure performed for correlator (29), we then obtain

$$\begin{aligned}
\frac{m_{f_0}^2 \bar{f}_{f_0}^2(s\bar{s})}{M^2 e^{m_{f_0}^2/M^2}} \langle \xi_{3;f_0}^{p(s\bar{s}),0} \rangle^2 &= \frac{1}{\pi M^2} \int_{4m_s^2}^{s_{f_0}} ds e^{-s/M^2} \text{Im } I_{3;f_0}^{\text{pert}}(s) - \frac{3m_s}{M^2} \langle \bar{s}s \rangle + \frac{2}{27M^4} \left(4 + \frac{5m_s^2}{M^2} \right) \\
& \times \langle g_s \bar{s}s \rangle^2 + \frac{1}{\pi M^2} \left\{ \frac{1}{8} + \frac{m_s^2}{M^2} \left[\frac{1}{3} + \frac{1}{2} \left(\ln \left(\frac{M^2}{\mu^2} \right) + 1 - \gamma_E \right) \right] \right\} \langle \alpha_s G^2 \rangle + \frac{m_s^2}{\pi^2 M^6} \\
& \times \left[-\frac{97}{162} + \frac{4}{27} \left(\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right) \right] \langle g_s^3 f G^3 \rangle + \frac{(2+\kappa^2)}{M^4} \left\{ \frac{5}{54\pi^2} + \frac{m_s^2}{\pi^2 M^2} \right. \\
& \times \left[\frac{119}{729} - \frac{30}{253} \left(\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right) \right] \Big\} \langle g_s^2 \bar{q}q \rangle^2 + \frac{m_s}{M^4} \langle g_s \bar{s}\sigma T G s \rangle, \quad (36)
\end{aligned}$$

$$\text{Im } I_{3;f_0}^{\text{pert}}(s) = \frac{3s}{8\pi} \left\{ \left(1 - \frac{m_s^2}{s} \right) \left[\frac{s^2 - (s - m_s^2)^2}{2s^2} + 1 \right] \right\} + \frac{3m_s^2}{4\pi} \left(\frac{2(1-s)^2 v^2}{s^2} \right), \quad (37)$$

where $v^2 = 1 - \frac{4m_s^2}{s}$. As suggested in Ref. [70], a better sum rules for $\langle \xi_{2,f_0}^{(s\bar{s}),n} \rangle$ should be

$$\langle \xi_{2,f_0}^{(s\bar{s}),n} \rangle_\mu = \frac{(\langle \xi_{3,f_0}^{p(s\bar{s}),0} \rangle \langle \xi_{2,f_0}^{(s\bar{s}),n} \rangle)|_{\text{Form Eq.[32]}}}{\sqrt{\langle \xi_{3,f_0}^{p(s\bar{s}),0} \rangle^2}|_{\text{Form Eq.[36]}}} \quad (38)$$

Then we deal with the ξ -moments of $\phi_{2,f_0}^{(n\bar{n})}(x, \mu)$ for the second part. Its corresponding correlation function is

$$\Pi_{2,f_0}^{(n\bar{n})}(z, q) = i \int d^4x e^{iq \cdot x} \langle 0 | T \{ J_n^{(n\bar{n})}(x), \hat{J}_0^{(n\bar{n}),\dagger}(0) \} | 0 \rangle = (z \cdot q)^{n+1} I_{2,f_0}^{(n\bar{n})}(q^2) \quad (39)$$

with

$$J_n^{(n\bar{n})}(x) = \frac{1}{\sqrt{2}} [\bar{u}(x) \not{x} (i\vec{z} \cdot \vec{D})^n u(x) + \bar{d}(x) \not{x} (i\vec{z} \cdot \vec{D})^n d(x)] \hat{J}_0^{(n\bar{n}),\dagger}(0) = \frac{1}{\sqrt{2}} [\bar{u}(0)u(0) + \bar{d}(0)d(0)] \quad (40)$$

The detailed calculation process is consistent with the $s\bar{s}$ state, so there is no need to introduce it specifically here. The detailed calculation process can also refer to our previous work [60]. Since the current quark masses of u and d quarks are quite small, contributions from the u - and d -quark mass square terms can be safely neglected in the calculation. The final sum rules for the ξ -moments of $\phi_{2,f_0}^{(n\bar{n})}(x, \mu)$ becomes

$$\begin{aligned} \frac{m_{f_0} \bar{f}_{f_0}^{2(n\bar{n})} \langle \xi_{3,f_0}^{p(n\bar{n}),0} \rangle \langle \xi_{2,f_0}^{(n\bar{n}),n} \rangle}{M^2 e^{m_{f_0}^2/M^2}} &= \frac{[1 - (-1)^n]}{2} \left\{ \frac{1}{\pi M^2} \int_{(m_u+m_d)^2}^{s_{f_0}} ds e^{-s/M^2} \left[-\frac{3(m_u+m_d)}{16\pi(n+1)(n+2)} \right. \right. \\ &\times [2n + (-1)^n + 3] + \frac{(\langle \bar{u}u \rangle + \langle \bar{d}d \rangle)}{M^2} - \frac{n(\langle g_s \bar{u} \sigma T G u \rangle + \langle g_s \bar{d} \sigma T G d \rangle)}{2M^4} + \frac{2m_u \langle g_s \bar{u}u \rangle^2}{27M^6} \\ &\left. \left. + \frac{2m_d \langle g_s \bar{d}d \rangle^2}{27M^6} + L_{M^2} I_{\langle G^2 \rangle} + L_{M^2} I_{\langle G^3 \rangle} + L_{M^2} I_{\langle q^4 \rangle} \right\} \quad (41) \end{aligned}$$

$$\begin{aligned} L_{M^2} I_{\langle G^2 \rangle} &= -\frac{(m_u+m_d) \langle \alpha_s G^2 \rangle}{48\pi M^4} \left\{ 4 \left[\ln \left(\frac{M^2}{\mu^2} \right) + 1 - \gamma_E \right] + 2 \frac{n+2}{n+1} - 4(\psi^{(0)}(n+1) + \gamma_E) \right. \\ &+ \theta(n-2) \left[(6n+4)\psi_3(n) + \frac{4n^3+10n^2-6n-8}{n(n+1)} \right] + \theta(n-1) \left[-\frac{2}{n+1} [3n^2+5n \right. \\ &\left. \left. + 3 + (n+1)\psi_2(n)] - 8(n+1) \left[\psi^{(0)}(n+1) + 2\gamma_E - \ln \left(\frac{M^2}{\mu^2} \right) - 1 \right] \right] \right\} \quad (42) \end{aligned}$$

