

# Subcritical bubble prehistory in a weak first-order phase transition

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Standard calculations of cosmological first-order phase transitions usually assume critical bubbles to nucleate on a homogeneous symmetric vacuum background. However, this assumption can fail in weak transitions, where thermal fluctuations trigger subcritical bubbles before the standard nucleation temperature  $T_n$ . Motivated by this possibility, we systematically examine whether the homogeneous nucleation background approximation is self-consistent. By evolving the Gelmini-Gleiser subcritical bubble kinetics and comparing it with the standard critical bubble nucleation picture, we identify the parameter regions in which the background becomes apparently mixed. A detailed scan of these regions shows that sizable subcritical volume fractions arise when the two phases are nearly degenerate at  $T_n$ , the potential barrier is low, the difference of free energy between the symmetric and broken phases is moderate and the transition strength is weak. Within this simplified phenomenological kinetic framework, this scan further calibrates a practical reference criterion,  $\log_{10} \hat{f}_\xi(T_n) \simeq -1.95$ , for percent level subcritical bubble volume fractions. Parameter points above this boundary should be treated as mixed background candidates rather than as ordinary homogeneous bounce points.

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## I. INTRODUCTION

The electroweak phase transition is an important event in the thermal history of the early Universe. In the Standard Model of particle physics, the observed Higgs mass makes this transition a smooth crossover rather than a first-order phase transition [1]. Extensions of the Higgs sector can change this situation, because additional scalar degrees of freedom may generate a potential barrier and turn the transition into a first-order one [2–7]. Such a transition leads to vacuum bubble nucleation, whose subsequent collisions, plasma sound waves, and turbulence may source a stochastic gravitational wave background within the projected reach of space-based interferometers [7–17]. The moving bubble walls can also provide the out-of-equilibrium condition required for electroweak baryogenesis [18–21], while the associated thermal history may affect dark matter production or primordial black hole

formation (PBH) [22–28]. For this reason, first-order electroweak phase transitions provide a direct link between Higgs sector dynamics and cosmological observables.

The standard description of a first-order transition begins with the nucleation and expansion of critical bubbles. In Coleman's formulation and most finite temperature applications, the critical bubble is computed on a homogeneous symmetric vacuum background [29,30]. At finite temperature, however, this background is dynamical. Thermal fluctuations can push the field across the barrier toward the broken-phase side of the potential, while reverse fluctuations can erase these regions before they reach the critical size [31,32]. The prenucleation state may therefore contain short-lived broken regions that are continuously created and destroyed, as illustrated in Fig. 1. Since their sizes are controlled by thermal correlation scales and remain below the critical radius, these objects are usually referred to as subcritical bubbles [32].

The possible impact of subcritical bubbles was recognized in Refs. [31–33]. These works argued that, when a transition is sufficiently weak, thermal fluctuations can populate both phases and replace the usual homogeneous metastable state by a two-phase emulsion. The idea was later developed into a kinetic framework in Refs. [34–36], where subcritical bubbles are formed and removed by shrinking, thermal noise, and reverse fluctuations. Real time

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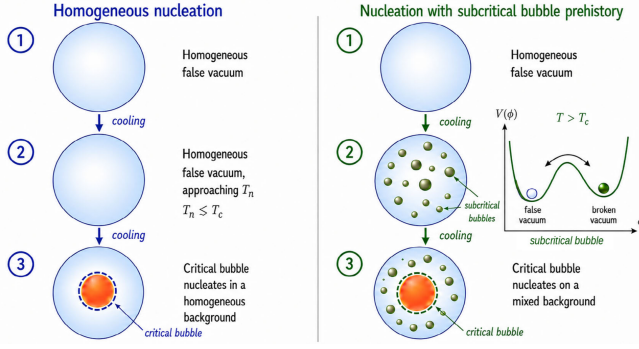


FIG. 1. A conceptual comparison of the homogeneous nucleation and the nucleation with subcritical bubble prehistory. In the latter case, the critical bubble forms on a mixed background rather than in a purely homogeneous false vacuum. This diagram was created with the assistance of ChatGPT.

Langevin studies further showed that thermal fluctuations can affect both the onset of the transition and the subsequent evolution of critical bubbles [34,37]. These studies therefore indicate that subcritical bubbles can constitute an important component of phase transition dynamics, rather than a negligible microscopic correction to the critical bubble picture.

However, there is no universal criterion for determining how weak a first-order transition must be before the subcritical population becomes large enough to challenge the homogeneous background approximation. It is important because weak transitions are often not expected to generate sizable macroscopic signals, as their released energy can be limited. If subcritical bubbles generate an appreciable broken phase population before ordinary critical bubble nucleation, a subset of weak transitions may begin from modified initial conditions, which can affect the subsequent transition history and its phenomenological consequences. As one possible consequence, the additional small scale structure introduced by subcritical bubbles may modify the high frequency part of the gravitational wave spectrum [38].

In this work, we study subcritical bubble prehistory as a consistency test of the standard homogeneous nucleation framework. A birth-death kinetic equation is evolved along a radiation-dominated cooling history, and the resulting true broken-phase volume fraction is compared with the standard nucleation picture. A parameter space scan is performed to determine how the subcritical fraction depends on the transition strength, the free energy splitting, the barrier height, and the nucleation temperature. Guided by this scan, we calibrate a practical reference criterion for deciding when the homogeneous symmetric vacuum background remains a self-consistent starting point and when mixed background effects should be included.

The paper is organized as follows. Section II A introduces the finite temperature potential and the standard critical bubble calculation. Section II B formulates the

subcritical kinetic equation in an expanding Universe. Section III describes the details of the parameter scan and the numerical inputs. Section IV presents and discusses the scan results, including representative histories, the parameter space dependence of the subcritical bubble volume fraction, and the resulting practical criterion. It also discusses how a subcritical prehistory may induce phenomenological corrections to the subsequent phase transition dynamics.

