

Holographic Krylov complexity for charged, composite, and extended probes

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(Received 13 July 2026; accepted 16 September 2026; published 1 October 2026)

We study a proposed holographic representation of spread/Krylov complexity for states or operators dual to charged, composite, and extended probes. Our analysis builds on the relation between the rate of spread complexity and the proper momentum of an infalling bulk excitation. A conserved Noether charge requires particular care: unitary spread complexity satisfies $\dot{\mathcal{C}}_K(0) = 0$, and therefore the velocity conjugate to a fixed charge cannot be counted as an independent spreading direction. We consequently eliminate cyclic coordinates by a partial Legendre transform and compute the proper momentum from the fixed-charge Routhian. We first apply this prescription to an R -charged particle in $\text{AdS}_5 \times S^5$. The charge enters through the reduced effective mass and energy, while the resulting complexity has the required quadratic onset. We next study probes that are pointlike from the boundary perspective but possess nontrivial bulk structure. These are baryon-vertex configurations and giant gravitons. For the giant graviton the fixed-charge prescription retains the effects of the Born-Infeld and Wess-Zumino couplings without reintroducing the cyclic angular velocity into the proper momentum. We also analyze a fundamental string falling in anti-de Sitter (AdS) while stretched along a boundary spatial direction as a collective model of a nonlocal excitation. Within the rigid-probe Poincaré-AdS examples considered here, the proper-momentum rate is linear at late times, whereas charges, compositeness, and spatial extension affect its normalization and its subleading or intermediate-time structure. We interpret this as a robust pattern in the tested class (not as a general universality theorem). The fixed-charge Routhian provides a candidate bulk counterpart of sector selection. An explicit symmetry-resolved boundary Lanczos construction remains an open problem.

DOI: [10.1103/PhysRevD.114.086001](https://doi.org/10.1103/PhysRevD.114.086001)

I. INTRODUCTION

Quantum complexity has become a useful meeting point between quantum information, many-body dynamics, quantum chaos, and holography. In recent years it has become increasingly clear that the word “complexity” covers several related but genuinely different notions, ranging from circuit and Nielsen complexity to tensor-network, path-integral, and holographic proposals. Among these, Krylov complexity has emerged as an especially natural diagnostic of operator growth. Starting from a reference operator and repeatedly acting with the Liouvillian, one generates a distinguished orthonormal basis through the Lanczos algorithm; the corresponding Krylov complexity measures how far the

time-evolved operator has spread along this dynamically generated chain. One of its main appeals is precisely that the basis is not imposed externally, but is instead fixed by the Hamiltonian and by the operator under consideration. In this sense, Krylov complexity provides a canonical notion of spreading, well suited to questions about scrambling, universality, and the detailed operator dependence of quantum evolution [1–9].

The modern development of the subject was strongly shaped by the operator-growth perspective of [1], who proposed that in generic many-body systems the Lanczos coefficients display an asymptotically linear growth. This viewpoint connected the structure of the Krylov chain to universal properties of chaotic dynamics and stimulated a rapidly expanding literature on late-time growth, bounds, integrability, quantum field theory, and geometric formulations of Krylov space [1–6,10–12]. A particularly important lesson from this body of work is that Krylov complexity is not merely a coarse measure of chaos. It is a probe of how a specific operator explores the available

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Hilbert-space directions, and it is therefore sensitive to the initial operator, to symmetries, to conserved quantum numbers, and to the detailed dynamical sector in which the evolution takes place. This sensitivity is especially attractive from a holographic point of view, where one would like to understand which bulk observables capture not only the overall growth of complexity, but also its dependence on the nature of the probe.

On the holographic side, the earliest and most influential proposals for complexity were the complexity-volume and complexity-action conjectures, together with their many generalizations and refinements. In these frameworks, one relates the complexity of the boundary state to an extremal volume or to the action evaluated on a Wheeler-DeWitt patch, thereby extending to complexity the geometric spirit that proved so fruitful in other holographic observables [7,13,14]. More recently, a complementary line of development has aimed at connecting spread or Krylov complexity to bulk kinematics more directly [6,15]. In particular, Krylov spread complexity admits an explicit bulk realization in Jackiw-Teitelboim gravity in suitable limits, while for local operator excitations in $\text{AdS}_3/\text{CFT}_2$ the proposal of [16] relates the rate of spread complexity to the proper radial momentum of an infalling probe [16,17]. This sharpened the idea that bulk motion can encode the boundary growth of complexity in a precise and calculable manner.

There is a necessary consistency condition on any proposed bulk representation of spread complexity. Given a normalized seed $|K_0\rangle$, the Lanczos expansion,

$$e^{-iHt}|K_0\rangle = \sum_{n \geq 0} \varphi_n(t)|K_n\rangle, \quad C_K(t) = \sum_{n \geq 0} n|\varphi_n(t)|^2, \quad (1.1)$$

implies $\varphi_1(t) = -ib_1 t + \mathcal{O}(t^2)$ and $\varphi_n(t) = \mathcal{O}(t^n)$. For unitary evolution generated by a Hermitian Hamiltonian, the corresponding Jacobi matrix may be chosen real, so that $|\varphi_n(t)|^2$ and $C_K(t)$ are even functions of time. Hence

$$C_K(t) = b_1^2 t^2 + \mathcal{O}(t^4), \quad \dot{C}_K(t) = 2b_1^2 t + \mathcal{O}(t^3), \quad \dot{C}_K(0) = 0. \quad (1.2)$$

This is a structural property of the Krylov construction, rather than a model-dependent expectation [18–20]. A bulk proper momentum containing a nonzero constant at the reference time therefore cannot directly represent the standard unitary Krylov-complexity rate. In the examples below this issue arises when the probe moves along a cyclic direction carrying a conserved Noether charge. Our prescription is to fix the charge first by a partial Legendre transform and to compute the proper momentum from the resulting Routhian [21]. Throughout the paper we use the orientation-independent convention (this makes the complexity of falling probes grow with time)

$$\dot{C}(t) = \kappa |P_y(t)|, \quad \kappa > 0, \quad t > 0, \quad (1.3)$$

where κ allows for a convention-dependent normalization.