$$\begin{aligned} L_{M^2} I_{\langle G^3 \rangle} &= \frac{(m_u+m_d) \langle g_s^3 f G^3 \rangle}{23040\pi^2 M^6} \left\{ (1584 - 7776n) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] + [72(108n-22) \right. \\ &- 96(81n-5)](\psi^{(0)}(n+1) + \gamma_E) - \frac{12(-1458n^2-1471n-121)}{n+1} + \theta(n-2) \left[72 \right. \\ &\times (n-1) \frac{-51n^3-35n^2+80n+52}{n(n+1)} + 144(9n+13)\psi_3(n) + 864n(n-1) \left[\ln \left(\frac{M^2}{\mu^2} \right) \right. \\ &\left. \left. + \frac{3}{2} - \gamma_E \right] \right] + \theta(n-1) \left[288(5n+8) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] - (1440n+1824)(\psi^{(0)} \right. \\ &\times (n+1) + \gamma_E) - \frac{12}{n+1} (431n^2-1281n-1030) - 96\psi_2(n) \right] + \theta(n-3) \left[\frac{54}{n(n^2-1)} \right. \\ &\times (-77n^5-249n^4+67n^3-95n^2-294n-72) - 216(20n^2+16n+9)\psi_1(n) - 324 \\ &\left. \left. \times (n^2+5n+6) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] + 648(n^2+5n+6)(\psi^{(0)}(n+1) + \gamma_E) \right] \right\}, \quad (43) \end{aligned}$$

$$\begin{aligned}
L_{M^2 I_{\langle q^4 \rangle}} = & \frac{(m_u + m_d)(2 + \kappa^2) \langle g_s^2 q \bar{q} \rangle^2}{233280 \pi^2 M^6} \left\{ 648(66n - 23) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] + 14904 \right. \\
& \times (\psi^{(0)}(n+1) + \gamma_E) + \frac{216(27n^2 + 40n + 82)}{n+1} + \theta(n-3) \left[3564(n^2 + 5n + 6) \right. \\
& \times \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] + 1188(20n^2 + 16n + 9) \psi_1(n) - 3564(n^2 + 5n + 6) (\psi^{(0)} \\
& \times (n+1) + \gamma_E) + \frac{54}{n(n^2 - 1)} (581n^5 + 2355n^4 + 1013n^3 - 991n^2 + 606n + 396) \left. \right] \\
& + \theta(n-2) \left[-4752n(n-1) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] + 72(46n - 45) \psi_3(n) - 108 \right. \\
& \times (n-1) \frac{-82n^3 - 35n^2 + 83n + 60}{n^2(n+1)} \left. \right] + \theta(n-1) \left\{ 414 \psi_2(n) - 648(3n - 10) \right. \\
& \times [(\psi^{(0)}(n+1) + \gamma_E)] + 648(3n - 10) \left[\ln \left(\frac{M^2}{\mu^2} \right) + \frac{3}{2} - \gamma_E \right] - \frac{18}{n+1} (1555n^2 + 2283n + 368) \left. \right\} \left. \right\}. \quad (44)
\end{aligned}$$

The 0_{th} moment expression of the twist-3 LCDA is

$$\begin{aligned}
\frac{m_{f_0}^2 \tilde{f}_{f_0}^{2(n\bar{n})} \langle \xi_{3;f_0}^{p(n\bar{n}),0} \rangle^2}{M^2 e^{m_{f_0}^2/M^2}} = & \frac{1}{2} \left\{ \frac{1}{\pi M^2} \int_{(m_u+m_d)^2}^{s_{f_0}} \left(\frac{3s}{4\pi} \right) ds e^{-s/M^2} - \frac{3(\langle \bar{u}u \rangle m_u + \langle \bar{d}d \rangle m_d)}{M^2} + \frac{\langle \alpha_s G^2 \rangle}{4\pi M^2} \right. \\
& \left. + \frac{5(2 + \kappa^2) \langle g_s^2 \bar{q}q \rangle^2}{27\pi^2 M^4} + \frac{8(\langle g_s \bar{u}u \rangle^2 + \langle g_s \bar{d}d \rangle^2)}{27M^4} + \frac{(\langle g_s \bar{u}\sigma TGu \rangle m_u + \langle g_s \bar{d}\sigma TGd \rangle m_d)}{M^4} \right\} \quad (45)
\end{aligned}$$

And similarly, the sum rules of $\langle \xi_{2;f_0}^{(n\bar{n}),n} \rangle$ is

$$\langle \xi_{2;f_0}^{(n\bar{n}),n} \rangle_\mu = \frac{(\langle \xi_{3;f_0}^{p(n\bar{n}),0} \rangle \langle \xi_{2;f_0}^{(n\bar{n}),n} \rangle)|_{\text{Form Eq.[41]}}}{\sqrt{\langle \xi_{3;f_0}^{p(n\bar{n}),0} \rangle^2}|_{\text{Form Eq.[45]}}} \quad (46)$$

Therefore, the complete ξ -moment expression corresponding to the $f_0(980)$ -meson twist-2 LCDA is

$$\langle \xi_{2;f_0}^n \rangle_\mu = \cos \theta \langle \xi_{2;f_0}^{(s\bar{s}),n} \rangle_\mu + \sin \theta \langle \xi_{2;f_0}^{(n\bar{n}),n} \rangle_\mu. \quad (47)$$

The symmetry of the twist-3 LCDA for the $f_0(980)$ -meson ensures the normalization of its zeroth moment, thereby enabling determination of $f_0(980)$ -meson's decay constant by Eqs. (36) and (45). Analogous to the twist-2 LCDA analysis, this decay constant calculation requires separate calculation of two distinct components, i.e.,

$$\tilde{f}_{f_0} = \cos \theta \tilde{f}_{f_0}^{(s\bar{s})} + \sin \theta \tilde{f}_{f_0}^{(n\bar{n})}. \quad (48)$$

In Eqs. (32)–(45), the scale μ is the renormalization scale and M^2 is the Borel parameter. m_s , m_u , and m_d are current quark masses of s -, u -, and d -quark, respectively. $\langle \bar{s}s \rangle$, $\langle \bar{u}u \rangle$, $\langle \bar{d}d \rangle$, and $\langle g_s s \bar{s} \rangle^2$, $\langle g_s u \bar{u} \rangle^2$, $\langle g_s d \bar{d} \rangle^2$ are double-quark condensates, $\langle g_s \bar{s}\sigma TGs \rangle$, $\langle g_s \bar{u}\sigma TGu \rangle$, $\langle g_s \bar{d}\sigma TGd \rangle$ are quark-gluon mixed condensates, $\langle g_s^2 s \bar{s} \rangle^2$, $\langle g_s^2 u \bar{u} \rangle^2$, $\langle g_s^2 d \bar{d} \rangle^2$ are four-quark condensates, $\langle \alpha_s G^2 \rangle$ is double-gluon condensate, and $\langle g_s^3 fG^3 \rangle$ is triple-gluon condensate, respectively. The 0_{th} -other derivative of the digamma function is given by $\psi^{(0)}(n+1) = \sum_{k=1}^n 1/k - \gamma_E$ with the Euler constant $\gamma_E = 0.577216$. As for the digamma function $\psi(n)$, we have

$$\psi_1(n) = \psi\left(\frac{n}{2}\right) - \psi\left(\frac{n-1}{2}\right) - (-1)^n \ln 4, \quad (49)$$

$$\psi_2(n) = \psi\left(\frac{n-1}{2}\right) - \psi\left(\frac{n}{2}\right) + (-1)^n \ln 4, \quad (50)$$

$$\psi_3(n) = \psi\left(\frac{n+1}{2}\right) - \psi\left(\frac{n}{2}\right) - (-1)^n \ln 4. \quad (51)$$

There is a linear relationship between $\langle \xi_{2,f_0}^n \rangle$ and B_n [35], that is

$$B_1(\mu) = \frac{5}{3} \langle \xi_{2,f_0}^1 \rangle|_\mu, \quad (52)$$

$$B_3(\mu) = \frac{3}{4} (7 \langle \xi_{2,f_0}^3 \rangle|_\mu - 3 \langle \xi_{2,f_0}^1 \rangle|_\mu), \quad (53)$$

$$\dots \quad (54)$$

Combining Eqs. (25) and (47), the analytical expression of twists-2 LCDA can be obtained.

