

One-loop renormalization of anisotropic two-scalar quantum field theories

Dmitry S. Ageev^{1,2,*} and Yulia A. Ageeva^{2,3,4,†}

¹*Department of Mathematical Methods for Quantum Technologies, [Steklov Mathematical Institute of Russian Academy of Sciences](#), Gubkina street 8, Moscow 119991, Russia*

²*Institute for Theoretical and Mathematical Physics, [Lomonosov Moscow State University](#), 119991 Moscow, Russia*

³*Institute for Nuclear Research of the Russian Academy of Sciences, 60th October Anniversary Prospect, 7a, 117312 Moscow, Russia*

⁴*Department of Particle Physics and Cosmology, Physics Faculty, M.V. [Lomonosov Moscow State University](#), Leninskie Gory 1-2, 119991 Moscow, Russia*



(Received 30 January 2026; accepted 12 May 2026; published 8 June 2026)

We develop a basis-covariant one-loop renormalization framework for two interacting real scalars in $D = 4 - \epsilon$ with the most general parity even two-derivative Lorentz-violating quadratic form, allowing anisotropic spatial gradients and direction-dependent kinetic mixing, together with general cubic and quartic interactions and the relevant lower-dimensional operators forming a renormalization group complete set of operators at one-loop. In dimensional regularization with minimal subtraction we compute the full set of one-loop UV divergences and obtain closed beta functions for quartic and cubic couplings, masses, and linear couplings. The pole coefficients admit a universal spectral representation as angular averages over the direction-dependent eigenvalues and projectors of the UV kinetic matrix; all anisotropy dependence enters through a single universal kernel admitting two-particle phase-space interpretation. We classify fixed points and fixed manifolds and show, in particular, that anisotropy restricts the existence of the coupled Wilson-Fisher-type fixed point. When the cross-gradients are turned on, the coefficients in beta functions are governed by six phase-space weights admitting interpretation in terms of encoding “populations” and “coherences” of the UV normal modes.

DOI: [10.1103/3ryd-3hx4](#)

I. INTRODUCTION

Lorentz invariance is one of the organizing principles of high-energy physics: it underlies the locality and analyticity properties of relativistic quantum field theory (QFT), and it severely constrains the spectrum of operators and provides the kinematic backbone for essentially all perturbative calculations. At the same time, there are many reasons—both conceptual and practical—to study controlled departures from exact Lorentz symmetry. On the one hand, Lorentz invariance may be an *emergent* symmetry of an interacting many-body system at long distances: in condensed-matter realizations, different fields typically propagate with different “speeds,” and the ultraviolet (UV) theory is anisotropic, while approximate relativistic invariance can appear only in the deep infrared (IR) region [1–3]. On the

other hand, in particle physics and quantum gravity it is natural to ask how robust the Lorentz-invariant fixed points are under Lorentz-violating (LV) perturbations, and whether small violations can remain technically natural under renormalization [4–6] [as explored, for instance, in effective field theory (EFT) frameworks such as the Standard-Model Extension and its gravitational generalizations [7–11] and in the broader phenomenological literature [12–14]].

There are (at least) two complementary ways in which Lorentz invariance can be violated in a continuum field theory. The first is to change the *scaling* structure by introducing higher spatial derivatives (Lifshitz-type theories with dynamical exponent $z \neq 1$,¹ including the Hořava-Lifshitz construction in gravity) [16–24]. The second—and in some sense the minimal one—is to keep the standard two-derivative structure (so that the canonical scaling is still relativistic, $z = 1$) but to allow different fields, or different linear combinations of fields, to propagate on different effective “light cones” (or, in Euclidean language, with different positive-definite quadratic forms in spatial gradients and/or time derivatives). This second class of models

*Contact author: ageev@mi-ras.ru

†Contact author: ageeva@inr.ac.ru

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](#) license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI. Funded by SCOAP³.

¹See, for example, [15] and references therein.

is ubiquitous: it appears as the low-energy theory of coupled order parameters with different stiffnesses, as multifield EFTs with distinct sound speeds [25–27], and as toy models for “multimetric” matter sectors in Lorentz-violating extensions of relativistic QFT [28,29].

Lorentz violation has been systematized from several complementary angles. In the EFT approach, the Standard-Model Extension provides a general local operator basis [7–10], while simplified “different limiting velocities” parametrizations were emphasized in [28,30,31]. See also classic LV electrodynamics examples such as [32]; more on LV in such setups can also be found in [33,34]. Renormalization and operator mixing in LV gauge theories have been investigated in a number of sectors, including one-loop analysis in Lorentz-violating scalar-spinor interactions [35–39], quantum electrodynamics (QED)–type settings [40,41], and quantum chromodynamics (QCD), in [42]. Reviews and continually updated experimental constraints can be found in [11–14]. Naturalness issues for small LV under radiative corrections have been highlighted in [4,5], with possible protection mechanisms in supersymmetric settings [6]. Moreover, higher-loop results for Lorentz-violating $O(N)$ scalar theories were obtained in Refs. [43–45], including the explicit three-loop field renormalization, an all-loop renormalization-group analysis, and an all-loop treatment of ultraviolet divergences. Another interesting Lorentz-violating setup, involving spontaneous Lorentz violation with an antisymmetric rank-two tensor, was studied in Ref. [46]. On the many-body side, early renormalization group (RG) arguments for the emergence of a common effective light cone go back to [1] (see also [2,3]), and in Lorentz-violating gravity, controlled preferred-frame frameworks include Einstein-aether and Hořava-Lifshitz models [17–20,29,47]. In [48] authors consider the renormalization of velocities in non-Lorentz-invariant sigma models, showing that the velocity differences flow to zero at low enough energies and it is possible to restore the Lorentz invariance.

Related setups—where scalar perturbations propagate with reduced or field-dependent sound speeds—arise in single- and multifield inflation [26,49,50] and in other contexts [51,52]. The study of multiscalar theories with different sound speeds was proposed in Refs. [53,54], where the authors analyze partial-wave unitarity.

The present work complements these studies by providing a compact, basis-covariant one-loop renormalization machinery tailored to multimetric two-derivative scalar theories with direction-dependent mixing. Related renormalization analyses for spatially anisotropic QFTs include Refs. [55,56]. Another similar setup is $O(N)$ self-interacting scalar field theory with Lorentz violation from [57], where a mass renormalization up to the two-loop level has been proceeded. Two closely related predecessors are Refs. [36,37], both of which analyze Lorentz-violating scalar theories in the special case where all field

components share a single quadratic form, so that the Lorentz-violating dependence of the RG functions is controlled by one determinant factor Π . Our results reproduce that structure in the common-metric $O(2)$ subsector of the present model. The novelty of our analysis is that we allow the two scalars to carry unequivalent quadratic forms, together with field-space time-derivative mixing and cross-gradients, and in this genuinely multimetric two-field regime the anisotropy no longer reduces to a single universal factor but instead enters through channel-dependent weights.

A basic structural question in this setting is the following: *how do interactions renormalize when the quadratic kinetic operator is anisotropic and mixes fields in a direction-dependent way?* Even for scalar theories, a fully systematic answer requires some care. First, the propagator is matrix-valued in field space, and kinetic mixing terms imply that the normal modes are not the original fields but momentum-dependent linear combinations. For example, in [58] there is a setup similar to ours with both kinetic and mass mixing: it was investigated whether the kinetic term can be kept diagonal under renormalization group evolution, even though the fields mix. Second, in dimensional regularization and minimal subtraction [59–61] one must separate UV poles from possible IR subtleties of massless vacuum diagrams. Third, if one wants results that are robust under field redefinitions, it is desirable to express UV divergences not in a particular diagonal basis but directly in terms of invariant spectral data of the quadratic operator [62].

One might naively think just to introduce anisotropy in spatial derivatives and study such theory with kinetic term as follows:

$$\mathcal{L}_{\text{kin}}^0 = \frac{1}{2}(\partial_0 \boldsymbol{\phi})^\top (\partial_0 \boldsymbol{\phi}) + \frac{1}{2}(\nabla \boldsymbol{\phi})^\top \mathbf{C}_\phi (\nabla \boldsymbol{\phi}) + \frac{1}{2}(\nabla \chi)^\top \mathbf{C}_\chi (\nabla \chi).$$

However, to keep the theory RG complete (in the presence of couplings that break the field-reflection symmetry, e.g., the $\boldsymbol{\phi} \rightarrow -\boldsymbol{\phi}$ symmetry) one must consider a more complicated version of such a theory with the addition of a cross-derivative term and keeping arbitrary normalization, i.e., adding to the theory terms of the form

$$\frac{1}{2}(\partial_0 \boldsymbol{\phi})^\top \mathbf{Z}_t (\partial_0 \boldsymbol{\phi}) + (\nabla \boldsymbol{\phi})^\top \mathbf{Y} (\nabla \chi). \quad (1)$$

The latter term with gradient mixing may not be optional and arises in different physical systems, for example, in superfluids solutions [63]. In this work we develop a one-loop renormalization of the theory for the simplest interacting model that contains all of these ingredients: two real scalar fields $\boldsymbol{\phi}$ and χ in $D = 4 - \epsilon$ Euclidean dimensions with the most general quadratic form built from two derivatives within the parity even sector, i.e., including time-derivative mixing and cross-gradient mixing, together with the most general cubic and quartic interactions. Mixed

time-gradient terms are excluded because they are parity odd and, hence, are not generated radiatively. As a result, the kinetic sector is closed under the RG flow.

Concretely, the kinetic terms are governed by (i) a positive-definite 2×2 matrix $\mathbf{Z}_t$ in field space multiplying $\partial_0 \varphi_i \partial_0 \varphi_j$, and (ii) three real symmetric $d \times d$ matrices $\mathbf{C}_\phi$, $\mathbf{C}_\chi$, $\mathbf{Y}$ multiplying spatial gradients, including the cross term $Y^{ab} \partial_a \phi \partial_b \chi$. We also allow a general symmetric mass matrix $\mathbf{M}^2$, and we keep linear couplings j_i because they are generated whenever cubic couplings and masses are present. The resulting quadratic operator defines, for every direction $\hat{\mathbf{n}} \in S^{d-1}$, the Z_t -normalized UV kernel $\mathbf{C}_Z(\hat{\mathbf{n}})$, a positive-definite 2×2 matrix in field space with positive eigenvalues $\mathbf{c}_\alpha(\hat{\mathbf{n}})$ and orthogonal projectors $\Pi_\alpha(\hat{\mathbf{n}})$. Physically, $v_\alpha(\hat{\mathbf{n}}) \equiv \sqrt{\mathbf{c}_\alpha(\hat{\mathbf{n}})}$ are the *directional sound speeds* (or, in Lorentzian signature, limiting velocities) of the UV normal modes.

We show that the one-loop UV poles of the theory can be written in a remarkably universal form. At $D = 4 - \epsilon$ all one-loop divergences are controlled by two “master integrals”: the tadpole [which carries one insertion of the mass operator in the minimal subtraction (MS) renormalization scheme] and the bubble at zero external momentum. After performing the frequency integral, both reduce to angular averages over $\hat{\mathbf{n}}$ dressed by a single positive kernel

$$\mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) \equiv \frac{1}{\sqrt{\mathbf{c}_\alpha} \sqrt{\mathbf{c}_\beta} (\sqrt{\mathbf{c}_\alpha} + \sqrt{\mathbf{c}_\beta})} = \frac{1}{v_\alpha v_\beta (v_\alpha + v_\beta)}. \quad (2)$$

This kernel admits a direct physical interpretation: it is the two-particle energy denominator (and, equivalently, the two-particle phase-space density per logarithmic interval in the RG scale) for intermediate UV modes with linear dispersion $E_\alpha(\mathbf{p}) = |\mathbf{p}| v_\alpha(\hat{\mathbf{n}})$. Anisotropy and mixing enter only through the directional velocities $v_\alpha(\hat{\mathbf{n}})$ and through the “coherence factors” encoded in $\Pi_\alpha(\hat{\mathbf{n}})$, which determine how strongly the UV normal modes overlap with the original fields.

A. Summary of results

The paper contains both general structural results and explicit fixed-point analyses in consistent one-loop truncations. The main results are as follows:

- (1) *General one-loop MS renormalization for a two-scalar anisotropic QFT.* We compute the complete

set of one-loop UV divergences in dimensional regularization with minimal subtraction near $D = 4 - \epsilon$ for the theory (10) and (11). All counterterms and beta functions are expressed in compact tensor form in terms of two finite objects: the tadpole residue $\mathcal{T}_{ij}$ and the bubble residue $\mathcal{B}_{ij;kl}$.

- (2) *Spectral representation of UV pole coefficients.* We show that both $\mathcal{T}$ and $\mathcal{B}$ can be written as angular averages over the UV spectral data of some operator $\mathbf{C}_Z(\hat{\mathbf{n}})$:

$$\mathcal{B}_{ij;kl} = \frac{1}{4\pi^2} \left\langle \sum_{\alpha,\beta} \mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) \mathbf{P}_{Z,\alpha}(\hat{\mathbf{n}})_{ik} \mathbf{P}_{Z,\beta}(\hat{\mathbf{n}})_{jl} \right\rangle_{S^2},$$

$$\mathbf{P}_{Z,\alpha}(\hat{\mathbf{n}}) \equiv \mathbf{Z}_t^{-1/2} \Pi_\alpha(\hat{\mathbf{n}}) \mathbf{Z}_t^{-1/2},$$

and similarly for $\mathcal{T}$ with a single insertion of the mass operator [see Eqs. (41) and (45) in the main text]. This makes the dependence on the anisotropy data $(\mathbf{Z}_t, \mathbf{C}_{\phi,\chi}, \mathbf{Y})$ explicit and basis covariant.

- (3) *Locality of one-loop poles and nonrunning of anisotropy matrices in MS.* A key structural simplification is that in $D = 4 - \epsilon$ the one-loop UV poles in two-point functions are momentum-independent (compare the standard Lorentz-invariant ϕ^4 case, where wave function renormalization starts at two loops in MS; see, e.g., [64,65]). As a consequence, in dimensional regularization (DR) + MS at one-loop

$$\beta_{\mathbf{Z}_t} = \beta_{\mathbf{C}_\phi} = \beta_{\mathbf{C}_\chi} = \beta_{\mathbf{Y}} = 0 \quad (\text{one-loop, MS}), \quad (3)$$

while the couplings $(\lambda_{ijkl}, h_{ijk}, M_{ij}^2, j_i)$ do run. Notice that while there are no divergent parts to get nonvanishing beta functions, finite pieces may be generated at the one-loop level, and this is the reason to include them obtaining a RG closed set of operators. In other words, at this order the anisotropy data act as *external parameters* that weight the RG flow of interactions but do not themselves evolve in MS. Finite parts, other schemes, or higher loops can induce running of these matrices; our formalism is designed to extend to those settings.

- (4) *Closed form beta functions in terms of $\mathcal{B}$ and $\mathcal{T}$.* We obtain universal tensor beta functions,

$$\beta_{\lambda_{ijkl}} = -\epsilon \lambda_{ijkl} + \frac{1}{2} (\lambda_{ijmn} \mathcal{B}_{mn;pq} \lambda_{klpq} + \lambda_{ikmn} \mathcal{B}_{mn;pq} \lambda_{jlpq} + \lambda_{ilmn} \mathcal{B}_{mn;pq} \lambda_{jkpq}), \quad (4)$$

$$\beta_{h_{ijk}} = -\left(1 + \frac{\epsilon}{2}\right) h_{ijk} + \frac{1}{2} (\lambda_{ijmn} \mathcal{B}_{mn;pq} h_{kpq} + \lambda_{ikmn} \mathcal{B}_{mn;pq} h_{jpq} + \lambda_{jkmn} \mathcal{B}_{mn;pq} h_{ipq}), \quad (5)$$

together with the mass and linear beta functions (69) and (70). These expressions make it transparent how anisotropy enters: *only* through the residues $\mathcal{B}$ and $\mathcal{T}$.

- (5) *Three “weights” for $\mathbf{Z}_t = \mathbf{1}$ and $\mathbf{Y} = 0$ and their meaning.* In the truncation with no kinetic mixing in field space,

$$\mathbf{Z}_t = \mathbf{1}, \quad \mathbf{Y} = 0, \quad (6)$$

the field-space projectors are constant, and all anisotropy dependence collapses into three scalar weights

$$J_{11} = \langle (\hat{\mathbf{n}}^\top \mathbf{C}_\phi \hat{\mathbf{n}})^{-3/2} \rangle_{S^2}, \quad J_{22} = \langle (\hat{\mathbf{n}}^\top \mathbf{C}_\chi \hat{\mathbf{n}})^{-3/2} \rangle_{S^2}, \quad (7)$$

$$J_{12} = 2 \langle \mathcal{K}(c_\phi(\hat{\mathbf{n}}), c_\chi(\hat{\mathbf{n}})) \rangle_{S^2}. \quad (8)$$

They have a clean interpretation: they are the *UV two-particle phase-space weights* of the normal modes, averaged over directions, and they reduce to the familiar isotropic factors when $\mathbf{C}_\phi = c_\phi \mathbf{1}$, $\mathbf{C}_\chi = c_\chi \mathbf{1}$. We show that in $d = 3$ the diagonal weights are purely geometric, while the mixed weight admits an interpolating determinant representation

$$J_{12} = \int_0^1 d\tau (\det[(1-\tau)\mathbf{C}_\phi + \tau\mathbf{C}_\chi])^{-1/2}, \quad (9)$$

and obeys the universal inequality $J_{12}^2 \leq J_{11}J_{22}$. Introducing the mismatch ratio $\rho \equiv J_{12}/\sqrt{J_{11}J_{22}} \in (0, 1]$ makes many RG statements manifest. In the RG closed sector corresponding to the reflection-symmetric quartic couplings case, beta functions have the simple form

$$\begin{aligned} \beta_{\lambda_1} &= -\epsilon\lambda_1 + \frac{3}{16\pi^2} [J_{11}\lambda_1^2 + J_{22}\lambda_3^2], \\ \beta_{\lambda_2} &= -\epsilon\lambda_2 + \frac{3}{16\pi^2} [J_{22}\lambda_2^2 + J_{11}\lambda_3^2], \\ \beta_{\lambda_3} &= -\epsilon\lambda_3 + \frac{1}{16\pi^2} [J_{11}\lambda_1\lambda_3 + 4J_{12}\lambda_3^2 + J_{22}\lambda_2\lambda_3], \end{aligned}$$

structurally similar to the isotropic case.

- (6) *Fixed points, existence bounds, and cubic couplings.* In the reflection-symmetric quartic sector ($\lambda_{4,5} = 0$) we find coupled fixed points generalizing the standard two-scalar Wilson-Fisher point [66–68] and the coupled-order-parameter multicritical analyses in $O(n_1) \oplus O(n_2)$ -type theories [69–71]. Their very existence depends on anisotropy through ρ : the IR-stable coupled point exists only for a sufficiently small mismatch, $\rho \geq \sqrt{3}/2$ (equivalently $4J_{12}^2 \geq 3J_{11}J_{22}$), and disappears when the mismatch is too large. In the fully coupled quartic sector ($\lambda_{4,5} \neq 0$) the one-loop equations admit a fixed line

(a marginal direction at this order), which we analyze and show to be of saddle type. Cubic couplings are relevant already at tree level in $4 - \epsilon$ dimensions, and at one-loop the cubic flow remains linear in h (see, e.g., [64,65]). In particular, within the perturbative $4 - \epsilon$ expansion there is no interacting fixed point with $h^* \neq 0$; cubic perturbations destabilize the quartic critical loci unless they are tuned to zero (or forbidden by a symmetry).

- (7) *Six “weights” when $\mathbf{Y} \neq 0$ and the interpretation of mixing.* When cross-gradients are turned on ($\mathbf{Z}_t = \mathbf{1}$, $\mathbf{Y} \neq 0$), the UV eigenmodes are direction-dependent mixtures of ϕ and χ and the bubble residue no longer reduces to three numbers. Instead, six independent angular weights [Eq. (111)] control the quartic flow. These weights admit a natural interpretation as phase-space-weighted moments of *populations* and *coherences* of the UV normal modes (the projector entries), and they quantify precisely how kinetic mixing opens additional virtual channels in one-loop scattering. In the isotropic homogeneous limit $\mathbf{C}_{\phi,\chi} \propto \mathbf{1}$, $\mathbf{Y} \propto \mathbf{1}$ they reduce to elementary functions of the two eigenspeeds and the global mixing angle.

Beyond providing a compact set of beta functions for a particular two-scalar model, we view the main utility of the present analysis as structural.

The spectral-decomposition method packages anisotropy into invariant data $\{c_\alpha(\hat{\mathbf{n}}), \Pi_\alpha(\hat{\mathbf{n}})\}$ and isolates a universal kernel (2). This is well suited for systematic generalizations: more fields, other interaction tensors, and ultimately gauge or fermionic sectors where species propagate on different effective quadratic forms.

Also many continuum descriptions of phase transitions involve multiple scalar order parameters with distinct stiffness matrices and gradient mixing. Our weights provide a sharp measure of “phase-space overlap” between such sectors and determine when coupled Wilson-Fisher-type fixed points can exist.

An important thing to notice is that even though in MS at one-loop the anisotropy matrices do not run, the interaction flow is already nontrivially reshaped by anisotropy through the weights. This clarifies what part of “emergent Lorentz” questions is scheme or loop-order sensitive: the *distribution of UV phase space* among channels is already visible at one-loop, while the actual running of velocities is deferred to higher loops and/or nonminimal schemes.

The fact that the same kernel $\mathcal{K}$ controls both tadpoles and bubbles makes it easy to build intuition: J weights are not arbitrary integrals but encode how many UV intermediate states are available in each channel, dressed by mixing coherence factors. This viewpoint is especially helpful when $\mathbf{Y} \neq 0$, where direction-dependent mixing makes a naive “two speeds” picture inadequate.

This paper is organized as follows. In Sec. III we evaluate the one-loop master integrals (tadpole and bubble)

in a form adapted to a matrix-valued anisotropic propagator and extract their UV pole parts in DR + MS [59–61]. In Sec. IV we assemble these results into compact tensor beta functions for the general theory. In Sec. V we analyze several consistent truncations, including the decoupled gradient limit $\mathbf{Y} = 0$ (where anisotropy enters through three weights) and the mixed gradient case $\mathbf{Y} \neq 0$ (where six weights appear), and we discuss the associated fixed points and their stability. Technical derivations and useful identities for the weights are collected in Appendixes A, B, and C, while in Appendix D we write the useful comparison of the results from Refs. [36,37] with our paper's results.

II. SETUP: THEORY AND ITS DECOMPOSITION

A. The lagrangian

In this paper we consider the theory of two real interacting scalar fields ϕ and χ in $D \equiv d + 1$ Euclidean spacetime dimensions. The Lagrangian under consideration is given by

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_0 \boldsymbol{\varphi})^\top \mathbf{Z}_t (\partial_0 \boldsymbol{\varphi}) + \frac{1}{2}(\nabla \boldsymbol{\varphi})^\top \mathbf{C}_\phi (\nabla \boldsymbol{\varphi}) + \frac{1}{2}(\nabla \chi)^\top \mathbf{C}_\chi (\nabla \chi) \\ & + (\nabla \boldsymbol{\varphi})^\top \mathbf{Y} (\nabla \chi) + \frac{1}{2} \boldsymbol{\varphi}^\top \mathbf{M}^2 \boldsymbol{\varphi} + \mathcal{L}_1 + \mathcal{L}_3 + \mathcal{L}_4 \\ \equiv & \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}, \end{aligned} \quad (10)$$

where² $\boldsymbol{\varphi} \equiv (\phi, \chi)^\top$ and the interactions have the form

$$\mathcal{L}_1 = \mu^{\frac{D+2}{2}} (j_\phi \phi + j_\chi \chi), \quad (11a)$$

$$\mathcal{L}_3 = \mu^{\frac{6-D}{2}} \left(\frac{h_1}{3!} \phi^3 + \frac{h_2}{3!} \chi^3 + \frac{h_3}{2} \phi^2 \chi + \frac{h_4}{2} \phi \chi^2 \right), \quad (11b)$$

$$\mathcal{L}_4 = \mu^{4-D} \left(\frac{\lambda_1}{4!} \phi^4 + \frac{\lambda_2}{4!} \chi^4 + \frac{\lambda_3}{4} \phi^2 \chi^2 + \frac{\lambda_4}{6} \phi^3 \chi + \frac{\lambda_5}{6} \phi \chi^3 \right). \quad (11c)$$

The interaction part could also be rewritten using compact notation

$$\begin{aligned} \mathcal{L}_1 + \mathcal{L}_3 + \mathcal{L}_4 = & \mu^{\frac{D+2}{2}} j_i \varphi_i + \frac{\mu^{\frac{6-D}{2}}}{3!} h_{ijk} \varphi_i \varphi_j \varphi_k \\ & + \frac{\mu^{4-D}}{4!} \lambda_{ijkl} \varphi_i \varphi_j \varphi_k \varphi_l. \end{aligned} \quad (12)$$

The counterterms for renormalization in this theory have the form

²Equivalently, the quadratic derivative terms could be written in index notation as $\mathcal{L}_{\text{kin}} = \frac{1}{2} Z_t^{ij} \partial_0 \varphi_i \partial_0 \varphi_j + \frac{1}{2} C_\phi^{ab} \partial_a \phi \partial_b \phi + \frac{1}{2} C_\chi^{ab} \partial_a \chi \partial_b \chi + Y^{ab} \partial_a \phi \partial_b \chi$.

$$\begin{aligned} \mathcal{L}_{\text{ct}} = & \frac{1}{2}(\partial_0 \boldsymbol{\varphi})^\top \delta \mathbf{Z}_t (\partial_0 \boldsymbol{\varphi}) + \frac{1}{2}(\nabla \boldsymbol{\varphi})^\top \delta \mathbf{C}_\phi (\nabla \boldsymbol{\varphi}) \\ & + \frac{1}{2}(\nabla \chi)^\top \delta \mathbf{C}_\chi (\nabla \chi) + (\nabla \boldsymbol{\varphi})^\top \delta \mathbf{Y} (\nabla \chi) + \frac{1}{2} \boldsymbol{\varphi}^\top \delta \mathbf{M}^2 \boldsymbol{\varphi} \\ & + \mu^{\frac{D+2}{2}} \delta j_i \varphi_i + \frac{\mu^{\frac{6-D}{2}}}{3!} \delta h_{ijk} \varphi_i \varphi_j \varphi_k + \frac{\mu^{4-D}}{4!} \delta \lambda_{ijkl} \varphi_i \varphi_j \varphi_k \varphi_l. \end{aligned} \quad (13)$$

We use $i, j \in \{1, 2\}$ for field-space indices and $a, b \in \{1, \dots, d\}$ for spatial indices. The matrices in (10) characterize the anisotropy and read

- (i) $\mathbf{Z}_t$ is a 2×2 positive-definite matrix in field space. Positivity is required for a well-defined Euclidean path integral and for stability. This matrix is defined as

$$\mathbf{Z}_t \equiv \begin{pmatrix} z_{\phi\phi} & z_{\phi\chi} \\ z_{\phi\chi} & z_{\chi\chi} \end{pmatrix}, \quad (14)$$

in a field space mixing the time derivatives and allowing cross time-derivative mixing $\partial_0 \phi \partial_0 \chi$. We denote the eigenvalues of this matrix as z_1 and z_2 in what follows.

- (ii) $\mathbf{C}_\phi, \mathbf{C}_\chi$ are “sound-speed”³ real symmetric positive-definite $d \times d$ matrices acting on spatial indices. For example, if one considers the $d = 2$ case, then $\mathbf{C}_\phi$ is

$$\mathbf{C}_\phi = \begin{pmatrix} C_{xx}^{(\phi)} & C_{xy}^{(\phi)} \\ C_{xy}^{(\phi)} & C_{yy}^{(\phi)} \end{pmatrix}. \quad (15)$$

- (iii) $\mathbf{Y}$ is a real symmetric $d \times d$ matrix that mixes spatial gradients, i.e., $Y^{ab} \partial_a \phi \partial_b \chi$.
 (iv) $\mathbf{M}^2$ is a real symmetric 2×2 mass matrix, and it has a form similar to (14).
 (v) j_i are linear couplings. They are generated by renormalization whenever cubic couplings and masses are present [one-point one-particle-irreducible (1PI) divergences], so they must be included formally for closure of RG flow.

We keep the most general quadratic form in this parity even subclass, including unusual terms like $\mathbf{Z}_t$ and $\mathbf{Y}$, because it provides an RG-closed basis at one-loop.

Even if the spatial-gradient mixing matrix $\mathbf{Y}$ is set to zero at tree level, it is generically generated at one-loop as a finite contribution to the off-diagonal two-point function. In particular, mixed cubic interactions $\phi^2 \chi$ and $\phi \chi^2$ induce a momentum-dependent self-energy $\Sigma_{\phi\chi}(p)$ whose low-momentum expansion contains a term proportional to $p_a p_b$, corresponding to the operator $(\nabla \boldsymbol{\varphi})^\top \mathbf{Y} (\nabla \chi)$. Near

³We tacitly assume that they are sound-speed matrices in the Lorentzian signature; in the Euclidean signature they could be called stiffness matrices.

$D = 4 - \epsilon$, the UV divergent part of $\Sigma_{\phi\chi}$ is momentum independent and renormalizes only the mass matrix, while the gradient mixing arises solely from finite terms. In the special case of equal velocities and isotropic kinetic terms, the generated contribution is proportional to $p^2 = p_0^2 + \mathbf{p}^2$ and can be removed by a linear field redefinition; however, in the general anisotropic theory it represents an independent operator. Consequently, inclusion of $\mathbf{Y}$ is required for closure of the effective action under renormalization.

Finally, we restrict attention to the parity even, two-derivative sector and thus exclude time/space mixed terms of the form $R_a^{ij} \partial_0 \phi_i \partial_a \phi_j$. Since the R_a -terms are odd under time reflection and spatial inversion, they are not generated by renormalization once these symmetries are imposed, and the sector considered here remains closed under the RG flow. Including such terms in our setup leads to substantial technical complications, and the case $R_a \neq 0$ requires a separate analysis, which we leave for future work. While excluding these terms reduces the full generality of the setup, the resulting sector remains nontrivial, closed under RG flow, and sufficiently broad to describe a significant class of Lorentz-violating theories.