The purpose of this paper is to explore how far this proposal can be pushed when the probe is no longer a completely structureless particle. Our viewpoint is that this question is particularly natural in holography. A local operator may be represented in the bulk by a pointlike object, but that object can still carry nontrivial internal data, like conserved charges, worldvolume fluxes, induced couplings, or a composite origin in terms of higher-dimensional branes. Conversely, one may also consider genuinely extended probes, whose dual boundary interpretation is manifestly nonlocal. It is then natural to ask which features of the Krylov-complexity story are universal, which are sensitive to the operator being probed, and which are artifacts of the simplest point-particle approximation.

A. General idea of this paper

Most holographic discussions of complexity growth based on infalling probes use objects whose dynamics is effectively pointlike from the boundary point of view. In the simplest setups one studies a massive particle falling along a radial geodesic, and the proper momentum of this probe reproduces the expected complexity growth [16,22–24]. This is both elegant and physically transparent. But it also leaves open an important issue: whether the same dictionary continues to hold when the operator is still local for the field-theory observer, yet carries an internal structure that is nontrivial in the bulk.

The first aim of this work is to study precisely this class of probes. We consider, on the one hand, a particle carrying a conserved R -charge, whose motion involves both the AdS radial direction and the internal space, in the spirit of BMN-like sectors [25]. We then turn to probes that are pointlike for the boundary observer but arise from higher-dimensional branes with nontrivial worldvolume physics, like baryon-vertex configurations [26] and giant gravitons [27,28]. These are natural laboratories in which to test the dependence of spread complexity on conserved quantum numbers, compositeness, induced charges, and Wess-Zumino couplings, while remaining in a kinematical regime that is still close to the original point-particle proposal.

The examples are selected so as to vary the nature of the probe while keeping the central Poincaré- AdS geometry fixed. The ordinary particle supplies the baseline; the R -charged particle isolates a conserved Noether sector. The baryon vertex adds compositeness, attached strings, and induced charge. The giant graviton introduces wrapped-brane dynamics and a Wess-Zumino coupling. Finally, the spatially stretched fundamental string changes the effective radial action of a nonlocal excitation. This organization allows us to distinguish effects of probe structure from those of the background geometry. It also makes clear why the

late-time agreement found below is evidence within a controlled class, rather than a proof of universality.

Our second aim is to move beyond this effectively pointlike sector and consider a genuinely extended probe, namely a fundamental string falling in AdS while stretched along a spatial direction. From the boundary viewpoint, this is intended to model a nonlocal operator. This case is conceptually more delicate, and for that reason also more interesting. Following the prescription advocated in [16], we assume that a suitably defined generalized proper momentum still captures the rate of spread complexity. For rigid or consistently truncated configurations, this is a natural semiclassical working hypothesis. Nevertheless, its status should be stated with some care when extended objects are involved.

Indeed, several possible criticisms become relevant in the extended case. First, a string or brane supports infinitely many internal modes, and a collective-coordinate description need not capture the full operator growth of the dual nonlocal excitation. Second, worldsheet or worldvolume gauge choices, Wess-Zumino terms, endpoint contributions, and conserved charges may affect the definition of the canonical momenta entering the complexity prescription. Third, for a nonlocal operator it is less obvious that the resulting quantity should be identified with the same notion of Krylov complexity that is naturally associated with local operators. For this reason, throughout the paper we interpret the extended-probe calculation conservatively: it should be viewed as a controlled proposal for a collective notion of spread complexity, rather than as a complete first-principles derivation. In our view this is not a weakness but a useful diagnostic. By comparing structured pointlike probes with genuinely extended ones, one can isolate which aspects of the momentum/complexity correspondence are robust and which call for a more refined boundary understanding.

With this perspective in mind, the goal of the paper is not to claim a final holographic dictionary for every version of Krylov complexity, but rather to enlarge the class of probes for which the proposal can be tested explicitly. Our emphasis will therefore be on solvable semiclassical dynamics, on the role of conserved quantities and induced interactions, and on the contrast between local/composite probes and nonlocal/extended probes. This also helps to identify the main open problem on the field-theory side. That is, to isolate the boundary sectors and operators for which an independent Krylov or spread-complexity computation can be matched to the bulk results discussed here.

The paper is organized as follows. In Sec. II we study a particle falling in AdS₅ while carrying a conserved R -charge. We separate the unreduced momentum of the full bulk trajectory from the proper momentum of the fixed-charge Routhian, and identify only the latter with the candidate Krylov-complexity rate. In Sec. III we turn to baryon-vertex-like operators, which are composite but effectively pointlike

probes, and derive their dynamics and associated proper momentum, including the effect of the attached fundamental strings. In Sec. IV we consider a falling giant graviton. Its Born-Infeld and Wess-Zumino couplings provide a second, more stringent application of the fixed-charge prescription. Section V studies a genuinely extended fundamental string as a collective model of the spread of a nonlocal excitation. We conclude in Sec. VI. Appendix A extends the baryon-vertex analysis to AdS₃ × S³ × CY₂, while Appendix B rederives the baryon-vertex dynamics using a Routhian formulation. Appendix C gives the fixed-charge short-time expansion for the giant graviton, and Appendix D collects additional details of the Routhian prescription.