III. NUMERICAL ANALYSIS

A. Behavior of ξ -moments, f_{f_0} and ϕ_{2,f_0} for the $f_0(980)$ -meson

We are ready to derive the ξ -moments, the decay constant f_{f_0} and the twist-2 LCDA ϕ_{2,f_0} of $f_0(980)$.

In calculation, we take the mass of $f_0(980)$ as $m_{f_0} = 0.990 \pm 0.020$ GeV, the c current quark mass $m_c(\bar{m}_c) = 1.27 \pm 0.02$ GeV [5], the u, d , and s current quark mass are adopted as $m_u = 2.16^{+0.49}_{-0.26}$ MeV, $m_d = 4.67^{+0.48}_{-0.17}$ MeV and $m_s = 93^{+11}_{-5}$ MeV at scale $\mu = 2$ GeV [5], respectively. The values of the scale dependent vacuum condensates are [70–72]: $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = (-2.417^{+0.227}_{-0.114}) \times 10^{-2} \text{ GeV}^3$, $\langle \bar{s}s \rangle = \kappa \langle \bar{u}u \rangle$ with $\kappa = \langle \bar{s}s \rangle / \langle \bar{u}u \rangle = 0.74 \pm 0.03$, $\langle g_s \bar{u}u \rangle^2 = \langle g_s \bar{d}d \rangle^2 = (2.082^{+0.734}_{-0.679}) \times 10^{-3} \text{ GeV}^6$, $\langle g_s \bar{s}s \rangle^2 = \kappa^2 \langle g_s \bar{u}u \rangle^2$, $\langle g_s \bar{u}\sigma TGu \rangle = (-1.934^{+0.188}_{-0.103}) \times 10^{-2} \text{ GeV}^5$, $\langle g_s \sigma T G \bar{s} \rangle = \kappa \langle g_s \bar{u}\sigma TGu \rangle$, $\langle g_s^2 q\bar{q} \rangle^2 = (7.420^{+2.614}_{-2.483}) \times 10^{-3} \text{ GeV}^6$ at $\mu = 2$ GeV. the values of the scale independent gluon condensates are: $\langle \alpha_s G^2 \rangle = 0.038 \pm 0.011 \text{ GeV}^4$ and $\langle g_s^3 f G^3 \rangle = 0.045 \text{ GeV}^6$. When doing the numerical calculation, each vacuum condensates and current quark masses should be run from their initial values at an initial scale to the required scale by applying the renormalization group equations (RGEs) [70,73].

The continuum threshold parameter s_{f_0} and Borel parameter M^2 are two important parameters for determining the decay constant f_{f_0} . Usually, the continuum threshold s_{f_0} is taken as the value that is close to the squared mass of the first excited state of $f_0(980)$. As a conservative prediction, we set $s_{f_0} = 1.9 \pm 0.1 \text{ GeV}^2$ to do our discussion, whose central value is set by $f_0(1370)$ [5]. In addition, it is also necessary to require that $\langle \xi_{3,f_0}^n \rangle$ be normalized. To determine the allowable M^2 range, we require that f_{f_0} is flat within the allowable Borel window.

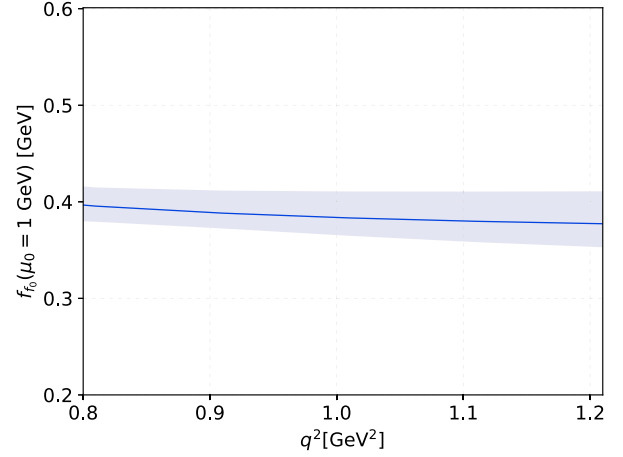


FIG. 2. The curve of the $f_0(980)$ decay constant f_{f_0} within the Borel parameter M^2 with $\mu_0 = 1$ GeV, where the shaded band indicated the uncertainties from the input parameters.

The $f_0(980)$ decay constant f_{f_0} versus the Borel parameter M^2 is shown in Fig. 2. Besides, the contribution of the continuum state does not exceed 30%, and the six-dimensional condensation does not exceed 1%. Therefore, we obtain the range of the Borel parameter as $M^2 \in [0.816, 1.150] \text{ GeV}^2$. Since f_{f_0} exhibits scale dependence [40], we present here only the results evaluated at the initial energy scale ($\mu_0 = 1$ GeV).

For comparison, theoretical predictions from two other approaches are also listed in Table I. Reference [40] employs a method based on ground state and excited state separation for its calculations. Furthermore, the associated correlation function they constructed is different from ours. Notably, the $f_0(980)$ -meson has been treated as a pure $s\bar{s}$ state in their frameworks. It can be found that our prediction result is close to that of QCD SR'06 [40] and there is a gap compared with the prediction results of 3PSR [41]. This further verifies the predicted results of the experiment, e.g., the $s\bar{s}$ component dominates, while the $n\bar{n}$ component cannot be ignored.

After determining the decay constant, we then calculate the moments of the twist-2 LCDA ϕ_{2,f_0} . Given the antisymmetric nature of the twist-2 LCDA, its continuum threshold must be calibrated using the zeroth moment of the twist-3 LCDA, maintaining consistency with the continuum threshold employed in the calculation of decay

TABLE I. The decay constant f_{f_0} (in unit: GeV) using the BFTSR approach. As a comparison, typical result derived from other approaches have also been presented.

