

Is the shear induced spin polarization nondissipative?

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(Received 13 September 2025; accepted 1 May 2026; published 15 June 2026)

The shear-induced polarization plays a crucial role in understanding the local polarization of Λ and $\bar{\Lambda}$ hyperons. A key puzzle is whether the shear-induced polarization is nondissipative or not. In this work, we analyzed the shear-induced polarization and the anomalous Hall effects using the entropy flow and H theorem introduced from quantum (chiral) kinetic theory. While the shear-induced polarization and the anomalous Hall effect do not directly contribute to the entropy production rate, the perturbations associated with the shear tensor lead to an increase in entropy, similar to the role of the shear tensor in classical kinetic theory. We also examined these effects within the framework of linear response theory using Zubarev's approach. Our analysis based on Zubarev's approach suggests that these effects may indeed possess a dissipative nature. These findings highlight the limitations of the current theoretical framework in fully characterizing the dissipative properties of these phenomena.

DOI: [10.1103/3rd5-nwv2](https://doi.org/10.1103/3rd5-nwv2)

I. INTRODUCTION

Recently, the global polarization of Λ and $\bar{\Lambda}$ hyperons, first proposed in the pioneering works [1–3], has been observed in relativistic heavy-ion collisions [4–6]. By employing models based on statistical field theory and analyzing the experimental data, it was found that the quark-gluon plasma created in these collisions represents the most vortical fluid observed to date. The global polarization of hyperons is well described by various phenomenological models [7–23]. In most of these models, the global polarization of hyperons is induced by the thermal vorticity of the fluid, under the assumption of global equilibrium in the system. In addition to global polarization, experiments have also observed the local polarization of Λ and $\bar{\Lambda}$ hyperons. However, the experimental data for local polarization is inconsistent with the predictions of the aforementioned models. Understanding local

polarization requires the inclusion of additional hydrodynamic effects beyond global equilibrium, such as polarization induced by the shear viscous tensor, fluid acceleration, and gradients of the chemical potential over temperature [24–39]. By accounting for all these hydrodynamic effects, the local polarization observed in high-energy collisions has been successfully explained [39]. Very recently, it has been observed that the local spin polarization in $p + A$ collisions [40] cannot be explained by the combined effects of polarization induced by thermal vorticity and the shear viscous tensor [41]. On the other hand, the spin alignment of vector mesons has also been observed in relativistic heavy-ion collisions [42,43]. Notably, the spin alignment of ϕ mesons is significantly larger than expected. This large spin alignment of ϕ mesons has been investigated using several theoretical models [44–55]. One successful approach is based on the effective vector meson strong force field [45,46]. This model successfully describes the transverse momentum dependence of the spin alignment of ϕ mesons and has also accurately predicted its rapidity dependence [56]. Interestingly, alternative models attribute the spin alignment to effects induced by the shear viscous tensor [51,52], rather than relying on effective vector meson strong force fields. Such contributions from hydrodynamical field, however, were shown to be very small by some follow-up works [57–60]. For a more comprehensive

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discussion of these polarization effects, we refer readers to recent reviews [61–63] and the references therein.

Shear-induced polarization is key to understanding the local polarization of hyperons and may also play an important role in resolving the puzzles associated with the spin alignment of vector mesons. Therefore, it is essential to investigate this effect within a fundamental theoretical framework. Broadly speaking, the shear-induced polarization of hyperons corresponds to a spin current (more precisely, the axial-vector current of fermions in its canonical form) generated by the shear viscous tensor in phase space. It is a well-known fact that the shear viscous tensor, as the symmetric part of the energy-momentum tensor in relativistic hydrodynamics, is dissipative, e.g., see the early studies by Israel-Stewart [64] or the modern review [65] and the reference therein. It is straightforward to show that the shear viscous tensor contributes to an increase in entropy. Consequently, one might expect the spin current induced by the shear viscous tensor to also be dissipative. However, spin currents are often absent from the entropy production rate, which suggests that shear-induced polarization could potentially be nondissipative. This leads to an intriguing question that has garnered significant attention: Is shear-induced polarization dissipative or nondissipative?

In this work, we aim to address the above questions. Since the spin current induced by the shear viscous tensor exists in phase space, it is natural to employ quantum kinetic theory to investigate this problem. The main difficulty in this study lies in the lack of a strict definition of entropy density or entropy flow with quantum corrections. Although some progress has been made in Refs. [66–69], it remains unclear how to define entropy density with quantum corrections in the framework of kinetic theory. To avoid these ambiguities in the definition of entropy, we adopt the following strategy. First, we utilize the chiral kinetic theory for massless fermions instead of the quantum kinetic theory for massive fermions. For massive fermions, linking the axial current associated with the polarization pseudovector to the entropy production rate is challenging. Consequently, applying entropy analysis to shear-induced polarization in such cases becomes difficult. In chiral kinetic theory, the axial-vector current is conserved in the absence of electromagnetic fields, and directly contributes to the entropy production rate. This approach helps us clarify the underlying puzzles. Second, we restrict our analysis to entropy density up to the order of $\hbar$, as only the leading-log order of the collisional kernel is known so far [70–75]. Furthermore, the entropy flow at the second order in the gradient expansion (or $\sim \hbar^2$) in relativistic hydrodynamics also suffers from ambiguities [64]. Third, we extend the classical kinetic theory to introduce the H theorem within the quantum kinetic theory framework and verify its consistency with hydrodynamic approaches. Using this formalism, we then analyze the dissipative nature of shear-induced polarization.

The structure of this work is organized as follows. In Sec. II, we briefly review the H theorem and entropy currents in classical kinetic theory. In Sec. III, we extend these studies to the chiral kinetic theory. Following that, we apply the H theorem to the chiral kinetic theory using the relaxation-time approach. In Sec. IV, we compute the entropy production rate associated with the shear viscous tensor. In Sec. V, we discuss the shear induced polarization in the linear response theory in Zubarev’s approaches. We summarize this work and discuss the results in Sec. VI. Throughout this work, we adopt the metric $g_{\mu\nu} = \{+, -, -, -\}$ and the projector $\Theta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$, where u^μ denotes the fluid velocity. For an arbitrary tensor $A^{\mu\nu}$, we also introduce its traceless part, $A^{(\mu\nu)} = [\frac{1}{2}(\Theta^\mu_\alpha \Theta^\nu_\beta + \Theta^\mu_\beta \Theta^\nu_\alpha) - \frac{1}{3}\Theta^{\mu\nu}\Theta_{\alpha\beta}]A^{\alpha\beta}$.

II. H THEOREM AND ENTROPY CURRENT IN CLASSICAL KINETIC THEORY

In this section, we will briefly review the H theorem and entropy current in the classical kinetic theory. Let us start from the relativistic Boltzmann equation for fermions,

$$\frac{d}{dt}f_p \equiv \frac{p^\mu}{E_p} \left(\frac{\partial}{\partial x^\mu} - F_{\mu\nu} \frac{\partial}{\partial p_\nu} \right) f_p = C[f_p], \quad (1)$$

where $f_p = f_p(t, \mathbf{x}, \mathbf{p})$ is the distribution function for on-shell particles with $p^\mu = (E_p, \mathbf{p})$ and $E_p = \sqrt{\mathbf{p}^2 + m^2}$, $C[f]$ stands for the collision term and $F_{\mu\nu}$ denotes the electromagnetic strength tensor. We further consider the typical 2-2 scatterings for the collisions,

$$C[f_p] = \frac{1}{2E_p} \int d\Gamma_{kk' \rightarrow pp'} [f_k f_{k'} (1 - f_p)(1 - f_{p'}) - (1 - f_k)(1 - f_{k'}) f_p f_{p'}], \quad (2)$$

where $d\Gamma_{kk' \rightarrow pp'} = \frac{d^3 p'}{(2\pi)^3 (2E_{p'})} \frac{d^3 k}{(2\pi)^3 (2E_k)} \frac{d^3 k'}{(2\pi)^3 (2E_{k'})} (2\pi)^4 \delta(k + k' - p - p') |\mathcal{M}_{kk' \rightarrow pp'}|^2$ with $\mathcal{M}_{kk' \rightarrow pp'}$ being the amplitude for scatterings. The detailed balance condition gives, $|\mathcal{M}_{kk' \rightarrow pp'}|^2 = |\mathcal{M}_{pp' \rightarrow kk'}|^2$.

Next, we introduce the global and local equilibrium in the kinetic theory. It is straightforward to prove that the collision term vanishes when $f_p = f_p^{(0)}$, where $f_p^{(0)}$ is the Fermi-Dirac distribution at thermal equilibrium,

$$f_p^{(0)} = \frac{1}{e^{(u \cdot p - \mu)/T} + 1}, \quad (3)$$

with u^μ, μ, T being fluid velocity, chemical potential, and temperature. We emphasize that the distribution function that can make $C[f] = 0$ does not have to be the solution of the Boltzmann equation (1). Therefore, the distribution function for global and local equilibrium is commonly defined as follows.