II. CONSERVED R -CHARGE AND THE FIXED-CHARGE PRESCRIPTION

In this section we consider a massive particle that falls radially in AdS₅ and carries angular momentum J along an equator of S⁵. The angular coordinate is cyclic, so J labels a conserved R -charge sector. Our aim is to determine the proper momentum appropriate to that *fixed-charge sector*. We first display the full bulk trajectory, but apply the complexity prescription only after eliminating the cyclic angle by a partial Legendre transform. This order of operations is required by the universal short-time law $C_K(t) = b_1^2 t^2 + \mathcal{O}(t^4)$ in Eq. (1.2). The configuration on the Quantum Field theory (QFT) side may be viewed as a BMN-like excitation built from adjoint fields and covariant derivatives [25]. The generalized proper-momentum constructions of [29–32] will be useful below, but a conserved cyclic charge requires the additional fixed-sector reduction described in [21].

The ten-dimensional metric and 5-form are

$$ds^2 = \frac{r^2}{l^2} (-dt^2 + d\vec{x}_3^2) + \frac{l^2 dr^2}{r^2} + l^2 d\Omega_5^2, \\ F_5 = N_c (1 + *) \text{Vol}_{S^5}. \quad (2.1)$$

The five sphere has line element

$$d\Omega_5^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2 \\ + \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3 d\theta_4^2 \\ + \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3 \sin^2 \theta_4 d\psi^2.$$

Let us consider a point particle probing AdS₅ × S⁵ in the configuration

$$r = r(t), \quad \psi = \psi(t). \quad (2.2)$$

This is a consistent embedding subjected to the fact that one takes the various angles $\theta_i = \frac{\pi}{2}$, on the five sphere. The Lagrangian for this particle is

$$L = -m\sqrt{\frac{r^2}{l^2} - \frac{l^2\dot{\psi}^2}{r^2}} - l^2\dot{\psi}^2. \quad (2.3)$$

There are two equations of motion and two conserved quantities,

$$-\frac{d}{dt}\left[\frac{l^2\dot{r}}{r^2L}\right] = \frac{(r^4 + l^4\dot{\psi}^2)}{l^2r^3L}, \quad \frac{d}{dt}\left[\frac{\dot{\psi}}{L}\right] = 0, \quad (2.4)$$

$$-J = \frac{m^2l^2}{L}\dot{\psi}, \quad -H = \frac{m^2r^2}{l^2L}. \quad (2.5)$$

One can invert Eq. (2.5) to obtain expressions for $\dot{r}$ and $\dot{\psi}$,

$$\dot{\psi} = \frac{Jr^2}{Hl^4}, \quad \dot{r} = \pm \frac{r^2}{Hl^4} \sqrt{H^2l^4 - (J^2 + m^2l^2)r^2}. \quad (2.6)$$

These first-order equations solve the second-order ones in Eq. (2.4). The solution to the first-order equations is

$$r(t) = \pm \frac{Hl^2}{\sqrt{H^2l^2 + J^2 + l^2m^2}}; \quad \psi(t) = \frac{J}{\sqrt{J^2 + m^2l^2}} \tan^{-1}\left(\frac{Ht}{\sqrt{J^2 + m^2l^2}}\right). \quad (2.7)$$

Notice that at $t = 0$, the particle starts from the UV-cutoff radial position $r_{UV} = \frac{Hl^2}{\sqrt{J^2 + m^2l^2}}$ and zero initial velocity. Also, for $t = 0$ the angle starts from $\psi = 0$ with initial angular velocity $\frac{JH}{J^2 + m^2l^2}$.

We now distinguish two different quantities. The first is the momentum associated with the full two-dimensional trajectory (r, ψ) . It is obtained by introducing

$$dy_{\text{all}}^2 = \frac{l^2}{r^2} dr^2 + l^2 d\psi^2, \quad \dot{y}_{\text{all}} = \sqrt{\frac{l^2\dot{r}^2}{r^2} + l^2\dot{\psi}^2}, \quad (2.8)$$

and gives, on the solution (2.7),

$$\mathfrak{P}_{\text{all}}(t) = \frac{1}{l} \sqrt{H^2t^2 + J^2}. \quad (2.9)$$

This quantity measures the magnitude of the total semi-classical motion, including the velocity conjugate to J . Since $\mathfrak{P}_{\text{all}}(0) = |J|/l$, it cannot equal the unitary fixed-sector Krylov-complexity rate. We therefore do not associate it with $\dot{C}$. Its expansions,

$$\mathfrak{P}_{\text{all}}(t) = \frac{|J|}{l} + \frac{H^2}{2|J|l} t^2 + \mathcal{O}(t^4), \quad t \rightarrow 0, \quad (2.10)$$

$$\mathfrak{P}_{\text{all}}(t) = \frac{|H|}{l} |t| + \frac{J^2}{2|H|l|t|} + \mathcal{O}(t^{-3}), \quad |t| \rightarrow \infty, \quad (2.11)$$

remain useful kinematical diagnostics, but they should not be interpreted as the short-time or subleading behavior of a fixed-charge Krylov complexity.

