Area law for entanglement entropy in particle scattering

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The scattering cross section is the effective area of collision when two particles collide. Quantum mechanically, it is a measure of the probability for a specific process to take place. Employing wave packets to describe the scattering process, we compute the entanglement entropy in 2-to-2 scattering of particles in a general setting using the S-matrix formalism. Applying the optical theorem, we show that the linear entropy $\mathcal{E}_2$ is given by the elastic cross section $\sigma_{\rm el}$ in unit of the transverse size L^2 of the wave packet, $\mathcal{E}_2 \sim \sigma_{\rm el}/L^2$, when the initial states are not entangled. The result allows for dual interpretations of the entanglement entropy as an area and as a probability. Since $\sigma_{\rm el}$ is generally believed, and observed experimentally, to grow with the collision energy $\sqrt{s}$ in the high energy regime, the result suggests a "second law" of entanglement entropy for high energy collisions. Furthermore, the Froissart bound places an upper limit on the entropy growth.

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I. INTRODUCTION

Entropy is among the most fundamental concepts in physics and quantifies the number of possible microstates giving rise to a particular macrostate. It is directly related to probability—a macrostate containing more microstates has a higher probability to occur and hence a higher entropy. A physical system naturally evolves into more probable states, thereby increasing the entropy, which defines an arrow of time.

Entropy can also be understood as a measure of randomness, or disorder, since a more probable state is considered to be more disordered. As such the entropy is also used to quantify the amount of information in a physical system, since a highly disordered system is capable of carrying a large amount of information. Indeed, a single sentence assembled from letters of a small alphabet is very ordered but conveys little information. In classical information theory the Shannon entropy, which measures the average amount of information in an event drawn from a particular probability distribution [1,2], features prominently.

One usually refers to the entropy defined in classical systems as coarse-grained entropy, which focuses on a subset of simple observables without knowing the full microscopic configuration space. Even though the underlying dynamics is deterministic, the ignorance of the microstates necessitates the use of probability. Quantum mechanics, on the other hand, is inherently probabilistic and introduces intrinsic randomness in a quantum system. Such randomness is described by the quantum, or fine-grained, entropy [3,4]. The most prominent quantum entropy is the von Neumann entropy [2], which can be generalized to two classes of quantum entropies: the Rényi entropy [5] and the Tsallis entropy [6].

Similar to their classical counterpart, quantum entropies also quantify the amount of information present in a quantum system, as well as quantum correlations, or entanglement, between systems [2]. The latter, computed from the (reduced) density matrices, are called entanglement entropies. While by definition the entropy of a closed system is invariant under unitary transformation, and hence time evolution, this is not always the case for an open quantum system interacting with an environment. In fact, one often divides a closed system into a bipartite system whose Hilbert space is $\mathcal{H} = \mathcal{H}_{\rm A} \otimes \mathcal{H}_{\rm B}$. The subsystem entropy for A is then computed from the reduced density matrix $\rho_{\rm A}$ by tracing over B, and can be nonzero even when the total system has vanishing entropy.

In many-body systems the entropy is usually an extensive quantity and scales with the size of the system. Therefore it is quite mysterious that in gravitational systems, the black hole in particular, the Bekenstein-Hawking entropy scales with the area of horizon [7,8]. Since then there have been more examples of area laws for entropy both in gravitational and nongravitational many-body systems [9–12], some of which involves holography [4,13]. In these examples the areas involved are often computed from a "boundary" between two "regions."

In this work we will consider the entanglement entropy in a very different setting: The 2-to-2 scattering of particles in quantum field theories, using the S-matrix formalism. This is hardly a many-body system, and it is not obvious at all if there is a "boundary" for which to define an area. Nevertheless, using the wave packet formalism to describe the particle scattering, we show that for the final state entanglement, the linear entropy is given by the elastic cross section in unit of the transverse size of the wave packet when the initial states are not entangled; here the linear entropy $\mathcal{E}_2$ for density matrix ρ,

$$\mathcal{E}_2 = 1 - \mathrm{Tr}(\rho)^2, \tag{1}$$

is the $n = 2$ case of the Tsallis entropy $\mathcal{E}_n = (1 - \mathrm{Tr}\rho^n)/(n-1)$ [6]. From the von Neumann entropy $\mathcal{E}_{\mathrm{vN}} = -\mathrm{Tr}(\rho\log\rho)$, $\mathcal{E}_2$ can be obtained by expanding with respect to $\rho = 1$ and keeping the leading term. Recently the entanglement entropy in particle scattering has attracted considerable attention, although the calculation has mostly been done in specific quantum field theories and differing contexts [14–37]. Here we will not commit to a specific theory and will instead use the S-matrix formalism to make general statements which are nonperturbative in the couplings. Importantly, the wave packet formalism, mentioned recently in [36,37], gives a clear physical interpretation of the outcome.

Our result allows for novel, dual interpretations of the entanglement entropy as an area, in the cross-sectional sense, and a probability, in the information-theoretic sense. Moreover, it is generally accepted that the elastic cross section grows with the collision center-of-mass (c.m.) energy $\sqrt{s}$ [38,39], which has been verified experimentally [40,41]. This suggests a version of the second law of thermodynamics where the entanglement entropy grows with energy, and the Froissart bound [42,43] now gives an upper limit on the growth.

II. THE SETUP

We consider the scattering of two distinguishable particles: A(lice) and B(ob). The kinematic data are labeled by the momentum $p_{\mathrm{A/B}}$, which can be either massive or massless. All other nonkinematic quantum numbers are denoted by $\{i\}$ for A and $\{\bar{i}\}$ for B, respectively. These could include spacetime quantum numbers, such as the spin, or the internal quantum numbers like the isospin.

The scattering process we focus on is $\mathrm{AB} \to \mathrm{AB}$, from which we construct a bipartite system $\mathcal{H}_{\mathrm{AB}} = \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$, where $\mathcal{H}_{\mathrm{A/B}}$ is the Hilbert space associated with particle-A/B. For a general incoming state $|\mathrm{in}\rangle$, the density matrix is $\rho^{\mathrm{i}} = |\mathrm{in}\rangle\langle\mathrm{in}|$ and the linear entropy $\mathcal{E}_2$ for A is computed from the reduced density matrix $\rho^{\mathrm{i}}_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}\rho^{\mathrm{i}}$,

$$\mathcal{E}^{\mathrm{i}}_2 = 1 - \mathrm{Tr}(\rho^{\mathrm{i}}_{\mathrm{A}})^2. \tag{2}$$