- (i) Global equilibrium in kinetic theory: the distribution function that makes both the collision term $C[f]$ and the free streaming part df/dt vanish. By inserting $f_p^{(0)}$ into the Boltzmann equation (1) in the absence of $F^{\mu\nu}$, it gives the Killing condition at global equilibrium [76]

$$\partial_\mu \left(\frac{u_\nu}{T} \right) + \partial_\nu \left(\frac{u_\mu}{T} \right) = 0. \quad (4)$$

- (ii) Local equilibrium in kinetic theory: the distribution function that makes $C[f] = 0$ but the free streaming part $df/dt \neq 0$, while, $\partial_\mu \left(\frac{u_\nu}{T} \right) + \partial_\nu \left(\frac{u_\mu}{T} \right) \neq 0$. The conventional Boltzmann's H function is defined as

$$H[f_p] = \int \frac{d^3 p}{(2\pi)^3} \mathcal{H}[f_p], \quad (5)$$

where the kernel $\mathcal{H}$ for fermions is given by

$$\mathcal{H}[f] = -f \ln f - (1-f) \ln(1-f). \quad (6)$$

To provide the physical interpretation of H , we first introduce the energy-momentum tensor $T^{\mu\nu}$ and current j^μ by using Boltzmann equations,

$$\begin{aligned} T^{\mu\nu} &= \int \frac{d^3 p}{(2\pi)^3 E_p} p^\mu p^\nu f_p, \\ j^\mu &= \int \frac{d^3 p}{(2\pi)^3 E_p} p^\mu f_p. \end{aligned} \quad (7)$$

The energy density ε , pressure P and number density n are given by,

$$\varepsilon = u_\mu u_\nu T^{\mu\nu}, \quad P = -\frac{1}{3} \Theta_{\mu\nu} T^{\mu\nu}, \quad n = u_\mu j^\mu. \quad (8)$$

Inserting the $f_p = f_p^{(0)}$ into definition in Eq. (5), we then notice that the H is the entropy density at the thermal equilibrium. Analogous to the current, one can define the entropy current as

$$s^\mu = \int \frac{d^3 p}{(2\pi)^3 E_p} p^\mu \mathcal{H}[f_p], \quad (9)$$

whose time component is the entropy density $\mathcal{H}$. Taking the space-time divergence to s^μ in the kinetic theory, we can compute the covariant derivative of H ,

$$\begin{aligned} \partial_\mu s^\mu &= - \int \frac{d^3 p}{(2\pi)^3} \ln \left(\frac{f_p}{1-f_p} \right) C[f_p] + \int d^3 p \ln \left(\frac{f_p}{1-f_p} \right) \frac{p^\mu}{E_p} F_{\mu\nu} \partial_p^\nu f_p \\ &= - \int \frac{d^3 p}{(2\pi)^3} \ln \left(\frac{f_p}{1-f_p} \right) C[f_p] + 2 \int d^4 p \partial_p^\mu \left[\delta(p^2 - m^2) \theta(p^0) \ln \left(\frac{f_p}{1-f_p} \right) p^\mu F_{\mu\nu} f_p \right] \\ &= - \int \frac{d^3 p}{(2\pi)^3} \ln \left(\frac{f_p}{1-f_p} \right) C[f_p], \end{aligned} \quad (10)$$

where we have dropped the surface terms in the integration. Due to the symmetry, we find

$$\partial_\mu s^\mu = \frac{1}{4} \int \frac{d^3 p}{2E_p} d\Gamma_{kk' \rightarrow pp'} f_k f_{k'} (1-f_p)(1-f_{p'}) [(X-1) \ln X] \geq 0, \quad X > 0, \quad (11)$$

where

$$X = \frac{f_p f_{p'} (1-f_k)(1-f_{k'})}{(1-f_p)(1-f_{p'}) f_k f_{k'}}. \quad (12)$$

We notice that when we take the $f_p = f_p^{(0)}$ then it is straightforward to get $X \rightarrow 1$, and the $\partial_\mu s^\mu = 0$.

On the other hand, we can also obtain the entropy current satisfying the relativistic extended thermodynamic relation,

$$s^\mu = \frac{P}{T} u^\mu + \frac{1}{T} u_\nu T^{\mu\nu} - \frac{\mu}{T} j^\mu, \quad (13)$$

by employing $f_p = f_p^{(0)}$. Equation (13) is the generalization of the conventional thermodynamic relation, $s = (P + \varepsilon - \mu n)/T$, which has been widely discussed in many hydrodynamic studies [64]. In general, it is not guaranteed that Eq. (13) holds for an arbitrary distribution function f_p . To address this, we focus on systems that are very close to thermal equilibrium, i.e.,

$$f_p = f_p^{(0)} + \delta f_p, \quad (14)$$

where $\delta f_p \sim \mathcal{O}(\partial)$. Under these conditions, Eq. (13) is valid. In relativistic hydrodynamics, this relation holds up to $\mathcal{O}(\partial^2)$, which is consistent with results from kinetic

theory [64]. By using Eq. (14), we can insert the expression for the energy-momentum tensor and current into Eq. (10) and obtain,

$$\partial_\mu s^\mu = (T^{\mu\nu} - T_{(0)}^{\mu\nu}) \partial_\mu \left(\frac{u_\nu}{T} \right) - (j^\mu - j_{(0)}^\mu) \partial_\mu \left(\frac{\mu}{T} \right), \quad (15)$$

where $T_{(0)}^{\mu\nu}$ and $j_{(0)}^\mu$ are the energy-momentum tensor and current derived with $f_p = f_p^{(0)}$. The Eq. (15) is consistent with the entropy production rate up to $\mathcal{O}(\partial^2)$ in the relativistic hydrodynamics [64] and statistical field theory [77]. We emphasize that $dH/dt = 0$ at both global and local equilibrium in the classical kinetic theory, i.e., the entropy is at its maximum value.

At last, we consider the shear viscous tensor as an example to show how to implement the H theorem in the classical kinetic theory. If the system is close to the equilibrium, we can consider the perturbation around the distribution function at the thermal equilibrium as in Eq. (14), where one usually assumes that

$$\delta f_p = f_p^{(0)} (1 - f_p^{(0)}) \chi_p, \quad (16)$$

for convenience. Since f_p must obey the Boltzmann equation (1), the χ_p can be expressed as the moment coupled to dissipative effects. Here, we consider the shear viscous tensor as an example. Inserting Eq. (16) into Eq. (7), we can obtain the shear viscous tensor in terms of integration of δf_p over momentum,

$$\begin{aligned} \pi^{\mu\nu} &= T^{\langle\mu\nu\rangle} = \int \frac{d^3 p}{(2\pi)^3 E_p} p^{\langle\mu} p^{\nu\rangle} f_p^{(0)} (1 - f_p^{(0)}) \chi_p \\ &\equiv 2\eta \partial^{\langle\mu} u^{\nu\rangle}, \end{aligned} \quad (17)$$

where η is shear viscosity. By mapping $\frac{d}{dt} f_p^{(0)} = C[\delta f_p]$, we obtain the terms related to the shear viscous tensor,

$$\chi_p = A(\mathbf{p}) p^i p^j \partial_{\langle i} u_{j\rangle} = A(\mathbf{p}) p^i p^j \frac{\pi_{ij}}{2\eta}, \quad (18)$$

where $A(\mathbf{p})$ needs to be solved through the following constraint equation,

$$\begin{aligned} &f_p^{(0)} (1 - f_p^{(0)}) p^i p^j \\ &= \frac{T}{2E_p} \int d\Gamma_{kk' \rightarrow pp'} f_k^{(0)} f_{k'}^{(0)} f_p^{(0)} f_{p'}^{(0)} [B^{ij}(\mathbf{p}) + B^{ij}(\mathbf{p}') \\ &\quad - B^{ij}(\mathbf{k}) - B^{ij}(\mathbf{k}')], \end{aligned} \quad (19)$$

with $B^{ij}(\mathbf{k}) = A(\mathbf{k}) k^i k^j$. We can compute the correction to entropy flow,

$$\begin{aligned} s_{(1)}^\mu &= \frac{\pi_{ij}}{2\eta} \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu}{E_p} \ln \left(\frac{1 - f_p^{(0)}}{f_p^{(0)}} \right) \\ &\quad \times A(\mathbf{p}) f_p^{(0)} (1 - f_p^{(0)}) p^i p^j + \mathcal{O}(\chi^2), \end{aligned} \quad (20)$$

and entropy production rate in this case,

$$\begin{aligned} \partial_\mu s_{(1)}^\mu &= \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu}{E_p} \delta f_p \partial_\mu \ln \left(\frac{1 - f_p^{(0)}}{f_p^{(0)}} \right) = \frac{\partial_\mu u_\nu}{T} T^{\mu\nu}_{(1)} \\ &= \frac{\pi^{\mu\nu} \pi_{\mu\nu}}{2\eta T} \geq 0, \end{aligned} \quad (21)$$

which shows that shear viscous tensor is dissipative. We refer to Refs. [78–81] and the recent review [65] and references therein.

Before concluding this section, we summarize the procedure for constructing the entropy flow in classical kinetic theory. To define the entropy flow s^μ , one replaces the distribution function f_p in the current with the $\mathcal{H}$ function. Next, it is necessary to verify the positivity of the entropy production rate, $\partial_\mu s^\mu$, as defined in kinetic theory. Additionally, one must confirm that the entropy flow, defined through the $\mathcal{H}$ function, satisfies the macroscopic hydrodynamic equation (13) up to $\mathcal{O}(\partial)$. Finally, we note that $\mathcal{H}$, when constructed from an arbitrary distribution function, does not necessarily represent the entropy density satisfying the thermodynamic relations in Eqs. (13), (15). As a result, we will implement the H theorem when the system is close to equilibrium.

III. H THEOREM AND ENTROPY CURRENT IN CHIRAL KINETIC EQUATION

In this section, we extend our study to quantum kinetic theory. As previously mentioned, the axial-vector current for massless fermions directly corresponds to the spin current without any ambiguities. For simplicity, we will focus on the chiral kinetic theory for massless fermions. More details can be found in the recent review [62].