The physical fixed- J prescription is obtained by eliminating ψ before defining the proper coordinate. Solving $J = \partial L / \partial \dot{\psi}$ and performing the partial Legendre transform gives

$$R_J = J\dot{\psi} - L = \hat{m}_J \sqrt{\frac{r^2}{l^2} - \frac{l^2\dot{r}^2}{r^2}}, \quad \hat{m}_J^2 = m^2 + \frac{J^2}{l^2}. \quad (2.12)$$

The charge has become part of the effective mass. We are now working at a *fixed value of J* . The reduced proper coordinate and its momentum are

$$\dot{y}_J = \frac{l\dot{r}}{r}, \quad P_y^{(J)} = \frac{\partial R_J}{\partial \dot{y}_J} = -\frac{\hat{m}_J \dot{y}_J}{\sqrt{r^2/l^2 - \dot{y}_J^2}}. \quad (2.13)$$

The conserved energy derived from R_J reproduces the radial trajectory in Eq. (2.7). Using [16] and Eq. (1.3), for the release of the probe from rest at r_{UV} one obtains

$$|P_y^{(J)}(t)| = \frac{|H_J|}{l} |t|, \quad C_J(t) - C_J(0) = \frac{\kappa|H_J|}{2l} t^2 + \mathcal{O}(t^4). \quad (2.14)$$

In particular, $P_y^{(J)}(0) = 0$, as required by Eq. (1.2). The charge dependence is retained through $\hat{m}_J$, the initial position, and H_J ; the conjugate angular velocity is not counted as a direction of spreading within a sector of definite J .

For completeness, let us discuss a possible point of contact with symmetry resolved Krylov complexity on the field theory side [33,34]. A genuine symmetry-resolved boundary construction would start from a charge projector Π_J and the normalized seed

$$|\Psi_{0,J}\rangle = \frac{\Pi_J |\Psi_0\rangle}{\sqrt{\langle \Psi_0 | \Pi_J | \Psi_0 \rangle}}, \quad H_J^{\text{QFT}} = \Pi_J H \Pi_J, \quad (2.15)$$

and would perform the Lanczos recursion with H_J^{QFT} entirely inside fixed charge Hilbert space [33,34]. The present paper does not construct Π_J , the sector Lanczos coefficients, or the sector Krylov wave functions. The Routhian prescription should therefore be viewed as a candidate bulk counterpart of working at fixed charge, not as an explicit computation of symmetry-resolved Krylov complexity. Establishing that correspondence directly in $\mathcal{N} = 4$ Super Yang-Mills (SYM) is an important problem for future work.

The coordinates $(t, \vec{x}, r, y_J)$ have units of length, while the angles are dimensionless. The parameters H_J and m

have units of inverse length, J is dimensionless, and $P_y^{(J)}$ has units of inverse length. Thus all expressions above are dimensionally consistent.

Let us now focus on a different “point particle” probe. The difference is that whilst these probes do not have an extra quantum number (like the R-charge studied above), the operators considered do have an interesting internal structure (being made out of many strongly interacting components).

III. COMPLEXITY FOR BARYON-VERTEX-LIKE OPERATORS

It is known that the Krylov/spread complexity is in principle dependent on the operator whose “spread” across the Hilbert space is measured; see for example [35,36]. In this section, we consider the spread/Krylov complexity of *baryon-vertex-like operators* [26]. In holography, the baryon vertex consists of a D-brane that pulls back on time and part of the internal space. There is also an excited gauge field on the brane. This differs from the pointlike probes previously considered in the literature. In fact the charge induced on the wrapped D-brane probe is canceled by a set of N F1-strings that end at infinity (where another D-brane is placed). In this sense, the baryon vertex is a heavy “composite” operator [the mass is proportional to N , the number of F1-strings, and the number of colors in the Super conformal field theory (SCFT)]. From the viewpoint of the field theory observer, it is still a heavy point like object with electric charge and composed by a large number of constituents (nondynamical quarks).

Holographically, the complexity is computed by a Dp-brane extended on $[t, \Omega_p]$ that falls along a geodesic in the AdS radial direction, parametrized by $r(t)$. We must consider the action of the N -fundamental strings that stretch from the baryon-vertex Dp-brane to the brane at infinity. These F1s contribute to the potential part of the Lagrangian of the system. As a first approximation, we do not add a kinetic term for the F1 strings, and consider them tension-full “rods.” This approximation is valid as long as the strings do not fluctuate very much in directions perpendicular to the radial-direction respect to their (radial) stretch. The whole object has a dynamics different from the single particle falling in AdS. It is interesting to calculate

the spread complexity for this pointlike operator. Before we study the geodesic fall of this composite object, let us write some words about the different descriptions of the baryon vertex, with the purpose of making the forthcoming calculations clearer.