In the present one-loop DR + MS analysis near $D = 4 - \epsilon$, the UV $1/\epsilon$ poles in two-point functions are momentum independent; hence, they renormalize only $\mathbf{M}^2$ and not $\mathbf{Z}_t$, $\mathbf{C}_{\phi\chi}$, or $\mathbf{Y}$. The form of the counterterms matrices $\delta\mathbf{Z}_t$, $\delta\mathbf{C}_{\phi\chi}$, $\delta\mathbf{Y}$, and $\delta\mathbf{M}^2$ is fixed by the operators they renormalize.

Next we turn to the interaction part of the renormalized Lagrangian (12). Here we have the totally symmetric tensors $h_{ijk} = h_{(ijk)}$ and $\lambda_{ijkl} = \lambda_{(ijkl)}$, where $i, j, k, l \in \{1, 2\}$. If we write (12) explicitly, we arrive at expression (11) where tensor notations of (12) are related to (11) as

$$\begin{aligned} h_{111} &\equiv h_1, & h_{222} &\equiv h_2, & h_{112} &= h_{121} = h_{211} \equiv h_3, \\ h_{122} &= h_{212} = h_{221} \equiv h_4, \end{aligned} \quad (16a)$$

and

$$\begin{aligned} \lambda_{1111} &= \lambda_1, & \lambda_{2222} &= \lambda_2, & \lambda_{1122} &= \lambda_3, \\ \lambda_{1112} &= \lambda_4, & \lambda_{1222} &= \lambda_5, \end{aligned} \quad (16b)$$

where all components related by permutations of (i, j, k, l) are equal by the total symmetry of λ_{ijkl} .

B. The propagator

For the Fourier transform of each field component,

$$\varphi_i(t, \mathbf{x}) = \int \frac{d\omega}{2\pi} \frac{d^d \mathbf{p}}{(2\pi)^d} e^{i(\omega t + \mathbf{p} \cdot \mathbf{x})} \varphi_i(\omega, \mathbf{p}) \equiv \int_p e^{ip \cdot x} \varphi_i(p), \quad (17)$$

where $p = (\omega, \mathbf{p})$ and $\int_p \equiv \int \frac{d^D p}{(2\pi)^D}$, the action with the free Lagrangian from (10) reads

$$\mathcal{S}_{\text{free}} = \int d^D x \mathcal{L}_{\text{free}} = \frac{1}{2} \int_p \boldsymbol{\varphi}(-p)^\top \mathbf{D}(p) \boldsymbol{\varphi}(p). \quad (18)$$

Here the 2×2 field-space kernel is

$$\mathbf{D}(p) \equiv \omega^2 \mathbf{Z}_t + |\mathbf{p}|^2 \mathbf{C}(\hat{\mathbf{n}}) + \mathbf{M}^2, \quad \hat{\mathbf{n}} \equiv \mathbf{p}/|\mathbf{p}| \in S^{d-1}, \quad (19)$$

where for later use it is convenient to introduce the directional contractions

$$c_\phi(\hat{\mathbf{n}}) \equiv \hat{\mathbf{n}}^\top \mathbf{C}_\phi \hat{\mathbf{n}}, \quad c_\chi(\hat{\mathbf{n}}) \equiv \hat{\mathbf{n}}^\top \mathbf{C}_\chi \hat{\mathbf{n}}, \quad y(\hat{\mathbf{n}}) \equiv \hat{\mathbf{n}}^\top \mathbf{Y} \hat{\mathbf{n}},$$

and the associated field-space matrix

$$\mathbf{C}(\hat{\mathbf{n}}) \equiv \hat{n}_a \begin{pmatrix} C_{ab}^{(\phi)} & Y_{ab} \\ Y_{ab} & C_{ab}^{(\chi)} \end{pmatrix} \hat{n}_b = \begin{pmatrix} c_\phi(\hat{\mathbf{n}}) & y(\hat{\mathbf{n}}) \\ y(\hat{\mathbf{n}}) & c_\chi(\hat{\mathbf{n}}) \end{pmatrix}.$$

For every direction $\hat{\mathbf{n}}$ the matrix $\mathbf{C}(\hat{\mathbf{n}})$ is assumed to be symmetric positive-definite to avoid gradient instabilities, i.e.,

$$c_\phi(\hat{\mathbf{n}}) > 0, \quad c_\chi(\hat{\mathbf{n}}) > 0, \quad \det \mathbf{C}(\hat{\mathbf{n}}) > 0. \quad (20)$$

Thus, within these notations the propagator has the form

$$\mathbf{G}(p) = \mathbf{Z}_t^{-1/2} (\omega^2 \mathbf{1} + \mathbf{K}(\mathbf{p}))^{-1} \mathbf{Z}_t^{-1/2}, \quad (21)$$

with

$$\mathbf{K}(\mathbf{p}) \equiv |\mathbf{p}|^2 \mathbf{C}_Z(\hat{\mathbf{n}}) + \mathbf{M}_Z^2, \quad (22)$$

$$\mathbf{C}_Z(\hat{\mathbf{n}}) \equiv \mathbf{Z}_t^{-1/2} \mathbf{C}(\hat{\mathbf{n}}) \mathbf{Z}_t^{-1/2}, \quad (23)$$

$$\mathbf{M}_Z^2 \equiv \mathbf{Z}_t^{-1/2} \mathbf{M}^2 \mathbf{Z}_t^{-1/2}. \quad (24)$$

Here $\mathbf{Z}_t^{1/2}$ is the principal symmetric square root of $\mathbf{Z}_t$, which is well-defined since $\mathbf{Z}_t$ is symmetric and positive-definite. Here $\mathbf{1}$ denotes the 2×2 identity matrix in field space, and we take

$$\mathbf{Z}_t^{1/2} = \mathbf{Z}_t^{1/2\top}.$$

C. Eigenvalue decomposition of propagator

The general form of propagator (21) is cumbersome for loop computations, and it is, therefore, useful to diagonalize the field-space kernel. With the notation introduced above, the matrix $\mathbf{K}(\mathbf{p})$ is real and symmetric for every $\mathbf{p}$. Under the stability assumptions $\mathbf{Z}_t = \mathbf{Z}_t^\top > 0$ and $\mathbf{C}(\hat{\mathbf{n}}) = \mathbf{C}(\hat{\mathbf{n}})^\top > 0$ for all directions $\hat{\mathbf{n}} \in S^{d-1}$, where $\hat{\mathbf{n}}$ is given by (19), the matrix $\mathbf{C}_Z(\hat{\mathbf{n}})$ from (23) is positive-definite, and, therefore,

$$\mathbf{K}(\mathbf{p}) = |\mathbf{p}|^2 \mathbf{C}_Z(\hat{\mathbf{n}}) + \mathbf{M}_Z^2$$

is positive-definite for sufficiently large $|\mathbf{p}|$. This UV regime is the only one relevant for the extraction of one-loop MS

poles.⁴ Accordingly, we may use the spectral decomposition of $\mathbf{K}(\mathbf{p})$ in the UV to define matrix functions such as $\mathbf{K}^{-1/2}$. There exist orthonormal eigenvectors $\mathbf{u}_\alpha(\mathbf{p})$ and eigenvalues $\kappa_\alpha(\mathbf{p})$ (with $\alpha = 1, 2$) such that

$$\mathbf{K}(\mathbf{p})\mathbf{u}_\alpha(\mathbf{p}) = \kappa_\alpha(\mathbf{p})\mathbf{u}_\alpha(\mathbf{p}), \quad \mathbf{u}_\alpha(\mathbf{p})^\top \mathbf{u}_\beta(\mathbf{p}) = \delta_{\alpha\beta}. \quad (25)$$

One may then form the Euclidean spectral projectors

$$\tilde{\Pi}_\alpha(\mathbf{p}) \equiv \mathbf{u}_\alpha(\mathbf{p})\mathbf{u}_\alpha(\mathbf{p})^\top, \quad \sum_{\alpha=1}^2 \tilde{\Pi}_\alpha = \mathbf{1}, \quad \tilde{\Pi}_\alpha \tilde{\Pi}_\beta = \delta_{\alpha\beta} \tilde{\Pi}_\alpha. \quad (26)$$

Then,

$$\mathbf{K}(\mathbf{p}) = \sum_{\alpha=1}^2 \kappa_\alpha(\mathbf{p}) \tilde{\Pi}_\alpha(\mathbf{p}), \quad (27)$$

and so for the propagator one has⁵

$$\mathbf{G}(p) = \sum_{\alpha=1}^2 \frac{\mathbf{Z}_t^{-1/2} \tilde{\Pi}_\alpha(\mathbf{p}) \mathbf{Z}_t^{-1/2}}{\omega^2 + \kappa_\alpha(\mathbf{p})} \equiv \sum_{\alpha=1}^2 \frac{\tilde{\mathbf{P}}_{Z,\alpha}(\mathbf{p})}{\omega^2 + \kappa_\alpha(\mathbf{p})}, \quad (28)$$

where we introduced

$$\tilde{\mathbf{P}}_{Z,\alpha}(\mathbf{p}) \equiv \mathbf{Z}_t^{-1/2} \tilde{\Pi}_\alpha(\mathbf{p}) \mathbf{Z}_t^{-1/2}. \quad (29)$$

Note that $\tilde{\mathbf{P}}_{Z,\alpha}(\mathbf{p})$ is rank 1 but is not a projector in general: $\tilde{\mathbf{P}}_{Z,\alpha}^2 \neq \tilde{\mathbf{P}}_{Z,\alpha}$ (it becomes a genuine projector only when $\mathbf{Z}_t = \mathbf{1}$). However, it satisfies

$$\sum_{\alpha=1}^2 \tilde{\mathbf{P}}_{Z,\alpha}(\mathbf{p}) = \mathbf{Z}_t^{-1}. \quad (30)$$

For the UV analysis it is also convenient to isolate the massless part of the kernel, i.e., to set $\mathbf{M}_Z^2 = 0$ in (22). In that case the overall factor $|\mathbf{p}|^2$ multiplies $\mathbf{C}_Z(\hat{\mathbf{n}})$, and we only need the eigenvalues and eigenvectors of $\mathbf{C}_Z(\hat{\mathbf{n}})$:

$$\mathbf{C}_Z(\hat{\mathbf{n}})\mathbf{v}_\alpha(\hat{\mathbf{n}}) = c_\alpha(\hat{\mathbf{n}})\mathbf{v}_\alpha(\hat{\mathbf{n}}), \quad \mathbf{v}_\alpha(\hat{\mathbf{n}})^\top \mathbf{v}_\beta(\hat{\mathbf{n}}) = \delta_{\alpha\beta}, \quad (31)$$

with projectors

⁴If the IR spectrum of $\mathbf{K}(\mathbf{p})$ is not strictly positive for some parameters (e.g., due to $\mathbf{M}^2$), one may temporarily work in the symmetric phase where $\mathbf{M}^2 > 0$, or introduce an auxiliary IR regulator $\mathbf{M}_Z^2 \rightarrow \mathbf{M}_Z^2 + \delta^2 \mathbf{1}$ with $\delta > 0$. The MS pole parts are local and UV-controlled, hence, independent of the IR regulator, and can be analytically continued back.

⁵The matrix function $f(\mathbf{K})$ (e.g., $\mathbf{K}^{-1/2}$ or $e^{-i\mathbf{K}}$) is defined spectrally: if $\mathbf{K}(\mathbf{p}) = \sum_{\alpha=1}^2 \kappa_\alpha(\mathbf{p}) \tilde{\Pi}_\alpha(\mathbf{p})$, then $f(\mathbf{K}(\mathbf{p})) = \sum_{\alpha=1}^2 f(\kappa_\alpha(\mathbf{p})) \tilde{\Pi}_\alpha(\mathbf{p})$. In particular, in the UV region (large $|\mathbf{p}|$) the eigenvalues $\kappa_\alpha(\mathbf{p})$ are positive under the assumptions $\mathbf{Z}_t > 0$ and $\mathbf{C}(\hat{\mathbf{n}}) > 0$.

$$\Pi_\alpha(\hat{\mathbf{n}}) \equiv \mathbf{v}_\alpha(\hat{\mathbf{n}})\mathbf{v}_\alpha(\hat{\mathbf{n}})^\top, \quad \sum_{\alpha=1}^2 \Pi_\alpha = \mathbf{1}, \quad \Pi_\alpha \Pi_\beta = \delta_{\alpha\beta} \Pi_\alpha, \quad (32)$$

so that

$$\mathbf{C}_Z(\hat{\mathbf{n}}) = \sum_{\alpha=1}^2 c_\alpha(\hat{\mathbf{n}}) \Pi_\alpha(\hat{\mathbf{n}}). \quad (33)$$

Also notice that in the UV ($|\mathbf{p}| \rightarrow \infty$) one has $\kappa_\alpha(\mathbf{p}) = |\mathbf{p}|^2 c_\alpha(\hat{\mathbf{n}}) + \mathcal{O}(1)$, and the corresponding spectral projectors approach those of $\mathbf{C}_Z(\hat{\mathbf{n}})$ (the $\mathbf{M}_Z^2$ -dependence affects only IR/finite parts and does not modify MS poles).

The resolution for $(\omega^2 \mathbf{1} + |\mathbf{p}|^2 \mathbf{C}_Z(\hat{\mathbf{n}}))^{-1}$ then reads

$$(\omega^2 \mathbf{1} + |\mathbf{p}|^2 \mathbf{C}_Z(\hat{\mathbf{n}}))^{-1} = \sum_{\alpha=1}^2 \frac{\Pi_\alpha(\hat{\mathbf{n}})}{\omega^2 + |\mathbf{p}|^2 c_\alpha(\hat{\mathbf{n}})}, \quad (34)$$

and finally, conjugating with $\mathbf{Z}_t^{-1/2}$, we get

$$\mathbf{G}(p) = \sum_{\alpha=1}^2 \frac{\mathbf{P}_{Z,\alpha}(\hat{\mathbf{n}})}{\omega^2 + |\mathbf{p}|^2 c_\alpha(\hat{\mathbf{n}})}, \quad \mathbf{P}_{Z,\alpha}(\hat{\mathbf{n}}) \equiv \mathbf{Z}_t^{-1/2} \Pi_\alpha(\hat{\mathbf{n}}) \mathbf{Z}_t^{-1/2}. \quad (35)$$

In this form of $\mathbf{G}(p)$ it is clear that $v_\alpha(\hat{\mathbf{n}}) \equiv \sqrt{c_\alpha(\hat{\mathbf{n}})}$ are the directional sound speeds of the normal modes of the system.

III. ONE-LOOP DIAGRAMS CALCULATION

In this section we calculate all one-loop diagrams necessary for the extraction of beta functions. First, we introduce two “master integrals” needed for this purpose. With the propagator written in the form (28) and the interactions given in (11), we need the tadpole and bubble integrals for different field species. We begin with the tadpole master integral and then turn to the bubble integral, writing the final answers for the pole parts needed to calculate the beta functions in terms of the data $\mathbf{Z}_t$, $\mathbf{Y}$, $\mathbf{C}_\phi$, and $\mathbf{C}_\chi$.⁶

A. Tadpole integral

The first master integral reads

$$\begin{aligned} \mathbf{T} &\equiv \int \frac{d\omega}{2\pi} \frac{d^d \mathbf{p}}{(2\pi)^d} \mathbf{G}(p) \\ &= \int \frac{d\omega}{2\pi} \frac{d^d \mathbf{p}}{(2\pi)^d} \mathbf{Z}_t^{-1/2} (\omega^2 \mathbf{1} + \mathbf{K}(\mathbf{p}))^{-1} \mathbf{Z}_t^{-1/2}, \end{aligned} \quad (36)$$

⁶In pure dimensional regularization, massless vacuum integrals at zero external momentum are scaleless and vanish, mixing UV and IR. To extract the UV MS pole parts unambiguously we introduce an auxiliary IR regulator $\delta_{\text{IR}} > 0$; the $1/\epsilon$ poles are UV-local and independent of δ_{IR} .

where we substitute (21). Let us first work out the integration over energy ω in (36) as follows:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\omega^2 \mathbf{1} + \mathbf{K})^{-1} = \frac{1}{2} \mathbf{K}^{-1/2}, \quad (37)$$

with the use of (27), so (36) reduces to

$$\mathbf{T} = \frac{1}{2} \int \frac{d^d \mathbf{p}}{(2\pi)^d} \mathbf{Z}_t^{-1/2} \mathbf{K}(\mathbf{p})^{-1/2} \mathbf{Z}_t^{-1/2}.$$

The leading term of $\mathbf{T}$ in the UV (obtained by setting $\mathbf{M}_Z^2 = 0$) is a scaleless power-divergent integral. In dimensional regularization such scaleless integrals vanish, so it produces no $1/\epsilon$ pole in the DR + MS scheme. Therefore, the MS pole starts at first order in $\mathbf{M}_Z^2$ (a mass insertion).

It is more convenient to rewrite the integral over momentum in spherical coordinates as

$$\begin{aligned} \int \frac{d^d \mathbf{p}}{(2\pi)^d} &= \frac{1}{(2\pi)^d} \int_0^\infty d\rho \rho^{d-1} \int_{S^{d-1}} d\Omega_{d-1} \\ &= \frac{\Omega_{d-1}}{2(2\pi)^d} \int_0^\infty dx x^{\frac{d}{2}-1} \langle \cdots \rangle_{\hat{\mathbf{n}}}, \end{aligned} \quad (38)$$

where $\mathbf{p} \equiv \rho \hat{\mathbf{n}}$ with $\rho \equiv |\mathbf{p}| \geq 0$, and $x \equiv \rho^2$; Ω_{d-1} is the solid angle on the unit sphere and $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(\frac{d}{2})$. The notation $\langle \cdots \rangle_{\hat{\mathbf{n}}}$ is given by

$$\langle f(\hat{\mathbf{n}}) \rangle_{\hat{\mathbf{n}}} \equiv \frac{1}{\Omega_{d-1}} \int_{S^{d-1}} d\Omega_{d-1} f(\hat{\mathbf{n}}). \quad (39)$$

In dimensional regularization with $D = 4 - \epsilon$ one has $d = 3 - \epsilon$. The $1/\epsilon$ poles in the master integrals come from the radial $|\mathbf{p}|$ integration (e.g., $\int_0^\infty dx x^{-1-\epsilon/2} \sim 1/\epsilon$), while the angular averages over S^{d-1} remain finite as $\epsilon \rightarrow 0$ under our positivity assumptions. Therefore, for MS pole parts we may evaluate all angular averages at $d = 3$ (i.e., over S^2); keeping the exact S^{d-1} measure affects only finite terms. Hence, we arrive at

$$\mathbf{T} = \frac{1}{4} \frac{\Omega_{d-1}}{(2\pi)^d} \int_0^\infty dx x^{\frac{d}{2}-1} \langle \mathbf{Z}_t^{-1/2} \mathbf{K}(x, \hat{\mathbf{n}})^{-1/2} \mathbf{Z}_t^{-1/2} \rangle_{\hat{\mathbf{n}}}. \quad (40)$$

To work out this integral and extract the divergence, we use the form of the matrix $\mathbf{K}$ given in (22), as well as the eigenvalue decomposition (27). Moreover, to regularize the UV divergence in (40), we use the expansion at large momentum, i.e., at large x . We put all explicit evaluations related to the integral (40) in Appendix A, and here we write the final answer for the pole part of (36):

$$\begin{aligned} \mathbb{T} &\equiv [\mathbf{T}]_{\text{pole}} \\ &= -\frac{1}{4\pi^2 \epsilon} \left\langle \sum_{\alpha, \beta} \mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) \mathbf{Z}_t^{-1/2} \mathbf{\Pi}_\alpha(\hat{\mathbf{n}}) \mathbf{M}_Z^2 \mathbf{\Pi}_\beta(\hat{\mathbf{n}}) \mathbf{Z}_t^{-1/2} \right\rangle_{\hat{\mathbf{n}}}, \end{aligned} \quad (41)$$

where we also introduce

$$\mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) \equiv \frac{1}{\sqrt{\mathbf{c}_\alpha} \sqrt{\mathbf{c}_\beta} (\sqrt{\mathbf{c}_\alpha} + \sqrt{\mathbf{c}_\beta})}. \quad (42)$$

B. Bubble integral

For the diagrams with two propagators we need another master integral, which reads

$$\begin{aligned} \mathbf{B}_{ij;kl}(k) &\equiv \int \frac{d\omega}{2\pi} \frac{d^d \mathbf{p}}{(2\pi)^d} \mathbf{G}_{ik}(p) \mathbf{G}_{jl}(p+k) \\ &= \int \frac{d\omega}{2\pi} \frac{d^d \mathbf{p}}{(2\pi)^d} [\mathbf{Z}_t^{-1/2} (\omega^2 \mathbf{1} + \mathbf{K}(\mathbf{p}))^{-1} \mathbf{Z}_t^{-1/2}]_{ik} \\ &\quad \times [\mathbf{Z}_t^{-1/2} ((\omega + \Omega)^2 \mathbf{1} + \mathbf{K}(\mathbf{p} + \mathbf{k}))^{-1} \mathbf{Z}_t^{-1/2}]_{jl}, \end{aligned} \quad (43)$$

where $k \equiv (\Omega, \mathbf{k})$ is an external energy momentum. At large-loop momentum $p = (\omega, \mathbf{p})$ one has $\mathbf{G}(p) = \mathcal{O}(p^{-2})$ because $\mathbf{K}(\mathbf{p}) \sim |\mathbf{p}|^2 \mathbf{C}_Z(\hat{\mathbf{n}})$. Hence, $\mathbf{B}(k)$ is superficially logarithmically divergent in $D = 4 - \epsilon$, $\mathbf{B}(k) \sim \int d^D p p^{-4}$, so any UV divergence must be local. Expanding the integrand in the external momentum k produces additional inverse powers of p and yields UV-finite contributions. Therefore, the $1/\epsilon$ pole is momentum-independent and is fully captured by $\mathbf{B}_{ij;kl}(0)$:

$$\begin{aligned} \mathbf{B}_{ij;kl}(0) &\equiv \int \frac{d\omega}{2\pi} \frac{d^d \mathbf{p}}{(2\pi)^d} \mathbf{G}_{ik}(p) \mathbf{G}_{jl}(p) \\ &= \int \frac{d\omega}{2\pi} \frac{d^d \mathbf{p}}{(2\pi)^d} [\mathbf{Z}_t^{-1/2} (\omega^2 \mathbf{1} + \mathbf{K}(\mathbf{p}))^{-1} \mathbf{Z}_t^{-1/2}]_{ik} [\mathbf{Z}_t^{-1/2} (\omega^2 \mathbf{1} + \mathbf{K}(\mathbf{p}))^{-1} \mathbf{Z}_t^{-1/2}]_{jl} \\ &= \int \frac{d^d \mathbf{p}}{(2\pi)^d} \sum_{\alpha, \beta} \tilde{\mathbf{P}}_{Z, \alpha}(\mathbf{p})_{ik} \tilde{\mathbf{P}}_{Z, \beta}(\mathbf{p})_{jl} \int \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + \kappa_\alpha(\mathbf{p}))(\omega^2 + \kappa_\beta(\mathbf{p}))} \\ &= \int \frac{d^d \mathbf{p}}{(2\pi)^d} \sum_{\alpha, \beta} \frac{\tilde{\mathbf{P}}_{Z, \alpha}(\mathbf{p})_{ik} \tilde{\mathbf{P}}_{Z, \beta}(\mathbf{p})_{jl}}{2\sqrt{\kappa_\alpha(\mathbf{p})\kappa_\beta(\mathbf{p})}(\sqrt{\kappa_\alpha(\mathbf{p})} + \sqrt{\kappa_\beta(\mathbf{p})})}, \end{aligned} \quad (44)$$

where in the last equality we take the ω integral with the use of decomposition (27) as well as (29). Next, using (39), we rewrite integration over spatial $\mathbf{p}$ as

$$\mathbf{B}_{ij;kl}(0) = \frac{\Omega_{d-1}}{2(2\pi)^d} \int_0^\infty dx x^{\frac{d}{2}-1} \left\langle \sum_{\alpha,\beta} \frac{\tilde{\mathbf{P}}_{Z,\alpha}(\hat{\mathbf{n}}, x)_{ik} \tilde{\mathbf{P}}_{Z,\beta}(\hat{\mathbf{n}}, x)_{jl}}{2\sqrt{\kappa_\alpha(\hat{\mathbf{n}}, x)} \kappa_\beta(\hat{\mathbf{n}}, x) (\sqrt{\kappa_\alpha(\hat{\mathbf{n}}, x)} + \sqrt{\kappa_\beta(\hat{\mathbf{n}}, x)})} \right\rangle_{\hat{\mathbf{n}}}.$$

Performing the expansion at large momentum $|\mathbf{p}|$, after some algebra one arrives at the following answer for the divergent part of the bubble:

$$\mathbb{B}_{ij;kl} \equiv [\mathbf{B}_{ij;kl}]_{\text{pole}} = \frac{1}{4\pi^2\epsilon} \left\langle \sum_{\alpha,\beta} \mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) \mathbf{P}_{Z,\alpha}(\hat{\mathbf{n}})_{ik} \mathbf{P}_{Z,\beta}(\hat{\mathbf{n}})_{jl} \right\rangle_{\hat{\mathbf{n}}}, \quad (45)$$

where we also have used (42). We put all detailed and step-by-step calculations of (45) into Appendix A.

C. Diagrams calculation

Finally let us express all divergent parts of diagrams in compact notation via tadpole and bubble master integral contributions. Diagrams relevant for our calculation are well known and depicted in Fig. 1.

In compact notation contributions from fish diagrams in different channels are given by

$$\Delta\Gamma_{ij;kl}^{(4)} = \frac{1}{2} \sum_{m,n,p,q} (\lambda_{ijmn} \mathbb{B}_{mn;pq} \lambda_{klpq} + \lambda_{ikmn} \mathbb{B}_{mn;pq} \lambda_{jlpq} + \lambda_{ilmn} \mathbb{B}_{mn;pq} \lambda_{jkpq}), \quad (46)$$

where each term corresponds to s , t , u channels, respectively.⁷ All indices we have: external ones are $i, j, k, l \in \{1, 2\}$, while internal are $m, n, p, q \in \{1, 2\}$ and the factor $1/2$ comes from the overall $\frac{1}{2!}$ in the second-order expansion. Next, decay diagrams contribution is given by

$$\Delta\Gamma_{ijk}^{(3)} = \frac{1}{2} \sum_{m,n,p,q} (\lambda_{ijmn} \mathbb{B}_{mn;pq} h_{kpq} + \lambda_{ikmn} \mathbb{B}_{mn;pq} h_{jpq} + \lambda_{jkmn} \mathbb{B}_{mn;pq} h_{ipq}), \quad (47)$$

⁷Let us briefly clarify our notations: for example, in the case of incoming and outgoing $\phi\phi$ -pairs we write for s -channel

$$\Delta\Gamma_{11;11}^{(4)} = \frac{1}{2} (\lambda_1^2 \mathbb{B}_{11;11} + \lambda_4^2 \mathbb{B}_{12;12} + \lambda_3^2 \mathbb{B}_{22;22} + 2\lambda_1 \lambda_4 \mathbb{B}_{11;12} + 2\lambda_1 \lambda_3 \mathbb{B}_{11;22} + 2\lambda_4 \lambda_3 \mathbb{B}_{12;22}),$$

where $i = j = k = l = 1$ and coefficients 2 come from two possibilities such as $\mathbb{B}_{11;12}$ and $\mathbb{B}_{12;11}$.

and for all indices we have: external ones are $i, j, k \in \{1, 2\}$, while internal are $m, n, p, q \in \{1, 2\}$.

For the pole part of the bubble diagram one finds

$$\Sigma_{ij}^{(h^2)} = \frac{1}{2} \sum_{m,n,p,q} h_{imn} \mathbb{B}_{mn;pq} h_{jpq}, \quad (48)$$

where external indices are $i, j \in \{1, 2\}$, while internal ones are $m, n, p, q \in \{1, 2\}$.

Finally, the tadpole diagram contributes as the sum

$$\Sigma_{ij}^{(\lambda)} = \frac{1}{2} \sum_{m,n} \lambda_{ijmn} \mathbb{T}_{mn}, \quad (49)$$

where external indices are $i, j \in \{1, 2\}$, while internal ones are $m, n \in \{1, 2\}$. If $h_{ijk} \neq 0$ and $M_{ij}^2 \neq 0$, there is an additional one-loop divergent 1PI diagram with one external leg (a tadpole). Its contribution is just

$$\Delta\Gamma_i^{(1)} = \frac{1}{2} \sum_{m,n} h_{imn} \mathbb{T}_{mn}, \quad (50)$$

where $\mathbb{T}_{mn}$ is the tadpole master integral (and the factor $1/2$ comes from the combinatorics $3/3!$ at the cubic vertex).

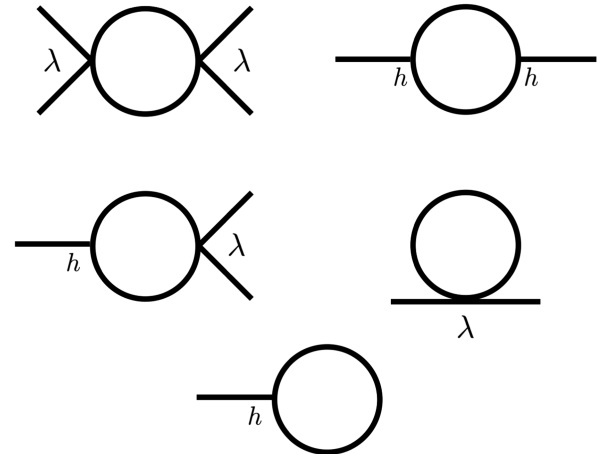


FIG. 1. All possible one-loop divergent diagrams for the model with the interaction Lagrangian (11). The notations λ and h are just schematic, and we specify couplings for each concrete diagram in the text. Since there are $\delta_{ji}\phi_i$ terms in the Lagrangian (11), we add the bottom diagram with one external leg, one propagator, and the h_{ijk} -type vertex as well.

This divergence requires the linear counterterm $\mu^{\frac{D+2}{2}} \delta j_i \varphi_i$ in II.