	$f_{f_0}(\mu_0 = 1 \text{ GeV})$
This work	0.386 ± 0.009
QCD SR'06 [40]	0.370 ± 0.020
3PSR [41]	0.180 ± 0.015

constant. Through analysis of contributions from continuum states and six-dimensional condensates, constraints on the M^2 parameter are established: for the first two moments of the twist-2 LCDA, the continuum contributes are less than 30% and 35%, respectively, while the six-dimensional condensate contributes are less than 1% and 1%, respectively. This leads to the ranges of M^2 as $[1.046, 1.689]\text{GeV}^2$ (for the first-order moment) and $[1.204, 2.141]\text{GeV}^2$ (for the second-order moment). By taking all uncertainty sources into consideration and applying the RGEs of the moments, $\langle \xi_{2,f_0}^n \rangle|_\mu$ at the initial scale μ_0 and the special scale $\mu_k = \sqrt{m_{D_s^+}^2 - m_c^2} \approx 1.5 \text{ GeV}$ are

$$\langle \xi_{2,f_0}^1 \rangle|_{\mu_0} = -0.693_{-0.081}^{+0.095}, \quad (55)$$

$$\langle \xi_{2,f_0}^1 \rangle|_{\mu_k} = -0.542_{-0.063}^{+0.074}, \quad (56)$$

$$\langle \xi_{2,f_0}^3 \rangle|_{\mu_0} = -0.261_{-0.095}^{+0.097}, \quad (57)$$

$$\langle \xi_{2,f_0}^3 \rangle|_{\mu_k} = -0.217_{-0.063}^{+0.079}. \quad (58)$$

And the corresponding Gegenbauer moments are

$$B_1(\mu_0) = -1.156_{-0.133}^{+0.157}, \quad (59)$$

$$B_1(\mu_k) = -0.903_{-0.104}^{+0.123}, \quad (60)$$

$$B_3(\mu_0) = 0.191_{-0.317}^{+0.294}, \quad (61)$$

$$B_3(\mu_k) = 0.079_{-0.269}^{+0.252}. \quad (62)$$

Substituting the determined parameters into Eq. (25), the analytical form of the twist-2 LCDA can be obtained. To facilitate visual comparison, Fig. 3 displays the curves of twist-2 LCDA $\phi_{2,f_0}(x, \mu_0)$ for both $n = 1$ and $n = 3$ cases, along with the theoretical predictions obtained from QCD

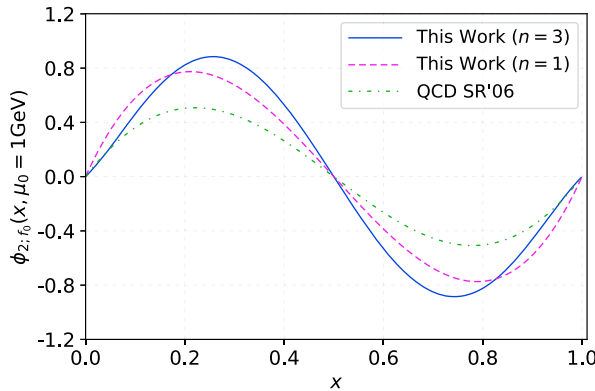


FIG. 3. The $f_0(980)$ twist-2 LCDA $\phi_{2,f_0}(x, \mu_0 = 1 \text{ GeV})$. As a comparison, the QCD SR'06 [40] predicted curve is also given.

SR'06 [40]. The first two Gegenbauer moments predicted by QCD SR'06 [40] are $b_1 = -0.78 \pm 0.08$ and $b_3 = 0.02 \pm 0.07$ respectively. Our predicted behavior of the twist-2 LCDA exhibits good consistency with the theoretical results from QCD SR'06, both of which exhibiting antisymmetric characteristics.

B. The TFF and branching fraction for the semileptonic $D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+ \pi^-) e^+ \nu_e$

In order to calculate the TFFs of the $D_s^+ \rightarrow f_0(980)$ decay in Eq. (21), we adopt the heavy meson mass $m_{D_s^+} = 1.9685 \text{ GeV}$ [5], the decay constants $f_{D_s^+} = 0.258 \pm 0.005 \text{ GeV}$ [74], and $f_{f_0}(\mu_k) = 0.440 \pm 0.010$ that is obtained by using the BFTSR. The parameter values needed when calculating the BF for the semileptonic $D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+ \pi^-) e^+ \nu_e$ are: $m_\pi \approx 0.140 \text{ GeV}$, $m_K \approx 0.494 \text{ GeV}$, $\tau_{D_s} = 0.5012 \text{ ps}$, $m_e = 0.511 \times 10^{-3} \text{ GeV}$, and $G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2}$. The above parameters are all from the PDG [5]. Besides, it is also necessary to know $g_1 = 165 \text{ MeV}/c^2$ and $g_2 = 4.21 g_1$ [54,75]. For the effective threshold parameter and Borel parameter, we take $s_0 = 6.5 \pm 0.5 \text{ GeV}^2$ and $M^2 = 7.0 \pm 1.0 \text{ GeV}^2$.

Figure 4 shows the curve of the TFF $f_+(0)$ as well as the contributions of twist-2,3 two-particle LCDAs. One can find that the twist-3 LCDAs ϕ_{3,f_0}^p and ϕ_{3,f_0}^s contribute significantly, whereas the contribution from the twist-2 LCDA ϕ_{2,f_0} is relatively minor. The latter arises from the antisymmetry of the $f_0(980)$ meson's twist-2 LCDA, which leads to a cancellation between its positive and negative components, thereby suppressing its overall contribution [76,77]. This situation is not contradictory to the OPE convergence. Unlike the traditional three-point QCD sum rules [78], the OPE convergence in LCSR is primarily reflected in the Fock components with multiplicities larger than two as well as the twist higher than 3 being so small that it can be ignored. Although the contribution of $f_0(980)$ twist-2 LCDA is small, based on the discussion behind

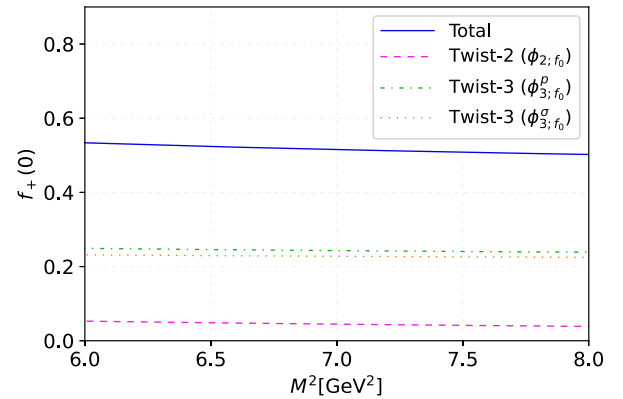


FIG. 4. LCSR predictions on the TFF $f_+(0)$ in the allowable M^2 range, which includes contributions from both twist-2 and twist-3 LCDAs.

TABLE II. Theoretical predictions on the TFF $f_+(0)$ at the large recoil point $q^2 = 0$.

	$f_+(0)$
This work (LCSR)	$0.516^{+0.027}_{-0.024}$
BESIII'24 [28]	$0.518 \pm 0.018 \pm 0.036$
LCSR'10 [29]	0.300 ± 0.030
3PSR'04 [32]	0.50 ± 0.13
3PSR'10 [33]	0.48 ± 0.23
LCSR'24-s1 [35]	0.580 ± 0.070
LCSR'24-s2 [35]	0.400 ± 0.060
LCSR'24-s3 [35]	$0.780^{+0.130}_{-0.100}$

Eqs. (11) and (12) and our previous research [60,61,79,80], our LCSR (21) for TFF $f_+(0)$ satisfies the OPE convergence.

With the above input and the $f_0(980)$ twist-2 LCDA determined in the previous subsection, the dependence of the TFFs for the semileptonic $D_s^+ \rightarrow f_0(980)$ decay on q^2 in the valid q^2 regions of the LCSR can be obtained. In particular, the values of these TFFs at the large recoil point are exhibited in Table II. As a comparison, we have also presented the results derived from various theoretical approaches and the recent BESIII result in Table II. Owing to substantial discrepancies among the predictions from various theoretical frameworks and experimental results, e.g., the LFQM calculation yielding $f_+(0) = 0.24 \pm 0.05$ [10], which deviates from experimental predictions by approximately 46% we primarily present the theoretical results derived from the LCSR and 3PSR methods in this table. It can be observed that our results are in close agreement with those of the BESIII Collaboration [28]. A recent LCSR study [35] shows that the predictions vary with different choices of current (s1, s2, s3). Compared with the s2 and s3 models, the prediction for s1 is closer to the BESIII result. This behavior is attributed to the fact that s1 uses the traditional current, which is consistent with the associated current we constructed.