## II. THEORETICAL FRAMEWORK

### A. Homogeneous background critical bubble nucleation

As a minimal benchmark for weak first-order phase transitions, we consider the finite temperature potential of a single scalar field.

$$V(\phi, T) = D(T^2 - T_0^2)\phi^2 - ET\phi^3 + \frac{\lambda}{4}\phi^4. \quad (1)$$

Its symmetric vacuum is located at

$$\phi_s = 0, \quad (2)$$

At high temperatures, this is the only local minimum. As the temperature decreases, two nonzero stationary points can appear, corresponding to the barrier location  $\phi_{\text{top}}$  and the broken phase  $\phi_b$

$$\phi_{\text{top},b}(T) = \frac{3ET \mp \sqrt{9E^2T^2 - 8\lambda D(T^2 - T_0^2)}}{2\lambda}. \quad (3)$$

The temperature  $T_1$  is the point at which these two nonzero stationary points first appear. Equivalently, it is the temperature where the square root in Eq. (3) vanishes. This gives

$$T_1^2 = \frac{8\lambda D}{8\lambda D - 9E^2} T_0^2. \quad (4)$$

For  $T > T_1$ , the nonzero stationary points are absent and the potential contains only the symmetric minimum. For  $T < T_1$ , the potential contains the ingredients of a first order transition: a symmetric minimum, a potential barrier, and a broken phase minimum.

The critical temperature  $T_c$  is defined by the degeneracy of the symmetric and broken phase,

$$V(0, T_c) = V(\phi_c, T_c), \quad V'(\phi_c, T_c) = 0. \quad (5)$$

These conditions give

$$\frac{\phi_c}{T_c} = \frac{2E}{\lambda} \equiv r_c, \quad T_c^2 = \frac{T_0^2}{1 - E^2/(\lambda D)}. \quad (6)$$

Given the finite temperature potential, the critical bubble is obtained from the finite temperature  $O(3)$  bounce equation [39,40],

$$\frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} = \frac{\partial V}{\partial \phi}, \quad \frac{d\phi}{dr} \Big|_{r=0} = 0, \quad \phi(r \rightarrow \infty) = 0, \quad (7)$$

where  $r$  is the radial coordinate of the bubble. The boundary condition at spatial infinity assumes that the field approaches the homogeneous symmetric vacuum. Then the corresponding three dimensional Euclidean action is

$$S_3(T) = 4\pi \int dr r^2 \left[ \frac{1}{2} \left( \frac{d\phi}{dr} \right)^2 + V(\phi, T) - V(0, T) \right]. \quad (8)$$

The nucleation rate is exponentially suppressed by this action [29],

$$\Gamma_c(T) = A_c(T) \exp[-S_3(T)/T], \quad (9)$$

where  $A_c$  is the fluctuation prefactor, estimated as  $A_c \simeq T^4$  in the present work. More refined treatments of the prefactor can be found in Ref. [41].

During radiation dominated period, the expected number of critical bubbles produced in one Hubble volume is

$$N_c(T) = \int_{t_c}^{t(T)} dt' \frac{\Gamma_c(t')}{H^3(t')} = \int_T^{T_c} \frac{dT'}{T'} \frac{\Gamma_c(T')}{H^4(T')}, \quad (10)$$

where  $dT/dt = -HT$  has been used and the nucleation temperature  $T_n$  is defined as

$$N_c(T_n) = 1. \quad (11)$$

To describe the progress of the transition, we also track the volume fraction converted by critical bubbles. Under the assumption that critical bubbles nucleate independently as a Poisson process [7], the probability that a given point has been converted to the broken phase is

$$p_{\text{crit}}(t) = 1 - \exp[-I_c(t)]. \quad (12)$$

Here  $I_c(t)$  is the spacetime volume swept out by bubbles nucleated at earlier times,

$$I_c(t) = \int_{t_i}^t dt' \Gamma_c(t') \left[ \frac{a(t')}{a(t)} \right]^3 \frac{4\pi}{3} R(t, t')^3, \quad (13)$$

where the factor  $[a(t')/a(t)]^3$  accounts for the dilution of the physical number density of bubbles due to cosmic expansion, and

$$R(t, t') = a(t) \int_{t'}^t \frac{v_w dt''}{a(t'')} \quad (14)$$

is the physical radius at time  $t$  of a bubble nucleated at time  $t'$ . In the numerical analysis, we take the bubble wall velocity  $v_w = 1$ . Modifying this quantity has little effect on the final result.

The transition strength is quantified by the ratio of vacuum energy released to the radiation energy density

$$\alpha(T) = \frac{\Delta\rho(T)}{\rho_{\text{rad}}(T)}, \quad \rho_{\text{rad}}(T) = \frac{\pi^2 g_* T^4}{30}, \quad (15)$$

where

$$\Delta\rho(T) = V(0, T) - V(\phi_b, T) - T[\partial_T V(0, T) - \partial_T V(\phi_b, T)], \quad (16)$$

$g_*$  is the number of relativistic degrees of freedom and  $\alpha_n \equiv \alpha(T_n)$  denotes the transition strength evaluated at the nucleation temperature,

## B. Subcritical bubble prehistory as a background consistency test

The standard bounce solution calculation assumes that the symmetric vacuum remains spatially homogeneous until critical bubbles nucleate. Subcritical bubbles challenge the validity of this assumption before  $T_n$ . Following the convention of Ref. [34], we denote  $n(R, t)$  as the number density of subcritical bubbles with fixed radius  $R$  at time  $t$ . The kinetic equation in radius space then reads

$$\frac{\partial n}{\partial t} + 3Hn = -\frac{\partial n}{\partial R} \frac{dR}{dt} + \frac{V_s}{V_{\text{tot}}} \Gamma_{s \rightarrow b}^{\text{sub}} - \frac{V_b^{(R)}}{V_{\text{tot}}} \Gamma_{b \rightarrow s}^{\text{sub}} - \frac{V_b^{(R)}}{V_{\text{tot}}} \Gamma_{\text{TN}}^{\text{sub}}, \quad (17)$$

where  $V_{\text{tot}}$  is the total physical volume,  $V_s$  is the symmetric vacuum portion, and  $V_b^{(R)}$  denotes the broken vacuum volume associated with the fixed radius  $R$  component of the subcritical population. The terms on the right-hand side describe shrinking in radius space, thermal production in the remaining symmetric volume, reverse subcritical nucleation, and erasure by thermal noise. The term  $3Hn$  accounts for the dilution caused by cosmic expansion. This contribution is absent in the original kinetic treatment, but it is required here because we follow the subcritical bubble population along a cosmological cooling history.