We will start with a pure initial state and comment on the mixed initial state later. We will also make the realistic assumption that there is no entanglement between p_{A}, p_{B}, and between the momenta and the other quantum numbers. The initial state is then written as

$$|\mathrm{in}\rangle = \sum_{i,\bar{i}} \Omega_{i\bar{i}} |\psi_{\mathrm{A}}\rangle \otimes |i\rangle \otimes |\psi_{\mathrm{B}}\rangle \otimes |\bar{i}\rangle, \tag{3}$$

where $|i\rangle$ and $|\bar{i}\rangle$ are normalized such that $\langle i|j\rangle = \delta^{ij}$, $\langle\bar{i}|\bar{j}\rangle = \delta^{\bar{i}\bar{j}}$, and Ω describes the nonkinematic configuration of the initial state, which is normalized such that $\mathrm{Tr}(\Omega\Omega^\dagger) = 1$. When the quantum numbers of two particles are not entangled, $\Omega_{i\bar{i}} = \omega_i \omega'_{\bar{i}}$ and $|\omega|^2 = |\omega'|^2 = 1$. On the other hand, $|\psi_{\mathrm{A}}\rangle$ and $|\psi_{\mathrm{B}}\rangle$ are described by the momentum wave packets of A and B, respectively,

$$|\psi_{\mathrm{A/B}}\rangle = \int_p \psi_{\mathrm{A/B}}(p)|p\rangle, \qquad \int_p \equiv \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}, \tag{4}$$

where the one-particle state is normalized as follows,

$$\langle p|q\rangle = (2\pi)^3 2E_p \delta^3(\vec{p} - \vec{q}). \tag{5}$$

The momentum wave packets are normalized such that

$$\langle\psi|\psi\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} |\psi(p)|^2 = 1. \tag{6}$$

With Eq. (6), the initial density matrix ρ^{i} is properly normalized, $\mathrm{Tr}\rho^{\mathrm{i}} = 1$. If we had used the one-particle states $|k_{\mathrm{A}}\rangle$ and $|k_{\mathrm{B}}\rangle$ in place of $|\psi_{\mathrm{A}}\rangle$ and $|\psi_{\mathrm{B}}\rangle$, the δ-function normalization in Eq. (5) would have introduced divergences in $\mathrm{Tr}\rho$, which is often regularized by introducing a finite box in spacetime. We will not be using such a box normalization. In the CM frame, $\vec{k}_{\mathrm{A}} = -\vec{k}_{\mathrm{B}} \equiv \vec{k}$.

In an experimental setup the incoming particles are prepared with fairly well-defined momenta, subject to experimental resolution. Therefore we expect $\psi(p)_{\mathrm{A/B}}$ to be sharply peaked at $\pm\vec{k}$. That is, if δ_{p} characterizes the linear width of the momentum wave packet, we expect

$$\delta_{\mathrm{p}}/|\vec{k}| \ll 1, \qquad \psi(k) \sim 1/\delta_{\mathrm{p}}{}^{3/2}. \tag{7}$$

We compute the entanglement entropy in the limit $\delta_{\mathrm{p}} \to 0$ and to the first nonvanishing order in $\delta_{\mathrm{p}}/|\vec{k}|$.

Now let us consider the final state produced by the scattering process, which is given by $|\mathrm{out}\rangle \equiv S|\mathrm{in}\rangle$, where $S = 1 + iT$ is the scattering matrix and T is the transition matrix, which is related to the scattering amplitude M

$$\langle \{k_\mathrm{f}\}, f_\mathrm{f}|T|\{k_\mathrm{i}\}, f_\mathrm{i}\rangle = (2\pi)^4\delta^4\left(\sum k_\mathrm{f} - \sum k_\mathrm{i}\right) M_{f_\mathrm{i},f_\mathrm{f}}(\{k_\mathrm{i}\};\{k_\mathrm{f}\}), \quad (8)$$

where $f_{\mathrm{i/f}}$ labels the discrete quantum numbers of the initial and final states. Unitarity of the S-matrix implies $2\,\mathrm{Im}\,T = T^\dagger T$, which gives rise to the optical theorem.

In general, $|\mathrm{out}\rangle$ is a linear superposition of all possible outcomes from the collision of A and B. Since we will focus on the elastic scattering AB → AB, the outgoing state is also consisted of A and B, which can be selected by manually applying a projection operator P_{AB} on $|\mathrm{out}\rangle$, $|\mathrm{out}\rangle_{\mathrm{el}} = P_{\mathrm{AB}}|\mathrm{out}\rangle$. Applying the Lüders rule [44], the properly normalized final state density matrix ρ^f after the projection is

$$\rho^\mathrm{f} = \frac{P_{\mathrm{AB}}|\mathrm{out}\rangle\langle\mathrm{out}|P_{\mathrm{AB}}}{\mathrm{Tr}(P_{\mathrm{AB}}|\mathrm{out}\rangle\langle\mathrm{out}|)} = \frac{1}{1-\mathcal{P}_{\mathrm{inel}}}|\mathrm{out}\rangle_{\mathrm{el}\;\mathrm{el}}\langle\mathrm{out}|, \quad (9)$$

where

$$\mathcal{P}_{\mathrm{inel}} = \langle\mathrm{out}|1-P_{\mathrm{AB}}|\mathrm{out}\rangle = \langle\mathrm{in}|T^\dagger(1-P_{\mathrm{AB}})T|\mathrm{in}\rangle \quad (10)$$

is the probability for AB to scatter inelastically into anything *but* AB.

III. ENTANGLEMENT ENTROPY

The initial state reduced density matrix for A is

$$\rho^\mathrm{i}_\mathrm{A} = \sum_{i,j,\bar{i},\bar{j}} \Omega_{i\bar{i}}(\Omega^\dagger)_{\bar{j}j}\langle\psi_B,\bar{j}|\psi_B,\bar{i}\rangle|\psi_A,i\rangle\langle\psi_A,j| = \sum_{i,j}(\Omega\Omega^\dagger)_{ij}|\psi_A,i\rangle\langle\psi_A,j|, \quad (11)$$

and the entanglement entropy is

$$\mathcal{E}^\mathrm{i}_2 = 1 - \mathrm{Tr}(\Omega^\dagger\Omega)^2. \quad (12)$$

Clearly, if there is no entanglement between the initial states, $(\Omega^\dagger\Omega)^2 = \Omega^\dagger\Omega$ and the entropy $\mathcal{E}^\mathrm{i}_2 = 0$.