Theoretically, the LCSR approach for $D_s^+ \rightarrow f_0(980)$ TFFs are reliable in low and intermediate q^2 -regions, which can be extrapolated to all the physically allowable region $m_\rho^2 \ll (m_{D_s^+} - m_{f_0})^2 \approx 0.96 \text{ GeV}^2$. In the context of analytic continuation, various technical approaches are available, including but not limited to series expansion (SE), simplified series expansion (SSE), single-pole (SP), and double-pole parametrization, etc. In the present paper, we adopt the approach of SSE to do the extrapolation. one of the advantages of this is the simplicity to translate the near-threshold behavior of the form factors into a useful constraint on the expansion coefficients. So the TFFs take the following form [81]

$$f_+(q^2) = \frac{1}{1 - q^2/m_{D_s^+}^2} \sum_{k=1}^2 \alpha_k z^k(t, t_0) \quad (63)$$

TABLE III. Fitting parameters α_1 and α_2 for the central TFF $f_+^{(C)}(q^2)$, the upper $f_+^{(U)}(q^2)$, and the lower $f_+^{(L)}(q^2)$. Δ is the measure of the quality of extrapolation.

	$f_+^{(C)}(q^2)$	$f_+^{(U)}(q^2)$	$f_+^{(L)}(q^2)$
α_1	-1.139	-1.646	-0.761
α_2	-65.777	-62.603	-72.413
Δ	0.090%	0.091%	0.097%

The function $z(t = q^2, t_0)$, which incorporates the parameters $t_\pm, t_0$ and t , is defined as

$$z(t, t_0) = \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}} \quad (64)$$

where $t_\pm = (m_{D_s^+} \pm m_{f_0})^2$ and $t_0 = t_+(1 - \sqrt{1 - t_-/t_+})$. In this approach, the simple pole $1 - q^2/m_{D_s^+}^2$ is used to account for the low-lying resonances, and $m_{D_s^+} = 2.3177 \text{ GeV}$ is D_s^+ meson resonances [5,35]. The free parameters α_1 and α_2 are fixed to make the Δ as small as possible, such as $\Delta < 1\%$, where Δ is used to measure the quality of extrapolation and is define as

$$\Delta = \frac{\sum_t |F_+(t) - F_+^{\text{fit}}(t)|}{\sum_t |F_+(t)|} \times 100 \quad (65)$$

where $t \in [0, \frac{1}{40}, \dots, \frac{40}{40}] \times 0.58 \text{ GeV}$. The fitting parameters α_1, α_2 for the TFF $f_+(q^2)$ and the quality-of-fit Δ in Table III. It shows that under those choices of parameters, all the Δ values are no more than 0.097%, indicating a good agreement of the extrapolated curves with the LCSR within the same q^2 -region of $q^2 \in [0, 0.58] \text{ GeV}^2$.

We present the TFF $f_+(q^2)$ across the entire momentum transfer q^2 range in Fig. 5, along with the latest experimental measurements for direct comparison. To facilitate a more comprehensive comparison with experimental results, we also include the results of analytic continuation using the SP parametrization [51,82], with the central value of $f_+(q^2)$ plotted in Fig. 5. While the SP parametrization yields predictions closer to experimental data than the SSE parametrization, it relies solely on endpoint values of the TFF and neglects information from the medium- and low-energy regions accessible within the LCSR framework. Consequently, we recommend the SSE parametrization for extrapolation. Across the entire q^2 range, our theoretical predictions represented by their central values are in agreement with the BESIII'24 fit data and lie within the uncertainty bands of the latest BESIII'24 results.

Then, we can calculate the differential decay width of the semileptonic decay $D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e$ by using Eq. (8). In calculation, we take CKM matrix element $|V_{cs}| = 0.974$ [28]. The resulted differential decay width for the semileptonic $D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e$ decay

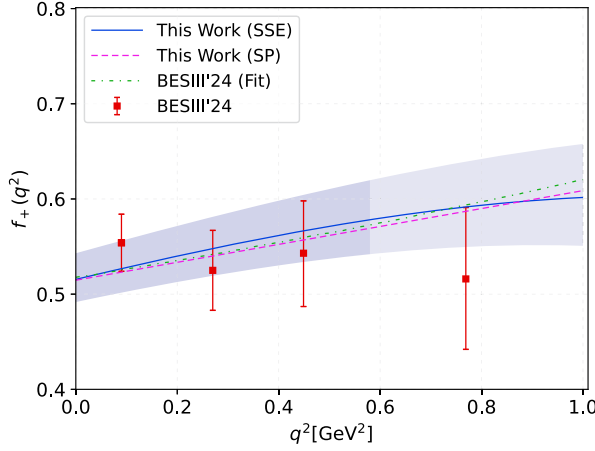


FIG. 5. The extrapolated TFF $f_+(q^2)$ in entire q^2 -region, where the solid line is the central value and the shaded band shows its uncertainty. The thicker shaded band shows the LCSR prediction. As a comparison, BESIII'24 and BESIII'24 fit are also presented.

versus q^2 is shown in Fig. 6, where the solid line is for the central value and the shaded band shows the corresponding error. As a comparison, BESIII'24 and BESIII'24 fit are also presented. Figure 6 shows that the central values and errors we calculated are all within the predicted error range of the latest BESIII'24 experiment. In different to BESIII'24 fit, our prediction tends to 0 when $q^2 \rightarrow (m_{D_s} - m_{f_0})^2 \approx 0.96 \text{ GeV}^2$, which is similar to most of the other semileptonic decay processes such as final state involving $a_0(980)$, $K_0^*(1430)$, π , and K , and etc. The primary reason for the discrepancy between our predicted differential decay width and the BESIII'24 fit results lies in the selection of $\pi^+\pi^-$ invariant mass (s) ranges. During the BESIII'24 fitting process, the variable $s \in [0.6, 1.6] \text{ GeV}/c^2$ [28], whereas our actual computational framework restricts this interval to $s \in [0.9, 0.96] \text{ GeV}/c^2$ [48,54].

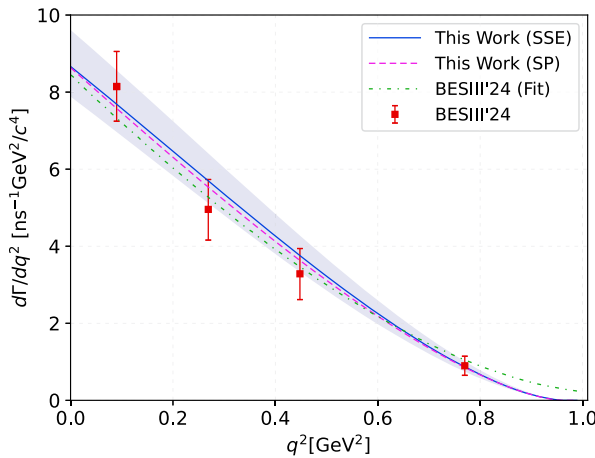


FIG. 6. Differential decay width of the semileptonic $D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e$ decay. As a comparison, BESIII'24 and BESIII'24 fit are also presented.

