

Renormalized quark masses using gradient flow

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We propose a new and simple method for determining the renormalized quark masses from lattice simulations. Renormalized quark masses are an important input to many phenomenological applications, including searching and modeling physics beyond the Standard Model. The nonperturbative renormalization is performed using gradient flow combined with the short-flow-time expansion that is improved by renormalization group (RG) running to match to the $\overline{\text{MS}}$ scheme. Implementing the RG running perturbatively, we demonstrate this method works reliably at least up to the charm-quark mass and exhibits an easily attainable “windowing condition.” Using RBC/UKQCD’s (2 + 1)-flavor Shamir domain-wall fermion ensembles with Iwasaki gauge action, we find $m_s^{\overline{\text{MS}}}(\mu = 2 \text{ GeV}) = 89(3) \text{ MeV}$ and $m_c^{\overline{\text{MS}}}(\mu = 3 \text{ GeV}) = 972(16) \text{ MeV}$. These results predict the scale-independent ratio $m_c/m_s = 12.1(4)$. Generalization to other observables is possible, providing an efficient approach to determine nonperturbatively renormalized fermionic observables like form factors or bag parameters from lattice simulations.

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I. INTRODUCTION

Theoretical predictions for Standard Model (SM) processes require that nonperturbative effects due to the strong force are accounted for. These are typically expressed in terms of decay constants, form factors, bag parameters, etc. An *ab initio* way to determine such nonperturbative quantities is the framework of lattice quantum chromodynamics (QCD) which allows us to improve the associated uncertainties systematically. Lattice quantities need to be renormalized and matched to a continuum scheme. This matching requires a renormalization scheme that can be implemented on the lattice as well as in perturbation theory (PT). Different methods are employed, each with distinct pros and cons: The Schrödinger functional [1–4] scheme uses gauge-invariant correlation functions, but the boundary conditions make perturbative calculations challenging.

The regularization-independent momentum subtraction (RI/MOM) schemes [5] break gauge invariance and have a “window problem” arising from the need to choose a matching scale μ that is large enough to control the perturbative expansion, $\mu \gg \Lambda_{\text{QCD}}$, but small enough to avoid large cutoff effects, $\mu \ll 1/a$. Position-space renormalization schemes [6,7] use gauge invariant correlation functions, where source-sink separation has to satisfy $a \ll t \ll \Lambda_{\text{QCD}}^{-1}$, creating a window problem.

In this work we develop a method to calculate nonperturbatively renormalized quark masses using lattice QCD simulations and match them to the $\overline{\text{MS}}$ scheme [8–10] using the short-flow-time expansion (SFTX) [8,11,12]. The procedure combines lattice and perturbative gradient flow (GF) [11,13–15], improved by renormalization group (RG) running. Our approach is gauge invariant, numerically precise, avoids most of the window problems, and is straightforward to implement.¹

We demonstrate our method by predicting renormalized strange- and charm-quark masses. Precise values for the quark masses are an important input to many phenomenological applications, helping us to explore potential physics

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¹An alternative approach based on flowed vacuum-expectation values has been recently presented in Ref. [16].

beyond the SM. We extract the renormalized quark-mass values in the GF scheme from the two-point correlation functions of flowed and regular pseudoscalar and axial-vector currents using the partially conserved axial-vector current (PCAC) relation. The lattice calculation is performed on a set of $(2+1)$ -flavor gauge-field configurations, and we use tuned strange- and charm-quark masses in the valence sector. After extrapolating to the continuum limit, we match the resulting GF renormalized quantities to the $\overline{\text{MS}}$ scheme by multiplying with the corresponding SFTX coefficients and taking the limit to zero flow time ($\tau \rightarrow 0$).

We aid this matching by RG running the flow time τ from the low-energy lattice scale to values where PT becomes reliable. Currently, this step is based on the perturbatively evaluated GF mass anomalous dimension [17], but we envision replacing this with a nonperturbative running in the future [18]. This RG running improves the $\tau \rightarrow 0$ limit, making it better controlled and more stable compared to the unimproved extrapolation.

II. NOTATION AND DEFINITIONS

A. Renormalized quark mass in the GF scheme

The gradient-flow equation for the gauge field reads

$$\partial_\tau B_\mu = D_\nu G_{\nu\mu}, \quad (1)$$

where the flowed gauge field is related to the regular gluon field by $B_\mu(\tau=0) = A_\mu$, and

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], \quad D_\mu = \partial_\mu + [B_\mu, \cdot].$$

The evolution of the quark fields is given on the gauge background:

$$\partial_\tau \chi = \Delta \chi, \quad \partial_\tau \bar{\chi} = \bar{\chi} \bar{\Delta}, \quad \Delta = D_\mu D_\mu. \quad (2)$$

The flow regularizes ultraviolet (UV) divergences in composite operators of flowed gauge fields. Composite operators which also involve flowed fermions are finite only after a multiplicative fermion wave function renormalization is taken into account, $\chi_R = Z_\chi^{1/2} \chi$ [15,19]. The perturbative approach to the flow equations has been discussed in Refs. [11,14,15,20].

On the lattice, the gradient flow is an invertible smoothing transformation. Its numerical evaluation on the gauge fields is straightforward [21]. Fermionic quantities are always expressed as expectation values of quark propagators, requiring the inversion of the Dirac operator on some source. While flowing both “ends” of the propagator is numerically challenging, flowing only one end is again straightforward [15]. In the following, we consider current correlators where only the sinks are flowed.

Specifically, we are interested in ratios

$$R_{rs}^\mathcal{O}(t; \tau) = - \frac{\langle A_0(t; \tau) \mathcal{O}(t=0; \tau=0) \rangle_{rs}}{\langle P(t; \tau) \mathcal{O}(t=0; \tau=0) \rangle_{rs}}, \quad (3)$$

where t denotes Euclidean time, $A_0^{rs}(t; \tau)$ and $P^{rs}(t; \tau)$ are the zero-momentum axial and pseudoscalar charge operators of (flowed) quark fields with flavors r and s , and $\mathcal{O}$ could be either P^{rs} or A_0^{rs} . In this paper, we only consider flavor-nonsinglet (connected) correlators. Since Z_χ as well as the divergences stemming from $\mathcal{O}(t=0; \tau=0)$ cancel in the ratio, $R_{rs}^\mathcal{O}(t; \tau)$ is UV-finite.

Both correlators in Eq. (3) couple to the same pseudoscalar state of mass M_{PS} . Thus, at (sufficiently) large source-sink separation and assuming t is also much larger than the footprint $\sqrt{8\tau}$ of the flow, the ground state dominates and the ratio $R_{rs}^\mathcal{O}(t; \tau)$ is independent of t . We define

$$\bar{R}^\mathcal{O}(\tau) = \lim_{t \rightarrow \infty} R^\mathcal{O}(t; \tau) \quad \text{with} \quad \sqrt{8\tau} \ll t. \quad (4)$$

The renormalized PCAC relation (Ward identity) then implies [22]

$$\left(m_{\text{GF}}^{(r)}(\tau) + m_{\text{GF}}^{(s)}(\tau) \right) = M_{\text{PS}}^{(rs)} \bar{R}_{rs}^\mathcal{O}(\tau), \quad (5)$$

where $m_{\text{GF}}^{(r,s)}(\tau)$ is the renormalized quark mass of flavor r or s at scale τ in the GF scheme, and $M_{\text{PS}}^{(rs)}$ is the pseudoscalar mass in the channel.

If the time extent of the lattice configuration is finite, the mixed correlators $\langle A_0(t; \tau) P(0; 0) \rangle_{rs}$ and $\langle P(t; \tau) A_0(0; 0) \rangle_{rs}$ depend on $\sinh(M_{\text{PS}}(T/2 - t))$, while the $\langle A_0(t; \tau) A_0(0; 0) \rangle_{rs}$ and $\langle P(t; \tau) P(0; 0) \rangle_{rs}$ correlators show a $\cosh(M_{\text{PS}}(T/2 - t))$ time dependence. Hence the time dependence in the ratios defined in Eq. (3) does not cancel for lattices with finite extent T/a and the $R^\mathcal{O}(t; \tau)$ in Eq. (4) should be replaced by the ratio of the corresponding amplitudes. This can be achieved by multiplying/dividing the ratio by $\cosh(M_{\text{PS}}(T/2 - t)) / \sinh(M_{\text{PS}}(T/2 - t))$, since aM_{PS} can be determined with high precision. Alternatively, we can avoid this difficulty by considering the product of $\bar{R}_{rs}^P(\tau)$ and $\bar{R}_{rs}^{A_0}(\tau)$

$$\left(m_{\text{GF}}^{(r)}(\tau) + m_{\text{GF}}^{(s)}(\tau) \right) = M_{\text{PS}}^{(rs)} \sqrt{\bar{R}_{rs}^{A_0}(\tau) \bar{R}_{rs}^P(\tau)}. \quad (6)$$

We have analyzed our data using both Eq. (5) as well as Eq. (6). For the range of flow times τ of interest, both ratios are consistent within our statistical uncertainties. To simplify the discussion later on we introduce

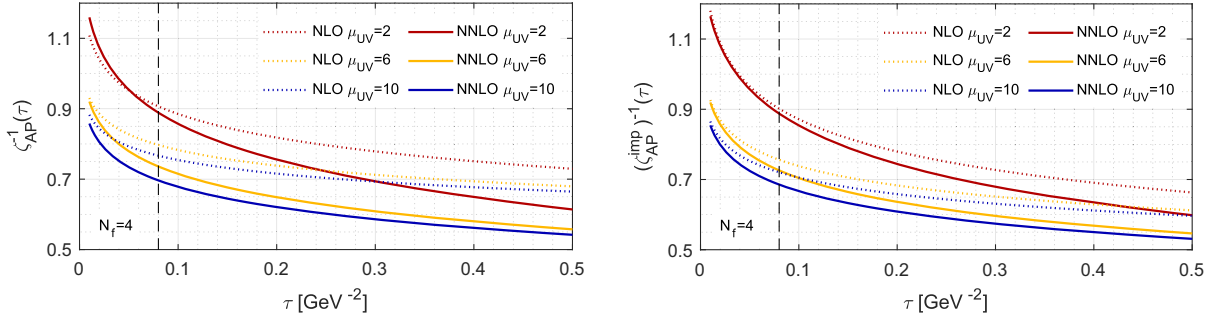


FIG. 1. The matching factor ζ_{AP}^{-1} as a function of the flow time τ for different values of the matching scale μ_{UV} . The left panel shows the NLO (dotted lines) and NNLO (solid lines) predictions without, the right panel with RG improvement. The vertical dashed lines indicate $\tau_{\min}$ used in our analysis. For consistency we use the three-loop γ_m^{GF} function when evaluating NNLO, but only the two-loop function for NLO. Using three-loop γ_m^{GF} for both NNLO and NLO, the differences would mostly disappear. Recalling that $\zeta_{AP}^{-1}|_{\text{LO}} = 1$, these plots demonstrate the perturbative validity of the approach for our choice of parameters.

$$\bar{R}^{AP}(\tau) \equiv \sqrt{\bar{R}_{rs}^{A_0}(\tau) \bar{R}_{rs}^P(\tau)}, \quad (7)$$

and also refer to $\bar{R}^{AP}$ as ratio.

B. Matching to the $\overline{\text{MS}}$ scheme

Finally, we need to connect GF renormalized quark masses $m_{\text{GF}}^{(r,s)}(\tau)$ to the renormalized quark mass in the $\overline{\text{MS}}$ scheme at some energy scale μ_{UV} . This matching can be performed in the SFTX

$$m_{\overline{\text{MS}}}(\mu_{UV}) = \lim_{\tau \rightarrow 0} \zeta_{AP}^{-1}(\mu_{UV}, \tau) m_{\text{GF}}(\tau), \quad (8)$$

where $\zeta_{AP}(\mu_{UV}, \tau) = \zeta_A(\mu_{UV}, \tau) \zeta_P^{-1}(\mu_{UV}, \tau)$. The $\tau \rightarrow 0$ limit implements the transition from the massive GF scheme to the massless $\overline{\text{MS}}$ scheme. ζ_A and ζ_P are the SFTX matching coefficients of the axial-vector and the pseudoscalar currents. They are known perturbatively to NNLO level [23], and depend on τ and μ_{UV} via $\alpha_s(\mu_{UV})$ and $\ln \tau \mu_{UV}^2$. For the strong coupling $\alpha_s(\mu_{UV})$ to be in the perturbative regime, we need to choose μ_{UV} sufficiently large. At the same time, the flow time τ should not be too different from $1/\mu_{UV}^2$, so that the logarithms do not spoil the perturbative series. On the other hand, the smearing radius $\sqrt{8\tau}$ induced by the gradient flow should be larger than the lattice spacing a in order to minimize cutoff effects. These conditions can be fulfilled on typical lattice ensembles, including the ones analyzed in this work. To illustrate this, the left panel of Fig. 1 compares the NLO and NNLO values of ζ_{AP}^{-1} as functions of the flow time τ at different values of μ_{UV} . For $\tau \rightarrow 0$, ζ_{AP}^{-1} diverges logarithmically as required to compensate the singularity from $m_{\text{GF}}(\tau)$ determined on the lattice. The gap between NLO and NNLO increases with τ , but the difference between the LO ($\zeta_{AP}^{-1} = 1$), NLO, and NNLO values progressively decreases in the entire τ range relevant for this analysis.

