

Semiclassical spectrum of the Ising CFT

Oleg Antipin^{1,*}, Jahmall Bersini^{2,†}, Jacob Hafjall^{3,‡}, Giulia Muco^{3,§} and Francesco Sannino^{3,4,||}

¹*Rudjer Boskovic Institute, Division of Theoretical Physics, Bijenička 54, 10000 Zagreb, Croatia*

²*Albert Einstein Center for Fundamental Physics, Bern University,
Sidlerstrasse 5, CH-3012 Bern, Switzerland*

³*Quantum Theory Center (hQTC) at IMADA & D-IAS, Southern Denmark University, Campusvej 55,
5230 Odense M, Denmark*

⁴*Department of Physics Ettore Pancini, Università di Napoli Federico II, via Cintia, 80126 Napoli, Italy*



(Received 18 November 2025; accepted 20 July 2026; published 2 September 2026)

We develop a semiclassical framework to determine scaling dimensions of neutral composite operators in scalar conformal field theories. For the critical Ising $\lambda\phi^4$ theory in $d = 4 - \epsilon$, we obtain the full spectrum of composite operators built out of n fields transforming according to the various Lorentz representations to next-to-leading order in the double-scaling limit $n \rightarrow \infty$ and $\lambda \rightarrow 0$ with λn fixed. At any given order, the semiclassical expansion resums an infinite number of Feynman diagrams. Combining our results with existing perturbative computations further yields the complete five-loop scaling dimensions in the ϵ -expansion for the family of ϕ^n operators. Finally, extrapolating the semiclassical results to three dimensions leads to a competitive alternative to existing methodologies.

DOI: [10.1103/43ck-2h6m](https://doi.org/10.1103/43ck-2h6m)

I. INTRODUCTION

Conformal field theories (CFT)s play a major role in describing critical phenomena in nature, from condensed matter to fundamental interactions. Scaling dimensions are relevant CFT data associated with the two-point function of the primary operators of the theory. In the past decades a number of methodologies have been developed to tackle the important dynamics related to composite operators. These range from the use of larger symmetries, such as supersymmetry, to large quantum numbers [1,2] expansions, bootstrap [3], as well as numerical approaches. These approaches, either perturbative or nonperturbative, fall short of determining the conformal dimensions of neutral composite operators built out of a large number n of fields. To better appreciate the challenge, even in a weakly-coupled field theory with a single coupling $\lambda \ll 1$, the perturbative expansion for the scaling dimension of these operators is controlled by λn and therefore it breaks down for λn of order unity [4]. Moreover, operator mixing under

renormalization group (RG) flow leads to an anomalous dimension matrix whose size grows fast with n , making it hard to determine the physical spectrum (scaling dimensions) of primary operators. Beyond CFTs, determining the RG functions associated with higher-dimension operators is an important problem in the renormalization of fundamental and effective theories. This is, for example, particularly relevant for the Standard Model effective field theory [5–7], where higher-dimension operators are included to parametrize new physics effects at colliders. Perturbation theory also fails when studying multiboson production processes, e.g., $1 \rightarrow n$, $2 \rightarrow n$, in generic weakly-coupled quantum field theories (QFT)s [8–10]. This problem also afflicts Standard Model processes involving a large number of W, Z, and Higgs bosons [11]. These issues can be traced back to the actual computation of multilegged amplitudes, where the large number of loops and legs leads to an exponential proliferation of Feynman diagrams, severely hampering the efforts in solving for the dynamics of QFTs.

The overarching goal of this paper is twofold. First, to introduce a semiclassical methodology apt at determining the scaling dimensions of composite operators beyond perturbation theory and without the use of Feynman diagrams, consolidating and going beyond the work in [12]. Specifically, while [12] calculated the leading semiclassical order, we here determine the leading quantum correction. Second, we will determine, to next-to-leading order in the semiclassical expansion, the full spectrum of operators in the critical (Ising) $\lambda\phi^4$ theory in $d = 4 - \epsilon$ dimensions,

*Contact author: oantipin@irb.hr

†Contact author: jahmall.bersini@unibe.ch

‡Contact author: jahaf21@student.sdu.dk

§Contact author: giulia@qtc.sdu.dk

||Contact author: sannino@qtc.sdu.dk

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4, \quad (1)$$

where the Wilson-Fisher fixed point coupling occurs at

$$\lambda_* = \frac{8\pi^2\epsilon}{9} + \frac{136\pi^2\epsilon^2}{243} + \mathcal{O}(\epsilon^3). \quad (2)$$

As we shall see, when extrapolated to $\epsilon = 1$, our results exhibit reasonable agreement with the conformal bootstrap for the low-lying spectrum of the three-dimensional Ising CFT while providing for the first time a controlled framework for the higher-dimensional operators. Remarkably, while at the leading semiclassical order, the full spectrum of the theory is degenerate [12], the inclusion of the next-to-leading order (NLO) lifts this degeneracy. This allows us to distinguish operators belonging to different Lorentz representations, unveiling the richness of the spectrum and its previously observed surprising degeneracies [13,14]. At the same time, the NLO calculation unveils the limitations of the approach as the relevant classical trajectory develops instabilities in the strongly coupled regime.

