

Yang-Mills interaction from boundary vector model

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We construct a non-Abelian spin-1 gauge theory with a cubic interaction in AdS_4 from the exact renormalization group (ERG) flow of a conformal field theory (CFT) in three dimensions. The latter is the $USp(2N)$ singlet sector of the free field theory of $2N$ massless complex scalars. The quadratic and cubic terms in the bulk action are those of a gauge fixed version of a (local) gauge invariant action. By construction, this bulk action is the evolution operator for the ERG equation of the boundary theory and thus is guaranteed to reproduce the correct boundary correlators using the usual AdS/CFT prescription. This work expands on the program initiated by one of the authors [A holographic form for Wilson's RG, *Nucl. Phys. B* **924**, 603 (2017)].

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I. INTRODUCTION

The idea of holography is one of the most interesting ideas in theoretical physics and it suggests that quantum gravity can be understood in terms of a lower dimensional nongravitational field theory [1,2]. A very concrete realization of this idea is encoded in the AdS/CFT conjecture, for which there is ample evidence, according to which gravitational theories in anti-de Sitter (AdS) space are dual to conformal field theories (CFTs) in the boundary of AdS [3–6]. The radial direction of the bulk in AdS/CFT corresponds to the scale of the boundary theory. Thus, the bulk radial evolution performs renormalization group (RG) on the boundary theory.¹ This idea is referred to as holographic renormalization group [7–19]. This also means the bulk dynamics is encoded in the RG of the boundary theory, and perhaps the bulk theory, and more importantly its gravitational dynamics, can be obtained from the RG of the boundary theory [20–29].

In [30], a prescription has been laid out to obtain a $D + 1$ dimensional dual bulk action from a D -dimensional

boundary theory using Polchinski's exact renormalization group (ERG) equation. In subsequent works, specializing to the $O(N)$ model [31,32], the bulk scalar action up to cubic order, and the leading order in $\frac{1}{\sqrt{N}}$ action for the spin-1 and spin-2 fields [33] has been obtained. In [34], the cubic coupling between the graviton and two scalars also has been obtained.

The bulk dual of the free $O(N)$ model is known to be the minimal bosonic Vasiliev theory in AdS_4 [35]. This bulk theory consists of fields of all even spins, and obey a higher spin algebra. These are dual to the conserved singlet currents of the $O(N)$ model, of which also there is one corresponding to each even spin, and they obey the same higher spin algebra. Vasiliev's theory is defined in terms of highly nonlinear equations of motion with a high degree of redundancy in terms of auxiliary fields [36]. Thus, obtaining an action for the theory is of use.

There are several works attempting to define the Vasiliev theory through the 3D free $O(N)$ model [25,26,37–47]. The higher spins of the theory are all characterized by gauge freedom, which must play a role in their interactions. Therefore, in this paper we work with the simplest gauge field in the bulk, setting the rest to zero, and obtain its action up to cubic interaction from the ERG of the boundary theory's spin-1 current.

The $O(N)$ model contains only even spin currents. To look at odd spins, one must work with the $U(N)$ model. The three-point function of the spin-1 current in the $U(N)$ model is zero, and, hence, the bulk theory does not have a cubic interaction for this theory. We look at the simplest non-Abelian case. When the boundary theory is the free

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¹The boundary theory is a CFT, but the boundary conditions can turn on deformations of the CFT action and then the boundary action will flow.

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$U(2N)$ vector model, and we consider the currents which are singlets of $USp(2N)$, the theory has conserved currents of all spins, one per even spin, and three per odd spin. The odd spin triplets are each in the adjoint representation of $\mathfrak{su}(2)$.

A. The prescription

We obtain the bulk AdS theory from the boundary theory in the following steps:

- (1) Introduce the singlet current of interest as an auxiliary field σ_s in the action. (The subscript s stands for all the indices the current may have.) This will need us to introduce another auxiliary field χ^s which is a Lagrange multiplier enforcing the condition that σ_s is the required current.
- (2) Integrate out the low energy modes of the fundamental vector field ϕ^I and obtain a low energy action depending on ϕ^I , σ_s , χ^s , and any sources. Set the high energy field $\phi^I_h = 0$ and integrate out χ^s as well. The resulting action would be a low energy action for the σ_s current. It gives the correlators for the current at low energies. We derive the ERG equation for the current from the Polchinski's ERG equation for the fundamental field.
- (3) The ERG equation is analogous to the heat equation and we can obtain the ERG evolution operator as a path integral. The fields in this path integral depend also on t (the scaling direction), $t = -\log(\Lambda/\Lambda_b)$, where Λ_b is the bare momentum cutoff and Λ is a generic cutoff. Therefore this is now a $D+1$ dimensional path integral.
- (4) The $D+1$ dimensional action from this path integral is nonlocal. We change the t to the Poincaré z coordinate by defining $z = \frac{1}{\Lambda_b} e^t = \frac{1}{\Lambda}$. We further redefine the field σ_s which now depends on the bulk radial direction z along with the boundary directions because of its scale dependence, $\sigma_s = f y_s$. f is a smooth field redefinition which we define to cancel the nonlocal factor in the kinetic term. We further require f to obey a differential equation that makes the quadratic term for y_s the standard action in AdS_{D+1} for a scalar field of mass $m^2 = D(D-\Delta)$. This set of scalar fields can be identified with a gauge-fixed spin- n field on defining $y_s = z^n a_s$. This quadratic action is obtained as the quadratic part of a gauge invariant kinetic term after gauge fixing.
- (5) The $D+1$ dimensional path integral action contains cubic and higher order terms in σ_s . After the mapping $\sigma_s = f y_s$, they become polynomials in y_s . The question addressed in this paper then is whether the cubic terms (after the mapping) are local and also part of a gauge invariant action. It is shown that it is part of a local gauge invariant action.

- (6) The differential equation imposed on f is solved by modified Bessel functions. This determines the regulating function in the propagator for the fundamental field we use in the ERG. This complicated propagator shows up in the cubic and higher order terms in loop integrals. For the cubic term, this loop integral is difficult to tackle directly, so we replace the complicated regulator with a simpler regulator and perform the integral. This modification does not affect the boundary correlators computed by the bulk action. Remarkably, the result of the simpler integral cancels the Bessel functions appearing in the cubic term except for some $|p|$ -like factors, where p is the momentum of one of the fields.
- (7) We systematically exchange these $|p|$ factors for z derivatives and p^2 factors using the classical form of the field a_s to obtain an action that we show to be a gauge-fixed action made from the gauge invariant field strength tensor F_{MN} .

Thus, our final outcome is an action with a Yang-Mills term as well as an $\text{Tr}(F_M^L F_N^M F_L^N)$ term. This action correctly reproduces the boundary correlators for the $O(N)$ model and the correlators match the form required by [48] to obey the conformal Ward identities.

II. THE ACTION

We begin with the $U(2N)$ action in D dimensions. Later we set $D = 3$,

$$S = \sqrt{N} \int_x \partial_\mu \varphi_A^* \partial^\mu \varphi^A, \quad (1)$$

and after separating the $USp(2N)$ and the $SU(2)$ indices the action becomes (see Appendix A for details)

$$S = \frac{1}{2} \sqrt{N} \int_x \partial_\mu \phi^{Ia} \partial^\mu \phi^{Jb} Q^{IJ} \epsilon_{ab}, \quad (2)$$

where $1 \leq I, J \leq 2N$ are $USp(2N)$ indices and a, b are the $j = 1/2$ $SU(2)$ indices and

$$Q \equiv \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}. \quad (3)$$

We need to introduce auxiliary fields to stand for the currents, but the currents will run under RG, so we first begin by introducing sources for the currents, A_μ^i . We introduce this background gauge field through covariantizing the derivative. A_μ^i lets us obtain the correlators of the current on differentiating the partition function with it:

$$S = \frac{1}{2} \sqrt{N} \int_x D_\mu \phi^{Ia} D^\mu \phi^{Jb} Q^{IJ} \epsilon_{ab}, \quad (4)$$

where

$$D_\mu \phi^{Ia} := \partial_\mu \phi^{Ia} + A_\mu^i (T_i)^a_b \phi^{Ib}, \quad (5)$$

and T_i are the generators of the $j = 1/2$ representation of $\mathfrak{su}(2)$. The index i is the $\mathfrak{su}(2)$ adjoint index.

We introduce the auxiliary field as a functional derivative with respect to the gauge field,

$$Z[A] = \int \mathcal{D}\sigma \delta\left(\sigma_\mu^i - \frac{1}{\sqrt{N}} \frac{\delta}{\delta A_\mu^i}\right) Z[A]. \quad (6)$$

We also introduce a Lagrange multiplier field to enforce the delta functional,

$$\begin{aligned} Z[A] &= \int \mathcal{D}\chi \mathcal{D}\sigma \mathcal{D}\phi \exp\left\{i \int_x \chi_i^\mu \left(\sigma_\mu^i - \frac{1}{\sqrt{N}} \frac{\delta}{\delta A_\mu^i}\right)\right\} \exp\{-S[\phi, A_\mu]\} \\ &= \int \mathcal{D}\chi \mathcal{D}\sigma \mathcal{D}\phi \exp\left\{i \int_x \chi_i^\mu \sigma_\mu^i\right\} \exp\left\{-S\left[\phi, A_\mu - \frac{1}{\sqrt{N}} i\chi_\mu\right]\right\} \\ &= \int \mathcal{D}\chi \mathcal{D}\sigma \mathcal{D}\phi \exp\left\{\int_x (i\chi_i^\mu + \sqrt{N} A_\mu^i) \sigma_\mu^i\right\} \exp\left\{-S\left[\phi, -\frac{1}{\sqrt{N}} i\chi_\mu\right]\right\}. \end{aligned} \quad (7)$$

In the last step, we have shifted χ_μ by the field A_μ .

III. ERG EQUATION FOR THE CURRENT

We first introduce a cutoff Λ_b in the kinetic term of the action that serves as the bare UV cutoff. It is immaterial to this presentation whether the UV cutoff is finite or ∞ , but we leave it unspecified for now. So the bare action becomes

$$\begin{aligned} S_B[\phi, A_\mu] &= \frac{1}{2} Q^{IJ} \epsilon_{ab} \sqrt{N} \int_x \{\phi^{Ia} \Delta^{-1} \phi^{Jb} \\ &\quad + 2A_\mu^i (T_i)^a_c \phi^{Ia} \partial_\mu \phi^{Jb} \\ &\quad + (T_i)^a_c (T_j)^b_d A_\mu^i A_\mu^j \phi^{Ic} \phi^{Jd}\}, \end{aligned} \quad (8)$$

where $\Delta = \frac{1}{p^2} K_{\Lambda_b}(p^2)$, i.e., the usual scalar field propagator multiplied by a cutoff function that we leave unspecified. The cutoff function will later be determined implicitly when we map the RG flow to AdS in Sec. V.

Polchinski's exact renormalization group equation [49] (see also [50–58]) gives the rate of change of the interacting part of an action with its cutoff. Here we work with the *flipped* ERG equation,² which occurs more naturally in holographic RG contexts [15]. The action is split into two,

²The flipped ERG equation is for an action obtained by partially integrating out the fundamental field from energy scale 0 to Λ . In contrast, Polchinski's equation is for an action obtained upon integrating out modes between Λ and ∞ . When we introduce auxiliary composite fields and integrate out low energy modes, the propagator lines get contributions from fundamental modes between energy scales 0 and Λ . This is the expected behavior for an effective action for the composite field. As is clear from the derivation of the flipped ERG equation in Appendix B, the contributions from fundamental modes above Λ are set to zero. Thus, the flow for the composite field action is not the usual Wilsonian ERG. There is information lost, which is recovered as $\Lambda \rightarrow \infty$, or from bulk perspective, as we move close to boundary. See Appendix C and [34] for more discussion.

one containing terms with low energy modes which we integrate out, and the other involving quadratic high energy terms,

$$\begin{aligned} S_l[\phi_l, \phi_h, \chi_\mu] &= Q^{IJ} \epsilon_{ab} \int_x \left\{ \frac{1}{2} \sqrt{N} \phi_l^{Ia} \Delta_l^{-1} \phi_l^{Jb} \right. \\ &\quad - i\chi_i^\mu (T_i)^a_c (\phi_l^{Ic} \partial_\mu \phi_l^{Jb} + \phi_h^{Ic} \partial_\mu \phi_l^{Jb} + \phi_l^{Ic} \partial_\mu \phi_h^{Jb}) \\ &\quad \left. - \frac{1}{2} \frac{1}{\sqrt{N}} (T_i)^a_c (T_j)^b_d \chi_i^\mu \chi_\mu^j (\phi_l^{Ic} \phi_l^{Jd} + 2\phi_h^{Ic} \phi_l^{Jd}) \right\} \end{aligned} \quad (9)$$

and

$$\begin{aligned} S_h[\phi_h, \chi_\mu] &= Q^{IJ} \epsilon_{ab} \int_x \left\{ \frac{1}{2} \sqrt{N} \phi_h^{Ia} \Delta_h^{-1} \phi_h^{Jb} \right. \\ &\quad - i\chi_i^\mu (T_i)^a_c \phi_h^{Ic} \partial_\mu \phi_h^{Jb} \\ &\quad \left. - \frac{1}{2} \frac{1}{\sqrt{N}} (T_i)^a_c (T_j)^b_d \chi_i^\mu \chi_\mu^j \phi_h^{Ic} \phi_h^{Jd} \right\}, \end{aligned} \quad (10)$$

i.e.,

$$\begin{aligned} S_B &= S_h + S_l, & \phi &= \phi_h + \phi_l, & \text{and the propagator} \\ \Delta &= \Delta_h + \Delta_l. \end{aligned}$$

S_B , i.e., the bare theory is defined at a cutoff Λ_b . We split the fields in this theory into two modes, with Δ_h only propagating modes between cutoff Λ and bare cutoff Λ_b , and Δ_l only propagating modes below Λ .