For concrete computation of the final state reduced density matrix from Eq. (9), it becomes necessary to specify the form of the initial wave packets. We choose the momentum-space wave packet to be a cube with a side length of $2\delta_\mathrm{p}$, as shown in Fig. 1(a). The wave packet is chosen to be uniform in the p_z-direction but not in the $p_{x,y}$-directions as we would like the wave packet to approach the plane wave in position space. Specifically, we define the regularized delta functions

$$\tilde{\delta}^3(\vec{p}) = \tilde{\delta}_0(p_x)\tilde{\delta}_0(p_y)\tilde{\delta}_z(p_z), \quad (13)$$

$$\tilde{\delta}_0(k) = \frac{\pi}{2}\frac{\Theta(k+\delta_\mathrm{p}) - \Theta(k-\delta_\mathrm{p})}{\mathrm{Si}(\delta_\mathrm{p}L/2)}\frac{L}{2\pi}\mathrm{sinc}\frac{kL}{2}, \quad (14)$$

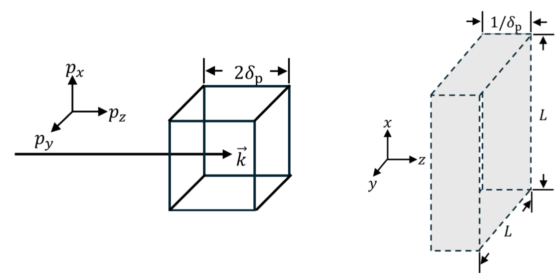
$$\tilde{\delta}_z(k) = \frac{\Theta(k+\delta_\mathrm{p}) - \Theta(k-\delta_\mathrm{p})}{2\delta_\mathrm{p}}, \quad (15)$$

where $\mathrm{sinc}(k) = (\sin k)/k$ is the Fourier transform of a uniform rectangular function and $\mathrm{Si}(x) = \int_0^x dy\,\mathrm{sinc}(y)$. The parameter L characterizes the linear size of the wave packet in the position space in the xy-plane. We assume $\delta_\mathrm{p}L \gg 1$ such that $\lim_{\delta_\mathrm{p}\to 0}\tilde{\delta}^{(3)}(k) = \delta^{(3)}(k)$. The correctly normalized wave function is

$$\psi_{\mathrm{A/B}}(\vec{p}) = \frac{8\sqrt{\pi^5\delta_\mathrm{p}}}{L}\tilde{\delta}^3(\vec{p} - \vec{k}_{\mathrm{A/B}}). \quad (16)$$

Choosing the form of the wave packet in the transverse direction to be given by Eq. (14), we are able to make the wave packet in the position space, shown in Fig. 1(b), to be transversely uniform inside a cross-sectional square of size L^2, confined within the shaded sides, up to $\mathcal{O}(1/(\delta_\mathrm{p}L))$ corrections; we will see in the following that this choice makes an additional, clear interpretation of our results possible. In the z-direction it is unbounded, with the width of the peak characterized by $1/\delta_\mathrm{p}$. It is useful to set $1/(\delta_\mathrm{p}L) \lesssim \delta_\mathrm{p}/|\vec{k}|$, so that taking the plane wave limit amounts to expanding around the single small parameter $\delta_\mathrm{p}/|\vec{k}|$. This requirement is equivalent to $L\lambda \gg 1/\delta_\mathrm{p}{}^2$, where λ is the Compton wavelength for momentum k.

Now let us apply the wave packets to computing the inelastic scattering probability $\mathcal{P}_{\mathrm{inel}}$ in Eq. (9). We can insert an identity matrix, $\sum_\mathrm{f}\int d\Pi_\mathrm{f}|\mathrm{f}\rangle\langle\mathrm{f}|$, between $T^\dagger$ and T in Eq. (10) and get



(a) (b)

FIG. 1. The wave packet (a) in momentum space and (b) in position space, respectively. In the limit $\delta_\mathrm{p}L \gg 1$ and $\delta_\mathrm{p} \to 0$, it approaches the plane wave in z-direction in the position space.

$$\begin{aligned}\mathcal{P}_{\text{inel}} = &\int_{p_1,p_2,q_1,q_2} \psi_{\text{A}}(p_1)\psi_{\text{B}}(p_2)\psi^*_{\text{A}}(q_1)\psi^*_{\text{B}}(q_2) \\ &\times (2\pi)^4\delta^4(q_1+q_2-p_1-p_2) \\ &\times \sum_{\text{f}\notin\{\text{AB}\}} \int d\Pi_{\text{f}}(2\pi)^4\delta^4\left(p_1+p_2-\sum k_{\text{f}}\right)\sum_{i,\bar{i},j,\bar{j}}\Omega_{i\bar{i}}\Omega^*_{j\bar{j}} \\ &\times M_{i\bar{i},f_f}(p_1,p_2;\{k_{\text{f}}\})M^*_{j\bar{j},f_f}(q_1,q_2;\{k_{\text{f}}\}), \end{aligned} \tag{17}$$

where the sum for the final state $|\text{f}\rangle$ include any outcome that is not AB, with $\int d\Pi_{\text{f}}$ being the phase space integral for $|\text{f}\rangle$. In the plane wave limit, the above integral only receives significant contributions when p_1, q_1 are close to k_{A} and p_2, q_2 to k_{B}, thus at the leading order one can pull the last two lines of Eq. (17) out and convert it to the inelastic cross section σ_{inel} for AB → anything but AB

$$\mathcal{P}_{\text{inel}} = I_0(|\vec{k}|)[\sigma_{\text{inel}} + \mathcal{O}(\delta_{\text{p}}/|\vec{k}|)], \tag{18}$$

where

$$\begin{aligned}I_0(|\vec{k}|) = 4|\vec{k}|\sqrt{s}&\int_{p_1,p_2,q_1,q_2} \psi_{\text{A}}(p_1)\psi_{\text{B}}(p_2)\psi^*_{\text{A}}(q_1) \\ &\times \psi^*_{\text{B}}(q_2)(2\pi)^4\delta^4(q_1+q_2-p_1-p_2)\end{aligned} \tag{19}$$

For our choice of wave packets in Eq. (16), we show in the Appendix that

$$I_0(|\vec{k}|) = \frac{1}{L^2}(1+\mathcal{O}(\delta_{\text{p}}/|\vec{k}|)). \tag{20}$$

In other words, $I_0(|\vec{k}|)$ is the inverse of the cross-sectional area of the wave packet at the leading order in $\delta_{\text{p}} \to 0$, in which limit $L \gg 1/\delta_{\text{p}}$. Then

$$\mathcal{P}_{\text{inel}} = \frac{\sigma_{\text{inel}}}{L^2} + \mathcal{O}(\delta_{\text{p}}{}^5/|\vec{k}|^5), \tag{21}$$

which can also be seen from the definition of cross section, $\sigma_{\text{inel}} = N_{\text{inel}}/(d_{\text{A}}l_{\text{A}}d_{\text{B}}l_{\text{B}}A)$, where $d_{\text{A/B}}$ are the number density of uniform beams, $l_{\text{A/B}}$ are the length of these beams, A is the cross sectional area of both beams and N_{inel} is the expectation value for the number of inelastic scattering events. In our case we are scattering two single particles head-to-head, thus $d_{\text{A}}l_{\text{A}}A = d_{\text{B}}l_{\text{B}}A = 1$, $A = L^2$ and $N_{\text{inel}} = \mathcal{P}_{\text{inel}}$. Such a direct interpretation of σ_{inel}/L^2 as $\mathcal{P}_{\text{inel}}$ is made possible by our choice of the initial wave packets to be transversely uniform, as guaranteed by Eq. (14).

