

# Log soft constraints on the Kosower-Maybee-O'Connell formalism

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The Kosower-Maybee-O'Connell (KMOC) formalism provides a systematic framework for extracting classical observables perturbatively from on-shell scattering amplitudes. In this work, we use this formalism to analyze electromagnetic infrared observables in four dimensions, focusing in particular on the linear memory and its tail contributions. Building on previous analysis of such observables using soft-photon theorems, we study the constraints implied by the logarithmic soft photon theorem on the corresponding soft radiation kernel. We show that demanding the expected nonperturbative properties of memory and tail effects imposes a nontrivial set of consistency conditions on the underlying S-matrix. We interpret these constraints as necessary consistency conditions, motivated by macroscopic causality, on the classical limit of the S-matrix as probed by inclusive observables.

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## I. INTRODUCTION

The recent advances in the understanding and computing scattering amplitudes using on-shell methods have also had rather striking ramifications on our ability to compute classical observables in electromagnetic and gravitational scattering in the post-Minkowskian (PM) approximation. The building blocks for an amplitude, namely the loop integrands, turn into building blocks for classical observables. A carefully defined classical limit, at the level of loop integrand, ensures that only a subset of basis integrands survive and hence the increasing sophistication in computing the S-matrix directly aids us in computing classical scattering observables [1–38]. In the gravitational theory, amplitude-based techniques can also be employed to analyze the bound orbits in post-Newtonian (PN) approximation [39–44].

Although there are several avenues by which the classical limit of the S matrix can be employed to analyze classical scattering, in this paper, we will focus on one such formalism proposed by Kosower, Maybee, and O'Connell (KMOC). The KMOC formalism is a rather remarkable development in the world of quantum observables, as it defines classical observables as “inclusive observables” obtained by using the so-called in-in formalism on shell.

More in detail, KMOC formulates the computation of classical observables first and foremost as a quantum observable, where for a fixed set of incoming coherent states, one computes the expectation value of any observable in the far future, irrespective of the outgoing state. Classical limit is defined in the large impact parameter approximation, where it can be shown that all the massless momenta become soft as they scale with  $\hbar$  [17–20,23,45–50].

In a nutshell, KMOC formalism provides us with a map from the space of scattering amplitudes, quantum observables to classical observables such as radiative flux at  $\mathcal{I}^+$ . The on-shell amplitudes thus replace equations of motion of the classical theory and lead to a catalog of classical observables that can be computed for large impact parameter scattering involving the initial state of two particles. Thus, in the KMOC formalism, classical observables are obtained by taking the  $\hbar \rightarrow 0$  limit of inclusive on-shell observables in the quantum theory, and as a result, we obtain infrared finite quantities regardless of the fact that the perturbative S-matrix is ill-defined due to infrared catastrophe.

Classical theory is macrocausal, and at higher orders in perturbation theory, dissipative effects on scattering objects, as well as radiation reaction effects, need to be incorporated in computing the observables. The on-shell approach naturally incorporates both of these effects thanks to unitarity and locality of the S-matrix. However, classical observables satisfy macroscopic causality, and it is an interesting question to ask, in what sense does this macroscopic causality emerge from the locality of the quantum S-matrix. Thus, we view the macroscopic causality constraints as the necessary conditions that should be met by

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S-matrix in the classical limit for the propagation to be retarded in the classical limit. This notion of macroscopic causality was used in [51] to relate the soft limits of amplitudes to the classical waveform.

The requirement that classical scattering satisfies macroscopic causality is independent of the perturbative expansion and is encapsulated in the fact that in classical theory, propagation happens in the closure of the future light cone. As the on-shell approach is necessarily perturbative in nature, one may wonder as to how the nonperturbative constraint on classical observables (satisfaction of macroscopic causality) interfaces with the derivation of these observables via the quantum S matrix (which satisfies microcausality). Our contention in this paper is that one approach to analyzing this question is to demand that radiative observables obtained in the KMOC formalism satisfy classical soft theorems to all orders in the perturbative expansion.

More in detail, in the case of electromagnetic scattering, classical causality and Poincaré invariance imply that late-time electromagnetic radiation behaves as follows:

$$A_\mu = A_\mu^{(0)} + \frac{1}{u} A_\mu^{(\text{ln})} + \dots \quad (1)$$

where the leading-order term  $A_\mu^{(0)}$  in this expansion is known as electromagnetic memory, and the subleading term  $A_\mu^{(\text{ln})}$  is known as the log soft factor [51,52]. The actual form of both the memory and the log soft factor is independent of the details of the classical scattering, and both of these observables are completely fixed by the asymptotic momenta and masses of the scattering objects. Clearly, these properties imply that  $A_\mu^{(0)}, A_\mu^{(\text{ln})}$  are completely fixed by macroscopic causality and Poincaré invariance. Hence, demanding that the classical limit of the radiative field in the KMOC formalism equal the universal tails to all orders in perturbative expansion can reveal to us how macroscopic causality arises in on-shell approaches.

In this paper, we initiate an inquiry into this question by subjecting the KMOC formalism to the following criteria: Under what conditions does the classical limit of on-shell amplitude reproduce the electromagnetic memory and the classical log soft photon factor to all orders in perturbative expansion. The first of these constraints was already analyzed in [53], where it was shown that certain moments of the perturbative amplitudes must take a specific form in the classical limit so that electromagnetic memory is reproduced nonperturbatively via on-shell methods. In this paper, we first prove that the local and unitary S-matrix of QED is sufficient to ensure that constraints formulated in [53] are satisfied. As a result, the macroscopic causality probed by the existence of the memory observable is, in fact, guaranteed by the microcausality of the S-matrix. However, we then derive the constraints that

arise from the universality of the log soft factor. And here, we discover something surprising. Namely, that if on-shell methods are to reproduce the universal log soft factor, then the perturbative amplitudes have to satisfy a set of constraints at all orders in the loop expansion. These constraints are summarized in (60). Following [53], we refer to these constraints as subleading soft constraints on the KMOC formalism, and as such, they may be viewed as causality-motivated constraints on the classical limit of the S-matrix.

This paper is organized as follows: in Sec. II we briefly review the KMOC formalism and the aspects of soft theorems which we have used in this paper. Section III describes how the classical limit and the soft limit have been taken to define the soft radiation kernel. In Sec. IV we derive the constraints on the leading soft radiation kernel due to the memory term, to all orders in perturbation. Sections V and VI describe how the vanishing of the quantum log soft factor puts constraints on the subleading soft radiation kernel as well as on the S-matrix. Finally, in Sec. VII we verify these subleading constraints at one loop.

We now summarize the notations used for various quantities throughout the paper.

(i) We define the following measures:

$$\begin{aligned} d\Phi(p) &= \hat{d}^4 p \Theta(p^0) \hat{\delta}(p^2 - m^2), \\ d\mu(q_i) &= \hat{d}^4 q_i \hat{\delta}(2p_i \cdot q_i + q_i^2), \\ d\mu_q &= \hat{d}^4 q \hat{\delta}(p_1 \cdot q) \hat{\delta}(p_2 \cdot q), \end{aligned} \quad (2)$$

where  $\hat{d}^4 q = d^4 q / (2\pi)^4$  and  $\hat{\delta} = \delta / 2\pi$ . Moreover,  $d\mu(\bar{q}_i)$  is defined for scaled  $q_i = \hbar \bar{q}_i$

$$d\mu(\bar{q}_i) = \hat{d}^4 \bar{q}_i \hat{\delta}(2p_i \cdot \bar{q}_i + \hbar \bar{q}_i^2). \quad (3)$$

(ii) We write  $L$ -loop amplitude compactly as

$$\mathcal{A}_{(L)}(P|P') = \mathcal{A}_{(L)}(p_1, p_2 \rightarrow p'_1, p'_2). \quad (4)$$

(iii) We denote  $\tilde{\mathcal{A}}_{(L)}$  as the amplitude with its coupling stripped off and the most negative power of  $\hbar$  factored out, which comes from the ladder diagrams.

## II. A BRIEF REVIEW ON KMOC FORMALISM AND SOFT THEOREMS

### A. KMOC

In the KMOC formalism [17], classical observables are calculated by taking the classical limit of inclusive on-shell quantum amplitudes. The change in an observable  $\hat{O}$  in quantum theory is defined as

$$\Delta O = {}_{\text{in}}\langle\Psi|S^\dagger\hat{O}S|\Psi\rangle_{\text{in-in}}\langle\Psi|\hat{O}|\Psi\rangle_{\text{in}}. \quad (5)$$

Within this framework, we compute the observables for  $2 \rightarrow 2$  scattering in the large impact parameter regime. The particles are in the initial state  $|\Psi\rangle_{\text{in}}$ , and they evolve to a final state  $|\Psi\rangle_{\text{out}}$  by the action of the  $S$  matrix. In a frame in which the second particle is at the origin,  $|\Psi\rangle_{\text{in}}$  is defined as

$$|\Psi\rangle_{\text{in}} = \int d\Phi(p_1, p_2) \phi_1(p_1) \phi_2(p_2) e^{ib \cdot p_1} |p_1, p_2\rangle, \quad (6)$$

where the measure  $d\Phi(p_i)$  imposes on-shell constraints on the momentum eigenstates  $|p_1, p_2\rangle$  and the wave packets  $\phi_i(p_i)$ . These wave packets are chosen with appropriate normalization, and they are peaked around the classical momenta of the particles. The width of the wave packets is regulated by a dimensionless ratio  $\xi = l_c/l_w$ , where  $l_c$  and  $l_w$  are respectively the Compton length and position-space wave packet spread of the particle. Using dimensional analysis, we restore the  $\hbar$  factors in the coupling (we consider scalar QED interaction) and scale the momentum transfer and loop momenta with  $\hbar$ , i.e.,  $l = \hbar \bar{l}$ , where  $\bar{l}$  is the wave number. The  $\hbar$  scaling introduces terms that scale as negative powers of  $\hbar$ , which are the so-called superclassical terms. However, these terms cancel among each other in the classical limit for physical observables. For the physical observable of our interest, i.e., the radiation kernel, the cancellation of superclassical terms to all orders in perturbation was shown in [54] using  $N$ -operator formalism. So, in our calculations, we will ignore these superclassical terms.

The classical limit is taken by imposing the limit  $\hbar \rightarrow 0$ . For massive spinning particles, the spin  $S$  should also be taken  $S \rightarrow \infty$  such that  $S\hbar = \text{constant}$  [19]. We can take  $\hat{O}$  to be the momentum operator  $\mathbb{P}_1^\mu$  in (5), which gives us the linear impulse  $\Delta p_1^\mu$  for the first particle [17]. Using  $|\Psi\rangle_{\text{in}}$  we have the following expression for linear impulse:

$$\begin{aligned} \Delta p_1^\mu = & \int d\mu(q_1, q_2) e^{-ib \cdot q_1/\hbar} \hat{\delta}^4(q_1 + q_2) \left[ i q_1^\mu \mathcal{A}(P|P+Q) \right. \\ & + \sum_X \int \prod_{m=1}^X d\Phi(k_m) d\mu(w_1, w_2) \hat{\delta}^4(w_1 + w_2 + \Sigma_{i=1}^X k_i) \\ & \left. \times w_1^\mu \mathcal{A}^*(P+Q|P+W, \{k_X\}) \mathcal{A}(P|P+W, \{k_X\}) \right]. \end{aligned} \quad (7)$$

We have suppressed the double bracket notation of [17] that denotes the wave packet integration, which in the classical limit, viz.  $\xi \rightarrow 0$  amounts to replacing the external momenta with the classical initial momenta. In this paper, we focus on the radiative gauge field at  $\mathcal{I}^+$ , which is the radiation kernel introduced in [17] for the computation of radiation flux in an inelastic scattering process, and studied

in great detail in [18]. We substitute the expression for gauge field in terms of creation and annihilation operators in (5) and, using (6), we write

$$\begin{aligned} \mathcal{R}(\bar{k}) = & \hbar^{3/2} \int d\mu(q_1, q_2) e^{-ib \cdot q_1/\hbar} \hat{\delta}^4(q_1 + q_2 + k) \\ & \times \left[ i \mathcal{A}(P|P+Q, k) + \sum_X \int \prod_{m=1}^X d\Phi(k_m) d\mu(w_1, w_2) \right. \\ & \times \hat{\delta}^4(w_1 + w_2 + k + \Sigma_{i=1}^X k_i) \\ & \left. \times \mathcal{A}^*(P+Q|P+W, \{k_X\}) \mathcal{A}(P|P+W, k, \{k_X\}) \right]. \end{aligned} \quad (8)$$

The radiation kernel itself is not a physical observable; rather, the radiation flux is what we observe. The flux is the on-shell integral over the photon momentum weighted by the factor  $|\mathcal{R}(k)|^2$ . The overall scaling of the kernel is such that the radiation flux is finite in the classical limit.

## B. Soft theorems

Consider a process where  $M$  particles interact with each other and produce  $N$  particles and an additional photon in the final state. When the photon momentum is taken to be soft compared to the momentum of the hard particles, the amplitude for this process factorizes into a soft factor and a lower-point amplitude. These soft factors capture the radiative effects, and they are defined as the ratio between the  $M+N+1$  point amplitude and the  $M+N$  point amplitude. These factors have universal and nonuniversal theory-dependent terms. In  $D=4$  as was proved in [51], the universal part of the quantum soft photon theorem has the following structure

$$\begin{aligned} \mathcal{A}_{M+N+1}(\{p_I\}, \{p'_I\}, k, \epsilon) \\ = \sum_{n=-1}^{\infty} \omega^n (\ln \omega)^{n+1} \mathcal{S}^{(n)}(\{p_I\}, \{p'_I\}, k, \epsilon) \mathcal{A}_{M+N}(\{p_I\}, \{p'_I\}). \end{aligned} \quad (9)$$

Here, all the momenta are ingoing. The soft factor  $\mathcal{S}^{(n)}$  first appears at  $(n+1)$ th loop and is  $(n+1)$ -loop exact. The  $\omega^m (\ln \omega)^{m+1}$  factor for  $m \leq n$  comes after performing the loop integration in the  $(n+1)$ -loop amplitude. In [52,51], the subleading ( $n=0$ ) and subsubleading ( $n=1$ ) soft factors were shown to be universal and subsequently one-loop and two-loop exact. The subleading soft factor contains a quantum term (10) in addition to the classical term [51], whereas the leading soft factor,  $\mathcal{S}^{(-1)}$  receives no such contribution,

$$\Delta S^{(0)} = -\frac{1}{8\pi^2} \sum_{\substack{a,b \\ b \neq a}} \frac{Q_a^2 Q_b}{p_a \cdot k} \varepsilon_\mu k_\nu (p_a \wedge p_b)^{\mu\nu} \left[ \ln \left( \frac{p_a \cdot p_b + \sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}}{p_a \cdot p_b - \sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}} \right) \frac{m_a^2 m_b^2}{((p_a \cdot p_b)^2 - m_a^2 m_b^2)^{3/2}} - \frac{2p_a \cdot p_b}{(p_a \cdot p_b)^2 - m_a^2 m_b^2} \right]. \quad (10)$$

In [55], the quantum soft factors up to subleading order were shown to be related to the radiative field by defining an appropriate classical limit. More precisely, the soft theorem constrains the Fourier mode of the radiative gauge field at null infinity. We denote this radiative mode by  $\mathcal{A}_{\text{rad}}^\mu(\omega, \hat{k})$ , with  $k^\mu = \omega \hat{k}^\mu$ . Its low-frequency (soft) expansion is

$$\int dt e^{i\omega t} \mathcal{A}_{\text{rad}}^\mu(\vec{x}, t) = -\frac{i}{4\pi R} e^{i\omega R} \left[ \frac{1}{\omega} \mathcal{S}_{\text{classical}}^{(-1),\mu} + \ln \omega \mathcal{S}_{\text{classical}}^{(0),\mu} + O(\omega^0) \right]. \quad (11)$$

up to the choice of Fourier convention. Here,  $k = \omega(1, \hat{\mathbf{x}})$  and  $R = |\mathbf{x}|$  is the distance of the detector. The leading and subleading terms [51] in (11) are

$$\mathcal{S}_{\text{classical}}^{(-1),\mu} = \sum_a Q_a \frac{p_a^\mu}{p_a \cdot \hat{k}}, \quad (12)$$

$$\mathcal{S}_{\text{classical}}^{(0),\mu} = i \sum_a Q_a \frac{k_\nu}{p_a \cdot k} \sum_{\substack{b \neq a \\ \eta_a \eta_b = 1}} \frac{Q_a Q_b}{4\pi} \frac{m_a^2 m_b^2 (p_a \wedge p_b)^{\mu\nu}}{((p_a \cdot p_b)^2 - m_a^2 m_b^2)^{3/2}}. \quad (13)$$

The universal structure of the classical soft factors was conjectured in [51] and was proved successively in [56]. We will use these expressions to evaluate the constraints in later sections.

### III. SOFT LIMIT IN KMOC FORMALISM

KMOC formalism provides a natural framework to study the connection between classical and quantum soft theorems. Computing the gauge field in KMOC involves taking the classical limit and performing the loop integration, taking the classical limit before the soft limit is not equivalent to taking them in the reverse order. The reason for this noncommutativity is the infrared sensitivity of four-dimensional QED. The loop integrations contain regions in which the loop momentum becomes soft, and these regions can generate logarithmic dependence on the soft photon frequency or on an infrared regulator. As a result, the expansion in the emitted photon frequency is not uniform

over the full loop integration domain. Therefore taking the soft limit at the integrand level can differ from first performing the loop integrations and only then expanding in the soft frequency. Since the classical limit in KMOC also involves a definite scaling of the momentum transfers and radiation momentum with  $\hbar$ , the loop integration can mix the soft expansion with the classical expansion. For this reason we treat the “soft-first” limit as a definite prescription rather than assuming commutativity of limits and our results should therefore be interpreted as a matching between the KMOC radiation kernel and the classical logarithmic soft theorem within the soft-first prescription shown in Fig. 1.

The radiation kernel (8) in the KMOC formalism has a five-point inelastic amplitude in it. We take the soft limit of this five-point amplitude at the integrand level. The tree-level soft expansion of an  $(M + N + 1)$ -point amplitude is given by

$$\mathcal{M}_{M+N+1}(\{p_I\}, \{p'_I\}, k, \epsilon) = \sum_{n=-1}^{\infty} \omega^n \left[ \hat{\mathcal{S}}^{(n)}(\{p_I\}, \{p'_I\}, \hat{k}, \epsilon) \mathcal{M}_{M+N}(\{p_I\}, \{p'_I\}) + R_n(\hat{k}, \epsilon) \right], \quad (14)$$

where  $R_n(\hat{k}, \epsilon)$  has the form

$$R_n(\hat{k}, \epsilon) = \epsilon^\mu \hat{k}^{\nu_1} \cdots \hat{k}^{\nu_n} A_{\mu\nu_1 \cdots \nu_n}. \quad (15)$$



FIG. 1. The sequence of classical and soft limits. CSPT designates classical soft photon theorem; QSPT designates quantum soft photon theorem.