The flipped ERG involves integrating out ϕ_l^{Ia} , then using the ERG for ϕ_h^{Ia} to derive an ERG for σ_μ^i by setting ϕ_h^{Ia} to zero and integrating out χ_μ^i . We do this in Appendix B to

obtain the equation describing change in $S_\Lambda[\sigma]$ with Λ ,

$$\begin{aligned} \frac{\partial}{\partial t} e^{-S_\Lambda[\sigma_i^\mu]} &= \left\{ \frac{1}{2} \int_p \dot{I}(p^2) \frac{\delta^2}{\delta \sigma_i^\mu(p) \delta \sigma_\mu^i(-p)} \right. \\ &+ \frac{2}{\sqrt{N}} \epsilon_{ijk} \int_p \int_q \int_r \delta^D(p+q+r) \\ &\times \int_k \frac{\Delta_{l,k+p} \Delta_{l,k-q} \dot{\Delta}_{l,k}}{I(p^2) I(q^2) I(r^2)} (k+p)_\mu k_\nu (k-q)_\rho \\ &\times \sigma_i^\mu(p) \sigma_j^\nu(q) \sigma_k^\rho(r) \left. \right\} e^{-S_\Lambda[\sigma_i^\mu]} + \dots \end{aligned} \quad (11)$$

IV. HOLOGRAPHIC ACTION

The flipped ERG equation (B29) is of the form

$$\frac{\partial}{\partial t} \psi = \left(\frac{1}{2} \dot{G} \frac{\partial^2}{\partial x^2} + \frac{1}{\sqrt{N}} V(x) \right) \psi, \quad (12)$$

where $\mathcal{G} = I(p^2)$, given in (B21). The evolution operator is then given in analogy with the Feynman path integral solution to the Schrödinger equation:

$$\begin{aligned} \psi(x_f, T) &= \int_{x(0)=x_i}^{x(T)=x_f} dx_i \int \mathcal{D}x(t) \\ &\times \exp \left\{ \int_0^T dt \left[-\frac{\dot{x}^2}{2\mathcal{G}} + \frac{1}{\sqrt{N}} V(x(t)) \right] \right\} \psi(x_i, 0) \end{aligned} \quad (13)$$

$$\equiv \int dx_i U(x_f, T; x_i, 0) \psi(x_i, 0), \quad (14)$$

where U is the evolution operator.

The evolution operator takes $S_\Lambda[\sigma_i^\mu]$ from $\Lambda = 0$, where it is 0, to $\Lambda = \Lambda_b$ where it is the full action for σ .

For the present case, we have the evolution operator as

$$\begin{aligned} U[\sigma_f, t_f; \sigma_i, t_i] &= \int_{\sigma(t_i)=\sigma_i}^{\sigma(t_f)=\sigma_f} \mathcal{D}\sigma^\mu \exp \left\{ \int dt \left[-\frac{1}{2} \int_p \frac{1}{\dot{I}(p^2)} \dot{\sigma}_i^\mu \dot{\sigma}_\mu^i \right. \right. \\ &+ \frac{2}{\sqrt{N}} \epsilon_{ijk} \int_p \int_q \int_r \delta^D(p+q+r) \int_k \frac{\Delta_{l,k+p} \Delta_{l,k-q} \dot{\Delta}_{l,k}}{I(p^2) I(q^2) I(r^2)} (k+p)_\mu k_\nu (k-q)_\rho \\ &\times \sigma_i^\mu(p) \sigma_j^\nu(q) \sigma_k^\rho(r) \left. \right] \left. \right\}. \end{aligned} \quad (15)$$

We interpret this as the functional integral for a $D+1$ dimensional field theory, with t interpreted as an extra spacelike coordinate. We can read off the action for this theory from this functional integral. This “bulk” action is given by

$$\begin{aligned} S[\sigma^\mu] &= \int dt \left[\frac{1}{2} \int_p \frac{1}{\dot{I}(p^2)} \dot{\sigma}_i^\mu(p) \dot{\sigma}_\mu^i(-p) \right. \\ &- \frac{2}{\sqrt{N}} \epsilon_{ijk} \int_p \int_q \int_r \delta^D(p+q+r) \int_k \frac{\Delta_{l,k+p} \Delta_{l,k-q} \dot{\Delta}_{l,k}}{I(p^2) I(q^2) I(r^2)} (k+p)_\mu k_\nu (k-q)_\rho \\ &\times \sigma_i^\mu(p) \sigma_j^\nu(q) \sigma_k^\rho(r) \left. \right]. \end{aligned} \quad (16)$$

V. MAPPING TO ADS

The action obtained above (16) is nonlocal. With a field redefinition, we can bring the quadratic term into the standard AdS kinetic form.

A. The field redefinition

We suppress the $\mathfrak{su}(2)$ adjoint indices i, j, k in the following. The field redefinition function f differs a bit from [33] for the kinetic term,

$$f(p, t) = \sqrt{-\dot{I}(p^2, t) e^{Dt}}. \quad (17)$$

We define

$$z \equiv \frac{1}{\Lambda_b} e^{-t} = \frac{1}{\Lambda}, \quad (18)$$

$$f(p, z) = \sqrt{z^{-D+1} \partial_z I(p^2, z)} \quad (19)$$

$$\sigma_\mu(p, z) \equiv z f(p, z) a_\mu(p, z). \quad (20)$$

If we choose an $f(z, p)$ that solves the following differential equation, then the kinetic term of the action can be put in standard AdS form:

$$\frac{\partial}{\partial z} \left(z^{-D+1} \frac{\partial}{\partial z} \frac{1}{f} \right) = z^{-D+1} (p^2 z^2 + m^2) \frac{1}{f}, \quad (21)$$

where $m^2 = 1 - D$ for the vector.

The kinetic term in the action then looks like

$$S_K = \frac{1}{2} \int_p \int \frac{dz}{z^{D-3}} (\partial_z a_i^\mu(p, z) \partial_z a_\mu^i(-p, z) + p^2 a_i^\mu(p, z) a_\mu^i(-p, z)). \quad (22)$$

The general solution to Eq. (21) that satisfies the constraints of analyticity is described in [33],

$$1/f = A(p) z^{\frac{D}{2}} K_\nu(pz) + B(p) z^{\frac{D}{2}} I_\nu(pz), \quad (23)$$

where $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel functions. As described in [30], $y_\mu = z a_\mu$ satisfies the same differential equation as the equation of motion (EOM). From (20) and (D5), thus, I/f must also satisfy the same differential equation, and must be given in terms of the modified Bessel functions as $I/f = z^{\frac{D}{2}}(C(p)K_\nu(pz) + D(p)I_\nu(pz))$, for some $C(p)$, $D(p)$. [That I/f must satisfy the same differential equation can also be seen by substituting $-z^{D-1}f^2$ for $\partial_z I$ in the differential equation, on which it reduces to (21).] Hence,

$$I(p, z) = \frac{C(p)K_\nu(pz) + D(p)I_\nu(pz)}{A(p)K_\nu(pz) + B(p)I_\nu(pz)}. \quad (24)$$

Further,

$$AD - BC = 1, \quad (25)$$

which results from taking the Wronskian of I/f and $1/f$, which are two independent solutions of the differential equation (21).

The functions A , B , C , and D are fixed by the boundary behaviors of f and I ,

$$I(p, 0) \equiv -\gamma p^{2\nu} \Big|_{D=3} = -\gamma p; \quad \text{and} \quad I(p, \infty) = 0, \quad (26)$$

where γp is the full Green's function at the boundary in $D = 3$, and where $\nu^2 = m^2 + D^2/4$, which comes from solving the differential equation (21) [59], with $m^2 = 1 - D$ for the vector. We choose the positive root

$$\nu = \frac{D}{2} - 1 = \frac{3}{2} - 1 = \frac{1}{2}. \quad (27)$$

The boundary behavior of f is determined as follows. Near the boundary, the leading behavior of the bulk AdS field y_μ is given by

$$y_\mu(x, z) \approx z^{D-\Delta} A_\mu(x), \quad (28)$$

where A_μ is the boundary source for the current $\sigma_\mu(x)$. $\Delta = \frac{D}{2} + \nu = 2$ is the scaling dimension of $\sigma_\mu(x)$. The EOM for the bulk field $\sigma_\mu(z, x)$ is given by (D5)

$$\sigma_\mu(p, z) = b_\mu(p) I(p, z), \quad (29)$$

for some $b_\mu(p)$. The boundary action is given by [refer (B27)]

$$S_b[\sigma_{b\mu}] = \frac{1}{2} \int_p \frac{1}{I_b(p)} \sigma_{b\mu}(p) \sigma_b^\mu(-p) + \int_p J_\mu(p) \sigma_b^\mu(-p), \quad (30)$$

where the subscript b indicates $\sigma_{b\mu}(p) \equiv \sigma_\mu(p, z = \epsilon)$. The EOM given by this action is

$$\sigma_{b\mu}(p) = -A_\mu(p) I(p). \quad (31)$$

Therefore, b^μ from (29) is the boundary source $-A^\mu$ and $\sigma^\mu(p, z) = -A_\mu(p) I(p, z)$. Then, from the mapping equation (20) and boundary behaviors of y_μ (28) and I (26), we have

$$\lim_{z \rightarrow \epsilon} f(p, z) \approx \gamma z^{-D/2+\nu} p^{2\nu}. \quad (32)$$

Now that we have three additional constraints on A , B , C , and D given by boundary behaviors of I (26) and f (32), along with the Wronskian (25), we can determine their values.

The asymptotic behavior of the modified Bessel functions is

$$I_\nu(x) \xrightarrow{x \rightarrow \infty} \frac{1}{\sqrt{2\pi x}} e^x, \quad (33)$$

$$K_\nu(x) \xrightarrow{x \rightarrow \infty} \sqrt{\frac{\pi}{2x}} e^{-x}, \quad (34)$$

therefore for the second boundary condition of I to be satisfied,

$$D(p) = 0. \quad (35)$$

Thus, (25) becomes

$$BC = -1. \quad (36)$$

At the other limit, $pz \rightarrow 0$,

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + \nu + 1) k!} \left(\frac{x}{2}\right)^{2k+\nu}, \quad (37)$$

$$K_\nu(x) = \frac{1}{2} \left[\Gamma(\nu) \left(\frac{x}{2} \right)^{-\nu} \left(1 + \frac{x^2}{4(1-\nu)} + \dots \right) + \Gamma(-\nu) \left(\frac{x}{2} \right)^\nu \left(1 + \frac{x^2}{4(1+\nu)} + \dots \right) \right], \quad \nu \notin \mathbb{Z}, \quad (38)$$

$$C(p) = -\frac{\gamma}{\Gamma(\nu)} 2^{1-\nu} p^\nu; \quad B(p) = \frac{1}{\gamma} \Gamma(\nu) 2^{-1+\nu} p^{-\nu}. \quad (41)$$

therefore we have from (32) and (23),

$$A(p) = \frac{1}{\Gamma(\nu)} 2^{1-\nu} p^{-\nu}. \quad (39)$$

Equation (26) gives

$$C(p)/A(p) = -\gamma p^{2\nu}. \quad (40)$$

Then we have

$$1/f = p^{-\nu} z^{\frac{D}{2}} \left(\frac{1}{\Gamma(\nu)} 2^{1-\nu} K_\nu(pz) + \frac{1}{\gamma} \Gamma(\nu) 2^{-1+\nu} I_\nu(pz) \right), \quad (42)$$

$$I(p, z) = -\frac{\gamma^2 p^{2\nu} K_\nu(pz)}{\gamma K_\nu(pz) + 2^{2\nu-2} \Gamma(\nu)^2 I_\nu(pz)}. \quad (43)$$

B. The cubic term

The $O(\frac{1}{N})$ term is (16), substituting a_μ for σ_μ ,

$$\begin{aligned} S_3 &= -\frac{2}{\sqrt{N}} \epsilon_{ijk} \int dt \int_p \int_q \int_r \delta^D(p+q+r) \int_k \frac{\Delta_{l,k+p} \Delta_{l,k-q} \dot{\Delta}_{l,k}}{I(p^2) I(q^2) I(r^2)} k_\mu k_\nu (k+p)_\rho \sigma_i^\mu(p) \sigma_j^\nu(q) \sigma_k^\rho(r) \\ &= -\frac{2}{\sqrt{N}} \epsilon_{ijk} \int \frac{dz}{z} \int_p \int_q \int_r \delta^D(p+q+r) \int_k \frac{\Delta_{l,k+p} \Delta_{l,k-q} z \partial_z \Delta_{l,k}}{I(p^2) I(q^2) I(r^2)} k_\mu k_\nu (k+p)_\rho \\ &\quad \times z^3 f(p) f(q) f(r) a_i^\mu(p) a_j^\nu(q) a_k^\rho(r). \end{aligned}$$

Note here that only z -dependent part of this whole term is $\Delta_{l,k+p} \Delta_{l,k-q} \partial_z \Delta_{l,k}$. The rest is a z independent considering (D4).