A. Review on the chiral kinetic theory

We introduce the gauge invariant Wigner function defined as the Wigner transformation of two point functions for right-handed fermions, [82],

$$\dot{S}^<(p, X) \equiv \int d^4 Y e^{\frac{ip \cdot Y}{\hbar}} S^<(x, y), \quad (22)$$

where $S_{ab}^<(x, y) = U(y, x) \langle \chi_b^\dagger(y) \chi_a(x) \rangle$ with χ being the Pauli spinor, and $U(y, x)$ is the gauge link to ensure the gauge invariance. Here we define $X = (x + y)/2$ and $Y = x - y$. The perturbative solution up to $\mathcal{O}(\hbar\partial)$ for Wigner function of right-handed fermions is given by [62,82],

$$\begin{aligned} \dot{S}_R^{<\mu} = 2\pi\bar{e}_{(n)} \left[\delta(p^2)(p^\mu + \hbar S_{(n)}^{\mu\nu} \mathcal{D}_\nu) \right. \\ \left. + \hbar \epsilon^{\mu\nu\alpha\beta} p_\nu F_{\alpha\beta} \frac{\partial \delta(p^2)}{2\partial p^2} \right] f_p^{(n)}, \end{aligned} \quad (23)$$

where

$$\mathcal{D}_\nu f_p^{(n)} = \Delta_\nu f_p^{(n)} - C_\nu, \quad \Delta_\mu = \partial_\mu + F_{\nu\mu} \partial_\nu^\mu, \quad (24)$$

with C_ν denoting the collisional corrections, $\bar{e}_{(n)}$ is the sign of $p \cdot n$ with n^μ being the frame vector. The general expression for C_ν can be found in Ref. [82] and also see Ref. [73] for the $U(1)$ gauge interactions under the hard thermal loop approximation. Here, the spin tensor is given by

$$S_{(n)}^{\mu\nu} = \frac{\epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta}{2(p \cdot n)}. \quad (25)$$

Different from Eq. (3), the global equilibrium distribution function for chiral kinetic theory contains the quantum corrections [24,66],

$$f_p^{eq} = \frac{1}{e^g + 1}, \quad g = \beta p \cdot u - \beta \mu_R + \frac{\hbar S_{(u)}^{\mu\nu}}{2} \partial_\mu (\beta u_\nu), \quad (26)$$

where $\beta \equiv 1/T$ and μ_R denotes the chemical potential for right-handed fermions. In global equilibrium, the distribution function is frame-independent, and we typically set $n^\mu = u^\mu$. Similarly, in local equilibrium, we also set $n^\mu = u^\mu$ for simplicity.

The energy momentum tensor and current are defined as follows,

$$\begin{aligned} T^{\mu\nu} &= \int \frac{d^4 p}{(2\pi)^4} (p^\mu \dot{S}^{<\nu} + p^\nu \dot{S}^{<\mu}) \\ &= \epsilon u^\mu u^\nu - P \Theta^{\mu\nu} + \Pi_{non}^{\mu\nu} + \Pi_{dis}^{\mu\nu}, \\ J^\mu &= 2 \int \frac{d^4 p}{(2\pi)^4} \dot{S}^{<\mu} = n u^\mu + \nu_{non}^\mu + \nu_{dis}^\mu, \end{aligned} \quad (27)$$

where the nondissipative effects from first-order anomalous transport are related to the chiral transports,

$$\begin{aligned} \Pi_{non}^{\mu\nu} &= \hbar \xi_\omega (\omega^\mu u^\nu + \omega^\nu u^\mu) + \hbar \xi_B (B^\mu u^\nu + B^\nu u^\mu), \\ \nu_{non}^\mu &= \hbar \sigma_B B^\mu + \hbar \sigma_\omega \omega^\mu. \end{aligned} \quad (28)$$

Here, the transport coefficients $\xi_\omega, \xi_B, \sigma, \sigma_B$ can be computed systematically under quantum kinetic theory, e.g., see Refs. [24,83] as an example. The B^μ is the magnetic field defined as $B^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$. The $\Pi_{dis}^{\mu\nu}$ and ν_{dis}^μ stand for the dissipative terms derived by inserting the perturbations near the thermal equilibrium $\delta f_p^{(u)} = f_p^{(u)} - f_p^{eq}$ into the

general expression for $T^{\mu\nu}$ and J^μ . By using the distribution function in the thermal equilibrium, one can also obtain the conservation equations with chiral anomaly,

$$\partial_\mu J^\mu = -\frac{\hbar}{4\pi^2} E_\mu B^\mu, \quad \partial_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu. \quad (29)$$

By using the solution of Wigner function in Eq. (23), one can further derive the following kinetic equation for $f_p^{(n)}$ [82],

$$\begin{aligned} \delta \left(p^2 - \hbar \frac{B \cdot p}{p \cdot u} \right) \left[p \cdot \Delta + \hbar \frac{S_{(u)}^{\mu\nu} E_\mu}{p \cdot u} \Delta_\nu + \hbar S_{(u)}^{\mu\nu} (\partial_\mu F_{\rho\nu}) \partial_p^\rho \right. \\ \left. + \hbar (\partial_\mu S_{(u)}^{\mu\nu}) \Delta_\nu \right] f_p^{(u)} \\ = \delta \left(p^2 - \hbar \frac{B \cdot p}{p \cdot u} \right) \bar{C}, \end{aligned} \quad (30)$$

where $E^\mu = F^{\mu\nu} u_\nu$ and

$$\bar{C} = \left(p^\mu + \hbar \frac{S_{(u)}^{\nu\mu} E_\nu}{p \cdot u} + \hbar \partial_\rho S_{(u)}^{\rho\mu} \right) \tilde{C}_\mu, \quad (31)$$

is the collision term with $\tilde{C}_\mu$ being a vector related to the self-energies of the theory. We refer to Refs. [70,73] for the systematical discussion on collision terms. Equation (30) is considered as the classical Boltzmann equation (1) with quantum corrections. From now on, we will use Eq. (30) as our starting point to investigate the entropy flow instead of the Wigner function.

B. Entropy flow in chiral kinetic theory

Following the same strategy for the classical kinetic theory, we introduce the entropy flow in the chiral kinetic theory by replacing $f_p^{(u)}$ with the Boltzmann's $\mathcal{H}$ function,

$$\begin{aligned} s_R^\mu &= 2 \int \frac{d^4 p}{(2\pi)^3} \bar{e}_{(u)} \left[\delta(p^2)(p^\mu + \hbar S_{(u)}^{\mu\nu} \mathcal{D}_\nu) \right. \\ &\quad \left. + \hbar \epsilon^{\mu\nu\alpha\beta} p_\nu F_{\alpha\beta} \frac{\partial \delta(p^2)}{2\partial p^2} \right] \mathcal{H}(f_p^{(u)}). \end{aligned} \quad (32)$$

We notice that the above equation has already been assumed in the early studies for the Wigner function done by some of us [67]. Here, for simplicity, we consider the entropy flow for the right-handed fermions only.

Next, let us examine whether the entropy flow defined in Eq. (32) is consistent with the one defined in relativistic hydrodynamics. We can compute the entropy flow at local equilibrium. Inserting f_p^{eq} in Eq. (26) with $n^\mu = u^\mu$ into Eq. (32), yields,

$$s_{R,leq}^\mu = \beta (u^\mu P + T_{leq}^{\mu\nu} u_\nu - \mu_R J_{leq}^\mu + \xi_B B^\mu + 2\xi_\omega \omega^\mu), \quad (33)$$

which agrees with the entropy flow in the chiral anomalous hydrodynamics [84,85]. The terms $\xi_B B^\mu + 2\xi_\omega \omega^\mu$ are quantum corrections to the classical entropy flow in the presence of electromagnetic fields. Following the Refs. [67,84,85], one can obtain that

$$\partial_\mu s_{R,leq}^\mu = 0, \quad (34)$$

as expected. We emphasize that to obtain the above result, we have assumed that the $\mathcal{C}_\nu[f_p^{eq}] = 0$ in Eq. (32).

At last, let us check the positivity of the entropy production rate. For simplicity, we will neglect the background electromagnetic field. Similar to Eq. (16), we consider a perturbation in the distribution function near the local equilibrium,

$$\begin{aligned} f_p^{(u)} &= f_p^{eq} + \delta f_p, \\ \delta f_p &= f_p^{eq}(1 - f_p^{eq})\chi_p, \quad |\chi_p| \ll 1. \end{aligned} \quad (35)$$

In the absence of electromagnetic fields, the Eq. (30) reduces to

$$[p \cdot \partial + \hbar(\partial_\mu S_{(u)}^{\mu\nu})\partial_\nu]f_p^{(u)} = \bar{\mathcal{C}}[f_p^{(u)}]. \quad (36)$$

The collision term $\bar{\mathcal{C}}$ can be computed for a given microscopic theory, e.g., see Refs. [70,73] for the one given by the quantum electrodynamics in the hard thermal loop approximation. Here, we focus on the general properties of the entropy flow rather than its specifics within a particular theory. To simplify the collision term, we adopt the relaxation-time approximation. In general, the collision term can be written as

$$\bar{\mathcal{C}}[f_p^{(u)}] = -\frac{1}{\tau_R}(p \cdot u + \hbar p \cdot \mathcal{R})\delta f_p, \quad (37)$$

where τ_R is the relaxation time. $\mathcal{R}^\mu$ is an arbitrary spacelike vector of $\mathcal{O}(\partial^0)$ and, therefore, can be parametrized as

$$\mathcal{R}^\mu = c_1 u^\mu + \frac{1}{u \cdot p} c_2 \bar{p}^\mu + c_3 \bar{R}^\mu, \quad (38)$$

where $\bar{p}^\mu = p_\nu \Theta^{\mu\nu}$ and $\bar{R}^\mu = (\Theta^{\mu\nu} - \bar{p}^\mu \bar{p}^\nu / \bar{p}^2)R_\nu$. Here, $c_{i=1,2,3}$ are dimensionless coefficients at $\mathcal{O}(\partial^0)$. We emphasize that the relaxation time is expanded in powers of $\hbar$ in Eq. (37). The power counting leads to the condition $1 + \hbar c_1 > 0$.

In the absence of the external field, the entropy current in Eq. (32) reduces to

$$\begin{aligned} s_R^\mu &= 2 \int \frac{d^4 p}{(2\pi)^3} \bar{e}_{(u)} \delta(p^2) \{ (p^\mu + \hbar S_{(u)}^{\mu\nu} \partial_\nu) \mathcal{H}[f_p^{(u)}] \\ &\quad - \hbar S_{(u)}^{\mu\nu} \mathcal{C}_\nu[\mathcal{H}[f_p^{(u)}]] \}. \end{aligned} \quad (39)$$

We can further assume that $\mathcal{C}_\nu[\mathcal{H}[f_p]]$ is linear in f_p and consider that s^μ up to $\mathcal{O}(\partial^1)$. Under these two assumptions, we find that the last terms proportional to $\mathcal{C}_\nu$ will not contribute to the entropy flow. More details can be found in Appendix A.