TABLE IV. The branching fraction of $D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e$, where the BESIII and CLEO results are given as a comparison.

	$\mathcal{B}(D_s \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e)$
This work (SSE)	$(1.783^{+0.227}_{-0.189}) \times 10^{-3}$
This work (SP)	$(1.743^{+0.190}_{-0.157}) \times 10^{-3}$
CLEO'09 [24]	$(1.3 \pm 0.4 \pm 0.1) \times 10^{-3}$
CLEO'09 [25]	$(2.0 \pm 0.3 \pm 0.1) \times 10^{-3}$
BESIII'24 [28]	$(1.72 \pm 0.13 \pm 0.10) \times 10^{-3}$

Finally, by using the lifetimes of D_s^+ -meson, we calculate the branching fraction of $D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e$ by using the extrapolated TFFs under the SSE and SP methods, respectively, which are listed in Table IV. Meanwhile, the BESIII and CLEO experimental predictions [24,25,28] are also given as a comparison. Our results are consistent with the latest experimental results within errors, with only a 3.7% shift of the central value for the SSE extrapolation, and a 1.3% shift of the central value for the SP extrapolation, accordingly.

IV. SUMMARY

In this work, we treat the $f_0(980)$ as a mixed state composed of distinct quark components, $|s\bar{s}\rangle \cos \theta + |n\bar{n}\rangle \sin \theta$, and employ the BFTSR method to compute its decay constant and twist-2 LCDA. The predicted decay constant, $f_{f_0}|_{1 \text{ GeV}} = 0.386 \pm 0.009 \text{ GeV}$, aligns closely to the QCD SR'06 result, which also validates the latest BESIII data. For the $f_0(980)$ -meson, its $s\bar{s}$ component dominates, while the $n\bar{n}$ component remains substantial and non-negligible. In Fig. 3, we present the twist-2 LCDAs for $n=1$ and $n=3$ at the initial scale of 1 GeV, both of which exhibit antisymmetry.

Subsequently, we calculate the TFF $f_+(0)$ for $D_s^+ \rightarrow f_0(980)$, observing that the endpoint results align precisely with the latest BESIII experimental predictions. By further extending the TFF analysis to the full momentum transfer q^2 range through SSE parametrization, we derive the differential decay width. The final predictions are consistent with the BESIII's latest theoretical prediction across the entire q^2 region. Since experimental results are presented using the SP parametrization, we also include the SP predictions here. As observed, the SP parametrization predicted TFF and BF values closer to experimental measurements. However, due to the inherent limitations of the SP approach, we recommend prioritizing the SSE parametrization results for theoretical analysis.

Finally, based on the width formula given in the latest BESIII experiment, which includes the Flatté factor, we calculated the differential decay width of $D_s^+ \rightarrow f_0(980)(\rightarrow \pi^+\pi^-)e^+\nu_e$ (Fig. 6) and the branching fraction (Table IV). It is found that there is a gap between our final prediction result and the experiment, mainly because the

s -range is different. However, our predicted value is within the error range of the experiment. Among them, the central value of the branching fraction predicted by the SSE parametrization has an error of only 3.7% compared with the BESIII 24's predictions. It is hoped that these results can provide important insights into the intrinsic properties of the light scalar $f_0(980)$ -meson and the non-perturbation mechanism controlling the decay dynamics of the charm meson.

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DATA AVAILABILITY

The data are available from the authors upon reasonable request.

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- [1] D. Black, A. H. Fariborz, F. Sannino, and J. Schechter, Putative light scalar nonet, *Phys. Rev. D* **59**, 074026 (1999).
 - [2] N. N. Achasov and A. V. Kiselev, Light scalars in semileptonic decays of heavy quarkonia, *Phys. Rev. D* **86**, 114010 (2012).
 - [3] T. Sekihara and E. Oset, Investigating the nature of light scalar mesons with semileptonic decays of D mesons, *Phys. Rev. D* **92**, 054038 (2015).
 - [4] N. R. Soni, A. N. Gadaria, J. J. Patel, and J. N. Pandya, Semileptonic decays of charmed mesons to light scalar mesons, *Phys. Rev. D* **102**, 016013 (2020).
 - [5] S. Navas *et al.* (Particle Data Group), Review of particle physics, *Phys. Rev. D* **110**, 030001 (2024).
 - [6] N. A. Tornqvist, Understanding the scalar meson $q\bar{q}$ nonet, *Z. Phys. C* **68**, 647 (1995).
 - [7] E. van Beveren and G. Rupp, Comment on 'Understanding the scalar meson $q\bar{q}$ nonet', *Eur. Phys. J. C* **10**, 469 (1999).
 - [8] D. Morgan and M. R. Pennington, New data on the $K\bar{K}$ threshold region and the nature of the $f_0(S^*)$, *Phys. Rev. D* **48**, 1185 (1993).
 - [9] N. A. Tornqvist and M. Roos, Resurrection of the sigma meson, *Phys. Rev. Lett.* **76**, 1575 (1996).
 - [10] H. W. Ke, X. Q. Li, and Z. T. Wei, Whether new data on $D_s \rightarrow f_0(980)e^+\nu_e$ can be understood if $f_0(980)$ consists of only the conventional $q\bar{q}$ structure, *Phys. Rev. D* **80**, 074030 (2009).
 - [11] R. L. Jaffe, Multi-quark hadrons. 1. The phenomenology of $Q^2\bar{Q}^2$ mesons, *Phys. Rev. D* **15**, 267 (1977).
 - [12] M. G. Alford and R. L. Jaffe, Insight into the scalar mesons from a lattice calculation, *Nucl. Phys. B* **578**, 367 (2000).
 - [13] A. H. Fariborz, R. Jora, and J. Schechter, Global aspects of the scalar meson puzzle, *Phys. Rev. D* **79**, 074014 (2009).
 - [14] T. V. Brito, F. S. Navarra, M. Nielsen, and M. E. Bracco, QCD sum rule approach for the light scalar mesons as four-quark states, *Phys. Lett. B* **608**, 69 (2005).
 - [15] N. N. Achasov and A. V. Kiselev, The analytical $\pi\pi$ scattering amplitude and the light scalars, *Phys. Rev. D* **83**, 054008 (2011).
 - [16] H. Kim, K. S. Kim, M. K. Cheoun, and M. Oka, Tetraquark mixing framework for isoscalar resonances in light mesons, *Phys. Rev. D* **97**, 094005 (2018).
 - [17] V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, and A. E. Kudryavtsev, Evidence that the $a_0(980)$ and $f_0(980)$ are not elementary particles, *Phys. Lett. B* **586**, 53 (2004).
 - [18] T. Branz, T. Gutsche, and V. E. Lyubovitskij, $f_0(980)$ -meson as a $K\bar{K}$ molecule in a phenomenological Lagrangian approach, *Eur. Phys. J. A* **37**, 303 (2008).
 - [19] H. J. Lee, N. I. Kochelev, and Y. Oh, QCD sum rule study on the $f_0(980)$ structure as a pure $K\bar{K}$ bound state, *Phys. Rev. D* **87**, 117901 (2013).
 - [20] D. Robson, A basic guide for the glueball spotter, *Nucl. Phys. B* **130**, 328 (1977).
 - [21] S. Narison, Masses, decays and mixings of gluonia in QCD, *Nucl. Phys. B* **509**, 312 (1998).
 - [22] P. Lebiedowicz, R. Maciula, and A. Szczurek, Production of $f_0(980)$ meson at the LHC: Color evaporation versus color-singlet gluon-gluon fusion, *Phys. Lett. B* **806**, 135475 (2020).
 - [23] B. Aubert *et al.* (BABAR Collaboration), Study of the decay $D_s^+ \rightarrow K^+ K^- e^+ \nu_e$, *Phys. Rev. D* **78**, 051101 (2008).
 - [24] J. Yelton *et al.* (CLEO Collaboration), Absolute branching fraction measurements for exclusive $D_{(s)}$ semileptonic decays, *Phys. Rev. D* **80**, 052007 (2009).
 - [25] K. M. Ecklund *et al.* (CLEO Collaboration), Study of the semileptonic decay $D_s^+ \rightarrow f_0(980)e^+\nu_e$ and implications for $B_s^0 \rightarrow J/\psi f_0$, *Phys. Rev. D* **80**, 052009 (2009).
 - [26] M. Ablikim *et al.* (BESIII Collaboration), Observation of $D^+ \rightarrow f_0(500)e^+\nu_e$ and improved measurements of $D \rightarrow \rho e^+\nu_e$, *Phys. Rev. Lett.* **122**, 062001 (2019).
 - [27] M. Ablikim *et al.* (BESIII Collaboration), Study of light scalar mesons through $D_s^+ \rightarrow \pi^0 \pi^0 e^+ \nu_e$ and $K_S^0 K_S^0 e^+ \nu_e$ decays, *Phys. Rev. D* **105**, L031101 (2022).
 - [28] M. Ablikim *et al.* (BESIII Collaboration), Study of the $f_0(980)$ and $f_0(500)$ scalar mesons through the decay $D_s^+ \rightarrow \pi^+ \pi^- e^+ \nu_e$, *Phys. Rev. Lett.* **132**, 141901 (2024).