We can substitute from (43)

$$\frac{f(p, z)}{I(p, z)} = -\frac{2^{-1+\nu} \Gamma(\nu) z^{-D/2}}{\gamma p^\nu K_\nu(pz)} = -\sqrt{\frac{\pi}{2}} \frac{z^{-3/2}}{\gamma p^\nu K_\nu(pz)}, \quad (44)$$

giving

$$\begin{aligned} S_3 &= \frac{2}{\sqrt{N} \gamma^3} \epsilon_{ijk} \left(\frac{\pi}{2} \right)^{3/2} \int z^{-3/2} dz \int_p \int_q \int_r \delta^D(p+q+r) \int_k \Delta_{l,k+p} \Delta_{l,k-q} \partial_z \Delta_{l,k} \\ &\quad \times k_\mu k_\nu (k+p)_\rho \frac{p^{-\nu}}{K_\nu(pz)} \frac{q^{-\nu}}{K_\nu(qz)} \frac{r^{-\nu}}{K_\nu(rz)} \times a_i^\mu(p) a_j^\nu(q) a_k^\rho(r) \\ &= \frac{2}{3\sqrt{N} \gamma^3} \epsilon_{ijk} \left(\frac{\pi}{2} \right)^{3/2} \int z^{-3/2} dz \int_p \int_q \int_r \delta^D(p+q+r) \int_k \partial_z (\Delta_{l,k+p} \Delta_{l,k-q} \Delta_{l,k}) \\ &\quad \times k_\mu k_\nu (k+p)_\rho \frac{p^{-\nu}}{K_\nu(pz)} \frac{q^{-\nu}}{K_\nu(qz)} \frac{r^{-\nu}}{K_\nu(rz)} \times a_i^\mu(p) a_j^\nu(q) a_k^\rho(r). \end{aligned} \quad (45)$$

As mentioned before, the quantities outside ∂_z parentheses are z independent, hence the integrand is a total derivative in z . Thus, we could evaluate it at the limits of the integral. We can make use of this fact to simplify the evaluation of the integral. The low energy propagator is determined by the RG regularization chosen,³ but this is a complicated propagator. We instead choose to work with a simple low energy propagator that is more tractable. The error in correlators calculated with this is $O(p/\Lambda_b)$. Thus, in the limit of $\epsilon \rightarrow 0$, or $\Lambda_b \rightarrow \infty$, we get the correct correlators. The k integral is done in Appendix E.

³In [33], we discussed that when we map the ERG equation to an AdS bulk action, it constrains the regulator we can use for the ERG equation. Since this must be done for each operator that deforms the boundary CFT, there are conflicting constraints on the high energy propagator. This was resolved in [33] and the constraint imposed by the ERG equation of the scalar composite operator discussed in [32] was chosen. We follow the same choice here.

The result is (E12),

$$\int_k \partial_z (\Delta_{l,k+p} \Delta_{l,k-q} \Delta_{l,k}) k_\mu k_\nu (k+p)_\rho \approx \frac{1}{16(2\pi)^{5/2}} z^3 [-q_\mu r_\nu p_\rho z^2 - \delta_{\mu\nu} p_\rho (1+rz) - q_\mu \delta_{\nu\rho} (1+pz) - r_\nu \delta_{\mu\rho} (1+qz)] \left(\frac{r}{z}\right)^\nu \left(\frac{p}{z}\right)^\nu \left(\frac{q}{z}\right)^\nu K_\nu(rz) K_\nu(pz) K_\nu(qz). \quad (46)$$

The interaction term is then

$$\begin{aligned} S_3 &= \frac{1}{\sqrt{N} 384 \pi \gamma^3} \epsilon_{ijk} \int dz \int_p \int_q \int_r \delta^D(p+q+r) [-z^2 q_\mu r_\nu p_\rho - \delta_{\mu\nu} p_\rho (1+rz) - q_\mu \delta_{\nu\rho} (1+pz) - r_\nu \delta_{\mu\rho} (1+qz)] \\ &\quad \times a_i^\mu(p) a_j^\nu(q) a_k^\rho(r) \\ &= \frac{1}{\sqrt{N} 384 \pi \gamma^3} \epsilon_{ijk} \int dz \int_p \int_q \int_r \delta^D(p+q+r) [-z^2 q_\mu r_\nu p_\rho a_i^\mu(p) a_j^\nu(q) a_k^\rho(r) - 3(1+rz) a_i^\mu(p) a_\mu^j(q) \times p_\rho a_k^\rho(r)]. \end{aligned} \quad (47)$$

This action looks unfamiliar and has no gauge invariance manifest, but the gauge invariance is hidden. In the next section, we show that this action is equivalent to the gauge fixed version of a manifestly gauge invariant action.

VI. GAUGE INVARIANCE

Even though our cubic term is nonstandard and the gauge invariance is not manifest, it is equivalent classically to the cubic term in a gauge invariant action. In this section we show that, upon using the equations of motion, the following gauge invariant action reduces to (47):

$$\begin{aligned} S'[a^\mu] &= \frac{1}{4} \int d^{D+1}x \sqrt{G} F^{MNi}(x, z) F_{MN}^i(x, z) \\ &\quad + g' \epsilon^{ijk} \int d^{D+1}x \sqrt{G} F_M^{Li}(x, z) F_N^{Mj}(x, z) F_L^{Nk}(x, z), \end{aligned} \quad (48)$$

where L, M, N are $D+1$ dimensional indices that run over $\{x^\mu, z\}$; x^μ are the boundary coordinates,

$$\begin{aligned} F_{MN}^i(x, z) &= \partial_M a_N^i(x, z) - \partial_N a_M^i(x, z) \\ &\quad - i g \epsilon^{ijk} a_M^j(x, z) a_N^k(x, z), \end{aligned} \quad (49)$$

and g and g' are the Yang-Mills (YM) coupling and the FFF coupling, respectively. The metric is

$$ds^2 = \frac{1}{z^2} (dz^2 + \delta_{\mu\nu} dx^\mu dx^\nu), \quad (50)$$

therefore $\sqrt{G} = z^{-D-1}$.

We fix the gauge to be in “radial,” i.e.,

$$a_z^i(x, z) = 0, \quad (51)$$

and we use the residual gauge freedom to fix

$$\partial_\mu a^{\mu i}(x, z) = 0. \quad (52)$$

The fields obtained from ERG of the boundary theory are in this gauge.

On Fourier transforming the boundary coordinates, the cubic terms from this gauge fixed action are

$$S_3^{YM} = \frac{1}{2} g \epsilon^{ijk} \int_{p,q,r} \delta^D(p+q+r) z^{-D-1} z^4 (p_\mu a_\nu^i(p, z) - p_\nu a_\mu^i(p, z)) a^{\mu j}(q, z) a^{\nu k}(r, z) \quad (53)$$

$$\begin{aligned} S_3^{FFF} &= -i g' \epsilon^{ijk} \int_{p,q,r} \delta^D(p+q+r) z^{-D-1} z^6 [(p_\mu a_\nu^i(p, z) - p_\nu a_\mu^i(p, z)) (q_\nu a_\rho^j(q, z) - q_\rho a_\nu^j(q, z)) (r_\rho a_\mu^k(r, z) - r^\mu a_\rho^k(r, z)) \\ &\quad + (p_\mu a_\nu^i(p, z) - p_\nu a_\mu^i(p, z)) \partial_z a_\nu^j(q, z) \partial_z a^{\mu k}(r, z) + (q_\mu a_\nu^i(q, z) - q_\nu a_\mu^i(q, z)) \partial_z a_\nu^j(r, z) \partial_z a^{\mu k}(p, z) \\ &\quad + (r_\mu a_\nu^i(r, z) - r^\nu a_\mu^i(r, z)) \partial_z a_\nu^j(p, z) \partial_z a^{\mu k}(q, z)]. \end{aligned} \quad (54)$$

Since p, q, r are dummy variables, the terms which are the same up to permutations of these are the same, but we have written them out for clarity.

With some manipulation using $p^\mu + q^\mu + r^\mu = 0$, and that the fields are transverse, we have

$$S'_3 = -\epsilon^{ijk} \int_{p,q,r} \delta^D(p+q+r) [a_\mu^i(p, z) a_\nu^j(q, z) a_\rho^k(r, z) (gp^\rho \delta^{\mu\nu} + ig'z^2(2p^\rho q^\mu r^\nu - 3(p \cdot r + r \cdot q)p^\rho \delta^{\mu\nu})) \\ + 3ig'z^2 p^\rho \delta^{\mu\nu} (a_\mu^i(p, z) \partial_z a_\nu^j(q, z) \partial_z a_\rho^k(r, z) + \partial_z a_\mu^i(p, z) a_\nu^j(q, z) \partial_z a_\rho^k(r, z))] \quad (55)$$

$$= -\epsilon^{ijk} \int_{p,q,r} \delta^D(p+q+r) [a_\mu^i(p, z) a_\nu^j(q, z) a_\rho^k(r, z) (gp^\rho \delta^{\mu\nu} + ig'z^2(2p^\rho q^\mu r^\nu + 3r^2 p^\rho \delta^{\mu\nu})) \\ + 3ig'z^2 p^\rho \delta^{\mu\nu} \partial_z (a_\mu^i(p, z) a_\nu^j(q, z)) \partial_z a_\rho^k(r, z)]. \quad (56)$$

Now we show that *on shell*, the action we have obtained from ERG in Sec. V, in (47) is equivalent to this expression resulting from gauge fixing a gauge invariant action. The evaluation of correlators is done using the large N semi-classical limit and so using the on-shell form of the action is not an additional restriction.

In VA, we have shown that the redefined bulk field a^μ is given by

$$a_\mu^i(p, z) = \gamma \frac{2^{1-\nu}}{\Gamma(\nu)} A_\mu^i(p) z^{\frac{D}{2}-1} |p|^\nu K_\nu(pz), \quad (57)$$

where $A_\mu^i(p)$ is the source for the boundary current. Now for the boundary current, $\nu = \frac{D}{2} - 1 = \frac{1}{2}$, and $K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$. Thus, $a_\mu^i(p, z)$ only depends on z via e^{-pz} , and we have

$$\partial_z a_\mu^i(p, z) = -p a_\mu^i(p, z). \quad (58)$$

This means that we could exchange factors of p in the action for ∂_z acting on the gauge fields. Then the ERG derived action of (47) becomes

$$S_3 = \frac{1}{\sqrt{N}} \frac{1}{384\pi\gamma^3} \epsilon_{ijk} \int dz \int_p \int_q \int_r \delta^D(p+q+r) [-z^2 q_\mu r_\nu p_\rho a_i^\mu(p, z) a_j^\nu(q, z) a_k^\rho(r, z) \\ - 3a_i^\mu(p, z) a_\mu^j(q, z) \times p_\rho a_k^\rho(r, z) + 3za_i^\mu(p, z) a_\mu^j(q, z) \times p_\rho \partial_z a_k^\rho(r, z)]. \quad (59)$$

This matches the action in (56) except for the third term, $3za_i^\mu(p, z) a_\mu^j(q, z) \times p_\rho \partial_z a_k^\rho(r, z)$. We add the surface term $-\frac{3}{2} \partial_z (z^2 a_i^\mu(p, z) a_\mu^j(q, z) \times p_\rho \partial_z a_k^\rho(r, z))$, which goes to 0 as $z \rightarrow 0$, and we are left with

$$-\frac{3}{2} z^2 \partial_z (a_i^\mu(p, z) a_\mu^j(q, z)) \times p_\rho \partial_z a_k^\rho(r, z) \\ - \frac{3}{2} z^2 a_i^\mu(p, z) a_\mu^j(q, z) \times p_\rho \partial_z^2 a_k^\rho(r, z). \quad (60)$$

Finally, we exchange the two derivatives in the second term for r^2 to get

$$-\frac{3}{2} z^2 \partial_z (a_i^\mu(p, z) a_\mu^j(q, z)) \times p_\rho \partial_z a_k^\rho(r, z) \\ - \frac{3}{2} p^2 z^2 a_i^\mu(p, z) a_\mu^j(q, z) \times p_\rho a_k^\rho(r, z). \quad (61)$$

Comparing this with (56), we see that they both match with g and g' given by

$$g = \frac{1}{\sqrt{N}} \frac{1}{128\pi\gamma^3}; \quad g' = \frac{ig}{6}. \quad (62)$$

VII. SUMMARY AND CONCLUSIONS

One of the intriguing issues in holography is the emergence of gauge symmetry in the bulk theory. This is definitely required for consistency of the bulk theory as a quantum field theory. If the ERG approach of constructing the bulk action from first principles, described in [30], is to be applicable to all such theories, it is imperative that the mechanism that ensures this gauge invariance be understood better. In earlier papers that followed this approach, only the Abelian part had been explored [34]. The non-Abelian structure was not explored. This paper is a step in that direction.

In this paper we start from a boundary scalar field theory that possesses a global $SU(2)$ symmetry. The correlators of the corresponding currents have a well-defined form fixed by the Ward identities. We construct a dual bulk action in AdS space from first principles using the ERG approach described in [30]. The result is an action for a Yang-Mills gauge field (in holographic gauge $A_z = 0$) in the bulk. This action is shown to be the cubic part of the usual gauge invariant Yang-Mills action quadratic in field strengths along with a term cubic in Yang-Mills field strength. The action is given in (48) and verifies the consistency of this method.

There are many open questions. One is to extend this to the cubic interaction in gravity and indeed for all massless higher spins. It would be interesting if this can be done in a way that makes manifest the gauge symmetry so that the resulting bulk action is guaranteed to have the required symmetry.

The other issue is that of locality. It is not clear what notion of locality a “good” bulk dual should have. As emphasized earlier, the ERG approach to AdS/CFT guarantees that the bulk dual reproduces all correlators of the boundary theory. It also is designed to give a local kinetic term in AdS space for the bulk fields. The locality of cubic and higher order terms is not guaranteed. It was verified for the cubic scalar self-interaction in [32] and for the cubic minimal coupling of the gravitational perturbation to the scalar in [34] (and in the present paper for the Yang-Mills gauge field self-coupling). A better understanding of this issue is certainly required.