Nevertheless, we can further improve the matching in Eq. (8) by resumming the logarithmic terms with the help of the RG equation in the GF scheme,

$$\tau \partial_\tau \mathcal{O}(\tau) = \gamma_{\mathcal{O}}^{\text{GF}} \mathcal{O}(\tau), \quad (9)$$

where, in our case, $\mathcal{O} \in \{A_0, P\}$. As shown in Refs. [20,24], the perturbative flowed anomalous dimension can be expressed in terms of the matching coefficients as $\gamma_{\mathcal{O}}^{\text{GF}} = \tau(\partial_\tau \zeta_{\mathcal{O}}) \zeta_{\mathcal{O}}^{-1}$, and—under certain conditions—knowledge of $\zeta_{\mathcal{O}}$ to order α_s^n allows to obtain $\gamma_{\mathcal{O}}^{\text{GF}}$ to order α_s^{n+1} . Integrating Eq. (9) leads to

$$m_{\overline{\text{MS}}}(\mu_{UV}) = \lim_{\tau \rightarrow 0} (\zeta_{AP}^{\text{imp}}(\mu_{UV}, \tau))^{-1} m_{\text{GF}}(\tau), \quad (10)$$

with

$$(\zeta_{AP}^{\text{imp}}(\mu_{UV}, \tau))^{-1} = \zeta_{AP}^{-1}(\mu_{UV}, \tau_\mu) \times \exp\left(-\int_{\tau_\mu}^{\tau} d\tau' \frac{\gamma_m^{\text{GF}}(\tau')}{\tau'}\right), \quad (11)$$

where it follows from the previous discussion that $\gamma_m^{\text{GF}} = \gamma_A^{\text{GF}} - \gamma_P^{\text{GF}}$. The upper limit of the integral in Eq. (11) is the lattice flow time, $\tau \geq \tau_{\min} \approx 0.08$ in our setup, while the lower limit is $\tau_\mu = e^{-\gamma_E}/(2\mu_{UV}^2)$ [25]. As μ_{UV} increases from 2 to 6 to 10 GeV, τ_μ decreases from ≈ 0.07 to ≈ 0.008 to ≈ 0.003 . The corresponding increase of the integration range requires increasingly precise determination of γ_m^{GF} .

Ideally, the nonperturbative expression for $\gamma_m^{\text{GF}}(\tau)$ should be used in Eq. (11) and a way to obtain it is demonstrated in Refs. [18,26,27]. For two-flavor QCD γ_m^{GF} was found to be close to the perturbative prediction [18]. In our analysis, we use the perturbative expression, which is known to α_s^3 [23]. In this case, the integral in Eq. (11) can be obtained analytically.

The right panel of Fig. 1 compares the NLO and NNLO values of ζ_{AP}^{-1} after the logarithmic resummation according to Eq. (11). We indeed observe an improved perturbative convergence for all values of μ_{UV} under consideration. Although the final result should be independent of the choice of μ_{UV} in Eqs. (8) and (10), the $\tau \rightarrow 0$ extrapolation in practice might not be. To account for systematic uncertainties arising from the integral in Eq. (11), while still ensuring convergence of the SFTX, we consider $\mu_{UV} \in (2, 6)$ GeV in the following. At the end, we convert the result for the masses to the commonly preferred reference scales $\mu = 2$ or 3 GeV by regular four-loop running in the $\overline{\text{MS}}$ scheme.

III. NUMERICAL SETUP, ANALYSIS, AND RESULTS

We determine the strange and charm quark masses m_s and m_c , and obtain their ratio m_c/m_s using RBC/UKQCD's set of $(2+1)$ -flavor Shamir domain-wall fermion (DWF) and Iwasaki gauge field ensembles [28–32]. These ensembles feature three different bare gauge couplings, corresponding to inverse lattice spacings a^{-1} ranging from 1.78 to 2.78 GeV, as well as different unitary pion masses (due to different light sea-quark masses). Details of the ensembles are provided in Table I. Bare physical strange-quark masses and inverse lattice spacings are taken from Refs. [30,32], and bare physical charm-quark masses are tuned using the D_s meson. Our lattice calculations are based on Grid [33] and Hadrons [34,35], and we implemented fermionic gradient flow [9,10,36,37] in Hadrons.²

We calculate the correlation functions of Eq. (3) at $\tau = 0$ by placing Z_2 wall sources [38,39] on several time slices per configuration and individually invert the Dirac operator to obtain quark propagators. For the strange quark, we apply Gaussian source smearing [40] before inverting Shamir DWF [41–43], whereas for the charm quark we directly invert stout-smear Möbius DWF [44–46]. Our calculation proceeds by evolving the gauge-field configuration together with the strange- and charm-quark propagators with GF as specified by Eqs. (1) and (2). We choose a step size $\epsilon = 0.01$ for the GF evolution and contract the quark propagators imposing a point sink every 10^{th} (for larger flow times every 40^{th}) step to obtain the corresponding two-point correlators as functions of the GF time τ . Sources and sinks are projected to zero momentum.

We average correlators calculated with different source positions on the same gauge field and afterwards use jackknife resampling to propagate statistical uncertainties. We extract the pseudoscalar meson mass M_{PS} and verify that it is indeed independent of the flow time τ . The corresponding values are listed in Table II. Next we extract $R^P(t/a; \tau/a^2)$ as

defined in Eq. (3) with $\mathcal{O} = P^{rs}$ for a range of lattice flow times $0 \leq \tau/a^2 \leq 3.5$. We repeat this analysis using $R^{AP}(t/a; \tau/a^2) = \sqrt{R^{A_0}(t/a; \tau/a^2)} R^P(t/a; \tau/a^2)$ cf. Eq. (7).

As an example, we show both resulting correlator ratios for the D_s meson on the F1S ensemble for a selection of GF times in Fig. 2. The values of the ratios are extracted by performing correlated fits to the plateaus and we report the results for the relevant flow times τ in Table III for $\bar{R}^P(\tau/a^2)$ and in Table IV for $\bar{R}^{AP}(\tau/a^2)$. Both ratios lead to consistent results within our statistical uncertainties. We proceed with our analysis using $\bar{R}^{AP}(\tau/a^2)$ because we observe better p -values in the extraction of $\bar{R}^{AP}(\tau/a^2)$ than for $\bar{R}^P(\tau/a^2)$. After repeating this step for all six ensembles, we show $\bar{R}$ as a function of the lattice flow time τ/a^2 in the upper panel of Fig. 3. Next, we multiply our ratios with the corresponding lattice value of $aM_{PS}/2$ and convert both aM_{PS} and the lattice flow time τ/a^2 to physical units using the inverse lattice spacings listed in Table I. The result is shown in the lower panel of Fig. 3. In the upper panel, data corresponding to the different coarse and medium ensembles sit on top of each other, indicating that sea-quark effects are not resolved in the $\bar{R}$ ratios. In the lower panel, the six ratios form a unique curve, showing deviation due to cutoff effects only in the small- τ region. We interpolate the flow times in the coarse and medium ensembles to match the values on the F1S ensemble and take the continuum limit $a \rightarrow 0$ for the different flow times τ . Since all actions used in this calculation are $\mathcal{O}(a)$ -improved and no sea-quark effects are resolved, we simply perform a linear extrapolation in a^2 to arrive at the continuum-limit results shown by the black crosses in the lower panel of Fig. 3. Details of the $a \rightarrow 0$ continuum limit extrapolation are shown for selected flow times in Fig. 4 and in Table V.

The continuum limit results for $m_{GF}(\tau)$ are renormalized in the GF scheme. We have to take the $\tau \rightarrow 0$ limit to connect to the mass-independent $\overline{\text{MS}}$ scheme following Eq. (8). We can improve the matching using RG running according to Eq. (11). To maintain consistency of our results in the RG running, we use γ_m^{GF} at order α_s^2 for the NLO improvement, and at order α_s^3 for the NNLO improvement. The larger right panel of Fig. 5 shows $\zeta_{AP}^{-1}(\tau, \mu)m_{GF}(\tau)$ and the corresponding RG improved values for the D_s system as a function of the flow time τ both at NLO and NNLO level. We use $N_f = 4$ for the $\zeta_{AP}(\tau, \mu)$ factor and the RG running.

At small flow time $\sqrt{8\tau} \lesssim a$, the lattice data are contaminated by lattice artifacts. On the coarsest ensembles, this corresponds to $\tau < 0.039$ GeV⁻². To obtain the $\tau \rightarrow 0$ limit we consider two fit *Ansätze*:³ quadratic $f_2(\tau) = \tau^2 c_2 + \tau c_1 + c_0$ and linear-log $f_l(\tau) = \tau(c_l \log(\tau\mu^2) + c_1) + c_0$. Moreover, we vary the fit range and always account for the maximal variation using $\tau_{\min} \in (0.08, 0.2)$ GeV⁻² with

²Cf. <https://github.com/aportelli/Hadrons/pull/137> and examples given at <https://github.com/mbr-phys/HeavyMesonLifetimes>.

³Alternative approaches are discussed in Ref. [49].

TABLE I. RBC/UKQCD's $N_f = 2 + 1$ Shamir DWF ensembles with Iwasaki gauge action [28–32] specified by the inverse lattice spacing (a^{-1}), unitary pion mass (M_π), light and strange sea quark masses (am_ℓ^{sea} and am_s^{sea}), valence strange (am_s^{val}) and charm (am_c) quark masses. For the coarse (C1, C2), medium (M1, M2, M3), and fine ensemble (F1S) we use N_{conf} number of configurations and place N_{src} Z_2 wall-sources per configuration.

	L/a	T/a	a^{-1} [GeV]	M_π [MeV]	am_ℓ^{sea}	am_s^{sea}	am_s^{val}	am_c	$N_{\text{src}} \times N_{\text{conf}}$
C1	24	64	1.7848(50)	339.8(1.2)	0.005	0.04	0.03224	0.64	32×101
C2	24	64	1.7848(50)	430.6(1.4)	0.010	0.04	0.03224	0.64	32×101
M1	32	64	2.3833(86)	303.6(1.4)	0.004	0.03	0.02477	0.45	32×79
M2	32	64	2.3833(86)	360.7(1.6)	0.006	0.03	0.02477	0.45	32×89
M3	32	64	2.3833(86)	410.8(1.7)	0.008	0.03	0.02477	0.45	32×68
F1S	48	96	2.785(11)	267.6(1.3)	0.002144	0.02144	0.02167	0.37	24×98

TABLE II. Values of the pseudoscalar meson masses M_{η_s} , M_{D_s} , and M_{η_c} obtained for our six ensembles. First we list each mass in lattice units with statistical uncertainty only. Next we convert the value to MeV using a^{-1} quoted in Table I and the statistical uncertainty as well as the uncertainty propagated from a^{-1} .

	aM_{η_s}	M_{η_s} [MeV]	aM_{D_s}	M_{D_s} [MeV]	aM_{η_c}	M_{η_c} [MeV]
C1	0.38961(51)	695.38(0.9)(1.9)	1.10198(66)	1966.8(1.2)(5.5)	1.64713(21)	2939.81(0.4)(8.2)
C2	0.39204(44)	699.71(0.8)(2.0)	1.10408(63)	1970.6(1.1)(5.5)	1.64812(17)	2941.57(0.3)(8.2)
M1	0.29175(35)	695.33(0.8)(2.5)	0.82889(48)	1975.5(1.1)(7.1)	1.24895(16)	2976.61(0.4)(10.7)
M2	0.29214(41)	696.26(1.0)(2.5)	0.82933(49)	1976.5(1.2)(7.1)	1.24923(16)	2977.28(0.4)(10.7)
M3	0.29281(36)	697.86(0.9)(2.5)	0.82995(57)	1978.0(1.4)(7.1)	1.24964(20)	2978.26(0.5)(10.7)
F1S	0.25512(19)	710.50(0.5)(2.8)	0.70717(37)	1969.5(1.0)(7.8)	1.065287(87)	2966.82(0.2)(11.7)

$\tau_{\text{max}} = 0.3 \text{ GeV}^{-2}$, and $\tau_{\text{max}} \in (0.25, 0.35) \text{ GeV}^{-2}$ with $\tau_{\text{min}} = 0.14 \text{ GeV}^{-2}$ to obtain the color-shaded bands in Fig. 5. Because successive data in GF time tend to be highly correlated, the full variance-covariance matrix is nearly

singular and cannot reliably be inverted. For one choice of $(\tau_{\text{min}}, \tau_{\text{max}})$, we therefore take the $\tau \rightarrow 0$ limit by performing two uncorrelated fits where all central values are coherently shifted by plus/minus their uncertainties. The full spread

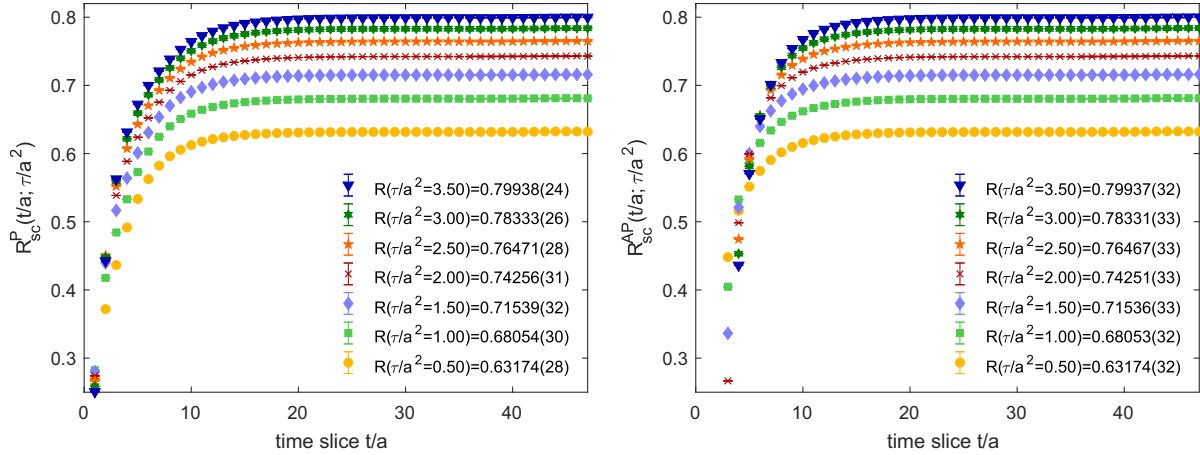


FIG. 2. On the left we present the ratios $R^P(t/a; \tau/a^2)$ defined in Eq. (3) and on the right $R^{AP}(t/a; \tau/a^2)$ introduced in Eq. (7) for selected flow times τ/a^2 for the D_s meson on the F1S ensemble. Statistical errors are within the symbol sizes. Due to the finite extent T/a of our gauge-field configurations and the use of antiperiodic boundary conditions for fermions in the time direction, we need to explicitly account for the “around-the-world effect” in $R^P(t/a; \tau/a^2)$ which enters with a different sign for $\langle AP \rangle$ than for the $\langle PP \rangle$ correlators. We therefore obtain $\bar{R}$ in Eq. (4) from the combination $R(t/a; \tau/a^2) \cosh(M_{\text{PS}}(T/2 - t)) / \sinh(M_{\text{PS}}(T/2 - t))$. Since the pseudoscalar mass M_{PS} can be measured very precisely (see Table II), this combination shows a long, flat plateau. The alternative combination $R^{AP}(t/a; \tau/a^2)$ avoids the problem of different “around-the-world” contributions and directly shows long, flat plateaus, as illustrated on the right panel. For both quantities we perform correlated fits to the plateaus from time slice 36 to 46, extracting the values of $\bar{R}(\tau/a^2)$ shown in the legend. Magnified plateau regions are shown in Fig. 10.