- [29] P. Colangelo, F. De Fazio, and W. Wang, $B_s \rightarrow f_0(980)$ form factors and B_s decays into $f_0(980)$, *Phys. Rev. D* **81**, 074001 (2010).
- [30] Y.J. Shi and W. Wang, Chiral dynamics and S-wave contributions in semileptonic D_s/B_s decays into $\pi^+\pi^-$, *Phys. Rev. D* **92**, 074038 (2015).
- [31] B. El-Bennich, O. Leitner, J. P. Dedonder, and B. Loiseau, The scalar meson $f_0(980)$ in heavy-meson decays, *Phys. Rev. D* **79**, 076004 (2009).
- [32] I. Bediaga, F. S. Navarra, and M. Nielsen, The structure of $f_0(980)$ from charmed mesons decays, *Phys. Lett. B* **579**, 59 (2004).
- [33] T. M. Aliev and M. Savci, Semileptonic decays of pseudo-scalar mesons to scalar f_0 meson, *Europhys. Lett.* **90**, 61001 (2010).
- [34] I. Bediaga and M. Nielsen, D_s decays into ϕ and $f_0(980)$ -mesons, *Phys. Rev. D* **68**, 036001 (2003).
- [35] S. Cheng and S. L. Zhang, $D_s \rightarrow f_0(980)$ form factors and the $D_s^+ \rightarrow (f_0(980) \rightarrow [\pi\pi]_S^{I=0}) e^+ \nu_e$ decay from light-cone sum rules, *Eur. Phys. J. C* **84**, 379 (2024).
- [36] V. M. Braun and I. E. Filyanov, QCD sum rules in exclusive kinematics and pion wave function, *Z. Phys. C* **44**, 157 (1989).
- [37] I. I. Balitsky, V. M. Braun, and A. V. Kolesnichenko, Radiative decay $\Sigma^+ \rightarrow p\gamma$ in quantum chromodynamics, *Nucl. Phys. B* **312**, 509 (1989).
- [38] V. L. Chernyak and I. R. Zhitnitsky, B meson exclusive decays into baryons, *Nucl. Phys. B* **345**, 137 (1990).
- [39] P. Ball, V. M. Braun, and H. G. Dosch, Form-factors of semi-leptonic D decays from QCD sum rules, *Phys. Rev. D* **44**, 3567 (1991).
- [40] H. Y. Cheng, C. K. Chua, and K. C. Yang, Charmless hadronic B decays involving scalar mesons: Implications to the nature of light scalar mesons, *Phys. Rev. D* **73**, 014017 (2006).
- [41] F. De Fazio and M. R. Pennington, Probing the structure of $f_0(980)$ through radiative ϕ decays, *Phys. Lett. B* **521**, 15 (2001).
- [42] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, QCD and resonance physics. Theoretical foundations, *Nucl. Phys. B* **147**, 385 (1979).
- [43] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, QCD and resonance physics: Applications *Nucl. Phys. B* **147**, 448 (1979).
- [44] J. Govaerts, F. de Viron, D. Gusbin, and J. Weyers, Exotic mesons from QCD sum rules, *Phys. Lett. B* **128B**, 262 (1983).
- [45] J. Govaerts, F. de Viron, D. Gusbin, and J. Weyers, QCD sum rules and hybrid mesons, *Nucl. Phys. B* **248**, 1 (1984).
- [46] T. Huang and Z. Huang, Quantum chromodynamics in background fields, *Phys. Rev. D* **39**, 1213 (1989).
- [47] T. Zhong, X. G. Wu, Z. G. Wang, T. Huang, H. B. Fu, and H. Y. Han, Revisiting the pion leading-twist distribution amplitude within the QCD background field theory, *Phys. Rev. D* **90**, 016004 (2014).
- [48] W. Wang, Search for the $a_0(980) - f_0(980)$ mixing in weak decays of D_s/B_s mesons, *Phys. Lett. B* **759**, 501 (2016).
- [49] S. M. Flatte, On the nature of 0^+ mesons, *Phys. Lett. B* **63**, 228 (1976).
- [50] R. Aaij *et al.* (LHCb Collaboration), Measurement of resonant and CP components in $\bar{B}_s^0 \rightarrow J/\psi \pi^+ \pi^-$ decays, *Phys. Rev. D* **89**, 092006 (2014).
- [51] N. N. Achasov, A. V. Kiselev, and G. N. Shestakov, Semileptonic decays $D \rightarrow \pi^+ \pi^- e^+ \nu_e$ and $D_s \rightarrow \pi^+ \pi^- e^+ \nu_e$ as the probe of constituent quark-antiquark pairs in the light scalar mesons, *Phys. Rev. D* **102**, 016022 (2020).
- [52] Y. J. Sun, Z. H. Li, and T. Huang, $B_{(s)} \rightarrow S$ transitions in the light cone sum rules with the chiral current, *Phys. Rev. D* **83**, 025024 (2011).
- [53] M. Artuso, B. Meadows, and A. A. Petrov, Charm meson decays, *Annu. Rev. Nucl. Part. Sci.* **58**, 249 (2008).
- [54] M. Ablikim *et al.* (BES Collaboration), Resonances in $J/\Psi \rightarrow \phi \pi^+ \pi^-$ and $\phi K^+ K^-$, *Phys. Lett. B* **607**, 243 (2005).
- [55] G. Duplancic, A. Khodjamirian, T. Mannel, B. Melic, and N. Offen, Light-cone sum rules for $B \rightarrow \pi$ form factors revisited, *J. High Energy Phys.* **04** (2008) 014.
- [56] H. Y. Han, X. G. Wu, H. B. Fu, Q. L. Zhang, and T. Zhong, Twist-3 distribution amplitudes of scalar mesons within the QCD sum rules and its application to the $B \rightarrow S$ transition form factors, *Eur. Phys. J. A* **49**, 78 (2013).
- [57] G. Duplancic and B. Melic, $B, B_s \rightarrow K$ form factors: An update of light-cone sum rule results, *Phys. Rev. D* **78**, 054015 (2008).
- [58] P. Ball and G. W. Jones, $B \rightarrow \eta^{(\prime)}$ form factors in QCD, *J. High Energy Phys.* **08** (2007) 025.
- [59] P. Ball and R. Zwicky, $B_{d,s} \rightarrow \rho, \omega, K^*, \phi$ decay form-factors from light-cone sum rules revisited, *Phys. Rev. D* **71**, 014029 (2005).
- [60] D. D. Hu, X. G. Wu, H. B. Fu, T. Zhong, Z. H. Wu, and L. Zeng, Properties of the η_q leading-twist distribution amplitude and its effects to the $B/D^+ \rightarrow \eta^{(\prime)} e^+ \nu_e$ decays, *Eur. Phys. J. C* **84**, 15 (2024).
- [61] H. B. Fu, L. Zeng, R. Lü, W. Cheng, and X. G. Wu, The $D \rightarrow \rho$ semileptonic and radiative decays within the light-cone sum rules, *Eur. Phys. J. C* **80**, 194 (2020).
- [62] V. M. Belyaev, A. Khodjamirian, and R. Ruckl, QCD calculation of the $B \rightarrow \pi, K$ form-factors, *Z. Phys. C* **60**, 349 (1993).
- [63] L. Maiani, F. Piccinini, A. D. Polosa, and V. Riquer, A new look at scalar mesons, *Phys. Rev. Lett.* **93**, 212002 (2004).
- [64] H. Kim, K. S. Kim, M. K. Cheoun, D. Jido, and M. Oka, Further signatures to support the tetraquark mixing framework for the two light-meson nonets, *Phys. Rev. D* **99**, 014005 (2019).
- [65] R. D. Matheus, F. S. Navarra, M. Nielsen, and R. Rodrigues da Silva, Do the QCD sum rules support four-quark states?, *Phys. Rev. D* **76**, 056005 (2007).
- [66] H. Y. Cheng, Hadronic D decays involving scalar mesons, *Phys. Rev. D* **67**, 034024 (2003).
- [67] A. Gokalp, Y. Sarac, and O. Yilmaz, An analysis of $f_0 - \sigma$ mixing in light cone QCD sum rules, *Phys. Lett. B* **609**, 291 (2005).
- [68] V. M. Braun, G. P. Korchemsky, and D. Müller, The uses of conformal symmetry in QCD, *Prog. Part. Nucl. Phys.* **51**, 311 (2003).