Now, Eq. (21) suggests that the inelastic scattering probability is an $\mathcal{O}(|\vec{k}|^{-2}/L^2) = \mathcal{O}(\delta_{\text{p}}{}^4/|\vec{k}|^4)$ small quantity. The physical picture here is that the wave packet is traveling at a high speed in the z-direction, which is Lorentz-contracted, so in the CM frame the two wave packets look like two thin pancakes whose linear size in the transverse direction is much bigger than that in the longitudinal direction. In the strict plane wave limit, when $L \to \infty$, the two incoming wave packets are so spread out in the transverse direction in position space that the probability of them interacting with each other is very small. This consideration means that we can expand $\mathcal{P}_{\text{inel}}$ in Eq. (9) as a small parameter.

Similarly, we can express the total scattering probability $\mathcal{P}_{\text{tot}}$ and the elastic scattering probability $\mathcal{P}_{\text{el}}$ in terms of the total cross section σ_{tot} and the elastic cross section σ_{el}

$$\mathcal{P}_{\text{tot}} = \langle\text{in}|T^\dagger T|\text{in}\rangle = \frac{\sigma_{\text{tot}}}{L^2} + \mathcal{O}(\delta_{\text{p}}{}^5/|\vec{k}|^5), \tag{22}$$

$$\mathcal{P}_{\text{el}} = \langle\text{in}|T^\dagger P_{\text{AB}} T|\text{in}\rangle = \frac{\sigma_{\text{el}}}{L^2} + \mathcal{O}(\delta_{\text{p}}{}^5/|\vec{k}|^5). \tag{23}$$

Clearly, $\mathcal{P}_{\text{el}}$ as well as $\mathcal{P}_{\text{tot}} = \mathcal{P}_{\text{el}} + \mathcal{P}_{\text{inel}}$ are also suppressed by $1/L^2$ in the plane wave limit, $L \to \infty$; the overwhelmingly probable outcome, associated with the probability $1 - \mathcal{P}_{\text{tot}}$, is when no scattering happens.

Now we are ready to compute the final state entanglement entropy using the reduced density matrix computed from ρ_{f} in Eq. (9). In the $\delta_{\text{p}} \to 0$ limit, we have

$$\begin{aligned}\mathcal{E}_2^{\text{f}} &= 1 - \text{Tr}(\Omega^\dagger\Omega)^2(1+2\mathcal{P}_{\text{inel}}) \\ &\quad + \frac{I_0(|\vec{k}|)}{|\vec{k}|\sqrt{s}}\{\text{Im}\,\text{Tr}[\Omega^\dagger\Omega\Omega^\dagger M^{\text{F}}(\Omega)] + \mathcal{O}(\delta_{\text{p}}/|\vec{k}|)\} \\ &= 1 - \text{Tr}(\Omega^\dagger\Omega)^2 + \frac{I_0(|\vec{k}|)}{|\vec{k}|\sqrt{s}}\{\text{Im}\,\text{Tr}[\Omega^\dagger\Omega\Omega^\dagger M^{\text{F}}(\Omega)] \\ &\quad - 2|\vec{k}|\sqrt{s}\sigma_{\text{inel}}\text{Tr}(\Omega^\dagger\Omega)^2 + \mathcal{O}(\delta_{\text{p}}/|\vec{k}|)\},\end{aligned} \tag{24}$$

where $[M^{\text{F}}(\Omega)]_{i\bar{i}} = \sum_{j,\bar{j}}\Omega_{j\bar{j}}M^{\text{F}}_{j\bar{j},i\bar{i}}$ and

$$M^{\text{F}}_{i\bar{i},j\bar{j}} = M_{i\bar{i},j\bar{j}}(k_{\text{A}},k_{\text{B}};k_{\text{A}},k_{\text{B}}) \tag{25}$$

is the elastic forward scattering amplitude for $\text{A}(i)\text{B}(\bar{i}) \to \text{A}(j)\text{B}(\bar{j})$. We demonstrate in the Appendix that terms dropped in Eq. (24) are higher orders in $\delta_{\text{p}}/|\vec{k}|$. Now consider the case where the initial states are not entangled; specifically, $\Omega_{i\bar{i}} = \omega_i\omega'_{\bar{i}}$, where

$$\omega = (1,0,0,\ldots), \qquad \omega' = (1,0,0,\ldots). \tag{26}$$

Then

$$\begin{aligned}\mathcal{E}_2^{\mathrm{f}} &= I_0(|\vec{k}|)\frac{\mathrm{Im}(M^{\mathrm{F}}_{1\bar{1},1\bar{1}}) - 2|\vec{k}|\sqrt{s}\sigma_{\mathrm{inel}} + \mathcal{O}(\delta_{\mathrm{p}}/|\vec{k}|)}{|\vec{k}|\sqrt{s}} \\ &= 2I_0(|\vec{k}|)[\sigma_{\mathrm{tot}} - \sigma_{\mathrm{inel}} + \mathcal{O}(\delta_{\mathrm{p}}/|\vec{k}|)] \\ &= 2\frac{\sigma_{\mathrm{el}}}{L^2} + \mathcal{O}(\delta_{\mathrm{p}}{}^5/|\vec{k}|^5), \end{aligned} \tag{27}$$

where we have used the optical theorem to rewrite $\mathrm{Im}(M^{\mathrm{F}}_{1\bar{1},1\bar{1}})$ in terms of the total cross section σ_{tot}. Comparing with Eq. (23), we observe that the entropy $\mathcal{E}_2^{\mathrm{f}}$ is just $2\mathcal{P}_{\mathrm{el}}$ (at the leading order); such a relation is actually insensitive to the details of the wave packets, as long as we stay in the plane wave limit [45].