The interaction Lagrangian (11) contains explicit factors of μ which make h_{ijk} and λ_{ijkl} dimensionless in $D = 4 - \epsilon$. In the diagrammatic expressions we write only the *reduced* tensor contractions built from (h, λ) and the master integrals $(\mathbb{T}, \mathbb{B})$. The corresponding full one-loop 1PI contributions that enter the renormalization of couplings are obtained by restoring the overall μ -factors from the vertices:

$$\Gamma_{1\text{-loop}}^{(4)} = \mu^{2(4-D)} \Delta \Gamma^{(4)}, \quad (51)$$

$$\Gamma_{1\text{-loop}}^{(3)} = \mu^{(4-D)+\frac{6-D}{2}} \Delta \Gamma^{(3)}, \quad (52)$$

$$\Sigma_{1\text{-loop}}^{(\lambda)} = \mu^{4-D} \Sigma^{(\lambda)}, \quad (53)$$

$$\Sigma_{1\text{-loop}}^{(h^2)} = \mu^{6-D} \Sigma^{(h^2)}, \quad (54)$$

$$\Gamma_{1\text{-loop}}^{(1)} = \mu^{\frac{6-D}{2}} \Delta \Gamma^{(1)}. \quad (55)$$

For quartic and cubic counterterms the extra factor $\mu^{4-D} = \mu^\epsilon = 1 + \mathcal{O}(\epsilon)$ does not affect the MS pole part. In contrast, the self-energy generated by two cubic vertices carries $\mu^{6-D} = \mu^{2+\epsilon}$, and the overall μ^2 is essential for dimensional consistency of the mass counterterm.

IV. GENERAL EXPRESSIONS FOR BETA FUNCTIONS

Now summarize what we obtained to get the renormalization-group flow in our compact notation. All one-loop UV divergences of the theory have been analyzed in detail in Sec. III, where the relevant master integrals were evaluated, and the tensor structure of the 1PI vertices was classified. Here we only assemble those results into a compact and systematic set of beta functions.

At fixed bare couplings beta functions are defined as

$$\beta_g \equiv \mu \frac{dg}{d\mu} \Big|_{g_0 \text{ fixed}}. \quad (56)$$

All one-loop UV poles are controlled by two objects:

- (i) the tadpole matrix $\mathbb{T}_{mn}$, and
 - (ii) the zero-momentum bubble tensor $\mathbb{B}_{ij;kl}(0)$.
- Their pole parts define finite tensors

$$\mathcal{T}_{mn} \equiv \epsilon \mathbb{T}_{mn}, \quad \mathcal{B}_{ij;kl} \equiv \epsilon \mathbb{B}_{ij;kl}(0), \quad (57)$$

whose explicit expressions, in terms of the UV spectral data of the kinetic operator and angular averages over $\hat{n}$, are given in Eqs. (41) and (45).

Importantly, the UV poles are local and momentum-independent. As a consequence,

- (i) only the couplings (λ, h, M^2, j) are renormalized at one-loop, and
- (ii) the kinetic and anisotropy matrices do *not* run in MS at this order.

A. Minimal subtraction and counterterms

In the MS scheme all counterterms are chosen to cancel only the UV pole parts in $\epsilon = 4 - D$ and to contain no finite contributions. Concretely, for every renormalized parameter g we write the corresponding bare one as

$$g_0 = \mu^{d_g} (g + \delta g), \quad (58)$$

where d_g is the canonical mass dimension (in $D = 4 - \epsilon$) of the bare coupling g_0 in our normalization, fixed by the explicit powers of μ in the interaction Lagrangian (12). For the couplings in (11) this means

$$\begin{aligned} \lambda_{ijkl}^{(0)} &= \mu^{4-D} (\lambda_{ijkl} + \delta \lambda_{ijkl}) = \mu^\epsilon (\lambda_{ijkl} + \delta \lambda_{ijkl}), \\ h_{ijk}^{(0)} &= \mu^{\frac{6-D}{2}} (h_{ijk} + \delta h_{ijk}), \end{aligned} \quad (59)$$

$$\begin{aligned} j_i^{(0)} &= \mu^{\frac{D+2}{2}} (j_i + \delta j_i) = \mu^{3-\epsilon/2} (j_i + \delta j_i), \\ (M_{ij}^2)^{(0)} &= M_{ij}^2 + \delta M_{ij}^2, \end{aligned} \quad (60)$$

in agreement with the counterterm Lagrangian (II).

Denoting the simple-pole coefficients as

$$\delta \lambda_{ijkl} = \frac{A_{ijkl}}{\epsilon}, \quad \delta h_{ijk} = \frac{B_{ijk}}{\epsilon}, \quad \delta j_i = \frac{C_i}{\epsilon}, \quad \delta M_{ij}^2 = \frac{D_{ij}}{\epsilon}, \quad (61)$$

where

$$\begin{aligned} A_{ijkl} &= [\Delta \Gamma_{ij;kl}^{(4)}]_{\text{pole}}, \\ B_{ijk} &= [\Delta \Gamma_{ijk}^{(3)}]_{\text{pole}}, \\ D_{ij} &= [\mu^{4-D} \Sigma_{ij}^{(\lambda)} + \mu^{6-D} \Sigma_{ij}^{(h^2)}]_{\text{pole}}, \\ C_i &= [\mu^{-\frac{D+2}{2}} \Gamma_{i,1\text{-loop}}^{(1)}]_{\text{pole}} = -[\mu^{2-D} \Delta \Gamma_i^{(1)}]_{\text{pole}}, \end{aligned} \quad (62)$$

then the expressions for beta functions have the form

$$\beta_{\lambda_{ijkl}} = -\epsilon \lambda_{ijkl} + A_{ijkl}, \quad \beta_{h_{ijk}} = -(1 + \epsilon/2) h_{ijk} + B_{ijk}, \quad (63)$$

$$\beta_{j_i} = -(3 - \epsilon/2) j_i + C_i, \quad \beta_{M_{ij}^2} = D_{ij}, \quad (64)$$

and, for the dimensionless masses $m_{ij}^2 \equiv M_{ij}^2/\mu^2$,

$$\beta_{m_{ij}^2} = -2m_{ij}^2 + \frac{1}{\mu^2} \beta_{M_{ij}^2}. \quad (65)$$

Thus, once the pole residues (A, B, C, D) are known from the 1PI diagrams, the beta functions are fixed immediately without keeping any finite parts. Now let us express beta functions in terms of tadpoles and bubbles.

B. Quartic couplings

The quartic vertex receives contributions from the three one-loop fish diagrams. Using the pole structure of the four-point function derived in Sec. III, the beta function takes the universal tensor form

$$\beta_{\lambda_{ijkl}} = -\epsilon \lambda_{ijkl} + \frac{1}{2} (\lambda_{ijmn} \mathcal{B}_{mn;pq} \lambda_{klpq} + \lambda_{ikmn} \mathcal{B}_{mn;pq} \lambda_{jlpq} + \lambda_{ilmn} \mathcal{B}_{mn;pq} \lambda_{jkpq}). \quad (66)$$

All anisotropy dependence enters exclusively through the bubble residue $\mathcal{B}$.

C. Cubic couplings

The cubic couplings renormalize only in the presence of quartic interactions. At one-loop the flow is linear in h , reflecting the absence of UV-divergent purely cubic one-loop diagrams near $D = 4$. From the three-point function pole computed in Sec. III one finds

$$\beta_{h_{ijk}} = -\left(1 + \frac{\epsilon}{2}\right) h_{ijk} + \frac{1}{2} (\lambda_{ijmn} \mathcal{B}_{mn;pq} h_{kpq} + \lambda_{ikmn} \mathcal{B}_{mn;pq} h_{jppq} + \lambda_{jkmn} \mathcal{B}_{mn;pq} h_{ipq}). \quad (67)$$

D. Mass matrix

The quadratic couplings are conveniently expressed in terms of the dimensionless mass matrix

$$m_{ij}^2 \equiv \frac{M_{ij}^2}{\mu^2}. \quad (68)$$

Its beta function receives a homogeneous contribution proportional to m^2 , as well as inhomogeneous terms induced by quartic and cubic interactions. Combining the tadpole and bubble contributions obtained in Sec. III, one finds

$$\beta_{m_{ij}^2} = -2m_{ij}^2 - \frac{1}{2} \lambda_{ijmn} \tilde{T}_{mn} + \frac{1}{2} h_{imn} \mathcal{B}_{mn;pq} h_{jpq}, \quad (69)$$

where $\tilde{T}_{ij} \equiv \frac{T_{ij}}{\mu^2}$ denotes the dimensionless tadpole residue, while $\mathcal{B}_{mn;pq}$ is already dimensionless. In particular, for $h \neq 0$ the surface $m^2 = 0$ is not preserved by the RG flow.

E. Linear couplings

When both cubic couplings and masses are present, one-loop tadpole diagrams generate a divergence linear in the

fields. As a result, the beta function for the linear couplings reads

$$\beta_{j_i} = -\left(3 - \frac{\epsilon}{2}\right) j_i - \frac{1}{2} h_{imn} \tilde{T}_{mn}. \quad (70)$$

Thus, the hyperplane $j_i = 0$ is RG invariant only if either $h = 0$ or $m^2 = 0$.

F. Absence of one-loop running of anisotropy matrices

Finally, since all one-loop UV divergences in the two-point function are local and momentum-independent, they renormalize only the mass operator. Therefore, in DR + MS,

$$\beta_{\mathbf{Z}_t} = \beta_{\mathbf{C}_\phi} = \beta_{\mathbf{C}_\chi} = \beta_{\mathbf{Y}} = 0 \quad (\text{one-loop}). \quad (71)$$

Possible running of these matrices arises only beyond one-loop or in nonminimal schemes.

The remainder of the paper is devoted to evaluating the tensors $\mathcal{B}$ and $\mathcal{T}$ in specific truncations of the anisotropy data and to analyzing the resulting RG flows and fixed points.

V. EXPLICIT FORM OF BETA FUNCTIONS, FIXED POINTS, AND PHYSICAL INTERPRETATION

So far we obtained the general expression for beta functions when the velocity tensors $\mathbf{C}_\phi, \mathbf{C}_\chi$ as well as $\mathbf{Z}_t$ and $\mathbf{Y}$ are nontrivial. Now let us explore several consistent truncations of the general one-loop beta functions. Some truncations are closed at one-loop in MS (because no momentum-dependent poles are generated), but they are not guaranteed to remain closed beyond one-loop or in other schemes. In addition, if cubic couplings h_{ijk} and masses M_{ij}^2 are nonzero, the RG flow generates linear (tadpole) couplings j_i , so in this section we include their beta functions as well.

A. Cross-gradients turned off: $\mathbf{Y} = 0$

We begin with the particularly simple truncation

$$\mathbf{Z}_t = \mathbf{1}, \quad \mathbf{Y} = 0. \quad (72)$$

This sector (as well as another case in the next subsection) should be viewed as a special subclass of the general theory, rather than as its generic representative. Indeed, setting $\mathbf{Z}_t = \mathbf{1}$ is a convenient normalization choice at one-loop in the present scheme, whereas the additional condition $\mathbf{Y} = 0$ is a genuine restriction on the spatial kinetic mixing. For this reason, the analysis below is best interpreted as the simplest benchmark subsector of the general RG system. In this case, the UV kinetic operator is diagonal in field space and all anisotropy is encoded solely in the two matrices $\mathbf{C}_\phi$ and $\mathbf{C}_\chi$. The details of the calculation for the formulas in

this subsection can be found in Appendix B 1. In this limit the direction-dependent matrix $\mathbf{C}(\hat{n})$ is diagonal in field space, so its spectral projectors reduce to the constant matrices $\Pi_\phi = \text{diag}(1, 0)$ and $\Pi_\chi = \text{diag}(0, 1)$ (and we will associate ϕ with index “1” and χ with “2” in field-space index notation below). As a result, all one-loop pole coefficients depend on anisotropy only through three scalar angular weights J_{11}, J_{22}, J_{12} :

$$J_{11} = \langle [\hat{n}^\top \mathbf{C}_\phi \hat{n}]^{-3/2} \rangle_{S^2}, \quad J_{22} = \langle [\hat{n}^\top \mathbf{C}_\chi \hat{n}]^{-3/2} \rangle_{S^2},$$

$$J_{12} = \int_0^1 (\det[(1-\tau)\mathbf{C}_\phi + \tau\mathbf{C}_\chi])^{-1/2} d\tau, \quad (73)$$

with the coefficients $\mathcal{B}_{mn;pq}$ given by

$$\mathcal{B}_{mn;pq} \equiv \epsilon \mathbb{B}_{mn;pq}$$

$$= \frac{1}{8\pi^2} (J_{11}\delta_{m1}\delta_{n1}\delta_{p1}\delta_{q1} + J_{22}\delta_{m2}\delta_{n2}\delta_{p2}\delta_{q2}$$

$$+ J_{12}(\delta_{m1}\delta_{n2}\delta_{p1}\delta_{q2} + \delta_{m2}\delta_{n1}\delta_{p2}\delta_{q1})) \quad (74)$$

and

$$\mathcal{T} \equiv \epsilon \mathbb{T} = -\frac{1}{8\pi^2} \begin{pmatrix} J_{11}M_{11}^2 & J_{12}M_{12}^2 \\ J_{12}M_{12}^2 & J_{22}M_{22}^2 \end{pmatrix}. \quad (75)$$

Throughout this subsection we set $\epsilon \rightarrow 0$ in angular averages and use $\langle \cdot \rangle_{S^2}$ for the normalized average over S^2 .

After some algebra one can obtain the explicit form of J_{ij} in terms of the eigenvalue data of $\mathbf{C}_\phi$ and $\mathbf{C}_\chi$. In dimensional regularization with $d = 3 - \epsilon$, the MS pole coefficients require the angular averages only at $\epsilon = 0$ (equivalently, one may keep S^{d-1} and set $\epsilon \rightarrow 0$ at the end; the difference contributes only to finite parts).

One can show that the integrals simplify according to $\langle (\hat{n}^\top \mathbf{C} \hat{n})^{-3/2} \rangle_{S^2} = (\det \mathbf{C})^{-1/2}$ in our case. Let $\mathbf{C}_\phi$ have (positive) eigenvalues $\{c_{\phi,1}, c_{\phi,2}, c_{\phi,3}\}$ and $\mathbf{C}_\chi$ have (positive) eigenvalues $\{c_{\chi,1}, c_{\chi,2}, c_{\chi,3}\}$. Then,

$$J_{11} = (\det \mathbf{C}_\phi)^{-1/2} = (c_{\phi,1}c_{\phi,2}c_{\phi,3})^{-1/2},$$

$$J_{22} = (\det \mathbf{C}_\chi)^{-1/2} = (c_{\chi,1}c_{\chi,2}c_{\chi,3})^{-1/2},$$

while the cross term J_{12} is defined by the generalized eigenvalues of the pair $(\mathbf{C}_\chi, \mathbf{C}_\phi)$, denoted by $\gamma_1, \gamma_2, \gamma_3 > 0$, i.e.,

$$\mathbf{C}_\chi v = \gamma \mathbf{C}_\phi v \Leftrightarrow \mathbf{A} \equiv \mathbf{C}_\phi^{-1/2} \mathbf{C}_\chi \mathbf{C}_\phi^{-1/2} = U \text{diag}(\gamma_i) U^\top.$$

Then,

$$J_{12} = (\det \mathbf{C}_\phi)^{-1/2} \int_0^1 \prod_{i=1}^3 ((1-\tau) + \tau\gamma_i)^{-1/2} d\tau,$$

and this integral can be expressed as the Lauricella F_D hypergeometric function

$$J_{12} = (\det \mathbf{C}_\phi)^{-1/2} F_D^{(3)} \left(1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 2; 1-\gamma_1, 1-\gamma_2, 1-\gamma_3 \right).$$

These weights arise from the universal kernel $\mathcal{K}(c_\phi(\hat{n}), c_\chi(\hat{n})) = 1/(\sqrt{c_\phi(\hat{n})}\sqrt{c_\chi(\hat{n})}(\sqrt{c_\phi(\hat{n})} + \sqrt{c_\chi(\hat{n})}))$ via $J_{12} = 2\langle \mathcal{K}(c_\phi(\hat{n}), c_\chi(\hat{n})) \rangle_{S^2}$, and how these formulas follow from the simple τ -representation (B21) is derived in Appendix B.

Also, let us introduce

$$\rho^2 = \frac{J_{12}^2}{J_{11}J_{22}}, \quad \rho \in (0, 1]. \quad (76)$$

Another important fact, valid for general positive-definite matrices $\mathbf{C}_\phi, \mathbf{C}_\chi$, is the inequality

$$J_{12}^2 \leq J_{11}J_{22}, \quad (77)$$

and that is why $\rho \in (0, 1]$; the derivation of this (77) inequality is presented in Appendix B 2. In these notations we obtain that beta functions can be written down explicitly in the form resembling the well-known answer for the two-field beta function, but weighted by J_{ij} in appropriate places. Namely, for the λ_i set of couplings we obtain

$$\beta_{\lambda_1} = -\epsilon\lambda_1 + \frac{3}{16\pi^2} [J_{11}\lambda_1^2 + 2J_{12}\lambda_4^2 + J_{22}\lambda_3^2], \quad (78a)$$

$$\beta_{\lambda_2} = -\epsilon\lambda_2 + \frac{3}{16\pi^2} [J_{22}\lambda_2^2 + 2J_{12}\lambda_5^2 + J_{11}\lambda_3^2], \quad (78b)$$

$$\beta_{\lambda_3} = -\epsilon\lambda_3 + \frac{1}{16\pi^2} [J_{11}(\lambda_1\lambda_3 + 2\lambda_4^2)$$

$$+ 4J_{12}\lambda_3^2 + J_{22}(\lambda_2\lambda_3 + 2\lambda_5^2) + 2J_{12}\lambda_4\lambda_5], \quad (78c)$$

$$\beta_{\lambda_4} = -\epsilon\lambda_4 + \frac{3}{16\pi^2} [J_{11}\lambda_1\lambda_4 + 2J_{12}\lambda_3\lambda_4 + J_{22}\lambda_3\lambda_5], \quad (78d)$$

$$\beta_{\lambda_5} = -\epsilon\lambda_5 + \frac{3}{16\pi^2} [J_{22}\lambda_2\lambda_5 + 2J_{12}\lambda_3\lambda_5 + J_{11}\lambda_3\lambda_4], \quad (78e)$$

with h_i running described by

$$\beta_{h_1} = -\left(1 + \frac{\epsilon}{2}\right)h_1 + \frac{3}{16\pi^2} [J_{11}\lambda_1h_1 + 2J_{12}\lambda_4h_3 + J_{22}\lambda_3h_4], \quad (79a)$$

$$\beta_{h_2} = -\left(1 + \frac{\epsilon}{2}\right)h_2 + \frac{3}{16\pi^2} [J_{22}\lambda_2h_2 + 2J_{12}\lambda_5h_4 + J_{11}\lambda_3h_3], \quad (79b)$$

$$\beta_{h_3} = -\left(1 + \frac{\epsilon}{2}\right)h_3 + \frac{1}{16\pi^2}[J_{11}(\lambda_1 h_3 + 2\lambda_4 h_1) + 4J_{12}\lambda_3 h_3 + 2J_{12}\lambda_4 h_4 + J_{22}(\lambda_3 h_2 + 2\lambda_5 h_4)], \quad (79c)$$

$$\beta_{h_4} = -\left(1 + \frac{\epsilon}{2}\right)h_4 + \frac{1}{16\pi^2}[J_{22}(\lambda_2 h_4 + 2\lambda_5 h_2) + 4J_{12}\lambda_3 h_4 + 2J_{12}\lambda_5 h_3 + J_{11}(\lambda_3 h_1 + 2\lambda_4 h_3)]. \quad (79d)$$

Beta functions for masses also obtain a similar form

$$\beta_{m_{11}^2} = -2m_{11}^2 + \frac{1}{16\pi^2}(\lambda_1 J_{11}m_{11}^2 + 2\lambda_4 J_{12}m_{12}^2 + \lambda_3 J_{22}m_{22}^2) + \frac{1}{16\pi^2}(J_{11}h_1^2 + 2J_{12}h_3^2 + J_{22}h_4^2), \quad (80a)$$

$$\beta_{m_{22}^2} = -2m_{22}^2 + \frac{1}{16\pi^2}(\lambda_2 J_{22}m_{22}^2 + 2\lambda_5 J_{12}m_{12}^2 + \lambda_3 J_{11}m_{11}^2) + \frac{1}{16\pi^2}(J_{22}h_2^2 + 2J_{12}h_4^2 + J_{11}h_3^2), \quad (80b)$$

$$\beta_{m_{12}^2} = -2m_{12}^2 + \frac{1}{16\pi^2}(\lambda_4 J_{11}m_{11}^2 + \lambda_5 J_{22}m_{22}^2 + 2\lambda_3 J_{12}m_{12}^2) + \frac{1}{16\pi^2}(J_{11}h_1 h_3 + J_{22}h_2 h_4 + 2J_{12}h_3 h_4), \quad (80c)$$

and, finally, for $j_\phi \equiv j_1$ and $j_\chi \equiv j_2$,

$$\beta_{j_\phi} = -\left(3 - \frac{\epsilon}{2}\right)j_\phi + \frac{1}{16\pi^2}(J_{11}h_1 m_{11}^2 + 2J_{12}h_3 m_{12}^2 + J_{22}h_4 m_{22}^2), \quad (81a)$$

$$\beta_{j_\chi} = -\left(3 - \frac{\epsilon}{2}\right)j_\chi + \frac{1}{16\pi^2}(J_{11}h_3 m_{11}^2 + 2J_{12}h_4 m_{12}^2 + J_{22}h_2 m_{22}^2), \quad (81b)$$

where we have introduced $m_{ij}^2 \equiv M_{ij}^2/\mu^2$. In particular, for $h \neq 0$ and $M^2 \neq 0$, the plane $j_\phi = j_\chi = 0$ is not RG invariant.

1. Sector even under field reflections: $\lambda_{4,5} = 0$ and $h_i = 0$

Now consider the renormalization group in such a model. Let us specialize first to the simplest case keeping only $\lambda_{1,2,3}$ and setting $\lambda_{4,5} = 0$. In this case beta functions have the form

$$\begin{aligned} \beta_{\lambda_1} &= -\epsilon\lambda_1 + \frac{3}{16\pi^2}[J_{11}\lambda_1^2 + J_{22}\lambda_3^2], \\ \beta_{\lambda_2} &= -\epsilon\lambda_2 + \frac{3}{16\pi^2}[J_{22}\lambda_2^2 + J_{11}\lambda_3^2], \\ \beta_{\lambda_3} &= -\epsilon\lambda_3 + \frac{1}{16\pi^2}[J_{11}\lambda_1\lambda_3 + 4J_{12}\lambda_3^2 + J_{22}\lambda_2\lambda_3], \end{aligned}$$

and formal solutions to the equations $\beta_{\lambda_i} = 0$ correspond to different fixed points of our model. Namely, we have two solutions we call $S_\pm$ (corresponding to $\pm$ sign choices):

$$\lambda_1^* = \frac{\frac{16\pi^2}{3}\epsilon \cdot 3 - 4J_{12}\lambda_3^*}{2J_{11}}, \quad (82a)$$

$$\lambda_2^* = \frac{\frac{16\pi^2}{3}\epsilon \cdot 3 - 4J_{12}\lambda_3^*}{2J_{22}}, \quad (82b)$$

$$\lambda_3^* = \frac{\frac{16\pi^2}{3}\epsilon(4J_{12} \pm \sqrt{4J_{12}^2 - 3J_{11}J_{22}})}{2(4J_{12}^2 + J_{11}J_{22})}, \quad (82c)$$

which are real-valued only when $4J_{12}^2 \geq 3J_{11}J_{22}$. Besides the coupled solutions $S_\pm$ with $\lambda_3^* \neq 0$, the equations $\beta_{\lambda_1} = \beta_{\lambda_2} = \beta_{\lambda_3} = 0$ also admit the following standard fixed points:

- (i) Gaussian: $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (0, 0, 0)$.
- (ii) Single-field Wilson-Fisher points (with $\lambda_3^* = 0$):

$$(\lambda_1^*, \lambda_2^*, \lambda_3^*) = \left(\frac{16\pi^2}{3} \frac{\epsilon}{J_{11}}, 0, 0\right),$$

$$(\lambda_1^*, \lambda_2^*, \lambda_3^*) = \left(0, \frac{16\pi^2}{3} \frac{\epsilon}{J_{22}}, 0\right).$$

- (iii) Decoupled two-field Wilson-Fisher point (with $\lambda_3^* = 0$):

$$(\lambda_1^*, \lambda_2^*, \lambda_3^*) = \left(\frac{16\pi^2}{3} \frac{\epsilon}{J_{11}}, \frac{16\pi^2}{3} \frac{\epsilon}{J_{22}}, 0\right).$$

We focus below on the coupled fixed points with $\lambda_3^* \neq 0$.

Combining with (77) we obtain that this fixed point exists when the mismatch ratio ρ is restricted to

$$\rho^2 = \frac{J_{12}^2}{J_{11}J_{22}} \in [3/4, 1], \quad (83)$$

while in general it belongs to $\rho \in (0, 1]$. Solution corresponding to S_- is the IR-stable attractor generalizing the ordinary Wilson-Fisher fixed point and reducing to it at the unit-normalized isotropic point. The coupled fixed point exists only when the mismatch is sufficiently small, $\rho \geq \sqrt{3}/2$. When $\rho < \sqrt{3}/2$, the coupled solution becomes complex and disappears from the real coupling space. The

solution S_+ is the IR fixed point of the saddle type containing one repulsive direction.

The last formal solution of $\beta_{\lambda_i} = 0$ is given by

$$\lambda_1^* = \frac{8\pi^2}{3} \frac{\epsilon}{J_{11}} \left(1 + \sqrt{1 - \frac{J_{11}J_{22}}{J_{12}^2}} \right), \quad (84a)$$

$$\lambda_2^* = \frac{8\pi^2}{3} \frac{\epsilon}{J_{22}} \left(1 - \sqrt{1 - \frac{J_{11}J_{22}}{J_{12}^2}} \right), \quad (84b)$$

$$\lambda_3^* = \frac{8\pi^2}{3} \frac{\epsilon}{J_{12}}, \quad (84c)$$

which is nonreal unless the bound (77) is saturated. A necessary condition for a stable Euclidean path integral is that the quartic potential is bounded from below. In the $\lambda_{4,5} = 0$ sector this requires $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 > -\frac{1}{3}\sqrt{\lambda_1\lambda_2}$ (in our normalization). We will implicitly restrict to this domain when interpreting fixed points.

2. Fully coupled quartic sector $\lambda_{4,5} \neq 0$ and $h_i = 0$

Turning on $\lambda_{4,5}$ always admits the trivial solution $\lambda_4^* = \lambda_5^* = 0$, which reduces to the reflection-symmetric sector. To obtain a genuinely reflection-breaking fixed point with $\lambda_4^* \neq 0$ and $\lambda_5^* \neq 0$, one must impose *both* conditions $\beta_{\lambda_4} = 0$ and $\beta_{\lambda_5} = 0$. Dividing these equations by λ_4 and λ_5 yields a homogeneous linear system for (λ_4, λ_5) , which has a nontrivial solution iff its determinant vanishes, i.e., $\Delta = 0$ below.

At one-loop this produces a one-parameter family (a fixed line), reflecting a marginal direction in the quartic sector. Beyond one-loop this degeneracy is expected to be lifted. In other words, for λ_3 satisfying $\Delta_3 \geq 0$ (the determinant Δ_3 will also be defined below) with $\lambda_4^2, \lambda_5^2 \geq 0$ one can find $\lambda_{1,2,4,5}$ such that $\beta_{\lambda_i} = 0$. Introducing the notation

$$\mathcal{X} \equiv J_{11}\lambda_1 + 2J_{12}\lambda_3 - c, \quad \mathcal{Y} \equiv J_{22}\lambda_2 + 2J_{12}\lambda_3 - c, \quad (85)$$

we rewrite equations $\beta_{\lambda_{4,5}} = 0$ as

$$\mathcal{X}\lambda_4 + J_{22}\lambda_3\lambda_5 = 0, \quad \mathcal{Y}\lambda_5 + J_{11}\lambda_3\lambda_4 = 0, \quad (86)$$

and the determinant condition for a nontrivial solution is $\Delta = 0$, where

$$\Delta \equiv \mathcal{X}\mathcal{Y} - J_{11}J_{22}\lambda_3^2, \quad c \equiv \frac{16\pi^2}{3}\epsilon. \quad (87)$$

Thus, we see that a fully reflection-breaking fixed point (with $\lambda_4, \lambda_5 \neq 0$) must lie on the codimension-1 surface $\mathcal{X}\mathcal{Y} = J_{11}J_{22}\lambda_3^2$ in $(\lambda_1, \lambda_2, \lambda_3)$ -space. From $\beta_{\lambda_1} = \beta_{\lambda_2} = 0$ we get the relations between different couplings at the fixed point

$$\lambda_4^2 = \frac{c\lambda_1 - J_{11}\lambda_1^2 - J_{22}\lambda_3^2}{2J_{12}}, \quad \lambda_5^2 = \frac{c\lambda_2 - J_{22}\lambda_2^2 - J_{11}\lambda_3^2}{2J_{12}}, \quad (88)$$

which also immediately imply the constraints

$$c\lambda_1 \geq J_{11}\lambda_1^2 + J_{22}\lambda_3^2, \quad c\lambda_2 \geq J_{22}\lambda_2^2 + J_{11}\lambda_3^2.$$

Finally, using $\beta_{\lambda_3} = 0$ and $\Delta = 0$ one can obtain different branches of the solution for $\lambda_{1,2}$ as functions of λ_3 .