In the free field theory limit, the composite operators can be generically denoted as $\partial^s \square^p \phi^n$. With this notation, we intend the appropriate linear combination of terms with n fields, $2p$ fully contracted and s uncontracted derivatives, such that the resulting operator is a conformal primary. There can be different primaries for the same values of s , p , and n . Operators transforming according to nonfully symmetric representations of the Lorentz group will be further specified via a subscript in the partial derivative symbol. In the interacting theory, these operators undergo mixing under renormalization, and the primary operators become certain linear combinations of free field theory primaries that diagonalize the anomalous dimension matrix. For simplicity, we here schematically denote the interacting primaries by those which they reduce to in the corresponding $\epsilon \rightarrow 0$ free field theory limit. Via the state-operators correspondence, conformal dimensions can be mapped into the energy spectrum of the corresponding states on the cylinder $\mathbb{R} \times S^{d-1}$. Building on the initial intuition put forward in [12,15], we here demonstrate that the full spectrum of scalar composite operators maps into the energy spectrum $E_{n,\{q_\ell\}}$ of the states obtained by quantizing the periodic homogeneous solutions of the classical equation of motions on the cylinder. The energy spectrum is computed via a semiclassical expansion in the limit $n \rightarrow \infty$, $\lambda \rightarrow 0$, and λn fixed, leading to the desired scaling dimensions,

$$\Delta_{n,\{q_\ell\}} = r E_{n,\{q_\ell\}} = n \sum_{i=0} \frac{C_i(\lambda n, \{q_\ell\})}{n^i}, \quad (3)$$

where r is the radius of S^{d-1} and $\{q_\ell\}$ is an infinite set of integers specifying the states and the corresponding composite operators $\partial^s \square^p \phi^n$.

The leading coefficient C_0 has been computed in [12] and reads

$$nC_0 = \frac{2\pi^2 m(1-m)}{\lambda(1-2m)^2}, \quad (4)$$

where m is related to λn by inverting the Bohr-Sommerfeld quantization condition $I = 2\pi n$ where the action variable I reads

$$I = \frac{16\pi^2}{3\lambda(1-2m)^{3/2}} [(2m-1)\mathbb{E}(m) + (1-m)\mathbb{K}(m)],$$

with $\mathbb{K}(m)$ and $\mathbb{E}(m)$ the complete elliptic integrals of the first and second kind. The NLO contribution can be determined using the framework developed in [16–19] for the quantization of periodic solutions in QFT. Following the derivation presented in Appendix A, which includes Refs. [20–25], we arrive at

$$C_1 = \frac{5\pi\sqrt{1-2m} - 24\mathbb{E}(m)}{4(1-2m)\mathbb{K}(m)} + \frac{28 - 67m + 19m^2}{8(1-2m)^2} + \frac{1}{8\sqrt{1-2m}\mathbb{K}(m)} \left(\sum_{\ell=2}^{\infty} \sigma(\ell) + 2 \sum_{\ell=1}^{\infty} q_\ell \nu_\ell \right), \quad (5)$$

where

$$\begin{aligned} \sigma(\ell) = & (\ell+1)^2 \nu_\ell + \frac{2}{\ell(1-2m)^{3/2}} (6(\ell^2 + \ell + 1) \\ & \times (2m-1)\mathbb{E}(m) + (6+4\ell-2(\ell+3)\ell^3 \\ & + (2\ell(\ell+1)(2\ell-1)(2\ell+5) - 15)m \\ & + (9-4\ell(\ell+1)(2\ell(\ell+2)-1))m^2)\mathbb{K}(m)). \end{aligned} \quad (6)$$

Here, ν_ℓ are the *stability angles* of the fluctuations around the classical solution and read

$$\nu_\ell = 2\pi - 4i\mathbb{K}(m)(Z(\alpha_+|m) + Z(\alpha_-|m)), \quad (7)$$

where Z denotes the Jacobi Zeta function and $\alpha_\pm$ solve,

$$\begin{aligned} \text{sn}^2(\alpha_\pm|m) = & \pm \frac{1}{2} \sqrt{1 - \frac{1}{3}\ell(\ell+2)(\ell(\ell+2)-2)(1-2m)^2} \\ & + \frac{1}{2} + \frac{1}{6}\ell(\ell+2)(2m-1), \end{aligned} \quad (8)$$

with $\text{sn}(z|m)$ the Jacobi elliptic sine function. The above constitute the main technical result of this letter. The sums over ℓ in C_1 are convergent and can be evaluated numerically for any value of $m = m(\lambda n)$ or analytically in the small/large λn regime. At the leading order of the semiclassical expansion C_0 is q_ℓ independent. Remarkably, this implies that the full spectrum of composite operators of the theory is degenerate at this order. We now proceed by analyzing the implications of our main result.