The factor of 2 in Eq. (27) clearly comes from the fact that the linear entropy is computed from the second power of the density matrix. Then the computation can be directly generalized to, e.g., the n-Tsallis entropy $\mathcal{E}_n(\rho) = (1 - \mathrm{Tr}\rho^n)/(n-1)$ where $n \geq 2$ is an integer

$$\mathcal{E}_n(\rho_{\mathrm{A}}^{\mathrm{f}}) = \frac{n}{n-1}\frac{\sigma_{\mathrm{el}}}{L^2} + \mathcal{O}(\delta_{\mathrm{p}}{}^5/|\vec{k}|^5). \tag{28}$$

We have been focusing on pure initial states; now we discuss briefly the case of mixed initial states. Let us consider unentangled states characterized by the following density matrix:

$$\rho^{\mathrm{i}} = \rho^{\mathrm{i}}_{f_{\mathrm{A}}} \otimes \rho^{\mathrm{i}}_{p_{\mathrm{A}}} \otimes \rho^{\mathrm{i}}_{f_{\mathrm{B}}} \otimes \rho^{\mathrm{i}}_{p_{\mathrm{B}}}, \tag{29}$$

where $\rho^{\mathrm{i}}_{f_{\mathrm{A/B}}}$ and $\rho^{\mathrm{i}}_{p_{\mathrm{A/B}}}$ are density matrices for the discrete and momentum subsystems of the two initial particles, respectively. Let us consider mixed states in discrete quantum numbers, such as the spin

$$\rho^{\mathrm{i}}_{f_{\mathrm{A}}} = \sum_i \mathbf{f}_i |i\rangle\langle i|, \qquad \rho^{\mathrm{i}}_{f_{\mathrm{B}}} = \sum_{\bar{i}} \bar{\mathbf{f}}_{\bar{i}} |\bar{i}\rangle\langle\bar{i}|, \tag{30}$$

where $\mathbf{f}_i \geq 0$, $\bar{\mathbf{f}}_{\bar{i}} \geq 0$, and $\sum_i \mathbf{f}_i = \sum_{\bar{i}} \bar{\mathbf{f}}_{\bar{i}} = 1$. Suppose the momentum states are still pure: $\rho^{\mathrm{i}}_{p_{\mathrm{A}}} = |\psi_{\mathrm{A}}\rangle\langle\psi_{\mathrm{A}}|$, $\rho^{\mathrm{i}}_{p_{\mathrm{B}}} = |\psi_{\mathrm{B}}\rangle\langle\psi_{\mathrm{B}}|$, the subsystem entropy of A is

$$\mathcal{E}^{\mathrm{f}}_{2,\mathrm{A}} = 2I_0(|\vec{k}|)\sum_{i,\bar{i}} \mathbf{f}_i^2 \bar{\mathbf{f}}_{\bar{i}}[(\sigma_{\mathrm{el}})_{i\bar{i}} + \mathcal{O}(\delta_{\mathrm{p}}/|\vec{k}|)]. \tag{31}$$

where $(\sigma_{\mathrm{el}})_{i\bar{i}}$ is the total elastic cross section where the initial discrete quantum numbers are i and $\bar{i}$. On the other hand, the elastic scattering probability for colliding mixed states is

$$\begin{aligned}\mathcal{P}_{\mathrm{el}} &= \frac{1}{L^2}\sum_{i,\bar{i}} \mathbf{f}_i \bar{\mathbf{f}}_{\bar{i}} (\sigma_{\mathrm{el}})_{i\bar{i}} + \mathcal{O}(\delta_{\mathrm{p}}{}^5/|\vec{k}|^5) \\ &\equiv \frac{\overline{\sigma_{\mathrm{el}}}}{L^2} + \mathcal{O}(\delta_{\mathrm{p}}{}^5/|\vec{k}|^5), \end{aligned} \tag{32}$$

from which we see that, when the density matrix satisfies $(\rho^{\mathrm{i}}_{f_{\mathrm{A}}})^2 \propto \rho^{\mathrm{i}}_{f_{\mathrm{A}}}$, the subsystem entropy can again be interpreted as a cross section,

$$\mathcal{E}^{\mathrm{f}}_{2,\mathrm{A}} = \frac{2}{n_{\mathrm{A}}}\frac{\overline{\sigma_{\mathrm{el}}}}{L^2} + \mathcal{O}(\delta_{\mathrm{p}}{}^5/|\vec{k}|^5). \tag{33}$$

The condition $(\rho^{\mathrm{i}}_{f_{\mathrm{A}}})^2 \propto \rho^{\mathrm{i}}_{f_{\mathrm{A}}}$ is satisfied when $\mathbf{f}_i$ is zero or the nonzero entries are all given by the same constant value, $\mathbf{f}_i = 1/n_{\mathrm{A}}$, where n_{A} counts the number of nonzero entries in $\mathbf{f}_i$. The same criteria was proposed in Ref. [30], which ensures the subsystem entropy to not decrease upon time evolution. A realistic scenario fulfilling this criteria is the unpolarized scattering, in which case $\overline{\sigma_{\mathrm{el}}}$ is simply the unpolarized elastic cross section.

IV. DISCUSSIONS AND CONCLUSION

Having shown that the entanglement entropy in 2-to-2 particle scattering is given by the elastic cross section in units of the transverse size of the wave packets, several interesting observations follow.

Since the cross section is the effective area of collision when particles collide, we have essentially discovered an area law for the entanglement entropy in a two-body system. In other examples of area laws for the entropy, the area is often given by the boundary of a spatial region [7–12]. In the present case it is not clear if a similar interpretation could be given. We speculate that, perhaps, the cross section could be viewed as the area of the spacelike surface connecting the past and the future light cones of the two particle scattering. On the other hand, cross section measures the probability that a quantum mechanical process would happen, in accordance with the thermodynamic and information-theoretic interpretations of the entropy as probability.

Scaling properties of cross sections, in particular the total and elastic cross sections, with respect to the collision energy $\sqrt{s}$, have been studied intensively for high energy collisions many decades ago. Froissart and Martin showed that the total cross section is bounded from the above by $\log^2 s$ [42,43] in the S-matrix formalism. Cheng and Wu argued very generally in quantum field theory that both the total and the elastic cross sections rise when $\sqrt{s}$ grows [38,39,46], which is a somewhat surprising result because cross sections for elementary processes at a fixed order in perturbation always decrease with $\sqrt{s}$. However, in a high energy scattering process there is abundant energy to create radiations and splittings of soft and collinear particles, which necessitates the resummation of higher order diagrams, akin to the soft photon resummation in quantum electrodynamics. These effects enhance the cross section to increase with the energy. In fact, Ref. [39] proposed a "black disk" picture of a total absorption area of radius $\log s$ for high energy collisions, which is similar in spirit to the wave packet adopted in this work. The growth of the cross section with respect to $\sqrt{s}$ has been measured over several orders of magnitude in $\sqrt{s}$ in pp and $p\overline{p}$ collisions at the Tevatron and the Large Hadron Collider [40,41,47,48].[1] Even though many of the studies are based on hadron-hadron collisions, it has become clear recently in the study of a multi-TeV muon collider that radiations and splittings are also important in high energy collisions of leptons [49–51]. Therefore, similar considerations for cross sections for lepton-lepton collisions should apply.