We next specify the rates in Eq. (17). For the  $s \rightarrow b$  source, we adopt a Gaussian ansatz for the field profile,

$$\phi_b^{\text{sub}}(r, T) = \phi_b(T) \exp[-r^2/R^2].$$

Substituting this ansatz into the potential gives the free energy

$$F_b(R, T) = A_\nabla(T)R + B_b(T)R^3, \quad (18)$$

where

$$A_\nabla = \frac{3\pi^{3/2}}{4\sqrt{2}}\phi_b^2, \quad (19)$$

and

$$B_b = \pi^{3/2} \left[ \frac{D(T^2 - T_0^2)\phi_b^2}{2^{3/2}} - \frac{ET\phi_b^3}{3^{3/2}} + \frac{\lambda\phi_b^4}{4^{5/2}} \right]. \quad (20)$$

Then the source rate can be written as

$$\Gamma_{s \rightarrow b}^{\text{sub}}(R, T) = w_b(T) A_{\text{sc}} T^4 \exp[-F_b(R, T)/T] \times \Theta(R - \xi_s) \Theta(R_c - R). \quad (21)$$

The two step functions in Eq. (21) specify the radius interval included in the subcritical bubble distribution  $n(R, t)$ . The lower cutoff at  $\xi_s$  should be understood as a coarse-graining prescription rather than a sharp physical boundary: we use it to separate the bubble profiles represented by  $n(R, t)$  from the background thermal fluctuations. In practice, profiles with  $R \lesssim \xi_s$  are absorbed into the thermal background, while profiles with  $R \gtrsim \xi_s$  are evolved through  $n(R, t)$ . In the calculation we take  $R_{\min} = \xi_s$ , with  $\xi_s(T) = [V''(0, T)]^{-1/2}$  denoting the correlation length in the symmetric vacuum. Appendix B 2 varies this cutoff to estimate the sensitivity of the subcritical abundance. The numerical values shift, but the qualitative trend discussed below remains unchanged. We also require  $R < R_c$ , where  $R_c = \sqrt{-A_\nabla/(3B_b)}$  for  $B_b < 0$ , and  $R_c = \infty$  otherwise. Because a thermal fluctuation is characterized by both its size  $R$  and its amplitude, fixing the central field value as  $\phi_b(T)$  in the Gaussian ansatz effectively treats each counted fluctuation of radius  $R$  as a broken-phase patch. This would overestimate the source near  $T_1$ , where the broken-side interval  $\phi_b - \phi_{\text{top}}$  only just opened and remains small. We therefore include a phenomenological factor to account for the fraction of fluctuations that actually cross the barrier

$$w_b(T) = \left[ \frac{\phi_b(T) - \phi_{\text{top}}(T)}{\phi_b(T)} \right]^2 \Theta(T_1 - T). \quad (22)$$

This phenomenological factor estimates the fraction of amplitude fluctuations that cross the barrier, makes the source vanish smoothly as the broken side interval closes at  $T_1$ , and approaches order unity once a sizable broken side region is available.

The reverse process is described by a symmetric fluctuation inside a subcritical bubble. With the profile  $\phi_s^{\text{sub}}(r, T) = \phi_b(T)[1 - \exp(-r^2/R^2)]$ , its free energy is written as

$$F_s(R, T) = A_\nabla(T)R + B_s(T)R^3, \quad (23)$$

where

$$B_s \simeq \pi^{3/2} [-1.646D(T^2 - T_0^2)\phi_b^2 + 2.132ET\phi_b^3 - 0.631\lambda\phi_b^4]. \quad (24)$$

The corresponding reverse rate is also restricted to subcritical symmetric-phase fluctuations with  $R > \xi_b$ ,

$$\Gamma_{b \rightarrow s}^{\text{sub}}(R, T) = A_{\text{sc}} T^4 e^{-F_s(R, T)/T} \Theta(R - \xi_b) \Theta(R_{c,s} - R), \quad (25)$$

where  $\xi_b = [V''(\phi_b, T)]^{-1/2}$  is the correlation length in the broken phase. The upper cutoff is  $R_{c,s} = \sqrt{-A_\nabla/(3B_s)}$  when  $B_s < 0$ . If  $B_s \geq 0$ , no finite reverse critical radius is imposed.

Following Ref. [34], we estimate the thermal noise erasure rate as

$$\Gamma_{\text{TN}}^{\text{sub}}(R, T) = \frac{aT}{4\pi R^3/3}. \quad (26)$$

The remaining ingredient in Eq. (17) is the volume factors. In the dilute limit, and in the notation of Ref. [34], the fixed radius volume fraction is approximated by

$$\frac{V_b^{(R)}}{V_{\text{tot}}} \simeq \frac{4\pi R^3}{3} n(R, t). \quad (27)$$

This approximation is reliable only at the early stage of subcritical bubble production. At later times it can fail because collisions, mergers, and other nonlinear processes invalidate the dilute approximation.

With the definition of  $n(R, t)$ , the dilute broken-phase volume fraction carried by the subcritical bubbles is

$$f_b^{\text{sub}}(T) = \int dR \frac{4\pi R^3}{3} n(R, t). \quad (28)$$

This expression sums the geometric volume associated with the radius-dependent number density. It is adequate only when  $f_b^{\text{sub}} \ll 1$ , where overlap among subcritical bubbles can be neglected. When the occupied volume becomes sizable, however, the linear expression does not account for random overlap and can no longer be interpreted as a probability. To avoid this limitation while keeping the calculation fast, we use a Poisson empty volume probability. Analogous to the critical bubble probability in Eq. (12), the broken phase volume fraction generated by subcritical bubbles is written as

$$p_b^{\text{sub}}(T) = 1 - e^{-f_b^{\text{sub}}(T)}. \quad (29)$$

The exponential form in Eq. (29) accounts for random overlap among subcritical bubbles. At the early stage, where  $f_b^{\text{sub}} \ll 1$ , one may expand  $p_b^{\text{sub}} = 1 - e^{-f_b^{\text{sub}}} \simeq f_b^{\text{sub}}$ . In this dilute limit,  $f_b^{\text{sub}}$  itself is the broken-phase volume fraction carried by subcritical bubbles. The symmetric vacuum fraction entering the source term is then

$$\frac{V_s}{V_{\text{tot}}} \simeq p_s^{\text{sub}}(T) \equiv 1 - p_b^{\text{sub}}(T) = e^{-f_b(T)}. \quad (30)$$

Combining these ingredients and taking  $dR/dt = -v$  gives the final equation

$$\frac{\partial n}{\partial t} + 3Hn = v \frac{\partial n}{\partial R} + p_s^{\text{sub}} \Gamma_{s \rightarrow b}^{\text{sub}} - aTn - \frac{4\pi R^3}{3} \Gamma_{b \rightarrow s}^{\text{sub}} n. \quad (31)$$

Its solution value at the nucleation temperature is then used to test whether the homogeneous symmetric vacuum background is a self-consistent starting assumption.