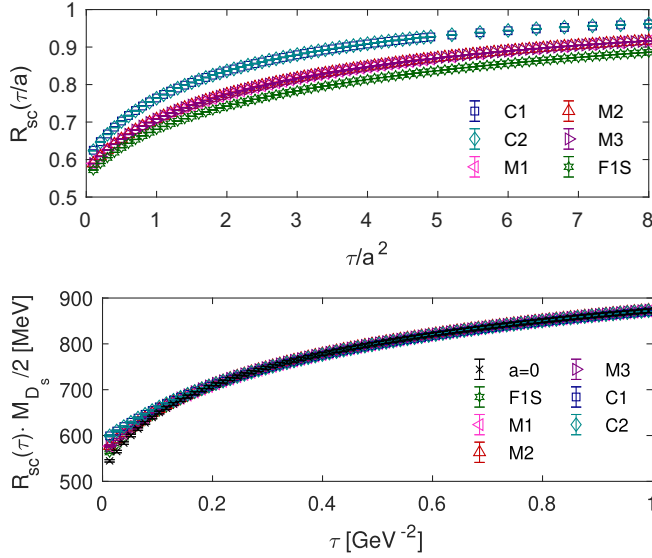


FIG. 3. Extracting $\bar{R}$ -ratios [see Eq. (4)] for D_s meson correlators. Top: dependence on the flow time τ/a in lattice units for our six ensembles. Bottom: converting to physical units and adding the $a \rightarrow 0$ continuum limit predictions.

across both fits feeds into our extrapolated result which is then taken as the symmetric uncertainty over the interval spanned by all variations of the fit range. For better visibility, we show the two final fit values for the four different approximations (NLO, NNLO, with and without RG improvement) in the small left panel of Fig. 5. While there is only little difference for the NNLO results with and without RG running, the NLO result improves substantially. The overall approach to zero becomes flatter, and the spread between our different fit *Ansätze* reduces. In Fig. 6 we show the dependence on the μ_{UV} scale. Our improved results show very little variation in the $3 \text{ GeV} < \mu_{UV} \leq 6 \text{ GeV}$ range. We therefore calculate correlated averages [50] for our NLO and NNLO determinations at $\mu_{UV} = 3, 4$, and 5 GeV and find

$$\begin{aligned} \left(\frac{m_c + m_s}{2} \right)^{\text{NLO}} &= 527.4(3.3)_{\text{GF}} \text{ MeV}, \\ \left(\frac{m_c + m_s}{2} \right)^{\text{NNLO}} &= 525.5(3.6)_{\text{GF}} \text{ MeV}, \end{aligned} \quad (12)$$

using the renormalization scale $\mu = 3 \text{ GeV}$. The uncertainty labeled “GF” covers both statistical and systematic effects from the $\tau \rightarrow 0$ extrapolation. As part of our analysis, we have propagated the uncertainty of a^{-1} and the tuning of the bare charm mass, whereas effects due to choosing the fit range to obtain $\bar{R}(\tau/a)$ or finite volume effects are negligible. Additional discretization errors may arise for the heavy charm quark that we generate in our partially quenched setup using stout-smear Möbius DWF. This smeared action has been explored and used by JLQCD and RBC/UKQCD for other calculations involving charm and bottom quarks

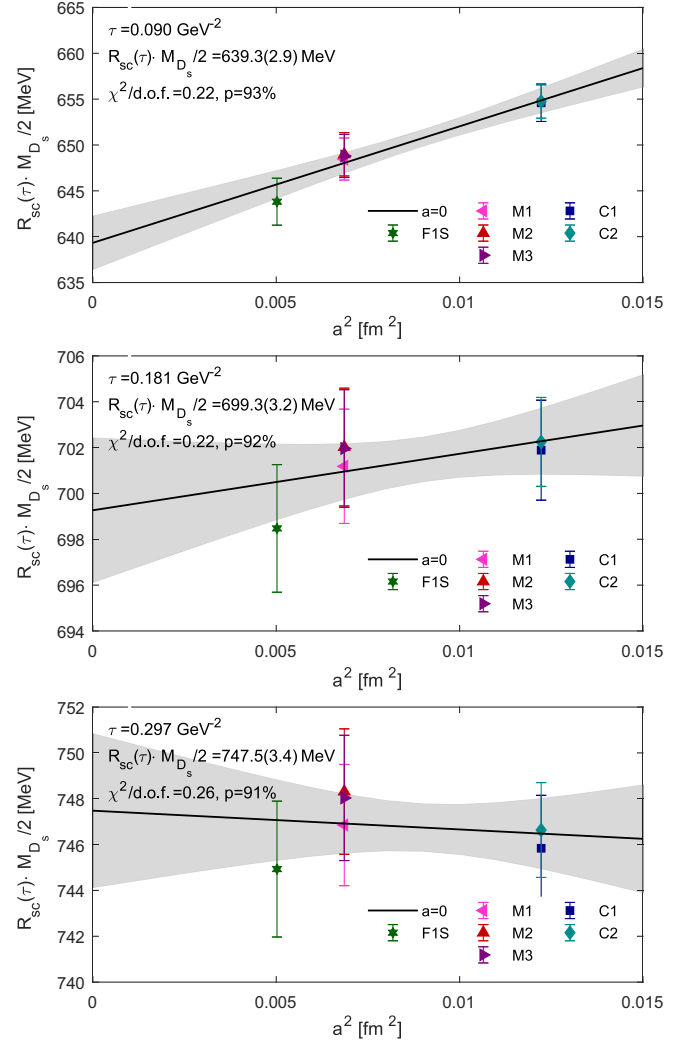


FIG. 4. Details of the continuum-limit extrapolation at selected flow times $\tau = 0.090, 0.181, 0.297 \text{ GeV}^{-2}$ for the D_s meson. Since the data points for M1, M2, M3 as well as C1 and C2 sit on top of each other, sea quark mass effects are not resolved. Hence we obtain the $a \rightarrow 0$ limit simply by fitting an *Ansatz* linear in a^2 .

(see, e.g., Refs. [32,51,52]). To check for discretization effects for DWF, we determine the residual mass for am_c . As can be seen in Appendix C, Fig. 11, we find small values and long plateaus for am_{res} on all ensembles for the bare quark-mass values considered. This indicates that charm discretization errors are negligible.

In order to estimate further systematic effects, we repeat our analysis discarding the two coarse ensembles (C1 and C2). On the one hand, this is a sensitive test of charm quark discretization errors, because the largest bare charm quark mass is reduced from 0.64 to 0.45; on the other hand, we substantially alter the $a \rightarrow 0$ continuum limit. For the remaining data at two lattice spacings, we perform a linear fit in a^2 as well as a fit to a constant to take the continuum limit. For the D_s correlators, the fit to a constant results in

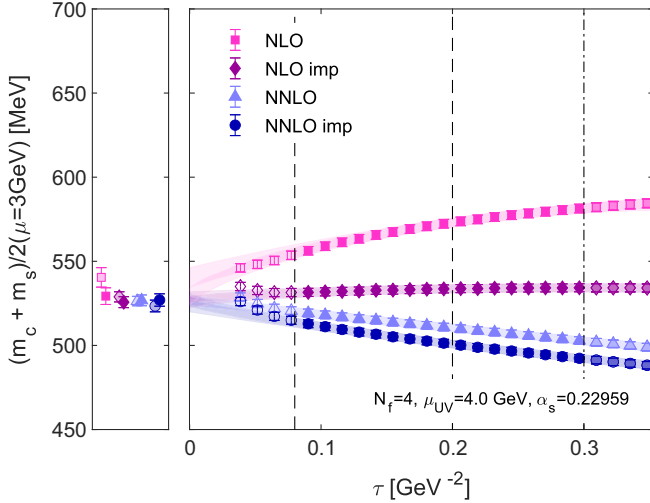


FIG. 5. The large panel illustrates extracting $(m_c + m_s)/2$ at the renormalization scale $\mu_{UV} = 4$ GeV, converted to $\mu = 3$ GeV using 4-loop $N_f = 4$ $\overline{\text{MS}}$ running [47,48]. Pink squares and light blue triangles denote the NLO and the NNLO values, respectively. Purple diamonds and dark blue circles show the corresponding RG improved NLO and NNLO values. The $\tau \rightarrow 0$ limit is taken in a $(\tau_{\min}, \tau_{\max})$ range as explained in the text. The shaded error bands show the maximal spread of quadratic or linear-log fits using the central values plus/minus their uncertainty. Only filled symbols enter the $\tau \rightarrow 0$ extrapolation. The small panel compares the final predictions using solid filled symbols for linear-log and shaded fillings for quadratic fits.

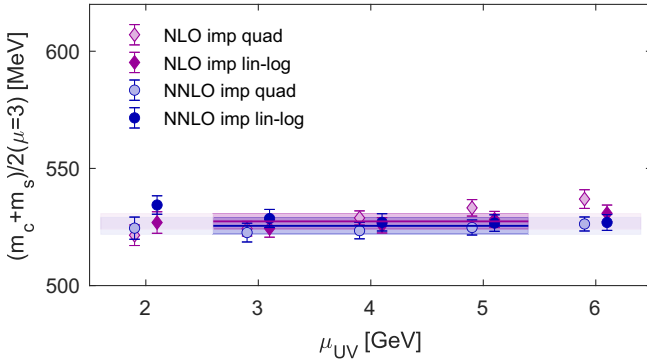


FIG. 6. Dependence of $(m_c + m_s)/2$ obtained from D_s correlators on μ_{UV} . Our final results are obtained by calculating the correlated average for NLO and NNLO using the values at $\mu_{UV} = 3, 4$, and 5 GeV.

the larger shift of the central value, amounting to 13.3 MeV at NNLO. We assign half of this difference as a systematic uncertainty to our final result, labeled by “CL.”

Similarly, we use half the difference between the central values of the NLO and the NNLO results in Eq. (12) in order to estimate the systematic uncertainty due to the truncation of the perturbative series (labeled “PT” in the final result). Thus, we obtain

$$\left(\frac{m_c + m_s}{2}\right) = 526(4)_{\text{GF}}(7)_{\text{CL}}(1)_{\text{PT}} \text{ MeV}, \quad (13)$$

At present, the CL-error estimate dominates our uncertainty, which reflects the fact that our dataset is currently quite limited. We expect a significant improvement from including a larger number of gauge field ensembles in the future.

Repeating the steps outlined above, we analyze data that describe the charmonium state η_c as well as the (unphysical) $(s\bar{s})$ -bound state, which we refer to as η_s . Details for η_c are presented in Fig. 7 and in Tables III–V. After the $\tau \rightarrow 0$ extrapolation we find

$$m_c^{\eta_c, \text{NLO}}(\mu = 3 \text{ GeV}) = 1001(11)_{\text{GF}} \text{ MeV},$$

$$m_c^{\eta_c, \text{NNLO}}(\mu = 3 \text{ GeV}) = 996(14)_{\text{GF}} \text{ MeV}. \quad (14)$$

Repeating the η_c analysis dropping the coarse ensembles, we observe that the linear $a \rightarrow 0$ continuum limit using only fine and medium ensembles results in a shift of the central NNLO value of 44 MeV, whereas using a fit to a constant barely changes the central values. Again we quote as final value our NNLO result and assign half of the shift of the central values as additional systematic uncertainties:

$$m_c^{\eta_c}(\mu = 3 \text{ GeV}) = 996(14)_{\text{GF}}(22)_{\text{CL}}(3)_{\text{PT}} \text{ MeV}. \quad (15)$$

In the case of η_s , we observe a tension in M_{η_s} on the F1S ensemble with respect to our determinations on the coarse and medium ensembles (cf. Table II) which we attribute to a slight misdetermination of the bare strange quark mass. Hence we discard the F1S data for this analysis, interpolate the flow times on the coarse ensembles to the values matching the medium ensembles, and show our final result based on data from the medium and coarse ensembles in Fig. 8. Since with only two lattice spacings a linear *Ansatz* for the $a \rightarrow 0$ limit is poorly constrained, we consider in addition fitting a constant. Both $a \rightarrow 0$ extrapolations have good p -values (cf. Table VI), so we repeat the remaining steps of the analysis by once using the linear in a^2 continuum limit (lin), and once the fit to a constant (cst). We obtain the following predictions for m_s :

$$m_s(\mu = 2 \text{ GeV})^{\text{NLO, cst}} = 91.24(42)_{\text{GF}} \text{ MeV},$$

$$m_s(\mu = 2 \text{ GeV})^{\text{NLO, lin}} = 88.0(1.3)_{\text{GF}} \text{ MeV},$$

$$m_s(\mu = 2 \text{ GeV})^{\text{NNLO, cst}} = 90.45(64)_{\text{GF}} \text{ MeV},$$

$$m_s(\mu = 2 \text{ GeV})^{\text{NNLO, lin}} = 88.0(1.1)_{\text{GF}} \text{ MeV}, \quad (16)$$

Taking the full spread between both continuum limits at the full MeV level, we find for both NLO and NNLO the same value. Hence, we quote as our final result for m_s in the $\overline{\text{MS}}$ scheme