Next, let us consider the entropy production rate. Before doing so, we emphasize that the collisional terms are subject to additional constraints imposed by the macroscopic hydrodynamic equations, $\partial_\mu J^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$, in the current approach. In classical kinetic theory, these hydrodynamic conservation equations can be automatically derived by applying the detailed balance conditions in the collisional kernel, which arise from time-reversal symmetry in microscopic theories [see the discussion around Eq. (2)]. However, in quantum kinetic theory, the time-reversal symmetry in the collisional term, including quantum corrections, may not be as straightforward. On the other hand, in the relaxation time approximation, the hydrodynamic conservation equations become additional constraints, even in classical kinetic theory. Therefore, in this work, we impose the requirement that the collisional term in chiral kinetic theory satisfies the conditions dictated by the hydrodynamic conservation equations,

$$\begin{aligned} 0 &= \partial_\mu J^\mu = 2 \int \frac{d^4 p}{(2\pi)^3} \bar{e}_{(u)} \delta(p^2) \bar{\mathcal{C}}[f_p^{(u)}], \\ 0 &= u_\nu \partial_\mu T^{\mu\nu} = \int \frac{d^4 p}{(2\pi)^3} \bar{e}_{(u)} \delta(p^2) \{ 2(p \cdot u) \bar{\mathcal{C}}[f_p^{(u)}] \\ &\quad - \hbar[(p \cdot u)(\partial_\mu S_{(u)}^{\mu\alpha})\partial_\alpha + p^\mu (\partial_\mu u_\nu) S_{(u)}^{\nu\alpha} \partial_\alpha] f_p^{(u)} \}. \end{aligned} \quad (40)$$

By using the above conditions, we can compute the entropy production rate,

$$\begin{aligned} \partial_\mu s_R^\mu &= \frac{1}{\tau_R} \int \frac{d^3 p}{(2\pi)^3} \{ [(1 + \hbar c_1) f_p^{eq}(1 - f_p^{eq}) \chi_p^2] |_{p_0=|p|} \\ &\quad + [(1 + \hbar c_1) f_p^{eq}(1 - f_p^{eq}) \chi_p^2] |_{p_0=-|p|} \} \geq 0, \end{aligned} \quad (41)$$

where c_1 is introduced in Eq. (38). Following the discussion below Eq. (38), the condition $1 + \hbar c_1 > 0$ ensures that the second law of thermodynamics is automatically satisfied. More details for deriving the above equations can be found in Appendix A.

IV. APPLICATION TO SPIN TRANSPORTS

In this section, we will implement the entropy flow and entropy production rate introduced in the previous section to analyze the shear induced polarization and anomalous Hall effects induced by shear tensor.

For the massless fermions, the spin vector in the phase space corresponds to the axial-vector current in the Wigner function. Since we focus on the right handed fermions, the axial-vector current reduces to the S_R^μ in phase space. Inserting the distribution function in the local equilibrium

in Eq. (26) into Eq. (23), we can obtain the spin vector induced by various hydrodynamic effects in the absence of electromagnetic fields [24,30,36],

$$S_{R,leq}^{<\mu} = S_{\text{thermal}}^{\mu} + S_{\text{shear}}^{\mu} + S_{\text{accT}}^{\mu} + S_{\text{chemical}}^{\mu}, \quad (42)$$

where the subscripts, “thermal,” “shear,” “accT,” and “chemical” stand for the terms related to the thermal vorticity, shear viscous tensor, fluid acceleration, and gradient of $\beta\mu$, respectively. Their expressions are as follows,

$$\begin{aligned} S_{\text{thermal}}^{\mu} &= a \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} p_{\nu} \partial_{\alpha} \frac{u_{\beta}}{T}, \\ S_{\text{shear}}^{\mu} &= -a \frac{1}{(u \cdot p) T} \epsilon^{\mu\nu\alpha\beta} p_{\alpha} u_{\beta} p^{\sigma} \partial_{(\sigma} u_{\nu)}, \\ S_{\text{accT}}^{\mu} &= -a \frac{1}{2T} \epsilon^{\mu\nu\alpha\beta} p_{\nu} u_{\alpha} \left[(u \cdot \partial) u_{\beta} - \frac{1}{T} \partial_{\beta} T \right], \\ S_{\text{chemical}}^{\mu} &= a \frac{1}{(u \cdot p)} \epsilon^{\mu\nu\alpha\beta} p_{\alpha} u_{\beta} \partial_{\nu} \frac{\mu}{T}, \end{aligned} \quad (43)$$

where $a = 4\pi\hbar\bar{e}_{(u)}\delta(p^2)f_p^{eq}(1 - f_p^{eq})$. Inserting Eq. (42) into the modified Cooper-Frye formulas [9,12], one can derive the spin polarization induced by the thermal vorticity, shear tensor, fluid acceleration, and the gradient of $\beta\mu$.

The interaction corrections to the spin vector in phase space have been systematically computed in Refs. [60,72–75,86–92]. Here, we focus on the interaction corrections related to the shear viscous tensor. As pointed out in Ref. [73], as well as in Refs. [74,89], one type of interaction correction associated with the shear viscous tensor originates from interactions but does not depend on the coupling constant. Similar terms have been widely discussed in condensed matter physics and are related to the anomalous and spin Hall effect [93–95]. To avoid complexity, we reproduce this term using the relaxation-time approximation for simplicity.

Inserting the perturbation of distribution function around the local equilibrium in Eq. (16) into the chiral kinetic equation (36) with relaxation-time approaches, we can derive the expression for the δf_p ,

$$\delta f_p = \frac{\tau_R}{u \cdot p} f_p^{(0)} (1 - f_p^{(0)}) \beta p^{\mu} p^{\nu} \partial_{(\mu} u_{\nu)} + \dots, \quad (44)$$

where ... stands for other terms irrelevant to shear tensor $\partial_{(\mu} u_{\nu)}$. Inserting δf_p into Eq. (23), we obtain the interaction corrections to $S_R^{<\mu}$,

$$\delta S_R^{<\mu} = 2\pi p^{\mu} \delta(p^2) \delta f_p + 2\pi\hbar \delta(p^2) S_{(u)}^{\mu\alpha} (\partial_{\alpha} \delta f_p - C_{\alpha}[\delta f_p]). \quad (45)$$

Analogous to the discussion in Appendix A, we parametrize the $C_{\nu}[\delta f_p]$ as

$$C_{\nu}[\delta f_p] = \frac{1}{\tau_R} (d_1 u_{\nu} + d_2 p_{\nu} + d_3 \partial_{\nu}^p) \delta f_p, \quad (46)$$

where $d_{1,2,3}$ and the other relaxation time τ'_R are at $\mathcal{O}(\partial^0)$. Under the above parametrization, we obtain,

$$\begin{aligned} \delta S_R^{<\mu} &= 2\pi p^{\mu} \delta(p^2) \frac{\tau_R}{u \cdot p} f_p^{(0)} (1 - f_p^{(0)}) \beta p^{\rho} p^{\sigma} \partial_{(\rho} u_{\sigma)} \\ &\quad - 4\pi\hbar \frac{1}{(u \cdot p)} \frac{\tau_R}{\tau'_R} \\ &\quad \times \delta(p^2) S_{(u)}^{\mu\nu} f_p^{(0)} (1 - f_p^{(0)}) d_3 \beta p^{\sigma} \partial_{(\nu} u_{\sigma)}. \end{aligned} \quad (47)$$

In general, τ_R and τ'_R can be different. However, in certain special cases, τ_R and τ'_R exhibit the same dependence on the coupling constant. As a result, the term proportional to τ_R/τ'_R does not explicitly depend on the coupling constant. For example, see the studies of quantum kinetic theory within the Chapman-Enskog expansion [73] and works in Refs. [74,89,92,96] and references therein. Such a term can be interpreted as the anomalous Hall effect by simply replacing the shear tensor with the electric force (see also related discussions in condensed matter physics [93–95,97]).

Now, we are ready to compute the entropy production rate related to the shear induced polarization and anomalous Hall effects. By using the δf_p in Eq. (44), the entropy flow related to the shear tensor, namely δs_R^{μ} , can be cast into a simple form,

$$\delta s_R^{\mu} = \beta u_{\nu} \delta T^{\mu\nu} - \beta \mu_R \delta J^{\mu}, \quad (48)$$

where

$$\begin{aligned} \delta T^{\mu\nu} &= \int \frac{d^4 p}{(2\pi)^4} (p^{\nu} \delta S_R^{<\mu} + p^{\mu} \delta S_R^{<\nu}) \\ &= \frac{4}{15} \tau_R \beta \partial^{\mu} u^{\nu} \int \frac{d^4 p}{(2\pi)^3} \delta(p^2) (p \cdot u)^3 f_p^{(0)} (1 - f_p^{(0)}), \end{aligned} \quad (49)$$

and

$$\begin{aligned} \delta J^{\mu} &= 2 \int \frac{d^4 p}{(2\pi)^4} \delta S_R^{<\mu} = 2\tau_R \beta \partial_{(\rho} u_{\sigma)} \\ &\quad \times \int \frac{d^4 p}{(2\pi)^3 (p \cdot u)} \delta(p^2) f_p^{(0)} (1 - f_p^{(0)}) p^{\mu} p^{\rho} p^{\sigma} \\ &= 0. \end{aligned} \quad (50)$$

We can further compute the entropy production rate,

$$\begin{aligned}
\partial_\mu s_R^\mu &= \delta T^{\mu\nu} \partial_\mu (\beta u_\nu) - \delta J^\mu \partial_\mu (\beta \mu) \\
&= \frac{4}{15} \tau_R \beta^2 \partial_{(\mu} u_{\nu)} \partial^{(\mu} u^{\nu)} \\
&\quad \times \int \frac{d^4 p}{(2\pi)^3} \delta(p^2) (p \cdot u)^3 f_p^{(0)} (1 - f_p^{(0)}) \geq 0. \quad (51)
\end{aligned}$$

The Eq. (51) is a natural extension of the entropy production rate up to $\mathcal{O}(\partial^2)$ in Eq. (15).