Here,  $A_{\mu\nu_1\cdots\nu_n}$ , antisymmetric in  $\mu$  and  $\nu_i$  indices, is a theory-dependent remainder term, which spoils the factorization beyond subleading order. The above soft factors capture the universal part of the soft expansion, and have the form

$$\hat{\mathcal{S}}^{(-1)}(\{p_I\}, \{p'_I\}, k, \epsilon) = \sum_{a=1}^M \mathcal{Q}_a \epsilon_\mu \frac{p_a^\mu}{p_a \cdot \hat{k}} - \sum_{a=1}^N \mathcal{Q}'_a \epsilon_\mu \frac{p'^\mu_a}{p'_a \cdot \hat{k}}, \quad (16)$$

$$\hat{\mathcal{S}}^{(n)}(\{p_I\}, \{p'_I\}, k, \epsilon) = \sum_{a=1}^M \mathcal{Q}_a \epsilon_\mu \hat{k}_\nu \frac{\hat{J}_a^{\mu\nu}}{p_a \cdot \hat{k}} \left( \hat{k} \cdot \frac{\partial}{\partial p_a} \right)^n - \sum_{a=1}^N \mathcal{Q}'_a \epsilon_\mu \hat{k}_\nu \frac{\hat{J}'^{\mu\nu}_a}{p'_a \cdot \hat{k}} \left( \hat{k} \cdot \frac{\partial}{\partial p'_a} \right)^n \quad (\forall n \geq 0). \quad (17)$$

In (14), the amplitudes are unstripped, which take into account the photon momentum in momentum conservation. We use the soft expansion of the five-point amplitude (14) up to subleading order in  $\omega$  in the kernel to write the soft radiation kernel as

$$\mathcal{R}(k) = \left\langle \mathcal{S}^{(-1)}(p_1, p_2, p_1 + q_1, p_2 + q_2, k) + \hat{\mathcal{S}}^{(0)}(p_1, p_2, p_1 + q_1, p_2 + q_2, k) \right\rangle. \quad (18)$$

$\hat{\mathcal{S}}^{(0)}$  is a differential operator that is linear in the angular momenta of external particles. In the interest of brevity, we will henceforth simply use  $\langle \rangle$  to denote the classical limit

$$\begin{aligned} \langle \hat{f} \rangle &= \lim_{\hbar \rightarrow 0} \hbar^{\beta_f} \int d\mu(\bar{q}_1, \bar{q}_2) e^{-ib \cdot \bar{q}_1} \left[ i \hat{f}(\bar{q}_1, \bar{q}_2) \mathcal{M}(P|P + \hbar \bar{Q}) + \sum_X \hbar^{2X} \int \prod_{m=1}^X d\Phi(\bar{k}_m) d\mu(\bar{w}_1, \bar{w}_2) \hat{f}(\bar{w}_1, \bar{w}_2) \right. \\ &\quad \left. \times \mathcal{M}^*(P + \hbar \bar{Q}|P + \hbar \bar{W}, \{\hbar \bar{k}_X\}) \mathcal{M}(P|P + \hbar \bar{W}, \{\hbar \bar{k}_X\}) \right]. \end{aligned} \quad (19)$$

In later sections, we will suppress the  $\hbar \rightarrow 0$  limit and will make it explicit only while specifying certain results. In classical theory, the contribution to the soft expansion (18) of the gauge field comes from the matter poles in the loop integrals. The contributions from photon poles cancel among each other in the perturbative expansion due to the retarded propagators in classical theory [51,57]. Each of the soft factors is built out of a certain combination of transfer momenta and angular momenta. This gives us two classes of constraints: one, due to the vanishing of terms coming from photon poles in the classical limit, and second, due to the fact that the contribution from matter poles matches the classical expression.

#### IV. LEADING SOFT CONSTRAINTS ON $\mathcal{R}(k)$

In [53], it was shown that the leading classical soft photon theorem imposes a hierarchy of constraints on the

conservative sector in KMOC formalism. More in detail, it was argued that the universal form of electromagnetic memory for a fixed set of initial and final configurations of charged particles leads to the following set of constraints on a set of observables

$$\langle q_a^{\mu_1} \cdots q_a^{\mu_N} \rangle = \Delta p_a^{\mu_1} \cdots \Delta p_a^{\mu_N}, \quad (20)$$

where  $\langle q_a^{\mu_1} \cdots q_a^{\mu_N} \rangle$  is defined as in (25). These constraints were verified to next-to-leading order (NLO) in [53]. In this section, we give an explicit proof for (20) to all orders in perturbation theory.

The leading soft factor for  $2 \rightarrow 2$  scattering (18) where the final momenta are given by  $p'_a = p_a + q_a$  can be written as

$$\begin{aligned} \mathcal{S}_a^{(-1)}(p_1, p_2, p_1 + q_1, p_2 + q_2, k) \\ = \mathcal{Q}_a \epsilon_\mu \left[ -\frac{q_a^\mu}{p_a \cdot k} + \sum_{N=1}^{\infty} (-1)^{N+1} \frac{(q_a \cdot k)^N}{(p_a \cdot k)^{N+1}} (p_a^\mu + q_a^\mu) \right]. \end{aligned} \quad (21)$$

We write the leading-order contribution to the kernel for particle  $a$  as

$$\mathcal{R}_a^{(-1)}(k) = \left\langle \mathcal{S}_a^{(-1)}(p_1, p_2, p_1 + \hbar \bar{q}_1, p_2 + \hbar \bar{q}_2, k) \right\rangle. \quad (22)$$

Our goal is to show that the above kernel (22) reduces to the classical LO soft factor, giving us the leading soft constraint

$$\mathcal{R}_a^{(-1)}(k) = \mathcal{S}_a^{(-1)}(p_1, p_2, p_1 + \Delta p_1, p_2 + \Delta p_2, k). \quad (23)$$

To show this, first we prove that for any rational function  $F(q_1, q_2 = -q_1)$ , we will have  $\langle F(q_1, q_2) \rangle = F(\Delta p_1, \Delta p_2)$ . In order to calculate this, we need the expectation value of the  $n$ -th moment of the transfer momentum  $\langle q_a^{\mu_1} \cdots q_a^{\mu_N} \rangle$ . We get the moment of degree 1 i.e.  $\langle q_a^\mu \rangle$  by taking the expectation value of the operator  $S^\dagger [\mathbb{P}_a^\mu, S]$  over the two particle initial state  $|\Psi\rangle_{\text{in}}$ . Similarly, it is straightforward to see that the  $n$ -th moment arises when we take the expectation value of the operator  $S^\dagger [\mathbb{P}_a^{\mu_1}, \dots, [\mathbb{P}_a^{\mu_N}, S] \dots]$ ,

$$\langle q_a^{\mu_1} \cdots q_a^{\mu_N} \rangle = \lim_{\hbar \rightarrow 0} \langle \Psi_{\text{in}} | S^\dagger [\mathbb{P}_a^{\mu_1}, \dots, [\mathbb{P}_a^{\mu_N}, S] \dots] | \Psi_{\text{in}} \rangle. \quad (24)$$

In terms of the phase-space integral, we can write (24) as

$$\begin{aligned} \langle q_a^{\mu_1} \cdots q_a^{\mu_N} \rangle &= \int d\mu(q_1, q_2) \hat{\delta}^4(q_1 + q_2) e^{-iq_1 \cdot b/\hbar} \left[ i q_a^{\mu_1} \cdots q_a^{\mu_N} \mathcal{A}(P|P + Q) \right. \\ &\quad + \sum_X \int \prod_{m=0}^X d\Phi(k_m) d\mu(w_1, w_2) \hat{\delta}^4(w_1 + w_2 + \sum_m k_m) w_a^{\mu_1} \cdots w_a^{\mu_N} \\ &\quad \left. \times \mathcal{A}(P|P + W, \{k_X\}) \mathcal{A}^*(P + Q|P + W, \{k_X\}) \right]. \end{aligned} \quad (25)$$

The above expression reduces to the product of impulses up to an additive quantum factor

$$\langle q_a^{\mu_1} \cdots q_a^{\mu_N} \rangle = \Delta p_a^{\mu_1} \cdots \Delta p_a^{\mu_N} + \lim_{\hbar \rightarrow 0} f_a^{\mu_1 \cdots \mu_N}. \quad (26)$$

We can separate  $f_a^{\mu_1 \cdots \mu_N}$  from the expectation value of the nested commutator in (24) by expanding the commutator brackets and then inserting the over-complete set of coherent states in between them. We then write the coherent states in terms of  $|\Psi_{\text{in}}\rangle$  and states that are orthogonal to it. The expectation values over the in-states separate to give the product of linear impulses, and we obtain the following expression for  $f_a^{\mu_1 \cdots \mu_N}$ :

$$\begin{aligned} f_a^{\mu_1 \cdots \mu_N} &= \sum_{\sigma \in S_N} C_1(\sigma) \Delta p_a^{\mu_{\sigma(1)}} \cdots \Delta p_a^{\mu_{\sigma(N-2)}} \mathfrak{R}_a^{\mu_{\sigma(N-1)} \mu_{\sigma(N)}} \\ &\quad + \sum_{\sigma \in S_N} C_2(\sigma) \Delta p_a^{\mu_{\sigma(1)}} \cdots \Delta p_a^{\mu_{\sigma(N-3)}} \mathfrak{R}_a^{\mu_{\sigma(N-2)} \mu_{\sigma(N-1)} \mu_{\sigma(N)}} \\ &\quad + \sum_{\sigma \in S_N} C_3(\sigma) \Delta p_a^{\mu_{\sigma(1)}} \cdots \Delta p_a^{\mu_{\sigma(N-3)}} \mathfrak{R}_a^{\mu_{\sigma(N-2)} \mu_{\sigma(N-1)} \mu_{\sigma(N)}} + \cdots + \mathfrak{R}_a^{\mu_1 \cdots \mu_N}. \end{aligned} \quad (27)$$

The factors  $\mathfrak{R}_a^{\mu_1 \cdots \mu_k} \ \forall \ k \geq 2$ , are found to be given by

$$\mathfrak{R}_a^{\mu_1 \cdots \mu_k} = \sum_{\alpha_1 \cdots \alpha_{k-1} \neq |\Psi_{\text{in}}\rangle} \langle \Psi_{\text{in}} | S^\dagger \mathbb{P}_a^{\mu_1} S | \alpha_1 \rangle \langle \alpha_1 | S^\dagger \mathbb{P}_a^{\mu_2} S | \alpha_2 \rangle \cdots \langle \alpha_{k-1} | S^\dagger \mathbb{P}_a^{\mu_k} S | \Psi_{\text{in}} \rangle. \quad (28)$$

Our next goal is to show that these are quantum, and vanish in the  $\hbar \rightarrow 0$  limit.

**A.  $\langle q_a^{\mu_1} \dots q_a^{\mu_N} \rangle$  and contact terms**

Let us start with the  $N = 2$  case. For the operator  $S^\dagger[\mathbb{P}_a^{\mu_1}, [\mathbb{P}_a^{\mu_2}, S]]$ , we use the over-complete set of states to write

$$\langle \Psi_{\text{in}} | S^\dagger[\mathbb{P}_a^{\mu_1}, [\mathbb{P}_a^{\mu_2}, S]] | \Psi_{\text{in}} \rangle = \Delta p_a^{\mu_1} \Delta p_a^{\mu_2} + \mathfrak{R}_a^{\mu_1 \mu_2}, \quad (29)$$

where the remainder term is given by

$$\mathfrak{R}_a^{\mu_1 \mu_2} = \sum_{\alpha \neq \Psi_{\text{in}}} \langle \Psi_{\text{in}} | S^\dagger \mathbb{P}_a^{\mu_1} S | \alpha \rangle \langle \alpha | S^\dagger \mathbb{P}_a^{\mu_2} S | \Psi_{\text{in}} \rangle. \quad (30)$$

The state  $|\alpha\rangle$  can be expressed as a sum over the states in which the momenta of the matter particles peak around their classical value, and the states for which the wave packets of the matter particles have negligible overlap with  $\phi_{\text{in}}$ ,

$$|\alpha\rangle = \sum_X \int \prod_{i=1}^2 d\Phi(p_i) \phi_{\text{in}}(p_i) e^{-i \frac{p_i \cdot b}{\hbar}} |p_1 p_2 X\rangle + \sum_Y \int \prod_{i=1}^2 d\Phi(r_i) \psi_i(r_i) e^{-i \frac{r_i \cdot b}{\hbar}} |r_1 r_2 Y\rangle.$$

The above expression with no intermediate photons ( $X = 0$ ) produces the product of impulses in (26). The negligible overlap of  $\psi_i$  and  $\phi_{\text{in}}$  translates to the following no-overlap condition in the classical limit:

$$\lim_{\hbar \rightarrow 0} \int d\Phi(p) \phi_{\text{in}}^*(p) \psi_i(p') = 0. \quad (31)$$

This ensures that these two wave packets are not peaked at the same momentum value, and they do not contribute classically. Now using the expression for  $|\alpha\rangle$  we evaluate (20)

$$\begin{aligned} \mathfrak{R}_a^{\mu_1 \mu_2} &= \sum_{X,Y} \int \prod_{j=1}^2 d\Phi(\tilde{p}_j) d\Phi(\tilde{p}'_j) d\Phi(p_j) d\Phi(p'_j) \phi_{\text{in}}(p_j) \phi_{\text{in}}^*(p'_j) \phi_{\text{in}}^*(\tilde{p}_j) \phi_{\text{in}}(\tilde{p}'_j) \\ &\times e^{-i \frac{(p_1 - p'_1) \cdot b}{\hbar}} \langle \tilde{p}_1 \tilde{p}_2 | S^\dagger \mathbb{P}_a^{\mu_1} S | p_1 p_2 X \rangle \langle p'_1 p'_2 Y | S^\dagger \mathbb{P}_a^{\mu_2} S | \tilde{p}'_1 \tilde{p}'_2 \rangle \\ &+ \sum_{X,Y} \int \prod_{j=1}^2 d\Phi(\tilde{p}_j) d\Phi(\tilde{p}'_j) d\Phi(p_j) d\Phi(r'_j) \phi_{\text{in}}^*(\tilde{p}_j) \phi_{\text{in}}(\tilde{p}'_j) \phi_{\text{in}}(p_j) \psi^*(r'_j) \\ &\times e^{-i \frac{(p_1 - p'_1) \cdot b}{\hbar}} \langle \tilde{p}_1 \tilde{p}_2 | S^\dagger \mathbb{P}_a^{\mu_1} S | p_1 p_2 X \rangle \langle r'_1 r'_2 Y | S^\dagger \mathbb{P}_a^{\mu_2} S | \tilde{p}'_1 \tilde{p}'_2 \rangle \\ &+ \sum_{X,Y} \int \prod_{j=1}^2 d\Phi(\tilde{p}_j) d\Phi(\tilde{p}'_j) d\Phi(p'_j) d\Phi(r_j) \phi_{\text{in}}^*(\tilde{p}_j) \phi_{\text{in}}(\tilde{p}'_j) \phi_{\text{in}}^*(p'_j) \psi(r_j) \\ &\times e^{-i \frac{(p'_1 - r_1) \cdot b}{\hbar}} \langle \tilde{p}_1 \tilde{p}_2 | S^\dagger \mathbb{P}_a^{\mu_1} S | r_1 r_2 Y \rangle \langle p'_1 p'_2 X | S^\dagger \mathbb{P}_a^{\mu_2} S | \tilde{p}'_1 \tilde{p}'_2 \rangle \\ &+ \sum_{X,Y} \int \prod_{j=1}^2 d\Phi(\tilde{p}_j) d\Phi(\tilde{p}'_j) d\Phi(r_j) d\Phi(r'_j) \phi_{\text{in}}^*(\tilde{p}_j) \phi_{\text{in}}(\tilde{p}'_j) \psi(r_j) \psi^*(r'_j) \\ &\times e^{-i \frac{(r_1 - r'_1) \cdot b}{\hbar}} \langle \tilde{p}_1 \tilde{p}_2 | S^\dagger \mathbb{P}_a^{\mu_1} S | r_1 r_2 X \rangle \langle r'_1 r'_2 Y | S^\dagger \mathbb{P}_a^{\mu_2} S | \tilde{p}'_1 \tilde{p}'_2 \rangle. \end{aligned} \quad (32)$$

In the classical limit, the last three terms vanish due to the no-overlap condition (31). The first term is quantum when we write  $p'_a, \tilde{p}'_a$  as  $p_a + \mathcal{O}(\hbar)$ , the measures  $d\Phi(p'_a)$  and  $d\Phi(\tilde{p}'_a)$  will contribute an additional  $\hbar^3$  factor each and for  $\langle \tilde{p}_1 \tilde{p}_2 | S^\dagger \mathbb{P}_a^{\mu_1} S | p_1 p_2 X \rangle$  we have the following expression:

$$\begin{aligned} \langle \tilde{p}_1 \tilde{p}_2 | S^\dagger \mathbb{P}_a^{\mu_1} S | p_1 p_2 X \rangle &= p_a^{\mu_1} \langle \tilde{p}_1 \tilde{p}_2 | p_1 p_2 X \rangle + i(\tilde{p}_a^{\mu_1} - p_a^{\mu_1}) \langle \tilde{p}_1 \tilde{p}_2 | T | p_1 p_2 X \rangle \\ &+ \sum_W \int \prod_{j=1}^2 d\Phi(s_j) (s_a^{\mu_1} - p_a^{\mu_1}) \langle \tilde{p}_1 \tilde{p}_2 | T^\dagger | s_1 s_2 W \rangle \langle s_1 s_2 W | T | p_1 p_2 X \rangle. \end{aligned}$$

$s_a^\mu$  are the momenta of the matter particles present in the intermediate states. When we scale these momenta with  $\hbar$ , the leading term in  $\langle \tilde{p}_1 \tilde{p}_2 | S^\dagger \mathbb{P}_a^{\mu_1} S | p_1 p_2 X \rangle$  is of  $\mathcal{O}(\hbar^{-3})$  which is not sufficient to compensate the  $\hbar$  factors coming from the measures. Hence

$$\lim_{\hbar \rightarrow 0} \mathfrak{R}_a^{\mu_1 \mu_2} = 0. \quad (33)$$

Thus we have shown that

$$\langle q_a^{\mu_1} q_a^{\mu_2} \rangle = \Delta p_a^{\mu_1} \Delta p_a^{\mu_2}. \quad (34)$$

It is straightforward to generalize the above result for moments containing  $q_1$  and  $q_2$ . Thus, we have  $\langle F(q_1, q_2) \rangle = F(\Delta p_1, \Delta p_2)$ . Similarly, for higher  $N = 3, 4$ , we have

$$\begin{aligned} \langle \Psi_{\text{in}} | S^\dagger [\mathbb{P}_a^{\mu_1}, [\mathbb{P}_a^{\mu_2}, [\mathbb{P}_a^{\mu_3}, S]]] | \Psi_{\text{in}} \rangle &= \Delta p_a^{\mu_1} \Delta p_a^{\mu_2} \Delta p_a^{\mu_3} + \Delta p_a^{\mu_3} \mathfrak{R}_a^{\mu_1 \mu_2} + \Delta p_a^{\mu_1} \mathfrak{R}_a^{\mu_2 \mu_3} - p_a^{\mu_2} \mathfrak{R}_a^{\mu_1 \mu_3} + \mathfrak{R}_a^{\mu_1 \mu_2 \mu_3}, \\ \langle \Psi_{\text{in}} | S^\dagger [\mathbb{P}_a^{\mu_1}, [\mathbb{P}_a^{\mu_2}, [\mathbb{P}_a^{\mu_3}, [\mathbb{P}_a^{\mu_4}, S]]]] | \Psi_{\text{in}} \rangle &= \Delta p_a^{\mu_1} \Delta p_a^{\mu_2} \Delta p_a^{\mu_3} \Delta p_a^{\mu_4} + \Delta p_a^{\mu_1} \Delta p_a^{\mu_2} \mathfrak{R}_a^{\mu_3 \mu_4} + \Delta p_a^{\mu_1} \Delta p_a^{\mu_4} \mathfrak{R}_a^{\mu_2 \mu_3} - \Delta p_a^{\mu_1} p_a^{\mu_3} \mathfrak{R}_a^{\mu_2 \mu_4} \\ &\quad - p_a^{\mu_2} \Delta p_a^{\mu_4} \mathfrak{R}_a^{\mu_1 \mu_3} + p_a^{\mu_2} p_a^{\mu_3} \mathfrak{R}_a^{\mu_1 \mu_4} + \Delta p_a^{\mu_1} \mathfrak{R}_a^{\mu_2 \mu_3 \mu_4} + \Delta p_a^{\mu_4} \mathfrak{R}_a^{\mu_1 \mu_2 \mu_3} - p_a^{\mu_3} \mathfrak{R}_a^{\mu_1 \mu_2 \mu_4} \\ &\quad - p_a^{\mu_2} \mathfrak{R}_a^{\mu_1 \mu_3 \mu_4} + \mathfrak{R}_a^{\mu_1 \mu_2} \mathfrak{R}_a^{\mu_3 \mu_4} + \mathfrak{R}_a^{\mu_1 \mu_2 \mu_3 \mu_4}. \end{aligned}$$

What one can check is that the remainder terms involving  $\mathfrak{R}_a^{\mu_1 \dots \mu_N}$  will contain one quantum term (which vanishes in the  $\hbar \rightarrow 0$  limit), while the remaining  $(4^{N-1} - 1)$  terms vanish due to the no-overlap condition. This continues to hold for all higher  $N$  and we find

$$\langle q_a^{\mu_1} \dots q_a^{\mu_N} \rangle = \Delta p_a^{\mu_1} \dots \Delta p_a^{\mu_N}. \quad (35)$$

## V. SUBLEADING SOFT RADIATION KERNEL

As reviewed in Sec. II B, classical soft photon theorems in  $D = 4$  are an infinite hierarchy of universal factorization theorems which state that in a soft expansion of the radiative field, coefficients of  $\omega^n (\ln \omega)^{n+1}$  are universal  $\forall n \geq -1$ . In this section, we subject the radiation kernel obtained using the KMOC formalism to the following consistency criteria: to all orders in the perturbative expansion, the subleading term in the kernel should precisely match the so-called log soft radiative mode. As we show below, this criterion leads to a set of nontrivial constraints that the hard amplitude must satisfy at any order in perturbative expansion. Before going into the details of the derivation, we outline the strategy used in deriving these consistency conditions.