II. THE SMALL λn EXPANSION

The general (asymptotic) perturbative expansion of $\Delta_{n,\{q_\ell\}}$ assumes the form,

$$\Delta_{n,\{q_\ell\}} \sim \sum_{k=0}^{\infty} P_k(n) \epsilon^k, \quad P_k(n) = \sum_{i=0}^{k+1} c_{ik} n^{k+1-i}, \quad (9)$$

where the k -loop coefficient $P_k(n)$ is a polynomial of degree $k+1$ in n . By comparing the above to Eq. (3), we deduce that the perturbative expansion of the C_i coefficients reads,

$$C_i \sim \sum_{k=0}^{\infty} c_{ik} (\epsilon n)^k, \quad (10)$$

and, therefore, the coefficients c_{ik} can be read off from the small ϵn (i.e., small λn) expansion of C_i . In other words, C_0 resums the terms with the leading power of n at any loop order, C_1 resums those with the next-to-leading power, and so on. By expanding our results in Eqs. (4) and (6) around $\epsilon n = 0$, we obtain

$$\begin{aligned} \Delta_{n,\{q_\ell\}} = & n + \sum_{\ell=1}^{\infty} q_\ell \ell + \frac{1}{6} \left[n^2 - 2 \left(2 + \sum_{\ell=1}^{\infty} \frac{(\ell-1)q_\ell}{\ell+1} \right) n + \mathcal{O}(n^0) \right] \epsilon \\ & - \frac{1}{324} \left[17n^3 - \left(67 + 3 \sum_{\ell=1}^{\infty} \frac{(\ell-1)(17\ell^4 + 78\ell^3 + 135\ell^2 + 98\ell + 12)q_\ell}{\ell(\ell+1)^3(\ell+2)} \right) n^2 + \mathcal{O}(n, n^0) \right] \epsilon^2 + \mathcal{O}(\epsilon^3), \end{aligned} \quad (11)$$

where we color-coded in red and blue the C_0 and C_1 contributions. Explicit seven-loop results (i.e., the values of the coefficients c_{0k} and c_{1k} for $k = 1, \dots, 7$) for any finite set of integers $\{q_\ell\}$ are available in the ancillary Mathematica file [26]. The family of primaries corresponding to the ground state (i.e., $q_\ell = 0 \ \forall \ \ell$) is typically denoted schematically as ϕ^n and includes all the lowest Lorentz scalars of the theory. For arbitrary n , their scaling dimensions, henceforth denoted as Δ_n , have been determined in the ϵ -expansion up to three-loop order in [12], and one can easily check that these results match Eq. (11) for $q_\ell = 0$. A relevant application of our findings is the possibility of

deriving new perturbative results by combining semiclassical and conventional diagrammatic computations. Specifically, at the k -loop order, our NLO calculation determines two of the c_{ik} coefficients of the polynomial $P_k(n)$ appearing in Eq. (9), and the remaining k coefficients can be fixed by matching onto existing k -loop perturbative data for k distinct values of n . In particular, by employing the most up-to-date diagrammatic results [27–31], we obtain the complete $\mathcal{O}(\epsilon^5)$ (five-loop) expression for the scaling dimension of ϕ^n which for all $n \geq 8$ improves upon previously available lower-order computations,

$$\begin{aligned} \Delta_n = & \left(1 - \frac{\epsilon}{2} \right) n + \frac{n}{6} (n-1) \epsilon - \frac{n}{324} (17n^2 - 67n + 47) \epsilon^2 + (0.0321502n^4 - 0.123379n^3 + 0.0734761n^2 + 0.0270975n) \epsilon^3 \\ & + (-0.0254558n^5 + 0.113814n^4 - 0.201973n^3 + 0.348649n^2 - 0.239199n) \epsilon^4 \\ & + (0.0231664n^6 - 0.121078n^5 + 0.327421n^4 - 0.53835n^3 - 0.0673349n^2 + 0.389004n) \epsilon^5. \end{aligned} \quad (12)$$

The analytic form can be found in the ancillary file. The stability angles are ordered as $\nu_\ell < \nu_{\ell+1}$, and therefore the first excited state is obtained by exciting a single $\ell = 1$ mode (i.e., setting $q_\ell = \delta_{1\ell}$, with $\delta_{1\ell}$ the Kronecker delta). Since $\nu_1 = \mathcal{T}$, the corresponding conformal dimension increases by one unit, yielding a descendant state corresponding to acting with the conformal raising operator ∂_μ . Therefore, a necessary criterion for an excited state to be a primary is $q_1 = 0$. Moreover, given any primary operator, the corresponding full conformal multiplet is generated by exciting the $\ell = 1$ mode repeatedly. Next, we observe that exciting a single mode with $\ell = s > 1$, one obtains families of spin s operators, schematically of the form $\delta^s \phi^n$. Their dimension is obtained by setting $q_\ell = \delta_{s\ell}$ in

Eq. (11) and we checked that the results match known one-loop calculations, which are available for $s = 2, 3, 4, 5$ [14] (see also [32] for a comprehensive review of the Ising CFT data). These operators are generically primaries, but for specific values of n and s (e.g., $n = 4, s = 5$), they may be descendants. By continuing this construction, one can see that by exciting multiple modes, we can reproduce the full spectrum of primary operators of the Ising CFT. In essence, exciting an ℓ mode amounts to adding ℓ derivatives in the $(\ell/2, \ell/2)$ representation of $SO(4)$ and, to identify the resulting operators, one has to consider the corresponding decomposition into irreducible Lorentz representations. For instance, by exciting two $\ell = 2$ modes we have to decompose the product of two $(1,1)$ representations using