These observations lead to a "second law" of entanglement entropy with respect to the CM energy $\sqrt{s}$, in which the area (cross section) grows with the energy and so does the entanglement entropy. The Froissart bound now constrains the growth of the entropy with respect to energy. It would be interesting to explore whether there is a holographic understanding of the relation between the entanglement entropy and the cross section.

Last but not least, certain other cross sections, for instance the deep inelastic scatteringlike cross section, can also be related to entanglement entropies corresponding to different ways to construct a bipartite system in 2-to-2 scatterings [45]. It will be interesting to see if all cross sections can be interpreted this way.

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APPENDIX: DETAILS OF IMPLEMENTING THE WAVE-PACKET FORMALISM

Here we detail the computations involving wave packets. First, let us see that in general, how the universal factor $I_0(|\vec{k}|)$ appears at the leading order in the relevant computations. For example, consider $\mathcal{P}_{\rm inel}$ in Eq. (10), which is given by

$$\mathcal{P}_{\rm inel} = \int_{p_1,p_2,q_1,q_2} \psi_{\rm A}(p_1)\psi_{\rm B}(p_2)\psi^*_{\rm A}(q_1)\psi^*_{\rm B}(q_2) \times (2\pi)^4\delta^4(q_1+q_2-p_1-p_2)\mathcal{F}_{\rm inel}(p_1,p_2,q_1,q_2), \tag{A1}$$

with

$$(2\pi)^4\delta^4(q_1+q_2-p_1-p_2)\mathcal{F}_{\rm inel}(p_1,p_2,q_1,q_2) = \sum_{i,\bar{i},j,\bar{j}} \Omega_{i\bar{i}}\Omega^*_{j\bar{j}}\langle q_1,j;q_2,\bar{j}|T^\dagger(1-P_{\rm AB})T|p_1,i;p_2,\bar{i}\rangle, \tag{A2}$$

which can be expressed in terms of inelastic amplitudes using Eq. (8). Now, without specifying the detail of the wave packets, we require that in momentum space the wave functions are bounded within an $\mathcal{O}(\delta_{\rm p})$ radius, thus

$$\mathcal{F}_{\rm inel}(p_1,p_2,q_1,q_2) = \mathcal{F}_{\rm inel}(k_{\rm A},k_{\rm B},k_{\rm A},k_{\rm B})[1+\mathcal{O}(\delta_{\rm p}/|\vec{k}|)] \tag{A3}$$

can be pulled out of the integral in Eq. (A1)

$$\begin{aligned}\mathcal{P}_{\rm inel} &= \mathcal{F}_{\rm inel}(k_{\rm A},k_{\rm B},k_{\rm A},k_{\rm B})[1+\mathcal{O}(\delta_{\rm p}/|\vec{k}|)] \\ &\quad\times \int_{p_1,p_2,q_1,q_2} \psi_{\rm A}(p_1)\psi_{\rm B}(p_2)\psi^*_{\rm A}(q_1)\psi^*_{\rm B}(q_2) \\ &\quad\times (2\pi)^4\delta^4(q_1+q_2-p_1-p_2) \\ &= I_0(|\vec{k}|)[\sigma_{\rm inel}+\mathcal{O}(\delta_{\rm p}/|\vec{k}|)],\end{aligned} \tag{A4}$$

with

$$\sigma_{\rm inel} = \frac{1}{4|\vec{k}|\sqrt{s}} \sum_{\rm f\notin\{AB\}} \int d\Pi_{\rm f} \left|\sum_{i,\bar{i}} \Omega_{i\bar{i}} M_{i\bar{i},f_{\rm f}}(k_{\rm A},k_{\rm B};\{k_{\rm f}\})\right|^2, \tag{A5}$$

and we arrive at Eq. (18). In general, we see in computations other wave packet integrals similar to $I_0(|\vec{k}|)$. However, we will argue that all other integrals encountered are higher in $\delta_{\rm p}$ and can be neglected. (See Ref. [37] for an argument in a similar spirit.)

Now let us consider the case of uniform wave functions in the transverse directions of the position space. Specifically, we define $\tilde{\delta}^3(\vec{p})$ as in Eq. (13) and the wave packet as in Eq. (16). The wave function is chosen in the following spirit: We first bound the wave function strictly inside a cube of size $(2\delta_{\rm p})^3$ in the momentum space, as shown in Fig. 1(a). Next, we treat the transverse directions separately compared to the z direction, as in Fig. 1(b): We make the wave function in the z direction uniform within the $2\delta_{\rm p}$ sized window in the momentum space. On the other hand, we require that the wave function is bounded and uniform within a square of size L^2 in the x- and y-direction of the position space, up to $\mathcal{O}(1/(\delta_{\rm p}L))$ corrections; we choose $L \gg 1/\delta_{\rm p}$, with $1/\delta_{\rm p}$ characterizing the width of the peak along the z direction in the position space. We also set $1/(\delta_{\rm p}L) \lesssim \delta_{\rm p}/|\vec{k}|$ so there is a single small parameter $\delta_{\rm p}/|\vec{k}|$ to expand.

[1]See, for example, Fig. 94 in Ref. [41], as well as Fig. 20.7 and Sec. 53.5 of [48].

Let us first consider the general integral below

$$\tilde{I}_0 = \int d^3\vec{p}_1 d^3\vec{p}_2 d^3\vec{q}_1 d^3\vec{q}_2 \delta^4(q_1 + q_2 - p_1 - p_2) \times \mathcal{F}_0(p_1, p_2, q_1, q_2)\tilde{\delta}^3(\vec{p}_1 - \vec{k})\tilde{\delta}^3(\vec{p}_2 + \vec{k}) \times \tilde{\delta}^3(\vec{q}_1 - \vec{k})\tilde{\delta}^3(\vec{q}_2 + \vec{k}). \tag{A6}$$

As previously argued, we can pull $\mathcal{F}_0$ out of the integral at the leading order (assuming $\mathcal{F}_0(k_\mathrm{A}, k_\mathrm{B}, k_\mathrm{A}, k_\mathrm{B})$ to be nonvanishing)

$$\tilde{I}_0 = \mathcal{F}_0(k_\mathrm{A}, k_\mathrm{B}, k_\mathrm{A}, k_\mathrm{B})[1 + \mathcal{O}(\delta_\mathrm{p}/|\vec{k}|)] \times \int d^3\vec{p}_1 d^3\vec{p}_2 d^3\vec{q}_1 d^3\vec{q}_2 \tilde{\delta}^3(\vec{p}_1 - \vec{k})\tilde{\delta}^3(\vec{p}_2 + \vec{k}) \times \tilde{\delta}^3(\vec{q}_1 - \vec{k})\tilde{\delta}^3(\vec{q}_2 + \vec{k})\delta^4(q_1 + q_2 - p_1 - p_2). \tag{A7}$$