### III. SCAN STRATEGY

To determine when subcritical prehistory can modify the standard nucleation description, we perform a parameter scan in the following region:

$$D \in [0.05, 0.35], \quad \lambda \in [0.03, 0.30], \quad r_c \in [0.035, 1.0]. \quad (32)$$

For each point these inputs are mapped to

$$E = \frac{\lambda r_c}{2}, \quad T_0 = v_0 \sqrt{\frac{\lambda}{2D}}, \quad v_0 = 246 \text{ GeV}. \quad (33)$$

We keep only points satisfying  $D > 0$ ,  $\lambda > 0$ ,  $E > 0$ ,  $E \lesssim 0.10$ , and  $E^2/(\lambda D) < 0.5$  to achieve a weak first-order phase transition. For each accepted point, we compute  $S_3(T)/T$  with the CosmoTransitions [42] and determine  $T_n$  from Eq. (10). Only points with a finite nucleation integral crossing, and finite  $S_3(T_n)/T_n$  are retained for the analysis. The evolution range is

$$T_{\text{start}} = 1.2T_c, \quad T_{\text{end}} = \max(0.8T_n, 1.001T_0), \quad (34)$$

Unless stated otherwise, we use  $g_* = 106.75$ ,  $A_{\text{sc}} = 1$ ,  $a = 0.5$ ,  $v = 1$ , and  $v_w = 1$ . Details of the numerical implementation, including the finite difference update scheme and numerical consistency checks, are summarized in Appendixes A and B 1.

### IV. RESULTS AND DISCUSSIONS

The scan described in Sec. III yields 3245 accepted points in the parameter space. For each point, we solve Eq. (31) and evaluate  $p_b^{\text{sub}}(T_n)$ , the subcritical bubble

volume fraction measured at the nucleation temperature. This quantity tests whether the homogeneous symmetric vacuum assumption remains valid. To summarize the scan, we classify the points using three reference thresholds,

$$p_b^{\text{sub}} \in \{10^{-3}, 10^{-2}, 10^{-1}\}. \quad (35)$$

The points with  $p_b^{\text{sub}}(T_n) < 10^{-3}$  are labeled as safe critical; those with  $10^{-3} < p_b^{\text{sub}}(T_n) < 10^{-2}$  are labeled as subcritical correction; those with  $10^{-2} < p_b^{\text{sub}}(T_n) < 10^{-1}$  are labeled as prehistory relevant; and those with  $p_b^{\text{sub}}(T_n) > 10^{-1}$  are labeled as dilute breakdown.

With this definition, the full kinetic setup gives 1787 safe critical points, 160 subcritical correction points, 352 prehistory relevant points, and 946 dilute breakdown points. In the first two classes, the homogeneous assumption remains self-consistent, albeit with possible small corrections. In the third class, the critical bubble will nucleate in a mixed local environment. The fourth class marks the regime where the dilute approximation used for  $V_b^{(R)}/V_{\text{tot}}$  in Eq. (27) is no longer valid. Thus, the large number of dilute approximation breakdown points should not be interpreted as evidence that subcritical bubbles reach percolation. Rather, these points identify a region where the assumption of a homogeneous false-vacuum background may no longer be reliable before standard critical-bubble nucleation. Confirming this possibility requires a treatment that includes overlap, mergers, and nonlinear mixed-background evolution, such as a real-time simulation beyond the present kinetic description.

The representative histories in Fig. 2 show how the above classes arise dynamically. For a safe critical point,  $p_{\text{crit}}$  rises while the subcritical bubble volume fraction remains negligible. For a prehistory relevant point, subcritical bubbles are already present before the temperature reaches  $T_n$ , while the dilute approximation for  $V_b^{(R)}/V_{\text{tot}}$  remains adequate. For a dilute breakdown point, the subcritical bubble volume fraction becomes so large that neither the homogeneous background nor the dilute approximation can be trusted as a complete description. The later part of the  $p_b^{\text{sub}}$  curve should be viewed as an extrapolation of the simplified kinetic equation rather than as a reliable prediction of the physical volume fraction.

A common feature of the three representative histories in Fig. 2 is the sharp turn-on of the subcritical bubble volume fraction at the appearance temperature  $T_1$ . For  $T > T_1$ , the broken phase is absent, and the subcritical source is kinematically forbidden. At  $T_1$ , the broken phase and the potential barrier first appear with  $\phi_b = \phi_{\text{top}}$ , so the amplitude factor  $w_b$  still vanishes. Immediately below  $T_1$ , however, the interval  $\phi_b - \phi_{\text{top}}$  opens and the source becomes nonzero. Because the microscopic formation and erasure timescales are much shorter than the Hubble cooling time, the subcritical bubble volume fraction rapidly