$$1 \otimes 1 = 2 \oplus 1 \oplus 0, \quad (13)$$

Accordingly, the Ising CFT contains the four operators $\partial^4 \phi^n$, $\partial^2 \square \phi^n$, $\square^2 \phi^n$, and $\partial_{\square}^4 \phi^n$, which, up to the computed semiclassical order, have the same scaling dimensions obtained by setting $q_\ell = 2\delta_{2\ell}$ in Eq. (11). This result was again checked by comparing it to the one-loop results in [13,14]. Intuitively, the spin-zero operator is built by contracting four derivatives (two for each $\ell = 2$ mode), and so will schematically be of the form $\square^2 \phi^n$, i.e., $s = 0$, $p = 2$. Analogously, the spin two and four operators have $s = 2$, $p = 1$ and $s = 4$, $p = 0$, respectively. Finally, we exhaust the list of available perturbative results by further comparing to [14] the dimension of the primaries obtained by exciting $\ell = 2$ and $\ell = 3$ mode ($q_\ell = \delta_{2\ell} + \delta_{3\ell}$), which, since $3/2 \otimes 1 = 5/2 \oplus 3/2 \oplus 1/2$, have $s = \{5, 3, 1\}$ uncontracted derivatives.

We remark that when rewriting ϵ in terms of the renormalized coupling, the results above for the scaling dimensions extend away from the conformal regime, where they become anomalous dimensions in dimensional regularization.

III. ON THE LARGE λn EXPANSION

By expanding our results in Eqs. (4) and (6) in the large- λn limit and focusing on the ground state ($q_\ell = 0 \ \forall \ \ell$), we obtain

$$\Delta_n = \left(\frac{\sqrt{\frac{3\pi}{2}} \Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} n \epsilon \right)^{\frac{4-\epsilon}{3-\epsilon}} \left(\frac{1}{\epsilon} + \alpha + \mathcal{O}(\epsilon) \right) + \mathcal{O}\left(n^{\frac{2-\epsilon}{3-\epsilon}}\right), \quad (14)$$

where α is an integral over a function stemming from the expansion of the stability angles in the large λn limit. Interestingly, the NLO correction modifies the large n behavior from $\Delta_n \sim n^{4/3}$ found in [12] to

$$\lim_{n \rightarrow \infty} \Delta_n \sim n^{\frac{d}{d-1}}. \quad (15)$$

Intriguingly, this result mimics the generic nonperturbative behavior of the scaling dimension of the lowest operator with charge n in generic CFTs with global symmetries [1]. However, in our case, there is more to it. In fact, the integrand defining α has a branch cut, which makes α multivalued. Its presence can be understood by noting that ν_ℓ in (7) becomes complex for any ℓ in the range,

$$\sqrt{\frac{4-5m}{1-2m}} \leq \ell + 1 \leq \sqrt{2} \sqrt{1 + \frac{\sqrt{1-m+m^2}}{1-2m}}. \quad (16)$$

Whenever an integer value ℓ falls in the above range, the corresponding stability angles become complex, signaling an instability of the associated classical orbit. In such a

case, the computation breaks down, and one may need to take into account additional (generically nonhomogeneous) saddle points. Physically, when ℓ satisfies the above, the spectral parameter of the finite-gap fluctuation operator around the classical solution takes values outside the allowed bands of the spectrum. For small values of m (i.e., small values of λn), no integer ℓ satisfies the above condition. However, as λn increases, the $\ell = 2$ mode eventually enters the unstable orbit region for $m = 3/8$, corresponding to $\lambda n \approx 50$. In general, for $3/8 < m < \frac{1}{65}(4\sqrt{61} + 1)$, there are ranges of values of m such that there is a single complex stability angle for a certain integer ℓ . For $m \geq \frac{1}{65}(4\sqrt{61} + 1)$ ($\lambda n \gtrsim 10856$), there is always at least one complex stability angle. Finally, in $m \rightarrow 1/2$ ($\lambda n \rightarrow \infty$) limit, where the stability angles tend to a continuum, the complex ones fill the whole region $\sqrt{3/2} < \ell \sqrt{1-2m} < 3^{1/4}$. Therefore, for certain values of λn in the range $50 \lesssim \lambda n \lesssim 10856$ and for all $\lambda n \gtrsim 10856$, one has to investigate further the saddle point analysis. Since a precise analysis of the instability would require performing an involved Lefschetz thimble analysis of the path integral [33], we leave it to future investigations. In particular, it would be interesting to understand the relation between the instability as a Stokes phenomenon, operator mixing, and chaotic properties of the CFT spectrum.

IV. ISING CFT IN THREE DIMENSIONS

As a relevant application of our methodology, we now present the conformal dimensions Δ_n associated with the ϕ^n operators for the physical Ising CFT in $d = 3$. In Fig. 1 we compare our NLO semiclassical predictions to the results obtained via the conformal bootstrap and the ϵ -expansion.