(i) For the first branch, the equation $\beta_{\lambda_3} = 0$ together with $\Delta = 0$ gives

$$\mathcal{X} + \mathcal{Y} = 2J_{12}\lambda_3 - c \quad (89)$$

and

$$\mathcal{X}\mathcal{Y} = J_{11}J_{22}\lambda_3^2. \quad (90)$$

Then, picking a real parameter $\lambda_3 \neq 0$ for $\sigma = \pm 1$, we solve these two equations together and also define

$$\Delta_3(\lambda_3) \equiv (c - 2J_{12}\lambda_3)^2 - 4J_{11}J_{22}\lambda_3^2 \geq 0,$$

$$\mathcal{X}_\sigma = \frac{-(c - 2J_{12}\lambda_3) + \sigma\sqrt{\Delta_3(\lambda_3)}}{2},$$

$$\mathcal{Y}_\sigma = \frac{-(c - 2J_{12}\lambda_3) - \sigma\sqrt{\Delta_3(\lambda_3)}}{2};$$

then, $\mathcal{X}_\sigma\mathcal{Y}_\sigma = J_{11}J_{22}\lambda_3^2$ and

$$\lambda_1 = -\frac{\mathcal{Y}_\sigma}{J_{11}}, \quad \lambda_2 = -\frac{\mathcal{X}_\sigma}{J_{22}}.$$

(ii) The second branch of the formal solution has the following form (with $\lambda_3 \neq 0$ and $\sigma = \pm 1$):

$$\mathcal{X}_\sigma = \lambda_3 \left(J_{12} + \sigma\sqrt{J_{12}^2 - J_{11}J_{22}} \right),$$

$$\mathcal{Y}_\sigma = \lambda_3 \left(J_{12} - \sigma\sqrt{J_{12}^2 - J_{11}J_{22}} \right),$$

so $\mathcal{X}_\sigma\mathcal{Y}_\sigma = J_{11}J_{22}\lambda_3^2$ and $\mathcal{X}_\sigma + \mathcal{Y}_\sigma = 2J_{12}\lambda_3$. Then, using (85),

$$\lambda_1 = \frac{\mathcal{X}_\sigma + c - 2J_{12}\lambda_3}{J_{11}}, \quad \lambda_2 = \frac{\mathcal{Y}_\sigma + c - 2J_{12}\lambda_3}{J_{22}},$$

and this branch is also nonreal unless the bound (77) is saturated.

The fully coupled fixed points for $\lambda_{4,5} > 0$ do not come as isolated points but as a family; i.e., there is a marginal eigendirection at one-loop.

Also, inequalities (88) impose for λ_1, λ_2 to lie in finite intervals for any given λ_3 . So if such fixed points exist, they live in a bounded region of coupling space (at order ϵ). The analysis of stability for these branches repeats the same line of reasoning as for the isotropic case and is not affected

significantly by anisotropy. In other words, the line of fixed points corresponding to the fully interacting λ_i set of couplings in general has the same stability properties as the isotropic one, changing only the location of the line. Anisotropy only restricts the existence of fixed points and shifts their location in coupling space. In particular, for generic initial conditions the RG flow runs away from the fully coupled fixed line.

3. Effect of cubic couplings

The cubic interactions in (11) are parametrized by the dimensionless couplings h_i (or, equivalently, h_{ijk}). In $D = 4 - \epsilon$ they are already *relevant* at tree level: their canonical RG contribution is

$$\beta_{h_{ijk}} \supset -\frac{6-D}{2}h_{ijk} = -(1 + \epsilon/2)h_{ijk}. \quad (91)$$

At one-loop in MS the beta functions remain *linear* in h [in the $\mathbf{Z}_t = \mathbf{1}$, $\mathbf{Y} = 0$ limit, the explicit formulas for $\beta_{h_1}, \dots, \beta_{h_4}$ are given by (79)]. Therefore, the hyperplane $h_{ijk} = 0$ is RG invariant, but it is not IR attractive: generic infinitesimal cubic perturbations grow toward the IR and destabilize the quartic critical manifolds unless h is tuned to zero (or forbidden by a $\mathbb{Z}_2$ symmetry).

In the decoupled-gradient limit $\mathbf{Z}_t = \mathbf{1}$, $\mathbf{Y} = 0$, the one-loop pole coefficients depend on anisotropy only through the three angular weights J_{11} , J_{22} , J_{12} (or equivalently through $\rho = J_{12}/\sqrt{J_{11}J_{22}}$). In particular, after the constant (i.e., μ -independent) rescalings, the cubic RG flow can be written as a linear system whose coefficients depend on the quartic couplings only via the rescaled variables u_i on the quartic fixed line (see Appendix C for details).

Concretely, we introduce the rescaled quartics

$$\begin{aligned} g_1 &\equiv J_{11}\lambda_1, & g_2 &\equiv J_{22}\lambda_2, & g_3 &\equiv \sqrt{J_{11}J_{22}}\lambda_3, & (92) \\ g_4 &\equiv \sqrt{J_{11}J_{12}}\lambda_4, & g_5 &\equiv \sqrt{J_{22}J_{12}}\lambda_5, & \rho &\equiv \frac{J_{12}}{\sqrt{J_{11}J_{22}}} \in (0, 1], & (93) \end{aligned}$$

and $u_i \equiv g_i/c$ with $c \equiv \frac{16\pi^2}{3}\epsilon$. Next, we rescale the cubics by the μ -independent transformation, Eq. (C46),

$$H_1 \equiv h_1, \quad H_2 \equiv \sqrt{\frac{J_{12}J_{22}}{J_{11}^2}}h_2, \quad H_3 \equiv \sqrt{\frac{J_{12}}{J_{11}}}h_3, \quad (94)$$

$$H_4 \equiv \sqrt{\frac{J_{22}}{J_{11}}}h_4, \quad \mathbf{H} \equiv (H_1, H_2, H_3, H_4)^\top. \quad (95)$$

Then, the one-loop cubic flow takes the compact matrix form (see Appendix C for more details)

$$\beta_{\mathbf{H}} = -\left(1 + \frac{\epsilon}{2}\right)\mathbf{H} + \frac{\epsilon}{3}\mathbb{M}_h(u, \rho)\mathbf{H}, \quad (96)$$

with the explicit 4×4 matrix $\mathbb{M}_h$ given in (C49).

Evaluating $\mathbb{M}_h$ on the fully coupled quartic fixed line $u = u^*(r)$ (Appendix C) one finds a strong universality statement:

$$\text{spec}(\mathbb{M}_h(u^*, \rho)) = \{0, 0, 1, 3\}, \quad (97)$$

where “spec” means spectrum and it is independent of the fixed-line parameter r and also independent of the anisotropy ratio ρ (Appendix C). The corresponding RG exponents are, therefore,

$$\omega_h(x) = -\left(1 + \frac{\epsilon}{2}\right) + \frac{\epsilon}{3}x, \quad x \in \{0, 0, 1, 3\}, \quad (98)$$

i.e.,

$$\omega_h = \left\{-1 - \frac{\epsilon}{2}, -1 - \frac{\epsilon}{2}, -1 - \frac{\epsilon}{6}, -1 + \frac{\epsilon}{2}\right\}. \quad (99)$$

With the convention $\beta_g = \mu \frac{dg}{d\mu}$ (with fixed bare coupling), the IR corresponds to $\mu \rightarrow 0$, so a perturbation scales as $\delta h \sim \mu^{\omega_h}$ and thus grows in the IR when $\omega_h < 0$. Hence, all cubic couplings are *relevant* perturbations of the quartic fixed line for $\epsilon > 0$. In particular, within the perturbative $4 - \epsilon$ expansion there is no interacting fixed point with $h_i^* \neq 0$: at one-loop the only fixed locus in the cubic sector is $h^* = 0$.

Two technical remarks sharpen the interpretation: (i) the two zero eigenvalues of $\mathbb{M}_h$ do *not* signal marginal cubic couplings; they merely mean that two cubic combinations receive no quartic correction at one-loop, yet they remain relevant due to the canonical term $-(1 + \epsilon/2)\mathbf{H}$; (ii) anisotropy affects the *eigenvectors* (which linear combinations of the original h_i form the scaling operators) through the J -dependent rescaling above, but it does not change the one-loop *relevance* pattern or the universal set of exponents.

At the unit-normalized isotropic point $\mathbf{C}_\phi = \mathbf{C}_\chi = \mathbf{1}$ one has $J_{11} = J_{22} = J_{12} = 1$ and $\rho = 1$, so all rescalings become trivial. The cubic beta functions then reduce to the familiar isotropic two-scalar results, and the linearized cubic exponents around the fully coupled quartic fixed line are exactly the same numbers (99). Thus, at one-loop in $4 - \epsilon$, *anisotropy does not generate new cubic fixed points and does not change the qualitative fate of nonzero cubic couplings*: it only changes the map between the microscopic couplings h_i and the scaling eigenoperators.

A slightly more general, but still isotropic, case is

$$\mathbf{C}_\phi = c_\phi \mathbf{1}, \quad \mathbf{C}_\chi = c_\chi \mathbf{1}, \quad \mathbf{Y} = 0, \quad c_\phi, c_\chi > 0.$$

Then,

$$J_{11} = c_\phi^{-3/2}, \quad J_{22} = c_\chi^{-3/2}, \quad J_{12} = \frac{2}{\sqrt{c_\phi} \sqrt{c_\chi} (\sqrt{c_\phi} + \sqrt{c_\chi})},$$

and

$$\rho = \frac{2(c_\phi c_\chi)^{1/4}}{\sqrt{c_\phi} + \sqrt{c_\chi}} \leq 1.$$

In the further common-metric subcase $c_\phi = c_\chi = c$ one has

$$J_{11} = J_{22} = J_{12} = c^{-3/2}, \quad \rho = 1,$$

so the unit-normalized isotropic point discussed above is simply the special choice $c = 1$. Thus a common overall isotropic normalization rescales all weights uniformly, while unequal but still isotropic velocities correspond to $\rho < 1$; nevertheless, after the J -dependent rescalings the one-loop cubic flow has the same qualitative behavior, and the universal exponents (99) remain unchanged.

The same conclusion is robust around the generalized Wilson-Fisher fixed point in the reflection-symmetric quartic sector: since $\lambda^* = \mathcal{O}(\epsilon)$, the loop mixing in β_h is $\mathcal{O}(\epsilon)h$ and cannot overcome the order-one canonical term $-(1 + \epsilon/2)h$ for small ϵ , so cubic perturbations remain relevant near $D = 4$.

Allowing $h \neq 0$ also activates the quadratic/linear sectors. First, the mass beta functions contain an *inhomogeneous* contribution $\beta_{m^2} \supset +h^2$ already at one-loop, so the critical surface $m^2 = 0$ is not RG invariant unless $h = 0$. Second, when both $h \neq 0$ and $m^2 \neq 0$ are present, the RG flow generates linear (tadpole) couplings j_i through the one-point divergence discussed around (50). Therefore, a quartic critical fixed point/fixed line describes the critical manifold only after tuning to the symmetry-preserving surface

$$h_i = 0, \quad m_{ij}^2 = 0, \quad j_i = 0, \quad (100)$$

while generic flows with $h \neq 0$ depart from criticality by generating m^2 (and then j). For completeness, the linearized mass spectrum around the fixed line at $h = 0$ is $\omega_m = \{-2, -2, -2 - \epsilon/3\}$ (see Appendix C), so all quadratic deformations are relevant, as expected for a critical theory in $4 - \epsilon$ dimensions.

B. Turning on cross-gradients: $\mathbf{Z}_t = \mathbf{1}$ and $\mathbf{Y} \neq 0$

In the previous subsection we imposed $\mathbf{Y} = 0$ and obtained a drastic simplification: the UV kinetic operator was diagonal in field space and all direction dependence collapsed into three scalar weights J_{11} , J_{22} , J_{12} . Here we relax this truncation and keep

$$\mathbf{Z}_t = \mathbf{1}, \quad \mathbf{Y} \neq 0, \quad (101)$$

while still working in the one-loop DR + MS scheme near $D = 4 - \epsilon$. We also set $h_i = 0$ throughout this subsection (the cubic sector can be reinstated later by substituting the

same bubble/tadpole residues into the general formulas of the previous section).

With $\mathbf{Z}_t = \mathbf{1}$ the Z_t -conjugated UV spatial kernel coincides with the original one, $\mathbf{C}_Z(\hat{n}) = \mathbf{C}(\hat{n})$. For every direction $\hat{n} \in S^2$ we define the three directional scalars

$$c_\phi(\hat{n}) \equiv \hat{n}^\top \mathbf{C}_\phi \hat{n}, \quad c_\chi(\hat{n}) \equiv \hat{n}^\top \mathbf{C}_\chi \hat{n}, \quad y(\hat{n}) \equiv \hat{n}^\top \mathbf{Y} \hat{n}, \quad (102)$$

so that

$$\mathbf{C}(\hat{n}) = \begin{pmatrix} c_\phi(\hat{n}) & y(\hat{n}) \\ y(\hat{n}) & c_\chi(\hat{n}) \end{pmatrix}. \quad (103)$$

The stability assumptions (no gradient instabilities) require $\mathbf{C}(\hat{n}) > 0$ for all $\hat{n}$, i.e.,

$$c_\phi(\hat{n}) > 0, \quad c_\chi(\hat{n}) > 0, \quad c_\phi(\hat{n})c_\chi(\hat{n}) - y(\hat{n})^2 > 0 \quad (\forall \hat{n} \in S^2). \quad (104)$$

Since $\mathbf{C}(\hat{n})$ is a real symmetric 2×2 matrix, it can be diagonalized explicitly. Introduce

$$\Delta(\hat{n}) \equiv c_\phi(\hat{n}) - c_\chi(\hat{n}), \quad \Lambda(\hat{n}) \equiv \sqrt{\Delta(\hat{n})^2 + 4y(\hat{n})^2} > 0, \quad (105)$$

and then the two (positive) directional eigenvalues are

$$\epsilon_\pm(\hat{n}) = \frac{1}{2}(c_\phi(\hat{n}) + c_\chi(\hat{n}) \pm \Lambda(\hat{n})), \quad \epsilon_\pm(\hat{n}) > 0. \quad (106)$$

Equivalently, one may parametrize the diagonalization by a mixing angle $\theta(\hat{n})$ defined by

$$\tan(2\theta(\hat{n})) = \frac{2y(\hat{n})}{c_\phi(\hat{n}) - c_\chi(\hat{n})}, \quad \cos(2\theta) = \frac{\Delta}{\Lambda}, \quad \sin(2\theta) = \frac{2y}{\Lambda}. \quad (107)$$

Writing $c_\theta(\hat{n}) \equiv \cos \theta(\hat{n})$ and $s_\theta(\hat{n}) \equiv \sin \theta(\hat{n})$, the (direction-dependent) orthogonal projectors onto the eigenmodes are

$$\Pi_+(\hat{n}) = \begin{pmatrix} c_\theta^2 & c_\theta s_\theta \\ c_\theta s_\theta & s_\theta^2 \end{pmatrix}, \quad \Pi_-(\hat{n}) = \begin{pmatrix} s_\theta^2 & -c_\theta s_\theta \\ -c_\theta s_\theta & c_\theta^2 \end{pmatrix}, \quad (108)$$

$$\Pi_\pm^2 = \Pi_\pm, \quad \Pi_+ \Pi_- = 0, \quad \Pi_+ + \Pi_- = \mathbf{1}. \quad (109)$$

In contrast to the $\mathbf{Y} = 0$ truncation, the projectors now *vary* with $\hat{n}$, and this is precisely what complicates the RG coefficients.

For the MS pole parts, it is sufficient to evaluate all angular averages at $d = 3$ (i.e., over S^2), as explained in Sec. III. With $\mathbf{Z}_t = \mathbf{1}$ the general one-loop bubble residue (45) becomes

$$\mathcal{B}_{ij;kl} \equiv \epsilon \mathbb{B}_{ij;kl} = \frac{1}{4\pi^2} \left\langle \sum_{\alpha, \beta \in \{+, -\}} \mathcal{K}(\mathbf{c}_\alpha(\hat{\mathbf{n}}), \mathbf{c}_\beta(\hat{\mathbf{n}})) \Pi_\alpha(\hat{\mathbf{n}})_{ik} \Pi_\beta(\hat{\mathbf{n}})_{jl} \right\rangle_{S^2}. \quad (110)$$

Because $\Pi_\pm(\hat{\mathbf{n}})$ are no longer constant, the tensor $\mathcal{B}_{ij;kl}$ does not reduce to three weights. Instead, in the original (ϕ, χ) basis one finds that six components are generally independent, and all others follow from the index symmetries $\mathcal{B}_{ij;kl} = \mathcal{B}_{ji;lk} = \mathcal{B}_{kl;ij} = \mathcal{B}_{kj;il} = \mathcal{B}_{il;kj}$.

To keep the notation parallel to the $\mathbf{Y} = 0$ discussion, we package them into six angular weights (with the conventional $8\pi^2$ normalization):

$$\begin{aligned} J_{11}^{(Y)} &\equiv 8\pi^2 \mathcal{B}_{11;11}, & J_{22}^{(Y)} &\equiv 8\pi^2 \mathcal{B}_{22;22}, & J_{12}^{(Y)} &\equiv 8\pi^2 \mathcal{B}_{12;12}, \\ J_{1122}^{(Y)} &\equiv 8\pi^2 \mathcal{B}_{11;22}, & J_{1112}^{(Y)} &\equiv 8\pi^2 \mathcal{B}_{11;12}, & J_{2212}^{(Y)} &\equiv 8\pi^2 \mathcal{B}_{22;12}. \end{aligned} \quad (111)$$

They can be written explicitly as S^2 averages of elementary functions of $\mathbf{c}_\pm(\hat{\mathbf{n}})$ and $\theta(\hat{\mathbf{n}})$. Introducing the shorthand

$$\mathcal{K}_{++} \equiv \mathcal{K}(\mathbf{c}_+, \mathbf{c}_+) = \frac{1}{2\mathbf{c}_+^{3/2}}, \quad \mathcal{K}_{--} \equiv \frac{1}{2\mathbf{c}_-^{3/2}}, \quad \mathcal{K}_{+-} \equiv \mathcal{K}(\mathbf{c}_+, \mathbf{c}_-), \quad (112)$$

one finds *pointwise* in $\hat{\mathbf{n}}$ the integrands

$$\begin{aligned} J_{11}^{(Y)} &= 2 \langle \mathcal{K}_{++} c_\theta^4 + \mathcal{K}_{--} s_\theta^4 + 2\mathcal{K}_{+-} c_\theta^2 s_\theta^2 \rangle_{S^2}, \\ J_{22}^{(Y)} &= 2 \langle \mathcal{K}_{++} s_\theta^4 + \mathcal{K}_{--} c_\theta^4 + 2\mathcal{K}_{+-} c_\theta^2 s_\theta^2 \rangle_{S^2}, \\ J_{1122}^{(Y)} &= 2 \langle (\mathcal{K}_{++} + \mathcal{K}_{--} - 2\mathcal{K}_{+-}) c_\theta^2 s_\theta^2 \rangle_{S^2}, \\ J_{12}^{(Y)} &= 2 \langle \mathcal{K}_{+-} + (\mathcal{K}_{++} + \mathcal{K}_{--} - 2\mathcal{K}_{+-}) c_\theta^2 s_\theta^2 \rangle_{S^2}, \\ J_{1112}^{(Y)} &= 2 \langle c_\theta s_\theta (\mathcal{K}_{++} c_\theta^2 - \mathcal{K}_{--} s_\theta^2 + \mathcal{K}_{+-} (s_\theta^2 - c_\theta^2)) \rangle_{S^2}, \\ J_{2212}^{(Y)} &= 2 \langle c_\theta s_\theta (\mathcal{K}_{++} s_\theta^2 - \mathcal{K}_{--} c_\theta^2 + \mathcal{K}_{+-} (c_\theta^2 - s_\theta^2)) \rangle_{S^2}. \end{aligned} \quad (113)$$

When $\mathbf{Y} \rightarrow 0$, the mixing angle becomes $\theta \rightarrow 0$ (or $\pi/2$), and, therefore, $c_\theta s_\theta \rightarrow 0$ and $c_\theta^2 s_\theta^2 \rightarrow 0$; in this limit

$$J_{1112}^{(Y)} \rightarrow 0, \quad J_{2212}^{(Y)} \rightarrow 0, \quad J_{1122}^{(Y)} \rightarrow 0, \quad (114)$$

while $J_{11}^{(Y)}, J_{22}^{(Y)}, J_{12}^{(Y)}$ reduce precisely to the three weights J_{11}, J_{22}, J_{12} of the $\mathbf{Y} = 0$ truncation.

With $h_i = 0$ the one-loop flow of quartic couplings is closed. For practical analysis it is convenient to absorb the overall “anisotropy weights” by a constant rescaling of couplings. Define the rescaled quartics

$$\begin{aligned} g_1 &\equiv J_{11}^{(Y)} \lambda_1, & g_2 &\equiv J_{22}^{(Y)} \lambda_2, & g_3 &\equiv \sqrt{J_{11}^{(Y)} J_{22}^{(Y)}} \lambda_3, \\ g_4 &\equiv \sqrt{J_{11}^{(Y)} (J_{12}^{(Y)} + J_{1122}^{(Y)})} \lambda_4, & g_5 &\equiv \sqrt{J_{22}^{(Y)} (J_{12}^{(Y)} + J_{1122}^{(Y)})} \lambda_5, \end{aligned} \quad (115)$$

and the dimensionless ratios

$$\rho \equiv \frac{J_{12}^{(Y)}}{\sqrt{J_{11}^{(Y)} J_{22}^{(Y)}}}, \quad \sigma \equiv \frac{J_{1122}^{(Y)}}{\sqrt{J_{11}^{(Y)} J_{22}^{(Y)}}}, \quad \tilde{\rho} \equiv \rho + \sigma = \frac{J_{12}^{(Y)} + J_{1122}^{(Y)}}{\sqrt{J_{11}^{(Y)} J_{22}^{(Y)}}}, \quad (116)$$

together with

$$\begin{aligned} \alpha &\equiv \frac{J_{1112}^{(Y)}}{\sqrt{J_{11}^{(Y)} (J_{12}^{(Y)} + J_{1122}^{(Y)})}}, & \beta &\equiv \frac{J_{2212}^{(Y)}}{\sqrt{J_{22}^{(Y)} (J_{12}^{(Y)} + J_{1122}^{(Y)})}}, \\ \kappa &\equiv \frac{\rho + 3\sigma}{\rho + \sigma}. \end{aligned} \quad (117)$$

In terms of $(g_1, \dots, g_5)$ the one-loop beta functions take the compact form

$$\beta_{g_1} = -\epsilon g_1 + \frac{3}{16\pi^2} [g_1^2 + g_3^2 + 2g_4^2 + 2\sigma g_1 g_3 + 4\alpha g_1 g_4 + 4\beta g_3 g_4], \quad (118a)$$

$$\beta_{g_2} = -\epsilon g_2 + \frac{3}{16\pi^2} [g_2^2 + g_3^2 + 2g_5^2 + 2\sigma g_2 g_3 + 4\beta g_2 g_5 + 4\alpha g_3 g_5], \quad (118b)$$

$$\beta_{g_3} = -\epsilon g_3 + \frac{1}{16\pi^2} \left[(g_1 + g_2) g_3 + \sigma g_1 g_2 + (4\rho + 5\sigma) g_3^2 + \frac{2}{\tilde{\rho}} (g_4^2 + g_5^2) + 2\kappa g_4 g_5 + 2\alpha (g_1 g_5 + 5g_3 g_4) + 2\beta (g_2 g_4 + 5g_3 g_5) \right], \quad (118c)$$

$$\beta_{g_4} = -\epsilon g_4 + \frac{3}{16\pi^2} [g_1 g_4 + \sigma g_1 g_5 + (2\rho + 3\sigma) g_3 g_4 + g_3 g_5 + 2\tilde{\rho} \alpha g_1 g_3 + 2\alpha g_4^2 + 2\tilde{\rho} \beta g_3^2 + 2\beta g_4 g_5], \quad (118d)$$

$$\beta_{g_5} = -\epsilon g_5 + \frac{3}{16\pi^2} [g_2 g_5 + \sigma g_2 g_4 + g_3 g_4 + (2\rho + 3\sigma) g_3 g_5 + 2\tilde{\rho} \beta g_2 g_3 + 2\beta g_5^2 + 2\tilde{\rho} \alpha g_3^2 + 2\alpha g_4 g_5]. \quad (118e)$$

Undoing the rescaling (115) gives the corresponding explicit β_{λ_i} expressions in the original λ_i variables. The $\mathbf{Y} = 0$ limit is obtained by setting $J_{112}^{(Y)} = J_{2212}^{(Y)} = J_{1122}^{(Y)} = 0$, i.e., $\sigma = \alpha = \beta = 0$ and $\tilde{\rho} = \rho$, which reduces (118) to the earlier $\mathbf{Y} = 0$ system.

In this limit the rescaled couplings reduce smoothly to the variables used in the $\mathbf{Y} = 0$ analysis,

$$\begin{aligned} g_1 &= J_{11}\lambda_1, & g_2 &= J_{22}\lambda_2, & g_3 &= \sqrt{J_{11}J_{22}}\lambda_3, \\ g_4 &= \sqrt{J_{11}J_{12}}\lambda_4, & g_5 &= \sqrt{J_{22}J_{12}}\lambda_5, \end{aligned}$$

and the beta functions written in terms of g_i reduce exactly to the $\mathbf{Y} = 0$ flow depending only on the J_{ij} weights.

C. Interpretation of weights and effect of anisotropy. Physical interpretation of the $Y \neq 0$ weights

In the $\mathbf{Z}_t = \mathbf{1}$ truncation the cross-gradient matrix $\mathbf{Y}$ does not run at one-loop in DR + MS (no momentum-dependent poles), but it *does* enter the RG coefficients through the bubble residue $\mathcal{B}_{ij;kl} \equiv \epsilon \mathbb{B}_{ij;kl}$. When $\mathbf{Y} \neq 0$, the matrix $\mathbf{C}(\hat{\mathbf{n}})$ is nondiagonal in the (ϕ, χ) basis for generic directions $\hat{\mathbf{n}} \in S^2$, so the one-loop coefficients are governed not by three but by six independent angular weights, defined in (111). Below we give a Wilsonian/phase-space interpretation of these weights.

Bubble pole as a sum over intermediate two-particle states. At zero external momentum the bubble master integral is

$$\mathbf{B}_{ij;kl}(0) = \int \frac{d\omega}{2\pi} \frac{d^3\mathbf{p}}{(2\pi)^3} G_{ik}(\omega, \mathbf{p}) G_{jl}(\omega, \mathbf{p}), \quad (119)$$

and in the UV (massless) regime we may use the spectral representation of the directional field-space matrix $\mathbf{C}(\hat{\mathbf{n}})$:

$$\begin{aligned} \mathbf{C}(\hat{\mathbf{n}}) &= \sum_{\alpha=\pm} \mathbf{c}_\alpha(\hat{\mathbf{n}}) \Pi_\alpha(\hat{\mathbf{n}}), & \mathbf{c}_\alpha(\hat{\mathbf{n}}) &> 0, \\ \Pi_\alpha \Pi_\beta &= \delta_{\alpha\beta} \Pi_\alpha, & \Pi_+ + \Pi_- &= \mathbf{1}. \end{aligned} \quad (120)$$

Defining the directional velocities $v_\alpha(\hat{\mathbf{n}}) \equiv \sqrt{\mathbf{c}_\alpha(\hat{\mathbf{n}})}$, the UV energies of the normal modes are

$$E_\alpha(\mathbf{p}) = |\mathbf{p}| v_\alpha(\hat{\mathbf{n}}), \quad (121)$$

recalling that $\hat{\mathbf{n}} \equiv \mathbf{p}/|\mathbf{p}|$ and the propagator decomposes as

$$G(\omega, \mathbf{p}) = \sum_{\alpha=\pm} \frac{\Pi_\alpha(\hat{\mathbf{n}})}{\omega^2 + E_\alpha(\mathbf{p})^2}. \quad (122)$$

Substituting this into the bubble (119) and integrating over ω gives the standard two-particle energy denominator:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + E_\alpha^2)(\omega^2 + E_\beta^2)} = \frac{1}{2E_\alpha E_\beta (E_\alpha + E_\beta)}. \quad (123)$$

Therefore, the bubble is a sum over intermediate two-particle normal-mode states (α, β) , weighted by the overlaps (“coherence factors”) encoded in the projectors:

$$\mathbf{B}_{ij;kl}(0) = \sum_{\alpha, \beta=\pm} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\Pi_\alpha(\hat{\mathbf{n}})_{ik} \Pi_\beta(\hat{\mathbf{n}})_{jl}}{2E_\alpha(\mathbf{p}) E_\beta(\mathbf{p}) (E_\alpha(\mathbf{p}) + E_\beta(\mathbf{p}))}. \quad (124)$$

This form makes the physical content transparent: the one-loop correction to a four-point amplitude is a second-order quantum process in which two external fields create an intermediate fast pair with total energy $E_\alpha + E_\beta$, which is then annihilated back into the external fields; the factor $1/(E_\alpha + E_\beta)$ is precisely the energy denominator of that virtual two-particle state.