- (i) Our approach to the soft limit of the radiation kernel starts with the subleading term in the soft expansion of the inelastic loop integrand. That is, we start by expanding the tree-level subleading soft factor in the limit of small exchange momenta. Thus, for the first particle, the soft factor can be written as

$$\begin{aligned} \hat{\mathcal{S}}_1^{(0)}(p_1, p_1 + q_1, k) \\ = Q_1 \varepsilon_\mu k_\nu \left[ -\frac{\Delta \hat{\mathcal{J}}_1^{\mu\nu}}{p_1 \cdot k} + \sum_{N=1}^{\infty} (-1)^{N+1} \frac{(q_1 \cdot k)^N}{(p_1 \cdot k)^{N+1}} \hat{\mathcal{J}}_{1,\text{out}}^{\mu\nu} \right]. \end{aligned} \quad (36)$$

With the substitution  $p'_1 = p_1 + q_1$ , the angular impulse and the final angular momentum are written as [24]

$$\Delta \hat{\mathcal{J}}_1^{\mu\nu} = -i\hbar(p_1 \wedge \partial_{p_1} + q_1 \wedge \partial_{q_1})^{\mu\nu}, \quad (37)$$

$$\hat{\mathcal{J}}_{1,\text{out}}^{\mu\nu} = -i\hbar((p_1 + q_1) \wedge \partial_{q_1})^{\mu\nu}. \quad (38)$$

- (ii) The subleading soft factor can be extracted from the radiation kernel (18) using the operator  $\lim_{\omega \rightarrow 0} \omega (\partial_\omega)^2 \omega$ ,

$$\mathcal{R}^{(0)}(k) = \lim_{\omega \rightarrow 0} \omega (\partial_\omega)^2 (\omega \mathcal{R}(k)). \quad (39)$$

The subleading kernel, as can be seen in (36), gets contribution from two quantities  $\langle \Delta \hat{\mathcal{J}}_1^{\mu\nu} \rangle$  and  $\langle q_1^{\mu_1} \dots q_1^{\mu_N} \hat{\mathcal{J}}_{1,\text{out}}^{\mu\nu} \rangle$ ,

$$\begin{aligned} \mathcal{R}^{(0)}(k) = Q_1 \varepsilon_\mu k_\nu \left[ -\frac{\langle \Delta \hat{\mathcal{J}}_1^{\mu\nu} \rangle}{p_1 \cdot k} + \sum_{N=1}^{\infty} (-1)^{N+1} \right. \\ \left. \times \frac{k_{\alpha_1} \dots k_{\alpha_N}}{(p_1 \cdot k)^{N+1}} \langle q_1^{\alpha_1} \dots q_1^{\alpha_N} \hat{\mathcal{J}}_{1,\text{out}}^{\mu\nu} \rangle \right]. \end{aligned} \quad (40)$$

Here we have suppressed the particle label.

- (iii) As discussed in [51] the logarithmic contribution to the radiation kernel arises from the region of integration for loop momenta to  $\omega < |\vec{l}| \ll |\vec{q}|$ , where  $\vec{l}$  and  $\vec{q}$  are scaled momenta.
- (iv) Using this we finally show that (40) leads to the classical log soft factor if and only if the  $2 \rightarrow 2$  scattering amplitude satisfies a set of constraints which can be schematically written as

$$(i) \mathcal{Z}_i(p_1, p_2, b; L) = 0, \quad \forall 1 \leq L(i = 1, 2, 3),$$

$$\langle q_1^{\alpha_1} \cdots q_1^{\alpha_N} \hat{\mathcal{J}}_{1,\text{out}}^{\mu\nu} \rangle_{(L),q} = 0,$$

$$(ii) \mathcal{R}_{\text{ln}}^{(0)}|_{\mathcal{Z}_i=0} = \text{classical log soft factor.} \quad (41)$$

Detailed expressions for  $\mathcal{Z}_i$  can be found in Sec. VI.

- (v) In Sec. VII, we show that any amplitude that satisfies the first set of vanishing conditions in (41) also satisfies the second condition at one loop, thus

showing that the KMOC formalism leads to the classical log soft factor.

### A. Computation of angular impulse

In this section, we compute the angular impulse  $\langle \Delta \hat{\mathcal{J}}_1^{\mu\nu} \rangle = R^{\mu\nu} + C^{\mu\nu}$ . It can be easily checked that  $\langle \Delta \hat{\mathcal{J}}_1^{\mu\nu} \rangle$  can be expressed in terms of the amplitude as follows:

$$\begin{aligned} \langle \Delta \hat{\mathcal{J}}_1^{\mu\nu} \rangle = & \hbar \int d\mu(q_1, q_2) e^{-ib \cdot q_1 / \hbar} \left[ (p_1 \wedge \partial_{p_1} + q_1 \wedge \partial_{q_1})^{\mu\nu} \{ \delta^4(q_1 + q_2) \mathcal{A}(P|P + Q) \} \right. \\ & - i \sum_X \int d\mu(w_1, w_2) \delta^4(q_1 + q_2 - w_1 - w_2 - \{k_X\}) \mathcal{A}^*(P + W, \{k_X\}|P + Q) \\ & \left. \times (p_1 \wedge \partial_{p_1} + w_1 \wedge \partial_{w_1})^{\mu\nu} \{ \delta^4(w_1 + w_2 + \{k_X\}) \mathcal{A}(P|P + W, \{k_X\}) \} \right]. \end{aligned} \quad (42)$$

Here,  $R^{\mu\nu}$  and  $C^{\mu\nu}$  are the linear and cut contributions respectively. The  $L$ th loop contribution to  $R^{\mu\nu}$  can be written in the form

$$\begin{aligned} R_{(L)}^{\mu\nu} = & \frac{e^{2L+2}}{4\hbar^L} \int d\mu_{\bar{q}} e^{-ib \cdot \bar{q}} \sum_{n=0}^L \frac{(-1)^n \hbar^n}{2^n n!} (\partial_{x_1} - \partial_{x_2})^n \left[ (\bar{q}^2)^n (p_1 \wedge \partial_{p_1} - ib \wedge \bar{q}) \tilde{\mathcal{A}}_{(L)}(P|P + \bar{Q}) \right. \\ & \left. - \partial_{x_1} ((\bar{q}^2)^n p_1 \wedge \bar{q} \tilde{\mathcal{A}}_{(L)}(P|P + \bar{Q})) \right] \quad (L \geq 0). \end{aligned} \quad (43)$$

The cut contribution  $C_{(L)}^{\mu\nu}$  can also be written in the same form as

$$C_{(L)}^{\mu\nu} = \frac{e^{2L+2}}{4\hbar^L} \int d\mu_{\bar{q}} e^{-ib \cdot \bar{q}} \sum_{n=0}^L \frac{(-1)^n \hbar^n}{2^n n!} \left( G_{1,n}^{\mu\nu(L)} - G_{2,n}^{\mu\nu(L)} \right) (L \geq 1). \quad (44)$$

The expressions for  $G_{i,n}^{(L)}$  are given in the Appendix A. Several comments are in order.

- (i) To write  $R_{(L)}^{\mu\nu}$  and  $C_{(L)}^{\mu\nu}$  in the current form, we have used integration by parts (IBP) to transfer derivatives from the momentum-conserving delta functions. Moreover, the derivatives  $\partial_{x_i}$  are taken after decomposing  $\bar{q}$  as

$$\bar{q}^\mu = \alpha_1 p_1^\mu + \alpha_2 p_2^\mu + q_\perp^\mu. \quad (45)$$

The coefficients  $\alpha_i$  are given by

$$\alpha_1 = \frac{1}{D} \left( (p_1 \cdot p_2) x_2 - m_2^2 x_1 \right), \quad (46)$$

$$\alpha_2 = \frac{1}{D} \left( (p_1 \cdot p_2) x_1 - m_1^2 x_2 \right), \quad (47)$$

where  $D = (p_1 \cdot p_2)^2 - m_1^2 m_2^2$  [53].

- (ii) To calculate, e.g.  $R_{(L)}$ , we expand the amplitude up to  $\mathcal{O}(\hbar^L)$ . The terms of  $\mathcal{O}(\hbar^{<L})$  will give rise to superclassical terms in addition to the classical terms when multiplied with appropriate powers of  $\hbar$ . The vanishing of superclassical terms for the radiation kernel was shown in [54] using  $N$ -operator formalism. The terms containing  $(\bar{q} \wedge b)^{\mu\nu}$  and  $(\bar{w} \wedge b)^{\mu\nu}$  in the linear and cut terms respectively combine to give  $(\Delta p \wedge b)^{\mu\nu}$ , which is  $\mathcal{O}(\omega^0)$  and does not contribute at  $\mathcal{O}(\log \omega)$ .

### B. Computation of $\langle q_1^{\alpha_1} \cdots q_1^{\alpha_N} \hat{J}_{1,\text{out}}^{\mu\nu} \rangle$

The second term in (40) has the following expression in terms of amplitude:

$$\begin{aligned} \langle q_1^{\alpha_1} \cdots q_1^{\alpha_N} \hat{J}_{1,\text{out}}^{\mu\nu} \rangle &= \hbar \int d\mu(q_1, q_2) e^{-ib \cdot q_1 / \hbar} \left[ q_1^{\alpha_1} \cdots q_1^{\alpha_N} ((p_1 + q_1) \wedge \partial_{q_1})^{\mu\nu} \left\{ \delta^4(q_1 + q_2) \mathcal{A}(P|P + Q) \right\} \right. \\ &\quad - i \sum_X \int d\mu(w_1, w_2) \delta^4(q_1 + q_2 - w_1 - w_2 - \{k_X\}) \mathcal{A}^*(P + W, \{k_X\}|P + Q) \\ &\quad \left. \times w_1^{\alpha_1} \cdots w_1^{\alpha_N} ((p_1 + w_1) \wedge \partial_{w_1})^{\mu\nu} \left\{ \delta^4(w_1 + w_2 + \{k_X\}) \mathcal{A}(P|P + W, \{k_X\}) \right\} \right]. \end{aligned} \quad (48)$$

Once again, the  $L$ th loop contribution can be written in the same form as for the angular impulse. The linear-amplitude contribution  $R'_{(L)}$  is

$$R'_{(L)}{}^{\mu\nu\alpha} = \frac{e^{2L+2}}{4\hbar^{L-N}} \int d\mu_{\bar{q}} e^{-ib \cdot \bar{q}} \sum_{n=0}^L \frac{(-1)^n \hbar^n}{2^n n!} H_{1,n}^{(L)\mu\nu\alpha}, \quad (L \geq 0), \quad (49)$$

Here the vector index is  $\alpha = \alpha_1 \cdots \alpha_N$ . The cut contribution is written as

$$C'_{(L)}{}^{\mu\nu\alpha} = \frac{e^{2L+2}}{4\hbar^{L-N}} \int d\mu_{\bar{q}} e^{-ib \cdot \bar{q}} \sum_{n=0}^L \frac{(-1)^n \hbar^n}{2^n n!} H_{2,n}^{(L)\mu\nu\alpha}, \quad (L \geq 0), \quad (50)$$

The expressions for  $H_{i,n}^{(L)\mu\nu\alpha}$  are also given in the Appendix A.

## VI. LOG SOFT CONSTRAINTS ON RADIATION KERNEL

In this section, we will derive a set of conditions that the “hard amplitude” has to satisfy so that the radiation kernel produces a tail to the electromagnetic memory at a generic order in perturbative expansion. One of these two conditions corresponds to the constraint that, in the classical limit, the S-matrix satisfies macroscopic causality (retarded propagation). We now outline our proposal to impose macroscopic causality in the context of KMOC observables and then use it as a necessary condition that ensures that the subleading soft radiation kernel matches with classical log soft theorem.

Let  $\langle O \rangle$  be an inclusive observable computed using KMOC. It can be written schematically as

$$\langle O \rangle = \sum_L \int_{\bar{q}} \prod_{i=1}^L \hat{d}^4 l_i \frac{\mathcal{N}(p_1, p_2, \bar{q}, \bar{k}, \{l_j\})}{\mathcal{D}(p_1, p_2, \bar{q}, \bar{k}, \{l_j\})}, \quad (51)$$

where  $1/\mathcal{D}$  denotes the product of propagators. In the classical limit, the radiation field obtained from the KMOC radiation kernel should correspond to the retarded classical solution. We therefore impose the following consistency

condition motivated from causality, the classical contribution to  $\langle O \rangle$  should be unchanged when the Feynman propagators appearing on the right-hand side of (51) are replaced by retarded propagators.<sup>1</sup> This condition is useful because the difference between the two prescriptions is localized on the pole contributions whose support is sensitive to the choice of Green’s function (boundary conditions). Thus, requiring the Feynman and retarded propagations to give the same classical observable amounts to demanding that the pole contributions incompatible with retarded classical propagation do not contribute to the classical result. In the present application, this criterion is used to exclude the photon pole contribution which would otherwise obstruct the identification of the radiation kernel with the retarded classical waveform.

Hence we demand, *one can always choose the contour of integration in the complex  $l_0^{(i)}$  plane  $\forall i \in \{1, \dots, L\}$ , such that only matter poles contribute*; alternatively, our formulation of macroscopic causality in the context of KMOC

<sup>1</sup>The traditional formulation of macroscopic causality in S-matrix theory by Stapp and Iagolnitzer is a requirement that only cut propagators contribute in the classical limit; in our analysis we showed at one loop this is indeed true, i.e. only the cut part of the angular impulse contributes classically, see Sec. VII A. While we do believe that the macroscopic causality criteria used in this paper are equivalent to the formulation in [58], a detailed comparison of the two is beyond the scope of this paper.

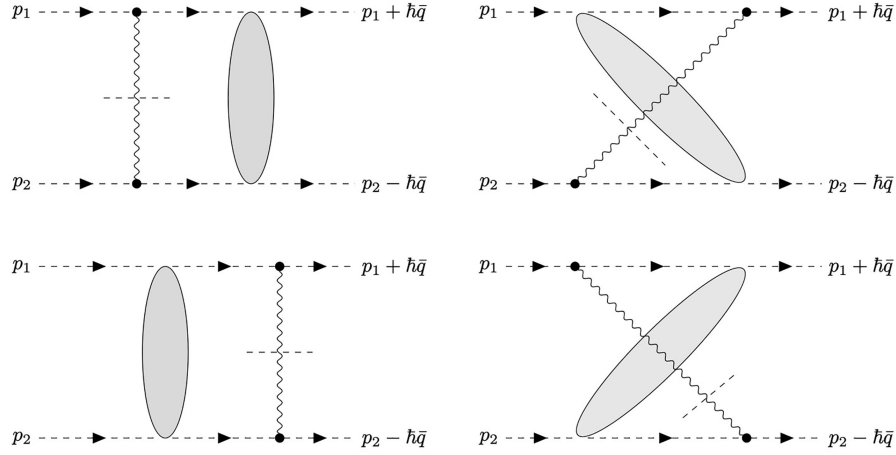


FIG. 2. The four classes of  $L$ -loop diagrams that give log-divergent contributions in the soft region of the photon separated from the  $L$  loop.

formalism is equivalent to showing that the contribution of photon pole in the evaluation of  $\langle O \rangle$  vanishes in the classical limit.

In the next section, we show how the above-stated condition for macroscopic causality is one of the constraints that amplitudes have to satisfy, so that the soft radiation kernel matches the tail to the electromagnetic memory at  $O(\log \omega)$ .

### A. First set of consistency conditions

To calculate the log tail from the subleading kernel, first we need to extract the  $O(\log \omega)$  contributions from (43) and (44). At  $L$  loop, log divergence arises when the loop momentum of one of the outermost photon lines, say  $\bar{l}_o$ , is restricted to the region  $\omega < |\bar{l}_o| \ll |\bar{q}|$ . It turns out such diagrams fall into four classes depending upon how the  $l_o$  photon attaches to the  $(L-1)$ -loop blob. Other diagrams do not produce  $O(\log \omega)$  terms. For example, the diagram 3 does not contribute to  $O(\log \omega)$  as there is only one internal matter propagator and power counting tells us the contribution from these diagrams will be  $O(\omega)$ .

In each of these diagrams, we take the outermost photon leg to be on-shell. If we consider an  $L$ -loop diagram, then Fig. 2 will be the four sets of diagrams that contribute to  $O(\log \omega)$ . By taking the outermost photon on-shell, we

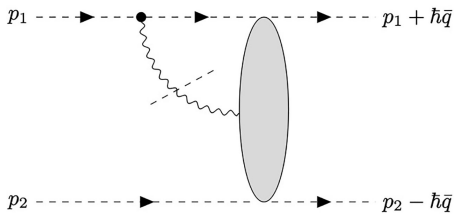


FIG. 3.  $L$ -loop diagram with the outer photon being in the soft region.

segregate the photon pole contribution from (40). Thus,  $\mathcal{R}^{(0)}$  can be written as

$$\mathcal{R}^{(0)} = \mathcal{R}_q^{(0)} + \mathcal{R}_{cl}^{(0)}, \quad (52)$$

where  $\mathcal{R}_q^{(0)}$  is the quantum term that has photon pole contribution, and  $\mathcal{R}_{cl}^{(0)}$  is the classical contribution coming from the matter poles. The vanishing of the photon pole contribution in the classical limit provides constraints on the perturbative amplitude

$$\mathcal{R}_q^{(0)} = 0. \quad (53)$$

$\mathcal{R}_q^{(0)}$  in (40) consists of two terms  $\langle \Delta \hat{J}_1^{\mu\nu} \rangle_q$  and  $\langle q_1^{\alpha_1} \dots q_1^{\alpha_N} \hat{J}_{1,\text{out}}^{\mu\nu} \rangle_q$  which vanish separately owing to their different tensor structures.