- [69] D. D. Hu, H. B. Fu, T. Zhong, L. Zeng, W. Cheng, and X. G. Wu, $\eta^{(\prime)}$ -meson twist-2 distribution amplitude within QCD sum rule approach and its application to the semi-leptonic decay $D_s^+ \rightarrow \eta^{(\prime)} \ell^+ \nu_\ell$, *Eur. Phys. J. C* **82**, 12 (2022).
- [70] T. Zhong, Z. H. Zhu, H. B. Fu, X. G. Wu, and T. Huang, Improved light-cone harmonic oscillator model for the pion leading-twist distribution amplitude, *Phys. Rev. D* **104**, 016021 (2021).
- [71] D. Huang, T. Zhong, H. B. Fu, Z. H. Wu, X. G. Wu, and H. Tong, $K_0^*(1430)$ twist-2 distribution amplitude and $B_s, D_s \rightarrow K_0^*(1430)$ transition form factors, *Eur. Phys. J. C* **83**, 680 (2023).
- [72] Z. H. Wu, H. B. Fu, T. Zhong, D. Huang, D. D. Hu, and X. G. Wu, $a_0(980)$ -meson twist-2 distribution amplitude within the QCD sum rules and investigation of $D \rightarrow a_0(980)(\rightarrow \eta\pi)e^+\nu_e$, *Nucl. Phys. A* **1036**, 122671 (2023).
- [73] K. C. Yang, W. Y. P. Hwang, E. M. Henley, and L. S. Kisslinger, QCD sum rules and neutron proton mass difference, *Phys. Rev. D* **47**, 3001 (1993).
- [74] G. Duplancic and B. Melic, Form factors of $B, B_s \rightarrow \eta^{(\prime)}$ and $D, D_s \rightarrow \eta^{(\prime)}$ transitions from QCD light-cone sum rules, *J. High Energy Phys.* **11** (2015) 138.
- [75] M. Ablikim *et al.* (BESIII Collaboration), Amplitude analysis of the $\chi_{c1} \rightarrow \eta\pi^+\pi^-$ decays, *Phys. Rev. D* **95**, 032002 (2017).
- [76] S. Cheng and J. M. Shen, $\bar{B}_s \rightarrow f_0(980)$ form factors and the width effect from light-cone sum rules, *Eur. Phys. J. C* **80**, 554 (2020).
- [77] L. Zeng, X. G. Wu, D. D. Hu, H. B. Fu, and T. Zhong, Longitudinal leading-twist distribution amplitude of the 1_1^P -state $b_1(1235)$ meson and its implications on $B \rightarrow b_1(1235)\ell^+\nu_\ell$ decays, *Phys. Rev. D* **111**, 056030 (2025).
- [78] H. G. Dosch, E. M. Ferreira, F. S. Navarra, and M. Nielsen, Semileptonic D decay into scalar mesons: A QCD sum rule approach, *Phys. Rev. D* **65**, 114002 (2002).
- [79] D. D. Hu, H. B. Fu, T. Zhong, Z. H. Wu, and X. G. Wu, $a_1(1260)$ -meson longitudinal twist-2 distribution amplitude and the $D \rightarrow a_1(1260)\ell^+\nu_\ell$ decay processes, *Eur. Phys. J. C* **82**, 603 (2022).
- [80] Y. X. Wang, D. D. Hu, W. B. Luo, T. Zhong, and H. B. Fu, Status of $D_s^+ \rightarrow \phi\ell^+\nu_\ell$ decay with chiral-odd ϕ -meson LCDA, *arXiv:2505.15014*.
- [81] A. Bharucha, T. Feldmann, and M. Wick, Theoretical and phenomenological constraints on form factors for radiative and semi-leptonic B-meson decays, *J. High Energy Phys.* **09** (2010) 090.
- [82] D. Becirevic and A. B. Kaidalov, Comment on the heavy $\rightarrow$ light form-factors, *Phys. Lett. B* **478**, 417 (2000).