Now, let us analyze the contributions of $\delta T^{\mu\nu}$ and δJ^μ to $\partial_\mu s_R^\mu$. Comparing with Eq. (17), we observe that $\delta T^{\mu\nu}$ corresponds to the shear viscous tensor in classical kinetic theory within the relaxation-time approximation. Accordingly, the entropy production rate $\partial_\mu s_R^\mu$ is nothing new and corresponds to Eq. (21) in the same approximation. However, we also note that the terms related to shear induced polarization and the anomalous Hall effect vanish in δJ^μ , as well as in $\partial_\mu s_R^\mu$ in the end, after momentum integration.

At this point, we encounter a dilemma. On the one hand, the entropy production rate is nonzero when $\partial_{(\mu} u_{\nu)} \partial^{(\mu} u^{\nu)} \neq 0$, which implies that the effects related to the shear tensor are dissipative. On the other hand, all spin transport terms related to the shear tensor do not contribute to the entropy production rate, suggesting that they are nondissipative at least for a momentum-integrated spin current.

V. DISSIPATIVE NATURE OF LOCAL EQUILIBRIUM WIGNER FUNCTION

In the previous section, we showed that entropy analysis within the framework of quantum kinetic theory might be inadequate for studying shear-induced polarization and the anomalous Hall effect, as these phenomena do not contribute to entropy flow after momentum integration. Consequently, it is essential to explore alternative approaches for analyzing entropy directly in phase space, rather than limiting the analysis exclusively to coordinate space. In this section, we analyze the shear-induced polarization using linear response theory within Zubarev's approach [77,98,99]. We consider the leading-order axial-vector Wigner function from linear response theory based on Zubarev's density operator and explicitly compute the axial-vector Wigner function using such linear response theory, incorporating hydrodynamic-gradient sources up to the first-order gradient. Our results suggest that the axial-vector Wigner function, induced by shear-stress tensor and gradient of $\beta\mu$, could be dissipative when the global equilibrium density operator is chosen as the basis density operator. Different from our previous discussions, in this section we focus on the general massive fermion fields.

For ordinary transport phenomena, the dissipative nature of a transport effect can often be known by using time-reversal symmetry analysis of its transport coefficient.

Consider a general Hall current $\mathbf{j}^i = \sigma^{ij} E^j$. We can estimate the entropy production rate in this case. The contribution from the current to the entropy production rate is $\partial_\mu s^\mu \sim \frac{1}{T} \mathbf{j} \cdot \mathbf{E} = \frac{1}{T} \sigma^{ij} E^i E^j$, which is analogous to Eq. (51), replacing $\partial_\mu (\beta\mu)$ with E_μ/T . We observe that the ordinary Hall conductivity, which corresponds to the symmetric part of σ^{ij} , contributes to the entropy production rate. On the other hand, the antisymmetric part, sometimes referred to as the anomalous Hall conductivity, does not contribute to the entropy production rate. Although both the ordinary and anomalous Hall conductivities follow the same rule under time-reversal transformation, the anomalous Hall effect is nondissipative [93]. It is well known that the anomalous Hall effect arises from three main mechanisms: the intrinsic, skew-scattering, and side-jump contributions. The intrinsic contribution is proportional to the Berry curvature of the bands and is independent of interaction details, which can be calculated in a perfect crystal. The other two mechanisms, however, primarily result from scattering processes. Although these mechanisms induce zero entropy production, they originate from dissipative processes. This is very similar to the situation discussed in the previous sections, where the leading scattering contribution to shear-induced polarization generates zero entropy production but originates from 2-to-2 scattering processes. More importantly, if one computes the entropy production for the entire system, it is found to be positive in both the anomalous Hall effect and spin transport. More discussion can be found in the recent review [93] and references therein. Therefore, we will study the shear-induced polarization by other methods instead.

We start from the linear response theory in Zubarev's approaches [77,100]. The global equilibrium density operator, which is independent of hypersurface and chosen as the reference state, is defined as,

$$\hat{\rho}_{\text{GE}} = \frac{1}{Z} \exp \left[-\beta_\nu \hat{P}^\nu + \alpha \hat{Q} + \frac{1}{2} \hat{J}^{\mu\nu} \varpi_{\mu\nu} \right], \quad (52)$$

where $\{\hat{P}^\mu, \hat{Q}, \hat{J}^{\mu\nu}\}$ are the momentum, $U(1)$ charge, and total angular momentum operators, respectively, and $\{\beta_\mu, \alpha, \varpi_{\mu\nu}\}$ are the according Lagrangian multipliers. Here one can define the thermodynamical quantities: temperature $T = 1/\sqrt{\beta^\mu \beta_\mu}$, chemical potential $\mu = \alpha T$, and spin chemical potential $\omega_{\mu\nu} = T \varpi_{\mu\nu}$. And the physical quantities can be defined by their densities after integration over its surrounding region,

$$\begin{aligned}
\hat{P}^\mu &= \int_{\Sigma_\tau} d\Sigma_\nu(x) \hat{T}^{\nu\mu}(x), & \hat{Q} &= \int_{\Sigma_\tau} d\Sigma_\mu(x) \hat{j}^\mu(x), \\
\hat{J}^{\mu\nu} &= \int_{\Sigma_\tau} d\Sigma_\alpha(x) \hat{J}^{\alpha\mu\nu}(x), \quad (53)
\end{aligned}$$

with $\{\hat{T}^{\nu\mu}(x), \hat{j}^\mu(x), \hat{J}^{\alpha\mu\nu}(x)\}$ their corresponding density operators. The partition function guarantees the normalization

of $\hat{\rho}_{\text{GE}}$ such that $\text{Tr}\hat{\rho}_{\text{GE}} = 1$. In general the whole nonequilibrium density operator $\hat{\rho}$ can be defined in a one-parameter family spacelike hypersurface Σ_τ , which is parametrized by τ defined as the proper time along the integral curves of the normal vector $n_\mu(x)$ [99]. The multipliers $\{\beta_\mu, \alpha, \varpi_{\mu\nu}\}$ can be promoted to be spacetime dependent. In such case, the local equilibrium modification to Eq. (52) induced by fluid field gradients can be obtained by generalizing the quantum Liouville equation,

$$\delta\hat{\rho}_{\text{GE}}(\tau) = \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\tau} d\tau' e^{\varepsilon(\tau'-\tau)} \int_{\Sigma_{\tau'}} d\Sigma_\lambda(x) n^\lambda \left[\int_0^1 dz \hat{\rho}_{\text{GE}}^z (\hat{T}^{\nu\mu}(x) \partial_\nu \beta_\mu(x) - \hat{j}^\nu(x) \partial_\nu \alpha(x)) \hat{\rho}_{\text{GE}}^{1-z} \right. \\ \left. - (\langle \hat{T}^{\nu\mu}(x) \rangle_{\text{GE}} \partial_\nu \beta_\mu(x) - \langle \hat{j}^\nu(x) \rangle_{\text{GE}} \partial_\nu \alpha(x)) \hat{\rho}_{\text{GE}} \right]. \quad (54)$$

A physical quantity is defined as $O = \langle \hat{O} \rangle = \text{Tr}(\hat{\rho} \hat{O})$ with $\hat{\rho} = \hat{\rho}_{\text{GE}} + \delta\hat{\rho}_{\text{GE}}$ and $\langle \hat{O} \rangle_{\text{GE}} = \text{Tr}(\hat{\rho}_{\text{GE}} \hat{O})$. Equation (54) can be further simplified if choosing the hypersurface as $d\Sigma_\mu = (d^3\mathbf{x}, 0)$ with $n^\mu = (1, \mathbf{0})$ [100],

$$O(x) - \langle \hat{O}(x) \rangle_{\text{GE}} \\ = \lim_{k \rightarrow 0} \text{Im} \frac{i}{\beta(x)k_0} \int_{t_0}^t d^4x' e^{-ik \cdot (x'-x)} \langle -[\hat{O}(x), \hat{T}^{\mu\nu}(x')] \partial_\mu \beta_\nu(x) + [\hat{O}(x), \hat{j}^\mu(x')] \partial_\mu \alpha(x) \rangle_{\text{GE}} + \mathcal{O}(\partial^2). \quad (55)$$

Equation (55) was originally derived using Zubarev's density operator, also see the discussion in the early pioneering work [98] later revisited in Refs. [77,99,101]. A rederivation of this result, similar to our Eq. (54), can also be found in Eqs. (30) and (31) of a recent review [100].

We emphasize that the correction coming from Eq. (54) or (55) induces positive entropy production and is thus dissipative [98]. It is worth noting that the corrections from Eq. (54) or (55) are deviations from their local equilibrium quantities. Generally, such a local-equilibrium baseline can be evaluated using a hypersurface-dependent local-equilibrium density operator. For instance, a shear-induced axial-vector Wigner function with explicit $\hat{t}^\mu$ dependence was evaluated in Ref. [26] and recently reproduced by [102]. In this section, we adopt the simplest case by using the global equilibrium as the baseline. In this case, the

polarization arising from the shear-stress tensor is purely from the dissipative source and remains independent of an extra timelike vector $\hat{t}^\mu$.