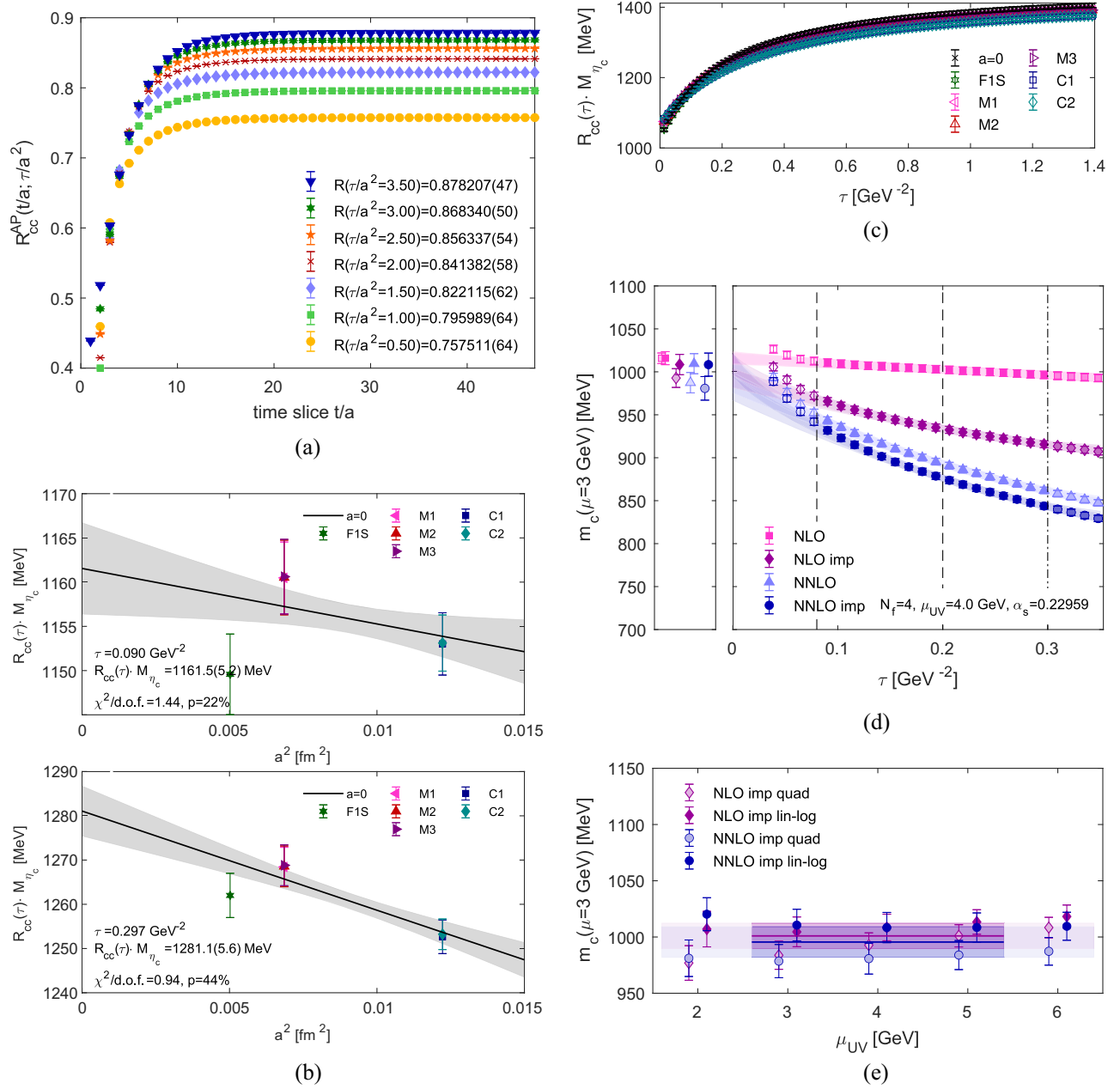


FIG. 7. Details of our charm-quark mass determination using η_c correlators. Figure (a) shows the extraction of the ratios $R^{AP}(t/a; \tau/a^2)$ by performing correlated fits using time slices 42 to 46 for selected flow times τ/a^2 on the F1S ensemble. For magnified plateau regions see Fig. 10. Interpolating results on the coarse and medium ensembles to flow times matching values on F1S, we perform $a \rightarrow 0$ continuum limit extrapolations as shown for $\tau = 0.090$ and 0.297 GeV $^{-2}$ in panels of (b) using an Ansatz linear in a^2 . Subsequently we present $\bar{R}^{AP}(\tau)$ in physical units and including the $a \rightarrow 0$ continuum limit in panel (c). The large panel in Figure (d) illustrates extracting m_c using η_c correlators at the renormalization scale $\mu_{UV} = 4$ GeV, converted to $\mu = 3$ GeV using 4-loop $N_f = 4$ $\overline{\text{MS}}$ running [47,48]. Pink squares and light blue triangles denote the NLO and NNLO values, respectively. Purple diamonds and dark blue circles show the corresponding RG improved NLO and NNLO values. The $\tau \rightarrow 0$ limit is taken in a $(\tau_{\min}, \tau_{\max})$ range as explained in the text. The shaded error bands show the maximal spread of quadratic or linear-log fits using the central values plus/minus their uncertainty. Only filled symbols enter the $\tau \rightarrow 0$ extrapolation. The small panel compares the final predictions using solid filled symbols for linear-log and shaded fillings for quadratic fits. Figure (e) demonstrates the dependence of the result for m_c on the scale μ_{UV} . We obtain our final values by calculating correlated averages for NLO and NNLO using values at $\mu_{UV} = 3, 4, 5$ GeV.

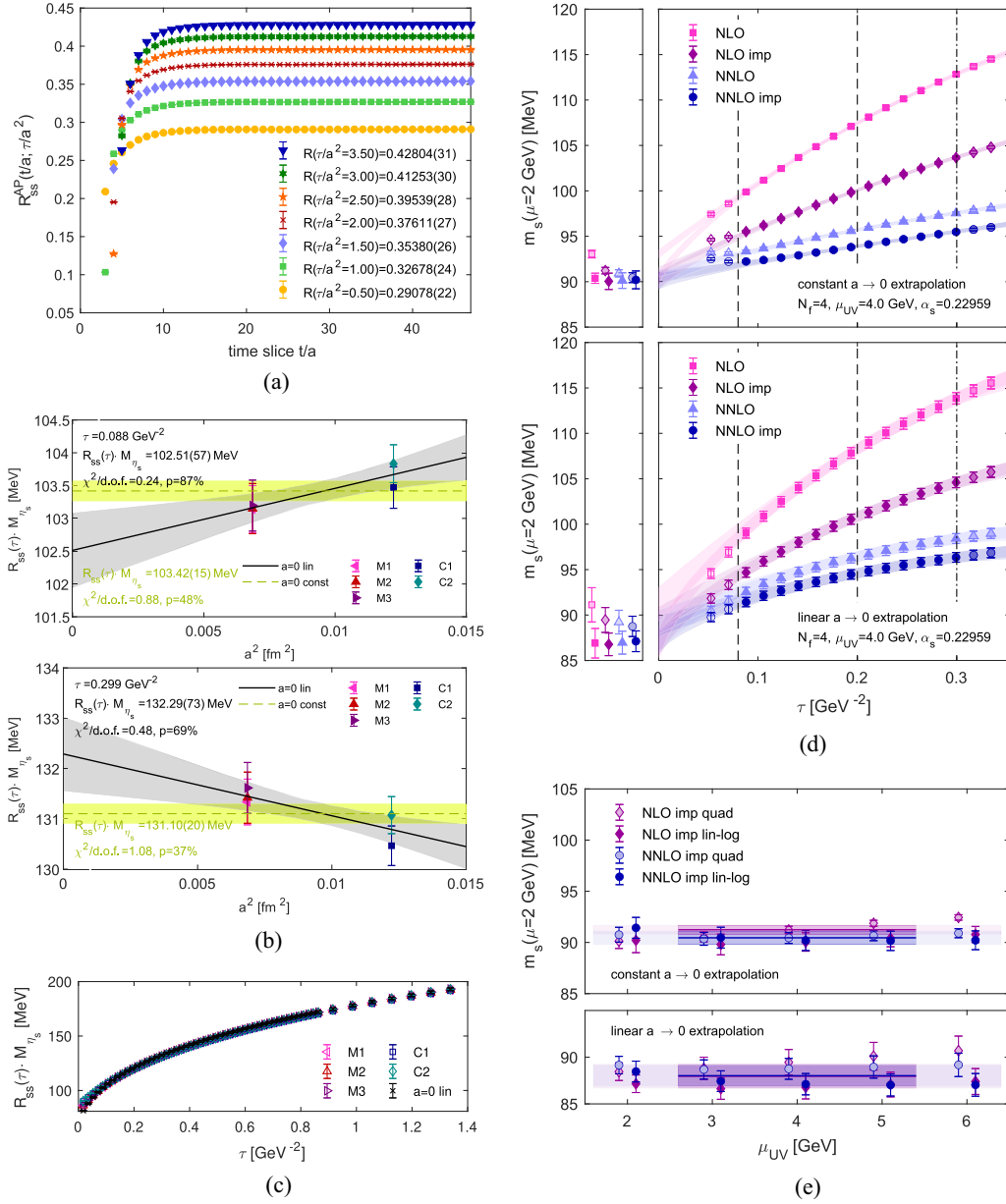


FIG. 8. Details of our strange quark mass determination using η_s correlators. Figure (a) shows the extraction of the ratios $R^{AP}(t/a; \tau/a^2)$ by performing correlated fits using time slices 36 to 46 for selected flow times τ/a^2 on the F1S ensemble. For magnified plateau regions see Fig. 10. As pointed out in the text, we observe a mistuning of the bare strange quark mass on F1S. We therefore perform $a \rightarrow 0$ continuum limit extrapolations as shown for $\tau = 0.088$ and 0.299 GeV^{-2} in panels (b) using only coarse and medium data but consider both an Ansatz linear in a^2 as well as a fit to a constant. Next we obtain $\bar{R}^{AP}(\tau)$ in physical units and including the $a \rightarrow 0$ continuum limit as shown in panel (c). The large panel in Figure (d) illustrates extracting m_s using η_s correlators at the renormalization scale $\mu_{UV} = 4 \text{ GeV}$, converted to $\mu = 2 \text{ GeV}$ using 4-loop $N_f = 4$ $\overline{\text{MS}}$ running [47,48]. Pink squares and light blue triangles denote the NLO and NNLO values, respectively. Purple diamonds and dark blue circles show the corresponding RG improved NLO and NNLO values. The $\tau \rightarrow 0$ limit is taken in a $(\tau_{\min}, \tau_{\max})$ range as explained in the text. The shaded error bands show the maximal spread of quadratic or linear-log fits using the central values plus/minus their uncertainty. Only filled symbols enter the $\tau \rightarrow 0$ extrapolation. The small panel compares the final predictions using solid filled symbols for linear-log and shaded fillings for quadratic fits. Figure (e) demonstrates the dependence of the result for m_s on the scale μ_{UV} . We obtain our final values by calculating correlated averages for NLO and NNLO using values at $\mu_{UV} = 3, 4, 5 \text{ GeV}$ and then accounting for the full spread due to different choices for the $a \rightarrow 0$ extrapolation.

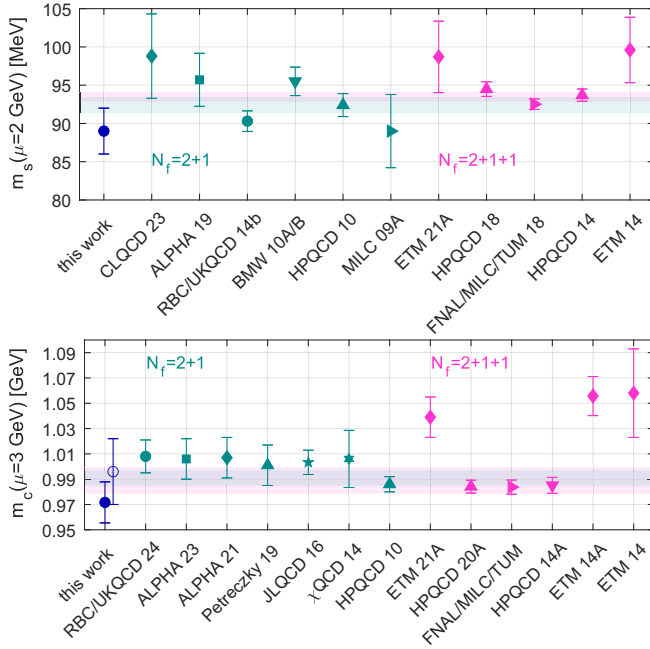


FIG. 9. Comparison of our strange and charm quark masses (blue circles) to the FLAG averages [53] (shaded bands) for $2 + 1$ and $2 + 1 + 1$ flavors. We show the direct prediction for m_c using the η_c correlators with an open circle, and the determination from the D_s meson with a filled circle. Also shown are results entering FLAG averages: strange $2 + 1$ [60–66], strange $2 + 1 + 1$ [56–59,67], charm $2 + 1$ [54,63,68–71], charm $2 + 1 + 1$ [56–59,72,73], as well as a new result not part of the average [74].

$$m_s(\mu = 2 \text{ GeV}) = 89(3)_{\text{GF+CL}}(0)_{\text{PT}} \text{ MeV}. \quad (17)$$

Adding the uncertainties of our result in Eq. (13) in quadrature and combining it with the value for m_c of Eq. (17) run up to $\mu = 3 \text{ GeV}$, we obtain the more precise value

$$m_c(\mu = 3 \text{ GeV}) = 972(16) \text{ MeV}, \quad (18)$$

which we also utilize to deduce the ratio

$$\frac{m_c}{m_s} = 12.1(4). \quad (19)$$

A comparison of our values for the strange and charm quark mass to the literature is shown in Fig. 9. Also, our value for the ratio m_c/m_s is in good agreement with the FLAG averages [53] for $2 + 1$ [54,55] and $2 + 1 + 1$ [56–59] flavors.

IV. SUMMARY

Gradient-flowed quark propagators provide a simple, nonperturbative method to renormalize fermionic quantities.

Matching to the perturbative $\overline{\text{MS}}$ scheme can be performed using SFTX with improved convergence by RG running in the GF scheme. We implement and test this idea to obtain renormalized strange- and charm-quark masses at percent-level accuracy. The accuracy can easily be improved by using state-of-the-art physical-point gauge field configurations and finer lattice spacings. The GF is a mass-dependent renormalization scheme and the matching to the mass-independent $\overline{\text{MS}}$ scheme requires taking the $\tau \rightarrow 0$ limit. Determination of a nonperturbative flowed anomalous dimension $\gamma_m^{\text{GF}}(\tau)$ for the RG improvement in Eq. (11) is an interesting task for the future.