Spectral (Källén-Lehmann-type) viewpoint. It is useful to view the bubble (43) as a Euclidean two-point function of a bilinear field *density*. To make the spectral structure explicit, consider the bubble (43) with a purely Euclidean external energy k_0 and vanishing external spatial momentum:

$$\mathbf{B}_{ij;kl}(k_0) \equiv \int \frac{d\omega}{2\pi} \frac{d^3\mathbf{p}}{(2\pi)^3} G_{ik}(\omega, \mathbf{p}) G_{jl}(\omega + k_0, \mathbf{p}). \quad (125)$$

In the massless UV regime we may use the normal-mode decomposition

$$G(\omega, \mathbf{p}) = \sum_{\alpha} \frac{\Pi_\alpha(\hat{\mathbf{n}})}{(\omega^2 + E_\alpha(\mathbf{p})^2)},$$

where $E_\alpha(\mathbf{p}) = |\mathbf{p}| v_\alpha(\hat{\mathbf{n}})$ and $\hat{\mathbf{n}} = \mathbf{p}/|\mathbf{p}|$. Performing the ω integral in (125) gives

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + E_\alpha^2)((\omega + k_0)^2 + E_\beta^2)} \\ &= \frac{E_\alpha + E_\beta}{2E_\alpha E_\beta (k_0^2 + (E_\alpha + E_\beta)^2)}, \end{aligned} \quad (126)$$

and, thus, we arrive at

$$\begin{aligned} \mathbf{B}_{ij;kl}(k_0) &= \sum_{\alpha, \beta} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \Pi_\alpha(\hat{\mathbf{n}})_{ik} \Pi_\beta(\hat{\mathbf{n}})_{jl} \\ &\quad \times \frac{E_\alpha + E_\beta}{2E_\alpha E_\beta (k_0^2 + (E_\alpha + E_\beta)^2)}. \end{aligned} \quad (127)$$

Next, introduce the positive two-particle spectral density

$$\rho_{ij;kl}^{(2)}(\Omega) \equiv \sum_{\alpha,\beta} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\Pi_\alpha(\hat{\mathbf{n}})_{ik} \Pi_\beta(\hat{\mathbf{n}})_{jl}}{4E_\alpha(\mathbf{p})E_\beta(\mathbf{p})} \\ \times \delta(\Omega - (E_\alpha(\mathbf{p}) + E_\beta(\mathbf{p}))), \\ \rho_{ij;kl}^{(2)}(\Omega) \geq 0,$$

and the positivity is immediate, since Π_α are Hermitian projectors and $E_\alpha > 0$. Then the bubble (125) admits the following Källén-Lehmann-type representation:

$$\mathbf{B}_{ij;kl}(k_0) = \int_0^\infty d\Omega^2 \frac{\rho_{ij;kl}^{(2)}(\Omega)}{k_0^2 + \Omega^2}. \quad (128)$$

At $k_0 = 0$ this reduces to $\mathbf{B}_{ij;kl}(0) = 2 \int_0^\infty \frac{d\Omega}{\Omega} \rho_{ij;kl}^{(2)}(\Omega)$, which makes explicit the logarithmic UV sensitivity ($\int d\Omega/\Omega$) that becomes a $1/\epsilon$ pole in DR + MS.

Evaluating $\rho_{ij;kl}^{(2)}$ in spherical variables $\mathbf{p} = \rho \hat{\mathbf{n}}$ gives an Ω -independent integrand and reproduces the universal kernel:

$$\rho_{ij;kl}^{(2)} = \frac{1}{8\pi^2} \left\langle \sum_{\alpha,\beta} \Pi_\alpha(\hat{\mathbf{n}})_{ik} \Pi_\beta(\hat{\mathbf{n}})_{jl} \mathcal{K}(\mathbf{c}_\alpha(\hat{\mathbf{n}}), \mathbf{c}_\beta(\hat{\mathbf{n}})) \right\rangle_{S^2}, \\ \mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) = \frac{1}{v_\alpha v_\beta (v_\alpha + v_\beta)}. \quad (129)$$

Thus, $\mathcal{K}$ controls the spectral weight of intermediate two-particle states, while the projector entries provide the coherence factors (overlaps with the microscopic fields). Moreover, using (128) one may straightforwardly show that in a Lorentzian-signature continuation $\rho_{ij;kl}^{(2)}$ becomes the discontinuity across the corresponding two-particle cut.

Direction-dependent mixing and coherence factors. Turning on $\mathbf{Y}$ makes $\mathbf{C}(\hat{\mathbf{n}})$ nondiagonal in (ϕ, χ) , so the fast normal modes are direction-dependent linear

combinations of ϕ and χ . It is convenient to parametrize the projectors by a mixing angle $\theta(\hat{\mathbf{n}})$:

$$\tan 2\theta(\hat{\mathbf{n}}) = \frac{2y(\hat{\mathbf{n}})}{c_\phi(\hat{\mathbf{n}}) - c_\chi(\hat{\mathbf{n}})}, \\ c_{\phi,\chi}(\hat{\mathbf{n}}) = \hat{\mathbf{n}}^\top \mathbf{C}_{\phi,\chi} \hat{\mathbf{n}}, \quad y(\hat{\mathbf{n}}) = \hat{\mathbf{n}}^\top \mathbf{Y} \hat{\mathbf{n}}, \quad (130)$$

so that

$$\Pi_+(\hat{\mathbf{n}}) = \begin{pmatrix} \cos^2\theta & \sin\theta \cos\theta \\ \sin\theta \cos\theta & \sin^2\theta \end{pmatrix}, \\ \Pi_-(\hat{\mathbf{n}}) = \begin{pmatrix} \sin^2\theta & -\sin\theta \cos\theta \\ -\sin\theta \cos\theta & \cos^2\theta \end{pmatrix}. \quad (131)$$

Define the “populations” $p_\alpha(\hat{\mathbf{n}})$, $q_\alpha(\hat{\mathbf{n}})$ and “coherence” $r_\alpha(\hat{\mathbf{n}})$ of mode α by

$$p_\alpha(\hat{\mathbf{n}}) \equiv (\Pi_\alpha)_{11}, \quad q_\alpha(\hat{\mathbf{n}}) \equiv (\Pi_\alpha)_{22}, \quad r_\alpha(\hat{\mathbf{n}}) \equiv (\Pi_\alpha)_{12}. \quad (132)$$

Then, $p_\alpha, q_\alpha \in [0, 1]$ measure how much of ϕ or χ participates in the fast mode α , while $r_\alpha = \pm \frac{1}{2} \sin 2\theta$ measures the off-diagonal overlap, i.e., coherence. Under $\mathbf{Y} \rightarrow -\mathbf{Y}$ one has $\theta \rightarrow -\theta$; hence, p_α, q_α are even while r_α is odd. This parity property will immediately explain which weights are even/odd in $\mathbf{Y}$.

The six weights as phase-space-weighted moments of populations and coherence. In the $\mathbf{Z}_t = \mathbf{1}$ truncation, the residue $\mathcal{B}$ can be viewed as a symmetric 3×3 matrix acting on the space of symmetric bilinears $(\phi^2, \chi^2, \sqrt{2}\phi\chi)$. Equivalently, the six independent entries of $\mathcal{B}$ may be packaged into the six scalar weights defined in (111). A convenient explicit form is

$$J_{11}^{(Y)} = 2 \left\langle \sum_{\alpha,\beta} \mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) p_\alpha p_\beta \right\rangle_{S^2}, \quad J_{22}^{(Y)} = 2 \left\langle \sum_{\alpha,\beta} \mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) q_\alpha q_\beta \right\rangle_{S^2}, \\ J_{12}^{(Y)} = 2 \left\langle \sum_{\alpha,\beta} \mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) p_\alpha q_\beta \right\rangle_{S^2}, \quad J_{1122}^{(Y)} = 2 \left\langle \sum_{\alpha,\beta} \mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) r_\alpha r_\beta \right\rangle_{S^2}, \\ J_{1112}^{(Y)} = 2 \left\langle \sum_{\alpha,\beta} \mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) p_\alpha r_\beta \right\rangle_{S^2}, \quad J_{2212}^{(Y)} = 2 \left\langle \sum_{\alpha,\beta} \mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) q_\alpha r_\beta \right\rangle_{S^2}.$$

These formulas make the interpretation immediate:

- (i) $J_{11}^{(Y)}, J_{22}^{(Y)}$ are *population-population* moments: they count how much two-particle UV phase space is “seen” as (ϕ, ϕ) or (χ, χ) on the internal lines. They control the strength of purely ϕ and purely χ -loop renormalizations in β_{λ_1} and β_{λ_2} .

- (ii) $J_{12}^{(Y)}$ is the *cross* population moment: it counts how often an intermediate fast pair looks ϕ -like on one internal line and χ -like on the other. It generalizes the familiar $Y = 0$ weight J_{12} .

- (iii) $J_{1122}^{(Y)}$ is the basic *coherence-coherence* moment: it measures the amount of two-particle phase space

carried by genuinely mixed fast modes (neither purely ϕ nor purely χ). It is even in $\mathbf{Y}$.

- (iv) $J_{1112}^{(Y)}, J_{2212}^{(Y)}$ are *population-coherence interference* moments: they measure a directed bias correlating the sign of mixing (the sign of $r_\alpha \propto \sin 2\theta$) with the ϕ - or χ -content of the fast modes. They are odd in $\mathbf{Y}$ and vanish whenever the phase-space average has no net mixing sign bias.

Positivity and useful inequalities. Since $\mathcal{K}(c_\alpha, c_\beta) > 0$ and the projectors are real, $\mathcal{B}$ defines a positive bilinear form on symmetric bilinears, and hence, its 3×3 matrix is positive semi-definite. As a result, the J -weights satisfy Cauchy-Bunyakovsky-Schwarz-type inequalities (useful as consistency checks), for example,

$$\begin{aligned} (J_{12}^{(Y)})^2 &\leq J_{11}^{(Y)} J_{22}^{(Y)}, & (J_{1112}^{(Y)})^2 &\leq J_{11}^{(Y)} J_{1122}^{(Y)}, \\ (J_{2212}^{(Y)})^2 &\leq J_{22}^{(Y)} J_{1122}^{(Y)}. \end{aligned} \quad (133)$$

Moreover, $J_{1122}^{(Y)} \geq 0$ is manifest from (133) together with the pointwise identity [with $w_{\alpha\beta} \equiv \mathcal{K}(c_\alpha, c_\beta)$ and $v_\alpha = \sqrt{c_\alpha}$]

$$\sum_{\alpha,\beta} w_{\alpha\beta} r_\alpha r_\beta = \frac{\sin^2 2\theta}{4} (w_{++} + w_{--} - 2w_{+-}), \quad (134)$$

where

$$w_{++} + w_{--} - 2w_{+-} = \frac{(v_+ - v_-)^2 (v_+^2 + 3v_+ v_- + v_-^2)}{2v_+^3 v_-^3 (v_+ + v_-)} \geq 0. \quad (135)$$

Thus, $J_{1122}^{(Y)}$ is enhanced both when mixing is strong ($|\sin 2\theta| \sim 1$) and when the two mode velocities are split ($|v_+ - v_-| \neq 0$), while it vanishes if either mixing is absent ($\sin 2\theta = 0$) or if the dispersions are degenerate ($v_+ = v_-$), reflecting the fact that “coherence” becomes unobservable when the kinetic operator is proportional to the identity in field space.

How the weights enter β -functions (diagrammatic meaning). At one-loop the quartic beta functions arise from fish diagrams and are quadratic in λ . At the tensor level

$$\begin{aligned} \beta_{\lambda_{ijkl}} &= -\epsilon \lambda_{ijkl} + \frac{1}{2} (\lambda_{ijmn} \mathcal{B}_{mn;pq} \lambda_{klpq} \\ &\quad + \lambda_{ikmn} \mathcal{B}_{mn;pq} \lambda_{jl pq} + \lambda_{ilmn} \mathcal{B}_{mn;pq} \lambda_{jk pq}), \end{aligned} \quad (136)$$

so every monomial in β can be read as a specific *two-step scattering process* weighted by a corresponding entry of $\mathcal{B}$. In particular:

- (i) Terms weighted by $J_{11}^{(Y)}, J_{22}^{(Y)}, J_{12}^{(Y)}$ are the familiar contributions from intermediate pairs that are predominantly $\phi\phi$, $\chi\chi$, or mixed $\phi\chi$ in the UV.
(ii) Terms weighted by $J_{1122}^{(Y)}$ originate from loops in which at least one internal contraction uses the

off-diagonal propagator component $G_{\phi\chi}$, i.e., from intermediate fast pairs that propagate as direction-dependent linear combinations of ϕ and χ . These are new quantum channels that are absent when $\mathbf{Y} = 0$.

- (iii) Terms weighted by $J_{1112}^{(Y)}$ and $J_{2212}^{(Y)}$ are interference contributions between “population” and “coherence” channels. Being odd in $\mathbf{Y}$, they quantify the extent to which the UV mixing texture has a directed bias and, therefore, can feed reflection-breaking couplings in the beta functions.

In this sense the six weights provide a compact book-keeping of how the kinetic mixing $\mathbf{Y}$ opens additional virtual intermediate states in one-loop quantum processes and redistributes the phase-space weight among them.

Isotropic simplification and the role of field redefinitions. If $\mathbf{C}_\phi = c_\phi \mathbf{1}$, $\mathbf{C}_\chi = c_\chi \mathbf{1}$, $\mathbf{Y} = y \mathbf{1}$, then $\mathbf{C}(\hat{n})$ is independent of $\hat{n}$, so $v_\pm$ and θ are constants and all angular averages become trivial. The weights (133) reduce to explicit algebraic combinations of the three numbers w_{++}, w_{--}, w_{+-} and the constant angle θ . Moreover, in this isotropic case there exists a *global* $SO(2)$ field rotation that diagonalizes $\mathbf{C}$ once and for all, removing y from the kinetic term; in that rotated basis the propagator is diagonal and the one-loop coefficients reduce to the $\mathbf{Y} = 0$ structure (three weights), while the effect of mixing is transferred entirely to the interaction tensors (h, λ) by the same rotation. For generic anisotropic matrices $\mathbf{C}_{\phi,\chi}$ and $\mathbf{Y}$, however, $\theta(\hat{n})$ depends on direction and no single global rotation diagonalizes $\mathbf{C}(\hat{n})$ for all $\hat{n}$; the six weights in (111) are then genuine invariants of the UV kinetic data and encode the physically relevant directional mixing of fast modes.

D. Isotropic limit for $\mathbf{Y} \neq 0$

A useful consistency check is the fully homogeneous (isotropic) case

$$\begin{aligned} \mathbf{C}_\phi &= c_\phi \mathbf{1}, & \mathbf{C}_\chi &= c_\chi \mathbf{1}, & \mathbf{Y} &= y \mathbf{1}, \\ c_\phi &> 0, & c_\chi &> 0, & c_\phi c_\chi - y^2 &> 0. \end{aligned} \quad (137)$$

Then for any $\hat{n} \in S^2$ one has $c_\phi(\hat{n}) = c_\phi$, $c_\chi(\hat{n}) = c_\chi$, $y(\hat{n}) = y$, and, therefore, the field-space matrix is direction-independent:

$$\mathbf{C}(\hat{n}) \equiv \mathbf{C} = \begin{pmatrix} c_\phi & y \\ y & c_\chi \end{pmatrix}. \quad (138)$$

Its eigenvalues are obtained from $\det(\mathbf{C} - c \mathbf{1}) = 0$:

$$\begin{aligned} \Delta &\equiv c_\phi - c_\chi, & \Lambda &\equiv \sqrt{\Delta^2 + 4y^2}, \\ c_\pm &= \frac{c_\phi + c_\chi}{2} \pm \frac{1}{2} \Lambda, & v_\pm &\equiv \sqrt{c_\pm} > 0. \end{aligned} \quad (139)$$

A single global $SO(2)$ rotation diagonalizes $\mathbf{C}$, $O(\theta)^\top \mathbf{C} O(\theta) = \text{diag}(\mathbf{c}_+, \mathbf{c}_-)$, with

$$\tan 2\theta = \frac{2y}{\Delta}, \quad \sin 2\theta = \frac{2y}{\Lambda}, \quad \cos 2\theta = \frac{\Delta}{\Lambda}. \quad (140)$$

In terms of $c_\theta \equiv \cos \theta$, $s_\theta \equiv \sin \theta$ the constant projectors read

$$\Pi_+ = \begin{pmatrix} c_\theta^2 & s_\theta c_\theta \\ s_\theta c_\theta & s_\theta^2 \end{pmatrix}, \quad \Pi_- = \begin{pmatrix} s_\theta^2 & -s_\theta c_\theta \\ -s_\theta c_\theta & c_\theta^2 \end{pmatrix}. \quad (141)$$

Since $\mathbf{c}_\pm$ and $\Pi_\pm$ are constant, all angular averages trivialize and the six weights (110) reduce to elementary

combinations of the three kernel values

$$w_{++} = \mathcal{K}(\mathbf{c}_+, \mathbf{c}_+) = \frac{1}{2\mathbf{c}_+^{3/2}}, \quad (142)$$

$$w_{--} = \mathcal{K}(\mathbf{c}_-, \mathbf{c}_-) = \frac{1}{2\mathbf{c}_-^{3/2}}, \quad (143)$$

$$w_{+-} = \mathcal{K}(\mathbf{c}_+, \mathbf{c}_-) = \frac{1}{\sqrt{\mathbf{c}_+} \sqrt{\mathbf{c}_-} (\sqrt{\mathbf{c}_+} + \sqrt{\mathbf{c}_-})}. \quad (144)$$

Explicitly, one finds

$$\begin{aligned} J_{11}^{(Y)} &= 2(w_{++}c_\theta^4 + w_{--}s_\theta^4 + 2w_{+-}s_\theta^2c_\theta^2), & J_{22}^{(Y)} &= 2(w_{++}s_\theta^4 + w_{--}c_\theta^4 + 2w_{+-}s_\theta^2c_\theta^2), \\ J_{1122}^{(Y)} &= 2s_\theta^2c_\theta^2(w_{++} + w_{--} - 2w_{+-}) = \frac{1}{2}\sin^2(2\theta)(w_{++} + w_{--} - 2w_{+-}), \\ J_{12}^{(Y)} &= 2(w_{+-} + s_\theta^2c_\theta^2(w_{++} + w_{--} - 2w_{+-})) = 2w_{+-} + J_{1122}^{(Y)}, \\ J_{1112}^{(Y)} &= 2s_\theta c_\theta(c_\theta^2(w_{++} - w_{+-}) - s_\theta^2(w_{--} - w_{+-})), \\ J_{2212}^{(Y)} &= 2s_\theta c_\theta(s_\theta^2(w_{++} - w_{+-}) - c_\theta^2(w_{--} - w_{+-})). \end{aligned}$$

These identities immediately imply nontrivial relations among the six weights:

$$J_{12}^{(Y)} - J_{1122}^{(Y)} = 2w_{+-}, \quad J_{11}^{(Y)} + J_{22}^{(Y)} + 2J_{1122}^{(Y)} = 2(w_{++} + w_{--}) = \mathbf{c}_+^{-3/2} + \mathbf{c}_-^{-3/2}, \quad (145)$$

showing that in the homogeneous case the apparent six parameters are in fact rigidly determined by the three kinetic constants (c_ϕ, c_χ, y) [equivalently $(\mathbf{c}_+, \mathbf{c}_-, \theta)$].

For linear dispersions $E_\pm(\mathbf{p}) = |\mathbf{p}|v_\pm$ the constant-energy surfaces $\Sigma_\Omega^{a\beta} = \{\mathbf{p} : E_a(\mathbf{p}) + E_\beta(\mathbf{p}) = \Omega\}$ are spheres of radius $|\mathbf{p}| = \Omega/(v_a + v_\beta)$. A direct evaluation gives the two-particle phase-space identity

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{4E_a E_\beta} \delta(\Omega - (E_a + E_\beta)) = \frac{1}{8\pi^2} \frac{1}{v_a v_\beta (v_a + v_\beta)} = \frac{1}{8\pi^2} \mathcal{K}(\mathbf{c}_a, \mathbf{c}_\beta), \quad (146)$$

so the kernel $\mathcal{K}$ is literally the (direction-independent) two-particle density of states per logarithmic interval in the RG scale, while the weights (145) are obtained by dressing this phase space by the constant coherence factors in the projectors $\Pi_\pm$.

In this isotropic case one may diagonalize the kinetic term by the global field rotation $(\varphi_+, \varphi_-)^\top = O(\theta)^\top (\phi, \chi)^\top$, which removes y from the quadratic action: $\frac{1}{2}c_\phi(\nabla\phi)^2 + \frac{1}{2}c_\chi(\nabla\chi)^2 + y\nabla\phi \cdot \nabla\chi = \frac{1}{2}\mathbf{c}_+(\nabla\varphi_+)^2 + \frac{1}{2}\mathbf{c}_-(\nabla\varphi_-)^2$. In the eigenbasis the bubble residue is of the three-weight $Y = 0$ form with constants $J_+ = \mathbf{c}_+^{-3/2}$, $J_- = \mathbf{c}_-^{-3/2}$, and $J_{+-} = 2/(\sqrt{\mathbf{c}_+}\sqrt{\mathbf{c}_-}(\sqrt{\mathbf{c}_+} + \sqrt{\mathbf{c}_-}))$, while the interaction couplings are simply rotated by the same $O(\theta)$. In particular, the homogeneous limit $y \rightarrow 0$ gives $s_\theta c_\theta \rightarrow 0$ and $J_{1122}^{(Y)} = J_{1112}^{(Y)} = J_{2212}^{(Y)} \rightarrow 0$, and one recovers the isotropic $Y = 0$ weights.

VI. CONCLUSION

In this work we have presented a systematic one-loop renormalization analysis of a two-scalar quantum field theory with the most general parity even two-derivative but Lorentz-violating quadratic structure. The theory allows for anisotropic spatial gradients, field-space mixing in time derivatives, and cross-gradient terms, while retaining standard ($z = 1$) power counting. Even in this minimal setting, the renormalization structure turns out to be considerably richer than in Lorentz-invariant scalar theories.

Although Lorentz violation is introduced exclusively through the kinetic sector, interactions inevitably generate an enlarged but closed set of couplings under renormalization, including masses and linear (tadpole) terms. As a consequence, a consistent renormalization-group treatment cannot be formulated solely in terms of modified kinetic

terms or “running velocities.” Instead, an RG-complete operator basis compatible with the reduced symmetry must be retained from the outset.

Technically, we have shown that in dimensional regularization with minimal subtraction near $D = 4 - \epsilon$ all one-loop ultraviolet divergences are governed by two universal master objects: the tadpole with a single mass insertion and the zero-momentum bubble. Using a spectral decomposition of the anisotropic quadratic operator, we derived compact, basis-covariant expressions for their pole parts in terms of the direction-dependent eigenvalues and projectors of the kinetic matrix. Both master residues are dressed by a single positive kernel, which admits a transparent interpretation as a two-particle energy denominator, or equivalently as a two-particle phase-space weight per logarithmic interval in the RG scale. This representation makes the physical origin of all anisotropy-dependent coefficients in the beta functions explicit.

A key structural consequence of locality at one-loop is that, in the MS scheme, all two-point UV poles are momentum independent. As a result, the anisotropy matrices in the quadratic action do not run at one-loop, while masses, linear couplings, and interaction tensors do. The full set of one-loop beta functions for cubic and quartic couplings, masses, and linear couplings was obtained in closed tensor form in terms of the master residues.

We then explored several consistent truncations of the general flow. In the absence of kinetic mixing in field space, all anisotropy dependence collapses into three geometric/phase-space weights, which control the existence and stability of interacting fixed points. In particular, we found that the coupled Wilson-Fisher-type fixed point survives only when the mismatch between the two kinetic sectors is sufficiently small and disappears beyond a universal threshold. In the fully coupled quartic sector the one-loop flow admits a fixed line, which we showed to be of saddle type. Cubic interactions remain relevant near four dimensions and do not lead to interacting fixed points at one-loop, destabilizing the quartic critical manifolds unless tuned away by symmetry. When cross-gradients are present, the renormalization structure is governed by six independent angular weights, which we interpreted as phase-space-weighted populations and coherences of the direction-dependent ultraviolet eigenmodes.

There are several natural directions for future work. First, extending the analysis beyond one-loop is essential to address the running of anisotropy matrices themselves and to assess the emergence (or lack thereof) of effective Lorentz symmetry in the infrared. Second, it would be interesting to generalize the present spectral framework to theories with more fields, with internal symmetries, or to fermionic and gauge sectors with multiple effective metrics. Third, the phase-space interpretation of the anisotropy weights suggests a close connection to Wilsonian momentum-shell RG, which could be made explicit in nonminimal

schemes or in real-space RG formulations. Finally, the framework developed here provides a controlled setting for studying anisotropic critical phenomena and multi-critical points in coupled systems, in both high-energy and condensed-matter contexts, where direction-dependent propagation and kinetic mixing are generic rather than exceptional.

ACKNOWLEDGMENTS

The authors are grateful to Irina Aref'eva and Andrei Kataev for useful comments and fruitful discussions. This work has been supported by Russian Science Foundation Grant No. 24-72-00121, [72].

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: EXPLICIT EVALUATIONS OF MASTER INTEGRALS

This appendix collects the explicit step-by-step regularization of the two master integrals from Sec. III. We begin with the tadpole master integral (36), and then turn to the bubble integral. In dimensional regularization, massless vacuum integrals at zero external momentum are scaleless. To isolate the UV $1/\epsilon$ poles in the MS scheme we evaluate the master integrals with an auxiliary IR regulator mass $\delta_{\text{IR}} > 0$, implemented by the shift

$$\omega^2 \mathbf{1} + \mathbf{K}(\mathbf{p}) \longrightarrow \omega^2 \mathbf{1} + \mathbf{K}(\mathbf{p}) + \delta_{\text{IR}}^2 \mathbf{1}. \quad (\text{A1})$$

Equivalently, each propagator denominator becomes $\omega^2 + \kappa_a(\mathbf{p}) + \delta_{\text{IR}}^2$ in the spectral representation. This makes all Schwinger/proper-time integrals convergent at large proper time and removes the $|\mathbf{p}| \rightarrow 0$ IR region from the radial integrals. The MS pole parts are UV-local and, therefore, independent of δ_{IR} ; all δ_{IR} -dependence resides in finite (IR) terms that we drop. At the end we may safely send $\delta_{\text{IR}} \rightarrow 0$.

1. Tadpole integral

To make all evaluations explicit, we first work out the integration over energy ω in (36):

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\omega^2 \mathbf{1} + \mathbf{K})^{-1} = \frac{1}{2} \mathbf{K}^{-1/2}, \quad (\text{A2})$$

and let us introduce the intermediate IR regulator δ_{IR} to have explicit control on UV/IR convergence:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\omega^2 \mathbf{1} + \mathbf{K} + \delta_{\text{IR}}^2 \mathbf{1})^{-1} = \frac{1}{2} (\mathbf{K} + \delta_{\text{IR}}^2 \mathbf{1})^{-1/2}. \quad (\text{A3})$$

Here $\mathbf{K}$ is given by (27). After introducing such a regulator into (36), the original integral

$$\mathbf{T} = \frac{1}{2} \int \frac{d^d \mathbf{p}}{(2\pi)^d} \mathbf{Z}_t^{-1/2} \mathbf{K}(\mathbf{p})^{-1/2} \mathbf{Z}_t^{-1/2}$$

turns into the regulated one

$$\mathbf{T}(\delta_{\text{IR}}) \equiv \int \frac{d\omega}{2\pi} \frac{d^d \mathbf{p}}{(2\pi)^d} \mathbf{Z}_t^{-1/2} (\omega^2 \mathbf{1} + \mathbf{K}(\mathbf{p}) + \delta_{\text{IR}}^2 \mathbf{1})^{-1} \mathbf{Z}_t^{-1/2}. \quad (\text{A4})$$

It is more convenient to rewrite the integral over momentum in spherical coordinates as

$$\begin{aligned} \int \frac{d^d \mathbf{p}}{(2\pi)^d} &= \frac{1}{(2\pi)^d} \int_0^\infty d\rho \rho^{d-1} \int_{S^{d-1}} d\Omega_{d-1} \\ &= \frac{\Omega_{d-1}}{2(2\pi)^d} \int_0^\infty dx x^{\frac{d}{2}-1} \langle \cdots \rangle_{\hat{\mathbf{n}}}, \end{aligned} \quad (\text{A5})$$

where $\mathbf{p} \equiv \rho \hat{\mathbf{n}}$ with $\rho \equiv |\mathbf{p}| \geq 0$, and $x \equiv \rho^2$; Ω_{d-1} is the solid angle on the unit sphere and $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(\frac{d}{2})$. We also remind the reader that $\langle \cdots \rangle_{\hat{\mathbf{n}}}$ is given by (39). Hence, we arrive at the regulated representation

$$\begin{aligned} \mathbf{T}(\delta_{\text{IR}}) &= \frac{1}{4} \frac{\Omega_{d-1}}{(2\pi)^d} \int_0^\infty dx x^{\frac{d}{2}-1} \\ &\times \langle \mathbf{Z}_t^{-1/2} (\mathbf{K}(x, \hat{\mathbf{n}}) + \delta_{\text{IR}}^2 \mathbf{1})^{-1/2} \mathbf{Z}_t^{-1/2} \rangle_{\hat{\mathbf{n}}}. \end{aligned} \quad (\text{A6})$$

Our aim is to regularize the divergences in the latter expression. To this end let us use the Schwinger parametrization

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)} \int_0^\infty u^{n-1} e^{-uA} du, \quad (\text{A7})$$

so

$$(\mathbf{K}(x, \hat{\mathbf{n}}) + \delta_{\text{IR}}^2 \mathbf{1})^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-t\delta_{\text{IR}}^2} e^{-t\mathbf{K}(x, \hat{\mathbf{n}})} dt. \quad (\text{A8})$$

Recalling that, due to (22), we have

$$\mathbf{K}(\mathbf{p}) = |\mathbf{p}|^2 \mathbf{C}_Z(\hat{\mathbf{n}}) + \mathbf{M}_Z^2,$$

and we obtain

$$\begin{aligned} \mathbf{T}(\delta_{\text{IR}}) &= \frac{1}{4} \frac{\Omega_{d-1}}{(2\pi)^d \sqrt{\pi}} \int_0^\infty t^{-1/2} dt \\ &\times \int_0^\infty x^{\frac{d}{2}-1} dx \langle \mathbf{Z}_t^{-1/2} e^{-t\delta_{\text{IR}}^2} e^{-t(x\mathbf{C}_Z + \mathbf{M}_Z^2)} \mathbf{Z}_t^{-1/2} \rangle_{\hat{\mathbf{n}}}. \end{aligned} \quad (\text{A9})$$

The factor $e^{-t\delta_{\text{IR}}^2}$ makes the t -integral absolutely convergent at $t \rightarrow \infty$. Therefore, we may use the Dyson expansion in $\mathbf{M}_Z^2$ under the integral sign to extract the UV pole, which comes entirely from the short-proper-time region $t \rightarrow 0$. As a next step we perform the Dyson expansion for the exponent $e^{-t(x\mathbf{C}_Z + \mathbf{M}_Z^2)}$ in the mass matrix $\mathbf{M}_Z^2$; this is legitimate since we extract the UV divergence at large momenta, i.e., large eigenvalues of $\mathbf{C}_Z$ and large x . Hence,

$$\begin{aligned} e^{-t(x\mathbf{C}_Z + \mathbf{M}_Z^2)} &= e^{-tx\mathbf{C}_Z} - \int_0^t d\tau e^{-(t-\tau)x\mathbf{C}_Z} \mathbf{M}_Z^2 e^{-\tau x\mathbf{C}_Z} \\ &+ \mathcal{O}((\mathbf{M}_Z^2)^2), \end{aligned} \quad (\text{A10})$$

and thus, up to $\mathcal{O}((\mathbf{M}_Z^2)^2)$, which we omit below,

$$\mathbf{T}(\delta_{\text{IR}}) \equiv \mathbf{T}_0(\delta_{\text{IR}}) + \mathbf{T}_1(\delta_{\text{IR}}) \quad (\text{A11})$$

$$\begin{aligned} &= \frac{1}{4} \frac{\Omega_{d-1}}{(2\pi)^d \sqrt{\pi}} \left[\int_0^\infty t^{-1/2} e^{-t\delta_{\text{IR}}^2} dt \int_0^\infty x^{\frac{d}{2}-1} dx \langle \mathbf{Z}_t^{-1/2} e^{-tx\mathbf{C}_Z} \mathbf{Z}_t^{-1/2} \rangle_{\hat{\mathbf{n}}} \right. \\ &\quad \left. - \int_0^\infty t^{-1/2} e^{-t\delta_{\text{IR}}^2} dt \int_0^\infty x^{\frac{d}{2}-1} dx \int_0^t d\tau \langle \mathbf{Z}_t^{-1/2} e^{-(t-\tau)x\mathbf{C}_Z} \mathbf{M}_Z^2 e^{-\tau x\mathbf{C}_Z} \mathbf{Z}_t^{-1/2} \rangle_{\hat{\mathbf{n}}} \right]. \end{aligned} \quad (\text{A12})$$

Beginning with $\mathbf{T}_0$, we use the decomposition $\mathbf{C}_Z(\hat{\mathbf{n}}) = \sum_{\alpha=1}^2 \mathbf{c}_\alpha(\hat{\mathbf{n}}) \mathbf{\Pi}_\alpha(\hat{\mathbf{n}})$ given by (33) and then take the x integral:

$$\mathbf{T}_0(\delta_{\text{IR}}) = \frac{1}{4} \frac{\Omega_{d-1}}{(2\pi)^d \sqrt{\pi}} \Gamma\left(\frac{d}{2}\right) \int_0^\infty t^{-1/2} e^{-t\delta_{\text{IR}}^2} dt t^{-d/2} \sum_\alpha \langle \mathbf{c}_\alpha(\hat{\mathbf{n}})^{-d/2} \mathbf{Z}_t^{-1/2} \mathbf{\Pi}_\alpha(\hat{\mathbf{n}}) \mathbf{Z}_t^{-1/2} \rangle_{\hat{\mathbf{n}}}, \quad (\text{A13})$$

and after some algebra this reduces to

$$\mathbf{T}_0(\delta_{\text{IR}}) = \frac{1}{2^{d+1} \pi^{d/2} \sqrt{\pi}} \int_0^\infty t^{-\frac{(d+1)}{2}} e^{-t\delta_{\text{IR}}^2} dt \sum_\alpha \langle \mathbf{c}_\alpha^{\frac{d}{2}} \mathbf{Z}_t^{-1/2} \mathbf{\Pi}_\alpha(\hat{\mathbf{n}}) \mathbf{Z}_t^{-1/2} \rangle_{\hat{\mathbf{n}}}, \quad (\text{A14})$$

where we used $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(\frac{d}{2})$. Using $d = 3 - \epsilon$, the $\mathbf{M}_Z^2 = 0$ contribution $\mathbf{T}_0$ is a scaleless power-divergent integral and vanishes in pure dimensional regularization. With the auxiliary IR mass $\delta_{\text{IR}} > 0$ kept finite one finds $\mathbf{T}_0(\delta_{\text{IR}}) \propto \delta_{\text{IR}}^2$ (and it may contain a pole $\propto \delta_{\text{IR}}^2/\epsilon$); hence, it vanishes as $\delta_{\text{IR}} \rightarrow 0$ and does not contribute to the MS counterterms of the original theory.⁸ Therefore, the MS pole relevant for renormalization starts at first order in $\mathbf{M}_Z^2$, i.e., comes from $\mathbf{T}_1$.