A. Comments on baryon vertex

A baryon is defined as the contraction of N quarks (fundamental fields) with the antisymmetric tensor of $SU(N)$. In the case of a field theory without fields in the fundamental representation of the gauge group (quarks), the closest object is a composite of N external quarks. In holography, one works with a combination of Dp-branes and F1 strings. The Dp-brane extends along $[t, \Omega_p]$ where Ω_p belongs to the internal space, making this object look like a particle for the field theory observer. This Dp-brane admits a fluctuation described by a gauge field a_1 with curvature f_2 . Aside from the Born-Infeld action of the brane $S_{BI} = T_{Dp} \int d^{p+1}x e^{-\Phi} \sqrt{-\det[g+f]}$ we have a Wess-Zumino term (associated with a background Ramond potential C_{p-1})

$$\begin{aligned} S_{WZ} &= -T_{Dp} \int_{(t, \Omega_p)} C_{p-1} \wedge f_2 = T_{Dp} \int_{(t, \Omega_p)} F_p \wedge a_1 \\ &= NT_{Dp} \int a_t(t) dt. \end{aligned} \quad (3.1)$$

In the cases we study below $\int_{\Omega_p} F_p = N$. We consider a gauge field on the brane $a_1 = a_t(t)dt$ that has zero curvature $f_2 = da_1 = 0$. The action describing this Dp-brane is

$$\begin{aligned} S_{BIWZ} &= T_{Dp} \int d^{p+1}x e^{-\Phi} \sqrt{-\det[g_{ind}]} \\ &\quad + NT_{Dp} \int a_t(t) dt. \end{aligned} \quad (3.2)$$

The equation of motion $\frac{\delta S_{BIWZ}}{\delta a_t} = N = 0$ forces us to add N -fundamental strings that carry the charge away to satisfy Gauss’ law in a compact space Ω_p (in other words a_1 acts as a Lagrange multiplier). The consistent Dp-F1 system has action

$$\begin{aligned} S_{BIWZ-F1} &= T_{Dp} \int d^{p+1}x e^{-\Phi} \sqrt{-\det[g_{ind}]} + NT_{Dp} \int a_t(t) dt + NS_{F1}, \\ S_{F1} &= S_{NG} - T_{F1} \int a_t(t) dt, \\ S_{NG} &= T_{F1} \int dt \int_{r(t)}^{r_{UV}} dr \sqrt{-g_{tt}g_{rr}} = T_{F1} \int dt [r_{UV} - r(t)]. \end{aligned} \quad (3.3)$$

This yields the complete action of the Dp-F1 system

$$S_{\text{BIWZ-F1}} = T_{Dp} \int d^{p+1}x e^{-\Phi} \sqrt{-\det[g_{\text{ind}}]} + NT_{F1} \int dt [r_{\text{UV}} - r(t)]. \quad (3.4)$$

In our calculation, we allow the probe Dp-brane to fall following a radial geodesic in an AdS geometry. We also approximate $S_{F1} \approx \int dt (r_{\text{UV}} - r(t))$ (the F1s extended along the radial direction do not fluctuate as a first approximation) contributing to the potential energy of the system. An alternative view of the baryon vertex (that we do not use here) is in terms of a soliton—a Born-Infeld solution—of the Born-Infeld-Wess-Zumino action, with a dynamical gauge field. In this configuration the charge of F1 is induced on the worldvolume of the Dp-brane. To compare the relevant bibliography explaining the different viewpoints see [26,37–39].

We consider the case of baryon-vertex operators measuring complexity in $\text{AdS}_5 \times S^5$. In Appendix A we extend the treatment to the case of $\text{AdS}_3 \times S^3 \times CY_2$. In Appendix B (for the case of $\text{AdS}_5 \times S^5$), we present a treatment that reproduces our results above using the Routhian (an intermediate functional between the Lagrangian and the Hamiltonian). This treatment does *not* need the gauge field on the brane a_1 to have zero curvature $f_2 = da_1$. The end result is that of Eq. (3.4).

B. The baryon vertex falling in $\text{AdS}_5 \times S^5$

We start studying the baryon vertex in $\mathcal{N} = 4$ SYM. The dual background is $\text{AdS}_5 \times S^5$. We consider a probe D5 that extends on $[t, \Omega_5]$, moves along the radial coordinate with $r(t)$, and has a gauge field switched on the brane. The induced metric is

$$ds_{\text{ind},D5}^2 = dt^2 \left(-\frac{r^2}{l^2} + \frac{l^2 \dot{r}^2}{r^2} \right) + l^2 d\Omega_5^2. \quad (3.5)$$

The Born-Infeld part of the action is

$$S_{\text{BI},D5} = T_{D5} l^5 \int dt d\Omega_5 \frac{r}{l} \sqrt{1 - \frac{l^4 \dot{r}^2}{r^4}} = T_{D5} l^5 \text{Vol}_{S^5} \int dt \frac{r}{l} \sqrt{1 - \frac{l^4 \dot{r}^2}{r^4}}. \quad (3.6)$$

In the absence of Ramond potentials $C_6 = C_2 = C_0 = 0$ and for NS $B_2 = 0$, but in the presence of nonzero C_4 , the Wess-Zumino part of the action is

$$S_{\text{WZ}} = -T_{D5} \int dt d\Omega_5 C_4 \wedge f_2 = +T_{F1} N \int a_t dt. \quad (3.7)$$

Following the treatment indicated above for the case of a background generated by a stack of D3-branes with the

baryon vertex being a D5-brane probe, we find for the action of the strings

$$\begin{aligned} S_{F1} &= S_{NG} + S_{\text{WZ-F1}}, \\ S_{NG} &= NT_{F1} \int dt \int_{r(t)}^{r_{\text{UV}}} dr \sqrt{-g_{tt} g_{rr}} \\ &= NT_{F1} \int dt [r_{\text{UV}} - r(t)], \\ S_{\text{WZ-F1}} &= -NT_{F1} \int a_t dt. \end{aligned} \quad (3.8)$$