The higher-spin-vector-model duality has been the playground for constructing bulk from boundary theory in several works. Some of these use the (E)RG approach [25,26,43], while others use collective fields or bilocal holography [39,40,42,44,45,47]. There is a need to understand the relation between these approaches and ours. Reference [45] also deals with obtaining a gauge theory in the bulk, albeit Abelian. The connection has been explored by [26] along with the question of obtaining an RG cutoff that naturally corresponds to the bulk radial direction. We believe our approach described in the series of papers [30–34,60], and the present one, answers this issue of the RG cutoff.

Finally, there is the question of generalizing the ERG approach to other spacetimes such as flat space [61] and de Sitter [60].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: THE CURRENT

We begin with the free $U(2N)$ invariant scalar model,

$$S_B = \sqrt{N} \int_x \partial_\mu \varphi_A^* \partial^\mu \varphi^A, \quad (\text{A1})$$

where A is the $U(2N)$ index. As long as we do not introduce any interactions, the action is also invariant under a larger group of transformations: $SO(4N)$, the group of rotations of the $4N$ real scalars that are the real and imaginary parts of the complex φ^A . $SO(4N)$ has a maximal subgroup $USp(2N) \times SU(2)$. Our objective is to separate the indices corresponding to these two subgroups, so that we may consider operators which are traced over the $USp(2N)$.

Now define new fields ϕ^{Ia} in terms of φ^A with $I \in \{1, 2, \dots, 2N\}$ and $a \in \{1, 2\}$ as follows. For $I = i \leq N$,

$$\phi^{I1} := \varphi^i; \quad \phi^{I2} := \varphi_{N+i}^*, \quad (\text{A2})$$

and for $I = N + i$, $1 \leq i \leq N$,

$$\phi^{Ia} := \epsilon^{ab} \phi_b^{i*}. \quad (\text{A3})$$

ϵ_{ab} is the two-dimensional Levi-Civita tensor and ϵ^{ab} is its inverse, $\epsilon_{12} = 1$. That is,

$$\phi^{I1} = \begin{pmatrix} \varphi^i \\ -\varphi^{N+i} \end{pmatrix} \quad \text{and} \quad \phi^{I2} = \begin{pmatrix} \varphi_{N+i}^* \\ \varphi_i^* \end{pmatrix}. \quad (\text{A4})$$

The action is now

$$S = \sqrt{N} \int_x \partial_\mu \varphi_A^* \partial^\mu \varphi^A \equiv \frac{1}{2} \sqrt{N} \int_x \partial_\mu \phi^{Ia} \partial^\mu \phi^{Jb} Q^{IJ} \epsilon_{ab}, \quad (\text{A5})$$

where

$$Q \equiv \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}. \quad (\text{A6})$$

In relabeling the fields ϕ instead of φ , we have only shuffled around the $4N$ real fields that are the real and imaginary parts of the complex φ^A . So, the action must still be $SO(4N)$ invariant. But, in this avatar, the index structure makes manifest the invariance of the action under the maximal subgroup $USp(2N) \times SU(2)$ of $SO(4N)$. I, J are $USp(2N)$ indices and a, b are $SU(2)$ indices. $USp(2N)$ is the set of unitary transformations of determinant 1 that leaves the matrix Q invariant, i.e., complex $2N \times 2N$ matrices S such that $SQS^T = Q$ and $\det S = 1$, and $SU(2)$ transformations leave ϵ invariant.

The global $SO(4N)$ symmetry means there are conserved currents,

$$J_\mu^{IJab} \equiv \phi^{Ia} \partial_\mu \phi^{Jb}, \quad (\text{A7})$$

which are in the adjoint representation of $SO(4N)$. We select the $USp(2N)$ singlets⁴ among them by weakly gauging the $USp(2N)$ and then taking the zero coupling limit [63]. With the index structure we have chosen, it is easy to make the currents $USp(2N)$ invariant. We contract them with Q_{IJ} . We then obtain three currents in the adjoint representation of $SU(2)$:

$$J_-^\mu \equiv Q^{IJ} \phi^{I1} \partial^\mu \phi^{J1}, \quad (\text{A8})$$

$$J_3^\mu \equiv \frac{1}{2} Q^{IJ} (\phi^{I1} \partial^\mu \phi^{J2} + \phi^{I2} \partial^\mu \phi^{J1}), \quad (\text{A9})$$

$$J_+^\mu \equiv Q^{IJ} \phi^{I2} \partial^\mu \phi^{J2}. \quad (\text{A10})$$

Written in terms of the original fields, these are

$$J_-^\mu = Q^{AB} \varphi^A \partial^\mu \varphi^B, \quad (\text{A11})$$

$$J_3^\mu = \frac{1}{2} (\varphi_A^* \partial^\mu \varphi^A - \varphi^A \partial^\mu \varphi_A^*), \quad (\text{A12})$$

$$J_+^\mu = Q_{AB} \varphi_A^* \partial^\mu \varphi_B^*. \quad (\text{A13})$$

APPENDIX B: DERIVING THE ERG FOR THE CURRENT

In this appendix we obtain the ERG equation for the current. Following the separation of the high energy and low energy modes as in Sec. III, we integrate out the low energy modes. To this end, we separate the quadratic and linear terms in the action. If no spacetime parameter is specified explicitly for any field, it is x .

The partition function is given by

$$Z[A] = \int \mathcal{D}\chi \mathcal{D}\sigma \mathcal{D}\phi_l \mathcal{D}\phi_h \exp \left\{ \int_x (i\chi_i^\mu + \sqrt{N} A_i^\mu) \sigma_\mu^i \right\} \exp \{ -S_h[\phi_h, \chi_\mu] - S_l[\phi_l, \phi_h, \chi_\mu] \}, \quad (\text{B1})$$

where the part of the S_l quadratic in ϕ_l is given by

$$\begin{aligned} S_l^{(2)} &= Q^{IJ} \epsilon_{ab} \int_x \left\{ \frac{1}{2} \sqrt{N} \phi_l^{Ia} \Delta_l^{-1} \phi_l^{Jb} - i\chi_i^\mu (T_i)_c^a \phi_l^{Ic} \partial_\mu \phi_l^{Jb} - \frac{1}{2\sqrt{N}} (T_i)_c^a (T_j)_d^b \chi_i^\mu \chi_j^\mu \phi_l^{Ic} \phi_l^{Jd} \right\} \\ &= \frac{1}{2} Q^{IJ} \int_x \int_y \phi_l^{Ia} [\sqrt{N} \Delta_l^{-1} (x-y) \epsilon_{ab} - \underbrace{(2i\epsilon_{cb} \chi_i^\mu (T_i)_a^c \frac{\partial}{\partial x^\mu} + \frac{1}{\sqrt{N}} \epsilon_{cd} \chi_i^\mu \chi_j^\mu (T_i)_a^c (T_j)_b^d \delta(x-y))}_{K_{ab}}] \phi_l^{Jb}(y) \\ &\equiv \frac{1}{2} Q^{IJ} \int_x \int_y \phi_l^{Ia} [\sqrt{N} \Delta_l^{-1} \epsilon_{ab} - K_{ab}]_{xy} \phi_l^{Jb}(y). \end{aligned} \quad (\text{B2})$$

We have defined K_{ab} :

$$K_{ab}(x, y) = \left[i\chi_i^\mu \{ \epsilon_{cb} (T_i)_a^c + \epsilon_{ac} (T_i)_b^c \} \frac{\partial}{\partial x^\mu} + \frac{1}{\sqrt{N}} \chi_i^\mu \chi_j^\mu \epsilon_{cd} (T_i)_a^c (T_j)_b^d \right] \delta(x-y). \quad (\text{B3})$$

Note that $K_{ab} = -K_{ba}$.

The linear terms,

$$S_l^{(1)} = -\frac{1}{2} Q^{IJ} \int_x \int_y \phi_l^{Ia} K_{ab}(x, y) \phi_h^{Jb}(y) - \frac{1}{2} Q^{IJ} \int_x \int_y \phi_h^{Ia} K_{ab}(x, y) \phi_l^{Jb}(y), \quad (\text{B4})$$

and

$$S_h[\phi_h, \chi_\mu] = \frac{1}{2} Q^{IJ} \int_x \int_y \phi_h^{Ia} [\sqrt{N} \Delta_l^{-1} \epsilon_{ab} - K_{ab}]_{xy} \phi_h^{Jb}(y). \quad (\text{B5})$$

The path integral is then

⁴We select the $USp(2N)$ singlets instead of the $SO(4N)$ singlet so that the bulk dual is a non-Abelian gauge field [62].

$$\begin{aligned}
 Z[A] &\equiv \int \mathcal{D}\chi \mathcal{D}\sigma \mathcal{D}\phi_h \exp \left\{ \int_x (i\chi_i^\mu + \sqrt{N} A_i^\mu) \sigma_\mu^i \right\} \exp \left\{ -Q^{IJ} \epsilon_{ab} \int_x \frac{1}{2} \sqrt{N} \phi_h^{Ia} \Delta_h^{-1} \phi_h^{Jb} - S_{I,\Lambda}[\phi_h, \chi_\mu] \right\} \\
 &\equiv \int \mathcal{D}\sigma \mathcal{D}\phi_h e^{-S_\Lambda[\phi_h, \sigma^\mu] + \sqrt{N} A_i^\mu \sigma_\mu^i}, \tag{B6}
 \end{aligned}$$

with

$$\begin{aligned}
 e^{-S_{I,\Lambda}} &\equiv \int \mathcal{D}\phi_l^{Ia} \exp \left\{ -S_l^{(1)} - S_l^{(2)} + \frac{1}{2} Q^{IJ} \int_x \int_y \phi_h^{Ia} K_{ab}(x, y) \phi_h^{Jb}(y) \right\} \\
 &= \exp \left\{ -\frac{1}{2} \text{Tr} \log [Q^{IJ} (\sqrt{N} \Delta_l^{-1} \epsilon_{ab} - K_{ab})] \right. \\
 &\quad \left. + \frac{1}{2} Q^{IJ} \int_u \int_v \left(\int_x \phi_h^{Ia}(x) K_{ac}(x, u) \right) [\sqrt{N} \Delta_l^{-1} - K]^{-1cd}(u, v) \left(\int_y K_{db}(v, y) \phi_h^{Jb}(y) \right) + \frac{1}{2} Q^{IJ} \int_x \int_y \phi_h^{Ia} K_{ab}(x, y) \phi_h^{Jb}(y) \right\} \\
 &= \exp \left\{ -\frac{1}{2} \text{Tr} \log [Q^{IJ} (\sqrt{N} \Delta_l^{-1} \epsilon_{ab} - K_{ab})] + \frac{1}{2} Q^{IJ} \int_u \int_x \int_y \phi_h^{Ia}(x) K_{ac}(x, u) \left[1 - \frac{1}{\sqrt{N}} \Delta_l \epsilon K \right]_b^{-1c}(u, y) \phi_h^{Jb}(y) \right\}. \tag{B8}
 \end{aligned}$$

The subscript Λ indicates the action has an IR cutoff Λ . Taking the limit $\Lambda \rightarrow \infty$ then implies the path integral is fully calculated. The subscript I stands for “interaction.”

The flipped Polchinski ERG equation governs how the quantity $S_{I,\Lambda}[\phi_h]$ changes with change in Λ . It is the counterpart of the Polchinski ERG equation. The rate of change is in terms of the “ERG time” $t \equiv \log(\Lambda/\Lambda_b)$. $t = -\infty$ corresponds to theory which has not been integrated over, and $t = 0$ corresponds to the complete theory being integrated out. The equation is

$$\frac{\partial}{\partial t} e^{-S_{I,\Lambda}} = -\frac{1}{2} Q^{IJ} \epsilon^{ab} \frac{1}{\sqrt{N}} \int_x \int_y \dot{\Delta}_l(x, y) \frac{\delta^2 e^{-S_{I,\Lambda}}}{\delta \phi_h^{Ia} \delta \phi_h^{Jb}(y)}. \tag{B9}$$

It is the usual Polchinski’s equation [55,57] generalized to the case of multiple fields with $USp(N)$ and $SU(2)$ indices.

The “flipped” nature of the equation is coming from choosing to integrate out ϕ_l and keeping ϕ_h instead of the other way around. Since the derivation of Polchinski’s equation is ignorant of the meaning of these labels, exchanging the labels in the standard derivation of Polchinski’s equation gives us the flipped version.

We will set ϕ_h^{Ia} to 0 after obtaining this expression, so we will only have to evaluate the following:

$$\frac{\delta^2 S_{I,\Lambda}}{\delta \phi_h^{Ia} \delta \phi_h^{Jb}(y)} = -Q^{IJ} \int_u K_{ac}(x, u) \left[1 - \frac{1}{\sqrt{N}} \Delta_l \epsilon K \right]_b^{-1c}(u, y). \tag{B10}$$

Expanding the inverse matrix in the center in a power series,

$$\begin{aligned}
 \frac{\delta^2 S_{I,\Lambda}}{\delta \phi_h^{Ia} \delta \phi_h^{Jb}(y)} &= -Q^{IJ} \int_u K_{ac}(x, u) \left[1 - \frac{1}{\sqrt{N}} \Delta_l \epsilon K \right]_b^{-1c}(u, y) \\
 &= -Q^{IJ} \int_u K_{ac}(x, u) \left[\delta_b^c \delta(u - y) + \frac{1}{\sqrt{N}} \epsilon^{cd} \int_v \Delta_l(u, v) K_{db}(v, y) + \frac{1}{N} [\Delta_l \epsilon K \Delta_l \epsilon K]_e^d(u, z) + \dots \right] \\
 &= -Q^{IJ} K_{ab}(x, y) - \frac{1}{\sqrt{N}} Q^{IJ} \epsilon^{cd} \int_u \int_v K_{ac}(x, u) \Delta_l(u, v) K_{db}(v, y) \\
 &\quad - \frac{1}{N} Q^{IJ} \epsilon^{ce} \epsilon^{df} \int_u \int_v K_{ac}(x, u) K_{db}(v, y) \int_z \int_w \Delta_l(u, w) K_{fe}(w, z) \Delta_l(z, y) + O(K^4).
 \end{aligned}$$

We look at the terms on the rhs of (B9) in detail. Note that $Q^{IJ} Q^{IJ} = 2N$.