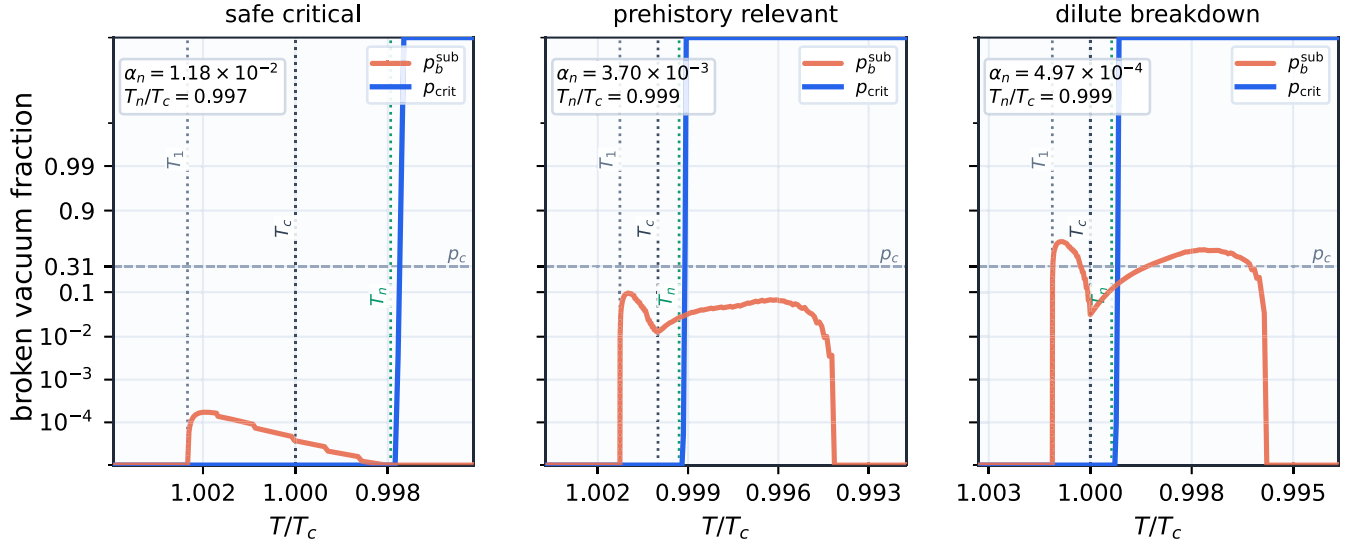


FIG. 2. The pretransition histories for different benchmark points. The orange curve gives the subcritical broken vacuum volume fraction  $p_b^{\text{sub}} = 1 - e^{-f_b^{\text{sub}}}$ , while the blue curve gives the critical bubble probability  $p_{\text{crit}}$ .

relaxes to its instantaneous kinetic balance. This produces the steep initial rise of  $p_b^{\text{sub}}$  seen in all three panels.

The related physical factors of the subcritical bubble volume fraction are shown in Fig. 3. Each accepted scan point is colored by  $p_b^{\text{sub}}(T_n)$  so the gradual growth of the subcritical bubble volume fraction is visible. We use

$$\Delta V_{\text{ft}}(T_n) \equiv V(\phi_s, T_n) - V(\phi_b, T_n), \quad (36)$$

for the finite temperature free energy splitting between the two phases, and

$$\Delta V_{\text{bar}}(T_n) \equiv V(\phi_{\text{top}}, T_n) - V(\phi_s, T_n), \quad (37)$$

for the barrier height above the false vacuum. The broken phase field value at nucleation is denoted by  $\phi_n \equiv \phi_b(T_n)$ .

Figure 3 shows that the subcritical bubble prehistory is controlled by the overall structure of the finite-temperature potential. Panel (a) shows that large subcritical bubble volume fractions arise when the two phases remain nearly degenerate at  $T_n$ , for which the free energy cost of occupying the broken side is small. Panel (b) presents the corresponding cooling history, where the largest fractions are concentrated near  $T_n/T_c \simeq 1$ .

Panels (c) and (d) further identify the other ingredients that make this behavior possible. A smaller  $\phi_n/T_n$  indicates that the broken phase lies closer to the symmetric point at  $T_n$ , reducing both the field excursion and the gradient energy cost of forming a subcritical bubble. A smaller  $\Delta V_{\text{bar}}(T_n)/T_n^4$  weakens the Boltzmann suppression in the formation rate. In each panel, however, the plotted quantity provides only a partial ordering of the final subcritical bubble volume fraction. No single quantity fully determines  $p_b^{\text{sub}}(T_n)$ . Instead, the formation and survival of

subcritical bubbles are governed by the combined effects of the finite temperature potential, the cooling history, and the microscopic dynamics.

These trends motivate a fast criterion, which can be obtained by reducing the kinetic equation to a single representative bin equation. Starting from Eq. (31), we first integrate over the bubble radius. The total number density of subcritical bubbles is defined as

$$\mu(t) = \int_0^\infty dR n(R, t). \quad (38)$$

This gives

$$\frac{d\mu}{dt} = \int_0^\infty dR v \frac{\partial n}{\partial R} + S_{\text{tot}}(T) - \int_0^\infty dR K(R, T) n(R, t), \quad (39)$$

where

$$S_{\text{tot}}(T) = \int_0^\infty dR p_s^{\text{sub}}(T) \Gamma_{s \rightarrow b}^{\text{sub}}(R, T), \quad (40)$$

and

$$K(R, T) = \frac{4\pi R^3}{3} [\Gamma_{b \rightarrow s}^{\text{sub}}(R, T) + \Gamma_{\text{TN}}^{\text{sub}}(R, T)] + 3H. \quad (41)$$

The first term on the right-hand side of Eq. (39) is a boundary term in radius space. For a constant shrinkage speed,

$$\int_0^\infty dR v \frac{\partial n}{\partial R} = v[n(\infty, t) - n(0, t)], \quad (42)$$