Next, let us consider the axial-vector Wigner functions. The axial-vector Wigner function is defined as the Wigner transformation of the two point functions,

$$\mathcal{A}^{<\mu}(q, X) = -\frac{1}{4} \int d^4Y e^{iq \cdot Y} \text{Tr}(\hat{\rho} \bar{\psi}(y) \gamma^5 \gamma_\mu \psi(x)), \quad (56)$$

where $\hat{\rho}$ denotes the density operator. The $\mathcal{A}^\mu$ with global equilibrium density operator gives the well-known polarization induced by thermal vorticity [103] and its radiative corrections [104–106]. Its modification from gradient of chemical potential reads

$$\delta\mathcal{A}_{\text{LE},\alpha}^{<\mu}(q, X) = \partial_\nu \alpha(X) \lim_{k \rightarrow 0} \text{Im} \frac{1}{\beta(X)k_0} \frac{1}{4} \int d^4Y e^{iq \cdot Y} \int_{-\infty}^{+\infty} d^4x' e^{-ik \cdot (x'-X)} (-i) \theta(X_0 - x'_0) \\ \times \langle \hat{\psi}_b(y) (\gamma^5 \gamma^\mu)_{ba} \hat{\psi}_a(x) \hat{j}^\nu(x') - \hat{j}^\nu(x') \hat{\psi}_b(y) (\gamma^5 \gamma^\mu)_{ba} \hat{\psi}_a(x) \rangle_{\text{GE}}, \quad (57)$$

where the subscript α denotes the axial-vector current induced by gradient of chemical potential.

We now explicitly evaluate the dissipative corrections to the phase-space axial-vector current based on Eq. (55). By calculating the various retarded correlators between $\hat{\mathcal{A}}^{<\mu}$ and $\{\hat{j}_{\rho\sigma\lambda}, \hat{T}_{\alpha\beta}, \hat{j}_\nu\}$, we demonstrate that it is possible to recover the local equilibrium corrections within the framework of linear response theory [27]. This implies the dissipative nature of the local equilibrium axial-vector current, particularly the contributions arising from the

shear-stress tensor and chemical potential gradients. On the other hand, we emphasize that our results differ from the shear contributions discussed in Ref. [26], where the polarization pseudovector was computed through an expansion on a general hypersurface. Here, we focus exclusively on the $\mathcal{O}(\partial^1)$ contribution, where $\varpi_{\mu\nu}$ does not play a role in Eq. (55). For second-order contributions based on a gradient expansion within kinetic theory, see Ref. [73] from quantum kinetic theory and Refs. [107–109] from Zubarev's approaches.

We focus on the free theory and substituting $\{\hat{j}^\mu, \hat{T}^{\mu\nu}\}$ of free fermion fields. After some calculation shown in Appendix B, we derive that

$$\begin{aligned} \delta\mathcal{A}_{\text{LE},\alpha}^{<,\mu}(q, X) &= 2\pi\delta(q^2 - m^2)T(\partial_\nu\alpha)\epsilon^{\mu\alpha\nu\beta}q_\beta u_\alpha \frac{1}{2|q_0|} \{-\partial_{q_0}f_q^{(0)}\} \\ &= 2\pi\delta(q^2 - m^2)(\partial_\nu\alpha)\frac{\epsilon^{\mu\beta\alpha}q_\beta u_\alpha}{2|q_0|}f_q^{(0)}(1 - f_q^{(0)}). \end{aligned} \quad (58)$$

The above result is consistent with those obtained using metric perturbation in the imaginary-time formalism [25,27] and aligns with the massless derivation based on chiral kinetic theory [30,110]. The possible observations related to the anomalous Hall effect are beyond the scope of this work and will be presented elsewhere.

Similarly, the axial-vector Wigner function related to the shear viscous tensor, i.e., the shear induced polarization, can also be computed,

$$\begin{aligned} \delta\mathcal{A}_{\text{LE},\xi}^{<,\mu}(q, X) &= -q^\beta\xi_{\alpha\beta}(X)2\pi\delta(q^2 - m^2)\frac{\epsilon^{\mu\alpha\rho\sigma}q_\rho u_\sigma}{2|q_0|}f_q^{(0)}(1 - f_q^{(0)}), \end{aligned} \quad (59)$$

where we have used the Belinfante pseudogauge for energy momentum tensor, such that we only need to focus on the symmetrized gradient $\xi_{\alpha\beta} = \frac{1}{2}\partial_{(\alpha}\beta_{\beta)}$ known as the thermal shear tensor. More details can be found in Appendix B.

In this work, we restrict our discussion to the case without interaction corrections. In principle, one can use perturbation theory to calculate interaction corrections in the weakly coupled limit, up to $\mathcal{O}(g^2)$. Unlike the global equilibrium scenario, where the angular momentum vertex is nonlocal and challenging to handle, all the correlators discussed in this section can be reliably computed using thermal field theory techniques [111,112]. It is particularly interesting to evaluate the correlators in Eq. (55) while accounting for scattering contributions, including both elastic and inelastic processes. This can be achieved using diagrammatic techniques commonly employed in the calculation of transport coefficients within real-time formalism [113–115]. The only difference here is that one of the vertices is of the axial-vector type, $\gamma^5\gamma^\mu$. This is the relativistic analog of calculations for anomalous (or spin) Hall conductivity [93,94,116] in a hot QCD plasma. We expect this to yield the so-called side-jump contribution, up to the leading coupling, as predicted from quantum kinetic theory [73–75] through solutions of the spin Boltzmann equation [70–72].

Now, let us comment on Eqs. (58), (59). In fact, these equations represent all possible first-order noninteracting local equilibrium corrections to the axial-vector Wigner function. We observe that these results are consistent with those reported in Ref. [27]. This property is rather unique for spin transport compared to electric conductivity in

particle number transport. We attribute this distinct feature to the additional $\gamma^5\gamma^\mu$ factor in the axial-vector current, which makes this result applicable only to the phase-space current. Notably, these off-equilibrium corrections, up to leading order in the coupling, contribute zero to the axial-vector current $J_5^\mu(x)$. This observation hints at the possible nonrenormalization of $J_5^\mu \sim \mathcal{O}(\partial^1)$ from off-equilibrium sources.

Before concluding this section, we note that Eqs. (58), (59) are derived based on the density operator shown in Eq. (54), which extends beyond global equilibrium. As discussed below Eq. (55), this suggests that the polarization induced by the shear tensor and $\partial_\mu\alpha$ could be dissipative. More systematic studies based on the linear response theory within the Zubarev's approaches become necessary and will be present in Ref. [106].

VI. SUMMARY AND DISCUSSION

In this work, we analyze the entropy production rate associated with shear-induced polarization and the anomalous Hall effect within the framework of quantum (chiral) kinetic theory. Following the strategy used in classical kinetic theory, we introduce entropy flow and the H theorem into the chiral kinetic theory. To simplify the complexities of the collision term, we adopt the relaxation time approximation. We have also verified that the newly introduced entropy flow from chiral kinetic theory is consistent with the one derived from relativistic hydrodynamics, and that the H theorem ensures entropy increases.

Our analysis reveals that the perturbation of the distribution function around equilibrium, related to the shear viscous tensor, leads to an increase in the entropy production rate, similar to the behavior of the conventional shear viscous tensor in classical kinetic theory. However, both shear-induced polarization and the anomalous Hall effect do not contribute to the entropy production rate after integrating over momentum, as is the case for the anomalous Hall effect in condensed matter physics. As a result, it remains challenging to definitively conclude whether shear-induced polarization and the anomalous Hall effect are nondissipative based solely on entropy flow and entropy production rate. These findings highlight the limitations of conventional entropy analysis in capturing quantum effects within the framework of quantum kinetic theory.

We have also studied the shear-induced polarization using linear response theory within the framework of Zubarev's approach. We have derived $\delta\mathcal{A}_{\text{LE}}^{<,\mu}$ induced by the thermal shear tensor and $\partial_\mu\alpha$. According to our derivation, this suggests that the polarization induced by the shear tensor and $\partial_\mu\alpha$ could be dissipative. Although many spin-related effects contribute to spin polarization, it remains challenging to disentangle them experimentally.

Finally, we would like to discuss entropy analysis in quantum kinetic theory. As mentioned, the current studies

highlight the limitations of conventional entropy analysis in addressing certain quantum effects within kinetic theory. A reasonable extension of the current entropy analysis involves decomposing the distribution function as $f_p = f_{\text{GE}} + \delta f_p$, where f_{GE} represents the global equilibrium distribution function, as defined in Sec. V and it can receive interaction corrections even in global equilibrium [60]. Roughly speaking, the effects arising from f_{GE} can be considered nondissipative, while those associated with δf_p could be dissipative. This decomposition aligns with the conclusions drawn from Zubarev's approach. Therefore, to further improve entropy analysis, more systematic studies based on Zubarev's approach are essential. However, such investigations are beyond the scope of this work and will be presented in a forthcoming study.

ACKNOWLEDGMENTS

We are thankful to Francesco Becattini, Matteo Buzzegoli, and Shu Lin for helpful discussions. This work is supported in part by the National Key Research and Development Program of China under Contract No. 2022YFA1605500, by the Chinese Academy of Sciences (CAS) under Grant No. YSBR-088, by the National Natural Science Foundation of China (NSFC) under Grants No. 12075235 and No. 12135011, by the National Science and Technology Council (Taiwan) under Grant No. NSTC 113-2628-M-001-009-MY4, and by the Academia Sinica under Project No. AS-CDA-114-M01.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: DERIVATION TO EQS. (39) AND (41)