K term. The first term we have is

$$\begin{aligned}
& -\frac{1}{2\sqrt{N}} \mathcal{Q}^{IJ} \mathcal{Q}^{IJ} \epsilon^{ab} \int_x \int_y K_{ab}(x, y) \dot{\Delta}_l(x, y) = -\sqrt{N} \int_x \int_y \dot{\Delta}_l(x, y) \epsilon^{ab} \left[i\chi_i^\mu \{ \epsilon_{cb}(T_i)_a^c + \epsilon_{ac}(T_i)_b^c \} \frac{\partial}{\partial x^\mu} \right. \\
& \quad \left. + \frac{1}{\sqrt{N}} \chi_i^\mu \chi_\mu^j \epsilon_{cd}(T_i)_a^c (T_j)_b^d \right] \delta(x - y) \\
& = -\sqrt{N} \int_x \left[-i\chi_i^\mu \text{Tr}(T_i) \frac{\partial}{\partial x^\mu} + \frac{1}{\sqrt{N}} \chi_i^\mu \chi_\mu^j \text{Tr}(T_i T_j) \right] \dot{\Delta}_l(x, x) \\
& = -\frac{1}{2} \int_x \chi_i^\mu \chi_\mu^j \dot{\Delta}_l(0), \tag{B11}
\end{aligned}$$

where we have used the trace formulas for the generators. Recall that T_i are the generators of the $j = 1/2$ representation of $\mathfrak{su}(2)$. That is, $T_i = \frac{1}{2}\sigma_i$ where σ_i are the Pauli matrices. We have also made use above of the fact that for the generators, $\epsilon_{ab} T_c^b \epsilon^{cd} = (T^{-1})_a^d = T_a^d$, since Pauli matrices square to identity. The trace formulas for T_i are

$$\text{Tr}(T_i) = 0, \tag{B12}$$

$$\text{Tr}(T_i T_j) = \frac{1}{2} \delta_{ij}, \tag{B13}$$

$$\text{Tr}(T_i T_j T_k) = \frac{i}{4} \epsilon_{ijk}, \tag{B14}$$

$$\text{Tr}(T_i T_j T_k T_l) = \frac{1}{8} (\delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}). \tag{B15}$$

K² term. The following are the terms resulting from the K^2 term:

$$\begin{aligned}
& -\epsilon^{ce} \epsilon^{ab} \int_x \int_y \int_u \int_v K_{ac}(x, u) \Delta_l(u, v) K_{eb}(v, y) \dot{\Delta}_l(x, y) \\
& = -\epsilon^{ce} \epsilon^{ab} \int_x \int_y \int_u \int_v \left[i\chi_i^\mu(x) \{ \epsilon_{dc}(T_i)_a^d + \epsilon_{ad}(T_i)_c^d \} \frac{\partial}{\partial x^\mu} + \frac{1}{\sqrt{N}} \chi_i^\mu(x) \chi_\mu^j(x) \epsilon_{df}(T_i)_a^d (T_j)_c^f \right] \delta(x - u) \Delta_l(u, v) \\
& \quad \times \left[i\chi_k^\nu(v) \{ \epsilon_{gb}(T_k)_e^g + \epsilon_{eg}(T_k)_b^g \} \frac{\partial}{\partial v^\nu} + \frac{1}{\sqrt{N}} \chi_k^\nu(v) \chi_\nu^l(v) \epsilon_{gh}(T_k)_e^g (T_l)_b^h \right] \delta(v - y) \dot{\Delta}_l(x, y) \\
& = -\epsilon^{ce} \epsilon^{ab} \int_x \int_v \left[i\chi_i^\mu(x) \{ \epsilon_{dc}(T_i)_a^d + \epsilon_{ad}(T_i)_c^d \} \frac{\partial}{\partial x^\mu} + \frac{1}{\sqrt{N}} \chi_i^\mu(x) \chi_\mu^j(x) \epsilon_{df}(T_i)_a^d (T_j)_c^f \right] \Delta_l(x, v) \\
& \quad \times \left[i\chi_k^\nu(v) \{ \epsilon_{gb}(T_k)_e^g + \epsilon_{eg}(T_k)_b^g \} \frac{\partial}{\partial v^\nu} + \frac{1}{\sqrt{N}} \chi_k^\nu(v) \chi_\nu^l(v) \epsilon_{gh}(T_k)_e^g (T_l)_b^h \right] \dot{\Delta}_l(x, v) \\
& = -\int_x \int_y \left[4\chi_i^\mu(x) \chi_k^\nu(y) \text{Tr}(T_i T_k) \frac{\partial}{\partial x^\mu} \Delta_l(x, y) \frac{\partial}{\partial y^\nu} \dot{\Delta}_l(x, y) \right. \\
& \quad - 2 \frac{i}{\sqrt{N}} \text{Tr}(T_i T_j T_k) \left(\chi_i^\mu(x) \chi_j^\nu(y) \chi_k^\mu(y) \dot{\Delta}_l(x, y) \frac{\partial}{\partial x^\mu} \Delta_l(x, y) + \chi_i^\mu(x) \chi_\mu^j(x) \chi_k^\nu(y) \Delta_l(x, y) \frac{\partial}{\partial y^\nu} \dot{\Delta}_l(x, y) \right) \\
& \quad \left. - \frac{1}{N} \chi_i^\mu(x) \chi_\mu^j(x) \chi_k^\nu(y) \chi_\nu^l(y) \text{Tr}(T_i T_j T_k T_l) \Delta_l(x, y) \dot{\Delta}_l(x, y) \right], \tag{B16}
\end{aligned}$$

where we have relabeled v to y in the third step. Thus, because the cubic terms in χ in (B16) are symmetric in two of the χ s, on contracting with the ϵ_{ijk} resulting from the traces, they vanish. Therefore, the K^2 term is

$$-\int_x \int_y \left[2\chi_i^\mu(x) \chi_i^\nu(y) \frac{\partial}{\partial x^\mu} \Delta_l(x, y) \frac{\partial}{\partial y^\nu} \dot{\Delta}_l(x, y) - \frac{1}{8N} \chi(x)^2 \chi(y)^2 \Delta_l(x, y) \dot{\Delta}_l(x, y) \right]. \tag{B17}$$

The latter term in this is $O(1/N)$, so we do not look at it. Thus, we have the quadratic term

$$-\frac{1}{2} \int_x \int_y \chi_i^\mu(x) \chi_i^\nu(y) \dot{I}_{\mu\nu}(x, y), \quad (\text{B18})$$

with

$$I_{\mu\nu}(x, y) = \Delta_l(0) \delta(x - y) \delta_{\mu\nu} + 2 \frac{\partial}{\partial x^\mu} \Delta_l(x, y) \frac{\partial}{\partial y^\nu} \Delta_l(y, x). \quad (\text{B19})$$

In momentum space,

$$\begin{aligned} I_{\mu\nu}(p, q) &= \delta_{\mu\nu} \delta(p + q) \int_r \Delta_{l,r} + 2\delta(p + q) \int_r r_\mu(r + p)_\nu \Delta_{l,r} \Delta_{l,p+r} \\ &= \delta_{\mu\nu} \delta(p + q) \int_r \Delta_{l,r} + 2\delta(p + q) \int_r r_\mu r_\nu \Delta_{l,r} \Delta_{l,p+r} \end{aligned} \quad (\text{B20})$$

$$\begin{aligned} &= \delta_{\mu\nu} \delta(p + q) \left\{ \int_r \Delta_{l,r} + \frac{2}{D} \int_r r^2 \Delta_{l,r} \Delta_{l,p+r} \right\} \\ &\equiv \delta_{\mu\nu} \delta(p + q) I(p^2). \end{aligned} \quad (\text{B21})$$

Here $\Delta_{l,r}$ is the Fourier transform of the Δ_l we had earlier, and D is the number of dimensions of the boundary manifold. We have also made use of transversality to get rid of p in the expression.

K^3 term. We look at the $O(1/\sqrt{N})$ term from the K^3 term:

$$\begin{aligned} &-\frac{1}{\sqrt{N}} \epsilon^{ab} \epsilon^{ce} \epsilon^{df} \int_x \int_y \int_z \int_u \int_v \int_w \dot{\Delta}_l(x, y) K_{ac}(x, u) \Delta_l(u, w) K_{fe}(w, z) \Delta_l(z, v) K_{db}(v, y) \\ &= -\frac{1}{\sqrt{N}} \epsilon^{ab} \epsilon^{ce} \epsilon^{df} i^3 \int_x \int_y \int_z \int_u \int_v \int_w \dot{\Delta}_l(x, y) \Delta_l(u, w) \Delta_l(z, v) (\epsilon_{gc}(T_i)_a^g + \epsilon_{ag}(T_i)_c^g) \\ &\quad \times (\epsilon_{hb}(T_j)_d^h + \epsilon_{dh}(T_j)_b^h) (\epsilon_{oe}(T_k)_f^o + \epsilon_{fo}(T_k)_e^o) \chi_i^\mu(u) \chi_j^\nu(v) \chi_k^\rho(w) \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial v^\nu} \frac{\partial}{\partial w^\rho} \delta(x - u) \delta(v - y) \delta(w - z) \\ &= -\frac{8i}{\sqrt{N}} \text{Tr}(T_i T_j T_k) \int_x \int_y \int_z \chi_i^\mu(x) \chi_j^\nu(y) \chi_k^\rho(z) \frac{\partial}{\partial x^\mu} \Delta_l(x, z) \frac{\partial}{\partial z^\rho} \Delta_l(z, y) \frac{\partial}{\partial y^\nu} \dot{\Delta}_l(x, y) \\ &= \frac{2}{\sqrt{N}} \epsilon_{ijk} \int_x \int_y \int_z \chi_i^\mu(x) \chi_j^\nu(y) \chi_k^\rho(z) \frac{\partial}{\partial x^\mu} \Delta_l(x, z) \frac{\partial}{\partial z^\rho} \Delta_l(z, y) \frac{\partial}{\partial y^\nu} \dot{\Delta}_l(x, y), \end{aligned} \quad (\text{B22})$$

where we have made use of the trace formula (B14).

In momentum space, the $O(1/\sqrt{N})$ term is

$$= \frac{2i}{\sqrt{N}} \epsilon_{ijk} \int_p \int_q \int_r \delta^D(p + q + r) \int_k \Delta_{l,k+p} \Delta_{l,k-q} \dot{\Delta}_{l,k}(k, \chi_k(q)) ((k + p) \cdot \chi_i(p)) (k - q) \cdot \chi_j(r). \quad (\text{B23})$$

We now integrate out χ . We proceed as usual by replacing all χ terms apart from the quadratic term resulting from the log in Eq. (B8) by derivatives with respect to its source. As can be seen from (B6), we can replace $\chi_\mu^i(x)$ with $-i\delta/\delta\sigma_\mu^i(x)$, [or $\chi_\mu^i(p)$ with $-i\delta/\delta\sigma_\mu^i(-p)$]⁵, so that the flipped ERG equation (B9) looks like this in terms of σ_μ^i , after setting ϕ_h to 0:

⁵Note that $S_\Lambda[\sigma_\mu^i]$ below does not include the linear term $\int A_\mu^i \sigma_\mu^i$, so it does not appear in the flow equation.

$$\begin{aligned}
\frac{\partial}{\partial t} e^{-S_\Lambda[\sigma_i^\mu]} &= \left\{ \frac{1}{2} \int_x \int_y \dot{I}_{\mu\nu}(x, y) \frac{\delta^2}{\delta\sigma_\mu^i(x) \delta\sigma_\nu^j(y)} \right. \\
&\quad \left. + \frac{2i}{\sqrt{N}} \epsilon_{ijk} \int_x \int_y \int_z \frac{\partial}{\partial x^\mu} \Delta_l(x, z) \frac{\partial}{\partial z^\rho} \Delta_l(z, y) \frac{\partial}{\partial y^\nu} \dot{\Delta}_l(x, y) \frac{\delta^3}{\delta\sigma_\mu^i(x) \delta\sigma_\nu^j(y) \delta\sigma_\rho^k(z)} \right\} e^{-S_\Lambda[\sigma_i^\mu]} + \dots \\
&= \left\{ \frac{1}{2} \int_p \dot{I}(p^2) \frac{\delta^2}{\delta\sigma_\mu^i(p) \delta\sigma_\mu^i(-p)} - \frac{2}{\sqrt{N}} \epsilon_{ijk} \int_p \int_q \int_r \delta^D(p+q+r) \int_k \Delta_{l,k+p} \Delta_{l,k-q} \dot{\Delta}_{l,k}(k+p)_\mu k_\nu(k-q)_\rho \right. \\
&\quad \left. \times \frac{\delta^3}{\delta\sigma_\mu^i(-p) \delta\sigma_\nu^j(-q) \delta\sigma_\rho^k(-r)} \right\} e^{-S_\Lambda[\sigma_i^\mu]} + \dots, \tag{B24}
\end{aligned}$$

where we have written down the leading and subleading order terms in $1/\sqrt{N}$.