For $n = 1, 2, 4$, Δ_n has been measured experimentally but with large uncertainties [32]. The theoretical estimates obtained via Monte Carlo simulations and conformal bootstrap are more accurate and fall within the experimental bounds. More generally, bootstrap results are available for $n = 1, 2, 4, 5, 6$ [34,35]. There is no primary operator with $n = 3$. The ϵ -expansion results have been obtained as

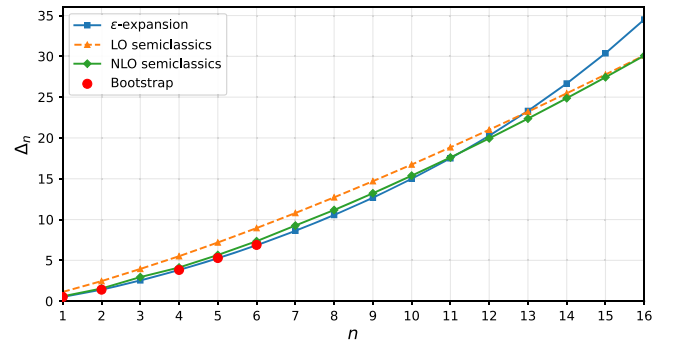


FIG. 1. Δ_n for $n \leq 16$ obtained via numerical bootstrap, ϵ -expansion, and semiclassics at LO and NLO.

follows; for $n = 1$ we considered the conformal-Borel resummed six-loop result of [31], for $n = 2, 4$, the hypergeometric–Meijer resummation of the seven-loop results of [28] performed in [36], while for $n > 5$ we consider the $[2/3]$ Padé approximant of our five-loop result given by Eq. (12). The latter was selected as it provides the closest agreement with the available bootstrap results for $n \leq 6$. Note that for $n = 6$ the NLO calculation fails as the $\ell = 2$ stability angle is complex.

As it should be evident from the figure, the resummed ϵ -expansion is in agreement with the available bootstrap results, and so no breakdown of the resummed perturbative theory is observed for $n < 6$. To understand this observation, in Fig. 2 we compare the results of the ϵ -expansion truncated to various loop orders with and without Padé resummation. Clearly, resummation methods play a key role in improving the convergence of the ϵ -expansion for larger n . In fact, the results of the ϵ -expansion without resummation indicate that the perturbative series is an asymptotic one controlled by the product ϵn . In particular, for $n > 2$, the optimal truncation order, and so the best possible result, is achieved at the one-loop level. Conversely, the sequence of Padé approximants exhibits convergence up to the five-loop level for any $n \leq 6$. From Fig. 1, it is clear that the semiclassical expansion performs well even when extrapolated to both $\epsilon = 1$ and low values of n , where numerical results are available. Crucially, the agreement with bootstrap results strengthens as n grows (from $\sim 14\%$ for $n = 1$ to $\sim 6\%$ for $n = 5$). Moreover, we observe that for $n < 12$ both the convergence of the semiclassical expansion and the agreement with the ϵ -expansion, which itself is known to provide reliable estimates in this model, improve systematically as n increases. This no longer holds for $n > 12$, where the LO semiclassical results are closer to the ϵ -expansion than the NLO ones, so the comparison ceases to be meaningful. Of course, it would be interesting to obtain precise numerical estimates for $n > 6$ to test the actual accuracy of the semiclassical predictions in

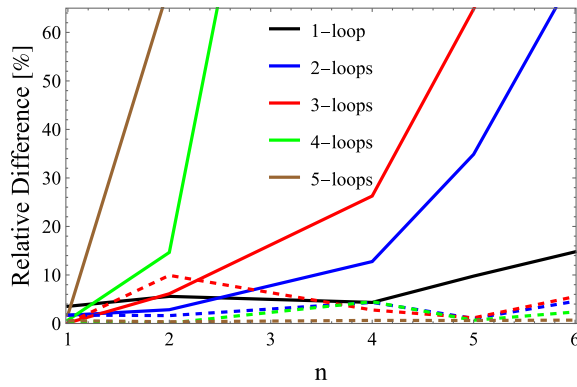


FIG. 2. *Solid lines*: Relative difference for Δ_n (in %) between the bootstrap [34,35] and the ϵ -expansion truncated to various loop orders. *Dashed lines*: same quantity evaluated for the corresponding Padé approximants.

$d = 3$. To this end, we collected our results for $n \leq 16$ in the Appendix B.

We conclude by remarking that while we focused on the Ising CFT, our framework can be employed to solve for the dynamics of composite operators stemming from different QFTs with applications ranging from condensed matter physics to particle physics and cosmology. For instance, we expect that gauge and Yukawa interactions can be incorporated within the same framework, allowing for the determination of the anomalous dimensions of composite operators that reduce, in the free field theory limit, to Higgs composites of the schematic form $\partial^s \square^p (H^\dagger H)^n$. In an interacting theory, such operators generally mix with operators built from fermionic and gauge degrees of freedom. Nevertheless, the semiclassical approach is expected to provide direct access to the eigenvalues of the anomalous dimension matrix, which correspond to the physical energy levels on the cylinder. This would provide a natural extension of our previous work on the Standard Model [37] beyond the charged sectors with immediate impact for RG functions of higher-dimensional operators. Further interesting generalizations include composite operators involving higher powers of gauge-field strengths as well as fermionic operators of the schematic form $(\bar{\psi}\psi)^n$. In the latter case, however, additional complications are expected due to the enlarged operator basis generated by the gamma-matrix algebra.