### B. Constraints at $L$ loop

We start by calculating the linear amplitude contribution to  $\langle \Delta \hat{J}_1^{\mu\nu} \rangle_q$  in (40) and  $R_{(L)q}$ , from (43). We denote by  $\tilde{\mathcal{K}}_{(L)}$  the sum of the diagrams over all four classes of diagrams without a cut on the outer photon across the  $L-1$ -loop blob.  $\tilde{\mathcal{K}}_{(L)q}$  then denotes the diagrams with a cut on the outer photon, the sum of diagrams in Fig. 2. The remaining term  $\tilde{\mathcal{K}}_{(L)} - \tilde{\mathcal{K}}_{(L)q}$  contributes to the  $\mathcal{R}_{cl}^{(0)}$  in (52),

$$\tilde{\mathcal{K}}_{(L)} = \tilde{\mathcal{K}}_{(L)q} + \tilde{\mathcal{K}}_{(L)cl}. \quad (54)$$

Photon poles can be isolated by simply replacing the photon propagator with the delta function  $1/\bar{l}^2 \rightarrow \hat{\delta}(\bar{l}^2)$  as shown by the cut in Fig. 2. We write the quantum contribution to  $R_{(L)}^{\mu\nu}$  using (43),

$$R_{(L),q}^{\mu\nu} = \frac{1}{4\hbar^L} \int d\mu_{\bar{q}} e^{-ib\cdot\bar{q}} \sum_{n=0}^L \frac{(-1)^n \hbar^n}{2^n n!} (\partial_{x_1} - \partial_{x_2})^n \times \left[ (\bar{q}^2)^n (p_1 \wedge \partial_{p_1})^{\mu\nu} \tilde{\mathcal{K}}_{(L),q} - \partial_{x_1} ((\bar{q}^2)^n (p_1 \wedge \bar{q})^{\mu\nu} \tilde{\mathcal{K}}_{(L),q}) \right]. \quad (55)$$

We separate the two tensor structures that appear in the above expression. The term  $(p_1 \wedge \partial_{p_1})^{\mu\nu} \tilde{\mathcal{K}}_{(L),q}$  gives rise to terms of the form

$$(p_1 \wedge \partial_{p_1})^{\mu\nu} \tilde{\mathcal{K}}_{(L),q} = (p_1 \wedge \bar{q})^{\mu\nu} \tilde{\mathcal{K}}_{(L),q}^{(p_1 \wedge \bar{q})} + (p_1 \wedge p_2)^{\mu\nu} \tilde{\mathcal{K}}_{(L),q}^{(p_1 \wedge p_2)}.$$

Here,  $\tilde{\mathcal{K}}_{(L),q}^{(p_1 \wedge \bar{q})}$  is the coefficient of  $(p_1 \wedge \bar{q})^{\mu\nu}$  term in  $(p_1 \wedge \partial_{p_1})^{\mu\nu} \tilde{\mathcal{K}}_{(L),q}$ . The intermediate term of the form  $(p_1 \wedge \bar{l})^{\mu\nu} \tilde{\mathcal{K}}_{(L),q}^{(p_1 \wedge \bar{l})}$ , where  $\bar{l}$  is the loop momentum, can be tensor decomposed into the above two terms.  $R_{(L),q}^{\mu\nu}$  in (55) can be written as

$$R_{(L),\text{ln}}^{\mu\nu} = \frac{1}{4\hbar^L} \int d\mu_{\bar{q}} e^{-ib\cdot\bar{q}} \sum_{n=0}^L \frac{(-1)^n \hbar^n}{2^n n!} (\partial_{x_1} - \partial_{x_2})^n \times \left[ (p_1 \wedge \bar{q})^{\mu\nu} (\bar{q}^2)^n \tilde{\mathcal{K}}_{(L),q}^{(p_1 \wedge \bar{q})} - (p_1 \wedge \bar{q})^{\mu\nu} \partial_{x_1} ((\bar{q}^2)^n \tilde{\mathcal{K}}_{(L),q}) \right. \\ \left. + (p_1 \wedge p_2)^{\mu\nu} (\bar{q}^2)^n \tilde{\mathcal{K}}_{(L),q}^{(p_1 \wedge p_2)} - \frac{P_1 \cdot P_2}{D} (p_1 \wedge p_2)^{\mu\nu} (\bar{q}^2)^n \tilde{\mathcal{K}}_{(L),q} \right]. \quad (56)$$

Next, we turn to calculate the cut contribution. Proceeding in the same spirit, we write the quantum terms in (44) as

$$C_{(L),q}^{\mu\nu} = \frac{i}{4\hbar^L} \int d\mu_{\bar{q}} e^{-ib\cdot\bar{q}} \sum_{n=0}^L \frac{(-1)^n \hbar^n}{2^n n!} (\partial_{x_1} - \partial_{x_2})^n \left[ (\bar{q}^2)^n \int d\mu(\bar{w}_1, \bar{w}_2) \left( -T_{1(L),q}^{\mu\nu} + T_{2(L),q}^{\mu\nu} \right) \right. \\ \left. - 2(\bar{q}^2)^n \int \hat{\mathcal{A}}(\bar{w}_1, \bar{w}_2) \hat{\mathcal{S}}'(2p_1 \cdot \bar{w}_1 + \hbar \bar{w}_1^2) \hat{\mathcal{S}}(2p_2 \cdot \bar{w}_2 + \hbar \bar{w}_2^2) (p_1 \wedge \bar{w}_1)^{\mu\nu} T_{3(L),q} \right. \\ \left. \times 2\partial_{x_1} \int d\mu(\bar{w}_1, \bar{w}_2) (\bar{q}^2)^n (\bar{w}_1 \wedge (p_1 + \hbar \bar{q}_1))^{\mu\nu} T_{3(L),q} \right]. \quad (57)$$

$\mathcal{T}_i$  are the quadratic amplitude terms after integrating out the intermediate photons. The explicit expressions for  $\mathcal{T}_i$  are given in the Appendix B. The subscript q on  $\mathcal{T}_i$  denotes the photon pole contribution from the diagrams constituting  $\mathcal{T}_i$ . As with the case of linear amplitude,  $C_{(L)}$  gets the log-divergent contribution from a certain class of diagrams. The sum of log-divergent diagrams,  $\tilde{\mathcal{K}}_{(L)}$ , is defined in a similar way to the previous case. The momentum of the outermost photon in  $\tilde{\mathcal{K}}_{(L)}$  is restricted to the region  $\omega < |\bar{l}| \ll |\bar{q}|$ . One such diagram with a cut on the outer photon is shown below, Fig. 4. The conjugate diagrams are defined in a similar way.

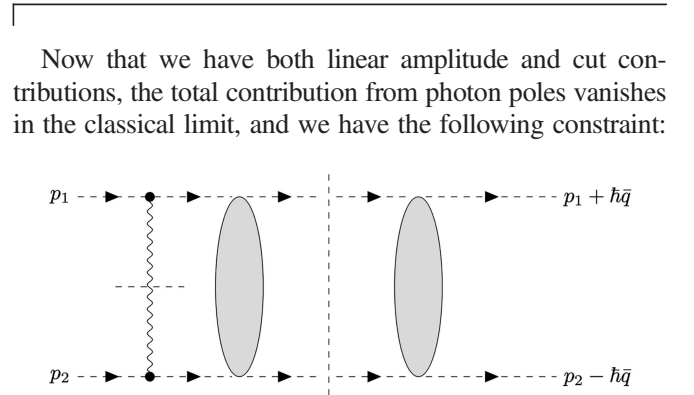


FIG. 4. Example of  $L$ -loop diagram appearing in the cut contribution.

$$\begin{aligned}
\langle \Delta \hat{J}_1^{\mu\nu} \rangle_{(L),q} &= \frac{1}{\hbar L} \int d\mu_{\bar{q}} e^{-ib \cdot \bar{q}} \sum_{n=0}^L \frac{(-1)^n \hbar^n}{2^n n!} (\partial_{x_1} - \partial_{x_2})^n \left[ (p_1 \wedge \bar{q})^{\mu\nu} ((\bar{q}^2)^n \tilde{\mathcal{K}}_{(L)q}^{p_1 \wedge \bar{q}} - \partial_{x_1} ((\bar{q}^2)^n \tilde{\mathcal{K}}_{(L)q})) \right. \\
&\quad + (p_1 \wedge p_2)^{\mu\nu} (\bar{q}^2)^n \left( \tilde{\mathcal{K}}_{(L)q}^{(p_1 \wedge p_2)} - \frac{p_1 \cdot p_2}{D} \tilde{\mathcal{K}}_{(L)q} \right) + i(\bar{q}^2)^n \int d\mu(\bar{w}_1, \bar{w}_2) \left( -\mathcal{T}_{1(L)q}^{\mu\nu} + \mathcal{T}_{2(L)q}^{\mu\nu} \right) \\
&\quad - 2i(\bar{q}^2)^n \int \hat{d}^4(\bar{w}_1, \bar{w}_2) \hat{\delta}'(2p_1 \cdot \bar{w}_1 + \hbar \bar{w}_1^2) \hat{\delta}(2p_2 \cdot \bar{w}_2 + \hbar \bar{w}_2^2) (p_1 \wedge \bar{w}_1)^{\mu\nu} \mathcal{T}_{3(L)q} \\
&\quad \left. + 2i\partial_{x_1} \int d\mu(\bar{w}_1, \bar{w}_2) (\bar{q}^2)^n (\bar{w}_1 \wedge (p_1 + \hbar \bar{q}))^{\mu\nu} \mathcal{T}_{3(L)q} \right] = 0. \tag{58}
\end{aligned}$$

The above constraint can further be reduced to three separate constraints.  $(R + C)_{(L),\text{ln}}^{\mu\nu}$  is antisymmetric, built out of vectors  $p_1^\mu$ ,  $p_2^\mu$ , and  $b^\mu$ . The tensor structure of the above expression has the form

$$\langle \Delta \hat{J}_1^{\mu\nu} \rangle_{(L),q} = \mathcal{Z}_1 (p_1 \wedge p_2)^{\mu\nu} + \mathcal{Z}_2 (p_1 \wedge b)^{\mu\nu} + \mathcal{Z}_3 (p_2 \wedge b)^{\mu\nu}. \tag{59}$$

Here,

$$\begin{aligned}
\mathcal{Z}_1 &= -\frac{1}{2D} \langle \Delta \hat{J}_1^{\mu\nu} \rangle_{(L),q} (p_1 \wedge p_2)_{\mu\nu}, \quad \mathcal{Z}_2 = \frac{1}{b^2 D} \langle \Delta \hat{J}_1^{\mu\nu} \rangle_{(L),q} ((p_1 \cdot p_2) p_2 - m_2^2 p_1) \wedge b)_{\mu\nu}, \\
\mathcal{Z}_3 &= \frac{1}{b^2 D} \langle \Delta \hat{J}_1^{\mu\nu} \rangle_{(L),q} ((p_1 \cdot p_2) p_1 - m_1^2 p_2) \wedge b)_{\mu\nu}.
\end{aligned}$$

Using the above decomposition, the constraint (53) gives

$$\mathcal{Z}_i(p_1, p_2, b; L) = 0. \tag{60}$$

The second constraint is obtained from the vanishing of the contributions from the photon poles from the second quantity in (40). Once again, for the log-divergent contribution in (48), we use the diagrams in Figs. 2 and 4 respectively in (49) and (50)

$$\begin{aligned}
&\langle q_1^{\alpha_1} \cdots q_1^{\alpha_N} \hat{J}_{1,\text{out}}^{\mu\nu} \rangle_{(L),q} \\
&= -\frac{ie^{2L+2}}{4\hbar^{L-N}} \int d\mu_{\bar{q}} e^{-ib \cdot \bar{q}} \sum_{n=0}^L \frac{(-1)^n \hbar^n}{2^n n!} (\partial_{x_1} - \partial_{x_2})^n \\
&\quad \times \left[ \int d\mu(\bar{w}_1, \bar{w}_2) \bar{w}_1^{\alpha_1} \cdots \bar{w}_1^{\alpha_N} \mathcal{T}_{4(L)q}^{\mu\nu} \right. \\
&\quad + \partial_{x_1} \int d\mu(\bar{w}_1, \bar{w}_2) ((p_1 + \hbar \bar{w}_1) \\
&\quad \left. \wedge (p_1 + \hbar \bar{q}))^{\mu\nu} \bar{w}_1^{\alpha_1} \cdots \bar{w}_1^{\alpha_N} \mathcal{T}_{5(L)q} \right]. \tag{61}
\end{aligned}$$

Again, the expressions for  $\mathcal{T}_{4(L)}^{\mu\nu}$  and  $\mathcal{T}_{5(L)}^{\mu\nu}$  are given in the Appendix B. The above expression appears within the kernel (40); consequently we have dropped terms in the above expression which vanish when contracted with  $\varepsilon_\mu k_\nu k_{\alpha_1}$ . The terms with tensors  $(p_1 + \hbar \bar{q})^{[\mu} \eta^{\nu]} (\alpha_1 \bar{q}^{\alpha_1} \cdots \bar{q}^{\alpha_N})$  and  $(p_1 + \hbar \bar{w}_1)^{[\mu} \eta^{\nu]} (\alpha_1 \bar{w}_1^{\alpha_1} \cdots \bar{w}_1^{\alpha_N})$

produce factors  $\varepsilon \cdot k$  and  $k^2$  when contracted with  $\varepsilon_\mu k_\nu k_{\alpha_1}$ , which are zero. Moreover, the terms of the form  $\frac{i}{\hbar} ((p_1 + \hbar \bar{q}) \wedge b)^{\mu\nu} \bar{q}^{\alpha_1} \cdots \bar{q}^{\alpha_N}$  and  $\frac{1}{\hbar} ((p_1 + \hbar \bar{w}_1) \wedge b)^{\mu\nu} \bar{w}_1^{\alpha_1} \cdots \bar{w}_1^{\alpha_N}$  combine, when integrated over the full region in loop integration, to a product of impulses using the result proved in Sec. IV; consequently, they are IR finite. The expression (40) contains only cut contributions as the linear-in-amplitude terms contain only the above-mentioned tensor structures. We obtain the constraint

$$\langle q_1^{\alpha_1} \cdots q_1^{\alpha_N} \hat{J}_{1,\text{out}}^{\mu\nu} \rangle_{(L),q} = 0. \tag{62}$$

The constraints (60) and (62) are the vanishing conditions in the classical limit for the quantum log terms found in the quantum calculation of the subleading soft factor [51].

### C. One-loop calculation

To calculate  $\mathcal{Z}_i$ 's, we first calculate the log-divergent contribution to (58) at one loop. First, we consider the cut contributions. Following from (B1), (B2), and (B3), we see that the amplitudes in these equations are tree-level amplitudes. Cutting one of the photon legs will introduce a delta function,  $\hat{\delta}(\bar{w}^2)$  or  $\hat{\delta}((\bar{w} - \bar{q})^2)$ .

For the first diagram in Fig. 5,  $\hat{\delta}(\bar{w}^2)$  together with the on-shell delta functions  $\hat{\delta}(p_i \cdot \bar{w})$  makes the  $\bar{w}$  integrals vanish and for the second diagram,  $\hat{\delta}((\bar{w} - \bar{q})^2)$  in the relevant region of integration  $\omega < |\bar{w}| \ll |\bar{q}|$ , becomes  $\hat{\delta}(\bar{q}^2)$ . By the same argument,  $\bar{q}$  integrals also vanish.

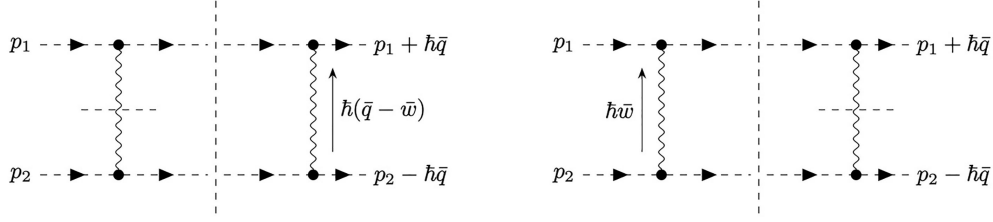


FIG. 5. The one-loop cut diagrams corresponding to the two possibilities of the photon leg being soft.

The same argument also shows that the cut contribution to the  $\langle q_1^{\alpha_1} \cdots q_1^{\alpha_N} \hat{J}_{1,\text{out}}^{\mu\nu} \rangle_{(L),q}$  in the second constraint (62) is zero, ensuring the validity of (62).

Now we turn to calculate the linear contribution in (58). The sum over four one-loop diagrams, i.e.,  $\tilde{\mathcal{K}}_{(1),q}$ , has contributions from the following diagrams in Fig. 6. The contribution from these diagrams has been calculated

in the Appendix C. The linear contribution has the following three tensor structures:  $(p_1 \wedge p_2)^{\mu\nu}$ ,  $(p_1 \wedge \bar{l})^{\mu\nu}$ ,  $(p_1 \wedge \bar{q})^{\mu\nu}$ . At one loop the  $(p_1 \wedge \bar{l})^{\mu\nu}$  terms give  $(p_1 \wedge p_2)^{\mu\nu}$  terms after integration over  $\bar{l}$  while the  $(p_1 \wedge \bar{q})^{\mu\nu}$  terms give  $(p_1 \wedge b)^{\mu\nu}$  terms when integrated over  $\bar{q}$ . From the calculation in Appendix C, we have the following results:

$$\begin{aligned}
 \mathcal{Z}_2 = & Q_1^2 Q_2^2 \frac{(p_1 \cdot p_2)^2}{D} \int d\mu_{\bar{q}} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{q})^{\mu\nu} \int \hat{d}\bar{l}^4 \frac{\hat{\delta}(\bar{l}^2)}{(\bar{q} - \bar{l})_+^2} \\
 & \times \left[ \frac{\bar{q}^2}{(\bar{q} - \bar{l})_+^2} \frac{m_2^2}{(p_1 \cdot \bar{l})_+} \left( \frac{1}{(p_2 \cdot \bar{l})_-} - \frac{1}{(p_2 \cdot \bar{l})_+} \right) \right. \\
 & + m_2^2 \left( \frac{1}{(p_1 \cdot \bar{l})_-} - \frac{1}{(p_1 \cdot \bar{l})_+} \right) \left( \frac{1}{(p_2 \cdot \bar{l})_-} - \frac{1}{(p_2 \cdot \bar{l})_+} \right) \\
 & + p_1 \cdot p_2 \left( \frac{1}{(p_1 \cdot \bar{l})_-} - \frac{1}{(p_1 \cdot \bar{l})_+} \right) \left( \frac{1}{(p_2 \cdot \bar{l})_-} - \frac{1}{(p_2 \cdot \bar{l})_+} \right) \\
 & \left. + \frac{\bar{q}^2}{(\bar{q} - \bar{l})_+^2} \frac{p_1 \cdot p_2}{(p_2 \cdot \bar{l})_-} \left( \frac{1}{(p_1 \cdot \bar{l})_+} - \frac{1}{(p_1 \cdot \bar{l})_-} \right) \right]. \quad (63)
 \end{aligned}$$

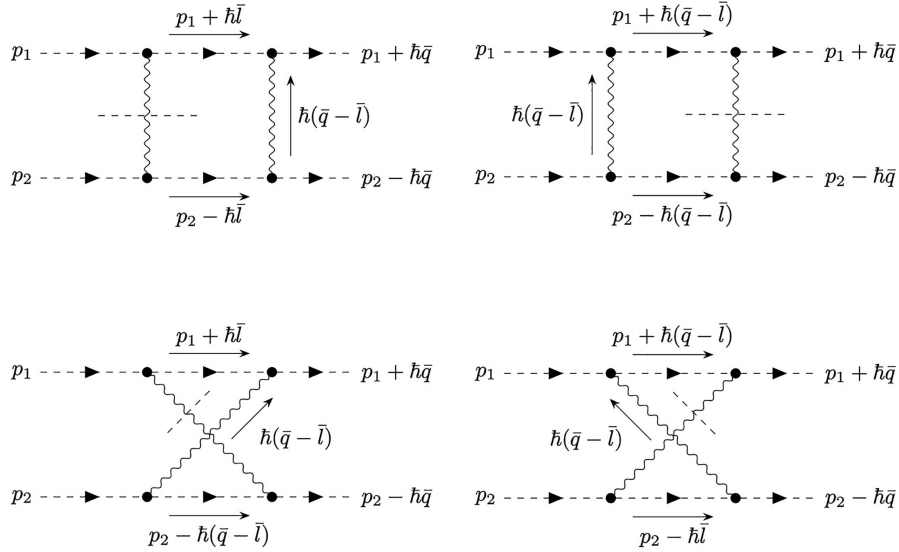


FIG. 6. The four one-loop diagrams corresponding to the four possibilities of the photon leg being soft.

In the region  $\omega < |\vec{l}| \ll |\vec{q}|$ ,  $\frac{\hat{q}^2}{(\vec{q}-\vec{l})_+^2} \approx 1 + \mathcal{O}(|\vec{l}|)$

$$\mathcal{Z}_2 = Q_1^2 Q_2^2 \frac{(p_1 \cdot p_2)^2}{D} \int d\mu_{\vec{q}} e^{-i\vec{q} \cdot b} (p_1 \wedge \vec{q})^{\mu\nu} \int \hat{d}\vec{l}^4 \frac{\hat{\delta}(\vec{l}^2)}{(\vec{q}-\vec{l})_+^2} \times \left[ \frac{m_2^2}{(p_1 \cdot \vec{l})_-} \left( \frac{1}{(p_2 \cdot \vec{l})_-} - \frac{1}{(p_2 \cdot \vec{l})_+} \right) - \frac{p_1 \cdot p_2}{(p_2 \cdot \vec{l})_+} \left( \frac{1}{(p_1 \cdot \vec{l})_-} - \frac{1}{(p_1 \cdot \vec{l})_+} \right) \right]. \quad (64)$$

$\mathcal{Z}_1$  is zero as there are no log-divergent terms present of the form  $(p_1 \wedge p_2)^{\mu\nu}$  as shown in C.  $\mathcal{Z}_3$  is also trivially zero as there are no terms that give  $(p_2 \wedge b)^{\mu\nu}$ . The above expression contains integrals of the following type:

$$\int \hat{d}^4 \vec{l} \frac{\hat{\delta}(p_1 \cdot \vec{l})}{(p_2 \cdot \vec{l})_{\pm}} \hat{\delta}(\vec{l}^2).$$

This integral vanishes because of the presence of  $\hat{\delta}(\vec{l}^2)$  and on-shell delta functions  $\hat{\delta}(p_i \cdot \vec{l})$ .

## VII. SECOND SET OF CONSISTENCY CONDITIONS

Having derived the constraints that probe macroscopic causality of the S-matrix thanks to the application of vanishing quantum log soft factor in the KMOC radiation kernel, we now analyze the constraints that *universal classical log soft factor* imposes on a class of conservative observables in the KMOC formalism. These constraints were anticipated in [53] and were proved for tree-level scattering in [59].