The solution of this equation, $S_\Lambda[\sigma]$, is an action for σ which gives approximate results for momenta below Λ . When $\Lambda \rightarrow \Lambda_b$, it becomes exact, i.e., the correlations of σ calculated with it are exact.

We need the form of $S_\Lambda[\sigma_i^\mu]$ for the functional derivative to act on. Remember that ϕ_h has been set to zero now. We have

$$e^{-S_\Lambda[\sigma^\mu]} = \int \mathcal{D}\chi \exp \left\{ i \int_x \chi_i^\mu \sigma_\mu^i \right\} \exp \{ -S_{I,\Lambda}[0, \chi_\mu] \}, \tag{B25}$$

where

$$\begin{aligned}
S_{I,\Lambda}[0, \chi_\mu] &= \frac{1}{2} \text{Tr} \log [Q^{IJ} (\sqrt{N} \Delta_l^{-1} \epsilon_{ab} - K_{ab})] \\
&= \frac{1}{2} \text{Tr} \left[\log (\sqrt{N} Q^{IJ} \Delta_l^{-1} \epsilon_{ab}) - \frac{1}{\sqrt{N}} \delta^{IJ} \Delta_l \epsilon^{ab} K_{bc} - \frac{1}{2N} \delta^{IJ} \Delta_l \epsilon^{ab} K_{bc} \Delta_l \epsilon^{cd} K_{de} - \dots \right] \\
&= -\sqrt{N} \int_x \int_y \Delta_l(x, y) \epsilon^{ab} K_{ba}(y, x) - \frac{1}{2} \int_x \int_y \int_z \int_w \Delta_l(x, y) \epsilon^{ab} K_{bc}(y, z) \Delta_l(z, w) \epsilon^{cd} K_{da}(w, x) + \dots \\
&= \frac{1}{2} \int_x \Delta_l(0) \chi_i^\mu \chi_i^\mu + \int_x \int_y \frac{\partial}{\partial x^\mu} \Delta_l(x, y) \frac{\partial}{\partial y^\nu} \Delta_l(y, x) \chi_i^\mu(x) \chi_i^\nu(y) + \dots \\
&\equiv \frac{1}{2} \int_x \int_y \chi_i^\mu(x) I_{\mu\nu}(x, y) \chi_i^\nu(y) + \dots, \tag{B26}
\end{aligned}$$

where we have kept only the kinetic term (higher order terms in χ_i^μ are also higher order in $1/\sqrt{N}$). Here, I is the *low energy* propagator for the χ field, because we get it after integrating out low energy modes ϕ_l^{Ia} .

Completing the Gaussian from (B25),

$$S_\Lambda[\sigma^\mu] = \frac{1}{2} \int_p \frac{1}{I(p^2)} \sigma_\mu^i(p) \sigma_\mu^i(-p) = \frac{1}{2} \int_x \int_y I_{\mu\nu}^{-1}(x, y) \sigma_\mu^i(x) \sigma_\nu^i(y). \tag{B27}$$

This obeys the ERG equation (B24) to leading order.

For the next order term in the flipped ERG equation, we have from (B24),

$$\begin{aligned}
& - \frac{2}{\sqrt{N}} \epsilon_{ijk} \int_p \int_q \int_r \delta^D(p+q+r) \int_k \Delta_{l,k+p} \Delta_{l,k-q} \dot{\Delta}_{l,k}(k+p)_\mu k_\nu(k-q)_\rho \frac{\delta^3}{\delta\sigma_\mu^i(-p) \delta\sigma_\nu^j(-q) \delta\sigma_\rho^k(-r)} \\
& \quad \times \exp \left\{ -\frac{1}{2} \int_s \frac{1}{I(s^2)} \sigma_{i'}^\kappa(s) \sigma_{\kappa'}^{i'}(-s) \right\} \\
& = \frac{2}{\sqrt{N}} \epsilon_{ijk} \int_p \int_q \int_r \delta^D(p+q+r) \int_k \frac{\Delta_{l,k+p} \Delta_{l,k-q} \dot{\Delta}_{l,k}}{I(p^2) I(q^2) I(r^2)} (k+p)_\mu k_\nu(k-q)_\rho \sigma_\mu^i(p) \sigma_\nu^j(q) \sigma_\rho^k(r) \\
& \quad \times \exp \left\{ -\frac{1}{2} \int_s \frac{1}{I(s^2)} \sigma_{i'}^\kappa(s) \sigma_{\kappa'}^{i'}(-s) \right\}. \tag{B28}
\end{aligned}$$

The flipped ERG equation in terms of σ is

$$\begin{aligned} \frac{\partial}{\partial t} e^{-S_\Lambda[\sigma_i^\mu]} = & \left\{ \frac{1}{2} \int_p \dot{I}(p^2) \frac{\delta^2}{\delta \sigma_i^\mu(p) \delta \sigma_\mu^i(-p)} + \frac{2}{\sqrt{N}} \epsilon_{ijk} \int_p \int_q \int_r \delta^D(p+q+r) \int_k \frac{\Delta_{l,k+p} \Delta_{l,k-q} \dot{\Delta}_{l,k}}{I(p^2) I(q^2) I(r^2)} (k+p)_\mu k_\nu (k-q)_\rho \right. \\ & \left. \times \sigma_i^\mu(p) \sigma_j^\nu(q) \sigma_k^\rho(r) \right\} e^{-S_\Lambda[\sigma_i^\mu]} + \dots \end{aligned} \quad (\text{B29})$$

APPENDIX C: SOLUTION OF THE FLIPPED ERG EQUATION

The flipped ERG equation does not have the physical interpretation of Wilsonian RG evolution. It is closer in spirit to ordinary perturbative field theory calculations where a UV regulator Λ is introduced and all momenta below Λ are integrated and eventually the limit $\Lambda \rightarrow \infty$ is taken (with possible addition of counterterms if there are divergences). It is also closer in spirit to AdS/CFT calculations where the bulk fields are integrated from $z = \infty$ to $z = \epsilon$ and subsequently the $\epsilon \rightarrow 0$ limit is taken.

In the present case, the ϕ fields below Λ are being integrated, keeping σ^μ fixed. When $\Lambda \rightarrow \infty$, one obtains an action for σ^μ that can be used for calculating correlations of

σ^μ . For finite Λ only momentum modes of ϕ below Λ have been integrated. Consequently, only low momentum correlations of σ^μ fields can be calculated. Note that this cannot be interpreted as a conventional Wilson action for σ^μ , since it has no information about physics above Λ . It is as if there is a physical cutoff at Λ . As $\Lambda \rightarrow \infty$ the full σ action is obtained. Note that this is to be interpreted as a bare action for σ^μ because it contains all momentum modes—nothing has been integrated. If one now integrates out the σ^μ modes with a source, one can obtain the generating functional $W[J^\mu]$, where J^μ is a source for σ^μ . This can be obtained by solving the bulk equations by adding a source for σ^μ at the boundary rather than fixing the value of σ^μ at the boundary.

Picking up from (B26), we can write down the action in terms of χ to next order in $1/N$:

$$\begin{aligned} S_{I,\Lambda}[0, \chi_\mu] = & \frac{1}{2} \text{Tr} \log [Q^{IJ} (\sqrt{N} \Delta_l^{-1} \epsilon_{ab} - K_{ab})] \\ = & \frac{1}{2} \text{Tr} \left[\log (\sqrt{N} Q^{IJ} \Delta_l^{-1} \epsilon_{ab}) - \frac{1}{\sqrt{N}} \delta^{IJ} \Delta_l \epsilon^{ab} K_{bc} - \frac{1}{2N} \delta^{IJ} \Delta_l \epsilon^{ab} K_{bc} \Delta_l \epsilon^{cd} K_{de} \right. \\ & \left. - \frac{1}{3N\sqrt{N}} \delta^{IJ} \Delta_l \epsilon^{ab} K_{bc} \Delta_l \epsilon^{cd} K_{de} \Delta_{lxy} \epsilon^{ef} K_{fg} - \dots \right]. \end{aligned} \quad (\text{C1})$$

The cubic term is

$$- \frac{1}{3\sqrt{N}} \int_{1-6} \Delta_{l12} \epsilon^{cd} K_{bc} [23] \Delta_{l34} \epsilon^{cd} K_{de} [45] \Delta_{l56} \epsilon^{ef} K_{fa} [61], \quad (\text{C2})$$

where we are representing coordinates x_1, x_2, x_3, x_4, x_5 , and x_6 by their subscripts. From the definition of K_{ab} in (B3), keeping only the $O(\frac{1}{\sqrt{N}})$ term,

$$\begin{aligned} = & - \frac{1}{3\sqrt{N}} \int_{1-6} \Delta_{l12} 2i\chi_i^\mu(x_2) (T_i)_c^a \partial_{2\mu} \delta_{23} \Delta_{l34} 2i\chi_j^\nu(x_4) (T_j)_e^c \partial_{4\nu} \delta_{45} \Delta_{l56} 2i\chi_k^\rho(x_6) (T_k)_a^e \partial_{6\rho} \delta_{61} \\ = & \frac{8i}{3\sqrt{N}} \text{Tr}(T_i T_j T_k) \int_x \int_y \int_z \chi_i^\mu(x) \chi_j^\nu(y) \chi_k^\rho(z) \frac{\partial}{\partial x^\mu} \Delta_{lxy}(x, y) \frac{\partial}{\partial y^\nu} \Delta_{lxy}(y, z) \frac{\partial}{\partial z^\rho} \Delta_{lxy}(z, x) \\ = & - \frac{2}{3\sqrt{N}} \epsilon_{ijk} \int_x \int_y \int_z \chi_i^\mu(x) \chi_j^\nu(y) \chi_k^\rho(z) \frac{\partial}{\partial x^\mu} \Delta_{lxy}(x, y) \frac{\partial}{\partial y^\nu} \Delta_{lxy}(y, z) \frac{\partial}{\partial z^\rho} \Delta_{lxy}(z, x) \\ = & - \frac{2i}{3\sqrt{N}} \epsilon_{ijk} \int_p \int_q \int_r \delta^D(p+q+r) \int_k \Delta_{l,k+p} \Delta_{l,k-q} \Delta_{h,k} (k+p)_\mu \chi_p^i(k, \chi_q^j) (k-q)_\nu \chi_r^k \\ = & \frac{2}{3\sqrt{N}} \epsilon_{ijk} \int_p \int_q \int_r \delta^D(p+q+r) \int_k \Delta_{l,k+p} \Delta_{l,k-q} \Delta_{h,k} (k+p)_\mu k_\nu (k-q)_\rho \frac{\delta^3}{\delta \sigma_\mu^i(-p) \delta \sigma_\nu^j(-q) \delta \sigma_\rho^k(-r)}. \end{aligned} \quad (\text{C3})$$

In the above, we have made use of the trace formula (B14), and substituted for $\chi_\mu^i(p) - i\delta/\delta\sigma_i^\mu(-p)$ which follows from (B6).

The action for σ_μ is then given from (B25) using usual perturbative methods by

$$e^{-S_\Lambda[\sigma^\mu]} = e^{-\text{cubic term in } \delta/\delta\sigma^\mu} e^{-\text{kinetic term (B27)}}, \quad (\text{C4})$$

so the cubic term in S_Λ is

$$-\frac{2}{3\sqrt{N}}\epsilon_{ijk} \int_p \int_q \int_r \delta^D(p+q+r) \int_k \frac{\Delta_{l,k+p}\Delta_{l,k-q}\Delta_{h,k}}{I(p^2)I(q^2)I(r^2)} (k+p)_\mu k_\nu (k-q)_\rho \sigma_i^\mu(p) \sigma_j^\nu(q) \sigma_k^\rho(r). \quad (\text{C5})$$

The flipped ERG action is therefore

$$S_\Lambda[\sigma^\mu] = \frac{1}{2} \int_p \frac{1}{I(p^2)} \sigma_i^\mu(p) \sigma_\mu^i(-p) - \frac{2}{3\sqrt{N}} \epsilon_{ijk} \int_p \int_q \int_r \delta^D(p+q+r) \int_k \frac{\Delta_{l,k+p}\Delta_{l,k-q}\Delta_{h,k}}{I(p^2)I(q^2)I(r^2)} (k+p)_\mu k_\nu (k-q)_\rho \sigma_i^\mu(p) \sigma_j^\nu(q) \sigma_k^\rho(r), \quad (\text{C6})$$

and it satisfies the flipped ERG equation (B29).

APPENDIX D: THE BOUNDARY ACTION FROM THE BULK ACTION

To verify the nonlocal holographic action given in (16), we solve it classically, and substitute the classical solution for the field in the action to obtain the boundary action.

To the leading order in N. The equations of motion are

$$\frac{\partial}{\partial t} \frac{\dot{\sigma}_i^\mu}{\dot{I}} = 0 \Rightarrow \frac{\dot{\sigma}_i^\mu(p)}{\dot{I}(p^2)} \equiv b_i^\mu(p), \quad (\text{D1})$$

for some constant b_i^μ . Thus,

$$b_i^\mu(p) = \frac{\sigma_i^\mu(t, p) - \sigma_{bi}^\mu(p)}{I(t, p^2) - I_b(p^2)}, \quad (\text{D2})$$

for any t . The subscript b stands for *bare*, and indicates quantities at the boundary.