ACKNOWLEDGMENTS

The authors would like to thank A.V. Bednyakov, D. Orlando, and S. Reffert for useful comments. The work of G.M. and F.S. is partially supported by the Carlsberg Foundation, semper ardens Grant No. CF22-0922. The work of J.B. is supported by the Swiss National Science Foundation under Grant No. 200021_219267.

DATA AVAILABILITY

The data that support the findings of this article are openly available [26].

APPENDIX A: SEMICLASSICAL CALCULATION

We here provide a derivation of our main result in Eq. (5). The Lagrangian on the cylinder reads

$$\mathcal{L}^{(\text{cyl})} = \frac{1}{2}(\partial\phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4, \quad (\text{A1})$$

with $\mu = (\frac{d-2}{2r})$ the conformal mass. Assuming a spatially homogeneous field configuration, the time-dependent equation of motion for $\langle\phi\rangle = v(t)$ takes the form of the quartic anharmonic oscillator,

$$\frac{d^2v}{dt^2} + \mu^2 v + \lambda v^3 = 0, \quad (\text{A2})$$

whose solution is [20]

$$v(t) = \mu \sqrt{\frac{2m}{\lambda(1-2m)}} \text{cn}\left(\frac{\mu t}{\sqrt{1-2m}} | m \right), \quad 0 < m < 1/2. \quad (\text{A3})$$

where $\text{cn}(\cdot|m)$ is the Jacobi elliptic cosine function. The period of the solution is

$$T = \frac{4}{\mu} \sqrt{1-2m} \mathbb{K}(m), \quad (\text{A4})$$

where $\mathbb{K}(m)$ denotes the complete elliptic integral of the first kind. The associated energy levels to the next-leading-order in the semiclassical expansion can be derived following [16–19]:

$$E_{n,\{q_\ell\}} = E_{cl} - \frac{\lambda_*}{2} \beta_0 E_{cl} + \frac{1}{T} \sum_{\nu_\ell > 0} \left(q_\ell + \frac{1}{2} \right) \nu_\ell. \quad (\text{A5})$$

Here E_{cl} is the classical energy of the solution (A3) and $\beta_0 = \frac{9}{8\pi^2}$ is the one-loop coefficient of the β function. The sum runs over all the positive stability angles ν_ℓ , with $\ell \in \mathbb{N}$, which emerge when analyzing the stability of the classical orbits by considering fluctuations around them. Mathematically, $e^{\pm i\nu_\ell}$ are the eigenvalues of the monodromy matrix associated with the eigenvalue equation of the fluctuation operator [16,21]. The detailed derivation of Eq. (A5) is lengthy and will be provided in a companion work [22]. The state-operator correspondence leads to

$$nC_0/r = E_{cl}, \quad C_1/r = -\frac{\lambda_*}{2} \beta_0 E_{cl} + \frac{1}{T} \sum_{\nu_\ell > 0} \left(q_\ell + \frac{1}{2} \right) \nu_\ell. \quad (\text{A6})$$

E_{cl} and ν_ℓ are functions of m where the latter is related to λn by inverting the Bohr-Sommerfeld quantization condition $I = 2\pi n$ where the action variable I reads

$$\begin{aligned} I &= \oint \Pi d\phi = \Omega_{d-1} r^{d-1} \int_0^T \left(\frac{dv}{dt} \right)^2 dt \\ &= \frac{16\pi^2}{3\lambda(1-2m)^{3/2}} [(2m-1)\mathbb{E}(m) + (1-m)\mathbb{K}(m)]. \end{aligned} \quad (\text{A7})$$

Here $\mathbb{E}(m)$ is the complete elliptic integral of the second kind, and we introduced the field momentum $\Pi = \Omega_{d-1} r^{d-1} \dot{\phi}$. The classical energy yields C_0 [12],

$$nC_0 = rE_{cl} = \frac{2\pi^2 m(1-m)}{\lambda(1-2m)^2}. \quad (\text{A8})$$

To determine C_1 , we expand the action around the classical trajectory as $\phi = v(t) + \eta(\vec{x}, t)$ obtaining the quadratic Lagrangian,

$$\mathcal{L}_2 = \frac{1}{2} \eta \mathcal{O}_2 \eta, \quad \mathcal{O}_2 = -\partial_t^2 + \Delta_{S^{d-1}} - \mu^2 - 3\lambda v^2(t), \quad (\text{A9})$$

with $\Delta_{S^{d-1}}$ the Laplacian on S^{d-1} . By changing variable as $z = \frac{\mu t}{\sqrt{1-2m}}$ we rewrite the fluctuation operator as

$$\mathcal{O}_2 = \frac{\mu^2}{1-2m} L_2, \quad L_2 = -\partial_z^2 + 6m \text{sn}^2(z|m) - \Lambda_\ell, \quad (\text{A10})$$

where $\text{sn}(z|m)$ is the Jacobi elliptic sine function and

$$\Lambda_\ell = 6m + (1-2m) \left(1 + \frac{2\ell}{d-2} \right)^2, \quad (\text{A11})$$

includes $\Delta_{S^{d-1}}$ in terms of its eigenvalues $J_\ell^2 = \ell(\ell+d-2)/r^2$. Remarkably, L_2 takes the form of the 2-gap Lamé operator whose stability angles are given by [23,24]

$$\nu_\ell = 2\pi - 4i\mathbb{K}(m)(Z(\alpha_+|m) + Z(\alpha_-|m)), \quad (\text{A12})$$

where Z denotes the Jacobi Zeta function and $\alpha_\pm$ solve

$$\begin{aligned} \text{sn}^2(\alpha_\pm|m) &= \pm \frac{1}{2m} \sqrt{\frac{4}{3}(1-m+m^2) - \frac{1}{3}(\Lambda_\ell - 2(1+m))^2} \\ &\quad + \frac{4(1+m) - \Lambda_\ell}{6m}. \end{aligned} \quad (\text{A13})$$