Since the GF directly renormalizes the fermionic operator, also “mixed action” setups using different fermion discretizations for light and heavy quarks (as we do for the D_s meson) can be nonperturbatively renormalized. PT only enters when matching to the continuum $\overline{\text{MS}}$ scheme using SFTX.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [75].

APPENDIX A: MAGNIFIED PLATEAU REGIONS FOR RATIO FITS

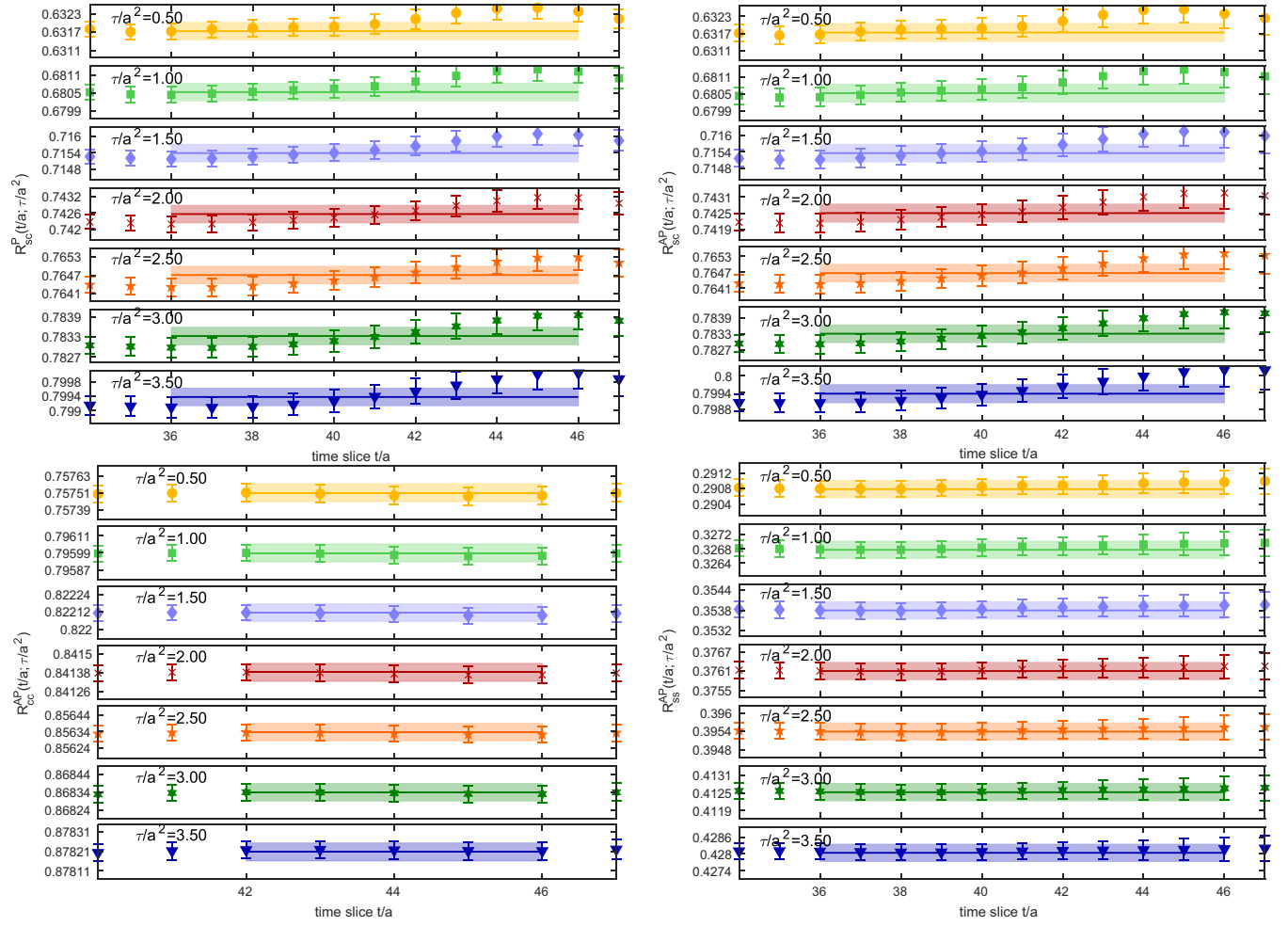


FIG. 10. Magnified plateau region demonstrating the extraction using ratios of correlators for the F1S ensemble. The top row shows the plateaus for the D_s meson correlators $R_{sc}^P(t/a; \tau/a^2)$ on the left and $R_{sc}^{AP}(t/a; \tau/a^2)$ on the right. The bottom left displays $R_{cc}^{AP}(t/a; \tau/a^2)$ using η_c correlators, and the bottom right $R_{ss}^{AP}(t/a; \tau/a^2)$ using η_s correlators. Correlated fits are performed to extract the shown central values and uncertainties. Details including the goodness-of-fit are listed in Tables III and IV, respectively.

APPENDIX B: RATIOS $\bar{R}^P(\tau/a^2)$ AND $\bar{R}^A(\tau/a^2)$ AND CONTINUUM LIMIT EXTRAPOLATIONSTABLE III. Values of the ratios $\bar{R}^P(\tau/a^2)$ for flow times τ/a^2 extracted from correlated fits of the Euclidean times slices in the specified range $[t_1 : t_2]$ with the corresponding $\chi^2/\text{d.o.f.}$ and p -value for all six ensembles analyzed. Each row lists our results using D_s -meson, η_c , and η_s correlators. Values corresponding to small τ not entering in our $\tau \rightarrow 0$ extrapolation are set in italics.

τ/a^2	D_s					η_c				η_s			
	$\bar{R}_{sc}^P(\tau/a^2)$	$[t_1 : t_2]$	χ^2/dof	$p\text{-val.}$		$\bar{R}_{cc}^P(\tau/a^2)$	$[t_1 : t_2]$	χ^2/dof	$p\text{-val.}$	$\bar{R}_{ss}^P(\tau/a^2)$	$[t_1 : t_2]$	χ^2/dof	$p\text{-val.}$
<i>FIS</i>	<i>0.10</i>	<i>0.57220(27)</i>	<i>[36:46]</i>	<i>1.67</i>	<i>8%</i>	<i>0.709447(62)</i>	<i>[42:46]</i>	<i>0.78</i>	<i>54%</i>	<i>0.24657(17)</i>	<i>[36:46]</i>	<i>0.90</i>	<i>54%</i>
<i>FIS</i>	<i>0.20</i>	<i>0.58958(28)</i>	<i>[36:46]</i>	<i>1.44</i>	<i>16%</i>	<i>0.723502(62)</i>	<i>[42:46]</i>	<i>0.78</i>	<i>54%</i>	<i>0.25982(18)</i>	<i>[36:46]</i>	<i>0.87</i>	<i>56%</i>
<i>FIS</i>	<i>0.30</i>	<i>0.60516(28)</i>	<i>[36:46]</i>	<i>1.26</i>	<i>25%</i>	<i>0.736088(62)</i>	<i>[42:46]</i>	<i>0.78</i>	<i>54%</i>	<i>0.27142(19)</i>	<i>[36:46]</i>	<i>0.85</i>	<i>58%</i>
<i>FIS</i>	<i>0.40</i>	<i>0.61914(28)</i>	<i>[36:46]</i>	<i>1.13</i>	<i>34%</i>	<i>0.747363(62)</i>	<i>[42:46]</i>	<i>0.78</i>	<i>54%</i>	<i>0.28168(19)</i>	<i>[36:46]</i>	<i>0.83</i>	<i>60%</i>
<i>FIS</i>	<i>0.50</i>	<i>0.63174(28)</i>	<i>[36:46]</i>	<i>1.03</i>	<i>41%</i>	<i>0.757495(62)</i>	<i>[42:46]</i>	<i>0.79</i>	<i>53%</i>	<i>0.29088(20)</i>	<i>[36:46]</i>	<i>0.82</i>	<i>61%</i>
<i>FIS</i>	<i>0.60</i>	<i>0.64318(28)</i>	<i>[36:46]</i>	<i>0.97</i>	<i>47%</i>	<i>0.766647(61)</i>	<i>[42:46]</i>	<i>0.79</i>	<i>53%</i>	<i>0.29922(20)</i>	<i>[36:46]</i>	<i>0.80</i>	<i>62%</i>
FIS	0.70	0.65364(29)	[36:46]	0.95	48%	0.774962(61)	[42:46]	0.79	53%	0.30689(20)	[36:46]	0.80	63%
FIS	0.80	0.66328(29)	[36:46]	0.98	46%	0.782559(60)	[42:46]	0.80	53%	0.31400(21)	[36:46]	0.79	64%
FIS	0.90	0.67222(29)	[36:46]	1.04	41%	0.789537(60)	[42:46]	0.80	52%	0.32065(21)	[36:46]	0.78	65%
FIS	1.00	0.68054(30)	[36:46]	1.13	34%	0.795978(59)	[42:46]	0.81	52%	0.32691(22)	[36:46]	0.77	65%
FIS	1.10	0.68834(30)	[36:46]	1.23	26%	0.801947(59)	[42:46]	0.81	52%	0.33283(22)	[36:46]	0.77	66%
FIS	1.20	0.69567(31)	[36:46]	1.34	20%	0.807501(58)	[42:46]	0.80	52%	0.33847(22)	[36:46]	0.76	67%
FIS	1.30	0.70260(31)	[36:46]	1.44	15%	0.812687(58)	[42:46]	0.80	53%	0.34385(22)	[36:46]	0.75	68%
FIS	1.40	0.70916(32)	[36:46]	1.53	12%	0.817544(57)	[42:46]	0.78	54%	0.34901(23)	[36:46]	0.74	69%
FIS	1.50	0.71539(32)	[36:46]	1.60	10%	0.822106(57)	[42:46]	0.77	55%	0.35396(23)	[36:46]	0.73	70%
FIS	1.60	0.72133(32)	[36:46]	1.66	8%	0.826401(56)	[42:46]	0.75	56%	0.35873(23)	[36:46]	0.72	71%
FIS	1.70	0.72699(32)	[36:46]	1.71	7%	0.830455(56)	[42:46]	0.73	57%	0.36333(23)	[36:46]	0.70	72%
FIS	1.80	0.73241(32)	[36:46]	1.74	7%	0.834290(55)	[42:46]	0.71	59%	0.36778(23)	[36:46]	0.69	74%
FIS	1.90	0.73759(32)	[36:46]	1.76	6%	0.837924(54)	[42:46]	0.68	60%	0.37209(24)	[36:46]	0.68	75%
FIS	2.00	0.74256(31)	[36:46]	1.79	6%	0.841374(54)	[42:46]	0.66	62%	0.37627(24)	[36:46]	0.66	76%
FIS	2.10	0.74734(31)	[36:46]	1.81	5%	0.844655(53)	[42:46]	0.65	63%	0.38034(24)	[36:46]	0.64	78%
FIS	2.20	0.75192(30)	[36:46]	1.83	5%	0.847780(52)	[42:46]	0.63	64%	0.38429(24)	[36:46]	0.63	79%
FIS	2.30	0.75634(30)	[36:46]	1.85	5%	0.850760(52)	[42:46]	0.62	65%	0.38814(25)	[36:46]	0.61	81%
FIS	2.40	0.76060(29)	[36:46]	1.87	4%	0.853606(51)	[42:46]	0.62	65%	0.39189(25)	[36:46]	0.60	82%
FIS	2.50	0.76471(28)	[36:46]	1.90	4%	0.856327(51)	[42:46]	0.63	64%	0.39555(25)	[36:46]	0.59	83%
FIS	2.60	0.76868(28)	[36:46]	1.92	4%	0.858932(50)	[42:46]	0.64	63%	0.39913(25)	[36:46]	0.58	83%
FIS	2.70	0.77252(27)	[36:46]	1.95	3%	0.861428(50)	[42:46]	0.66	62%	0.40263(26)	[36:46]	0.59	83%
<i>M1</i>	<i>0.10</i>	<i>0.58992(36)</i>	<i>[24:30]</i>	<i>0.60</i>	<i>73%</i>	<i>0.72703(10)</i>	<i>[28:31]</i>	<i>2.06</i>	<i>10%</i>	<i>0.24811(30)</i>	<i>[24:30]</i>	<i>0.62</i>	<i>72%</i>
<i>M1</i>	<i>0.20</i>	<i>0.60900(36)</i>	<i>[24:30]</i>	<i>0.64</i>	<i>70%</i>	<i>0.74227(10)</i>	<i>[28:31]</i>	<i>1.99</i>	<i>11%</i>	<i>0.26243(32)</i>	<i>[24:30]</i>	<i>0.58</i>	<i>75%</i>
<i>M1</i>	<i>0.30</i>	<i>0.62611(37)</i>	<i>[24:30]</i>	<i>0.68</i>	<i>67%</i>	<i>0.75579(10)</i>	<i>[28:31]</i>	<i>1.98</i>	<i>11%</i>	<i>0.27510(34)</i>	<i>[24:30]</i>	<i>0.58</i>	<i>75%</i>
<i>M1</i>	<i>0.40</i>	<i>0.64144(37)</i>	<i>[24:30]</i>	<i>0.69</i>	<i>66%</i>	<i>0.76780(10)</i>	<i>[28:31]</i>	<i>1.97</i>	<i>12%</i>	<i>0.28638(36)</i>	<i>[24:30]</i>	<i>0.60</i>	<i>73%</i>
M1	0.50	0.65522(37)	[24:30]	0.65	69%	0.778515(100)	[28:31]	1.96	12%	0.29654(37)	[24:30]	0.63	71%
M1	0.60	0.66769(37)	[24:30]	0.57	75%	0.788120(99)	[28:31]	1.95	12%	0.30579(38)	[24:30]	0.67	67%
M1	0.70	0.67906(37)	[24:30]	0.46	84%	0.796785(97)	[28:31]	1.92	12%	0.31432(39)	[24:30]	0.71	64%
M1	0.80	0.68948(37)	[24:30]	0.35	91%	0.804649(96)	[28:31]	1.90	13%	0.32225(40)	[24:30]	0.74	62%
M1	0.90	0.69911(37)	[24:30]	0.26	95%	0.811825(94)	[28:31]	1.88	13%	0.32968(40)	[24:30]	0.76	60%
M1	1.00	0.70805(36)	[24:30]	0.20	98%	0.818407(92)	[28:31]	1.86	13%	0.33668(40)	[24:30]	0.76	60%
M1	1.10	0.71638(36)	[24:30]	0.19	98%	0.824471(90)	[28:31]	1.84	14%	0.34332(41)	[24:30]	0.76	60%
M1	1.20	0.72420(35)	[24:30]	0.21	98%	0.830080(88)	[28:31]	1.81	14%	0.34963(41)	[24:30]	0.74	62%
M1	1.30	0.73155(35)	[24:30]	0.25	96%	0.835288(86)	[28:31]	1.78	15%	0.35567(41)	[24:30]	0.71	64%
M1	1.40	0.73849(34)	[24:30]	0.32	93%	0.840138(83)	[28:31]	1.75	15%	0.36146(41)	[24:30]	0.66	68%
M1	1.50	0.74507(34)	[24:30]	0.40	88%	0.844669(81)	[28:31]	1.72	16%	0.36702(41)	[24:30]	0.61	73%
M1	1.60	0.75132(33)	[24:30]	0.47	83%	0.848912(79)	[28:31]	1.69	17%	0.37239(41)	[24:30]	0.55	77%
M1	1.70	0.75728(33)	[24:30]	0.53	78%	0.852896(77)	[28:31]	1.66	17%	0.37756(41)	[24:30]	0.49	81%
M1	1.80	0.76297(33)	[24:30]	0.59	74%	0.856645(75)	[28:31]	1.65	18%	0.38258(41)	[24:30]	0.44	85%
M1	1.90	0.76841(33)	[24:30]	0.63	71%	0.860178(73)	[28:31]	1.65	17%	0.38743(41)	[24:30]	0.39	89%