As an aside, vanishing of the variation of action also implies the following condition:

$$\int_{t_0}^{\bar{t}} dt \int_p \frac{\partial}{\partial t} \left(\delta\sigma_i^\mu(t, p) \frac{\dot{\sigma}_i^\mu(t, p)}{\dot{I}(t, p^2)} \right) + \int_p \delta\sigma_{bi}^\mu(p) \frac{\sigma_{b\mu}^i(p)}{I_b(p^2)} = 0 \quad (\text{D3})$$

$$\begin{aligned} &\Rightarrow \int_{t_0}^{\bar{t}} dt \int_p \frac{\partial}{\partial t} (\delta\sigma_i^\mu(t, p) b_\mu^i(p)) + \int_p \delta\sigma_{bi}^\mu(p) \frac{\sigma_{b\mu}^i(p)}{I_b(p^2)} = 0 \\ &\Rightarrow - \int_p \delta\sigma_{bi}^\mu(p) b_\mu^i(p) + \int_p \delta\sigma_{bi}^\mu(p) \frac{\sigma_{b\mu}^i(p)}{I_b(p^2)} = 0 \\ &\Rightarrow b_i^\mu(p) = \frac{\sigma_{bi}^\mu(p)}{I_b(p^2)}. \end{aligned} \quad (\text{D4})$$

Then we have from (D2)

$$\sigma_i^\mu(p, z) = b_i^\mu(p) I(p, z). \quad (\text{D5})$$

The classical bulk action can then be written [in terms of $\sigma_i^\mu(t_f) \equiv \sigma_{fi}^\mu$ and $\sigma_i^\mu(t_0) \equiv \sigma_{bi}^\mu$] as

$$\begin{aligned} S_{cl}^{(0)}[\sigma_f^\mu, \sigma_b^\mu] &= \frac{1}{2} \int_{t_0}^{t_f} dt \int_p \frac{\dot{\sigma}_i^\mu(p)}{I(p^2)} \dot{\sigma}_\mu^i(-p) = \frac{1}{2} \int_{t_0}^{t_f} dt \int_p b_i^\mu \dot{\sigma}_\mu^i(-p) \\ &= \frac{1}{2} \int_p b_\mu^i(p) (\sigma_{fi}^\mu(-p) - \sigma_{bi}^\mu(-p)) = \frac{1}{2} \int_p \frac{(\sigma_{fi}^\mu(p) - \sigma_{bi}^\mu(p))^2}{I_f(p^2) - I_b(p^2)}, \end{aligned} \quad (D6)$$

from (D2). $I_f(p^2) \equiv I(t_f, p^2)$ and $I_b(p^2) = I(t_0, p^2)$. Since Δ_{lxy} vanishes for $t = t_0$, $I_b = 0$. We also choose to set $\sigma_{bi}^\mu = 0$, since it must vanish in the limit $\Lambda_0 \rightarrow \infty$, and we do not gain anything by choosing a general σ_{bi}^μ for a finite Λ_0 . Thus,

$$S_{cl}^{(0)}[\sigma_f^\mu] = \frac{1}{2} \int_p \frac{\sigma_{fi}^\mu(p) \sigma_{f\mu}^{ij}(-p)}{I_f(p^2)}, \quad (D7)$$

which matches with the Wilson action in (C6) to leading order.

$\mathcal{O}(1/\sqrt{N})$. Again we look at classical solution, including next order terms in EOM. First, we write the solution as

$$\sigma^\mu \equiv \sigma_0^\mu + \frac{1}{\sqrt{N}} \sigma_1^\mu + \mathcal{O}\left(\frac{1}{N}\right). \quad (D8)$$

The equation of motion at order $1/\sqrt{N}$ is [see (16)]

$$\frac{\partial}{\partial t} \frac{\dot{\sigma}_{1\mu}^i(-p)}{I(p^2)} = 6\epsilon_{ijk} \int_q \int_r \delta^D(p+q+r) \mathcal{V}_{\mu\nu\rho}(p, q, r; t) \sigma_{0j}^\nu(q) \sigma_{0k}^\rho(r), \quad (D9)$$

where we have decluttered the equations by introducing

$$\mathcal{V}_{\mu\nu\rho}(p, q, r; t) = \int_k \frac{\Delta_{l,k+p} \Delta_{l,k-q} \dot{\Delta}_{l,k}}{I(p^2) I(q^2) I(r^2)} (k-q)_\mu k_\nu (k+p)_\rho. \quad (D10)$$

We will also use

$$\mathcal{U}_{\mu\nu\rho}(p, q, r; t) = \int_k \frac{\Delta_{l,k+p} \Delta_{l,k-q} \Delta_{l,k}}{I(p^2) I(q^2) I(r^2)} (k-q)_\mu k_\nu (k+p)_\rho. \quad (D11)$$

Note that

$$\begin{aligned} &\frac{\partial}{\partial t} \int_p \int_q \int_r \delta^D(p+q+r) \mathcal{U}_{\mu\nu\rho}(p, q, r; t) \sigma^\mu(p) \sigma^\nu(q) \sigma^\rho(r) \\ &= 3 \int_p \int_q \int_r \delta^D(p+q+r) \mathcal{V}_{\mu\nu\rho}(p, q, r; t) \sigma^\mu(p) \sigma^\nu(q) \sigma^\rho(r), \end{aligned} \quad (D12)$$

when σ^μ is transverse.

We can use this relation to substitute for the order 1 field in terms of order 0 field in the action like so. The order 1 part of the kinetic term is

$$\int_{t_0}^{t_f} dt \int_p \frac{\dot{\sigma}_{1i}^\mu(-p)}{I(p^2)} \dot{\sigma}_{0\mu}^i(p) = - \int_{t_0}^{t_f} dt \int_p \frac{\partial}{\partial t} \left(\frac{\dot{\sigma}_{0i}^\mu(-p)}{I(p^2)} \right) \sigma_{1\mu}^i(p) + \left[\int_p \frac{\dot{\sigma}_{0i}^\mu(-p)}{I(p^2)} \sigma_{1\mu}^i(p) \right]_{t_0}^{t_f}. \quad (D13)$$

We choose the boundary conditions $\sigma_1^\mu = 0$ at both $t = t_0$ and $t = t_f$. Therefore the surface term vanishes. The first term also vanishes because $\frac{\dot{\sigma}_{0i}^\mu(-p)}{I(p^2)} = b^\mu$ is a constant.

Thus, the only term at this order in the EOM is the cubic term in the action at $O(1/\sqrt{N})$ [see (16)], given by

$$-2\epsilon_{ijk} \int_{t_0}^{t_f} dt \int_p \int_q \mathcal{V}_{\mu\nu\rho}(p, q; t) \sigma_{0i}^\mu(p) \sigma_{0j}^\nu(q) \sigma_{0k}^\rho(-p - q) = -\frac{2}{3} \epsilon_{ijk} \int_p \int_q \mathcal{U}_{\mu\nu\rho}(p, q; t_f) \sigma_{f0i}^{\mu i}(p) \sigma_{f0j}^\nu(q) \sigma_{f0k}^\rho(-p - q).$$

Thus, the classical bulk action is

$$S_{cl}[\sigma_{f0}^\mu] = \frac{1}{2} \int_p \frac{\sigma_{f0i}^\mu(p) \sigma_{f0\mu}^i(p)}{I_f(p^2)} - \frac{2}{3\sqrt{N}} \epsilon_{ijk} \int_p \int_q \mathcal{U}_{\mu\nu\rho}(p, q; t_f) \sigma_{f0i}^\mu(p) \sigma_{f0j}^\nu(q) \sigma_{f0k}^\rho(-p - q). \quad (\text{D14})$$

This matches exactly with the flipped ERG action (C6). The classical action is only in terms of σ_{f0}^μ because it is the only free parameter—all of the excitations in σ_f are completely determined in terms of σ_{f0}^μ because of the way we have chosen our boundary conditions.

APPENDIX E: CUBIC TERM z -DEPENDENT PART

The integral to be calculated is

$$\mathcal{K}_{\mu\nu\rho}(p, q, r) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^{2a_1}} \frac{1}{(k+p)^{2a_2}} \frac{1}{(k+p)^{2a_3}} (k-q)_\mu k_\nu (k-q)_\rho. \quad (\text{E1})$$

We will leave the regularization scheme unspecified for now. We are eventually interested in $a_i = 1$ and $D = 3$:

$$\mathcal{K}_{\mu\nu\rho}(p, q, r) = \int ds_1 \int ds_2 \int ds_3 \frac{s_1^{a_1-1} s_2^{a_2-1} s_3^{a_3-1}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int \frac{d^D k}{(2\pi)^D} e^{-(k-q)^2 s_3} e^{-(k+p)^2 s_2} e^{-k^2 s_1} (k+p)_\mu k_\nu (k-q)_\rho. \quad (\text{E2})$$

Setting $a_i = 1$,

$$\mathcal{K}_{\mu\nu\rho}(p, q, r) = \int ds_1 \int ds_2 \int ds_3 \int \frac{d^D k}{(2\pi)^D} \exp\{-(k-q)^2 s_3 - (k+p)^2 s_2 - k^2 s_1\} (k+p)_\mu k_\nu (k-q)_\rho.$$

We make a transformation of variables to get a known integral,

$$\begin{aligned} \ell &\equiv s_1 + s_2 + s_3; & \beta_1 &\equiv \frac{s_2 s_3}{\ell}; & \beta_2 &\equiv \frac{s_3 s_1}{\ell}; & \beta_3 &\equiv \frac{s_1 s_2}{\ell} \\ k \rightarrow k' &\equiv k + \ell^{-1/2} \sqrt{\beta_1 \beta_2 \beta_3} \left(\frac{p}{\beta_2} - \frac{q}{\beta_3} \right); & \ell &= \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3} \right)^2 \beta_1 \beta_2 \beta_3. \end{aligned} \quad (\text{E3})$$

The Jacobian for this transformation is

$$J^{-1} = \left| \det \frac{\partial \beta_i}{\partial s_j} \right| = \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3} \right) \frac{\beta_1 \beta_2 \beta_3}{\ell^2} = \frac{\sqrt{\beta_1 \beta_2 \beta_3}}{\ell^{3/2}}. \quad (\text{E4})$$

We had done this substitution combining two transformations of variables. The first one so we could simplify the k integral, and the second one so we could have the result in an easily identifiable standard integral:

$$\begin{aligned} \mathcal{K}_{\mu\nu\rho}(p, q, r) &= \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty d\beta_3 \int_{k'} \frac{\ell^{3/2}}{\sqrt{\beta_1 \beta_2 \beta_3}} \exp\{-\ell k'^2 - \beta_1 (p+q)^2 - \beta_2 q^2 - \beta_3 p^2\} \\ &\quad \times (k' + (1-x)p + yq)_\mu (k' - xp + yq)_\nu (k' - xp - (1-y)q)_\rho, \end{aligned} \quad (\text{E5})$$

where

$$x \equiv \sqrt{\frac{\beta_1 \beta_2 \beta_3}{\ell}} \frac{1}{\beta_2}; \quad y \equiv \sqrt{\frac{\beta_1 \beta_2 \beta_3}{\ell}} \frac{1}{\beta_3}; \quad 1 - x - y \equiv \sqrt{\frac{\beta_1 \beta_2 \beta_3}{\ell}} \frac{1}{\beta_1}.$$

Since this integral couples to fields $a^\mu(-p)a^\nu(-q)a^\rho(-r)\delta^D(p+q+r)$ which are transverse, and since $p^\mu + q^\mu + r^\mu = 0$,

$$\begin{aligned} \mathcal{K}_{\mu\nu\rho}(p, q, r) &= \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty d\beta_3 \int_k \frac{\ell^{3/2}}{\sqrt{\beta_1 \beta_2 \beta_3}} \exp\{-\ell k'^2 - \beta_1 r^2 - \beta_2 q^2 - \beta_3 p^2\} \\ &\times (k' + yq)_\mu (k' + xr)_\nu (k' + (1-x-y)p)_\rho. \end{aligned}$$

The k' integral only keeps the terms even in k' , and $\int_k f(k^2)k_\mu k_\nu = \delta_{\mu\nu}/D \int_k k^2 f(k^2)$. In performing the k' integral, note that

$$\ell^{3/2} \int \frac{d^D k'}{(2\pi)^D} k'^n \exp\{-\ell k'^2\} = \frac{1}{2(2\pi)^D} \ell^{\frac{3}{2} - \frac{D+n}{2}} \Gamma\left(\frac{D+n}{2}\right). \quad (\text{E6})$$

Term 1.

$$\begin{aligned} &q_\mu r_\nu p_\rho \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty d\beta_3 \int_k' \frac{\ell^{3/2}}{\sqrt{\beta_1 \beta_2 \beta_3}} xy(1-x-y) \exp\{-\ell k'^2 - \beta_1 r^2 - \beta_2 q^2 - \beta_3 p^2\} \\ &= \frac{\Gamma(3/2)}{2(2\pi)^3} q_\mu r_\nu p_\rho \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty d\beta_3 \frac{1}{\sqrt{\beta_1 \beta_2 \beta_3}} xy(1-x-y) \exp\{-\beta_1 r^2 - \beta_2 q^2 - \beta_3 p^2\} \\ &= \frac{\Gamma(3/2)}{2(2\pi)^3} q_\mu r_\nu p_\rho \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty d\beta_3 \frac{1}{\ell^{3/2}} \exp\{-\beta_1 r^2 - \beta_2 q^2 - \beta_3 p^2\} \\ &= \frac{\Gamma(3/2)}{2(2\pi)^3} q_\mu r_\nu p_\rho \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty d\beta_3 \exp\{-\beta_1 r^2 - \beta_2 q^2 - \beta_3 p^2\} (\beta_1 \beta_2 \beta_3)^{-3/2} \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3}\right)^{-3}. \end{aligned}$$

Term 2.