When the spectral parameter Λ_ℓ takes value inside one of the two gap region of L_2 , which are given by

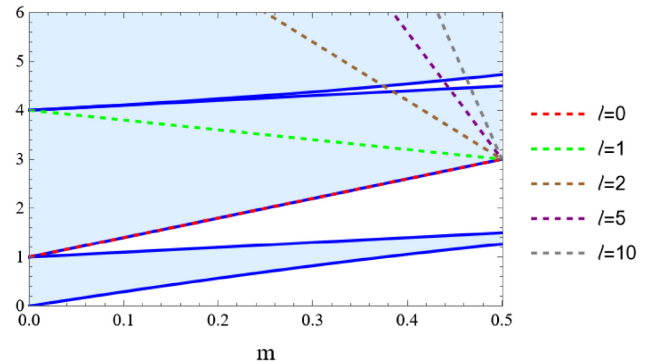


FIG. 3. Comparison of the band structure of the Lamé operator L_2 (allowed bands colored in light blue) and Λ_ℓ for $\ell = 0, 1, 2, 5, 10$ (dashed lines) as a function of m . Complex stability angles arise when Λ_ℓ takes values inside the forbidden zones i.e., when dashed lines are inside the uncolored zones.

$$\{1+m, 1+4m\}, \quad \{4+m, 2(m+\sqrt{1-m(1-m)}+1)\}, \quad (\text{A14})$$

the stability angles develop a nontrivial imaginary part. By solving this condition for ℓ one obtains Eq. (16). The known [25] band structure of L_2 as a function of $0 < m < 1/2$ is shown in Fig. 3 along with the values of Λ_ℓ for various values of ℓ . Finally, by regularizing the sum over ℓ in Eq. (A6), renormalizing the result via dimensional regularization, and setting $d = 4$ at the end of the calculation, we arrive at the expression for C_1 presented in the main text.

APPENDIX B: SEMICLASSICAL RESULTS IN THREE DIMENSIONS

In Table I, we compare our semiclassical predictions for Δ_n to the results obtained via the conformal bootstrap and the ϵ -expansion for $n = 1, 2, \dots, 16$.

TABLE I. Δ_n for $n \leq 16$ obtained via numerical bootstrap, ϵ -expansion, and semiclassics at LO and NLO.

n	Bootstrap	ϵ -expansion	LO	NLO
1	0.5181489(10) [34]	0.5181(6) [31]	1.133	0.5960
2	1.412625(23) [34]	1.4121(6) [28,36]	2.459	1.552
4	3.82968(23) [34]	3.8231(5) [28,36]	5.509	4.126
5	5.2906(11) [34]	5.257	7.190	5.660
6	6.8956(43) [34]	6.862	8.956	$\times$
7	$\times$	8.627	10.80	9.268
8	$\times$	10.56	12.72	11.17
9	$\times$	12.67	14.70	13.22
10	$\times$	15.0	16.74	15.38
11	$\times$	17.51	18.85	17.61
12	$\times$	20.27	21.00	19.95
13	$\times$	23.31	23.21	22.37
14	$\times$	26.67	25.47	24.87
15	$\times$	30.38	27.77	27.44
16	$\times$	34.51	30.12	30.08