Note that these N F1s cancel the charge induced by the gauge field on the compact part of the D5-brane (hence satisfying Gauss' law). In summary, the action of the system is

$$S_{D5-F1} = \int dt \left[\alpha \sqrt{\frac{r^2}{l^2} \left(1 - \frac{l^4 \dot{r}^2}{r^4} \right)} + \beta (r_{\text{UV}} - r(t)) \right]. \quad (3.9)$$

The coefficients α, β are

$$\alpha = T_{D5} l^5 \text{Vol}_{S^5}, \quad \beta = NT_{F1}. \quad (3.10)$$

In particular, notice that in a static situation $r(t) = r_0$, if $\alpha = l\beta$ we have a static supersymmetry object. Notice that we consider the D5 probe to be non-BPS (the tension is not equal to the charge). Our probe does feel an attractive force and falls along the radial coordinate. If the coefficient $\beta = 0$, we are working in a situation for which it is consistent to have no gauge field on the brane, and no need for the F1 strings. This can occur in situations for which charge is not induced on the Ω_p . In such cases the Dp-branes fall following a massive geodesic.

To continue, we study the dynamics of a composite point particle with generic action (3.9).

C. Dynamics of the infalling baryon vertex

As discussed above, we are interested in the point particle Lagrangian (describing the dynamics of the composite baryon vertex)

$$L = \alpha \sqrt{\frac{r^2}{l^2} \left(1 - \frac{l^4 \dot{r}^2}{r^4} \right)} + \beta (r_{\text{UV}} - r(t)). \quad (3.11)$$

The Euler-Lagrange equation of motion, canonical momentum, and Hamiltonian are

$$\frac{d}{dt} \left[\frac{\alpha l^3 \dot{r}}{r \sqrt{r^4 - l^4 \dot{r}^2}} \right] = \beta - \frac{\alpha}{l} \times \frac{(r^4 + l^4 \dot{r}^2)}{r^2 \sqrt{r^4 - l^4 \dot{r}^2}}, \quad (3.12)$$

$$P_r = \frac{\partial L}{\partial \dot{r}} = -\frac{\alpha l^3}{r} \frac{\dot{r}}{\sqrt{r^4 - l^4 \dot{r}^2}}, \quad (3.13)$$

$$H = P_r \dot{r} - L = -\frac{\alpha r^3}{l\sqrt{r^4 - l^4 \dot{r}^2}} - \beta[r_{\text{UV}} - r(t)]. \quad (3.14)$$

Using Eq. (3.14), we can invert the conserved Hamiltonian to obtain a first-order equation that solves the Euler-Lagrange equation (3.12). Integrating this equation gives

$$\dot{r} = -\frac{r^2 \sqrt{l^2 [H + \beta(r_{\text{UV}} - r)]^2 - \alpha^2 r^2}}{l^3 (H + \beta(r_{\text{UV}} - r))} = f(r), \quad (3.15)$$

$$\int \frac{dr}{f(r)} = t \Rightarrow r(t) = -\frac{l^2 (H + \beta r_{\text{UV}})}{\sqrt{(H + \beta r_{\text{UV}})^2 t^2 + \alpha^2 l^2 - \beta l^2}}. \quad (3.16)$$

We chose the negative sign for the velocity $\dot{r}$ to reflect that $r(t)$ decreases with time t and also the integration constant was chosen to be $t_0 = 0$.

Our solution in Eq. (3.16) describes the baryon vertex falling from r_{UV} (a UV cutoff) with zero initial velocity, in agreement with the proposal of [16,22,23]. First, we note that when evaluated at $t = 0$, Eq. (3.14) implies that the Hamiltonian has a value $H = -\frac{\alpha}{l} r_{\text{UV}}$. We then evaluate Eqs. (3.15) and (3.16) at $t = 0$, obtaining that the initial velocity and initial position are as required. Expanding for $t \simeq 0$, one finds

$$r(t) = r_{\text{UV}} - \frac{r_{\text{UV}}^3}{2\alpha l^4} (\alpha - \beta l) t^2 + \dots \quad (3.17)$$

which clearly indicates that for $\alpha = \beta l$, the brane does not fall. In other words, the tension of F1 string balances the gravitational pull on the D5-brane [the Bogomol'nyi Prasad Sommerfeld (BPS) situation]. On the other hand, for $\alpha > \beta l$, the non-BPS-D5 brane starts falling from r_{UV} towards the interior ($r(t) < r_{\text{UV}}$) of the bulk AdS_5 .

Following [16], we propose that there is a special radial coordinate ρ , such that, when all other coordinates are constant, satisfies $ds^2 = dy^2$. This special coordinate leads to the definition of *proper momentum* that is needed to calculate the complexity. In some detail,

$$ds^2 = \frac{r^2}{l^2} (-dt^2 + d\vec{x}_3^2) + \frac{l^2 dr^2}{r^2} + l^2 d\Omega_5^2,$$

$$d\Omega_5 = d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\Omega_3, \quad 4d\Omega_3 = \omega_1^2 + \omega_2^2 + \omega_3^2, \quad l^4 = g_s N_c \alpha'^2,$$

$$F_5 = \frac{4}{l} (1 + *) \text{Vol}_{S^5}, \quad C_4 = \frac{r^4}{l^4} dt \wedge dx_1 \wedge dx_2 \wedge dx_3 - \frac{l^4}{8} \cos^4 \theta d\phi \wedge \omega_1 \wedge \omega_2 \wedge \omega_3,$$

$$\omega_1 = \cos \psi d\alpha + \sin \psi \sin \alpha d\beta, \quad \omega_2 = -\sin \psi d\alpha + \cos \psi \sin \alpha d\beta, \quad \omega_3 = d\psi + \cos \alpha d\beta. \quad (4.1)$$