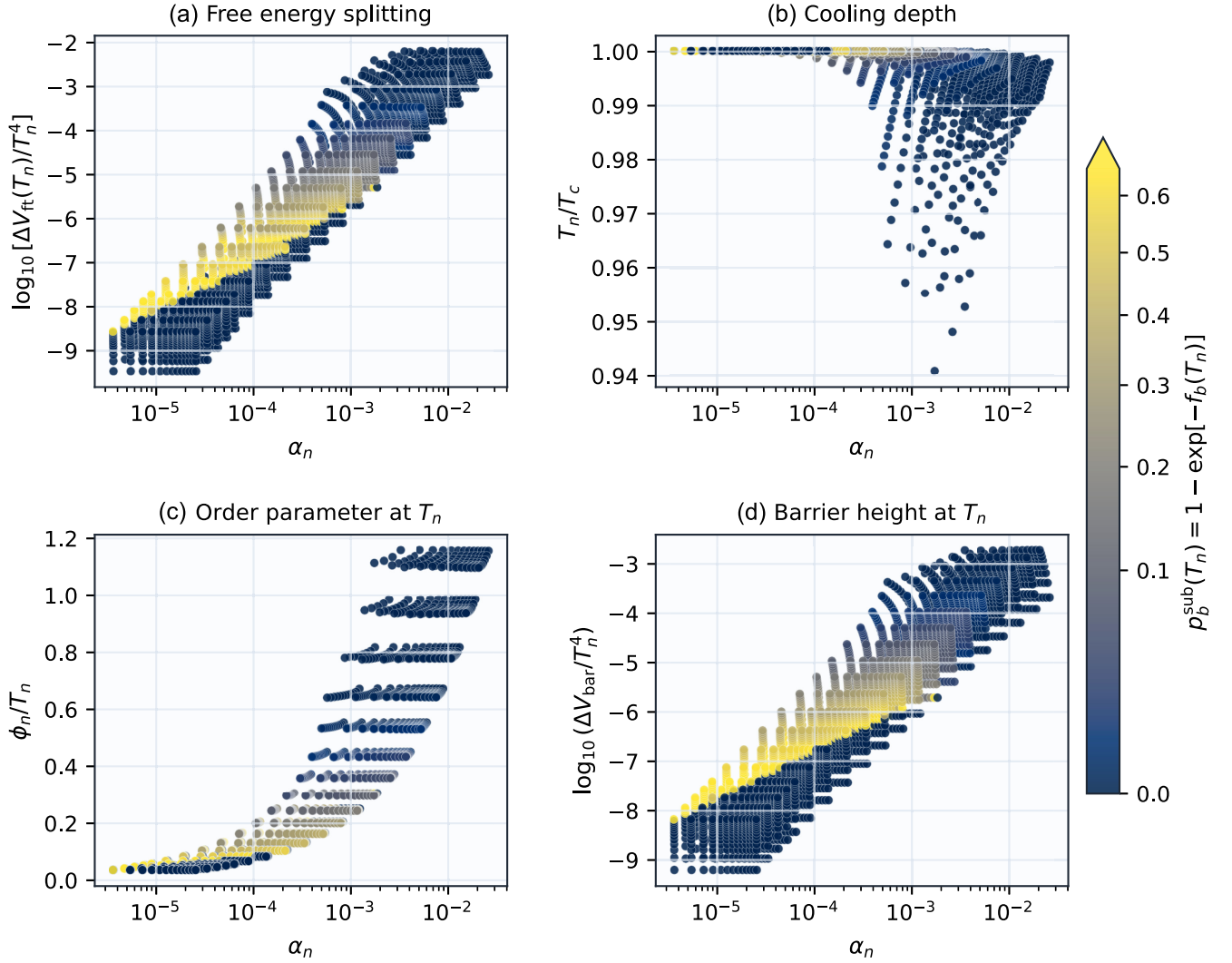


FIG. 3. The relationship between transition properties and the subcritical broken vacuum volume fraction. Each point represents one accepted scan point and is colored by  $p_b^{\text{sub}}(T_n)$ . Panel (a) compares the free energy splitting  $\Delta V_{\text{ft}}(T_n)$ , normalized by  $T_n^4$ , with  $\alpha_n$ . Panel (b) shows the cooling depth  $T_n/T_c$ . Panel (c) shows the order parameter value at nucleation,  $\phi_n/T_n$  with  $\phi_n = \phi_b(T_n)$ . Panel (d) compares the dimensionless barrier height  $\Delta V_{\text{bar}}(T_n)/T_n^4$  with  $\alpha_n$ .

which vanishes for the boundary conditions relevant to the distribution of subcritical bubble.

The remaining radius integral can be approximated by evaluating the integrand at a representative radius  $\xi \sim \xi_s = [V''(0, T)]^{-1/2}$ . In this single-bin approximation,

$$\int_0^\infty dR K(R, T) n(R, t) \simeq K(\xi_s, T) \mu(t), \quad (43)$$

and

$$\begin{aligned} S_{\text{tot}}(T) &\simeq p_s^{\text{sub}}(T) S^*(\xi_s, T) \\ &= p_s^{\text{sub}}(T) \Gamma_{s \rightarrow b}^{\text{sub}}(\xi_s, T) \Delta R_\xi(\xi_s, T), \end{aligned} \quad (44)$$

where

$$\Delta R_\xi(\xi, T) \equiv \frac{T}{F'_b(\xi, T)} \left[ 1 - \exp \left( -\frac{F'_b(\xi, T)}{T} (R_c - \xi) \right) \right] \quad (45)$$

The integrated kinetic equation therefore reduces to

$$\frac{d\mu}{dt} = S_{\text{tot}}(T) - K(\xi_s, T) \mu. \quad (46)$$

If the microscopic relaxation time is shorter than the Hubble cooling time, the system rapidly reaches a quasistationary state with  $d\mu/dt \simeq 0$ . Eq. (46) then gives

$$\mu_{\text{qs}}(T) \simeq \frac{S_{\text{tot}}(\xi_s, T)}{K(\xi_s, T)}. \quad (47)$$

At the early stage of subcritical bubble production, the broken phase volume fraction is negligible, and hence

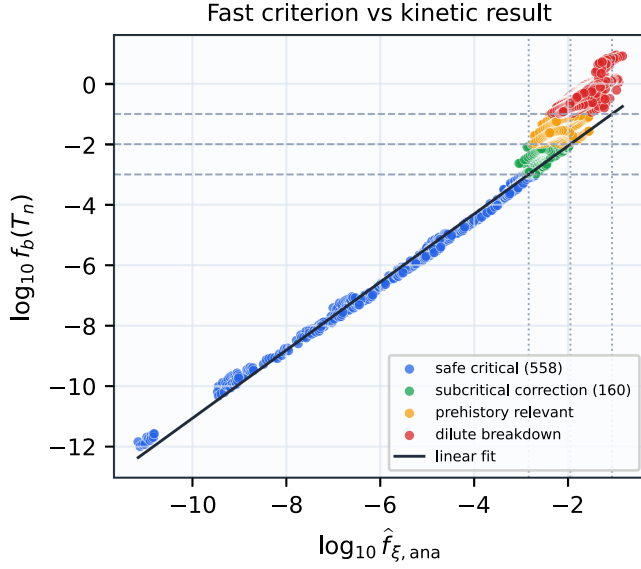


FIG. 4. Comparison between the fast criterion and the full kinetic result for the fitting sample. The displayed points have finite  $\hat{f}_\xi(T_n)$  and  $f_b(T_n) > 10^{-12}$ . The dashed lines show the  $10^{-3}$ ,  $10^{-2}$ , and  $10^{-1}$  volume fraction thresholds used in Eq. (35). The numbers in the legend give the selected sample size in each class.