(Table continued)

TABLE III. (Continued)

	τ/a^2	D_s				η_c				η_s			
		$\bar{R}_{sc}^P(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$	$\bar{R}_{cc}^P(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$	$\bar{R}_{ss}^P(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$
M2	0.10	0.58959(37)	[24:30]	1.76	10%	0.726823(100)	[28:31]	0.48	70%	0.24798(30)	[24:30]	1.44	19%
M2	0.20	0.60866(37)	[24:30]	1.55	16%	0.742078(99)	[28:31]	0.54	66%	0.26229(31)	[24:30]	1.49	18%
M2	0.30	0.62575(37)	[24:30]	1.43	20%	0.755607(99)	[28:31]	0.73	54%	0.27496(33)	[24:30]	1.54	16%
M2	0.40	0.64106(37)	[24:30]	1.38	22%	0.767622(99)	[28:31]	1.09	35%	0.28625(34)	[24:30]	1.58	15%
M2	0.50	0.65484(37)	[24:30]	1.36	23%	0.778333(100)	[28:31]	1.59	19%	0.29642(35)	[24:30]	1.62	14%
M2	0.60	0.66732(37)	[24:30]	1.36	23%	0.787935(99)	[28:31]	2.14	9%	0.30569(36)	[24:30]	1.67	12%
M2	0.70	0.67912(52)	[29:30]	0.50	48%	0.796642(90)	[29:31]	1.00	37%	0.31397(35)	[29:30]	0.86	35%
M2	0.80	0.68962(52)	[29:30]	0.33	56%	0.804509(89)	[29:31]	1.12	33%	0.32190(35)	[29:30]	0.89	34%
M2	0.90	0.69933(53)	[29:30]	0.20	65%	0.811689(87)	[29:31]	1.25	29%	0.32934(36)	[29:30]	0.93	33%
M2	1.00	0.70836(53)	[29:30]	0.11	74%	0.818274(86)	[29:31]	1.41	24%	0.33635(37)	[29:30]	0.98	32%
M2	1.10	0.71679(53)	[29:30]	0.04	83%	0.824342(85)	[29:31]	1.59	20%	0.34299(37)	[29:30]	1.03	31%
M2	1.20	0.72471(53)	[29:30]	0.01	92%	0.829954(84)	[29:31]	1.79	17%	0.34932(38)	[29:30]	1.10	29%
M2	1.30	0.73216(54)	[29:30]	0.00	99%	0.835165(83)	[29:31]	2.02	13%	0.35537(38)	[29:30]	1.16	28%
M2	1.40	0.73919(54)	[29:30]	0.01	90%	0.840019(82)	[29:31]	2.26	10%	0.36117(38)	[29:30]	1.24	27%
M2	1.50	0.74586(54)	[29:30]	0.05	82%	0.844553(81)	[29:31]	2.52	8%	0.36674(39)	[29:30]	1.31	25%
M2	1.60	0.75219(54)	[29:30]	0.11	74%	0.848800(80)	[29:31]	2.79	6%	0.37212(39)	[29:30]	1.38	24%
M2	1.70	0.75821(54)	[29:30]	0.18	67%	0.852788(79)	[29:31]	3.06	5%	0.37730(39)	[29:30]	1.46	23%
M2	1.80	0.76395(54)	[29:30]	0.27	61%	0.856540(78)	[29:31]	3.33	4%	0.38232(40)	[29:30]	1.53	22%
M2	1.90	0.76944(54)	[29:30]	0.37	55%	0.860078(77)	[29:31]	3.60	3%	0.38719(40)	[29:30]	1.61	21%
M3	0.10	0.58910(44)	[24:30]	1.08	37%	0.72658(12)	[28:31]	2.70	4%	0.24752(26)	[24:30]	1.59	15%
M3	0.20	0.60818(44)	[24:30]	1.08	37%	0.74185(12)	[28:31]	2.74	4%	0.26182(28)	[24:30]	1.50	17%
M3	0.30	0.62529(44)	[24:30]	1.08	37%	0.75539(12)	[28:31]	2.78	4%	0.27448(29)	[24:30]	1.42	20%
M3	0.40	0.64063(44)	[24:30]	1.10	36%	0.76742(12)	[28:31]	2.81	4%	0.28576(30)	[24:30]	1.36	23%
M3	0.50	0.65445(44)	[24:30]	1.13	34%	0.77815(11)	[28:31]	2.83	4%	0.29593(31)	[24:30]	1.31	25%
M3	0.60	0.66699(45)	[24:30]	1.18	31%	0.78777(11)	[28:31]	2.85	4%	0.30520(33)	[24:30]	1.29	26%
M3	0.70	0.67843(45)	[24:30]	1.24	28%	0.79645(11)	[28:31]	2.89	3%	0.31375(34)	[24:30]	1.28	26%
M3	0.80	0.68895(45)	[24:30]	1.31	25%	0.80432(11)	[28:31]	2.94	3%	0.32171(36)	[24:30]	1.26	27%
M3	0.90	0.69867(45)	[24:30]	1.39	21%	0.81151(11)	[28:31]	2.99	3%	0.32917(37)	[24:30]	1.24	28%
M3	1.00	0.70771(45)	[24:30]	1.47	18%	0.81810(10)	[28:31]	3.04	3%	0.33620(38)	[24:30]	1.20	30%
M3	1.10	0.71614(46)	[24:30]	1.56	15%	0.82418(10)	[28:31]	3.08	3%	0.34288(39)	[24:30]	1.16	32%
M3	1.20	0.72405(46)	[24:30]	1.65	13%	0.829795(99)	[28:31]	3.12	2%	0.34923(40)	[24:30]	1.13	34%
M3	1.30	0.73149(46)	[24:30]	1.72	11%	0.835011(97)	[28:31]	3.14	2%	0.35531(41)	[24:30]	1.11	35%
M3	1.40	0.73851(46)	[24:30]	1.78	10%	0.839871(95)	[28:31]	3.16	2%	0.36113(41)	[24:30]	1.11	35%
M3	1.50	0.74515(46)	[24:30]	1.82	9%	0.844411(93)	[28:31]	3.17	2%	0.36673(42)	[24:30]	1.14	34%
M3	1.60	0.75146(45)	[24:30]	1.85	9%	0.848663(91)	[28:31]	3.16	2%	0.37213(42)	[24:30]	1.18	32%
M3	1.70	0.75746(45)	[24:30]	1.87	8%	0.852656(89)	[28:31]	3.15	2%	0.37734(42)	[24:30]	1.22	29%
M3	1.80	0.76319(45)	[24:30]	1.88	8%	0.856414(87)	[28:31]	3.12	2%	0.38238(42)	[24:30]	1.28	26%
M3	1.90	0.76865(44)	[24:30]	1.89	8%	0.859957(85)	[28:31]	3.09	3%	0.38727(43)	[24:30]	1.33	24%
C1	0.10	0.62485(41)	[24:30]	0.31	93%	0.75068(12)	[28:31]	5.42	0%	0.26723(28)	[24:30]	0.60	73%
C1	0.20	0.64784(41)	[24:30]	0.28	95%	0.77005(11)	[28:31]	3.97	1%	0.28476(29)	[24:30]	0.47	83%
C1	0.30	0.66832(42)	[24:30]	0.28	95%	0.78659(10)	[28:31]	3.11	3%	0.30054(31)	[24:30]	0.38	89%
C1	0.40	0.68654(42)	[24:30]	0.28	95%	0.800809(96)	[28:31]	2.58	5%	0.31477(32)	[24:30]	0.31	93%
C1	0.50	0.70282(43)	[24:30]	0.28	95%	0.813127(91)	[28:31]	2.25	8%	0.32769(33)	[24:30]	0.25	96%
C1	0.60	0.71747(43)	[24:30]	0.28	95%	0.823888(87)	[28:31]	2.03	11%	0.33954(34)	[24:30]	0.21	97%
C1	0.70	0.73073(43)	[24:30]	0.29	94%	0.833371(84)	[28:31]	1.88	13%	0.35048(35)	[24:30]	0.18	98%
C1	0.80	0.74283(43)	[24:30]	0.32	92%	0.841793(81)	[28:31]	1.77	15%	0.36068(35)	[24:30]	0.16	99%
C1	0.90	0.75392(44)	[24:30]	0.39	88%	0.849326(79)	[28:31]	1.70	17%	0.37025(36)	[24:30]	0.13	99%
C1	1.00	0.76415(44)	[24:30]	0.50	81%	0.856106(76)	[28:31]	1.63	18%	0.37926(37)	[24:30]	0.12	99%
C1	1.10	0.77363(44)	[24:30]	0.63	70%	0.862241(75)	[28:31]	1.58	19%	0.38781(37)	[24:30]	0.10	100%
C2	0.10	0.62354(35)	[24:30]	1.71	11%	0.74987(12)	[28:31]	4.52	0%	0.26633(25)	[24:30]	0.89	50%
C2	0.20	0.64654(36)	[24:30]	1.67	12%	0.769353(98)	[28:31]	1.46	22%	0.28387(27)	[24:30]	1.05	39%
C2	0.30	0.66705(36)	[24:30]	1.60	14%	0.785979(88)	[28:31]	0.64	59%	0.29968(28)	[24:30]	1.19	31%
C2	0.40	0.68531(36)	[24:30]	1.52	17%	0.800265(82)	[28:31]	0.65	58%	0.31395(30)	[24:30]	1.31	25%

(Table continued)

TABLE III. (*Continued*)

	τ/a^2	D_s				η_c				η_s			
		$\bar{R}_{sc}^P(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$	$\bar{R}_{cc}^P(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$	$\bar{R}_{ss}^P(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$
C2	0.50	0.70164(36)	[24:30]	1.43	20%	0.812634(79)	[28:31]	0.85	47%	0.32692(31)	[24:30]	1.41	21%
C2	0.60	0.71635(37)	[24:30]	1.35	23%	0.823437(76)	[28:31]	1.03	38%	0.33882(32)	[24:30]	1.49	18%
C2	0.70	0.72969(37)	[24:30]	1.30	25%	0.832954(73)	[28:31]	1.16	32%	0.34982(32)	[24:30]	1.56	15%
C2	0.80	0.74187(37)	[24:30]	1.28	26%	0.841407(71)	[28:31]	1.28	28%	0.36009(33)	[24:30]	1.61	14%
C2	0.90	0.75307(37)	[24:30]	1.28	26%	0.848969(70)	[28:31]	1.39	24%	0.36971(34)	[24:30]	1.65	13%
C2	1.00	0.76341(37)	[24:30]	1.30	25%	0.855774(68)	[28:31]	1.51	21%	0.37880(34)	[24:30]	1.68	12%
C2	1.10	0.77300(37)	[24:30]	1.31	25%	0.861934(66)	[28:31]	1.64	18%	0.38740(34)	[24:30]	1.70	12%

TABLE IV. Values of $\bar{R}^{AP}(\tau/a^2) \equiv \sqrt{\bar{R}^P(\tau/a^2)\bar{R}^A(\tau/a^2)}$ for flow times τ/a^2 extracted from correlated fits of the Euclidean times slices in the specified range $[t_1:t_2]$ with the corresponding $\chi^2/\text{d.o.f.}$ and p -value for all six ensembles analyzed. Each row lists our results using D_s -meson, η_c , and η_s , correlators. Values corresponding to small τ not entering in our $\tau \rightarrow 0$ extrapolation are set in italics.