The subleading contribution to the radiation kernel in (40) at  $L$  loop can be obtained in a similar manner using similar diagrams as in Figs. 5 and 6 without the on-shell cut on the outermost photon line. As with the quantum case, the classical contribution to (40) produces two separate constraints,

$$\langle \Delta \hat{J}_i^{\mu\nu} \rangle_{\text{cl}} = \Delta J_{i,\log}^{\mu\nu}, \quad (65)$$

$$\langle q_i^{\alpha_1} \cdots q_i^{\alpha_N} \hat{J}_{i,\text{out}}^{\mu\nu} \rangle_{\text{cl}} = \Delta p_i^{\alpha_1} \cdots \Delta p_i^{\alpha_N} J_{i,\text{out},\log}^{\mu\nu}, \quad (66)$$

with the condition  $\mathcal{Z}_i = 0$  and  $\langle q_1^{\alpha_1} \cdots q_1^{\alpha_N} \hat{J}_{1,\text{out}}^{\mu\nu} \rangle_{(L),q} = 0$ . On the right side of the equations are the classical quantities.

As was mentioned previously, the classical contribution comes from the matter poles in the  $L$ -loop diagram, i.e., from the  $\tilde{\mathcal{K}}_{(L)\text{cl}}$  (54). Thus, we use  $\tilde{\mathcal{K}}_{(L)\text{cl}}$  in (43) and (44) to isolate the classical contribution to the angular impulse,

$$\begin{aligned} \langle \Delta \hat{J}_1^{\mu\nu} \rangle_{(L),\text{cl}} &= \frac{1}{\hbar^L} \int d\mu_{\vec{q}} e^{-i\vec{b} \cdot \vec{q}} \sum_{n=0}^L \frac{(-1)^n \hbar^n}{2^n n!} (\partial_{x_1} - \partial_{x_2})^n \\ &\times \left[ (p_1 \wedge \vec{q})^{\mu\nu} \left( (\vec{q}^2)^n \tilde{\mathcal{K}}_{(L)\text{cl}}^{p_1 \wedge \vec{q}} - \partial_{x_1}((\vec{q}^2)^n \tilde{\mathcal{K}}_{(L)\text{cl}}) \right) + (p_1 \wedge p_2)^{\mu\nu} (\vec{q}^2)^n \left( \tilde{\mathcal{K}}_{(L)\text{cl}}^{(p_1 \wedge p_2)} - \frac{p_1 \cdot p_2}{D} \tilde{\mathcal{K}}_{(L)\text{cl}} \right) \right. \\ &+ i(\vec{q}^2)^n \int d\mu(\vec{w}_1, \vec{w}_2) \left( -\mathcal{T}_{1(L)\text{cl}}^{\mu\nu} + \mathcal{T}_{2(L)\text{cl}}^{\mu\nu} \right) \\ &- 2i(\vec{q}^2)^n \int \hat{d}^4(\vec{w}_1, \vec{w}_2) \hat{\delta}'(2p_1 \cdot \vec{w}_1 + \hbar \vec{w}_1^2) \hat{\delta}(2p_2 \cdot \vec{w}_2 + \hbar \vec{w}_2^2) (p_1 \wedge \vec{w}_1)^{\mu\nu} \mathcal{T}_{3,(L)\text{cl}} \\ &\left. + 2i\partial_{x_1} \int d\mu(\vec{w}_1, \vec{w}_2) (\vec{q}^2)^n (\vec{w}_1 \wedge (p_1 + \hbar \vec{q}))^{\mu\nu} \mathcal{T}_{3,(L)\text{cl}} \right]. \quad (67) \end{aligned}$$

Given  $\mathcal{Z}_i(p_1, p_2, b; L) = 0$  (60), matching the above expression with the classical result [51] gives the following constraint:

$$\langle \Delta \hat{J}_1^{\mu\nu} \rangle_{(L),\text{cl}} = -\frac{iQ_1 Q_2 m_1^2 m_2^2}{4\pi} \log \omega \left[ \frac{(p_1 \wedge p_2)^{\mu\nu}}{((p_1 \cdot p_2)^2 - m_1^2 m_2^2)^{3/2}} + \frac{(p'_1 \wedge p'_2)^{\mu\nu}}{((p'_1 \cdot p'_2)^2 - m_1^2 m_2^2)^{3/2}} \right]. \quad (68)$$

Another constraint is obtained from the log-divergent classical contribution to  $\langle q_i^{\alpha_1} \cdots q_i^{\alpha_N} \hat{J}_{i,\text{out}}^{\mu\nu} \rangle$  in (40),

$$\begin{aligned} \langle q_i^{\alpha_1} \cdots q_i^{\alpha_N} \hat{J}_{i,\text{out}}^{\mu\nu} \rangle_{(L),\text{cl}} &= -\frac{ie^{2L+2}}{4\hbar^{L-N}} \int d\mu_{\bar{q}} e^{-ib\cdot\bar{q}} \sum_{n=0}^L \frac{(-1)^n \hbar^n}{2^n n!} (\partial_{x_1} - \partial_{x_2})^n \\ &\times \left[ \int d\mu(\bar{w}_1, \bar{w}_2) \bar{w}_1^{\alpha_1} \cdots \bar{w}_1^{\alpha_N} \mathcal{T}_{4(L)\text{cl}}^{\mu\nu} \right. \\ &\left. + \partial_{x_1} \int d\mu(\bar{w}_1, \bar{w}_2) ((p_1 + \hbar\bar{w}_1) \wedge (p_1 + \hbar\bar{q}))^{\mu\nu} \bar{w}_1^{\alpha_1} \cdots \bar{w}_1^{\alpha_N} \mathcal{T}_{5(L)\text{cl}} \right]. \end{aligned} \quad (69)$$

The subscript cl on  $\mathcal{T}^{\mu\nu}$  denotes the matter pole contribution. We now use the corresponding classical expression to write the constraint

$$\langle q_i^{\alpha_1} \cdots q_i^{\alpha_N} \hat{J}_{i,\text{out}}^{\mu\nu} \rangle_{(L),\text{cl}} = -\frac{iQ_1 Q_2 m_1^2 m_2^2}{4\pi} \log \omega \frac{(p'_1 \wedge p'_2)^{\mu\nu}}{((p'_1 \cdot p'_2)^2 - m_1^2 m_2^2)^{3/2}} \Delta p_1^{\alpha_1} \cdots \Delta p_1^{\alpha_N}, \quad (N \geq 1). \quad (70)$$

with the condition  $\langle q_i^{\alpha_1} \cdots q_i^{\alpha_N} \hat{J}_{i,\text{out}}^{\mu\nu} \rangle_{(L),\text{q}} = 0$ .

## A. One-loop calculation

### 1. Contribution from angular impulse

The linear-in-amplitude calculation for the angular impulse follows parallel to the quantum case. We take the outermost photon without an on-shell cut, as the photon pole contribution has already been shown to vanish. We rewrite the expressions obtained in Appendix C, considering only the log-divergent terms in our region of interest,

$$\begin{aligned} \langle \Delta \hat{J}_1^{\mu\nu} \rangle_{\text{cl}} &= iQ_1^2 Q_2^2 \frac{(p_1 \cdot p_2)^2}{D} \int d\mu_{\bar{q}} e^{-i\bar{q} \cdot b} \frac{(p_1 \wedge \bar{q})^{\mu\nu}}{\bar{q}^2} \int \frac{\hat{d}\bar{l}^4}{\bar{l}^2} \\ &\times \left[ \frac{m_2^2 \hat{\delta}(p_2 \cdot \bar{l})}{(p_1 \cdot \bar{l})_-} - \frac{(p_1 \cdot p_2) \hat{\delta}(p_1 \cdot \bar{l})}{(p_2 \cdot \bar{l})_+} \right]. \end{aligned} \quad (71)$$

It is easy to see that the above integrals vanish. Hence, we see that for the case of one loop, the  $\langle \Delta \hat{J}_1^{\mu\nu} \rangle_{\text{cl}}$  does not get any contribution from the linear amplitude. The classical log soft factor arises from the  $\mathcal{T}_{i(1)\text{cl}}$  in (67) when there is no cut on the photon leg carrying momentum  $\bar{w}$ . The cut contribution to (67) at one loop can be written as

$$\begin{aligned} C^{\mu\nu} &= i\hbar \int d\mu(q_1, q_2, w_1, w_2) e^{-ib \cdot q_1 / \hbar} \delta^4(w_1 + w_2 - q_1 - q_2) \\ &\times \mathcal{A}^*(P + Q|P + W) \{ (p_1 \wedge \partial_{p_1})^{\mu\nu} + (w_1 \wedge \partial_{w_1})^{\mu\nu} \} (\delta^4(w_1 + w_2) \mathcal{A}(P|P + W)). \end{aligned} \quad (72)$$

We substitute the tree-level amplitude  $\mathcal{A}_0$  in the above expression. After some algebra, we can write this expression in terms of  $C_1^{\mu\nu}$  and  $C_2^{\mu\nu}$ .  $C_1^{\mu\nu}$  consists of the terms where derivative operators act on the tree-level amplitude

$$\begin{aligned} C_1^{\mu\nu} &= i\hbar^5 \int \prod_{i=1}^2 \hat{d}^4 \bar{q}_i \hat{d}^4 \bar{w}_i \hat{\delta}(2p_i \cdot \bar{q}_i + \hbar \bar{q}_i^2) \hat{\delta}(2p_i \cdot \bar{w}_i + \hbar \bar{w}_i^2) e^{-i\bar{q}_1 \cdot b} \delta^4(\bar{w}_1 + \bar{w}_2 - \bar{q}_1 - \bar{q}_2) \\ &\times \delta^4(\bar{w}_1 + \bar{w}_2) \mathcal{A}_0^*(P + \bar{Q}|P + \bar{W}) (p_1 \wedge \partial_{p_1})^{\mu\nu} (\mathcal{A}_0(P|P + \bar{W})) \\ &- i\hbar^5 \int \prod_{i=1}^2 \hat{d}^4 \bar{q}_i \hat{d}^4 \bar{w}_i \hat{\delta}(2p_i \cdot \bar{q}_i + \hbar \bar{q}_i^2) \hat{\delta}(2p_i \cdot \bar{w}_i + \hbar \bar{w}_i^2) e^{-i\bar{q}_1 \cdot b} \delta^4(\bar{w}_1 + \bar{w}_2 - \bar{q}_1 - \bar{q}_2) \\ &\times \delta^4(\bar{w}_1 + \bar{w}_2) \mathcal{A}_0(P|P + \bar{W}) (\bar{w}_1 \wedge \partial_{\bar{w}_1})^{\mu\nu} (\mathcal{A}_0^*(P + \bar{Q}|P + \bar{W})) \end{aligned} \quad (73)$$

$$= C_{11}^{\mu\nu} + C_{12}^{\mu\nu}. \quad (74)$$

$C_2^{\mu\nu}$  contains the terms where derivative operators act on the on-shell delta functions

$$\begin{aligned}
C_2^{\mu\nu} = & 2i\hbar^5 \int \prod_{i=1}^2 \hat{d}^4 \bar{q}_i \hat{d}^4 \bar{w}_i (\hat{\delta}(2p_1 \cdot \bar{q}_1 + \hbar \bar{q}_1^2) \hat{\delta}'(2p_1 \cdot \bar{w}_1 + \hbar \bar{w}_1^2) + \hat{\delta}'(2p_1 \cdot \bar{q}_1 + \hbar \bar{q}_1^2) \\
& \times \hat{\delta}(2p_1 \cdot \bar{w}_1 + \hbar \bar{w}_1^2)) \hat{\delta}(2p_2 \cdot \bar{q}_2 + \hbar \bar{q}_2^2) \hat{\delta}(2p_2 \cdot \bar{w}_2 + \hbar \bar{w}_2^2) (p_1 \wedge \bar{w}_1)^{\mu\nu} e^{-i\bar{q}_1 \cdot b} \\
& \times \delta^4(\bar{w}_1 + \bar{w}_2 - \bar{q}_1 - \bar{q}_2) \delta^4(\bar{w}_1 + \bar{w}_2) \mathcal{A}_0(P|P + \bar{W}) \mathcal{A}_0^*(P + \bar{Q}|P + \bar{W}) \\
& + 2i\hbar^6 \int \prod_{i=1}^2 \hat{d}^4 \bar{q}_i \hat{d}^4 \bar{w}_i \hat{\delta}'(2p_1 \cdot \bar{q}_1 + \hbar \bar{q}_1^2) \hat{\delta}(2p_1 \cdot \bar{w}_1 + \hbar \bar{w}_1^2) \hat{\delta}(2p_2 \cdot \bar{q}_2 + \hbar \bar{q}_2^2) \hat{\delta}(2p_2 \cdot \bar{w}_2 + \hbar \bar{w}_2^2) \\
& \times (\bar{q}_1 \wedge \bar{w}_1)^{\mu\nu} e^{-i\bar{q}_1 \cdot b} \delta^4(\bar{w}_1 + \bar{w}_2 - \bar{q}_1 - \bar{q}_2) \delta^4(\bar{w}_1 + \bar{w}_2) \mathcal{A}_0(P|P + \bar{W}) \\
& \times \mathcal{A}_0^*(P + \bar{Q}|P + \bar{W}) \\
& - i\hbar^5 \int \prod_{i=1}^2 \hat{d}^4 \bar{q}_i \hat{d}^4 \bar{w}_i \hat{\delta}(2p_i \cdot \bar{q}_i + \hbar \bar{q}_i^2) \hat{\delta}(2p_i \cdot \bar{w}_i + \hbar \bar{w}_i^2) e^{-i\bar{q}_1 \cdot b} \delta^4(\bar{w}_1 + \bar{w}_2 - \bar{q}_1 - \bar{q}_2) \\
& \times \delta^4(\bar{w}_1 + \bar{w}_2) \mathcal{A}_0(P|P + \bar{W}) (\bar{w}_1 \wedge \partial_{\bar{w}_1})^{\mu\nu} (\mathcal{A}_0^*(P + \bar{Q}|P + \bar{W}))
\end{aligned} \tag{75}$$

$$= C_{21}^{\mu\nu} + C_{22}^{\mu\nu} + C_{23}^{\mu\nu}. \tag{76}$$

The only diagram that contributes in this case is Fig. 7. We will do the transformation  $\bar{w} \rightarrow \bar{q} - \bar{w}$  and average over it before evaluating the  $\bar{w}$  and  $\bar{q}$  integrals. The two tree-level amplitudes in this case are

$$\mathcal{A}_0^*(P + \bar{Q}|P + \bar{W}) = \frac{-iQ_1 Q_2}{\hbar^3} \left( \frac{4p_1 \cdot p_2 + 2\hbar(p_2 - p_1) \cdot (\bar{w}_1 + \bar{q}_1) + \mathcal{O}(\hbar^2)}{(\bar{q}_2 - \bar{w}_2)_+^2} \right), \tag{77}$$

$$\mathcal{A}_0(P|P + \bar{W}) = \frac{iQ_1 Q_2}{\hbar^3} \left( \frac{4p_1 \cdot p_2 + 2\hbar(p_2 - p_1) \cdot \bar{w}_1 + \mathcal{O}(\hbar^2)}{(\bar{w}_2)_+^2} \right). \tag{78}$$

Using these expressions, we will only write the classical contributions that are  $\mathcal{O}(\log \omega)$ . The calculation in its entirety has been carried out in Appendix D,

$$C_{11}^{\mu\nu} \Big|_{\log} = -\frac{iQ_1^2 Q_2^2}{4} (p_1 \cdot p_2) \int d\mu_{\bar{q}, \bar{w}} e^{-i\bar{q} \cdot b} \frac{(p_1 \wedge \bar{q})^{\mu\nu}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2}, \tag{79}$$

$$C_{12}^{\mu\nu} \Big|_{\log} = \frac{iQ_1^2 Q_2^2}{4} (p_1 \cdot p_2) \int d\mu_{\bar{q}, \bar{w}} e^{-i\bar{q} \cdot b} \frac{(p_2 \wedge \bar{q})^{\mu\nu}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2}. \tag{80}$$

Adding these two contributions, we have

$$C_1^{\mu\nu} \Big|_{\log} = \frac{Q_1 Q_2}{8\pi\sqrt{D}} ((p_2 - p_1) \wedge \Delta p)^{\mu\nu} \log \omega. \tag{81}$$

Similarly for  $C_2^{\mu\nu}$  we have

$$\begin{aligned}
C_{21}^{\mu\nu} \Big|_{\log} = & \frac{3iQ_1^2 Q_2^2}{2} (p_1 \cdot p_2) \int d\mu_{\bar{q}, \bar{w}} e^{-i\bar{q} \cdot b} \frac{(p_1 \wedge \bar{w})^{\mu\nu}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} \\
& - iQ_1^2 Q_2^2 \left( \frac{m_2^2 (p_1 \cdot p_2)^2 + (p_1 \cdot p_2)^3}{D} \right) \int d\mu_{\bar{q}, \bar{w}} e^{-i\bar{q} \cdot b} \frac{(p_1 \wedge \bar{w})^{\mu\nu}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} \\
& - iQ_1^2 Q_2^2 \left( \frac{m_2^2 (p_1 \cdot p_2)^2 + (p_1 \cdot p_2)^3}{D} \right) \int d\mu_{\bar{q}, \bar{w}} e^{-i\bar{q} \cdot b} \frac{(p_1 \wedge \bar{w})^{\mu\nu}}{((\bar{q} - \bar{w})_+^2)^2},
\end{aligned} \tag{82}$$

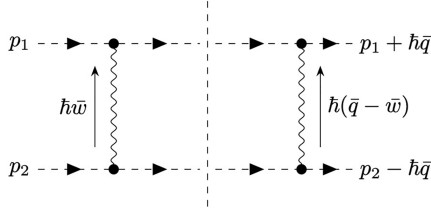


FIG. 7. The only possible diagrams for cut amplitude at one loop.

$$C_{22}^{\mu\nu} \Big|_{\log} = iQ_1^2 Q_2^2 \frac{m_2^2 (p_1 \cdot p_2)^2}{D} \int d\mu_{\bar{q}, \bar{w}} e^{-i\bar{q} \cdot b} \frac{(p_1 \wedge \bar{w})^{\mu\nu}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} - iQ_1^2 Q_2^2 \frac{(p_1 \cdot p_2)^3}{D} \int d\mu_{\bar{q}, \bar{w}} e^{-ib \cdot \bar{q}} \frac{(p_2 \wedge \bar{w})^{\mu\nu}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2}, \quad (83)$$

$$C_{23}^{\mu\nu} \Big|_{\log} = \frac{iQ_1^2 Q_2^2}{2} (p_1 \cdot p_2) \int d\mu_{\bar{q}, \bar{w}} e^{-ib \cdot \bar{q}} \frac{(p_2 \wedge \bar{w})^{\mu\nu}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2}. \quad (84)$$

Finally, adding all these terms we have

$$C_2^{\mu\nu} \Big|_{\log} = \left[ \frac{Q_1 Q_2}{8\pi\sqrt{D}} ((3p_1 + p_2) \wedge \Delta p)^{\mu\nu} - Q_1 Q_2 \frac{(p_1 \cdot p_2)^2}{4\pi D^{\frac{3}{2}}} ((p_1 + p_2) \wedge \Delta p)^{\mu\nu} \right] \log \omega. \quad (85)$$

And the cut contribution is

$$C^{\mu\nu} = -Q_1 Q_2 \frac{p_1^2 p_2^2}{4\pi D^{\frac{3}{2}}} ((p_1 + p_2) \wedge \Delta p)^{\mu\nu} \log \omega. \quad (86)$$

This is the angular impulse for one loop as can be seen using the expression for classical angular momentum from [51],

$$J_a^{\mu\nu} = -\log \omega \sum_{\substack{b \neq a \\ \eta_a \eta_b = 1}} \frac{Q_a Q_b}{4\pi} \frac{p_a^2 p_b^2 (p_a \wedge p_b)^{\mu\nu}}{((p_a \cdot p_b)^2 - p_a^2 p_b^2)^{3/2}}. \quad (87)$$

Here,  $a$  and  $b$  are the labels for the incoming and outgoing particles. Now for  $a = 1$  we can calculate the expression for angular impulse at  $\mathcal{O}(e^4)$ ,

$$\Delta J_1^{\mu\nu} |_{\mathcal{O}(e^4)} = -\log \omega \frac{Q_1 Q_2}{4\pi} \frac{p_1^2 p_2^2 ((p_1 + p_2) \wedge \Delta p)^{\mu\nu}}{((p_1 \cdot p_2)^2 - p_1^2 p_2^2)^{3/2}}. \quad (88)$$

Hence, (68) is verified at one loop.