Let us further simplify the last term $-\hbar S_{(u)}^{\mu\nu} C_\nu[\mathcal{H}[f_p^{(u)}]]$ in Eq. (39). Since the entropy flow is at $\mathcal{O}(\partial^1)$, we can decompose the collisional corrections C_ν into two parts,

$$C_\nu[\mathcal{H}(f_p^{(u)})] = C_\nu^{leq} + C_\nu^\delta, \quad (\text{A1})$$

where $C_\nu^{leq} \equiv C_\nu[\mathcal{H}[f_p^{eq}]]$ and $C_\nu^\delta \equiv C_\nu[\mathcal{H}[\delta f_p]]$. On the other hand, it is natural to assume that $C_\nu[\mathcal{H}[f]] \propto f$. For the first part, C_ν^{leq} can be parametrized as

$$C_\nu^{leq} = (b_1 u_\nu + b_2 p_\nu + b_3 \partial_\nu^p) f_p^{eq} + b_{4,\nu} f_p^{(0)}, \quad (\text{A2})$$

where $b_i \sim \mathcal{O}(\partial^0)$, $b_{4,\nu} \sim \mathcal{O}(\partial^1)$ and $f_p^{(0)}$ is given by Eq. (3). Here, $\partial_\nu^p \equiv \partial/\partial p^\nu$. Inserting the above decomposition into entropy flow, we find the term proportional to C_ν^{leq} vanishes after the momentum integration. Similarly, we can parametrize C_ν^δ as

$$C_\nu^\delta = \frac{1}{\tau_R''} (a_1 u_\nu + a_2 p_\nu + a_3 \partial_\nu^p) \delta f_p, \quad (\text{A3})$$

where $a_i \sim \mathcal{O}(\partial^0)$ and $\tau_R'' \sim \mathcal{O}(\partial^0)$ is another relaxation time. In general, τ_R'' is not necessarily the same as τ_R introduced in Eq. (37). Inserting the C_ν^δ into Eq. (39), we can further simplify the entropy flow as

$$s^\mu = 2 \int \frac{d^4 p}{(2\pi)^3} \bar{\epsilon}_{(u)} \delta(p^2) \left\{ (p^\mu + \hbar S_{(u)}^{\mu\nu} \partial_\nu) \mathcal{H}[f_p^{(u)}] - \hbar \frac{1}{\tau_R''} a_3 S_{(u)}^{\mu\nu} f_p^{eq} (1 - f_p^{eq}) \partial_\nu^p \chi_p \right\}. \quad (\text{A4})$$

Integrating by parts, the term proportional to $\partial_\nu^p \chi_p$ in the entropy flow becomes,

$$- \int \frac{d^4 p}{(2\pi)^3} \partial_\nu^p \left[\bar{\epsilon}_{(u)} \delta(p^2) \hbar \frac{1}{\tau_R''} a_3 f_p^{(0)} (1 - f_p^{(0)}) \right] S_{(u)}^{\mu\nu} \chi_p,$$

where we have dropped the total derivatives and $\partial_\nu^p S_{(u)}^{\mu\nu} = 0$. Since both a_3 and τ_R'' are scalar and at $\mathcal{O}(\partial^0)$, they can only be the function of $u \cdot p$ and $p_\mu p_\nu \Theta^{\mu\nu}$. After taking the momentum derivative and contracting with $S_{(u)}^{\mu\nu}$, the above integration vanishes. Therefore, the entropy flow reduces to,

$$s^\mu = 2 \int \frac{d^4 p}{(2\pi)^3} \bar{\epsilon}_{(u)} \delta(p^2) (p^\mu + \hbar S_{(u)}^{\mu\nu} \partial_\nu) \mathcal{H}[f_p^{(u)}]. \quad (\text{A5})$$

Next, let us compute the entropy production rate. In the chiral kinetic theory with collisions, the current and the energy momentum tensor in the absence of the external field are given by

$$J^\mu = 2 \int \frac{d^4 p}{(2\pi)^3} \bar{\epsilon}_{(u)} \delta(p^2) \{ (p^\mu + \hbar S_{(u)}^{\mu\nu} \partial_\nu) f_p^{(u)} - \hbar S_{(u)}^{\mu\nu} C_\nu[f_p^{(u)}] \},$$

$$T^{\mu\nu} = \int \frac{d^4 p}{(2\pi)^3} \bar{\epsilon}_{(u)} \delta(p^2) [p^\nu \{ (p^\mu + \hbar S_{(u)}^{\mu\alpha} \partial_\alpha) f_p^{(u)} - \hbar S_{(u)}^{\mu\alpha} C_\alpha[f_p^{(u)}] \} + p^\mu \{ (p^\nu + \hbar S_{(u)}^{\nu\alpha} \partial_\alpha) f_p^{(u)} - \hbar S_{(u)}^{\nu\alpha} C_\alpha[f_p^{(u)}] \}]. \quad (\text{A6})$$

Analogous to Eq. (A1), we rewrite the collision term as two parts, $C_\alpha[f_p^{(u)}] = C_\alpha[f_p^{eq}] + C_\alpha[\delta f_p]$. As mentioned, $C_\alpha[f_p^{eq}] = 0$ ensures that Eq. (10) is satisfied. For the part $C_\alpha[\delta f_p]$, we can further parametrize it as $C_\alpha[\delta f_p] = (b'_1 u_\alpha + b'_2 p_\alpha + b'_3 \partial_\alpha^p) \delta f_p$,

where $b'_i \sim \mathcal{O}(\partial^0)$. Inserting this decomposition back to Eq. (A6), we find that all terms proportional to $S_{(u)}^{\mu\nu} \mathcal{C}_\nu$ will not contribute.

By using the constraints given by the conservation equations shown in Eq. (40), we can compute the entropy production rate,

$$\begin{aligned} \partial_\mu s_R^\mu &= 2 \int \frac{d^4 p}{(2\pi)^3} \bar{e}_{(u)} \delta(p^2) \ln \left(\frac{1 - f_p^{(u)}}{f_p^{(u)}} \right) \bar{\mathcal{C}}[f_p^{(u)}] \\ &= 2 \int \frac{d^4 p}{(2\pi)^3} \bar{e}_{(u)} \delta(p^2) \left[\beta(u \cdot p) - \beta\mu + \beta \frac{\hbar p \cdot \omega}{2p \cdot u} - \chi_p \right] \bar{\mathcal{C}}[f_p^{(u)}] \\ &= 2\beta \int \frac{d^4 p}{(2\pi)^3} \bar{e}_{(u)} \delta(p^2) \left\{ \frac{\hbar}{2} [(p \cdot u)(\partial_\mu S_{(u)}^{\mu\alpha}) \partial_\alpha + p^\mu (\partial_\mu u_\nu) S_{(u)}^{\nu\alpha} \partial_\alpha] f_p^{(u)} + \frac{\hbar p \cdot \omega}{2p \cdot u} \bar{\mathcal{C}}[f_p^{(u)}] - \frac{1}{\beta} \chi_p \bar{\mathcal{C}}[f_p^{(u)}] \right\} \\ &\simeq -2 \int \frac{d^4 p}{(2\pi)^3} \bar{e}_{(u)} \delta(p^2) \chi_p \bar{\mathcal{C}}[f_p^{(u)}] + \mathcal{O}(\partial^3), \end{aligned} \quad (\text{A7})$$

where we have used the relation $(p \cdot u)(\partial_\mu S_{(u)}^{\mu\alpha}) + p^\mu (\partial_\mu u_\nu) S_{(u)}^{\nu\alpha} = -\frac{p \cdot \omega}{p \cdot u} p^\alpha$ and have used the spin Boltzmann equation (36) in the last step. Inserting the collision term in Eq. (37) into above equations, yields,

$$\partial_\mu s_R^\mu = I_1 + I_2, \quad (\text{A8})$$

where

$$\begin{aligned} I_1 &= \frac{2}{\tau_R} \int \frac{d^4 p}{(2\pi)^3} \bar{e}_{(u)} \delta(p^2) \chi_p (p \cdot u) \delta f_p \\ &= \frac{1}{\tau_R} \int \frac{d^3 p}{(2\pi)^3} \{ [f_p^{eq} (1 - f_p^{eq}) \chi_p^2] |_{p_0=|\mathbf{p}|} \\ &\quad + [f_p^{eq} (1 - f_p^{eq}) \chi_p^2] |_{p_0=-|\mathbf{p}|} \} \geq 0, \end{aligned} \quad (\text{A9})$$

and

$$I_2 = \frac{2\hbar}{\tau_R} \int \frac{d^4 p}{(2\pi)^3} \bar{e}_{(u)} \delta(p^2) \chi_p p^\mu \mathcal{R}_\mu \delta f_p. \quad (\text{A10})$$

Inserting the parametrization for $\mathcal{R}^\mu$ in Eq. (38), yields Eq. (41).

APPENDIX B: DERIVATION OF EQ. (59)

We will compute the shear induced polarization for Eq. (57). Substituting the integral representation of sign function,

$$\theta(X_0 - x'_0) = i \int_{-\infty}^{+\infty} \frac{dk'_0}{2\pi} \frac{e^{-ik'_0(X_0 - x'_0)}}{k'_0 + i\eta}, \quad (\text{B1})$$

Eq. (57) reduces to,

$$\begin{aligned} \delta \mathcal{A}_{\text{LE},\alpha}^{<\mu}(q, X) &= \partial_\nu \alpha(X) \lim_{k \rightarrow 0} \text{Im} \frac{T(X)}{k_0} \frac{1}{4} \int d^4 Y e^{iq \cdot Y} \int_{-\infty}^{+\infty} d^4 x' \int_{-\infty}^{+\infty} \frac{dk'_0}{2\pi} \frac{e^{-ik'_0(X_0 - x'_0)}}{k'_0 + i\eta} e^{-ik \cdot (x' - X)} \\ &\quad \times \text{Tr} \left\{ \gamma^5 \gamma^\mu S_{0,\text{GE}}^> \left(X + \frac{Y}{2}, x' \right) \gamma^\nu S_{0,\text{GE}}^< \left(x', X - \frac{Y}{2} \right) \right. \\ &\quad \left. - \gamma^5 \gamma^\mu S_{0,\text{GE}}^< \left(X + \frac{Y}{2}, x' \right) \gamma^\nu S_{0,\text{GE}}^> \left(x', X - \frac{Y}{2} \right) \right\}. \end{aligned} \quad (\text{B2})$$