	τ/a^2	D_s				η_c				η_s			
		$\bar{R}_{cs}^{AP}(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$	$\bar{R}_{cc}^{AP}(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$	$\bar{R}_{ss}^{AP}(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$
<i>FIS</i>	<i>0.10</i>	<i>0.57221(30)</i>	<i>[36:46]</i>	<i>1.42</i>	<i>16%</i>	<i>0.709463(63)</i>	<i>[42:46]</i>	<i>0.58</i>	<i>68%</i>	<i>0.24649(19)</i>	<i>[36:46]</i>	<i>0.38</i>	<i>96%</i>
<i>FIS</i>	<i>0.20</i>	<i>0.58958(30)</i>	<i>[36:46]</i>	<i>1.21</i>	<i>28%</i>	<i>0.723519(63)</i>	<i>[42:46]</i>	<i>0.61</i>	<i>65%</i>	<i>0.25974(20)</i>	<i>[36:46]</i>	<i>0.36</i>	<i>96%</i>
<i>FIS</i>	<i>0.30</i>	<i>0.60516(31)</i>	<i>[36:46]</i>	<i>1.07</i>	<i>38%</i>	<i>0.736105(63)</i>	<i>[42:46]</i>	<i>0.67</i>	<i>61%</i>	<i>0.27135(21)</i>	<i>[36:46]</i>	<i>0.35</i>	<i>97%</i>
<i>FIS</i>	<i>0.40</i>	<i>0.61914(31)</i>	<i>[36:46]</i>	<i>0.99</i>	<i>45%</i>	<i>0.747379(64)</i>	<i>[42:46]</i>	<i>0.75</i>	<i>56%</i>	<i>0.28160(22)</i>	<i>[36:46]</i>	<i>0.35</i>	<i>97%</i>
<i>FIS</i>	<i>0.50</i>	<i>0.63174(32)</i>	<i>[36:46]</i>	<i>0.95</i>	<i>48%</i>	<i>0.757511(64)</i>	<i>[42:46]</i>	<i>0.84</i>	<i>50%</i>	<i>0.29078(22)</i>	<i>[36:46]</i>	<i>0.36</i>	<i>96%</i>
<i>FIS</i>	<i>0.60</i>	<i>0.64318(32)</i>	<i>[36:46]</i>	<i>0.95</i>	<i>49%</i>	<i>0.766662(64)</i>	<i>[42:46]</i>	<i>0.94</i>	<i>44%</i>	<i>0.29912(23)</i>	<i>[36:46]</i>	<i>0.37</i>	<i>96%</i>
FIS	0.70	0.65364(32)	[36:46]	0.96	47%	0.774976(64)	[42:46]	1.05	38%	0.30678(23)	[36:46]	0.38	96%
FIS	0.80	0.66328(32)	[36:46]	0.99	45%	0.782572(64)	[42:46]	1.15	33%	0.31388(24)	[36:46]	0.39	95%
FIS	0.90	0.67221(32)	[36:46]	1.03	41%	0.789550(64)	[42:46]	1.25	29%	0.32052(24)	[36:46]	0.39	95%
FIS	1.00	0.68053(32)	[36:46]	1.07	38%	0.795989(64)	[42:46]	1.33	26%	0.32678(24)	[36:46]	0.39	95%
FIS	1.10	0.68832(32)	[36:46]	1.10	36%	0.801958(63)	[42:46]	1.41	23%	0.33269(25)	[36:46]	0.39	95%
FIS	1.20	0.69565(32)	[36:46]	1.13	33%	0.807512(63)	[42:46]	1.47	21%	0.33832(25)	[36:46]	0.39	95%
FIS	1.30	0.70257(33)	[36:46]	1.15	32%	0.812697(63)	[42:46]	1.51	19%	0.34370(25)	[36:46]	0.39	95%
FIS	1.40	0.70913(33)	[36:46]	1.17	30%	0.817554(62)	[42:46]	1.55	19%	0.34885(26)	[36:46]	0.38	96%
FIS	1.50	0.71536(33)	[36:46]	1.19	29%	0.822115(62)	[42:46]	1.56	18%	0.35380(26)	[36:46]	0.37	96%
FIS	1.60	0.72129(33)	[36:46]	1.20	28%	0.826410(61)	[42:46]	1.57	18%	0.35857(26)	[36:46]	0.37	96%
FIS	1.70	0.72695(33)	[36:46]	1.21	28%	0.830464(60)	[42:46]	1.55	18%	0.36317(26)	[36:46]	0.36	96%
FIS	1.80	0.73236(33)	[36:46]	1.22	27%	0.834298(60)	[42:46]	1.53	19%	0.36762(27)	[36:46]	0.36	96%
FIS	1.90	0.73754(33)	[36:46]	1.24	26%	0.837932(59)	[42:46]	1.50	20%	0.37192(27)	[36:46]	0.35	97%
FIS	2.00	0.74251(33)	[36:46]	1.25	26%	0.841382(58)	[42:46]	1.46	21%	0.37611(27)	[36:46]	0.35	97%
FIS	2.10	0.74729(33)	[36:46]	1.26	25%	0.844663(57)	[42:46]	1.42	22%	0.38017(27)	[36:46]	0.34	97%
FIS	2.20	0.75188(33)	[36:46]	1.27	24%	0.847788(56)	[42:46]	1.39	24%	0.38412(28)	[36:46]	0.34	97%
FIS	2.30	0.75630(33)	[36:46]	1.28	23%	0.850769(56)	[42:46]	1.35	25%	0.38797(28)	[36:46]	0.34	97%
FIS	2.40	0.76056(33)	[36:46]	1.30	22%	0.853615(55)	[42:46]	1.32	26%	0.39173(28)	[36:46]	0.34	97%
FIS	2.50	0.76467(33)	[36:46]	1.32	22%	0.856337(54)	[42:46]	1.30	27%	0.39539(28)	[36:46]	0.34	97%
FIS	2.60	0.76864(33)	[36:46]	1.33	21%	0.858942(53)	[42:46]	1.29	27%	0.39897(29)	[36:46]	0.34	97%
FIS	2.70	0.77249(33)	[36:46]	1.35	20%	0.861438(53)	[42:46]	1.28	27%	0.40247(29)	[36:46]	0.35	97%
<i>M1</i>	<i>0.10</i>	<i>0.58969(36)</i>	<i>[24:30]</i>	<i>0.27</i>	<i>95%</i>	<i>0.72706(10)</i>	<i>[28:31]</i>	<i>1.04</i>	<i>37%</i>	<i>0.24816(33)</i>	<i>[24:30]</i>	<i>0.63</i>	<i>71%</i>
<i>M1</i>	<i>0.20</i>	<i>0.60876(36)</i>	<i>[24:30]</i>	<i>0.35</i>	<i>91%</i>	<i>0.742299(99)</i>	<i>[28:31]</i>	<i>0.80</i>	<i>49%</i>	<i>0.26248(35)</i>	<i>[24:30]</i>	<i>0.59</i>	<i>74%</i>
<i>M1</i>	<i>0.30</i>	<i>0.62587(36)</i>	<i>[24:30]</i>	<i>0.41</i>	<i>87%</i>	<i>0.755820(97)</i>	<i>[28:31]</i>	<i>0.69</i>	<i>56%</i>	<i>0.27515(37)</i>	<i>[24:30]</i>	<i>0.57</i>	<i>76%</i>
<i>M1</i>	<i>0.40</i>	<i>0.64120(37)</i>	<i>[24:30]</i>	<i>0.44</i>	<i>85%</i>	<i>0.767831(96)</i>	<i>[28:31]</i>	<i>0.64</i>	<i>59%</i>	<i>0.28644(39)</i>	<i>[24:30]</i>	<i>0.57</i>	<i>75%</i>
M1	0.50	0.65499(38)	[24:30]	0.42	86%	0.778540(94)	[28:31]	0.61	61%	0.29661(40)	[24:30]	0.59	74%
M1	0.60	0.66747(39)	[24:30]	0.36	90%	0.788142(93)	[28:31]	0.59	62%	0.30587(41)	[24:30]	0.62	71%
M1	0.70	0.67883(39)	[24:30]	0.28	95%	0.796805(91)	[28:31]	0.60	61%	0.31441(42)	[24:30]	0.66	68%
M1	0.80	0.68926(39)	[24:30]	0.19	98%	0.804666(89)	[28:31]	0.63	60%	0.32234(42)	[24:30]	0.69	66%
M1	0.90	0.69889(39)	[24:30]	0.12	99%	0.811841(88)	[28:31]	0.68	57%	0.32977(43)	[24:30]	0.72	63%

(Table continued)

TABLE IV. (Continued)

	τ/a^2	D_s				η_c				η_s			
		$\bar{R}_{cs}^{AP}(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$	$\bar{R}_{cc}^{AP}(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$	$\bar{R}_{ss}^{AP}(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$
M1	1.00	0.70783(38)	[24:30]	0.07	100%	0.818423(86)	[28:31]	0.73	53%	0.33678(44)	[24:30]	0.75	61%
M1	1.10	0.71618(38)	[24:30]	0.05	100%	0.824487(84)	[28:31]	0.79	50%	0.34342(44)	[24:30]	0.76	60%
M1	1.20	0.72400(37)	[24:30]	0.07	100%	0.830097(82)	[28:31]	0.85	47%	0.34974(45)	[24:30]	0.77	60%
M1	1.30	0.73135(36)	[24:30]	0.10	100%	0.835306(80)	[28:31]	0.89	44%	0.35578(45)	[24:30]	0.77	59%
M1	1.40	0.73830(36)	[24:30]	0.15	99%	0.840158(79)	[28:31]	0.91	43%	0.36157(46)	[24:30]	0.77	59%
M1	1.50	0.74489(36)	[24:30]	0.21	97%	0.844691(77)	[28:31]	0.91	43%	0.36714(46)	[24:30]	0.77	59%
M1	1.60	0.75115(36)	[24:30]	0.27	95%	0.848936(76)	[28:31]	0.89	44%	0.37250(47)	[24:30]	0.78	59%
M1	1.70	0.75711(36)	[24:30]	0.32	93%	0.852922(74)	[28:31]	0.85	47%	0.37768(47)	[24:30]	0.78	58%
M1	1.80	0.76281(35)	[24:30]	0.36	91%	0.856672(73)	[28:31]	0.79	50%	0.38269(47)	[24:30]	0.80	57%
M1	1.90	0.76825(35)	[24:30]	0.39	89%	0.860207(72)	[28:31]	0.72	54%	0.38754(48)	[24:30]	0.81	56%
M2	0.10	0.58957(42)	[24:30]	1.69	12%	0.72685(10)	[29:31]	0.21	81%	0.24771(29)	[24:30]	0.64	70%
M2	0.20	0.60864(42)	[24:30]	1.46	19%	0.742111(100)	[29:31]	0.22	81%	0.26199(30)	[24:30]	0.67	67%
M2	0.30	0.62573(42)	[24:30]	1.33	24%	0.755645(98)	[29:31]	0.23	79%	0.27464(32)	[24:30]	0.71	64%
M2	0.40	0.64105(42)	[24:30]	1.27	27%	0.767668(96)	[29:31]	0.25	78%	0.28591(33)	[24:30]	0.75	61%
M2	0.50	0.65483(41)	[24:30]	1.25	28%	0.778386(94)	[29:31]	0.26	77%	0.29606(34)	[24:30]	0.79	58%
M2	0.60	0.66733(41)	[24:30]	1.27	27%	0.787996(93)	[29:31]	0.27	76%	0.30532(35)	[24:30]	0.83	54%
M2	0.70	0.67873(40)	[24:30]	1.31	25%	0.796664(91)	[29:31]	0.30	74%	0.31386(35)	[24:30]	0.88	51%
M2	0.80	0.68921(39)	[24:30]	1.34	24%	0.804530(90)	[29:31]	0.36	70%	0.32180(36)	[24:30]	0.92	48%
M2	0.90	0.69891(39)	[24:30]	1.35	23%	0.811709(89)	[29:31]	0.46	63%	0.32924(37)	[24:30]	0.97	45%
M2	1.00	0.70793(38)	[24:30]	1.35	23%	0.818293(87)	[29:31]	0.60	55%	0.33625(38)	[24:30]	1.01	42%
M2	1.10	0.71637(37)	[24:30]	1.32	24%	0.824360(86)	[29:31]	0.79	46%	0.34290(38)	[24:30]	1.05	39%
M2	1.20	0.72428(36)	[24:30]	1.28	26%	0.829972(85)	[29:31]	1.00	37%	0.34923(39)	[24:30]	1.09	37%
M2	1.30	0.73174(36)	[24:30]	1.24	28%	0.835183(84)	[29:31]	1.24	29%	0.35528(40)	[24:30]	1.12	35%
M2	1.40	0.73878(35)	[24:30]	1.21	30%	0.840036(83)	[29:31]	1.50	22%	0.36108(40)	[24:30]	1.15	33%
M2	1.50	0.74545(34)	[24:30]	1.18	31%	0.844571(82)	[29:31]	1.75	17%	0.36665(41)	[24:30]	1.17	32%
M2	1.60	0.75179(34)	[24:30]	1.17	32%	0.848818(81)	[29:31]	2.01	13%	0.37202(42)	[24:30]	1.20	30%
M2	1.70	0.75783(33)	[24:30]	1.16	33%	0.852805(80)	[29:31]	2.25	11%	0.37721(42)	[24:30]	1.22	29%
M2	1.80	0.76358(33)	[24:30]	1.15	33%	0.856558(79)	[29:31]	2.47	8%	0.38223(42)	[24:30]	1.23	28%
M2	1.90	0.76908(33)	[24:30]	1.15	33%	0.860096(78)	[29:31]	2.67	7%	0.38709(42)	[24:30]	1.25	28%
M3	0.10	0.58908(45)	[24:30]	1.28	26%	0.72660(14)	[28:31]	1.78	15%	0.24744(28)	[24:30]	1.19	31%
M3	0.20	0.60814(46)	[24:30]	1.28	26%	0.74187(14)	[28:31]	1.79	15%	0.26174(29)	[24:30]	1.12	35%
M3	0.30	0.62523(46)	[24:30]	1.27	27%	0.75542(14)	[28:31]	1.82	14%	0.27439(30)	[24:30]	1.05	39%
M3	0.40	0.64055(46)	[24:30]	1.28	26%	0.76745(13)	[28:31]	1.83	14%	0.28567(32)	[24:30]	0.97	45%
M3	0.50	0.65436(45)	[24:30]	1.30	25%	0.77818(13)	[28:31]	1.83	14%	0.29584(33)	[24:30]	0.89	50%
M3	0.60	0.66688(45)	[24:30]	1.32	24%	0.78780(13)	[28:31]	1.83	14%	0.30512(34)	[24:30]	0.81	56%
M3	0.70	0.67832(45)	[24:30]	1.35	23%	0.79648(13)	[28:31]	1.82	14%	0.31367(35)	[24:30]	0.75	61%
M3	0.80	0.68883(45)	[24:30]	1.37	22%	0.80435(13)	[28:31]	1.82	14%	0.32164(37)	[24:30]	0.69	66%
M3	0.90	0.69854(45)	[24:30]	1.37	22%	0.81154(12)	[28:31]	1.82	14%	0.32910(38)	[24:30]	0.65	69%
M3	1.00	0.70756(45)	[24:30]	1.35	23%	0.81813(12)	[28:31]	1.82	14%	0.33615(39)	[24:30]	0.63	70%
M3	1.10	0.71598(45)	[24:30]	1.32	24%	0.82420(11)	[28:31]	1.82	14%	0.34283(40)	[24:30]	0.64	70%
M3	1.20	0.72386(45)	[24:30]	1.27	27%	0.82982(11)	[28:31]	1.82	14%	0.34919(41)	[24:30]	0.66	68%
M3	1.30	0.73128(45)	[24:30]	1.22	29%	0.83504(11)	[28:31]	1.81	14%	0.35527(42)	[24:30]	0.70	65%
M3	1.40	0.73828(45)	[24:30]	1.17	32%	0.83990(10)	[28:31]	1.79	15%	0.36110(43)	[24:30]	0.74	62%
M3	1.50	0.74492(46)	[24:30]	1.12	35%	0.844436(100)	[28:31]	1.77	15%	0.36670(43)	[24:30]	0.79	58%
M3	1.60	0.75121(46)	[24:30]	1.08	37%	0.848688(97)	[28:31]	1.74	16%	0.37210(44)	[24:30]	0.83	54%
M3	1.70	0.75721(46)	[24:30]	1.05	39%	0.852682(94)	[28:31]	1.70	16%	0.37731(44)	[24:30]	0.88	51%
M3	1.80	0.76292(46)	[24:30]	1.02	41%	0.856440(91)	[28:31]	1.67	17%	0.38236(45)	[24:30]	0.93	47%
M3	1.90	0.76839(46)	[24:30]	1.00	42%	0.859983(89)	[28:31]	1.63	18%	0.38725(45)	[24:30]	0.97	44%
C1	0.10	0.62479(44)	[24:30]	0.76	60%	0.75072(11)	[28:31]	3.35	2%	0.26727(32)	[24:30]	0.60	73%
C1	0.20	0.64780(45)	[24:30]	0.66	68%	0.77009(10)	[28:31]	2.81	4%	0.28480(34)	[24:30]	0.64	70%
C1	0.30	0.66829(45)	[24:30]	0.61	72%	0.786622(98)	[28:31]	2.52	6%	0.30058(36)	[24:30]	0.69	66%
C1	0.40	0.68653(46)	[24:30]	0.59	74%	0.800837(93)	[28:31]	2.37	7%	0.31481(38)	[24:30]	0.73	62%