-
- [1] S. Hellerman, D. Orlando, S. Reffert, and M. Watanabe, On the CFT operator spectrum at large global charge, *J. High Energy Phys.* **12** (2015) 071.
 - [2] Z. Komargodski and A. Zhiboedov, Convexity and liberation at large spin, *J. High Energy Phys.* **11** (2013) 140.
 - [3] R. Rattazzi, V.S. Rychkov, E. Tonni, and A. Vichi, Bounding scalar operator dimensions in 4D CFT, *J. High Energy Phys.* **12** (2008) 031.
 - [4] G. Badel, G. Cuomo, A. Monin, and R. Rattazzi, The epsilon expansion meets semiclassics, *J. High Energy Phys.* **11** (2019) 110.
 - [5] W. Buchmuller and D. Wyler, Effective lagrangian analysis of new interactions and flavor conservation, *Nucl. Phys.* **B268**, 621 (1986).
 - [6] B. Grzadkowski, M. Iskrzynski, M. Misiak, and J. Rosiek, Dimension-six terms in the Standard Model lagrangian, *J. High Energy Phys.* **10** (2010) 085.
 - [7] I. Brivio and M. Trott, The Standard Model as an effective field theory, *Phys. Rep.* **793**, 1 (2019).
 - [8] H. Goldberg, Breakdown of perturbation theory at tree level in theories with scalars, *Phys. Lett. B* **246**, 445 (1990).
 - [9] J.M. Cornwall, On the high-energy behavior of weakly coupled gauge theories, *Phys. Lett. B* **243**, 271 (1990).
 - [10] D. T. Son, Semiclassical approach for multiparticle production in scalar theories, *Nucl. Phys.* **B477**, 378 (1996).
 - [11] V.V. Khoze, Multiparticle Higgs and vector boson amplitudes at threshold, *J. High Energy Phys.* **07** (2014) 008.
 - [12] O. Antipin, J. Bersini, and F. Sannino, Exact results for scaling dimensions of neutral operators in scalar conformal field theories, *Phys. Rev. D* **111**, L041701 (2025).
 - [13] S. Kehrein, F. Wegner, and Y. Pismak, Conformal symmetry and the spectrum of anomalous dimensions in the N vector model in four epsilon dimensions, *Nucl. Phys.* **B402**, 669 (1993).
 - [14] S. K. Kehrein and F. Wegner, The structure of the spectrum of anomalous dimensions in the N vector model in (4-epsilon)-dimensions, *Nucl. Phys.* **B424**, 521 (1994).
 - [15] G. Cuomo, L. Rastelli, and A. Sharon, Moduli spaces in CFT: Large charge operators, *J. High Energy Phys.* **09** (2024) 185.
 - [16] M. C. Gutzwiller, Periodic orbits and classical quantization conditions, *J. Math. Phys. (N.Y.)* **12**, 343 (1971).
 - [17] R. F. Dashen, B. Hasslacher, and A. Neveu, The particle spectrum in model field theories from semiclassical functional integral techniques, *Phys. Rev. D* **11**, 3424 (1975).
 - [18] R. F. Dashen, B. Hasslacher, and A. Neveu, Nonperturbative methods and extended hadron models in field theory 1. Semiclassical functional methods, *Phys. Rev. D* **10**, 4114 (1974).
 - [19] M. Beccaria, G. V. Dunne, G. Macorini, A. Tirziu, and A. A. Tseytlin, Exact computation of one-loop correction to energy of pulsating strings in $\text{AdS}_5 \times S^5$, *J. Phys. A* **44**, 015404 (2011).
 - [20] A. Martín Sánchez and J. Díaz Bejarano, Quantum anharmonic symmetrical oscillators using elliptic functions, *J. Phys. A* **19**, 887 (1986).
 - [21] W. Magnus and S. Winkler, *Hill's Equation* (Wiley, New York, 1966).
 - [22] O. Antipin, J. Bersini, J. Hafjall, G. Muco, and F. Sannino, Semiclassical canovaccio for composite operators, *J. High Energy Phys.* **05** (2026) 137.
 - [23] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1927).

- [24] M. Pawellek, Quantum mass correction for the twisted kink, *J. Math. Phys. (N.Y.)* **42**, 045404 (2009).
- [25] F. M. Arscott, *Periodic Differential Equations: An Introduction to Mathieu, Lamé, and Allied Functions* (Pergamon Press, New York, 1964).
- [26] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/43ck-2h6m> for *Mathematica* notebook Ising_Data.nb contains explicit seven-loop results for the spectrum of scaling dimensions in the critical ϕ^4 theory.
- [27] J. Henriksson, F. Herzog, S. R. Kousvos, and J. Roosmale Nepveu, Multi-loop spectra in general scalar EFTs and CFTs, *Phys. Lett. B* **874**, 140235 (2026).
- [28] O. Schnetz, Numbers and functions in quantum field theory, *Phys. Rev. D* **97**, 085018 (2018).
- [29] O. Schnetz, ϕ^4 theory at seven loops, *Phys. Rev. D* **107**, 036002 (2023).
- [30] A. Bednyakov and A. Pikelner, Six-loop beta functions in general scalar theory, *J. High Energy Phys.* **04** (2021) 233.
- [31] M. V. Kompaniets and E. Panzer, Minimally subtracted six loop renormalization of $O(n)$ -symmetric ϕ^4 theory and critical exponents, *Phys. Rev. D* **96**, 036016 (2017).
- [32] J. Henriksson, The critical $O(N)$ CFT: Methods and conformal data, *Phys. Rep.* **1002**, 1 (2023).
- [33] G. V. Dunne and M. Ünsal, What is QFT? Resurgent trans-series, Lefschetz thimbles, and new exact saddles, *Proc. Sci. LATTICE2015* (2016) 010 [arXiv:1511.05977].
- [34] D. Simmons-Duffin, The lightcone bootstrap and the spectrum of the 3D Ising CFT, *J. High Energy Phys.* **03** (2017) 086.
- [35] J. Henriksson, S. R. Kousvos, and M. Reehorst, Spectrum continuity and level repulsion: The Ising CFT from infinitesimal to finite ϵ , *J. High Energy Phys.* **02** (2023) 218.
- [36] A. M. Shalaby, Critical exponents of the $O(N)$ -symmetric ϕ^4 model from the ϵ^7 hypergeometric-Meijer resummation, *Eur. Phys. J. C* **81**, 87 (2021).
- [37] O. Antipin, J. Bersini, P. Panopoulos, F. Sannino, and Z. W. Wang, Infinite order results for charged sectors of the Standard Model, *J. High Energy Phys.* **02** (2024) 168.