$p_s^{\text{sub}} \simeq 1$ . Therefore,

$$\mu_{\text{qs}}(T) \simeq \frac{S^*(\xi_s, T)}{K(\xi_s, T)}. \quad (48)$$

Our goal is to estimate the volume occupied by subcritical bubbles,

$$f_b^{\text{sub}}(T) \equiv \frac{V_b}{V_{\text{tot}}} \simeq \int_0^\infty dR \frac{4\pi R^3}{3} n(R, t), \quad (49)$$

which provides the necessary input for estimating  $p_b^{\text{sub}}$ . Applying the same single-bin approximation gives the following:

$$f_b^{\text{sub}}(T) \approx \hat{f}_\xi(T) \simeq \frac{4\pi \xi_s^3}{3} \frac{S^*(\xi_s, T)}{K(\xi_s, T)}. \quad (50)$$

This fast estimator contains the same ingredients that control the full scan: the volume of a correlation-sized subcritical bubble, the Boltzmann suppression from the free energy, and the lifetime set by erasure.

To quantify how the theoretical estimate differs from the full numerical result, we fit the relation between  $\hat{f}_\xi(T_n)$  and the kinetic result in the logarithmic space. Since the fast criterion is based on the dilute approximation, only points with  $p_b^{\text{sub}}(T_n) < 10^{-2}$  are included in the fit. For the comparison shown in Fig. 4, we also require a finite  $\hat{f}_\xi(T_n)$  and a nonzero kinetic volume fraction,  $f_b(T_n) > 10^{-12}$ .

This avoids having the logarithmic fit dominated by near-zero numerical values that already lie in the safe region.

As shown in Fig. 4, the fast criterion tracks the full kinetic result over this selected sample. A fit gives

$$\log_{10} f_b^{\text{sub}}(T_n) \simeq 0.20 + 1.13 \log_{10} \hat{f}_\xi(T_n), \quad (51)$$

with Pearson correlation  $r \simeq 0.995$ . The percent level boundary  $f_b^{\text{sub}}(T_n) = 10^{-2}$  corresponds to

$$\log_{10} \hat{f}_\xi(T_n) \simeq -1.95. \quad (52)$$

Therefore  $\log_{10} \hat{f}_\xi(T_n) \gtrsim -1.95$  is a practical indication that the homogeneous bounce calculation should be accompanied by a subcritical prehistory calculation. Far below this boundary, the homogeneous description is self-consistent. Near or above it, the kinetic equation should be solved. Points close to a class boundary can move when the source normalization or erasure model is varied, as shown in Appendix B 2. Thus the numerical threshold in Eq. (52) should not be interpreted as a universal boundary. It is a computationally inexpensive decision rule calibrated within the present kinetic prescription.

For a more accurate fast estimate, one may use two- or even multibin approximation. In these approaches, the subcritical radius range is divided into several bins. The abundance in each bin evolves separately. The formation and erasure rates are evaluated bin by bin, while shrinking is treated as a flux from larger-radius bins to smaller-radius bins.

Finally, subcritical bubble may also have indirect observational consequences. For electroweak baryogenesis, the impact is likely limited in the present scan, because sizable subcritical populations mainly occur in weak transitions  $\phi_n/T_n < 1$ , while washout avoidance usually requires a stronger transition. The situation can be different for filtered dark matter and PBH. In filtered dark matter scenarios, the relic abundance can depend on how particles encounter, transmit through, or become trapped by phase boundaries [23]. A preexisting subcritical bubble can therefore change the early exposure of the plasma to broken vacuum regions by introducing additional stochastic phase boundaries. Particles reflected by an expanding critical wall may subsequently scatter from nearby subcritical bubbles, be redirected back toward the wall, or become temporarily confined between multiple interfaces. This modifies the effective filtering history and may enhance the final dark matter abundance.

Since PBH production is controlled by rare regions that stay in the symmetric phase for an unusually long time, a preexisting subcritical bubble can disturb these would-be delayed patches by adding small broken vacuum regions inside them. These fluctuations may trigger earlier local conversion, prevent delayed regions from remaining fully in the symmetric phase, and change the probability of very

late completion. As a result, the abundance, typical size, and clustering of PBH producing regions may be modified.

For gravitational waves, subcritical bubbles are most prominent in the small  $\alpha_n$  region, where the released vacuum energy is modest and the signal is expected to be dominated by sound waves rather than bubble collisions. Since the subcritical population turns on over a short interval before  $T_n$ , critical bubbles can be viewed as nucleating in a fluctuation background. These fluctuations introduce length scales smaller than the usual mean bubble separation, set by the correlation length and by the shrinking and erasure times. The main sound wave peak may therefore remain controlled by the standard hydrodynamics, while the high frequency part of the spectrum may receive corrections from the smaller subcritical scales [38].

The present analysis should be viewed as a fast consistency check rather than a complete real-time simulation. It uses a simplified benchmark potential, a Gaussian ansatz, phenomenological formation and erasure rates, and a dilute treatment of  $V_b^{(R)}/V_{\text{tot}}$ . In realistic particle model scans, such as the real singlet model or two Higgs doublet models, the same logic can be applied with the full finite temperature effective potential, while points near dilute breakdown require a more precise evolution beyond the birth-death equation.

## V. CONCLUSION

We studied subcritical bubble prehistory as a consistency test of the homogeneous symmetric vacuum background assumed in standard nucleation picture. By evolving a birth-death equation for the subcritical radius distribution along a cosmological cooling history, we quantified the broken vacuum volume fraction already present at the usual nucleation temperature.

Our parameter scan shows that the homogeneous nucleation background is most vulnerable when the symmetric and broken vacua remain nearly degenerate at  $T_n$ , the barrier is low on the thermal scale, and the free energy of a correlation-sized broken patch is small enough to overcome erasure. The largest subcritical bubble volume fractions are concentrated at small transition strengths, typically  $\alpha_n \lesssim 10^{-3}$ , and modest order parameter values, while the intermediate prehistory relevant band extends to larger  $\phi_n/T_n$  and  $\alpha_n$ . The classification is therefore set by the combined free energy landscape and cooling history rather than by a single number.

The same physical ingredients are captured by the fast estimator  $\hat{f}_\xi(T_n)$ , defined in Eq. (50) as a single-bin estimate of the geometric volume factor  $f_b^{\text{sub}}(T_n)$ . The percent level boundary in the scan corresponds to  $\log_{10} \hat{f}_\xi(T_n) \simeq -1.95$ . Values far below this scale indicate that the homogeneous bounce calculation is self-consistent, while values near or above it identify points for which a subcritical prehistory analysis is needed. Finally, the mixed

background induced by subcritical bubble may affect inhomogeneous nucleation, filtered dark matter, PBH, or the high frequency part of the gravitational wave spectrum.