We consider a (D3-brane-) giant graviton that falls in the radial direction $r(t)$ and rotates in $\theta(t)$, $\phi(t)$, which is a consistent configuration.

$$y = \pm l \log r. \quad (3.18)$$

The proposal of [16]—see also [29–31,40–42]—relates the complexity rate to the magnitude of the proper momentum. With the orientation-independent convention (1.3),

$$\dot{C}(t) = \kappa |P_y| = \kappa \left| \frac{\partial L}{\partial \dot{r}} \frac{dr}{dy} \right| = \kappa \frac{|H + \beta r_{\text{UV}}|}{l} t. \quad (3.19)$$

The rate is linear in time, as for an ordinary particle in Poincaré AdS , while the wrapped brane and the N attached F1 strings enter through the effective conserved energy. Thus this composite object provides evidence that, within the rigid-probe approximation, the leading quadratic growth is controlled by the collective radial motion. This is a result for the present class of AdS configurations, not a general theorem relating boundary locality to a universal Krylov law.

Two extensions would test rather than merely repeat the pattern found here. First, $\text{AdS}_3 \times S^3 \times CY_2$ connects more directly with the $\text{AdS}_3/\text{CFT}_2$ setting in which the momentum/complexity relation was originally derived and changes both the boundary dimension and the brane realization. The more elaborate $\text{AdS}_3 \times S^2 \times CY_2 \times I_\eta$ families of [43–47] additionally introduce nontrivial flux, dilaton, source, and quiver data; they can test which of these data enter the coefficient and subleading terms of the proper momentum. Second, $\text{AdS}_7 \times S^4$ replaces the D-brane/F1 construction by an M-theory M5/M2 realization and changes the dual theory to the six-dimensional (2, 0) theory. This would distinguish consequences of pointlikeness and AdS kinematics from features specific to Dirac-Born-Infeld (DBI) actions and fundamental strings.

IV. COMPLEXITY MEASURED BY A FALLING GIANT GRAVITON

We study another pointlike excitation (from the field theory point of view). We discuss the complexity calculated by a falling giant graviton [27]. This type of solution has been studied in various works (never applied to complexity); see for example [48,49]. We discuss the usual giant in $\text{AdS}_5 \times S^5$. We use the metric and 4-form potential

We calculate the induced metric, the Born-Infeld and the Wess-Zumino action for the giant, namely a D3-brane, that extends along $[t, S^3]$, with $r(t), \theta(t), \phi(t)$. We find

$$ds_{\text{ind}}^2 = -\left(\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2} - l^2 \dot{\theta}^2 - l^2 \sin^2 \theta \dot{\phi}^2\right) dt^2 + \frac{l^2}{4} \cos^2 \theta (\omega_1^2 + \omega_2^2 + \omega_3^2), \quad (4.2)$$

$$S_{BI} = \frac{\pi^2}{4} T_{D3} l^3 \int dt \cos^3 \theta \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2} - l^2 \dot{\theta}^2 - l^2 \sin^2 \theta \dot{\phi}^2}, \quad (4.3)$$

$$S_{WZ} = -\frac{\pi^2}{4} T_{D3} l^4 \int dt \cos^4 \theta \dot{\phi},$$

$$S_{BIWZ} = \nu \int dt \left[\cos^3 \theta \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2} - l^2 \dot{\theta}^2 - l^2 \sin^2 \theta \dot{\phi}^2} - \mu \cos^4 \theta \dot{\phi} \right],$$

$$\nu = \frac{\pi^2}{4} T_{D3} l^3, \mu = l. \quad (4.4)$$

For this dynamical system, we define

$$A = \frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2} - l^2 \dot{\theta}^2, \quad Q(\theta) = J + \mu \nu \cos^4 \theta. \quad (4.5)$$

The expression for the angular momentum $J = \frac{\partial L}{\partial \dot{\theta}}$ (where L is the above Lagrangian) leads to

$$Q(\theta) = -\frac{\nu l^2 \sin^2 \theta \cos^3 \theta \dot{\phi}}{\sqrt{A - l^2 \sin^2 \theta \dot{\phi}^2}}. \quad (4.6)$$

This fixes the physical branch that would be lost by simply squaring this relation. Solving it for $\dot{\phi}$ gives

$$\dot{\phi} = -\frac{Q(\theta)}{l \sin \theta} \sqrt{\frac{A}{Q^2(\theta) + \nu^2 l^2 \sin^2 \theta \cos^6 \theta}}. \quad (4.7)$$

Substitution into the partial Legendre transform yields

$$\begin{aligned} \tilde{R}_J &= -J\dot{\phi} + L = -\mathcal{M}_J(\theta) \sqrt{A} \\ &= \mathcal{M}_J(\theta) \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2} - l^2 \dot{\theta}^2} \end{aligned} \quad (4.8)$$

$$\mathcal{M}_J(\theta) = \frac{\sqrt{Q^2(\theta) + \nu^2 l^2 \sin^2 \theta \cos^6 \theta}}{l \sin \theta} > 0. \quad (4.9)$$

This is an involved dynamical system in the $r(t), \theta(t)$ space. We solve it below in generality. We present the solution in two steps. In the first step, we consider the possibility of setting $\theta = \theta_0$ (a constant). In the second step, we study the full $r(t) - \theta(t)$ system.