(Table continued)

TABLE IV. (*Continued*)

	τ/a^2	D_s				η_c				η_s			
		$\bar{R}_{cs}^{AP}(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$	$\bar{R}_{cc}^{AP}(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$	$\bar{R}_{ss}^{AP}(\tau/a^2)$	$[t_1:t_2]$	χ^2/dof	$p\text{-val.}$
C1	0.50	0.70282(47)	[24:30]	0.57	75%	0.813149(88)	[28:31]	2.27	8%	0.32774(39)	[24:30]	0.77	60%
C1	0.60	0.71748(47)	[24:30]	0.56	76%	0.823906(84)	[28:31]	2.18	9%	0.33958(40)	[24:30]	0.79	58%
C1	0.70	0.73075(48)	[24:30]	0.56	76%	0.833384(81)	[28:31]	2.08	10%	0.35053(40)	[24:30]	0.79	57%
C1	0.80	0.74285(48)	[24:30]	0.57	76%	0.841803(77)	[28:31]	1.98	11%	0.36073(41)	[24:30]	0.79	58%
C1	0.90	0.75394(48)	[24:30]	0.58	75%	0.849333(74)	[28:31]	1.90	13%	0.37029(41)	[24:30]	0.79	58%
C1	1.00	0.76418(48)	[24:30]	0.59	74%	0.856112(72)	[28:31]	1.83	14%	0.37930(41)	[24:30]	0.78	58%
C1	1.10	0.77366(49)	[24:30]	0.61	73%	0.862247(70)	[28:31]	1.78	15%	0.38784(42)	[24:30]	0.79	58%
C2	0.10	0.62355(40)	[24:30]	1.46	19%	0.74999(11)	[28:31]	3.78	1%	0.26660(24)	[24:30]	0.45	85%
C2	0.20	0.64655(40)	[24:30]	1.48	18%	0.769448(99)	[28:31]	2.77	4%	0.28416(25)	[24:30]	0.51	80%
C2	0.30	0.66705(41)	[24:30]	1.47	18%	0.786060(94)	[28:31]	2.16	9%	0.29998(27)	[24:30]	0.55	77%
C2	0.40	0.68532(41)	[24:30]	1.44	20%	0.800334(89)	[28:31]	1.76	15%	0.31425(28)	[24:30]	0.59	74%
C2	0.50	0.70166(42)	[24:30]	1.40	21%	0.812695(86)	[28:31]	1.49	22%	0.32721(29)	[24:30]	0.62	71%
C2	0.60	0.71639(43)	[24:30]	1.36	23%	0.823491(83)	[28:31]	1.30	27%	0.33909(30)	[24:30]	0.65	69%
C2	0.70	0.72974(43)	[24:30]	1.34	23%	0.833004(80)	[28:31]	1.17	32%	0.35007(31)	[24:30]	0.67	67%
C2	0.80	0.74193(44)	[24:30]	1.34	23%	0.841451(78)	[28:31]	1.08	36%	0.36032(32)	[24:30]	0.69	66%
C2	0.90	0.75314(44)	[24:30]	1.36	23%	0.849008(76)	[28:31]	1.02	38%	0.36992(33)	[24:30]	0.69	66%
C2	1.00	0.76348(44)	[24:30]	1.37	22%	0.855809(74)	[28:31]	0.97	40%	0.37899(34)	[24:30]	0.69	66%
C2	1.10	0.77306(44)	[24:30]	1.36	22%	0.861964(72)	[28:31]	0.95	42%	0.38758(34)	[24:30]	0.68	66%

TABLE V. Results of the $a \rightarrow 0$ continuum limit extrapolations for quantities based on D_s and η_c correlators at fixed flow time τ using an *Ansatz* linear in a^2 . Listed are the flow time τ in physical units, the continuum limit value as well as the goodness of fit expressed by the corresponding $\chi^2/\text{d.o.f.}$ and the p -value for both quantities. Values corresponding to small τ not entering in our $\tau \rightarrow 0$ extrapolation are set in italics.

τ [GeV $^{-2}$]	D_s			η_c		
	$R_{sc}^{AP}(\tau)M_{D_s}$ [MeV]	$\chi^2/\text{d.o.f.}$	$p\text{-val.}$	$R_{cc}^{AP}(\tau)M_{\eta_c}$ [MeV]	χ^2/dof	$p\text{-val.}$
0.039	583.5(2.7)	0.40	81%	1096.8(4.9)	2.33	5%
0.052	600.1(2.8)	0.32	86%	1116.1(5.0)	2.02	9%
0.064	614.2(2.8)	0.23	92%	1132.3(5.1)	1.68	15%
0.077	627.6(2.9)	0.22	93%	1148.0(5.1)	1.54	19%
0.090	639.3(2.9)	0.22	93%	1161.5(5.2)	1.44	22%
0.103	650.1(3.0)	0.21	93%	1174.1(5.2)	1.35	25%
0.116	659.9(3.0)	0.20	94%	1185.3(5.2)	1.26	28%
0.129	668.8(3.0)	0.20	94%	1195.5(5.3)	1.19	31%
0.142	677.4(3.1)	0.23	92%	1205.3(5.3)	1.18	32%
0.155	685.0(3.1)	0.21	93%	1213.8(5.4)	1.12	35%
0.168	692.4(3.1)	0.22	93%	1222.1(5.4)	1.09	36%
0.181	699.3(3.2)	0.22	92%	1229.8(5.4)	1.06	37%
0.193	705.8(3.2)	0.23	92%	1236.9(5.5)	1.05	38%
0.206	711.9(3.2)	0.23	92%	1243.6(5.5)	1.02	39%
0.219	717.6(3.2)	0.23	92%	1249.7(5.5)	0.99	41%
0.232	723.2(3.3)	0.24	92%	1255.8(5.5)	0.99	41%
0.245	728.5(3.3)	0.24	92%	1261.4(5.6)	0.98	42%
0.258	733.5(3.3)	0.24	92%	1266.7(5.6)	0.96	43%
0.271	738.4(3.3)	0.25	91%	1271.7(5.6)	0.95	43%
0.284	742.9(3.3)	0.25	91%	1276.4(5.6)	0.94	44%
0.297	747.5(3.4)	0.26	91%	1281.1(5.6)	0.94	44%
0.309	751.7(3.4)	0.26	90%	1285.3(5.7)	0.92	45%
0.322	755.9(3.4)	0.27	90%	1289.5(5.7)	0.92	45%
0.335	759.9(3.4)	0.28	89%	1293.5(5.7)	0.92	45%
0.348	763.7(3.4)	0.28	89%	1297.2(5.7)	0.90	46%

TABLE VI. Results of the $a \rightarrow 0$ continuum limit extrapolations for ratios based on η_s correlators at fixed flow time τ using either an *Ansatz* linear in a^2 or a fit to a constant. Listed are the flow time τ in physical units, the continuum limit value as well as the goodness of fit given by the corresponding $\chi^2/\text{d.o.f.}$ and the p -value for both continuum limits. Values corresponding to small τ not entering in our $\tau \rightarrow 0$ extrapolation are set in italics.

τ [GeV ⁻²]	$\eta_s^{a \rightarrow 0}$ linear			$\eta_s^{a \rightarrow 0}$ const		
	$R_{sc}^{AP}(\tau)M_{\eta_s}$ [MeV]	$\chi^2/\text{d.o.f.}$	$p\text{-val.}$	$R_{cc}^{AP}(\tau)M_{\eta_s}$ [MeV]	χ^2/dof	$p\text{-val.}$
0.035	87.99(50)	0.19	91%	92.56(14)	22.57	0%
0.053	93.65(53)	0.21	89%	96.50(14)	8.12	0%
0.070	98.40(55)	0.22	88%	100.10(15)	2.81	2%
0.088	102.51(57)	0.24	87%	103.42(15)	0.88	48%
0.106	106.17(58)	0.26	85%	106.48(16)	0.27	90%
0.123	109.38(60)	0.28	84%	109.36(16)	0.21	93%
0.141	112.41(62)	0.30	83%	112.02(17)	0.33	86%
0.158	115.10(63)	0.32	81%	114.57(17)	0.43	79%
0.176	117.70(65)	0.34	80%	116.95(17)	0.62	65%
0.194	120.08(66)	0.36	78%	119.25(18)	0.71	59%
0.211	122.37(67)	0.38	77%	121.42(18)	0.83	51%
0.229	124.53(68)	0.40	76%	123.51(18)	0.90	46%
0.246	126.58(70)	0.42	74%	125.52(19)	0.94	44%
0.264	128.57(71)	0.44	73%	127.44(19)	1.02	40%
0.282	130.44(72)	0.46	71%	129.31(19)	1.01	40%
0.299	132.29(73)	0.48	69%	131.10(20)	1.08	37%
0.317	134.03(74)	0.51	68%	132.85(20)	1.07	37%
0.335	135.75(75)	0.53	66%	134.54(20)	1.11	35%
0.352	137.40(76)	0.56	64%	136.18(20)	1.12	35%

APPENDIX C: RESIDUAL MASSES

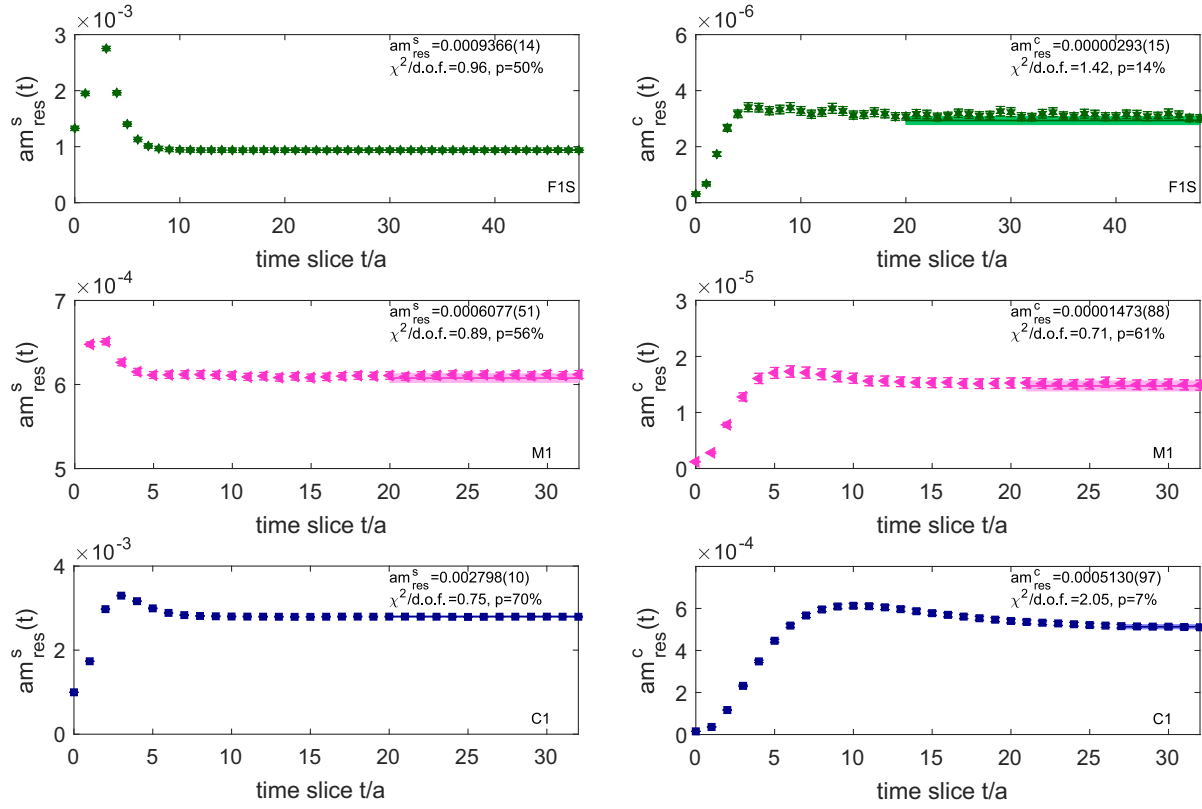


FIG. 11. Extraction of the residual mass using strange quark correlators (left) and charm quark correlators (right) on the F1S ensemble (top), M1 (center), and C1 (bottom). In all cases we observe the expected flat curve indicating that DWF discretization errors are well under control.

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