Turning to the second term $\mathbf{T}_1$ from (A11), we use (33) and rewrite the exponent as

$$e^{-(t-\tau)x\mathbf{C}_Z}\mathbf{M}_Z^2 e^{-\tau x\mathbf{C}_Z} = \sum_{\alpha,\beta} e^{-x[(t-\tau)\mathbf{c}_\alpha + \tau\mathbf{c}_\beta]}\mathbf{\Pi}_\alpha\mathbf{M}_Z^2\mathbf{\Pi}_\beta, \quad (\text{A15})$$

and thus,

$$\begin{aligned} \mathbf{T}_1(\delta_{\text{IR}}) = & -\frac{1}{4} \frac{\Omega_{d-1}}{(2\pi)^d \sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-t\delta_{\text{IR}}^2} dt \int_0^\infty x^{\frac{d}{2}-1} dx \int_0^t d\tau \\ & \times \left\langle \sum_{\alpha,\beta} e^{-x[(t-\tau)\mathbf{c}_\alpha + \tau\mathbf{c}_\beta]}\mathbf{Z}_t^{-1/2}\mathbf{\Pi}_\alpha\mathbf{M}_Z^2\mathbf{\Pi}_\beta\mathbf{Z}_t^{-1/2} \right\rangle_{\hat{\mathbf{n}}} . \end{aligned} \quad (\text{A16})$$

Taking the integral over x and using (A7), we obtain

$$\mathbf{T}_1(\delta_{\text{IR}}) = -\frac{1}{4} \frac{\Omega_{d-1}}{(2\pi)^d \sqrt{\pi}} \Gamma\left(\frac{d}{2}\right) \int_0^\infty t^{-1/2} e^{-t\delta_{\text{IR}}^2} dt \int_0^t d\tau \quad (\text{A17})$$

$$\times \left\langle \sum_{\alpha,\beta} [(t-\tau)\mathbf{c}_\alpha + \tau\mathbf{c}_\beta]^{-d/2} \mathbf{Z}_t^{-1/2} \mathbf{\Pi}_\alpha \mathbf{M}_Z^2 \mathbf{\Pi}_\beta \mathbf{Z}_t^{-1/2} \right\rangle_{\hat{\mathbf{n}}} . \quad (\text{A18})$$

Introducing the new variable $\tilde{\tau} \equiv \frac{\tau}{t}$, so that

$$(t-\tau)\mathbf{c}_\alpha + \tau\mathbf{c}_\beta = t[(1-\tilde{\tau})\mathbf{c}_\alpha + \tilde{\tau}\mathbf{c}_\beta], \quad (\text{A19})$$

the integrals over t and $\tilde{\tau}$ disentangle. The integral over $\tilde{\tau}$ for $d = 3$ is

$$\int_0^1 \frac{d\tilde{\tau}}{[(1-\tilde{\tau})\mathbf{c}_\alpha + \tilde{\tau}\mathbf{c}_\beta]^{d/2}} = \frac{2}{\mathbf{c}_\alpha \sqrt{\mathbf{c}_\beta} + \mathbf{c}_\beta \sqrt{\mathbf{c}_\alpha}}, \quad (\text{A20})$$

which is finite. The integral over t reads

⁸That is, in $d = 3 - \epsilon$ we use $\int_0^\infty dt t^{-\frac{d+1}{2}} e^{-t\delta_{\text{IR}}^2} = (\delta_{\text{IR}}^2)^{\frac{d+1}{2}} \Gamma(\frac{1-d}{2}) = (\delta_{\text{IR}}^2)^{1-\epsilon/2} \Gamma(-1 + \frac{\epsilon}{2}) \sim -\frac{2}{\epsilon} \delta_{\text{IR}}^2 + \mathcal{O}(\delta_{\text{IR}}^2 \ln \delta_{\text{IR}}^2)$.

$$\begin{aligned} \int_0^\infty t^{(1-d)/2} e^{-t\delta_{\text{IR}}^2} dt &= \int_0^\infty t^{-1+\frac{\epsilon}{2}} e^{-t\delta_{\text{IR}}^2} dt = (\delta_{\text{IR}}^2)^{-\epsilon/2} \Gamma\left(\frac{\epsilon}{2}\right) \\ &= \frac{2}{\epsilon} - \gamma_E - \ln(\delta_{\text{IR}}^2) + \mathcal{O}(\epsilon). \end{aligned} \quad (\text{A21})$$

Collecting everything together, the divergent part of the tadpole integral comes from $\mathbf{T}_1$ and is

$$[\mathbf{T}]_{\text{pole}} = -\frac{1}{4\pi^2\epsilon} \left\langle \sum_{\alpha,\beta} \mathcal{K}(\mathbf{c}_\alpha, \mathbf{c}_\beta) \mathbf{Z}_t^{-1/2} \mathbf{\Pi}_\alpha \mathbf{M}_Z^2 \mathbf{\Pi}_\beta \mathbf{Z}_t^{-1/2} \right\rangle_{\hat{\mathbf{n}}}, \quad (\text{A22})$$

which coincides with (41).

2. Bubble integrals

The bubble integral with nonzero external momentum is defined as

$$\begin{aligned} \mathbf{B}_{ij;kl}(k) &= \int \frac{d\omega}{2\pi} \frac{d^d \mathbf{p}}{(2\pi)^d} \mathbf{G}_{ik}(p) \mathbf{G}_{jl}(p+k) \\ &= \int \frac{d\omega}{2\pi} \frac{d^d \mathbf{p}}{(2\pi)^d} [\mathbf{Z}_t^{-1/2} (\omega^2 \mathbf{1} + \mathbf{K}(\mathbf{p}))^{-1} \mathbf{Z}_t^{-1/2}]_{ik} \\ &\quad \times [\mathbf{Z}_t^{-1/2} ((\omega + \Omega)^2 \mathbf{1} + \mathbf{K}(\mathbf{p} + \mathbf{k}))^{-1} \mathbf{Z}_t^{-1/2}]_{jl}. \end{aligned} \quad (\text{A23})$$

In the one-loop $D = 4 - \epsilon$ expansion the bubble diagram is logarithmically divergent; hence, its UV pole is local. Therefore, the MS pole is independent of the external momentum and can be extracted from the $k = 0$ integral.

To define the massless vacuum integral at $k = 0$ in dimensional regularization and to unambiguously separate UV from possible IR contributions, we introduce the same auxiliary IR regulator as in the tadpole calculation:

$$\omega^2 \mathbf{1} + \mathbf{K}(\mathbf{p}) \longrightarrow \omega^2 \mathbf{1} + \mathbf{K}(\mathbf{p}) + \delta_{\text{IR}}^2 \mathbf{1}, \quad \delta_{\text{IR}} > 0. \quad (\text{A24})$$

This produces the shift $\kappa_\alpha(\mathbf{p}) \rightarrow \kappa_\alpha(\mathbf{p}) + \delta_{\text{IR}}^2$ but does not change the eigenvectors or the projectors.

We consider the corresponding bubble integral at zero momentum with the IR regulator introduced:

$$\mathbf{B}_{ij;kl}(0; \delta_{\text{IR}}) \equiv \int \frac{d\omega}{2\pi} \frac{d^d \mathbf{p}}{(2\pi)^d} \mathbf{G}_{\delta,ik}(p) \mathbf{G}_{\delta,jl}(p), \quad (\text{A25})$$

$$\mathbf{G}_\delta(p) = \mathbf{Z}_t^{-1/2} (\omega^2 \mathbf{1} + \mathbf{K}(\mathbf{p}) + \delta_{\text{IR}}^2 \mathbf{1})^{-1} \mathbf{Z}_t^{-1/2}. \quad (\text{A26})$$

Using the spectral representation,

$$\mathbf{G}_\delta(p) = \sum_\alpha \frac{\tilde{\mathbf{P}}_{Z,\alpha}(\mathbf{p})}{\omega^2 + \kappa_\alpha(\mathbf{p}) + \delta_{\text{IR}}^2}, \quad (\text{A27})$$

and the integral (for $A, B > 0$)

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + A)(\omega^2 + B)} \\ &= \frac{1}{2\sqrt{A}\sqrt{B}(\sqrt{A} + \sqrt{B})}, \quad A, B > 0; \end{aligned} \quad (\text{A28})$$

hence,

$$\begin{aligned} \mathbf{B}_{ij;kl}(0; \delta_{\text{IR}}) &= \frac{1}{2} \int \frac{d^d \mathbf{p}}{(2\pi)^d} \sum_{\alpha, \beta} \tilde{\mathbf{P}}_{Z, \alpha}(\mathbf{p})_{ik} \tilde{\mathbf{P}}_{Z, \beta}(\mathbf{p})_{jl} \\ &\quad \times \mathcal{K}(\kappa_{\alpha}(\mathbf{p}) + \delta_{\text{IR}}^2, \kappa_{\beta}(\mathbf{p}) + \delta_{\text{IR}}^2), \end{aligned} \quad (\text{A29})$$

where we denote

$$\mathcal{K}(A, B) \equiv \frac{1}{\sqrt{A}\sqrt{B}(\sqrt{A} + \sqrt{B})}. \quad (\text{A30})$$

For the MS pole it is sufficient to keep the UV asymptotics $\kappa_{\alpha}(\mathbf{p}) = |\mathbf{p}|^2 \mathbf{c}_{\alpha}(\hat{\mathbf{n}}) + \mathcal{O}(1)$ and $\tilde{\mathbf{P}}_{Z, \alpha}(\mathbf{p}) = \mathbf{P}_{Z, \alpha}(\hat{\mathbf{n}}) + \mathcal{O}(1/|\mathbf{p}|^2)$. All $\mathcal{O}(1)$ and $\mathcal{O}(1/|\mathbf{p}|^2)$ corrections produce UV-finite contributions. Thus, the pole part equals the pole part of the UV-approximated integral.

Switching to $x \equiv |\mathbf{p}|^2$ and using $d^d \mathbf{p} = \frac{\Omega_{d-1}}{2} dx x^{\frac{d}{2}-1}$:

$$\begin{aligned} \mathbf{B}_{ij;kl}^{\text{UV}}(0; \delta_{\text{IR}}) &= \frac{\Omega_{d-1}}{4(2\pi)^d} \int_0^{\infty} dx x^{\frac{d}{2}-1} \\ &\quad \times \left\langle \sum_{\alpha, \beta} \mathbf{P}_{Z, \alpha}(\hat{\mathbf{n}})_{ik} \mathbf{P}_{Z, \beta}(\hat{\mathbf{n}})_{jl} \mathcal{I}_{\alpha\beta}(x; \delta_{\text{IR}}) \right\rangle_{\hat{\mathbf{n}}}, \end{aligned} \quad (\text{A31})$$

where

$$\mathcal{I}_{\alpha\beta}(x; \delta_{\text{IR}}) \equiv \mathcal{K}(x\mathbf{c}_{\alpha} + \delta_{\text{IR}}^2, x\mathbf{c}_{\beta} + \delta_{\text{IR}}^2). \quad (\text{A32})$$

Next, we use the identity (for $A, B > 0$)

$$\int_0^1 d\tau [(1-\tau)A + \tau B]^{-3/2} = \frac{2}{\sqrt{A}\sqrt{B}(\sqrt{A} + \sqrt{B})}, \quad (\text{A33})$$

which implies

$$\mathcal{I}_{\alpha\beta}(x; \delta_{\text{IR}}) = \frac{1}{2} \int_0^1 d\tau (x[(1-\tau)\mathbf{c}_{\alpha} + \tau\mathbf{c}_{\beta}] + \delta_{\text{IR}}^2)^{-3/2}. \quad (\text{A34})$$

The x -integral is then

$$\begin{aligned} \int_0^{\infty} dx x^{\frac{d}{2}-1} (x\mathbf{c} + \delta_{\text{IR}}^2)^{-3/2} &= c^{-d/2} \delta_{\text{IR}}^{d-3} \frac{\Gamma(d/2)\Gamma(\frac{3-d}{2})}{\Gamma(3/2)} \\ &= c^{-d/2} \delta_{\text{IR}}^{-\epsilon} \frac{\Gamma(d/2)\Gamma(\epsilon/2)}{\Gamma(3/2)}, \end{aligned} \quad (\text{A35})$$

where $c \equiv (1-\tau)\mathbf{c}_{\alpha} + \tau\mathbf{c}_{\beta}$. Taking the pole part and setting $d \rightarrow 3$ inside the finite angular/ τ integrals we obtain

$$[\mathbf{B}_{ij;kl}(0)]_{\text{pole}} = \frac{1}{4\pi^2 \epsilon} \left\langle \sum_{\alpha, \beta} \mathcal{K}(\mathbf{c}_{\alpha}, \mathbf{c}_{\beta}) \mathbf{P}_{Z, \alpha}(\hat{\mathbf{n}})_{ik} \mathbf{P}_{Z, \beta}(\hat{\mathbf{n}})_{jl} \right\rangle_{\hat{\mathbf{n}}}, \quad (\text{A36})$$

where

$$\mathcal{K}(\mathbf{c}_{\alpha}, \mathbf{c}_{\beta}) = \frac{1}{\sqrt{\mathbf{c}_{\alpha}}\sqrt{\mathbf{c}_{\beta}}(\sqrt{\mathbf{c}_{\alpha}} + \sqrt{\mathbf{c}_{\beta}})}, \quad (\text{A37})$$

which coincides with (45) in the main text. Now let us comment on the bubble integral with nonzero external momentum (A23).

For the purposes of MS renormalization in $D = 4 - \epsilon$, it is sufficient to know the UV pole part of the bubble diagram. The one-loop bubble is superficially logarithmically divergent; hence, any UV divergence is local. Expanding the integrand of (A23) in the external momentum k produces additional inverse powers of the loop momentum, so all terms proportional to Ω and/or $\mathbf{k}$ are UV finite. Therefore, the $1/\epsilon$ pole is independent of k and is completely captured by the $k = 0$ master integral computed above. Since this is a standard one-loop locality argument, we do not reproduce the full finite- k analysis here, because it does not produce divergent terms.

APPENDIX B: ANGULAR WEIGHTS J_{11} , J_{22} , AND J_{12} IN THE LIMIT $\mathbf{Z}_t = \mathbf{1}$ AND $\mathbf{Y} = \mathbf{0}$

In this appendix we collect the calculations concerning J_{11} , J_{22} , and J_{12} used in Sec. V in the limit $\mathbf{Z}_t = \mathbf{1}$ and $\mathbf{Y} = \mathbf{0}$, and prove several identities and inequalities employed there. All angular averages are taken over the unit sphere S^2 with the normalization

$$\langle f(\hat{\mathbf{n}}) \rangle_{S^2} \equiv \frac{1}{4\pi} \int_{S^2} d\Omega f(\hat{\mathbf{n}}), \quad \hat{\mathbf{n}} \in S^2. \quad (\text{B1})$$

We also specialize to the limit

$$\mathbf{Z}_t = \mathbf{1}, \quad \mathbf{Y} = \mathbf{0}, \quad (\text{B2})$$

so that the two scalar species propagate independently in the UV, while the spatial anisotropy is entirely encoded in the two positive-definite 3×3 matrices $\mathbf{C}_{\phi}$ and $\mathbf{C}_{\chi}$.

1. Weights J_{11} , J_{22} , and J_{12} in the limit $\mathbf{Z}_t = \mathbf{1}$, $\mathbf{Y} = \mathbf{0}$

In the limit (B2) the matrix $\mathbf{C}(\hat{\mathbf{n}})$ introduced in the main text becomes diagonal in field space,

$$\mathbf{C}(\hat{\mathbf{n}}) = \begin{pmatrix} \hat{\mathbf{n}}^{\top} \mathbf{C}_{\phi} \hat{\mathbf{n}} & 0 \\ 0 & \hat{\mathbf{n}}^{\top} \mathbf{C}_{\chi} \hat{\mathbf{n}} \end{pmatrix} \equiv \begin{pmatrix} c_{\phi}(\hat{\mathbf{n}}) & 0 \\ 0 & c_{\chi}(\hat{\mathbf{n}}) \end{pmatrix}, \quad (\text{B3})$$

where we recall that

$$c_\phi(\hat{n}) \equiv \hat{n}^\top \mathbf{C}_\phi \hat{n} > 0, \quad c_\chi(\hat{n}) \equiv \hat{n}^\top \mathbf{C}_\chi \hat{n} > 0. \quad (\text{B4})$$

Consequently, the spectral projectors in field space are constant:

$$\begin{aligned} \Pi_\phi &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \Pi_\chi &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ \Pi_\phi \Pi_\chi &= 0, & \Pi_\phi + \Pi_\chi &= \mathbf{1}. \end{aligned} \quad (\text{B5})$$

The general pole part of the bubble master integral at zero external momentum, Eq. (45), reduces in this limit to

$$\mathbb{B}_{ij;kl} = \frac{1}{4\pi^2\epsilon} \left\langle \sum_{\alpha,\beta \in \{\phi,\chi\}} \mathcal{K}(c_\alpha, c_\beta) \Pi_{\alpha ik} \Pi_{\beta jl} \right\rangle_{S^2}, \quad (\text{B6})$$

where

$$\mathcal{K}(c_\alpha, c_\beta) = \frac{1}{\sqrt{c_\alpha} \sqrt{c_\beta} (\sqrt{c_\alpha} + \sqrt{c_\beta})}. \quad (\text{B7})$$

In the limit (B2) one has $\mathbf{P}_{Z,\phi} = \Pi_\phi$ and $\mathbf{P}_{Z,\chi} = \Pi_\chi$, and explicitly

$$\mathcal{K}(c_\phi, c_\phi) = \frac{1}{2c_\phi^{3/2}}, \quad \mathcal{K}(c_\chi, c_\chi) = \frac{1}{2c_\chi^{3/2}}, \quad (\text{B8a})$$

$$\mathcal{K}(c_\phi, c_\chi) = \frac{1}{\sqrt{c_\phi} \sqrt{c_\chi} (\sqrt{c_\phi} + \sqrt{c_\chi})}. \quad (\text{B8b})$$

It is, therefore, natural to define

$$J_{11} \equiv \langle c_\phi(\hat{n})^{-3/2} \rangle_{S^2}, \quad J_{22} \equiv \langle c_\chi(\hat{n})^{-3/2} \rangle_{S^2}, \quad (\text{B9a})$$

$$J_{12} \equiv 2 \langle \mathcal{K}(c_\phi(\hat{n}), c_\chi(\hat{n})) \rangle_{S^2}. \quad (\text{B9b})$$

With these definitions one immediately obtains, for instance,

$$\begin{aligned} \mathbb{B}_{11;11} &= \frac{1}{4\pi^2\epsilon} \langle \mathcal{K}(c_\phi, c_\phi) \rangle_{S^2} = \frac{1}{4\pi^2\epsilon} \left\langle \frac{1}{2c_\phi^{3/2}} \right\rangle_{S^2} = \frac{J_{11}}{8\pi^2\epsilon}, \\ \mathbb{B}_{22;22} &= \frac{J_{22}}{8\pi^2\epsilon}, \\ \mathbb{B}_{12;12} &= \frac{1}{4\pi^2\epsilon} \langle \mathcal{K}(c_\phi, c_\chi) \rangle_{S^2} = \frac{1}{4\pi^2\epsilon} \cdot \frac{J_{12}}{2} = \frac{J_{12}}{8\pi^2\epsilon}, \end{aligned} \quad (\text{B10})$$

and similarly for $\mathbb{B}_{21;21}$. All other components vanish in this limit because the projectors (B5) kill cross-species structures. This explains why all one-loop RG coefficients in this limit are weighted only by J_{11} , J_{22} , and J_{12} .

Completely analogously, the general pole part of the tadpole master integral, Eq. (41), reduces in the limit (B2) to the same three weights (B9), because the same kernel $\mathcal{K}(c_\alpha, c_\beta)$ appears there. In particular, the diagonal entries are proportional to J_{11} and J_{22} , while the off-diagonal entry is proportional to J_{12} .

Now turn to the next simplification concerning J_{11} and J_{22} :

$$\langle (\hat{n}^\top \mathbf{C} \hat{n})^{-3/2} \rangle_{S^2} = (\det \mathbf{C})^{-1/2}, \quad \mathbf{C} = \mathbf{C}^\top > 0, \quad \mathbf{C} \in \mathbb{R}^{3 \times 3}. \quad (\text{B11})$$

The proof is elementary and follows from evaluating the same Gaussian integral in two different ways.

Consider

$$I(\mathbf{C}) \equiv \int_{\mathbb{R}^3} d^3p e^{-p^\top \mathbf{C} p}. \quad (\text{B12})$$

Since $\mathbf{C}$ is symmetric positive-definite, one may write $\mathbf{C} = \mathbf{U}^\top \mathbf{U}$ with $\mathbf{U} = \mathbf{C}^{1/2}$. Introduce $q = \mathbf{U}p$, so that

$$\begin{aligned} p^\top \mathbf{C} p &= p^\top \mathbf{U}^\top \mathbf{U} p = q^\top q = |q|^2, \\ d^3p &= (\det \mathbf{U})^{-1} d^3q = (\det \mathbf{C})^{-1/2} d^3q, \end{aligned} \quad (\text{B13})$$

and, therefore,

$$I(\mathbf{C}) = (\det \mathbf{C})^{-1/2} \int_{\mathbb{R}^3} d^3q e^{-|q|^2} = (\det \mathbf{C})^{-1/2} \pi^{3/2}. \quad (\text{B14})$$

Now parametrize $p = r\hat{n}$ with $r \geq 0$ and $\hat{n} \in S^2$. Then

$$p^\top \mathbf{C} p = r^2 (\hat{n}^\top \mathbf{C} \hat{n}), \quad d^3p = r^2 dr d\Omega. \quad (\text{B15})$$

Hence,

$$I(\mathbf{C}) = \int_{S^2} d\Omega \int_0^\infty r^2 dr \exp[-r^2 (\hat{n}^\top \mathbf{C} \hat{n})]. \quad (\text{B16})$$

Using the standard integral (valid for $a > 0$)

$$\int_0^\infty r^2 e^{-ar^2} dr = \frac{\sqrt{\pi}}{4} a^{-3/2}, \quad (\text{B17})$$

we obtain

$$I(\mathbf{C}) = \frac{\sqrt{\pi}}{4} \int_{S^2} d\Omega (\hat{n}^\top \mathbf{C} \hat{n})^{-3/2}. \quad (\text{B18})$$

Equating (B14) and (B18) and dividing by 4π , we find

$$\begin{aligned} \langle (\hat{\mathbf{n}}^\top \mathbf{C} \hat{\mathbf{n}})^{-3/2} \rangle_{S^2} &= \frac{1}{4\pi} \int_{S^2} d\Omega (\hat{\mathbf{n}}^\top \mathbf{C} \hat{\mathbf{n}})^{-3/2} \\ &= \frac{\pi^{3/2}}{(\sqrt{\pi}/4)4\pi} (\det \mathbf{C})^{-1/2} \\ &= (\det \mathbf{C})^{-1/2}, \end{aligned} \quad (\text{B19})$$

which proves (B11).

As an immediate corollary, for the weights (B9) one obtains

$$J_{11} = (\det \mathbf{C}_\phi)^{-1/2}, \quad J_{22} = (\det \mathbf{C}_\chi)^{-1/2}. \quad (\text{B20})$$

Now turn to the explicit form of J_{12} . In this subsection we derive the representation of J_{12} used in the main text:

$$J_{12} = \int_0^1 d\tau (\det[(1-\tau)\mathbf{C}_\phi + \tau\mathbf{C}_\chi])^{-1/2}. \quad (\text{B21})$$

We proceed in two steps: first, we rewrite the mixed kernel $\mathcal{K}(c_\phi, c_\chi)$ as a one-dimensional integral over τ , and then we use the identity (B11).

For any $u > 0$ and $v > 0$ one has the elementary identity

$$\int_0^1 d\tau [(1-\tau)u + \tau v]^{-3/2} = \frac{2}{\sqrt{u}\sqrt{v}(\sqrt{u} + \sqrt{v})}. \quad (\text{B22})$$

Now apply (B22) pointwise to

$$u = c_\phi(\hat{\mathbf{n}}) = \hat{\mathbf{n}}^\top \mathbf{C}_\phi \hat{\mathbf{n}}, \quad v = c_\chi(\hat{\mathbf{n}}) = \hat{\mathbf{n}}^\top \mathbf{C}_\chi \hat{\mathbf{n}}. \quad (\text{B23})$$

Using the definition (B9) of J_{12} we obtain

$$\begin{aligned} J_{12} &= 2 \left\langle \frac{1}{\sqrt{c_\phi} \sqrt{c_\chi} (\sqrt{c_\phi} + \sqrt{c_\chi})} \right\rangle_{S^2} \\ &= \left\langle \int_0^1 d\tau [(1-\tau)c_\phi(\hat{\mathbf{n}}) + \tau c_\chi(\hat{\mathbf{n}})]^{-3/2} \right\rangle_{S^2}. \end{aligned} \quad (\text{B24})$$

By linearity of the angular average, this becomes

$$J_{12} = \int_0^1 d\tau \langle [(1-\tau)\hat{\mathbf{n}}^\top \mathbf{C}_\phi \hat{\mathbf{n}} + \tau \hat{\mathbf{n}}^\top \mathbf{C}_\chi \hat{\mathbf{n}}]^{-3/2} \rangle_{S^2}. \quad (\text{B25})$$

Defining the interpolating matrix

$$\mathbf{C}_\tau \equiv (1-\tau)\mathbf{C}_\phi + \tau\mathbf{C}_\chi, \quad (\text{B26})$$

the expression in brackets in (B25) is simply $\hat{\mathbf{n}}^\top \mathbf{C}_\tau \hat{\mathbf{n}}$. Since $\mathbf{C}_\tau$ is positive-definite for all $\tau \in [0, 1]$ (convexity of the cone of positive-definite matrices), we may apply (B11) to obtain

$$\langle (\hat{\mathbf{n}}^\top \mathbf{C}_\tau \hat{\mathbf{n}})^{-3/2} \rangle_{S^2} = (\det \mathbf{C}_\tau)^{-1/2}. \quad (\text{B27})$$

Substituting this into (B25) we arrive at (B21):

$$J_{12} = \int_0^1 d\tau (\det[(1-\tau)\mathbf{C}_\phi + \tau\mathbf{C}_\chi])^{-1/2}. \quad (\text{B28})$$

This completes the derivation.