We emphasize that the θ function here only accounts for the boundary condition and the small imaginary part in the denominator should be taken to zero in the last. Here we have also introduced the following global equilibrium Wigner functions for the free fermions as follows

$$S_{0,\text{GE}}^{\lessgtr}(x, y) = \int \frac{d^4 q}{(2\pi)^4} e^{-iq \cdot (x - y)} S_{0,\text{GE}}^{\lessgtr} \left(q, \frac{x + y}{2} \right), \quad (\text{B3})$$

$$S_{0,\text{GE}}^{\lessgtr}(q, X) = 2\pi \bar{e}_{(u)} \delta(q^2 - m^2) (\gamma^\mu q_\mu + m) f^{\lessgtr,(0)}(q), \quad (\text{B4})$$

where up to $\mathcal{O}(\partial^0)$, the free fermion distribution function $f^{(0)}(q)$ is the same for massless and massive fermions and defined in Eq. (3) and $\bar{\epsilon}_{(u)}$ is the sign function. For convenience, we also introduce $f^> = 1 - f^<$. We will work in the β frame [117] with fluid velocity $u^\mu = \beta^\mu / \sqrt{\beta^2}$. Then Eq. (B2) reduces to

$$\begin{aligned}
& \delta\mathcal{A}_{\text{LE},\alpha}^{<,\mu}(q, X) \\
&= \partial_\nu \alpha(X) \lim_{k \rightarrow 0} \text{Im} \frac{T(X)}{k_0} \frac{1}{4} \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{dk'_0}{2\pi} \frac{1}{k'_0 + i\eta} 2\pi \delta(k_0 + p_{2,0} - p_{1,0} - k'_0) \\
&\quad \times (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{p}_2 - \mathbf{p}_1) (2\pi)^4 \delta^{(4)}\left(q - \frac{p_1 + p_2}{2}\right) \\
&\quad \times \text{Tr}\{\gamma^5 \gamma^\mu S_0^>(p_1) \gamma^\nu S_0^<(p_2) - \gamma^5 \gamma^\mu S_0^<(p_1) \gamma^\nu S_0^>(p_2)\} \\
&= \partial_\nu \alpha(X) \lim_{k \rightarrow 0} \text{Im} \frac{T(X)}{k_0} \frac{1}{4} \int \frac{d^4 Q}{(2\pi)^4} \frac{1}{k_0 - Q_0 + i\eta} (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{Q}) \\
&\quad \times \text{Tr}\left\{\gamma^5 \gamma^\mu S_0^>\left(q + \frac{Q}{2}\right) \gamma^\nu S_0^<\left(q - \frac{Q}{2}\right) - \gamma^5 \gamma^\mu S_0^<\left(q + \frac{Q}{2}\right) \gamma^\nu S_0^>\left(q - \frac{Q}{2}\right)\right\}. \tag{B5}
\end{aligned}$$

Inserting Eq. (B4) into the above equation, yields

$$\begin{aligned}
& \delta\mathcal{A}_{\text{LE},\alpha}^{<,\mu}(q, X) \\
&= \partial_\nu \alpha(X) \lim_{k_0 \rightarrow 0} \frac{\partial}{\partial k_0} \lim_{k_i \rightarrow 0} \text{Im} \frac{T(X)}{k_0} \frac{1}{4} \int \frac{d^4 Q}{(2\pi)^2} \frac{1}{k_0 - Q_0 + i\eta} (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{Q}) \delta\left(\left(q + \frac{Q}{2}\right)^2 - m^2\right) \\
&\quad \times \delta\left(\left(q - \frac{Q}{2}\right)^2 - m^2\right) \left\{f^{<,(0)}\left(q - \frac{Q}{2}\right) f^{>,(0)}\left(q + \frac{Q}{2}\right) - f^{<,(0)}\left(q + \frac{Q}{2}\right) f^{>,(0)}\left(q - \frac{Q}{2}\right)\right\} \\
&\quad \times \text{Tr}\left[\gamma^5 \gamma^\mu \left(\gamma^\alpha \left(q + \frac{Q}{2}\right)_\alpha + m\right) \gamma^\nu \left(\gamma^\beta \left(q - \frac{Q}{2}\right)_\beta + m\right)\right] \\
&= \partial_\nu \alpha(X) \lim_{k_0 \rightarrow 0} \lim_{k_i \rightarrow 0} \frac{T(X)}{k_0} \text{Im} \frac{1}{4} \int \frac{d^4 Q}{(2\pi)^2} \frac{1}{k_0 - Q_0 + i\eta} (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{Q}) (-4i\epsilon^{\mu\alpha\nu\beta} Q_\alpha q_\beta) \\
&\quad \times \delta\left(q^2 + \frac{Q^2}{4} - m^2\right) \delta(-2q \cdot Q) \left\{f^{<,(0)}\left(q - \frac{Q}{2}\right) - f^{<,(0)}\left(q + \frac{Q}{2}\right)\right\} \\
&= \partial_\nu \alpha(X) \lim_{k_0 \rightarrow 0} \frac{T(X)}{k_0} \lim_{k_i \rightarrow 0} \text{Im} \int \frac{dQ_0}{2\pi} (2\pi)^2 \frac{\delta(Q_0) Q_0^2}{k_0 - Q_0 + i\eta} (-i\epsilon^{\mu\alpha\nu\beta} q_\beta u_\alpha) \\
&\quad \times \delta\left(q^2 + \frac{Q_0^2}{4} - m^2\right) \frac{1}{2|q_0|} \{-\partial_{q_0} f^{<,(0)}(q) + \mathcal{O}(Q_0^2)\}, \tag{B6}
\end{aligned}$$

where $q_0 = u \cdot q$. In the $k \rightarrow 0$ limit,

$$\begin{aligned}
& \delta\mathcal{A}_{\text{LE},\alpha}^{<,\mu}(q, X) = T(X) \partial_\nu \alpha(X) \int \frac{dQ_0}{(2\pi)} \delta(Q_0) Q_0^2 \frac{Q_0^2 - \eta^2}{(Q_0^2 + \eta^2)^2} \epsilon^{\mu\alpha\nu\beta} q_\beta u_\alpha \\
&\quad \times (2\pi)^2 \delta\left(q^2 + \frac{Q_0^2}{4} - m^2\right) \frac{1}{2|q_0|} \{-\partial_{q_0} f^{<,(0)}(q) + \mathcal{O}(Q_0^2)\}, \tag{B7}
\end{aligned}$$

taking the $\eta \rightarrow 0^+$ introduced as boundary constraint for the θ function, we obtain,

$$\delta\mathcal{A}_{\text{LE},\alpha}^{<,\mu}(q, X) = 2\pi \delta(q^2 - m^2) T(\partial_\nu \alpha) \epsilon^{\mu\alpha\nu\beta} q_\beta u_\alpha \frac{1}{2|q_0|} \{-\partial_{q_0} f_q^{(0)}\}, \tag{B8}$$

which reduces to Eq. (58). It is found that the infinitesimal η only plays the role of imposing proper initial conditions for the retarded correlators, it can be simply taken as zero in the calculation of the correlators. Such observation is also noticed when calculating the transport coefficients, e.g., in Ref. [118].

The axial-vector Wigner function related to the shear viscous tensor is given by in the Belinfante pseudogauge,

$$\begin{aligned} \delta\mathcal{A}_{\text{LE},\xi}^{<,\mu}(q, X) = & -\frac{1}{8}\xi_{\alpha\beta}(X)\lim_{k_0\rightarrow 0}\lim_{k_i\rightarrow 0}\frac{T(X)}{k_0}\text{Im}\int_{-\infty}^{+\infty}d^4x'i\int_{-\infty}^{+\infty}\frac{dk'_0}{2\pi}\frac{e^{-ik'_0(X_0-x'_0)}}{k'_0+i\eta}e^{-ik\cdot(x'-X)} \\ & \times\int d^4ye^{iq\cdot y}\left\{(-1)^2\text{Tr}\left[\gamma^5\gamma^\mu S^>\left(X+\frac{y}{2}, x'\right)\gamma^\alpha(\vec{\partial}_{x'}^\beta-\vec{\partial}_{x'}^\beta)S^<\left(x', X-\frac{y}{2}\right)\right]\right. \\ & \left.-(-1)^2\text{Tr}\left[\gamma^5\gamma^\mu S^<\left(X+\frac{y}{2}, x'\right)\gamma^\alpha(\vec{\partial}_{x'}^\beta-\vec{\partial}_{x'}^\beta)S^>\left(x', X-\frac{y}{2}\right)\right]\right\}. \end{aligned} \quad (\text{B9})$$

We now insert the Fourier components,

$$\begin{aligned} \delta\mathcal{A}_{\text{LE},\xi}^{<,\mu}(q, X) = & \frac{1}{4}q^\beta\xi_{\alpha\beta}(X)\lim_{k_0\rightarrow 0}\lim_{k_i\rightarrow 0}\frac{T(X)}{k_0}\text{Im}\int\frac{d^4Q}{(2\pi)^4}\frac{1}{k_0-Q_0+i\eta}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{Q}) \\ & \times\left\{-\text{Tr}\left[\gamma^5\gamma^\mu S^>\left(q+\frac{Q}{2}\right)\gamma^\alpha S^<\left(q-\frac{Q}{2}\right)\right]+\text{Tr}\left[\gamma^5\gamma^\mu S^<\left(q+\frac{Q}{2}\right)\gamma^\alpha S^>\left(q-\frac{Q}{2}\right)\right]\right\}. \end{aligned} \quad (\text{B10})$$

It is interesting to note that the kernel is the same as the $\mathcal{A}_\alpha$ sector, where

$$\begin{aligned} & \lim_{k_0\rightarrow 0}\lim_{k_i\rightarrow 0}\text{Im}\frac{T(X)}{k_0}\frac{1}{4}\int\frac{d^4Q}{(2\pi)^4}\frac{1}{k_0-Q_0+i\eta}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{Q}) \\ & \times\text{Tr}\left\{\gamma^5\gamma^\mu S^>\left(q+\frac{Q}{2}\right)\gamma^\nu S^<\left(q-\frac{Q}{2}\right)-\gamma^5\gamma^\mu S^<\left(q+\frac{Q}{2}\right)\gamma^\nu S^>\left(q-\frac{Q}{2}\right)\right\} \\ & = 2\pi\delta(q^2-m^2)\frac{\epsilon^{\mu\nu\beta\alpha}q_\beta u_\alpha}{2|q_0|}f^{<,(0)}(q)f^{>,(0)}(q), \end{aligned} \quad (\text{B11})$$

which gives Eq. (59).

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