## 2. Contribution from $\frac{\epsilon_\mu k_\nu}{(p_1 \cdot k)^{N+1}} \langle (q_1 \cdot k)^N J_{1,\text{out}}^{\mu\nu} \rangle$

We consider the observable

$$N^{\mu\nu} = \langle (q_1 \cdot k)^N J_{1,\text{out}}^{\mu\nu} \rangle. \quad (89)$$

There is no classical contribution from the linear-in-amplitude part of this observable due to its tensor structure hence we will only consider the cut contribution. The cut contribution for the  $L$ -loop amplitude has the form

$$N_{(L)}^{\mu\nu} = ie^{2L+2} \hbar^{N-L-1} \sum_{X=0}^{L-1} \sum_{L'=0}^{L-X-1} \left[ \int \prod_{m=1}^X d\Phi(k_m) d\mu(\bar{q}_1, \bar{q}_2, \bar{w}_1, \bar{w}_2) e^{-i\bar{q}_1 \cdot b} \hat{\delta}^4(q_1 + q_2) \times \hat{\delta}^4(\bar{w}_1 + \bar{w}_2 + \sum_{i=1}^m k_i) (\bar{w}_1 \cdot k)^N \tilde{\mathcal{A}}_{L'}(P|P + \bar{W}, \{\bar{k}_X\}) \times ((p_1 + \hbar \bar{w}_1) \wedge (\partial_{\bar{w}_1} + \partial_{\bar{q}_1}))^{\mu\nu} \tilde{\mathcal{A}}_{L-L'-X-1}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) + 2\hbar \int \prod_{m=1}^X d\Phi(k_m) d\mu(\bar{w}_1, \bar{w}_2) \prod_{i=2}^2 \hat{\delta}^4 \bar{q}_i \hat{\delta}'(2p_1 \cdot \bar{q}_1 + \hbar \bar{q}_1^2) \hat{\delta}(2p_2 \cdot \bar{q}_2 + \hbar \bar{q}_2^2) e^{-i\bar{q}_1 \cdot b} \times \hat{\delta}^4(q_1 + q_2) \hat{\delta}^4(\bar{w}_1 + \bar{w}_2 + \sum_{i=1}^m k_i) (\bar{w}_1 \cdot k)^N ((p_1 + \hbar \bar{w}_1) \wedge (p_1 + \hbar \bar{q}_1))^{\mu\nu} \times \tilde{\mathcal{A}}_{L'}(P|P + \bar{W}, \{\bar{k}_X\}) \tilde{\mathcal{A}}_{L-L'-X-1}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \right]. \quad (90)$$

In this expression, we have scaled all the couplings and off-shell momenta so that the amplitude starts with  $\mathcal{O}(\hbar^0)$  terms. When the operators  $\partial_{\bar{w}_1}$  and  $\partial_{\bar{q}_1}$  act on  $\tilde{\mathcal{A}}^*$ , it produces an additional  $\hbar$  making the overall scaling as  $\hbar^{N-L}$ . So  $L$ -loop amplitude terms of order  $N > L$  will not contribute classically. Now for one loop we write this contribution as  $N^{\mu\nu} = N_1^{\mu\nu} + N_2^{\mu\nu}$ ,

$$\begin{aligned}
N_1^{\mu\nu} = & -i\hbar^5 \int d\mu(\bar{q}, -\bar{q}, \bar{w}, -\bar{w}) e^{-i\bar{q}.b} \\
& \times \bar{w}.k \mathcal{A}_0(P|P + \bar{W})((p_1 + \hbar\bar{w}) \wedge \partial_{\bar{w}})^{\mu\nu} \mathcal{A}_0^*(P + \bar{W}|P + \bar{Q}),
\end{aligned} \tag{91}$$

and

$$\begin{aligned}
N_2^{\mu\nu} = & -i\hbar^5 \int d\mu(\bar{q}, -\bar{q}, \bar{w}, -\bar{w}) e^{-i\bar{q}.b} \\
& \times \bar{w}.k \mathcal{A}_0(P|P + \bar{W})((p_1 + \hbar\bar{w}) \wedge \partial_{\bar{q}})^{\mu\nu} \mathcal{A}_0^*(P + \bar{W}|P + \bar{Q}) \\
& - 2i\hbar^5 \int d\mu(\bar{w}, -\bar{w}) \hat{d}^4 \bar{q} \hat{d}^4 \bar{w} \hat{\delta}'(2p_1.\bar{q} + \hbar\bar{q}^2) \hat{\delta}(2p_2.\bar{q} - \hbar\bar{q}^2) e^{-i\bar{q}.b} \\
& \times (\bar{w}.k)((p_1 + \hbar\bar{w}) \wedge (p_1 + \hbar\bar{q}))^{\mu\nu} \mathcal{A}_0(P|P + \bar{W}) \mathcal{A}_0^*(P + \bar{W}|P + \bar{Q}).
\end{aligned} \tag{92}$$

The classical contribution from  $N_1^{\mu\nu}$  reads

$$N_1^{\mu\nu} = -\frac{iQ_1^2 Q_2^2}{2} (p_1.p_2)(p_1 \wedge p_2)^{\mu\nu} \int d\mu_{\bar{q},\bar{w}} e^{-i\bar{q}.b} \frac{\bar{w}.k}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2}. \tag{93}$$

Taking the average after the transformation  $\bar{w} \rightarrow \bar{q} - \bar{w}$ , we have

$$N_1^{\mu\nu} = -\frac{iQ_1^2 Q_2^2}{4} (p_1.p_2)(p_1 \wedge p_2)^{\mu\nu} \int d\mu_{\bar{q},\bar{w}} e^{-i\bar{q}.b} \frac{\bar{q}.k}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2}. \tag{94}$$

After evaluating the integrals, we write only  $\mathcal{O}(\log \omega)$  terms

$$N_1^{\mu\nu} = -\frac{Q_1 Q_2}{4\pi} \frac{\Delta p.k}{2\sqrt{D}} (p_1 \wedge p_2)^{\mu\nu} \log \omega. \tag{95}$$

Similarly, the classical terms from  $N_2^{\mu\nu}$ ,

$$\begin{aligned}
N_2^{\mu\nu} = & -\frac{iQ_1^2 Q_2^2}{2} (p_1.p_2)(p_1 \wedge p_2)^{\mu\nu} \int d\mu_{\bar{q},\bar{w}} e^{-i\bar{q}.b} \frac{\bar{w}.k}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} \\
& - 2iQ_1^2 Q_2^2 (p_1.p_2)^2 \int d\mu_{\bar{q},\bar{w}} \hat{d}^4 \bar{q} e^{-i\bar{q}.b} \hat{\delta}'(p_1.\bar{q}) \hat{\delta}(p_2.\bar{q}) \frac{\bar{w}.k(p_1 \wedge (\bar{q} - \bar{w}))^{\mu\nu}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2}.
\end{aligned} \tag{96}$$

Averaging over and doing IBP, we have

$$\begin{aligned}
N_2^{\mu\nu} = & -\frac{iQ_1^2 Q_2^2}{4} (p_1.p_2)(p_1 \wedge p_2)^{\mu\nu} \int d\mu_{\bar{q},\bar{w}} e^{-i\bar{q}.b} \frac{\bar{q}.k}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} \\
& + iQ_1^2 Q_2^2 \frac{(p_1.p_2)^3}{D} (p_1 \wedge p_2)^{\mu\nu} \int d\mu_{\bar{q},\bar{w}} e^{-i\bar{q}.b} \frac{\bar{q}.k}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2}.
\end{aligned} \tag{97}$$

Evaluating these integrals we have the  $\mathcal{O}(\log \omega)$  terms

$$N_2^{\mu\nu} = -\left[ \frac{Q_1 Q_2}{4\pi} \frac{\Delta p.k}{2\sqrt{D}} (p_1 \wedge p_2)^{\mu\nu} - \frac{Q_1 Q_2}{4\pi} \frac{(p_1.p_2)^2}{D^{\frac{3}{2}}} (\Delta p.k)(p_1 \wedge p_2)^{\mu\nu} \right] \log \omega. \tag{98}$$

Adding these two results, we have

$$N^{\mu\nu} = -\frac{Q_1 Q_2}{4\pi} \frac{(\Delta p \cdot k)(p_1^2 p_2^2)}{D^{\frac{3}{2}}} (p_1 \wedge p_2)^{\mu\nu} \log \omega. \quad (99)$$

Finally, we have the expectation value

$$\begin{aligned} & \left\langle \frac{(q_1 \cdot k)}{(p_1 \cdot k)^2} \varepsilon_\mu k_\nu J_{1,\text{out}}^{\mu\nu} \right\rangle \\ &= -\frac{Q_1 Q_2 (p_1^2 p_2^2)}{4\pi D^{\frac{3}{2}}} \frac{\Delta p \cdot k}{(p_1 \cdot k)^2} \varepsilon_\mu k_\nu (p_1 \wedge p_2)^{\mu\nu} \log \omega. \end{aligned} \quad (100)$$

Hence, the second classical constraint (70) is verified at one loop.

### B. Subleading soft radiation kernel at one loop

Having calculated the two quantities in the expression (40), we can now write the constraints that follow from the universal classical subleading soft factor at one loop. The first of the two constraints is the usual angular impulse at one loop, viz.

$$\begin{aligned} \langle \Delta \hat{J}_1^{\mu\nu} \rangle|_{\mathcal{O}(\log \omega, e^4)} &= -\frac{Q_1 Q_2}{4\pi} \frac{p_1^2 p_2^2}{((p_1 \cdot p_2)^2 - p_1^2 p_2^2)^{\frac{3}{2}}} \\ &\times ((p_1 + p_2) \wedge \Delta p)^{\mu\nu} \log \omega. \end{aligned} \quad (101)$$

The second constraint is on the product of the first moment and the final angular momentum of the first particle

$$\begin{aligned} \langle q^\alpha \hat{J}_1'^{\mu\nu} \rangle|_{\mathcal{O}(\log \omega, e^4)} &= -\frac{Q_1 Q_2}{4\pi} \frac{p_1^2 p_2^2}{((p_1 \cdot p_2)^2 - p_1^2 p_2^2)^{\frac{3}{2}}} \\ &\times \Delta p^\alpha (p_1 \wedge p_2)^{\mu\nu} \log \omega. \end{aligned} \quad (102)$$

## VIII. DISCUSSION

Classical soft photon theorems produce a remarkable set of nonperturbative observables for classical electromagnetic scattering. These observables determine the electromagnetic memory, and its tails which decay as  $\frac{(\ln u)^m}{u^{m+1}}$ , where  $u$  is the affine parameter on  $\mathcal{I}^+$ . Universality of the classical soft theorems ensures that these observables only depend on the initial and final momenta, masses, and charges of the scattering particles and are independent of higher-order multipole moments. These theorems have a direct counterpart in S-matrix theory and are known as soft factorization theorems. But even though the electromagnetic memory and the Weinberg soft photon factor are simply “Fourier transforms” of each other [60], the tail to the electromagnetic memory does not equal the Fourier transform of the log soft factor that appears when we expand an amplitude in QED to subleading order in the soft expansion. As argued in [51,57], this distinction is directly tied to the fact

that the microcausality of the quantum S-matrix, which is encapsulated by the nature of the Feynman propagator, is structurally distinct from macroscopic causality of classical scattering which is encapsulated by the fact that the relationship between radiative field and source is governed by the retarded Green’s function. It is reasonable to expect that the classical limit of the S-matrix should satisfy macrocausality, but the precise quantification of this statement is hard to formulate sharply in analytic S-matrix theory [58].

In this paper, one can formulate a KMOC-level diagnostic of macroscopic causality for the classical limit of the QED S-matrix. More in detail, we prove that the difference between classical and quantum log soft factors can be used to write a hierarchy of constraints on QED S-matrix so that the radiation kernel computed using the KMOC formalism produces the leading classical log soft factor at all orders in the perturbative expansion. The constraints are summarized in (60). These constraints could also be thought of as providing subleading soft constraints on KMOC formalism in the spirit of [53], where it was shown that the soft radiative field computed using KMOC formalism equals electromagnetic memory if and only if a class of observables in the conservative sector satisfy a constraint. In Sec. IV, we prove the leading-order soft constraints of [53] to all orders in perturbative expansion.

Our line of inquiry opens up several avenues of investigation. The immediate task is to prove these constraints. While we have verified them at one loop, a proof at all loops remains open. It will also be interesting to know if these constraints can be used as a definition of macrocausality of the S-matrix. Such a proposal would be viable if, for example, the following were true: Assuming that an S-matrix involving photons and charged scalars satisfies (41), does it imply that to the subsubleading order in soft expansion, the radiation kernel equals the tail of tail derived by Sahoo in [52]?

Our results suggest that the universality of classical log soft radiation encodes genuinely new, nonperturbative information about the structure of the S-matrix, beyond that captured by standard soft theorems and leading-order memory effects. It would be interesting to investigate whether analogous constraints arise in gravitational scattering, where tail effects and infrared structure are even richer, and to explore whether the quadratic constraint admits an interpretation in terms of symmetry properties of amplitudes.

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### DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

### APPENDIX A: CALCULATION OF $G_{i,n}^{(L)}$ AND $H_{i,n}^{(L)}$

We start with the angular impulse in (42). We can use IBP and transfer the derivative  $\partial_{q_1}$  in the linear amplitude term. One such term, for example, when the derivative acts on the on-shell delta function  $\hat{\delta}(2p_1 \cdot q_1 + q_1^2)$ , is

$$\frac{2}{\hbar^L} \int \hat{d}^4(\bar{q}_1, \bar{q}_2) \hat{\delta}'(2p_1 \cdot \bar{q}_1 + \hbar \bar{q}_1^2) \hat{\delta}(2p_1 \cdot \bar{q}_2 + \hbar \bar{q}_2^2) \hat{\delta}^4(\bar{q}_1 + \bar{q}_2) e^{-ib \cdot \bar{q}_1} p_1 \wedge \bar{q}_1 \tilde{\mathcal{A}}_{(L)}(P|P + \hbar \bar{Q}).$$

We have scaled the transfer momenta by  $\hbar$  and have used the coupling-stripped amplitude  $\tilde{\mathcal{A}}_{(L)}$ . Performing  $\hbar$  expansion of the on-shell delta functions and, after the change of variable (45), transferring the derivatives at each order over to the amplitude, we obtain

$$-\frac{1}{4\hbar^L} \int d\mu_{\bar{q}} e^{-ib \cdot \bar{q}} \sum_{n=0}^L \frac{(-1)^n \hbar^n}{2^n n!} \partial_{x_1} (\partial_{x_1} - \partial_{x_2})^n ((\bar{q}^2)^n p_1 \wedge \bar{q}_1 \tilde{\mathcal{A}}_{(L)}(P|P + \hbar \bar{Q})), \quad (\text{A1})$$

Where  $x_i = p_i \cdot \bar{q}$ . Similarly, the other terms in (43) can be obtained. We write the expressions for  $G_{i,n}^{(L)}$  in (44) below

$$G_{2,n}^{(L)} = 2i \partial_{x_1} (\partial_{x_1} - \partial_{x_2})^n \sum_{X=0}^{L-1} \sum_{L'=0}^{L-X-1} (\bar{q}^2)^n \int \prod_{m=1}^X d\Phi(\bar{k}_m) d\mu(\bar{w}_1, \bar{w}_2) \hat{\delta}^4(\bar{w}_1 + \bar{w}_2 + \Sigma_{i=1}^X \bar{k}_i) \\ \times \bar{w}_1 \wedge (p_1 + \hbar \bar{q}_1) \tilde{\mathcal{A}}_{(L')} (P|P + \bar{W}, \{k_X\}) \tilde{\mathcal{A}}_{(L-L'-X-1)}^* (P + \bar{Q}|P + \bar{W}, \{k_X\})|_{\bar{q}_1 = -\bar{q}_2 = \bar{q}}. \quad (\text{A2})$$

$$G_{1,n}^{(L)} = i(\partial_{x_1} - \partial_{x_2})^n \sum_{X=0}^{L-1} \sum_{L'=0}^{L-X-1} (\bar{q}^2)^n \int \prod_{m=1}^X d\Phi(\bar{k}_m) [d\mu(\bar{w}_1, \bar{w}_2) \hat{\delta}^4(\bar{w}_1 + \bar{w}_2 + \Sigma_{i=1}^X \bar{k}_i) \\ \times \left( -p_1 \wedge \partial_{p_1} \tilde{\mathcal{A}}_{(L')} (P|P + \bar{W}, \{\bar{k}_X\}) \tilde{\mathcal{A}}_{(L-L'-X-1)}^* (P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \right. \\ \left. + \tilde{\mathcal{A}}_{(L')} (P|P + \bar{W}, \{\bar{k}_X\}) (\bar{w}_1 \wedge \partial_{\bar{w}_1} + \bar{q}_1 \wedge \partial_{\bar{q}_1}) \tilde{\mathcal{A}}_{(L-L'-X-1)}^* (P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \right. \\ \left. + ib \wedge \bar{w}_1 \tilde{\mathcal{A}}_{(L')} (P|P + \bar{W}, \{\bar{k}_X\}) \tilde{\mathcal{A}}_{(L-L'-X-1)}^* (P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \right) \\ \left. + 2 \int \hat{d}^4(\bar{w}_1, \bar{w}_2) \hat{\delta}'(2p_1 \cdot \bar{w}_1 + \hbar \bar{w}_1^2) \hat{\delta}(2p_2 \cdot \bar{w}_2 + \hbar \bar{w}_2^2) \hat{\delta}^4(\bar{w}_1 + \bar{w}_2 + \Sigma_{i=1}^X \bar{k}_i) \right. \\ \left. \times \bar{w}_1 \wedge p_1 \tilde{\mathcal{A}}_{(L')} (P|P + \bar{W}, \{\bar{k}_X\}) \tilde{\mathcal{A}}_{(L-L'-X-1)}^* (P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \right]|_{\bar{q}_1 = -\bar{q}_2 = \bar{q}}. \quad (\text{A3})$$

Calculation of  $H_{i,n}^{(L)}$  in (48) proceeds in a similar manner,

$$H_{1,n}^{(L)} = -(\partial_{x_1} - \partial_{x_2})^n \left( (\bar{q}^2)^n (p_1 + \hbar \bar{q}) \wedge \eta^{(\alpha_1} \bar{q}^{\alpha_2} \dots \bar{q}^{\alpha_N}) \tilde{\mathcal{A}}_{(L)}(P|P + \bar{Q}) \right. \\ \left. + \frac{i}{\hbar} (p_1 + \hbar \bar{q}) \wedge b \bar{q}^{\alpha_1} \dots \bar{q}^{\alpha_N} \tilde{\mathcal{A}}_{(L)}(P|P + \bar{Q}) \right), \quad (\text{A4})$$

and

$$\begin{aligned}
H_{2,n}^{(L)} = & (\partial_{x_1} - \partial_{x_2})^n \sum_{X=0}^{L-1} \sum_{L'=0}^{L-X-1} (\bar{q}^2)^n \int \prod_{m=1}^X d\Phi(\bar{k}_m) d\mu(\bar{w}_1, \bar{w}_2) \hat{\delta}^4(\bar{w}_1 + \bar{w}_2 + \Sigma_{i=1}^X \bar{k}_i) \\
& \times \left[ -i(p_1 + \hbar \bar{w}_1) \wedge \eta^{(\alpha_1 \bar{w}_1^{\alpha_2} \dots \bar{w}_1^{\alpha_N})} \right. \\
& \times \tilde{\mathcal{A}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \tilde{\mathcal{A}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \\
& - i\bar{w}_1^{\alpha_1} \dots \bar{w}_1^{\alpha_N} (p_1 + \hbar \bar{w}_1) \wedge (\partial_{\bar{q}_1} + \partial_{\bar{w}_1}) \\
& \times \tilde{\mathcal{A}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \tilde{\mathcal{A}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \\
& + \frac{1}{\hbar} (p_1 + \hbar \bar{w}_1) \wedge b\bar{w}_1^{\alpha_1} \dots \bar{w}_1^{\alpha_N} \\
& \times \tilde{\mathcal{A}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \tilde{\mathcal{A}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \Big]_{\bar{q}_1 = -\bar{q}_2 = \bar{q}} \\
& - 2i\partial_{x_1}(\partial_{x_1} - \partial_{x_2})^n \sum_{X=0}^{L-1} \sum_{L'=0}^{L-X-1} (\bar{q}^2)^n \int \prod_{m=1}^X d\Phi(\bar{k}_m) d\mu(\bar{w}_1, \bar{w}_2) \hat{\delta}^4(\bar{w}_1 + \bar{w}_2 + \Sigma_{i=1}^X \bar{k}_i) \\
& \times (p_1 + \hbar \bar{w}_1) \wedge (p_1 + \hbar \bar{q}_1) \bar{w}_1^{\alpha_1} \dots \bar{w}_1^{\alpha_N} \tilde{\mathcal{A}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \\
& \times \tilde{\mathcal{A}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \Big|_{\bar{q}_1 = -\bar{q}_2 = \bar{q}}.
\end{aligned} \tag{A5}$$