It is sometimes convenient to rewrite (B21) in terms of the generalized eigenvalues of the pair $(\mathbf{C}_\chi, \mathbf{C}_\phi)$. Introducing

$$\mathbf{A} \equiv \mathbf{C}_\phi^{-1/2} \mathbf{C}_\chi \mathbf{C}_\phi^{-1/2} = U \text{diag}(\gamma_1, \gamma_2, \gamma_3) U^\top, \quad \gamma_i > 0, \quad (\text{B29})$$

where U is orthogonal, one finds

$$J_{12} = (\det \mathbf{C}_\phi)^{-1/2} \int_0^1 d\tau \prod_{i=1}^3 ((1-\tau) + \tau\gamma_i)^{-1/2}, \quad (\text{B30})$$

and this can be expressed through Lauricella hypergeometric functions $F_D^{(3)}$ as quoted in the main text.

2. Proof of the inequality $J_{12}^2 \leq J_{11}J_{22}$

Finally, we prove an inequality which is useful for the existence analysis of fixed points in the reflection-symmetric sector:

$$J_{12}^2 \leq J_{11}J_{22}. \quad (\text{B31})$$

The proof uses only elementary inequalities and holds for any positive functions $c_\phi(\hat{\mathbf{n}}), c_\chi(\hat{\mathbf{n}})$ [in particular, for $c_\phi(\hat{\mathbf{n}}) = \hat{\mathbf{n}}^\top \mathbf{C}_\phi \hat{\mathbf{n}}$ and $c_\chi(\hat{\mathbf{n}}) = \hat{\mathbf{n}}^\top \mathbf{C}_\chi \hat{\mathbf{n}}$].

For any $a > 0$ and $b > 0$ the arithmetic-geometric mean inequality gives

$$\sqrt{a} + \sqrt{b} \geq 2(ab)^{1/4}. \quad (\text{B32})$$

Therefore, pointwise in $\hat{\mathbf{n}}$,

$$\frac{1}{\sqrt{c_\phi} \sqrt{c_\chi} (\sqrt{c_\phi} + \sqrt{c_\chi})} \leq \frac{1}{\sqrt{c_\phi} \sqrt{c_\chi} \cdot 2(c_\phi c_\chi)^{1/4}} = \frac{1}{2c_\phi^{3/4} c_\chi^{3/4}}. \quad (\text{B33})$$

Average both sides over S^2 and use Cauchy-Bunyakovsky-Schwarz:

$$\langle c_\phi^{-3/4} c_\chi^{-3/4} \rangle_{S^2}^2 \leq \langle c_\phi^{-3/2} \rangle_{S^2} \langle c_\chi^{-3/2} \rangle_{S^2} = J_{11}J_{22}. \quad (\text{B34})$$

Combining with the definition $J_{12} = 2 \langle 1/(\sqrt{c_\phi} \sqrt{c_\chi} (\sqrt{c_\phi} + \sqrt{c_\chi})) \rangle_{S^2}$, we obtain

$$J_{12} = 2 \left\langle \frac{1}{\sqrt{c_\phi} \sqrt{c_\chi} (\sqrt{c_\phi} + \sqrt{c_\chi})} \right\rangle_{S^2} \leq 2 \left\langle \frac{1}{2c_\phi^{3/4} c_\chi^{3/4}} \right\rangle_{S^2} = \langle c_\phi^{-3/4} c_\chi^{-3/4} \rangle_{S^2}; \quad (\text{B35})$$

hence,

$$J_{12}^2 \leq \langle c_\phi^{-3/4} c_\chi^{-3/4} \rangle_{S^2}^2 \leq J_{11} J_{22}, \quad (\text{B36})$$

which proves (B31).

Equality in (B31) requires equality both in arithmetic-geometric mean and in Cauchy-Bunyakovsky-Schwarz. The arithmetic-geometric mean equality forces $c_\phi(\hat{n}) = c_\chi(\hat{n})$ pointwise on S^2 , while equality in Cauchy-Bunyakovsky-Schwarz forces the functions $c_\phi^{-3/4}$ and $c_\chi^{-3/4}$ to be proportional. For $c_\phi(\hat{n}) = \hat{n}^\top \mathbf{C}_\phi \hat{n}$ and $c_\chi(\hat{n}) = \hat{n}^\top \mathbf{C}_\chi \hat{n}$ this implies $\hat{n}^\top \mathbf{C}_\phi \hat{n} = \hat{n}^\top \mathbf{C}_\chi \hat{n}$ for all $\hat{n}$, i.e., $\mathbf{C}_\phi = \mathbf{C}_\chi$.

APPENDIX C: LINEAR STABILITY OF THE GENERALIZED WILSON-FISHER FIXED POINT

In this appendix we analyze the one-loop linear stability of the coupled fixed point S_- (the “generalized Wilson-Fisher” point) found in Sec. V in the limit $\mathbf{Z}_t = \mathbf{1}$, $\mathbf{Y} = 0$ and in the reflection-symmetric quartic sector $\lambda_{4,5} = 0$ (we also set $h_i = 0$ throughout this appendix).

1. Beta functions and a convenient rescaling

In the sector $\lambda_{4,5} = 0$ the one-loop beta functions from Sec. V read

$$\begin{aligned} \beta_{\lambda_1} &= -\epsilon \lambda_1 + \frac{3}{16\pi^2} (J_{11} \lambda_1^2 + J_{22} \lambda_3^2), \\ \beta_{\lambda_2} &= -\epsilon \lambda_2 + \frac{3}{16\pi^2} (J_{22} \lambda_2^2 + J_{11} \lambda_3^2), \\ \beta_{\lambda_3} &= -\epsilon \lambda_3 + \frac{1}{16\pi^2} (J_{11} \lambda_1 \lambda_3 + 4J_{12} \lambda_3^2 + J_{22} \lambda_2 \lambda_3). \end{aligned} \quad (\text{C1})$$

It is useful to absorb the “weights” J_{11}, J_{22} by introducing rescaled couplings

$$\begin{aligned} g_1 &\equiv J_{11} \lambda_1, & g_2 &\equiv J_{22} \lambda_2, & g_3 &\equiv \sqrt{J_{11} J_{22}} \lambda_3, \\ \rho &\equiv \frac{J_{12}}{\sqrt{J_{11} J_{22}}} \in (0, 1], \end{aligned} \quad (\text{C2})$$

and also the standard one-loop coefficient

$$a \equiv \frac{1}{16\pi^2}, \quad c \equiv \frac{16\pi^2}{3} \epsilon = \frac{\epsilon}{3a}. \quad (\text{C3})$$

In terms of (g_1, g_2, g_3) the system (C1) becomes

$$\begin{aligned} \beta_{g_1} &\equiv J_{11} \beta_{\lambda_1} = -\epsilon g_1 + 3a(g_1^2 + g_3^2), \\ \beta_{g_2} &\equiv J_{22} \beta_{\lambda_2} = -\epsilon g_2 + 3a(g_2^2 + g_3^2), \\ \beta_{g_3} &\equiv \sqrt{J_{11} J_{22}} \beta_{\lambda_3} = -\epsilon g_3 + a(g_1 + g_2)g_3 + 4apg_3^2. \end{aligned} \quad (\text{C4})$$

Solving (C4) gives two branches $S_\pm$. It is convenient to introduce

$$\Delta \equiv \sqrt{4\rho^2 - 3}, \quad 4\rho^2 \geq 3 \Leftrightarrow \rho \geq \frac{\sqrt{3}}{2}. \quad (\text{C5})$$

The coupled fixed points exist if and only if $\Delta \in \mathbb{R}$, i.e., iff $\rho \geq \sqrt{3}/2$ (which is the same condition as $4J_{12}^2 \geq 3J_{11}J_{22}$ in the original variables). For the IR-stable branch S_- one finds that

$$g_1^* = g_2^* = g^*, \quad \lambda_1^* = \frac{g^*}{J_{11}}, \quad \lambda_2^* = \frac{g^*}{J_{22}}, \quad \lambda_3^* = \frac{g_3^*}{\sqrt{J_{11} J_{22}}}, \quad (\text{C6})$$

where

$$\begin{aligned} g_3^* &= \frac{c}{2(4\rho^2 + 1)} (4\rho - \Delta), \\ g^* &= \frac{c}{2(4\rho^2 + 1)} (4\rho^2 + 3 + 2\rho\Delta). \end{aligned} \quad (\text{C7})$$

At $\rho = 1$ one has the common-metric case

$$J_{11} = J_{22} = J_{12} \equiv J,$$

and, therefore,

$$g_1^* = g_2^* = \frac{24\pi^2}{5} \epsilon, \quad g_3^* = \frac{8\pi^2}{5} \epsilon.$$

At the unit-normalized isotropic point $J = 1$ (equivalently $\mathbf{C}_\phi = \mathbf{C}_\chi = \mathbf{1}$), this becomes

$$(\lambda_1^*, \lambda_2^*, \lambda_3^*) = \left(\frac{24\pi^2}{5} \epsilon, \frac{24\pi^2}{5} \epsilon, \frac{8\pi^2}{5} \epsilon \right), \quad (\text{C8})$$

i.e., the familiar two-scalar Wilson-Fisher fixed point written in our conventions.

2. Stability matrix and eigenvalues

Let $t \equiv \ln \mu$. Linearizing the RG flow around a fixed point g_a^* gives

$$\frac{d}{dt} \delta g_a = \sum_b \mathbb{M}_{ab} \delta g_b, \quad \mathbb{M}_{ab} \equiv \left. \frac{\partial \beta_{g_a}}{\partial g_b} \right|_{g=g^*}. \quad (\text{C9})$$

With our convention $\beta_g = \mu \frac{dg}{d\mu}$, the IR limit corresponds to $t \rightarrow -\infty$. Thus, the fixed point is IR-attractive iff all

eigenvalues ω of $\mathbb{M}$ are positive, because $\delta g \sim e^{\omega t}$ then decays as $t \rightarrow -\infty$.

From (C4) we obtain the Jacobian

$$\mathbb{M} = \begin{pmatrix} -\epsilon + 6ag_1 & 0 & 6ag_3 \\ 0 & -\epsilon + 6ag_2 & 6ag_3 \\ ag_3 & ag_3 & -\epsilon + a(g_1 + g_2) + 8apg_3 \end{pmatrix}_{g=g^*}. \quad (\text{C10})$$

At the coupled fixed points with $g_1^* = g_2^* = g^*$ and $g_3^* \neq 0$ one may simplify the diagonal entries to obtain

$$\mathbb{M} = \begin{pmatrix} 4a(g^* - \rho g_3^*) & 0 & 6ag_3^* \\ 0 & 4a(g^* - \rho g_3^*) & 6ag_3^* \\ ag_3^* & ag_3^* & 4apg_3^* \end{pmatrix}. \quad (\text{C11})$$

Introduce the combinations

$$\delta g_{\pm} \equiv \frac{1}{2}(\delta g_1 \pm \delta g_2), \quad (\text{C12})$$

so that δg_- is the “antisymmetric” [short notation “(as)”] deformation (splitting g_1 and g_2), while δg_+ is the “symmetric” [short notation “(sym)”] deformation (changing them equally). From (C11) one immediately sees that δg_- decouples because the third row contains $\delta g_1 + \delta g_2$ only. Explicitly,

$$\frac{d}{dt} \delta g_- = \omega_-^{(\text{as})} \delta g_-, \quad \omega_-^{(\text{as})} = 4a(g^* - \rho g_3^*). \quad (\text{C13})$$

Using (C7) and $ac = \epsilon/3$, this eigenvalue becomes

$$\omega_-^{(\text{as})} = \frac{2\epsilon}{4\rho^2 + 1} (1 + \rho\Delta) > 0 \quad (\rho > \sqrt{3}/2). \quad (\text{C14})$$

In the $(\delta g_+, \delta g_3)$ sector one gets a 2×2 matrix,

$$\frac{d}{dt} \begin{pmatrix} \delta g_+ \\ \delta g_3 \end{pmatrix} = \begin{pmatrix} 4a(g^* - \rho g_3^*) & 6ag_3^* \\ 2ag_3^* & 4apg_3^* \end{pmatrix} \begin{pmatrix} \delta g_+ \\ \delta g_3 \end{pmatrix}. \quad (\text{C15})$$

The eigenvalues are obtained by solving the characteristic polynomial. A convenient shortcut is to note that the trace equals

$$\text{tr } \mathbb{M}_{\text{sym}} = 4ag^*, \quad (\text{C16})$$

and after some algebra one finds that one eigenvalue is *exactly*

$$\omega_+^{(\text{sym})} = \epsilon, \quad (\text{C17})$$

independently of ρ . The second symmetric eigenvalue is

$$\omega_-^{(\text{sym})} = \frac{\epsilon}{3(4\rho^2 + 1)} (3 - 4\rho^2 + 4\rho\Delta). \quad (\text{C18})$$

In the allowed range $\rho \in (\sqrt{3}/2, 1]$ one has $\omega_-^{(\text{sym})} > 0$, and moreover, $\omega_-^{(\text{sym})} \rightarrow 0^+$ as $\rho \downarrow \sqrt{3}/2$, which reflects the merger of the two branches at the boundary of existence (C5). Collecting (C14), (C17), and (C18), the three (one-loop) stability eigenvalues of the coupled fixed point S_- in the subspace $(\lambda_1, \lambda_2, \lambda_3)$ are

$$\omega_1 = \epsilon, \quad (\text{C19a})$$

$$\omega_2 = \frac{\epsilon}{3(4\rho^2 + 1)} (3 - 4\rho^2 + 4\rho\sqrt{4\rho^2 - 3}), \quad (\text{C19b})$$

$$\omega_3 = \frac{2\epsilon}{4\rho^2 + 1} (1 + \rho\sqrt{4\rho^2 - 3}). \quad (\text{C19c})$$

Therefore, for $\epsilon > 0$ and $\rho > \sqrt{3}/2$ the fixed point S_- is IR-attractive within the quartic reflection-symmetric sector, in agreement with the statement in Sec. V. For $\rho = 1$ one recovers the familiar values $\{\omega_1, \omega_2, \omega_3\} = \{\epsilon, \epsilon/5, 4\epsilon/5\}$.

3. Stability with respect to reflection-breaking quartics $\lambda_{4,5}$

Still at $h_i = 0$, consider infinitesimal parity-breaking couplings $\lambda_{4,5}$ around the parity-symmetric fixed point $(\lambda_4^*, \lambda_5^*) = (0, 0)$. Linearizing the beta functions from Sec. V gives

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \lambda_4 \\ \lambda_5 \end{pmatrix} &= \mathbb{M}_{45} \begin{pmatrix} \lambda_4 \\ \lambda_5 \end{pmatrix}, \\ \mathbb{M}_{45} &= -\epsilon \mathbf{1} + \frac{3}{16\pi^2} \begin{pmatrix} J_{11}\lambda_1^* + 2J_{12}\lambda_3^* & J_{22}\lambda_3^* \\ J_{11}\lambda_3^* & J_{22}\lambda_2^* + 2J_{12}\lambda_3^* \end{pmatrix}. \end{aligned} \quad (\text{C20})$$

Using $g_1^* = g_2^* = g^*$ and $g_3^* \neq 0$ one finds that the diagonal entries coincide and simplify to $\epsilon/2$, while the off-diagonal product yields $\sqrt{(\mathbb{M}_{45})_{12}(\mathbb{M}_{45})_{21}} = 3ag_3^*$. Therefore, the two eigenvalues in the (λ_4, λ_5) sector are

$$\omega_{4,5}^{(\pm)} = \frac{\epsilon}{2} \pm 3a g_3^* = \frac{\epsilon}{2} \left(1 \pm \frac{4\rho - \sqrt{4\rho^2 - 3}}{4\rho^2 + 1} \right). \quad (\text{C21})$$

For $\rho \geq \sqrt{3}/2$ both $\omega_{4,5}^{(\pm)}$ are non-negative, and for $\rho > \sqrt{3}/2$ they are strictly positive. Hence, the generalized Wilson-Fisher fixed point S_- is also IR-stable with respect to turning on small $\lambda_{4,5}$ (at one-loop).

4. Stability of the fully coupled fixed line ($\lambda_i \neq 0$)

In this appendix we analyze the linear stability of the fully interacting sector in the $\mathbf{Z}_t = \mathbf{1}$, $\mathbf{Y} = 0$ limit, where all quartic couplings $\lambda_{1,\dots,5}$ are allowed to be nonzero. The one-loop beta functions read

$$\begin{aligned}
\beta_{\lambda_1} &= -\epsilon\lambda_1 + \frac{3}{16\pi^2}(J_{11}\lambda_1^2 + 2J_{12}\lambda_4^2 + J_{22}\lambda_3^2), \\
\beta_{\lambda_2} &= -\epsilon\lambda_2 + \frac{3}{16\pi^2}(J_{22}\lambda_2^2 + 2J_{12}\lambda_5^2 + J_{11}\lambda_3^2), \\
\beta_{\lambda_3} &= -\epsilon\lambda_3 + \frac{1}{16\pi^2}(J_{11}(\lambda_1\lambda_3 + 2\lambda_4^2) + 4J_{12}\lambda_3^2 + J_{22}(\lambda_2\lambda_3 + 2\lambda_5^2) + 2J_{12}\lambda_4\lambda_5), \\
\beta_{\lambda_4} &= -\epsilon\lambda_4 + \frac{3}{16\pi^2}(J_{11}\lambda_1\lambda_4 + 2J_{12}\lambda_3\lambda_4 + J_{22}\lambda_3\lambda_5), \\
\beta_{\lambda_5} &= -\epsilon\lambda_5 + \frac{3}{16\pi^2}(J_{22}\lambda_2\lambda_5 + 2J_{12}\lambda_3\lambda_5 + J_{11}\lambda_3\lambda_4).
\end{aligned} \tag{C22}$$

It is convenient to absorb the anisotropy weights by defining

$$\begin{aligned}
g_1 &\equiv J_{11}\lambda_1, & g_2 &\equiv J_{22}\lambda_2, & g_3 &\equiv \sqrt{J_{11}J_{22}}\lambda_3, \\
g_4 &\equiv \sqrt{J_{11}J_{12}}\lambda_4, & g_5 &\equiv \sqrt{J_{22}J_{12}}\lambda_5,
\end{aligned}$$

together with the single ratio

$$\rho \equiv \frac{J_{12}}{\sqrt{J_{11}J_{22}}}. \tag{C23}$$

In terms of g_i the beta functions depend on the anisotropy only through ρ :

$$\begin{aligned}
\beta_{g_1} &= -\epsilon g_1 + \frac{3}{16\pi^2}(g_1^2 + g_3^2 + 2g_4^2), \\
\beta_{g_2} &= -\epsilon g_2 + \frac{3}{16\pi^2}(g_2^2 + g_3^2 + 2g_5^2), \\
\beta_{g_3} &= -\epsilon g_3 + \frac{1}{16\pi^2}\left((g_1 + g_2)g_3 + 4\rho g_3^2 + \frac{2}{\rho}(g_4^2 + g_5^2) + 2g_4g_5\right), \\
\beta_{g_4} &= -\epsilon g_4 + \frac{3}{16\pi^2}(g_1g_4 + 2\rho g_3g_4 + g_3g_5), \\
\beta_{g_5} &= -\epsilon g_5 + \frac{3}{16\pi^2}(g_2g_5 + 2\rho g_3g_5 + g_3g_4).
\end{aligned} \tag{C24}$$

Introducing $c \equiv \frac{16\pi^2}{3}\epsilon$ and $u_i \equiv g_i/c$, the flow becomes $\beta_{u_i} = \epsilon\hat{\beta}_i(u)$ with

$$\begin{aligned}
\hat{\beta}_1 &= -u_1 + u_1^2 + u_3^2 + 2u_4^2, \\
\hat{\beta}_2 &= -u_2 + u_2^2 + u_3^2 + 2u_5^2, \\
\hat{\beta}_3 &= -u_3 + \frac{1}{3}\left((u_1 + u_2)u_3 + 4\rho u_3^2 + \frac{2}{\rho}(u_4^2 + u_5^2) + 2u_4u_5\right), \\
\hat{\beta}_4 &= -u_4 + u_1u_4 + 2\rho u_3u_4 + u_3u_5, \\
\hat{\beta}_5 &= -u_5 + u_2u_5 + 2\rho u_3u_5 + u_3u_4.
\end{aligned} \tag{C25}$$

Assuming $u_4, u_5 \neq 0$, define $r \equiv u_5/u_4$. The conditions $\hat{\beta}_4 = \hat{\beta}_5 = 0$ give

$$u_1 = 1 - u_3(2\rho + r), \quad u_2 = 1 - u_3\left(2\rho + \frac{1}{r}\right). \tag{C26}$$

Using $\hat{\beta}_1 = \hat{\beta}_2 = 0$ together with $u_5^2 = r^2 u_4^2$ yields

$$u_3 = \frac{1}{2\rho + r + \frac{1}{r}}. \quad (\text{C27})$$

Writing $D \equiv r^2 + 2\rho r + 1$, the fixed line can be parametrized as

$$\begin{aligned} u_1^* &= \frac{1}{D}, & u_2^* &= \frac{r^2}{D}, & u_3^* &= \frac{r}{D}, \\ u_4^* &= \sigma \frac{\sqrt{\rho r}}{D}, & u_5^* &= \sigma \frac{\sqrt{\rho r^3}}{D}, \end{aligned} \quad (\text{C28})$$

where

$$\sigma = \pm 1, \quad (\text{C29})$$

and where $\hat{\beta}_3 = 0$ enforces that u_4^* and u_5^* have the same sign (i.e., the common σ above).

Linearizing $u_i = u_i^* + \delta u_i$ gives

$$\frac{d}{d \ln \mu} \delta u_i = \epsilon \sum_j \Omega_{ij} \delta u_j, \quad \Omega_{ij} \equiv \left. \frac{\partial \hat{\beta}_i}{\partial u_j} \right|_{u=u^*}. \quad (\text{C30})$$

A direct differentiation of (C25) yields

$$\Omega = \begin{pmatrix} -1 + 2u_1 & 0 & 2u_3 & 4u_4 & 0 \\ 0 & -1 + 2u_2 & 2u_3 & 0 & 4u_5 \\ \frac{u_3}{3} & \frac{u_3}{3} & -1 + \frac{u_1 + u_2}{3} + \frac{8\rho u_3}{3} & \frac{1}{3} \left(\frac{4u_4}{\rho} + 2u_5 \right) & \frac{1}{3} \left(\frac{4u_5}{\rho} + 2u_4 \right) \\ u_4 & 0 & 2\rho u_4 + u_5 & -1 + u_1 + 2\rho u_3 & u_3 \\ 0 & u_5 & 2\rho u_5 + u_4 & u_3 & -1 + u_2 + 2\rho u_3 \end{pmatrix}_{u=u^*}. \quad (\text{C31})$$

Since $\hat{\beta}(u) = -u + Q(u)$ with Q homogeneous quadratic, Euler's theorem implies $\Omega u^* = u^*$, giving an eigenvalue $\hat{\omega} = 1$ (hence, $\omega = \epsilon$ in the original variables). Moreover, because the fixed points form a line $u^*(r)$, differentiating $\hat{\beta}(u^*(r)) = 0$ gives $\Omega \partial_r u^*(r) = 0$, i.e., a marginal eigenvalue $\hat{\omega} = 0$.

The remaining three eigenvalues can be extracted from trace invariants. Substituting (C28) into (C31) and evaluating $\text{tr } \Omega^n$ gives the universal results

$$\text{tr } \Omega = -\frac{5}{3}, \quad \text{tr } \Omega^2 = \frac{31}{9}, \quad \text{tr } \Omega^3 = -\frac{35}{27}, \quad (\text{C32})$$

independent of r , ρ , and σ . Writing the spectrum as $\{1, 0, \alpha, \beta, \gamma\}$, Newton identities yield

$$\alpha + \beta + \gamma = -\frac{8}{3}, \quad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{7}{3}, \quad \alpha\beta\gamma = -\frac{2}{3}, \quad (\text{C33})$$

so that

$$\hat{\omega}^3 + \frac{8}{3}\hat{\omega}^2 + \frac{7}{3}\hat{\omega} + \frac{2}{3} = 0 \Leftrightarrow (\hat{\omega} + 1)^2(3\hat{\omega} + 2) = 0. \quad (\text{C34})$$

Therefore, the full stability spectrum along the fixed line is

$$\hat{\omega} \in \left\{ 1, 0, -1, -1, -\frac{2}{3} \right\} \Rightarrow \omega \in \left\{ \epsilon, 0, -\epsilon, -\epsilon, -\frac{2}{3}\epsilon \right\}. \quad (\text{C35})$$

Hence, the fully coupled fixed line is a saddle line: one irrelevant direction ($+\epsilon$), one marginal direction (0), and three relevant directions ($-\epsilon, -\epsilon, -2\epsilon/3$). In particular, generic trajectories are repelled from the line, leading to runaway flow in the fully coupled sector.

5. Cubic and mass perturbations around the fully coupled fixed line

In this appendix we analyze the linearized RG flow of the cubic couplings h_i and the quadratic sector around the fully coupled fixed line found in the quartic sector (defined by $\mathbf{Z}_t = \mathbf{1}$ and $\mathbf{Y} = 0$). Throughout we work in $D = 4 - \epsilon$.

In the decoupled-gradient limit the one-loop quartic flow depends on the three positive angular averages J_{11}, J_{22}, J_{12} only through

$$\rho \equiv \frac{J_{12}}{\sqrt{J_{11}J_{22}}}. \quad (\text{C36})$$

We also use the rescaled quartic couplings

$$\begin{aligned} g_1 &= J_{11}\lambda_1, & g_2 &= J_{22}\lambda_2, & g_3 &= \sqrt{J_{11}J_{22}}\lambda_3, \\ g_4 &= \sqrt{J_{11}J_{12}}\lambda_4, & g_5 &= \sqrt{J_{22}J_{12}}\lambda_5, \end{aligned}$$

and the dimensionless variables

$$c \equiv \frac{16\pi^2}{3}\epsilon, \quad u_i \equiv \frac{g_i}{c}. \quad (\text{C37})$$

The fully coupled fixed line may be parametrized by $r > 0$ as

$$\begin{aligned} u_1^* &= \frac{1}{D}, & u_2^* &= \frac{r^2}{D}, & u_3^* &= \frac{r}{D}, \\ u_4^* &= \sigma \frac{\sqrt{\rho r}}{D}, & u_5^* &= \sigma \frac{\sqrt{\rho r^3}}{D}, \end{aligned} \quad (\text{C38})$$

where

$$D \equiv r^2 + 2\rho r + 1, \quad (\text{C39})$$

with $\sigma = \pm 1$ the sign choice $(u_4, u_5) \rightarrow -(u_4, u_5)$. In particular, the fixed-line variables obey the identities

$$u_2 = r^2 u_1, \quad u_3 = r u_1, \quad u_5 = r u_4, \quad (\text{C40a})$$

$$u_1 + u_2 + 2\rho u_3 = 1, \quad u_4^2 = \rho u_1 u_3. \quad (\text{C40b})$$

a. One-loop flow and matrix form

In the $\mathbf{Z}_l = \mathbf{1}$, $\mathbf{Y} = 0$ limit, the one-loop beta functions for the cubic couplings h_1, h_2, h_3, h_4 read

$$\beta_{h_1} = -\left(1 + \frac{\epsilon}{2}\right)h_1 + \frac{3}{16\pi^2}(J_{11}\lambda_1 h_1 + 2J_{12}\lambda_4 h_3 + J_{22}\lambda_3 h_4), \quad (\text{C41})$$

$$\beta_{h_2} = -\left(1 + \frac{\epsilon}{2}\right)h_2 + \frac{3}{16\pi^2}(J_{22}\lambda_2 h_2 + 2J_{12}\lambda_5 h_4 + J_{11}\lambda_3 h_3), \quad (\text{C42})$$

$$\beta_{h_3} = -\left(1 + \frac{\epsilon}{2}\right)h_3 + \frac{1}{16\pi^2}(J_{11}(\lambda_1 h_3 + 2\lambda_4 h_1) + 4J_{12}\lambda_3 h_3 + 2J_{12}\lambda_4 h_4 + J_{22}(\lambda_3 h_2 + 2\lambda_5 h_4)), \quad (\text{C43})$$

$$\beta_{h_4} = -\left(1 + \frac{\epsilon}{2}\right)h_4 + \frac{1}{16\pi^2}(J_{22}(\lambda_2 h_4 + 2\lambda_5 h_2) + 4J_{12}\lambda_3 h_4 + 2J_{12}\lambda_5 h_3 + J_{11}(\lambda_3 h_1 + 2\lambda_4 h_3)). \quad (\text{C44})$$

These equations are linear in the h_i and can be written in matrix form

$$\beta_{\mathbf{h}} = -\left(1 + \frac{\epsilon}{2}\right)\mathbf{h} + \frac{1}{16\pi^2}\mathbb{A}_h(\lambda, J)\mathbf{h}, \quad \mathbf{h} \equiv (h_1, h_2, h_3, h_4)^\top. \quad (\text{C45})$$

It is convenient to remove explicit factors of J_{11} and J_{22} that appear as square-root ratios in $\mathbb{A}_h$. Define the rescaled cubic variables

$$H_1 \equiv h_1, \quad H_2 \equiv \sqrt{\frac{J_{12}J_{22}}{J_{11}^2}}h_2, \quad H_3 \equiv \sqrt{\frac{J_{12}}{J_{11}}}h_3, \quad H_4 \equiv \sqrt{\frac{J_{22}}{J_{11}}}h_4, \quad (\text{C46})$$

so all together it reads

$$\mathbf{H} \equiv (H_1, H_2, H_3, H_4)^\top. \quad (\text{C47})$$

Since this is a μ -independent linear transformation, it only conjugates the linear flow matrix and does not affect its eigenvalues. In terms of the quartic variables $u_i = g_i/c$, the cubic flow becomes

$$\beta_{\mathbf{H}} = -\left(1 + \frac{\epsilon}{2}\right)\mathbf{H} + \frac{\epsilon}{3}\mathbb{M}_h(u, \rho)\mathbf{H}, \quad (\text{C48})$$

with

$$\mathbb{M}_h(u, \rho) = \begin{pmatrix} 3u_1 & 0 & 6u_4 & 3u_3 \\ 0 & 3u_2 & 3u_3 & 6\rho u_5 \\ 2u_4 & u_3 & u_1 + 4\rho u_3 & 2\rho u_4 + 2u_5 \\ u_3 & 2u_5/\rho & 2u_5 + 2u_4/\rho & u_2 + 4\rho u_3 \end{pmatrix}. \quad (\text{C49})$$

b. Evaluation on the fixed line and eigenvalues of $\mathbb{M}_h$

We now set $u = u^*(r)$ from (C38). The fixed-line identities (C40) imply that $\mathbb{M}_h(u^*, \rho)$ has two independent null vectors: hence, two zero eigenvalues. Concretely, using (C40) one checks

$$\mathbb{M}_h(u^*, \rho) \begin{pmatrix} -2\sqrt{\rho r} \\ -1/r \\ 1 \\ 0 \end{pmatrix} = 0, \quad \mathbb{M}_h(u^*, \rho) \begin{pmatrix} -r \\ -2\rho u_4^*/u_3^* \\ 0 \\ 1 \end{pmatrix} = 0, \quad (\text{C50})$$

so $\text{spec}(\mathbb{M}_h)$ contains $\{0, 0, \dots\}$.