$$\begin{aligned} &\frac{1}{D} \frac{\Gamma(5/2)}{2(2\pi)^3} \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty d\beta_3 \int_k \frac{\ell^{3/2}}{\sqrt{\beta_1 \beta_2 \beta_3}} k^2 \exp\{-\ell k^2 - \beta_1 r^2 - \beta_2 q^2 - \beta_3 p^2\} ((1-x-y)\delta_{\mu\nu} p_\rho + yq_\mu \delta_{\nu\rho} + xr_\nu \delta_{\mu\rho}) \\ &= \frac{1}{D} \frac{\Gamma(5/2)}{2(2\pi)^3} \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty d\beta_3 \frac{\ell^{-1}}{\sqrt{\beta_1 \beta_2 \beta_3}} \exp\{-\beta_1 r^2 - \beta_2 q^2 - \beta_3 p^2\} ((1-x-y)\delta_{\mu\nu} p_\rho + yq_\mu \delta_{\nu\rho} + xr_\nu \delta_{\mu\rho}) \\ &= \frac{1}{D} \frac{\Gamma(5/2)}{2(2\pi)^3} \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty d\beta_3 \ell^{-3/2} \exp\{-\beta_1 r^2 - \beta_2 q^2 - \beta_3 p^2\} \left(\frac{1}{\beta_1} \delta_{\mu\nu} p_\rho + \frac{1}{\beta_3} q_\mu \delta_{\nu\rho} + \frac{1}{\beta_2} r_\nu \delta_{\mu\rho}\right) \\ &= \frac{1}{D} \frac{\Gamma(5/2)}{2(2\pi)^3} \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty d\beta_3 \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3}\right)^{-3} (\beta_1 \beta_2 \beta_3)^{-3/2} \\ &\times \exp\{-\beta_1 r^2 - \beta_2 q^2 - \beta_3 p^2\} \left(\frac{1}{\beta_1} \delta_{\mu\nu} p_\rho + \frac{1}{\beta_3} q_\mu \delta_{\nu\rho} + \frac{1}{\beta_2} r_\nu \delta_{\mu\rho}\right). \end{aligned}$$

We use the following integral:

$$2^{-2a} \int_0^\infty dX \frac{X^{a-1}}{\Gamma(a)} e^{-XT/4} = T^{-a}. \quad (\text{E7})$$

Then

Term 1

$$= \frac{\Gamma(3/2)}{2^7 \Gamma(3) (2\pi)^3} q_\mu r_\nu p_\rho \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty d\beta_3 \int_0^\infty dX X^2 (\beta_1 \beta_2 \beta_3)^{-3/2} \exp \left\{ -\beta_1 r^2 - \beta_2 q^2 - \beta_3 p^2 - \frac{X}{4} \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3} \right) \right\}.$$

Term 2

$$= \frac{1}{D} \frac{\Gamma(5/2)}{2^7 \Gamma(3) (2\pi)^3} \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \int_0^\infty d\beta_3 \int_0^\infty dX \frac{X^2}{\Gamma(3)} (\beta_1 \beta_2 \beta_3)^{-D/2} \exp \left\{ -\beta_1 r^2 - \beta_2 q^2 - \beta_3 p^2 - \frac{X}{4} \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3} \right) \right\} \left(\frac{1}{\beta_1} \delta_{\mu\nu} p_\rho + \frac{1}{\beta_3} q_\mu \delta_{\nu\rho} + \frac{1}{\beta_2} r_\nu \delta_{\mu\rho} \right).$$

These β integrals are of the form

$$\int_0^\infty d\beta \beta^{-\nu-1} \exp \left\{ -k^2 \beta - \frac{X}{4\beta} \right\} = 2^{1+\nu} (k/\sqrt{X})^\nu K_\nu(k\sqrt{X}). \quad (\text{E8})$$

In this integral, X acts as the UV cutoff for β , which is of the dimension -2 , and k^2 acts as the IR cutoff. At this stage, we identify the lower bound of the X integral with the inverse square of the UV cutoff $1/\Lambda^2 = z^2$. Thus, we have (apologies for using the same symbol ν as the quantity $\Delta - D/2 = 1/2$ and as a spacetime index for the momenta)

Term 1

$$= \frac{2^{3\nu} \Gamma(3/2)}{2^5 (2\pi)^3} q_\mu r_\nu p_\rho \int_{z^2}^\infty dX X^2 \left(\frac{p}{\sqrt{X}} \right)^\nu K_\nu(p\sqrt{X}) \left(\frac{q}{\sqrt{X}} \right)^\nu K_\nu(q\sqrt{X}) \left(\frac{r}{\sqrt{X}} \right)^\nu K_\nu(r\sqrt{X}).$$

Term 2

$$= \frac{1}{D} \frac{2^{3\nu} \Gamma(5/2)}{2^4 (2\pi)^3} \int_{z^2}^\infty dX \frac{X^2}{2} \left[\delta_{\mu\nu} p_\rho \left(\frac{r}{\sqrt{X}} \right)^{1+\nu} \left(\frac{p}{\sqrt{X}} \right)^\nu \left(\frac{q}{\sqrt{X}} \right)^\nu K_{1+\nu}(r\sqrt{X}) K_\nu(p\sqrt{X}) K_\nu(q\sqrt{X}) \right. \\ \left. + q_\mu \delta_{\nu\rho} \left(\frac{r}{\sqrt{X}} \right)^\nu \left(\frac{p}{\sqrt{X}} \right)^{1+\nu} \left(\frac{q}{\sqrt{X}} \right)^\nu K_\nu(r\sqrt{X}) K_{1+\nu}(p\sqrt{X}) K_\nu(q\sqrt{X}) \right. \\ \left. + r_\nu \delta_{\mu\rho} \left(\frac{r}{\sqrt{X}} \right)^\nu \left(\frac{p}{\sqrt{X}} \right)^\nu \left(\frac{q}{\sqrt{X}} \right)^{1+\nu} K_\nu(r\sqrt{X}) K_\nu(p\sqrt{X}) K_{1+\nu}(q\sqrt{X}) \right].$$

In the above, when the momenta p, q, r appear without their index, it refers to their magnitude.

The derivative of this with respect to z appears in our cubic term,

$$\frac{\partial}{\partial z} \mathcal{K}_{\mu\nu\rho}(p, q, r; z) = -\frac{2^{3\nu} \Gamma(3/2)}{2^4 (2\pi)^3} z^5 q_\mu r_\nu p_\rho \left(\frac{r}{z} \right)^\nu \left(\frac{p}{z} \right)^\nu \left(\frac{q}{z} \right)^\nu K_\nu(rz) K_\nu(pz) K_\nu(qz) \\ - \frac{1}{D} \frac{2^{3\nu} \Gamma(5/2)}{2^3 (2\pi)^3} z^5 \left[\delta_{\mu\nu} p_\rho \left(\frac{r}{z} \right)^{1+\nu} \left(\frac{p}{z} \right)^\nu \left(\frac{q}{z} \right)^\nu K_{1+\nu}(rz) K_\nu(pz) K_\nu(qz) \right. \\ \left. + q_\mu \delta_{\nu\rho} \left(\frac{r}{z} \right)^\nu \left(\frac{p}{z} \right)^{1+\nu} \left(\frac{q}{z} \right)^\nu K_\nu(rz) K_{1+\nu}(pz) K_\nu(qz) \right. \\ \left. + r_\nu \delta_{\mu\rho} \left(\frac{r}{z} \right)^\nu \left(\frac{p}{z} \right)^\nu \left(\frac{q}{z} \right)^{1+\nu} K_\nu(rz) K_\nu(pz) K_{1+\nu}(qz) \right]. \quad (\text{E9})$$

We make use of the following property of modified Bessel functions⁶ to reduce the parameter of the Bessel functions to ν :

⁶See the entry at Wolfram Functions site: <https://web.archive.org/web/20081013054654/https://functions.wolfram.com/Bessel-TypeFunctions/BesselK/introductions/Bessels/05/>.

$$K_{1+\nu}(x) = K_{\nu-1}(x) + \frac{2\nu}{x}K_{\nu}(x). \quad (\text{E10})$$

For our purposes, $\nu = 1/2$ [because $K_{1/2}(x) = K_{-1/2}(x)$],

$$K_{3/2}(pz) = (1 + 1/pz)K_{1/2}(pz). \quad (\text{E11})$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial z} \mathcal{K}_{\mu\nu\rho}(p, q, r; z) &= -\frac{1}{16(2\pi)^{5/2}} z^5 q_{\mu} r_{\nu} p_{\rho} \left(\frac{r}{z}\right)^{\nu} \left(\frac{p}{z}\right)^{\nu} \left(\frac{q}{z}\right)^{\nu} K_{\nu}(rz) K_{\nu}(pz) K_{\nu}(qz) \\ &\quad - \frac{1}{16(2\pi)^{5/2}} z^5 \left[\delta_{\mu\nu} p_{\rho} \left(\frac{r}{z}\right)^{1+\nu} \left(\frac{p}{z}\right)^{\nu} \left(\frac{q}{z}\right)^{\nu} \left(1 + \frac{2\nu}{rz}\right) + q_{\mu} \delta_{\nu\rho} \left(\frac{r}{z}\right)^{\nu} \left(\frac{p}{z}\right)^{1+\nu} \left(\frac{q}{z}\right)^{\nu} \left(1 + \frac{2\nu}{pz}\right) \right. \\ &\quad \left. - p_{\nu} \delta_{\mu\rho} \left(\frac{r}{z}\right)^{\nu} \left(\frac{p}{z}\right)^{\nu} \left(\frac{q}{z}\right)^{1+\nu} \left(1 + \frac{2\nu}{qz}\right) \right] K_{\nu}(rz) K_{\nu}(pz) K_{\nu}(qz) \\ &= -\frac{1}{16(2\pi)^{5/2}} z^3 [q_{\mu} r_{\nu} p_{\rho} z^2 + \delta_{\mu\nu} p_{\rho} (1 + rz) + q_{\mu} \delta_{\nu\rho} (1 + pz) + r_{\nu} \delta_{\mu\rho} (1 + qz)] \\ &\quad \times \left(\frac{r}{z}\right)^{\nu} \left(\frac{p}{z}\right)^{\nu} \left(\frac{q}{z}\right)^{\nu} K_{\nu}(rz) K_{\nu}(pz) K_{\nu}(qz). \end{aligned} \quad (\text{E12})$$

In [VB](#), we see that the factor $(\frac{r}{z})^{\nu} (\frac{p}{z})^{\nu} (\frac{q}{z})^{\nu} K_{\nu}(rz) K_{\nu}(pz) K_{\nu}(qz)$ is canceled.

APPENDIX F: BOUNDARY THREE-POINT FUNCTION FROM THE BULK ACTION

In this appendix, we verify that the three-point correlator calculated by the bulk AdS action in [\(48\)](#) obeys the conformal ward identities by matching it with [\[48\]](#).

In [Sec. VI](#), we showed the action [\(48\)](#) is equivalent to the action in [\(47\)](#) on shell,

$$S_3 = -\epsilon_{ijk} \int dz \int_p \int_q \int_r \delta^D(p + q + r) [2g' z^2 q_{\mu} r_{\nu} p_{\rho} a_i^{\mu}(p) a_j^{\nu}(q) a_k^{\rho}(r) + (g + 6g' |r| z) a_i^{\mu}(p) a_{\mu}^j(q) \times p_{\rho} a_k^{\rho}(r)]. \quad (\text{F1})$$

The solution to the equations of motion to $O(g^0, g'^0)$ is given in [\(57\)](#):

$$\bar{a}_{\mu}^i(p, z) = A_{\mu}^i(p) \sqrt{|p|z} K_{\frac{1}{2}}(pz), \quad (\text{F2})$$

where $A_{\mu}^i(p)$ is the boundary source. Substituting this into [\(F1\)](#), we have

$$\begin{aligned} S_{\text{cl.3}} &= -\epsilon_{ijk} \int_p \int_q \int_r \delta^D(p + q + r) A_i^{\mu}(p) A_j^{\nu}(q) A_k^{\rho}(r) \sqrt{|p||q||r|} \\ &\quad \times \left[2g' q_{\mu} r_{\nu} p_{\rho} \int dz z^{7/2} K_{\frac{1}{2}}(pz) K_{\frac{1}{2}}(qz) K_{\frac{1}{2}}(rz) + p_{\rho} \delta_{\mu\nu} \left(g \int dz z^{3/2} K_{\frac{1}{2}}(pz) K_{\frac{1}{2}}(qz) K_{\frac{1}{2}}(rz) \right. \right. \\ &\quad \left. \left. + 6g' |r| \int dz z^{5/2} K_{\frac{1}{2}}(pz) K_{\frac{1}{2}}(qz) K_{\frac{1}{2}}(rz) \right) \right]. \end{aligned} \quad (\text{F3})$$

The integrals of the bessel functions can be computed to get

$$\begin{aligned} S_{\text{cl.3}} &= -\epsilon_{ijk} \left(\frac{\pi}{2}\right)^{3/2} \int_p \int_q \int_r \delta^D(p + q + r) A_i^{\mu}(p) A_j^{\nu}(q) A_k^{\rho}(r) \left[4g' q_{\mu} r_{\nu} p_{\rho} \frac{1}{(|p| + |q| + |r|)^3} \right. \\ &\quad \left. + p_{\rho} \delta_{\mu\nu} \left(\frac{g}{|p| + |q| + |r|} + 6g' \frac{|r|}{(|p| + |q| + |r|)^2} \right) \right]. \end{aligned} \quad (\text{F4})$$

This matches with the result for three-point functions obeying conformal Ward identities in [Sec. 9.5 of \[48\]](#).

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