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## DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

## APPENDIX A: IMPLICIT UPDATE OF THE SUBCRITICAL DISTRIBUTION

The kinetic equation is stiff whenever the microscopic erasure time is shorter than the Hubble cooling time. We therefore use an implicit upwind update in radius space. For the shrink flow  $dR/dt = -v$ , the discretized equation is

$$\frac{n_i^{m+1} - n_i^m}{\Delta t} = v \frac{n_{i+1}^{m+1} - n_i^{m+1}}{\Delta R_i} + S_i^{m+1} - K_i^{m+1} n_i^{m+1}, \quad (\text{A1})$$

where  $S_i = p_s^{\text{sub}}(T) \Gamma_{s \rightarrow b}^{\text{sub}}(R_i, T)$ . In our kinetic setup  $K_i = 3H + aT + (4\pi R_i^3/3) \Gamma_{b \rightarrow s}^{\text{sub}}(R_i, T)$ .

For a sufficiently small time step, we treat  $S_i$  and  $K_i$  as constant over the interval  $[t, t + \Delta t]$ . The resulting equation,

$$\frac{dn_i}{dt} = S_i - K_i n_i, \quad (\text{A2})$$

can then be integrated analytically, giving

$$n_i(t + \Delta t) = n_i(t) e^{-K_i \Delta t} + \frac{S_i}{K_i} (1 - e^{-K_i \Delta t}). \quad (\text{A3})$$

The empty volume factor is solved implicitly rather than taken from the previous step. Since the update is linear in  $S_i$ , we first compute the no source solution  $n_i^{(0)}$  and the full source solution  $n_i^{(1)}$ , and then solve

$$p_s^{\text{sub}} = \exp[-f_b[n^{(0)} + p_s^{\text{sub}}(n^{(1)} - n^{(0)})]. \quad (\text{A4})$$

by bisection. This implicit Poisson step avoids spurious binary feedback when the subcritical population relaxes

much faster than the temperature changes. The time step follows from  $dt = -dT/(HT)$ .

## APPENDIX B: NUMERICAL VALIDATION

### 1. Grid convergence

Figure 5 summarizes two checks on the solver. For the benchmark point  $(D, \lambda, r_c) = (0.35, 0.21, 0.40)$ , varying the cooling grid between 120 and 320 points leaves  $f_b(T_n)$  unchanged at the precision shown. The radius grid is more demanding because the direct volume integral weights the radius distribution by  $R^3$ : the coarsest radius grid shifts the result at the ten percent level, while the finer grids reduce the change to a few percent.

### 2. Robustness of kinetic assumptions

The baseline scan uses  $a = 0.5$ ,  $A_{sc} = 1$ ,  $v = 1$ , the reverse erasure cut  $R_{rev} = \xi_b$ , and the source coarse-graining cut  $R_{min} = \xi_s$ . Figure 6 tests the robustness of these choices on a 65-point validation sample, selected from  $\alpha_n$  quantiles and from points close to the classification thresholds. For each variation, we recompute the action curve and  $T_n$  before evolving the subcritical kinetic equation. Since  $v$  is a physical shrinkage speed, we vary it only within the causal range  $0 < v \leq 1$ , using  $v = 0.5$  and  $v = 0.8$ .

The resulting shifts follow the expected physical trends. Increasing  $A_{sc}$ , or weakening the erasure channels, increases the subcritical fraction, while faster shrinkage within the causal range decreases it. We also vary the reverse erasure cut between  $0.7\xi_b$  and  $1.3\xi_b$ , and the source coarse-graining cut between  $0.7\xi_s$  and  $1.3\xi_s$ . The latter variation directly probes the ambiguity in separating subcritical bubbles from background thermal fluctuations. Lowering the cutoff admits more correlation-scale fluctuations and therefore increases the dilute breakdown fraction, whereas raising the cutoff moves more points toward the safe critical and subcritical correction classes.

The classification should therefore be used as a decision band. Points deep in the safe critical or dilute breakdown

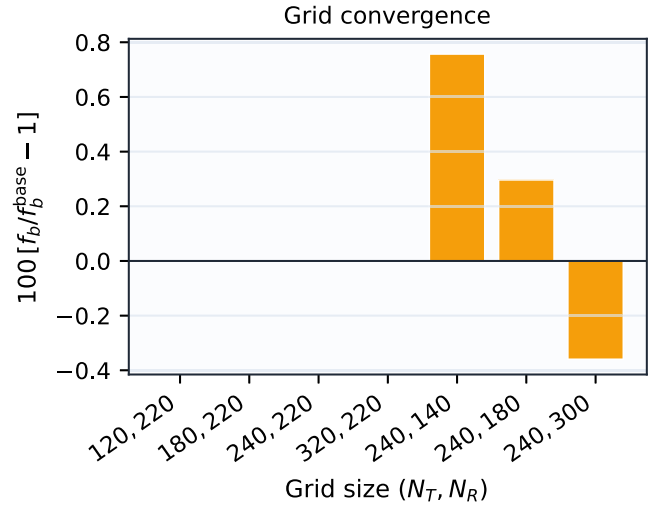


FIG. 5. Numerical checks about grid dependence of  $f_b(T_n)$  for a prehistory relevant benchmark.

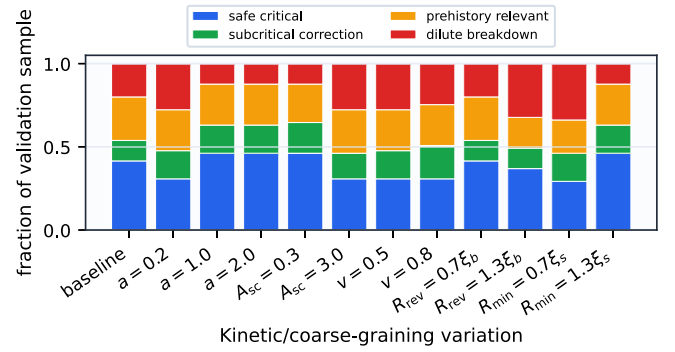


FIG. 6. Class fractions under variations of thermal erasure, source normalization, shrink speed, the reverse erasure radius cut, and the source coarse-graining cut.

regions are stable under these variations, whereas points close to  $10^{-2}$  or  $10^{-1}$  should be recomputed with more refined frameworks.

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