To determine the remaining two eigenvalues it is enough to compute two invariants. First, the trace of (C49) is

$$\begin{aligned} \text{tr } \mathbb{M}_h &= 3u_1 + 3u_2 + (u_1 + 4\rho u_3) + (u_2 + 4\rho u_3) \\ &= 4(u_1 + u_2 + 2\rho u_3) = 4, \end{aligned} \quad (\text{C51})$$

where we used $u_1 + u_2 + 2\rho u_3 = 1$ on the fixed line. Second, we compute $\text{tr } \mathbb{M}_h^2 = \sum_{i,j} (\mathbb{M}_h)_{ij} (\mathbb{M}_h)_{ji}$. From (C49) one finds

$$\begin{aligned} \text{tr } \mathbb{M}_h^2 &= (3u_1)^2 + (3u_2)^2 + (u_1 + 4\rho u_3)^2 + (u_2 + 4\rho u_3)^2 \\ &\quad + (6u_4)(2u_4) + (3u_3)(u_3) + (3u_3)(u_3) \\ &\quad + (6\rho u_5)(2u_5/\rho) + (2\rho u_4 + 2u_5)(2u_5 + 2u_4/\rho). \end{aligned}$$

Now imposing the fixed-line relations (C40) (in particular, $u_2 = r^2 u_1$, $u_3 = r u_1$, $u_5 = r u_4$, and $u_4^2 = \rho u_1 u_3$) reduces (C52) to the constant

$$\text{tr } \mathbb{M}_h^2 = 10. \quad (\text{C52})$$

Let the two nonzero eigenvalues be α, β . From (C51) and (C52) we obtain

$$\begin{aligned} \alpha + \beta &= 4, \quad \alpha^2 + \beta^2 = 10 \\ \Rightarrow \alpha\beta &= \frac{(\alpha + \beta)^2 - (\alpha^2 + \beta^2)}{2} = \frac{16 - 10}{2} = 3, \end{aligned} \quad (\text{C53})$$

so α, β are the roots of $x^2 - 4x + 3 = 0$, i.e.,

$$\text{spec}(\mathbb{M}_h(u^*, \rho)) = \{0, 0, 1, 3\}. \quad (\text{C54})$$

Equivalently, the characteristic polynomial factorizes as

$$\det(\mathbb{M}_h - x\mathbf{1}) = x^2(x - 1)(x - 3). \quad (\text{C55})$$

Importantly, the spectrum (C54) is independent of the line parameter r and of ρ .

Linearizing (C48) around $\mathbf{H} = 0$ on the fixed line, an eigenmode of $\mathbb{M}_h$ with eigenvalue $x \in \{0, 0, 1, 3\}$ scales with RG exponent

$$\omega_h(x) = -\left(1 + \frac{\epsilon}{2}\right) + \frac{\epsilon}{3}x \quad (x \in \{0, 0, 1, 3\}). \quad (\text{C56})$$

Thus, the four cubic eigenvalues are

$$\omega_h = \left\{-1 - \frac{\epsilon}{2}, -1 - \frac{\epsilon}{2}, -1 - \frac{\epsilon}{6}, -1 + \frac{\epsilon}{2}\right\}. \quad (\text{C57})$$

All are negative for small $\epsilon > 0$, so any nonzero cubic couplings are relevant perturbations of the quartic fixed line in $4 - \epsilon$ dimensions. In particular, there is no perturbative fixed point with $h_i \neq 0$ controlled by the $4 - \epsilon$ expansion: the equation $\beta_{\mathbf{H}} = 0$ admits only $\mathbf{H} = 0$ at this order.

To discuss stability it is natural to use dimensionless mass parameters

$$m_{ij}^2 \equiv \frac{M_{ij}^2}{\mu^2}. \quad (\text{C58})$$

The one-loop RG equations in the $\mathbf{Z}_t = \mathbf{1}$, $\mathbf{Y} = 0$ limit can be written as

$$\begin{aligned} \beta_{m_{ij}^2} &= -2m_{ij}^2 - \frac{1}{16\pi^2} [(\text{linear terms in } m^2 \text{ proportional to } \lambda)] \\ &\quad + \frac{1}{16\pi^2} [(\text{terms quadratic in } h)]. \end{aligned} \quad (\text{C59})$$

For the stability matrix around the critical manifold $m^2 = 0$, $h = 0$ we may drop the inhomogeneous h^2 contribution and keep only the linear terms in m^2 .

As in the cubic sector, we perform a constant rescaling that removes square-root ratios of the J 's. Define the vector

$$\mathcal{M} \equiv \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \end{pmatrix} \equiv \begin{pmatrix} m_{11}^2 \\ \sqrt{\frac{J_{22}}{J_{11}}} m_{22}^2 \\ \sqrt{2} \sqrt{\frac{J_{12}}{J_{11}}} m_{12}^2 \end{pmatrix}. \quad (\text{C60})$$

Then the linearized mass flow takes the form

$$\begin{aligned} \beta_{\mathcal{M}} &= -2\mathcal{M} - \frac{\epsilon}{3} \mathbb{M}_m(u, \rho) \mathcal{M} \\ &\quad (\text{at linear order in } m^2 \text{ and with } h = 0), \end{aligned} \quad (\text{C61})$$

where the 3×3 matrix $\mathbb{M}_m$ is

$$\mathbb{M}_m(u, \rho) = \begin{pmatrix} u_1 & u_3 & \sqrt{2}u_4 \\ u_3 & u_2 & \sqrt{2}u_5 \\ \sqrt{2}u_4 & \sqrt{2}u_5 & 2\rho u_3 \end{pmatrix}. \quad (\text{C62})$$

Setting $u = u^*(r)$ and using (C40), we first exhibit two null vectors:

$$\mathbb{M}_m(u^*, \rho) \begin{pmatrix} -r \\ 1 \\ 0 \end{pmatrix} = 0, \quad \mathbb{M}_m(u^*, \rho) \begin{pmatrix} -\sqrt{2}\sqrt{\rho r} \\ 0 \\ 1 \end{pmatrix} = 0. \quad (\text{C63})$$

Hence, $\mathbb{M}_m$ has two zero eigenvalues. The remaining eigenvalue is fixed by the trace

$$\text{tr } \mathbb{M}_m = u_1 + u_2 + 2\rho u_3 = 1, \quad (\text{C64})$$

so

$$\text{spec}(\mathbb{M}_m(u^*, \rho)) = \{0, 0, 1\}, \quad \det(\mathbb{M}_m - x\mathbf{1}) = x^2(x-1). \quad (\text{C65})$$

Again, the spectrum is independent of r and ρ .

Combining (C61) and (C65), the three mass eigenvalues are

$$\omega_m(x) = -2 - \frac{\epsilon}{3}x, \quad x \in \{0, 0, 1\}, \quad (\text{C66})$$

i.e.,

$$\omega_m = \left\{ -2, -2, -2 - \frac{\epsilon}{3} \right\}. \quad (\text{C67})$$

Therefore, all quadratic deformations are relevant near the quartic fixed line, as expected for a critical manifold in $4 - \epsilon$ dimensions.

For completeness we note that the full one-loop mass beta functions contain inhomogeneous terms quadratic in the cubic couplings (schematically $\beta_{m^2} \supset +h^2$), so the subspace $m^2 = 0$ is not invariant if $h \neq 0$. Consequently, remaining on the critical surface of the quartic fixed line requires tuning both m^2 and h to zero.

APPENDIX D: COMPARISON WITH OTHER LORENTZ-VIOLATING FIELD THEORY EXAMPLES

In this appendix we compare our results for one-loop quartic flow with the results of Refs. [36,37].

The scalar sector of Ref. [36] starts from the $O(N)$ -symmetric theory

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu \phi_i)(\partial^\mu \phi_i) + \frac{1}{2} \sum_{i=1}^N K_{\mu\nu}^i \partial^\mu \phi_i \partial^\nu \phi_i - \frac{1}{2} \mu^2 \phi_i^2 \\ & - \frac{\hat{\lambda}}{4!} (\phi_i^2)^2 + \sum_i u_i^\beta \phi_i \partial_\beta \phi_i + \sum_{i,j} \phi_i^2 v_j^\beta \partial_\beta \phi_j, \end{aligned} \quad (\text{D1})$$

where we use the notation $\hat{\lambda}$ to distinguish this coupling from ours. The following $O(N)$ -preserving case was considered in [36]

$$K_{\mu\nu}^i = K_{\mu\nu}, \quad (\text{D2})$$

and the terms containing u_i^β and v_j^β are omitted because they lead to total derivatives. Thus, the nontrivial scalar

overlap with our work is the common-metric $O(N) \phi^4$ theory with a single tensor $K_{\mu\nu}$. In that scalar sector, Ref. [36] finds that the Lorentz-violating dependence of the pure-scalar renormalization group functions is entirely captured by the determinant factor

$$\Pi = \frac{1}{\sqrt{\det(\mathbf{1} + K)}}. \quad (\text{D3})$$

In particular, at one-loop the quartic beta function is

$$\beta(\hat{\lambda}) = -\epsilon \hat{\lambda} + \frac{N+8}{3(4\pi)^2} \Pi \hat{\lambda}^2 + O(\hat{\lambda}^3), \quad (\text{D4})$$

and, more generally, the scalar-sector flow is obtained from the Lorentz-invariant one by the replacement

$$\beta(\lambda) = -\epsilon \lambda + \sum_{n \geq 2} \beta_n^{(0)} \lambda^n [\det(\mathbf{1} + K)]^{-(n-1)/2}. \quad (\text{D5})$$

Thus, in the common-metric scalar theory, Lorentz violation enters only through the single Jacobian factor Π .

To compare with our theory, one must restrict to the case of no mixed time-space terms, a single quadratic form shared by both scalars, no cubic couplings, and an $O(2)$ -symmetric quartic interaction. In our notation, this means

$$\mathbf{Z}_t = \mathbf{1}, \quad \mathbf{Y} = 0, \quad \mathbf{C}_\phi = \mathbf{C}_\chi \equiv \mathbf{C}, \quad h_i = 0. \quad (\text{D6})$$

In the quartic sector, $O(2)$ symmetry requires

$$\lambda_1 = \lambda_2 \equiv \lambda, \quad \lambda_3 = \frac{\lambda}{3}, \quad \lambda_4 = \lambda_5 = 0. \quad (\text{D7})$$

Indeed,

$$\frac{\lambda}{4!} (\phi^2 + \chi^2)^2 = \frac{\lambda}{4!} \phi^4 + \frac{\lambda}{4!} \chi^4 + \frac{\lambda}{12} \phi^2 \chi^2, \quad (\text{D8})$$

which matches our convention

$$\mathcal{L}_4 = \mu^{4-D} \left(\frac{\lambda_1}{4!} \phi^4 + \frac{\lambda_2}{4!} \chi^4 + \frac{\lambda_3}{4} \phi^2 \chi^2 + \frac{\lambda_4}{6} \phi^3 \chi + \frac{\lambda_5}{6} \phi \chi^3 \right) \quad (\text{D9})$$

precisely when $\lambda_3 = \lambda/3$. In this common sector, our three angular weights collapse to a single factor,

$$J_{11} = J_{22} = J_{12} \equiv J. \quad (\text{D10})$$

If the common time-derivative coefficient is normalized to unity, then the remaining Lorentz-violating data are entirely contained in the common spatial matrix $\mathbf{C}$, and one has

$$J = (\det \mathbf{C})^{-1/2}, \quad (\text{D11})$$

which is the direct analog of the factor Π in Ref. [36]. In the purely spatial subsector $\mathbf{1} + K = \text{diag}(1, \mathbf{C})$, hence,

$$\Pi = [\det(\mathbf{1} + K)]^{-1/2} = [\det \mathbf{C}]^{-1/2} = J.$$

Substituting the $O(2)$ -symmetric identifications above into our one-loop quartic beta functions,

$$\beta_{\lambda_1} = -\epsilon\lambda_1 + \frac{3}{16\pi^2} (J_{11}\lambda_1^2 + J_{22}\lambda_3^2), \quad (\text{D12})$$

$$\beta_{\lambda_2} = -\epsilon\lambda_2 + \frac{3}{16\pi^2} (J_{22}\lambda_2^2 + J_{11}\lambda_3^2), \quad (\text{D13})$$

$$\beta_{\lambda_3} = -\epsilon\lambda_3 + \frac{1}{16\pi^2} (J_{11}\lambda_1\lambda_3 + 4J_{12}\lambda_3^2 + J_{22}\lambda_2\lambda_3), \quad (\text{D14})$$

one obtains

$$\beta_\lambda = -\epsilon\lambda + \frac{10}{3} \frac{J\lambda^2}{16\pi^2}, \quad (\text{D15})$$

which is obviously equal to

$$\beta_\lambda = -\epsilon\lambda + \frac{N+8}{3(4\pi)^2} J\lambda^2, \quad \text{for } N=2. \quad (\text{D16})$$

Therefore, in the actual overlap sector,

$$J \leftrightarrow \Pi, \quad (\text{D17})$$

and our one-loop quartic flow agrees exactly with the $N=2$ specialization of the scalar result of Ref. [36]. The genuine novelty of our analysis begins when one leaves this common-metric sector, namely when $\mathbf{C}_\phi \neq \mathbf{C}_\chi$, $\mathbf{Z}_t$ is a nontrivial field-space matrix, or $\mathbf{Y} \neq 0$. In that case the anisotropy dependence no longer collapses to a single determinant factor, but is resolved into channel-dependent weights.

Next, Ref. [37] studies a massless $O(N)$ scalar theory with a single Lorentz-violating tensor common to all field components. In the notations of [37], the bare Lagrangian is

$$\mathcal{L}_B = \frac{1}{2} \partial_\mu \phi_B \partial^\mu \phi_B + \frac{1}{2} K_{\mu\nu} \partial^\mu \phi_B \partial^\nu \phi_B + \frac{\lambda_B}{4!} \phi_B^4 + \frac{1}{2} t_B \phi_B^2, \quad (\text{D18})$$

where ϕ_B is an N -component field and $K_{\mu\nu}$ is symmetric and identical for all components. The renormalized dimensionless quartic coupling is denoted by u there.

The beta function derived in [37] takes the form

$$\beta(u) = -\epsilon u + \frac{N+8}{6} \Pi u^2 - \frac{3N+14}{12} \Pi^2 u^3 + \dots, \quad (\text{D19})$$

$$\Pi = \frac{1}{\sqrt{\det(\mathbf{1} + K)}}.$$

More generally, their all-order structure may be written schematically as

$$\beta(u) = -\epsilon u + \sum_{n \geq 2} \beta_n^{(0)} \Pi^{n-1} u^n, \quad (\text{D20})$$

so that the nontrivial fixed point satisfies

$$u_* = \frac{u_*^{(0)}}{\Pi}. \quad (\text{D21})$$

Accordingly, in the common-metric $O(N)$ theory, Lorentz violation rescales the location of the fixed point but does not change the universal critical exponents.

To compare the results of [37] with our theory, we again stick to the same overlap sector:

$$\mathbf{Z}_t = \mathbf{1}, \quad \mathbf{Y} = 0, \quad \mathbf{C}_\phi = \mathbf{C}_\chi \equiv \mathbf{C}, \quad h_i = 0. \quad (\text{D22})$$

In the quartic sector one imposes the same $O(2)$ -symmetric notations

$$\lambda_1 = \lambda_2 \equiv \lambda, \quad \lambda_3 = \frac{\lambda}{3}, \quad \lambda_4 = \lambda_5 = 0. \quad (\text{D23})$$

Then,

$$J_{11} = J_{22} = J_{12} \equiv J. \quad (\text{D24})$$

To match their dimensionless coupling u , one uses the convention

$$u = \frac{\lambda}{8\pi^2}. \quad (\text{D25})$$

In this overlap sector, the single determinant factor of Ref. [37] is identified with our common angular weight,

$$J \leftrightarrow \Pi. \quad (\text{D26})$$

From our one-loop reduction in the $O(2)$ -symmetric common-metric sector,

$$\beta_\lambda = -\epsilon\lambda + \frac{10}{3} \frac{J\lambda^2}{16\pi^2}, \quad (\text{D27})$$

and with (D25) one immediately finds

$$\beta_u = \frac{1}{8\pi^2}\beta_\lambda = -\epsilon u + \frac{10}{6}Ju^2. \quad (\text{D28})$$

Since $N = 2$ in the overlap sector, this is precisely

$$\beta_u = -\epsilon u + \frac{N+8}{6}\Pi u^2, \quad \text{for } N = 2, \quad (\text{D29})$$

upon identifying $J = \Pi$. Thus, our one-loop quartic flow confirms the one-loop result of Ref. [37] in the

common-metric $O(2)$ subsector. The difference is that Ref. [37] remains within the single- Π regime, whereas in our genuinely two-field anisotropic problem the Lorentz-violating dependence resolves into the three weights J_{11} , J_{22} , J_{12} (or six weights when $\mathbf{Y} \neq 0$). In that more general setting the flow is no longer reducible to the Lorentz-invariant one by a single universal Jacobian factor, and this is precisely the regime in which our fixed-point analysis is new.

-
- [1] S. Chadha and H. B. Nielsen, Lorentz invariance as a low energy phenomenon, *Nucl. Phys.* **B217**, 125 (1983).
 - [2] G. E. Volovik, *The Universe in a Helium Droplet*, International Series of Monographs on Physics Vol. 117 (Clarendon Press, Oxford, 2003).
 - [3] X. G. Wen, *Quantum Field Theory of Many-body Systems: From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, Oxford, 2004).
 - [4] R. C. Myers and M. Pospelov, Ultraviolet modifications of dispersion relations in effective field theory, *Phys. Rev. Lett.* **90**, 211601 (2003).
 - [5] J. Collins, A. Perez, D. Sudarsky, L. Urrutia, and H. Vucetich, Lorentz invariance and quantum gravity: An additional fine-tuning problem?, *Phys. Rev. Lett.* **93**, 191301 (2004).
 - [6] S. G. Nibbelink and M. Pospelov, Lorentz violation in supersymmetric field theories, *Phys. Rev. Lett.* **94**, 081601 (2005).
 - [7] V. A. Kostelecký and S. Samuel, Spontaneous breaking of Lorentz symmetry in string theory, *Phys. Rev. D* **39**, 683 (1989).
 - [8] D. Colladay and V. A. Kostelecký, CPT violation and the standard model, *Phys. Rev. D* **55**, 6760 (1997).
 - [9] D. Colladay and V. A. Kostelecký, Lorentz-violating extension of the standard model, *Phys. Rev. D* **58**, 116002 (1998).
 - [10] V. A. Kostelecký, Gravity, Lorentz violation, and the standard model, *Phys. Rev. D* **69**, 105009 (2004).
 - [11] V. A. Kostelecký and N. Russell, Data tables for Lorentz and CPT violation, *Rev. Mod. Phys.* **83**, 11 (2011).
 - [12] D. Mattingly, Modern tests of Lorentz invariance, *Living Rev. Relativity* **8**, 5 (2005).
 - [13] T. Jacobson, S. Liberati, and D. Mattingly, Lorentz violation at high energy: Concepts, phenomena and astrophysical constraints, *Ann. Phys. (Amsterdam)* **321**, 150 (2006).
 - [14] S. Liberati, Tests of Lorentz invariance: A 2013 update, *Classical Quantum Gravity* **30**, 133001 (2013).
 - [15] S. Kachru, X. Liu, and M. Mulligan, Gravity duals of Lifshitz-like fixed points, *Phys. Rev. D* **78**, 106005 (2008).
 - [16] E. M. Lifshitz, On the theory of second-order phase transitions, *Zh. Eksp. Teor. Fiz.* **11**, 255 (1941).
 - [17] P. Hořava, Quantum gravity at a Lifshitz point, *Phys. Rev. D* **79**, 084008 (2009).
 - [18] T. P. Sotiriou, M. Visser, and S. Weinfurter, Phenomenologically viable Lorentz-violating quantum gravity, *Phys. Rev. Lett.* **102**, 251601 (2009).
 - [19] D. Blas, O. Pujolàs, and S. Sibiryakov, Consistent extension of Hořava gravity, *Phys. Rev. Lett.* **104**, 181302 (2010).
 - [20] D. Blas, O. Pujolàs, and S. Sibiryakov, Models of non-relativistic quantum gravity: The good, the bad and the healthy, *J. High Energy Phys.* **04** (2011) 018.
 - [21] D. Anselmi, Weighted power counting and Lorentz violating gauge theories. I: General properties, *Ann. Phys. (Amsterdam)* **324**, 874 (2009).
 - [22] A. O. Barvinsky, D. Blas, M. Herrero-Valea, S. M. Sibiryakov, and C. F. Steinwachs, Renormalization of Hořava gravity, *Phys. Rev. D* **93**, 064022 (2016).
 - [23] A. O. Barvinsky, A. V. Kurov, and S. M. Sibiryakov, Beta functions of $(3+1)$ -dimensional projectable Hořava gravity, *Phys. Rev. D* **105**, 044009 (2022).
 - [24] A. O. Barvinsky, A. V. Kurov, and S. M. Sibiryakov, Asymptotic freedom in $(3+1)$ -dimensional projectable Hořava gravity: Connecting the ultraviolet and infrared domains, *Phys. Rev. D* **108**, L121503 (2023).
 - [25] J. Garriga and V. F. Mukhanov, Perturbations in k-inflation, *Phys. Lett. B* **458**, 219 (1999).
 - [26] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan, and L. Senatore, The effective field theory of inflation, *J. High Energy Phys.* **03** (2008) 014.
 - [27] S. Weinberg, Effective field theory for inflation, *Phys. Rev. D* **77**, 123541 (2008).
 - [28] S. R. Coleman and S. L. Glashow, High-energy tests of Lorentz invariance, *Phys. Rev. D* **59**, 116008 (1999).
 - [29] T. Jacobson and D. Mattingly, Gravity with a dynamical preferred frame, *Phys. Rev. D* **64**, 024028 (2001).
 - [30] R. Iengo, J. G. Russo, and M. Serone, Renormalization group in Lifshitz-type theories, *J. High Energy Phys.* **11** (2009) 020.
 - [31] M. M. Anber and J. F. Donoghue, The emergence of a universal limiting speed, *Phys. Rev. D* **83**, 105027 (2011).
 - [32] S. M. Carroll, G. B. Field, and R. Jackiw, Limits on a Lorentz and parity violating modification of electrodynamics, *Phys. Rev. D* **41**, 1231 (1990).
 - [33] G. Rubtsov, P. Satunin, and S. Sibiryakov, On calculation of cross sections in Lorentz violating theories, *Phys. Rev. D* **86**, 085012 (2012).

- [34] P. Satunin, One-loop correction to the photon velocity in Lorentz-violating QED, *Phys. Rev. D* **97**, 125016 (2018).
- [35] B. Altschul, Lorentz violation and the Yukawa potential, *Phys. Lett. B* **639**, 679 (2006).
- [36] A. Ferrero and B. Altschul, Renormalization of scalar and Yukawa field theories with Lorentz violation, *Phys. Rev. D* **84**, 065030 (2011).
- [37] P. R. S. Carvalho and M. I. Sena-Junior, Critical properties of scalar field theory with Lorentz violation: Exact treatment of Lorentz-violating mechanism, *Ann. Phys. (Amsterdam)* **387**, 290 (2017).
- [38] B. Altschul, Lorentz and CPT violation in scalar-mediated potentials, *Phys. Rev. D* **87**, 045012 (2013).
- [39] B. Altschul, Lorentz violation in fermion-antifermion decays of spinless particles, *Phys. Rev. D* **89**, 116007 (2014).
- [40] B. Altschul, L. C. T. Brito, J. C. C. Felipe, S. Karki, A. C. Lehum, and A. Y. Petrov, Three- and four-point functions in CPT-even Lorentz-violating scalar QED, *Phys. Rev. D* **107**, 045005 (2023).
- [41] V. A. Kostelecký, C. D. Lane, and A. G. M. Pickering, One-loop renormalization of Lorentz-violating electrodynamics, *Phys. Rev. D* **65**, 056006 (2002).
- [42] B. Altschul, L. C. T. Brito, J. C. C. Felipe, S. Karki, A. C. Lehum, and A. Y. Petrov, Perturbative aspects of CPT-even Lorentz-violating scalar chromodynamics, *Phys. Rev. D* **107**, 115002 (2023).
- [43] P. R. S. Carvalho, Three-loop field renormalization for scalar field theory with Lorentz violation, *Phys. Lett. B* **730**, 320 (2014).
- [44] W. De Carvalho Vieira and P. R. Silva De Carvalho, Renormalization group approach for Lorentz-violating scalar field theory at all loop orders, *Int. J. Geom. Methods Mod. Phys.* **13**, 1650049 (2016).
- [45] P. R. S. Carvalho and M. I. Sena-Junior, Exact Lorentz-violating all-loop ultraviolet divergences in scalar field theories, *Eur. Phys. J. C* **77**, 753 (2017).
- [46] B. Altschul, Q. G. Bailey, and V. A. Kostelecky, Lorentz violation with an antisymmetric tensor, *Phys. Rev. D* **81**, 065028 (2010).
- [47] T. Jacobson, Einstein-aether gravity: A status report, *Proc. Sci., QG-PH2007* (2007) 020 [arXiv:0801.1547].
- [48] I. Affleck, D. Bykov, and K. Wamer, Flag manifold sigma models: Spin chains and integrable theories, *Phys. Rep.* **953**, 1 (2022).
- [49] C. Armendariz-Picon, T. Damour, and V. F. Mukhanov, k—inflation, *Phys. Lett. B* **458**, 209 (1999).
- [50] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, G-inflation: Inflation driven by the Galileon field, *Phys. Rev. Lett.* **105**, 231302 (2010).
- [51] S. Mironov, V. Rubakov, and V. Volkova, Bounce beyond Horndeski with GR asymptotics and γ -crossing, *J. Cosmol. Astropart. Phys.* **10** (2018) 050.
- [52] Y. Ageeva, P. Petrov, and V. Rubakov, Generating cosmological perturbations in non-singular Horndeski cosmologies, *J. High Energy Phys.* **01** (2023) 026.
- [53] Y. A. Ageeva and P. K. Petrov, Unitarity relation and unitarity bounds for scalars with different sound speeds, *Phys. Usp.* **66**, 1134 (2023).
- [54] D. S. Ageev and Y. A. Ageeva, Disparity in sound speeds: Implications for unitarity and effective potential in quantum field theory, *arXiv:2505.07794*.
- [55] I. Y. Arefeva and I. V. Volovich, Anisotropic asymptotics and high-energy scattering, *arXiv:hep-th/9412155*.
- [56] V. Periwal, Anisotropy beta functions, *Phys. Rev. D* **52**, 7328 (1995).
- [57] P. R. S. Carvalho, Mass renormalization in Lorentz-violating scalar field theory, *Phys. Lett. B* **726**, 850 (2013).
- [58] J. Bijnens, J. Oredsson, and J. Rathsmann, Scalar kinetic mixing and the renormalization group, *Phys. Lett. B* **792**, 238 (2019).
- [59] C. G. Bollini and J. J. Giambiagi, Dimensional renormalization: The number of dimensions as a regularizing parameter, *Phys. Lett.* **40B**, 566 (1972).
- [60] G. 't Hooft and M. Veltman, Regularization and renormalization of gauge fields, *Nucl. Phys.* **B44**, 189 (1972).
- [61] G. 't Hooft, Dimensional regularization and the renormalization group, *Nucl. Phys.* **B61**, 455 (1973).
- [62] S. Kamefuchi, L. O'Raifeartaigh, and A. Salam, Change of variables and equivalence theorems in quantum field theories, *Nucl. Phys.* **28**, 529 (1961).
- [63] A. F. Andreev and E. P. Bashkin, Three-velocity hydrodynamics of superfluid solutions, *Zh. Eksp. Teor. Fiz.* **69**, 319 (1975).
- [64] D. J. Amit and V. Martin-Mayor, *Field Theory, the Renormalization Group, and Critical Phenomena*, 3rd ed. (World Scientific, Singapore, 2005).
- [65] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 4th ed. (Oxford University Press, Oxford, 2002).
- [66] K. G. Wilson, Renormalization group and critical phenomena. I. Renormalization group and the Kadanoff scaling picture, *Phys. Rev. B* **4**, 3174 (1971).
- [67] K. G. Wilson, Renormalization group and critical phenomena. II. Phase-space cell analysis of critical behavior, *Phys. Rev. B* **4**, 3184 (1971).
- [68] K. G. Wilson and M. E. Fisher, Critical exponents in 3.99 dimensions, *Phys. Rev. Lett.* **28**, 240 (1972).
- [69] A. Aharony, Critical behavior of anisotropic cubic systems, *Phys. Rev. B* **8**, 4270 (1973).
- [70] A. Pelissetto and E. Vicari, Critical phenomena and renormalization-group theory, *Phys. Rep.* **368**, 549 (2002).
- [71] P. Calabrese, A. Pelissetto, and E. Vicari, Multicritical behavior in $O(n_1) \oplus O(n_2)$ symmetric theories, *Phys. Rev. B* **67**, 054505 (2003).
- [72] <https://rscf.ru/project/24-72-00121/>.