## APPENDIX B: CALCULATING $\mathcal{T}$

In this section, we proceed to write the cut integrals in  $\langle \Delta \hat{J}_i^{\mu\nu} \rangle$  and  $\langle q_i^{\alpha_1} \dots q_i^{\alpha_N} \hat{J}_{i,\text{out}}^{\mu\nu} \rangle$ . The type of diagrams that contribute to the cut (57) are defined in Sec. VI. We use those diagrams in the expressions (A2) and (A3) and define the following terms:

$$\begin{aligned}
\mathcal{T}_{1(L)}^{\mu\nu} = & \sum_{X=0}^{L-1} \sum_{L'=0}^{L-X-1} \int \prod_{m=1}^X d\Phi(\bar{k}_m) \hat{\delta}^4(\bar{w}_1 + \bar{w}_2 + \Sigma_{i=1}^X \bar{k}_i) \Big[ (p_1 \wedge \partial_{p_1})^{\mu\nu} \tilde{\mathcal{K}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \\
& \times \tilde{\mathcal{A}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) + (p_1 \wedge \partial_{p_1})^{\mu\nu} \tilde{\mathcal{A}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \\
& \times \tilde{\mathcal{K}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \Big]_{\bar{q}_1 = -\bar{q}_2 = \bar{q}},
\end{aligned} \tag{B1}$$

$$\begin{aligned}
\mathcal{T}_{2(L)}^{\mu\nu} = & \sum_{X=0}^{L-1} \sum_{L'=0}^{L-X-1} \int \prod_{m=1}^X d\Phi(\bar{k}_m) \hat{\delta}^4(\bar{w}_1 + \bar{w}_2 + \Sigma_{i=1}^X \bar{k}_i) \Big[ \tilde{\mathcal{K}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \\
& \times (\bar{w}_1 \wedge (\partial_{\bar{w}_1} + \partial_{\bar{q}_1}))^{\mu\nu} \tilde{\mathcal{A}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) + \tilde{\mathcal{A}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \\
& \times (\bar{w}_1 \wedge (\partial_{\bar{w}_1} + \partial_{\bar{q}_1}))^{\mu\nu} \tilde{\mathcal{K}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \Big]_{\bar{q}_1 = -\bar{q}_2 = \bar{q}},
\end{aligned} \tag{B2}$$

and

$$\begin{aligned}
\mathcal{T}_{3(L)} = & \sum_{X=0}^{L-1} \sum_{L'=0}^{L-X-1} \int \prod_{m=1}^X d\Phi(\bar{k}_m) \hat{\delta}^4(\bar{w}_1 + \bar{w}_2 + \Sigma_{i=1}^X \bar{k}_i) \Big[ \tilde{\mathcal{K}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \\
& \times \tilde{\mathcal{A}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) + \tilde{\mathcal{A}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \\
& \times \tilde{\mathcal{K}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \Big]_{\bar{q}_1 = -\bar{q}_2 = \bar{q}}.
\end{aligned} \tag{B3}$$

We also define similar terms for the cut (61),

$$\begin{aligned} T_{4(L)}^{\mu\nu} = & \sum_{X=0}^{L-1} \sum_{L'=0}^{L-X-1} (\bar{q}^2)^n \int \prod_{m=1}^X d\Phi(\bar{k}_m) \hat{\delta}^4(\bar{w}_1 + \bar{w}_2 + \sum_{i=1}^X \bar{k}_i) ((p_1 + \hbar \bar{w}_1) \wedge (\partial_{\bar{q}_1} + \partial_{\bar{w}_1}))^{\mu\nu} \\ & \times \left( \tilde{\mathcal{K}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \tilde{\mathcal{A}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \right. \\ & \left. + \tilde{\mathcal{A}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \tilde{\mathcal{K}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \right)_{\bar{q}_1 = -\bar{q}_2 = \bar{q}}, \end{aligned} \quad (\text{B4})$$

$$\begin{aligned} T_{5(L)} = & 2 \sum_{X=0}^{L-1} \sum_{L'=0}^{L-X-1} (\bar{q}^2)^n \int \prod_{m=1}^X d\Phi(\bar{k}_m) \hat{\delta}^4(\bar{w}_1 + \bar{w}_2 + \sum_{i=1}^X \bar{k}_i) \\ & \times \left( \tilde{\mathcal{K}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \tilde{\mathcal{A}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \right. \\ & \left. + \tilde{\mathcal{A}}_{(L-L'-X-1)}^*(P + \bar{Q}|P + \bar{W}, \{\bar{k}_X\}) \tilde{\mathcal{K}}_{(L')}(P|P + \bar{W}, \{\bar{k}_X\}) \right)_{\bar{q}_1 = -\bar{q}_2 = \bar{q}}. \end{aligned} \quad (\text{B5})$$

### APPENDIX C: $R^{\mu\nu}$ AT ONE LOOP

In order to extract the classical pieces, we scale the loop momenta and  $q$  by  $\hbar$  in the diagrams (4). The terms that give finite contributions in the classical limit are  $O(\hbar^0)$  and  $O(\hbar)$  in the amplitude. We write both terms separately:

$$\begin{aligned} \tilde{B}_0 + \tilde{C}_0 = & 4iQ_1^2 Q_2^2 (p_1 \cdot p_2)^2 \int \frac{\hat{d}^4 \bar{l}}{\bar{l}_+^2 (\bar{q} - \bar{l})_+^2} \left( \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_-} - \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot (\bar{q} - \bar{l}))_+} \right. \\ & \left. + \frac{1}{(p_1 \cdot (\bar{q} - \bar{l}))_- (p_2 \cdot (\bar{q} - \bar{l}))_+} - \frac{1}{(p_1 \cdot (\bar{q} - \bar{l}))_- (p_2 \cdot \bar{l})_-} \right), \end{aligned} \quad (\text{C1})$$

$$\begin{aligned} \tilde{B}_1 + \tilde{C}_1 = & -i\hbar Q_1^2 Q_2^2 \left[ (p_1 \cdot p_2) \int \frac{\hat{d}^4 \bar{l}}{\bar{l}_+^2 (\bar{q} - \bar{l})_+^2} \left( \frac{6(p_1 - p_2) \cdot \bar{q} + 4(p_2 - p_1) \cdot \bar{l}}{(p_1 \cdot (\bar{q} - \bar{l}))_- (p_2 \cdot (\bar{q} - \bar{l}))_+} \right. \right. \\ & + \frac{2(p_1 - p_2) \cdot \bar{q} + 4(p_1 - p_2) \cdot \bar{l}}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_-} + \frac{(2p_2 - 6p_1) \cdot \bar{q} + 4(p_1 + p_2) \cdot \bar{l}}{(p_1 \cdot \bar{l})_+ (p_2 \cdot (\bar{q} - \bar{l}))_+} \\ & + \frac{(6p_2 - 2p_1) \cdot \bar{q} - 4(p_1 + p_2) \cdot \bar{l}}{(p_1 \cdot (\bar{q} - \bar{l}))_- (p_2 \cdot \bar{l})_-} \left. \right) - 2(p_1 \cdot p_2)^2 \int \frac{\hat{d}^4 \bar{l}}{(\bar{q} - \bar{l})_+^2} \left( \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_-^2} \right. \\ & + \frac{1}{(p_1 \cdot \bar{l})_+^2 (p_2 \cdot (\bar{q} - \bar{l}))_+} - \frac{1}{(p_1 \cdot \bar{l})_+^2 (p_2 \cdot \bar{l})_-} - \frac{1}{(p_1 \cdot (\bar{q} - \bar{l}))_- (p_2 \cdot \bar{l})_-^2} \left. \right) \\ & - 2(p_1 \cdot p_2)^2 \int \frac{\hat{d}^4 \bar{l}}{\bar{l}_+^2} \left( \frac{1}{(p_1 \cdot (\bar{q} - \bar{l}))_-^2 (p_2 \cdot (\bar{q} - \bar{l}))_+} - \frac{1}{(p_1 \cdot (\bar{q} - \bar{l}))_- (p_2 \cdot (\bar{q} - \bar{l}))_+^2} \right. \\ & \left. \left. + \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot (\bar{q} - \bar{l}))_+^2} - \frac{1}{(p_1 \cdot (\bar{q} - \bar{l}))_-^2 (p_2 \cdot \bar{l})_-} \right) \right]. \end{aligned} \quad (\text{C2})$$

$\tilde{B}$  and  $\tilde{C}$  respectively represent the box and crossed-box diagrams in Fig. 4. In the intermediate step, we have terms with tensor  $(p_1 \wedge \bar{l})^{\mu\nu}$ . As discussed before, this can be decomposed and absorbed into the two tensor types  $(p_1 \wedge \bar{p}_2)^{\mu\nu}$  and  $(p_1 \wedge \bar{q})^{\mu\nu}$  to obtain the tensor split written in (59). For the purpose of our calculation at one loop, we do not perform such a decomposition, as it turns out at one loop; considering terms of the form  $(p_1 \wedge \bar{l})^{\mu\nu}$  separately is sufficient. So we write

$$R_{(1)}^{\mu\nu} = \frac{i}{4\hbar} \int \hat{d}^4 \bar{q} \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) e^{-ib \cdot \bar{q}} \left[ (p_1 \wedge \partial_{p_1})^{\mu\nu} (\tilde{\mathcal{B}}_1 + \tilde{\mathcal{C}}_1) - \partial_{x_1}((p_1 \wedge \bar{q})^{\mu\nu} (\tilde{\mathcal{B}}_1 + \tilde{\mathcal{C}}_1)) \right. \\ \left. - \frac{\hbar}{2} (\partial_{x_1} - \partial_{x_2})(\bar{q}^2 (p_1 \wedge \partial_{p_1})^{\mu\nu} (\tilde{\mathcal{B}}_0 + \tilde{\mathcal{C}}_0) - \partial_{x_1}(\bar{q}^2 (p_1 \wedge \bar{q})^{\mu\nu} (\tilde{\mathcal{B}}_0 + \tilde{\mathcal{C}}_0))) \right]. \quad (C3)$$

The linear contribution can be further simplified as

$$R_{(1)}^{\mu\nu} = \frac{i}{4\hbar} \int \hat{d}^4 \bar{q} \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) e^{-i\bar{q}b} (p_1 \wedge \partial_{p_1})^{\mu\nu} (\tilde{\mathcal{B}}_1 + \tilde{\mathcal{C}}_1) \\ - \frac{i}{8} \int \hat{d}^4 \bar{q} \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) e^{-i\bar{q}b} (\partial_{x_1} - \partial_{x_2})(p_1 \wedge \partial_{p_1})^{\mu\nu} (\tilde{\mathcal{B}}_0 + \tilde{\mathcal{C}}_0) \\ - \frac{i}{4\hbar} \int \hat{d}^4 \bar{q} \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) e^{-i\bar{q}b} \partial_{x_1} \left( (p_1 \wedge \bar{q})^{\mu\nu} (\tilde{\mathcal{B}}_1 + \tilde{\mathcal{C}}_1) \right) \\ + \frac{i}{8} \int \hat{d}^4 \bar{q} \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) e^{-i\bar{q}b} (\partial_{x_1} - \partial_{x_2}) \partial_{x_1} \left( (p_1 \wedge \bar{q})^{\mu\nu} (\tilde{\mathcal{B}}_0 + \tilde{\mathcal{C}}_0) \right) \\ = R_{1,\tilde{\mathcal{B}}_1+\tilde{\mathcal{C}}_1}^{\mu\nu} + R_{1,\tilde{\mathcal{B}}_0+\tilde{\mathcal{C}}_0}^{\mu\nu} + R_{2,\tilde{\mathcal{B}}_1+\tilde{\mathcal{C}}_1}^{\mu\nu} + R_{2,\tilde{\mathcal{B}}_0+\tilde{\mathcal{C}}_0}^{\mu\nu}. \quad (C4)$$

We consider the three tensors in  $R_{(1)}^{\mu\nu}$ ,  $(p_1 \wedge p_2)^{\mu\nu}$ ,  $(p_1 \wedge \bar{q})^{\mu\nu}$ ,  $(p_1 \wedge \bar{l})^{\mu\nu}$ , separately:

$$(R_{1,\tilde{\mathcal{B}}_1+\tilde{\mathcal{C}}_1}^{\mu\nu})_{(p_1 \wedge p_2)} = -iQ_1^2 Q_2^2 (p_1 \cdot p_2) (p_1 \wedge p_2)^{\mu\nu} \int d\mu_{\bar{q}} e^{-i\bar{q} \cdot b} \int \hat{d}^4 \bar{l} \\ \times \left[ \frac{\hat{\delta}(p_2 \cdot \bar{l})}{\bar{l}_+^2 (p_1 \cdot \bar{l})_-^2} + \frac{\hat{\delta}(p_1 \cdot \bar{l})}{\bar{l}_+^2 (p_2 \cdot \bar{l})_+^2} + \frac{\hat{\delta}(p_2 \cdot \bar{l})}{(\bar{q} - \bar{l})_+^2 (p_1 \cdot \bar{l})_+^2} + \frac{\hat{\delta}(p_1 \cdot \bar{l})}{(\bar{q} - \bar{l})_+^2 (p_2 \cdot \bar{l})_-^2} \right]. \quad (C5)$$

Next, we have

$$(R_{1,\tilde{\mathcal{B}}_1+\tilde{\mathcal{C}}_1}^{\mu\nu})_{(p_1 \wedge \bar{q})} = -\frac{Q_1^2 Q_2^2}{2} (p_1 \cdot p_2) \int d\mu_{\bar{q}} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{q})^{\mu\nu} \int \frac{\hat{d}^4 \bar{l}}{\bar{l}_+^2 (\bar{q} - \bar{l})_+^2} \\ \times \left[ -\left( \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_-} + \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_+} \right) \right. \\ \left. + 3 \left( \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_+} + \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_-} \right) \right] \\ - Q_1^2 Q_2^2 (p_1 \cdot p_2)^2 \int d\mu_{\bar{q}} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{q})^{\mu\nu} \int \hat{d}^4 \bar{l} \\ \times \left[ \frac{i\hat{\delta}(p_2 \cdot \bar{l})}{\bar{l}^2 (p_1 \cdot \bar{l})_-^3} + \frac{1}{2\bar{l}^2 (p_1 \cdot \bar{l})_-^2 (p_2 \cdot \bar{l})_+^2} + \frac{1}{2(\bar{q} - \bar{l})^2 (p_1 \cdot \bar{l})_-^2 (p_2 \cdot \bar{l})_-^2} \right]. \quad (C6)$$

Finally, the  $(p_1 \wedge \bar{l})$  term is

$$(R_{1,\tilde{\mathcal{B}}_1+\tilde{\mathcal{C}}_1}^{\mu\nu})_{(p_1 \wedge \bar{l})} = iQ_1^2 Q_2^2 (p_1 \cdot p_2)^2 \int d\mu_{\bar{q}} e^{-i\bar{q} \cdot b} \int \hat{d}^4 \bar{l} (p_1 \wedge \bar{l})^{\mu\nu} \\ \times \left[ \left( -\frac{\hat{\delta}(p_2 \cdot \bar{l})}{\bar{l}_+^2 (p_1 \cdot \bar{l})_-^3} - \frac{\hat{\delta}(p_2 \cdot \bar{l})}{(\bar{q} - \bar{l})_+^2 (p_1 \cdot \bar{l})_+^3} + \frac{\hat{\delta}'(p_1 \cdot \bar{l})}{2(\bar{q} - \bar{l})_+^2 (p_2 \cdot \bar{l})_-^2} + \frac{\hat{\delta}'(p_1 \cdot \bar{l})}{2\bar{l}_+^2 (p_2 \cdot \bar{l})_+^2} \right) \right]. \quad (C7)$$

Moving onto  $(R_{1,\tilde{B}_0+\tilde{C}_0}^{\mu\nu})$  we have

$$(R_{1,\tilde{B}_0+\tilde{C}_0}^{\mu\nu})_{(p_1 \wedge p_2)} = Q_1^2 Q_2^2 (p_1 \cdot p_2) (p_1 \wedge p_2)^{\mu\nu} \int d\mu_{\bar{q}} \bar{q}^2 e^{-i\bar{q} \cdot b} \int \frac{\hat{d}^4 \bar{l}}{\bar{l}_+^2 (\bar{q} - \bar{l})_+^2} \times \left[ \frac{(p_1 \cdot p_2) m_1^2}{2D} \frac{i\hat{\delta}(p_1 \cdot \bar{l})}{(p_1 \cdot \bar{l})_-^2} + \frac{i\hat{\delta}(p_2 \cdot \bar{l})}{(p_1 \cdot \bar{l})_-^2} + \frac{i\hat{\delta}(p_1 \cdot \bar{l})}{(p_2 \cdot \bar{l})_+^2} + \frac{(p_1 \cdot p_2)^2}{2D} \frac{i\hat{\delta}(p_2 \cdot \bar{l})}{(p_1 \cdot \bar{l})_-^2} \right]. \quad (C8)$$

The  $(p_1 \wedge \bar{l})^{\mu\nu}$  terms are

$$(R_{1,\tilde{B}_0+\tilde{C}_0}^{\mu\nu})_{(p_1 \wedge \bar{l})} = Q_1^2 Q_2^2 \left( \frac{m_2^2 (p_1 \cdot p_2)^2 + (p_1 \cdot p_2)^3}{D} \right) \int d\mu_{\bar{q}} \bar{q}^2 e^{-i\bar{q} \cdot b} \times \int \frac{\hat{d}^4 \bar{l}}{\bar{l}_+^2 ((\bar{q} - \bar{l})_+^2)^2} (p_1 \wedge \bar{l})^{\mu\nu} \hat{\delta}(p_1 \cdot \bar{l}) \hat{\delta}(p_2 \cdot \bar{l}) + i Q_1^2 Q_2^2 (p_1 \cdot p_2)^2 \int d\mu_{\bar{q}} \bar{q}^2 e^{-i\bar{q} \cdot b} \times \int \frac{\hat{d}^4 \bar{l}}{\bar{l}_+^2 (\bar{q} - \bar{l})_+^2} (p_1 \wedge \bar{l})^{\mu\nu} \left[ -\frac{\hat{\delta}'(p_1 \cdot \bar{l})}{2(p_2 \cdot \bar{l})_+^2} + \frac{\hat{\delta}(p_2 \cdot \bar{l})}{(p_1 \cdot \bar{l})_-^3} \right]. \quad (C9)$$

Contributions from terms containing the  $(p_1 \wedge \bar{q})^{\mu\nu}$  are

$$(R_{1,\tilde{B}_0+\tilde{C}_0}^{\mu\nu})_{(p_1 \wedge \bar{q})} = Q_1^2 Q_2^2 (p_1 \cdot p_2)^2 \int d\mu_{\bar{q}} \bar{q}^2 e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{q})^{\mu\nu} \int \frac{\hat{d}^4 \bar{l}}{\bar{l}_+^2 ((\bar{q} - \bar{l})_+^2)^2} \left[ \frac{m_2^2}{D} \left( -\frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_-} + \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_+} \right) + \frac{(p_1 \cdot p_2)}{D} \left( -\frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_-} + \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_+} \right) + \frac{(\bar{q} - \bar{l})^2}{(p_1 \cdot \bar{l})_-^3} \left( \frac{1}{(p_2 \cdot \bar{l})_-} - \frac{1}{(p_2 \cdot \bar{l})_+} \right) + \frac{(\bar{q} - \bar{l})^2}{(p_1 \cdot \bar{l})_-^2 (p_2 \cdot \bar{l})_+^2} \right]. \quad (C10)$$