We can first integrate out the spatial δ-functions along the $\vec{p}_1$ direction, and then integrate out the energy δ-function in the $(p_2)_z$ direction

$$\delta(E_{p_1} + E_{p_2} - E_{p_3} - E_{p_4}) \to \left(\frac{E_{k_\mathrm{A}} E_{k_\mathrm{B}}}{|\vec{k}|\sqrt{s}} + \mathcal{O}(\delta_\mathrm{p}/|\vec{k}|)\right)\delta((p_2)_z - r_0), \tag{A8}$$

where the factor in front of the δ-function on the right hand side of the above can be pulled out of the integral like $\mathcal{F}_0$, and r_0 is the root of $(p_2)_z$ in

$$\sqrt{(\vec{q}_1 + \vec{q}_2 - \vec{p}_2)^2 + m_\mathrm{A}^2} + \sqrt{\vec{p}_2^2 + m_\mathrm{B}^2} - \sqrt{\vec{q}_1^2 + m_\mathrm{A}^2} - \sqrt{\vec{q}_2^2 + m_\mathrm{B}^2} = 0. \tag{A9}$$

Then the integration we need in the $(p_2)_z$ direction is

$$\int d(p_2)_z \delta((p_2)_z - r_0)\tilde{\delta}_z((p_2)_z + |\vec{k}|) = \frac{\Theta(r_0 + |\vec{k}| + \delta_\mathrm{p}) - \Theta(r_0 + |\vec{k}| - \delta_\mathrm{p})}{2\delta_\mathrm{p}}, \tag{A10}$$

which is nonvanishing within a $2\delta_p$ sized window if we vary r_0 in the above. Therefore, to get leading order effects in δ_p, we need to solve r_0 up to linear order in δ_p, which turns out to be $r_0 = (q_2)_z + \mathcal{O}(\delta_\mathrm{p}{}^2/|\vec{k}|^2)$.[2] Then one sees that the integration of the different spatial dimensions actually decouples.

In the transverse directions, notice that if we take $1/(\delta_\mathrm{p} L) \to 0$ in Eq. (14), we have

$$\int dk\, \tilde{\delta}_0(k) \to \int_{-\infty}^{\infty} d(kL)\frac{1}{2\pi}\operatorname{sinc}\frac{kL}{2}. \tag{A11}$$

Therefore, as we are assuming $\delta_\mathrm{p} \gg 1/L$, at the leading order the Θ-functions as well as the normalization factor related to $\mathrm{Si}(x)$ in Eq. (14) can be effectively omitted from our computation

$$\tilde{\delta}_0(k) = \frac{\pi}{2}\frac{\Theta(k + \delta_\mathrm{p}) - \Theta(k - \delta_\mathrm{p})}{\mathrm{Si}(\delta_\mathrm{p} L/2)}\frac{L}{2\pi}\operatorname{sinc}\frac{kL}{2} \sim \frac{L}{2\pi}\operatorname{sinc}\frac{kL}{2}. \tag{A12}$$

The above approximation is valid up to $\mathcal{O}(1/(\delta_\mathrm{p} L))$, and as we would like an expansion of small $\delta_\mathrm{p}/|\vec{k}|$, it is then desirable to have $1/(\delta_\mathrm{p} L) \lesssim \delta_\mathrm{p}/|\vec{k}|$. The useful integral in the transverse directions is then

$$\int d(p_1)_x \int d(p_2)_x \int d(q_1)_x \int d(q_2)_x \times \delta[(\vec{p}_1 + \vec{p}_2 - \vec{q}_1 - \vec{q}_2)_x]\operatorname{sinc}\frac{L(p_1)_x}{2}\operatorname{sinc}\frac{L(p_2)_x}{2} \times \operatorname{sinc}\frac{L(q_1)_x}{2}\operatorname{sinc}\frac{L(q_2)_x}{2} = \left(\frac{2\pi}{L}\right)^3. \tag{A13}$$

In the z direction we need

$$\int_{k-\delta_\mathrm{p}}^{k+\delta_\mathrm{p}} d(p_1)_z \int_{-k-\delta_\mathrm{p}}^{k+\delta_\mathrm{p}} d(p_2)_z \int_{k-\delta_\mathrm{p}}^{k+\delta_\mathrm{p}} d(q_1)_z \times \int_{-k-\delta_\mathrm{p}}^{-k+\delta_\mathrm{p}} d(q_2)_z \delta[(\vec{p}_1 + \vec{p}_2 - \vec{q}_1 - \vec{q}_2)_z]\delta[(\vec{p}_2 - \vec{q}_2)_z] = 4\delta_\mathrm{p}{}^2. \tag{A14}$$

Then

$$\tilde{I}_0 = \frac{1}{16\pi^2}\frac{E_{k_\mathrm{A}} E_{k_\mathrm{B}}}{|\vec{k}|\sqrt{s}}\frac{L^2}{\delta_\mathrm{p}{}^2}\mathcal{F}_0(k_\mathrm{A}, k_\mathrm{B}, k_\mathrm{A}, k_\mathrm{B}) \times [1 + \mathcal{O}(\delta_\mathrm{p}/|\vec{k}|)]. \tag{A15}$$

Now we can compute $I_0(|\vec{k}|)$ from $\tilde{I}_0$. Let us plug in the explicit form of the wave packet from Eq. (16) into $I_0(|\vec{k}|)$ in Eq. (19), and compare with $\tilde{I}_0$ in Eq. (A6), we see $I_0(|\vec{k}|)$ is given by

[2]One can consider characterizing the size of the wave packet in the p_z direction by some δ_z, which is independent of δ_p. Then the solution r_0 needs to be computed up to $\mathcal{O}(\delta_z/|\vec{k}|)$, but it will admit an expansion of $\max(\delta_\mathrm{p}, \delta_z)$. Therefore, we need to require $\delta_z \gtrsim \delta_\mathrm{p}$ for our computation to still hold. In other words, $1/\delta_\mathrm{p}$ is the largest possible length scale which characterizes the spread of the wave packet in the direction of the momentum. As $1/\delta_\mathrm{p} \ll L$, the wave packet is much more compressed in the z direction compared to the transverse directions.

$$\mathcal{F}_0(k_\mathrm{A},k_\mathrm{B},k_\mathrm{A},k_\mathrm{B}) \to \frac{2^{12}\pi^{10}\delta_\mathrm{p}{}^2 4|\vec{k}|\sqrt{s}}{(2\pi)^8 2E_{k_\mathrm{A}}2E_{k_\mathrm{B}}L^4}. \tag{A16}$$

In the end we arrive at

$$I_0(|\vec{k}|) = \frac{1}{L^2}(1+\mathcal{O}(\delta_\mathrm{p}/|\vec{k}|)). \tag{A17}$$

We see clearly here that at the leading order, $I_0(|\vec{k}|)$ is exactly the inverse of the cross-sectional area of the wave packet.

Now let us see concretely other types integrals involving the overlap of wave packets which appear in the entanglement entropy calculation. We will show that they are higher order in $\delta_\mathrm{p}/|\vec{k}|$ and can be neglected. For example, $\mathcal{E}_2^\mathrm{f}$ in Eq. (24) contains the following terms:

$$2\,\mathrm{Re}\,\mathrm{tr}(\Omega^\dagger\tilde{M}(\Omega))^2 - 2\,\mathrm{Tr}[\Omega^\dagger\Omega\tilde{M}_1^{(2)}(\Omega)], \tag{A18}$$

where

$$\begin{aligned}[\tilde{M}(\Omega)]_{\bar{i}i} &= \sum_{j,\bar{j}}\Omega_{j\bar{j}}\int_{p_1,p_2,q_1,q_2}\psi_\mathrm{A}(p_1)\psi_\mathrm{B}(p_2)\psi_\mathrm{A}^*(q_1)\psi_\mathrm{B}^*(q_2)(2\pi)^4\delta^4(q_1+q_2-p_1-p_2)M_{j\bar{j},i\bar{i}}(p_1,p_2;q_1,q_2)\\ &= \frac{1}{4|\vec{k}|\sqrt{s}L^2}\left[\sum_{j,\bar{j}}\Omega_{j\bar{j}}M^\mathrm{F}_{j\bar{j},i\bar{i}} + \mathcal{O}(\delta_\mathrm{p}/|\vec{k}|)\right],\end{aligned} \tag{A19}$$

$$\begin{aligned}[\tilde{M}_1^{(2)}(\Omega)]_{\bar{i}_1\bar{i}_2} = \sum_{i,j,\bar{j},k,\bar{k}}&\int_{p_1,p_2,p_3,p_4,q_1,q_2,q_4}\frac{1}{\sqrt{2E_{p_3}}}\Omega^*_{j\bar{j}}\Omega_{k\bar{k}}\psi_\mathrm{A}^*(p_1)\psi_\mathrm{B}^*(p_2)\psi_\mathrm{B}(p_4)\psi_\mathrm{A}(q_1)\psi_\mathrm{B}(q_2)\psi_\mathrm{B}^*(q_4)\\ &\times(2\pi)^8\delta^4(p_3+p_4-p_1-p_2)\delta^4(q_3+q_4-q_1-q_2)\\ &\times M^*_{j\bar{j},i\bar{i}_1}(p_1,p_2;p_3,p_4)M_{k\bar{k},i\bar{i}_2}(q_1,q_2;q_3,q_4)|_{\vec{q}_3=\vec{q}_1+\vec{q}_2-\vec{q}_4}.\end{aligned} \tag{A20}$$

Then the first term in Eq. (A18) is $\mathcal{O}(1/(|\vec{k}|L)^4)\lesssim\mathcal{O}(\delta_\mathrm{p}{}^8/|\vec{k}|^8)$ and can be dropped. (Recall we demand $1/(\delta_\mathrm{p}L)\lesssim\delta_\mathrm{p}/|\vec{k}|$.) The second term in Eq. (A18) involves the following integral:

$$\begin{aligned}\tilde{I}_1 = \int & d^3\vec{p}_1d^3\vec{p}_2d^3\vec{p}_3d^3\vec{p}_4d^3\vec{q}_1d^3\vec{q}_2d^3\vec{q}_4\delta^4(q_1+q_2-p_3-q_4)\delta^4(p_1+p_2-p_3-p_4)\\ &\times\mathcal{F}_1(p_1,p_2,p_3,p_4,q_1,q_2,q_4)\tilde{\delta}^3(\vec{p}_1-\vec{k})\tilde{\delta}^3(\vec{p}_2+\vec{k})\tilde{\delta}^3(\vec{p}_4+\vec{k})\tilde{\delta}^3(\vec{q}_1-\vec{k})\tilde{\delta}^3(\vec{q}_2+\vec{k})\tilde{\delta}^3(\vec{q}_4+\vec{k}).\end{aligned} \tag{A21}$$

Here, p_3 is constrained by the on-shell condition of type-A particles, and the integrand in the above will constrain it into a range of radius $\mathcal{O}(\delta_\mathrm{p})$ around k_A. Similar to $\tilde{I}_0$, we pull out $\mathcal{F}_1$ first. Then we integrate out $\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$ in the direction of $\vec{p}_3$, and the other spatial δ-function in the direction of $\vec{q}_1$. Next, integrating out $\delta(E_{q_1}+E_{q_2}-E_{p_3}-E_{q_4})$ in the $(q_2)_z$ direction, we convert it to $\delta[(\vec{q}_2-\vec{q}_4)_z]$; integrating out $\delta(E_{p_1}+E_{p_2}-E_{p_3}-E_{p_4})$ in the $(p_4)_z$ direction, we convert it to $\delta[(\vec{p}_4-\vec{p}_2)_z]$. Again, the integrals in the spatial dimensions decouple, and upon integration we have

$$\begin{aligned}\tilde{I}_1 &= \mathcal{F}_1(k_\mathrm{A},k_\mathrm{B},k_\mathrm{A},k_\mathrm{B},k_\mathrm{A},k_\mathrm{B},k_\mathrm{B})\\ &\quad\times[1+\mathcal{O}(\delta_\mathrm{p}/|\vec{k}|)]\frac{1}{32\pi^2}\left(\frac{E_{k_\mathrm{A}}E_{k_\mathrm{B}}}{|\vec{k}|\sqrt{s}}\right)^2\frac{L^2}{\delta_\mathrm{p}{}^3}.\end{aligned} \tag{A22}$$

Therefore,

$$\begin{aligned}[\tilde{M}_1^{(2)}(\Omega)]_{\bar{i}_1\bar{i}_2} &= \left(\frac{1}{4|\vec{k}|\sqrt{s}L^2}\right)^2\\ &\quad\times\left[\sum_{i,j,\bar{j},k,\bar{k}}\Omega^*_{j\bar{j}}\Omega_{k\bar{k}}M^{\mathrm{F}*}_{j\bar{j},i\bar{i}_1}M^\mathrm{F}_{k\bar{k},i\bar{i}_2}+\mathcal{O}(\delta_\mathrm{p}/|\vec{k}|)\right],\end{aligned} \tag{A23}$$

thus the second term in Eq. (A18) is also $\mathcal{O}(1/(|\vec{k}|L)^4)$ and can be dropped. Notice that we are dropping these higher order terms because we are taking the plane wave limit $\delta_{\mathrm{p}} \to 0$, instead of any weak coupling limit as in, e.g., Ref. [36]. Therefore, our computation is valid to all orders in the coupling strength.

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