$R_2^{\mu\nu}$  contains only  $(p_1 \wedge p_2)^{\mu\nu}$  and  $(p_1 \wedge \bar{q})^{\mu\nu}$  terms. First, we write the terms with  $(p_1 \wedge p_2)^{\mu\nu}$ ,

$$(R_{2,\tilde{B}_1+\tilde{C}_1}^{\mu\nu})_{p_1 \wedge p_2} = \frac{i Q_1^2 Q_2^2 (p_1 \cdot p_2)^3}{2D} (p_1 \wedge p_2)^{\mu\nu} \int d\mu_{\bar{q}} e^{-i\bar{q} \cdot b} \int \frac{\hat{d}^4 \bar{l}}{\bar{l}_+^2 (\bar{q} - \bar{l})_+^2} \times \left[ \frac{\bar{l}^2}{(p_1 \cdot \bar{l})_+^2} \hat{\delta}(p_2 \cdot \bar{l}) + \frac{(\bar{q} - \bar{l})^2}{(p_1 \cdot \bar{l})_-^2} \hat{\delta}(p_2 \cdot \bar{l}) + \frac{\bar{l}^2}{(p_2 \cdot \bar{l})_-^2} \hat{\delta}(p_1 \cdot \bar{l}) + \frac{(\bar{q} - \bar{l})^2}{(p_2 \cdot \bar{l})_+^2} \hat{\delta}(p_1 \cdot \bar{l}) \right]. \quad (C11)$$

Now, considering terms containing  $(p_1 \wedge \bar{q})^{\mu\nu}$ , we have

$$(R_{2,\tilde{B}_1+\tilde{C}_1}^{\mu\nu})_{(p_1 \wedge \bar{q})} = Q_1^2 Q_2^2 (p_1 \cdot p_2) \int d\mu_{\bar{q}} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{q})^{\mu\nu} \int \hat{d}^4 \bar{l} \times \left[ \frac{(p_1 \cdot p_2) m_2^2}{D} \frac{1}{((\bar{q} - \bar{l})_+^2)^2 (p_1 \cdot \bar{l})_+} \left( -\frac{1}{(p_2 \cdot \bar{l})_-} + \frac{1}{(p_2 \cdot \bar{l})_+} \right) + \frac{(p_1 \cdot p_2)}{\bar{l}^2 (p_1 \cdot \bar{l})_-^3} \left( \frac{1}{(p_2 \cdot \bar{l})_-} - \frac{1}{(p_2 \cdot \bar{l})_+} \right) + \frac{(p_1 \cdot p_2)^2}{D ((\bar{q} - \bar{l})_+^2)^2 (p_2 \cdot \bar{l})_-} \left( \frac{1}{(p_1 \cdot \bar{l})_-} - \frac{1}{(p_1 \cdot \bar{l})_+} \right) - \frac{3}{2\bar{l}^2 (\bar{q} - \bar{l})_+^2} \left( \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_-} + \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_+} \right) + \frac{1}{2\bar{l}^2 (\bar{q} - \bar{l})_+^2} \left( \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_-} + \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_+} \right) + \frac{(p_1 \cdot p_2)}{2\bar{l}^2 (p_1 \cdot \bar{l})_-^2 (p_2 \cdot \bar{l})_+^2} + \frac{(p_1 \cdot p_2)}{2(\bar{q} - \bar{l})_+^2 (p_1 \cdot \bar{l})_-^2 (p_2 \cdot \bar{l})_-^2} \right]. \quad (C12)$$

Note that the log divergent pieces in (C6) and (C12) cancel among each other:

$$(R_{1,\tilde{B}_1+\tilde{C}_1}^{\mu\nu})_{(p_1\wedge\bar{q})}|_{\ln} + (R_{2,\tilde{B}_1+C_1}^{\mu\nu})_{(p_1\wedge\bar{q})}|_{\ln} = 0. \quad (C13)$$

Finally, the terms containing  $(p_1 \wedge p_2)^{\mu\nu}$  and  $(p_1 \wedge \bar{q})^{\mu\nu}$  in  $R_{2,\tilde{B}_0+\tilde{C}_0}^{\mu\nu}$  are as follows:

$$\begin{aligned} (R_{2,\tilde{B}_0+\tilde{C}_0}^{\mu\nu})_{(p_1\wedge p_2)} &= -\frac{Q_1^2 Q_2^2 (p_1 \cdot p_2)^3}{2D} (p_1 \wedge p_2)^{\mu\nu} \int d\mu_{\bar{q}} \bar{q}^2 e^{-i\bar{q} \cdot b} \int \frac{\hat{d}^4 \bar{l}}{\bar{l}_+^2 (\bar{q} - \bar{l})_+^2} \\ &\times \left( \frac{i\hat{\delta}(p_2 \cdot \bar{l})}{(p_1 \cdot \bar{l})_+^2} \right) - \frac{Q_1^2 Q_2^2}{2} \left( \frac{(p_1 \cdot p_2)^3 + (p_1 \cdot p_2)^2 m_1^2}{D} \right) (p_1 \wedge p_2)^{\mu\nu} \int d\mu_{\bar{q}} \bar{q}^2 e^{-i\bar{q} \cdot b} \\ &\times \int \frac{\hat{d}^4 \bar{l}}{\bar{l}_+^2 (\bar{q} - \bar{l})_+^2} \left( \frac{i\hat{\delta}(p_2 \cdot \bar{l})}{(p_1 \cdot \bar{l})_-^2} \right). \end{aligned} \quad (C14)$$

The terms with  $(p_1 \wedge \bar{q})^{\mu\nu}$  are

$$\begin{aligned} (R_{2,\tilde{B}_0+\tilde{C}_0}^{\mu\nu})_{(p_1\wedge\bar{q})} &= Q_1^2 Q_2^2 \int d\mu_{\bar{q}} e^{-i\bar{q} \cdot b} \left( p_1 \wedge \bar{q} \right)^{\mu\nu} \int \frac{\hat{d}^4 \bar{l}}{\bar{l}_+^2 (\bar{q} - \bar{l})_+^2} \\ &\times \left[ \frac{(p_1 \cdot p_2)^2 m_2^2 \bar{q}^2}{D(\bar{q} - \bar{l})_+^2} \left( \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_-} + \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_-} - \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_+} \right. \right. \\ &- \left. \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_+} \right) + \frac{(p_1 \cdot p_2)^2 m_2^2}{D} \left( \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_-} - \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_-} \right. \\ &- \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_+} + \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_+} \left. \right) + \frac{(p_1 \cdot p_2)^3}{D} \left( \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_-} \right. \\ &- \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_-} - \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_+} + \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_+} \left. \right) \\ &+ \frac{(p_1 \cdot p_2)^3 \bar{q}^2}{D(\bar{q} - \bar{l})_+^2} \left( \frac{1}{(p_1 \cdot \bar{l})_+ (p_2 \cdot \bar{l})_-} - \frac{1}{(p_1 \cdot \bar{l})_- (p_2 \cdot \bar{l})_+} \right) \\ &\left. + \frac{(p_1 \cdot p_2)^2 \bar{q}^2}{(p_1 \cdot \bar{l})_-^3} \left( -\frac{1}{(p_2 \cdot \bar{l})_-} + \frac{1}{(p_2 \cdot \bar{l})_+} \right) - \frac{(p_1 \cdot p_2)^2 \bar{q}^2}{2(p_1 \cdot \bar{l})_-^2 (p_2 \cdot \bar{l})_+^2} \right]. \end{aligned} \quad (C15)$$

#### APPENDIX D: $C^{\mu\nu}$ AT ONE LOOP

Here we explicitly evaluate terms contained in the cut contribution, beginning with  $C_{11}^{\mu\nu}$ ,

$$\begin{aligned} C_{11}^{\mu\nu} &= \frac{iQ_1^2 Q_2^2 \hbar^{-1}}{16} \int \hat{d}^4 \bar{q} \hat{d}^4 \bar{w} e^{-i\bar{q} \cdot b} \left[ \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{w}) \right. \\ &+ \hbar (\hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) (\hat{\delta}'(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{w}) - \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}'(p_2 \cdot \bar{w})) \\ &+ (\hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) - \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}'(p_2 \cdot \bar{q})) \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{w})) \left. \right] \\ &\times \left( \frac{4p_1 \cdot p_2 + 2\hbar(p_2 - p_1) \cdot (\bar{w} + \bar{q})}{(\bar{q} - \bar{w})_+^2} \right) \left( \frac{4(p_1 \wedge p_2)^{\mu\nu} - 2\hbar(p_1 \wedge \bar{w})^{\mu\nu}}{\bar{w}_+^2} \right). \end{aligned} \quad (D1)$$

The classical term here is

$$\begin{aligned}
C_{11}^{\mu\nu} = & iQ_1^2 Q_2^2 (p_1 \cdot p_2) (p_1 \wedge p_2)^{\mu\nu} \int \frac{\hat{d}^4 \bar{q} \hat{d}^4 \bar{w}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} \\
& \times \left[ \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) \left( \hat{\delta}'(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{w}) - \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}'(p_2 \cdot \bar{w}) \right) \right. \\
& \left. + \left( \hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) - \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}'(p_2 \cdot \bar{q}) \right) \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{w}) \right] \\
& - \frac{iQ_1^2 Q_2^2}{2} (p_1 \cdot p_2) \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{w})^{\mu\nu}.
\end{aligned} \tag{D2}$$

The first term here vanishes. Now, in this integral, we do the following transformation  $\bar{w} \rightarrow \bar{q} - \bar{w}$  which exchanges the photon momenta between the two legs and then average over them. After averaging, we have

$$\begin{aligned}
C_{11}^{\mu\nu} = & -\frac{iQ_1^2 Q_2^2}{4} (p_1 \cdot p_2) \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{q})^{\mu\nu} \\
= & \frac{-Q_1 Q_2}{8\pi\sqrt{D}} (p_1 \wedge \Delta p)^{\mu\nu} \log \omega.
\end{aligned} \tag{D3}$$

The term  $C_{12}^{\mu\nu}$  has the expression

$$\begin{aligned}
C_{12}^{\mu\nu} = & \frac{-iQ_1^2 Q_2^2}{8} \int \hat{d}^4 \bar{q} \hat{d}^4 \bar{w} e^{-i\bar{q} \cdot b} \left[ \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{w}) \right. \\
& + \hbar (\hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) \left( \hat{\delta}'(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{w}) - \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}'(p_2 \cdot \bar{w}) \right) \\
& \left. + \left( \hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) - \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}'(p_2 \cdot \bar{q}) \right) \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{w}) \right] \frac{\bar{w} \wedge p_2}{(\bar{q} - \bar{w})_+^2} \frac{4p_1 \cdot p_2 + 2\hbar(p_2 - p_1) \cdot \bar{w}}{\bar{w}_+^2}.
\end{aligned} \tag{D4}$$

The classical term that survives is

$$C_{12}^{\mu\nu} = \frac{iQ_1^2 Q_2^2}{2} (p_1 \cdot p_2) \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_2 \wedge \bar{w})^{\mu\nu}. \tag{D5}$$

Similarly, for this integral, after averaging over the two, we have

$$\begin{aligned}
C_{12}^{\mu\nu} = & \frac{iQ_1^2 Q_2^2}{4} (p_1 \cdot p_2) \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_2 \wedge \bar{q})^{\mu\nu} \\
= & \frac{Q_1 Q_2}{8\pi\sqrt{D}} (p_2 \wedge \Delta p)^{\mu\nu} \log \omega.
\end{aligned} \tag{D6}$$

Next we move onto calculating  $C_2^{\mu\nu}$ ; expanding  $C_{21}^{\mu\nu}$  in  $\hbar$  we have

$$\begin{aligned}
C_{21}^{\mu\nu} = & \frac{iQ_1^2 Q_2^2 \hbar^{-1}}{16} \int \hat{d}^4 \bar{q} \hat{d}^4 \bar{w} \left\{ \hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}(p_1 \cdot \bar{w}) + \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}'(p_1 \cdot \bar{w}) + \frac{\hbar}{2} \left( \bar{w}^2 (\hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}'(p_1 \cdot \bar{w}) \right. \right. \\
& \left. \left. + \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}''(p_1 \cdot \bar{w}) \right) + \bar{q}^2 (\hat{\delta}''(p_1 \cdot \bar{q}) \hat{\delta}(p_1 \cdot \bar{w}) + \hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}'(p_1 \cdot \bar{w})) \right\} \left\{ \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{w}) \right. \\
& \left. - \frac{\hbar}{2} \left( \bar{w}^2 \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}'(p_2 \cdot \bar{w}) + \bar{q}^2 \hat{\delta}'(p_2 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{w}) \right) \right\} (p_1 \wedge \bar{w})^{\mu\nu} e^{-i\bar{q} \cdot b} \left( \frac{4p_1 \cdot p_2}{\bar{w}_+^2} \right. \\
& \left. + \frac{2\hbar(p_2 - p_1) \cdot \bar{w}}{\bar{w}_+^2} \right) \left( \frac{4p_1 \cdot p_2 + 2\hbar(p_2 - p_1) \cdot (\bar{w} + \bar{q})}{(\bar{q} - \bar{w})_+^2} \right).
\end{aligned} \tag{D7}$$

The classical components that survive from  $C_{21}^{\mu\nu}$  are the following:

$$\begin{aligned}
C_{21}^{\mu\nu} = & -iQ_1^2 Q_2^2 (p_1 \cdot p_2) \int \frac{\hat{d}^4 \bar{q} \hat{d}^4 \bar{w}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{w})^{\mu\nu} \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}'(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{w}) (p_1 \cdot \bar{w}) \\
& - \frac{iQ_1^2 Q_2^2}{2} (p_1 \cdot p_2) \int \frac{\hat{d}^4 \bar{q} \hat{d}^4 \bar{w}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{w})^{\mu\nu} \hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{w}) (p_1 \cdot \bar{q}) \\
& + \frac{iQ_1^2 Q_2^2}{2} (p_1 \cdot p_2)^2 \int \frac{\hat{d}^4 \bar{q} \hat{d}^4 \bar{w}}{(\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{w})^{\mu\nu} \left( \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}''(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{w}) \right. \\
& + \hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}'(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{w}) - \hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}'(p_2 \cdot \bar{w}) \\
& \left. - \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}'(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}'(p_2 \cdot \bar{w}) \right) + \frac{iQ_1^2 Q_2^2}{2} (p_1 \cdot p_2)^2 \int \frac{\hat{d}^4 \bar{q} \hat{d}^4 \bar{w}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} \bar{q}^2 e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{w})^{\mu\nu} \\
& \times \left( \hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}'(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{w}) + \hat{\delta}''(p_1 \cdot \bar{q}) \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{w}) \right. \\
& \left. - \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}'(p_1 \cdot \bar{w}) \hat{\delta}'(p_2 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{w}) - \hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}'(p_1 \cdot \bar{w}) \hat{\delta}'(p_2 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{w}) \right). \tag{D8}
\end{aligned}$$

After doing IBP none of the terms containing  $(p_1 \wedge p_2)^{\mu\nu}$  survive from here; the remaining terms contain only  $(p_1 \wedge \bar{w})^{\mu\nu}$ , so

$$\begin{aligned}
(C_{21}^{\mu\nu})_{p_1 \wedge \bar{w}} = & \frac{3iQ_1^2 Q_2^2}{2} (p_1 \cdot p_2) \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{w})^{\mu\nu} \\
& - iQ_1^2 Q_2^2 \left( \frac{m_2^2 (p_1 \cdot p_2)^2 + (p_1 \cdot p_2)^3}{D} \right) \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{w})^{\mu\nu} \\
& - iQ_1^2 Q_2^2 \left( \frac{m_2^2 (p_1 \cdot p_2)^2 + (p_1 \cdot p_2)^3}{D} \right) \int \frac{d\mu_{\bar{q}, \bar{w}}}{((\bar{q} - \bar{w})_+^2)^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{w})^{\mu\nu}. \tag{D9}
\end{aligned}$$

Now the term  $C_{22}^{\mu\nu}$  has only one classical term which is

$$C_{22}^{\mu\nu} = iQ_1^2 Q_2^2 (p_1 \cdot p_2)^2 \int \frac{\hat{d}^4 \bar{q} \hat{d}^4 \bar{w}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (\bar{q} \wedge \bar{w})^{\mu\nu} \hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{w}). \tag{D10}$$

There is no  $(p_1 \wedge p_2)^{\mu\nu}$  term here so

$$\begin{aligned}
(C_{22}^{\mu\nu})_{p_1 \wedge \bar{w}} = & iQ_1^2 Q_2^2 \frac{m_2^2 (p_1 \cdot p_2)^2}{D} \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{w})^{\mu\nu} \\
& - iQ_1^2 Q_2^2 \frac{(p_1 \cdot p_2)^3}{D} \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_2 \wedge \bar{w})^{\mu\nu}. \tag{D11}
\end{aligned}$$

Finally,  $C_{23}^{\mu\nu}$  has the following form:

$$\begin{aligned}
C_{23}^{\mu\nu} = & \frac{-iQ_1^2 Q_2^2}{8} \int \hat{d}^4 \bar{q} \hat{d}^4 \bar{w} \left( \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) + \frac{\hbar}{2} \left( \hat{\delta}'(p_1 \cdot \bar{q}) \hat{\delta}(p_2 \cdot \bar{q}) - \hat{\delta}(p_1 \cdot \bar{q}) \hat{\delta}'(p_2 \cdot \bar{q}) \right) \right) \\
& \times \left( \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{w}) + \frac{\hbar}{2} \left( \hat{\delta}'(p_1 \cdot \bar{w}) \hat{\delta}(p_2 \cdot \bar{w}) - \hat{\delta}(p_1 \cdot \bar{w}) \hat{\delta}'(p_2 \cdot \bar{w}) \right) \right) e^{-i\bar{q} \cdot b} \\
& \times \frac{(\bar{w} \wedge p_2)^{\mu\nu}}{(\bar{w} - \bar{q})_+^2} \left( \frac{4p_1 \cdot p_2 + 2\hbar(p_2 - p_1) \cdot \bar{w}}{\bar{w}_+^2} \right). \tag{D12}
\end{aligned}$$

The only classical term from this part is

$$(C_{23}^{\mu\nu})_{p_1 \wedge \bar{w}} = \frac{iQ_1^2 Q_2^2}{2} (p_1 \cdot p_2) \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_2 \wedge \bar{w})^{\mu\nu}. \quad (D13)$$

The terms containing  $\frac{1}{((\bar{q}-\bar{w})_+^2)}$  in  $C_{21}^{\mu\nu}$  do not contribute as they are not log divergent, so the final result is

$$\begin{aligned} C_2^{\mu\nu} = & \frac{3iQ_1^2 Q_2^2}{2} (p_1 \cdot p_2) \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{w})^{\mu\nu} \\ & + \frac{iQ_1^2 Q_2^2}{2} (p_1 \cdot p_2) \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_2 \wedge \bar{w})^{\mu\nu} \\ & - iQ_1^2 Q_2^2 \frac{(p_1 \cdot p_2)^3}{D} \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_2 \wedge \bar{w})^{\mu\nu} \\ & - iQ_1^2 Q_2^2 \frac{(p_1 \cdot p_2)^3}{D} \int \frac{d\mu_{\bar{q}, \bar{w}}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{w})^{\mu\nu}. \end{aligned} \quad (D14)$$

Finally, averaging over the transformation  $\bar{w} \rightarrow \bar{q} - \bar{w}$  we have

$$\begin{aligned} C_2^{\mu\nu} = & \frac{3iQ_1^2 Q_2^2}{4} (p_1 \cdot p_2) \int \frac{\hat{d}^4 \bar{q} \hat{d}^4 \bar{w}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{q})^{\mu\nu} \\ & + \frac{iQ_1^2 Q_2^2}{4} (p_1 \cdot p_2) \int \frac{\hat{d}^4 \bar{q} \hat{d}^4 \bar{w}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_2 \wedge \bar{q})^{\mu\nu} \\ & - iQ_1^2 Q_2^2 \frac{(p_1 \cdot p_2)^3}{2D} \int \frac{\hat{d}^4 \bar{q} \hat{d}^4 \bar{w}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_2 \wedge \bar{q})^{\mu\nu} \\ & - iQ_1^2 Q_2^2 \frac{(p_1 \cdot p_2)^3}{2D} \int \frac{\hat{d}^4 \bar{q} \hat{d}^4 \bar{w}}{\bar{w}_+^2 (\bar{q} - \bar{w})_+^2} e^{-i\bar{q} \cdot b} (p_1 \wedge \bar{q})^{\mu\nu} \\ = & \left[ \frac{Q_1 Q_2}{8\pi\sqrt{D}} ((3p_1 + p_2) \wedge \Delta p)^{\mu\nu} - Q_1 Q_2 \frac{(p_1 \cdot p_2)^2}{4\pi D^{\frac{3}{2}}} ((p_1 + p_2) \wedge \Delta p)^{\mu\nu} \right] \log \omega. \end{aligned} \quad (D15)$$

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