A. Constant- θ reduction

A constant configuration $\theta(t) = \theta_0$ is a consistent reduction only if $\mathcal{M}'_J(\theta_0) = 0$. Whether an interior stationary point exists depends on the charge and brane parameters and should not be inferred from the spurious squared branch removed above. Whenever such a stationary point exists, the reduced dynamics is

$$\tilde{R}_J = \mathcal{M}_J(\theta_0) \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2}}, \quad (4.10)$$

which is the action of a particle of positive effective mass $\mathcal{M}_J(\theta_0)$ in AdS₅. Its fixed-charge proper momentum therefore gives $C_J(t) - C_J(0) \propto t^2$. We shall not assume that this special branch exists for generic parameters.

Let us now study the complete system with $r(t)$ and $\theta(t)$.

B. The complete system $r(t), \theta(t)$

We use the positive reduced Routhian in Eq. (4.8). The momentum conjugate to θ , its equation of motion and the conserved reduced energy are

$$\begin{aligned} P_\theta &= \frac{\partial \tilde{R}_J}{\partial \dot{\theta}} = -\frac{l^2 \mathcal{M}_J(\theta) \dot{\theta}}{\sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2} - l^2 \dot{\theta}^2}}, \\ \frac{d}{dt} \left(\frac{\partial \tilde{R}_J}{\partial \dot{\theta}} \right) &= \frac{\partial \tilde{R}_J}{\partial \theta} \longrightarrow \dot{P}_\theta = \mathcal{M}'_J(\theta) \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2} - l^2 \dot{\theta}^2}, \\ \mathcal{E}_J &= \tilde{R}_J - P_\theta \dot{r} - P_\theta \dot{\theta} = \frac{\mathcal{M}_J(\theta) r^2}{l^2 \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2} - l^2 \dot{\theta}^2}} > 0. \end{aligned} \quad (4.11)$$

A second conserved quantity is

$$K = P_\theta^2 + l^2 \mathcal{M}_J^2(\theta), \quad \dot{K} = 0. \quad (4.12)$$

The two constants $(\mathcal{E}_J, K)$ give

$$\begin{aligned} \dot{r} &= \pm \frac{r^2}{\mathcal{E}_J l^4} \sqrt{\mathcal{E}_J^2 l^4 - K r^2}, \\ r(t) &= \frac{\mathcal{E}_J l^2}{\sqrt{K + \mathcal{E}_J^2 (t - t_0)^2}}, \end{aligned} \quad (4.13)$$

$$l\dot{\theta} = \pm \sqrt{1 - \frac{l^2 \mathcal{M}_J^2(\theta)}{K}} \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2}}, \quad (4.14)$$

$$\int_{\theta_0}^{\theta(t)} \frac{d\theta}{\sqrt{1 - l^2 \mathcal{M}_J^2(\theta)/K}} = \pm \arctan \left[\frac{\mathcal{E}_J}{\sqrt{K}} (t - t_0) \right]. \quad (4.15)$$

Together with the physical angular velocity in Eq. (4.7), these equations give the full solution, implicitly in $\theta(t)$, of the original system in Eq. (4.4). The radial coordinate retains the particlelike form, while the charge and the wrapped-brane dynamics enter through $\mathcal{M}_J(\theta)$ and K .

For the comparison with spread complexity we choose the reference time $t_0 = 0$ and impose rest in the complete reduced configuration space,

$$\begin{aligned} r(0) &= r_{UV}, & \dot{r}(0) &= 0, \\ \theta(0) &= \theta_0, & \dot{\theta}(0) &= 0. \end{aligned} \quad (4.16)$$

These conditions fix

$$K = l^2 \mathcal{M}_J^2(\theta_0), \quad \mathcal{E}_J = \frac{\mathcal{M}_J(\theta_0) r_{UV}}{l}. \quad (4.17)$$

They are the bulk counterpart of choosing the state at the reference time as the initial Krylov seed. Initial data with $\dot{\theta}(0) \neq 0$ describe a different bulk preparation and are not identified here with the standard Krylov construction about $t = 0$.

Let us take a small detour to write the units of all the quantities that appear in this section. We find

$$\begin{aligned} [r] &= [x_i] = [l] = \text{length}, & [\theta] &= [\phi] = [\omega_i] = 1, & [\text{Lagrangian and Routhian}] &= \frac{1}{\text{length}}, \\ [T_{D3}] &= \frac{1}{\text{length}^4}, & [\mu] &= \text{length}, & [\mathcal{M}_J(\theta)] &= [\nu] = [\mathcal{E}_J] = \frac{1}{\text{length}}, & [P_\theta] &= [K] = 1. \end{aligned} \quad (4.18)$$

With these assignments of units, all the equations in this section are dimensionally consistent.

Let us calculate the complexity associated with the falling giant graviton.

C. Complexity of the falling giant

The coordinate ϕ was eliminated at fixed J in constructing the Routhian and must not be restored as an independent direction in the proper-distance element. The reduced configuration space is parametrized by (r, θ) , with

$$\dot{y}_J^2 = \frac{l^2 \dot{r}^2}{r^2} + l^2 \dot{\theta}^2, \quad \tilde{R}_J = \mathcal{M}_J(\theta) \sqrt{\frac{r^2}{l^2} - \dot{y}_J^2}. \quad (4.19)$$

The fixed-charge proper momentum is therefore computed from the same Routhian that generates the dynamics,

$$P_y^{(J)} = \frac{\partial \tilde{R}_J}{\partial \dot{y}_J} = - \frac{\mathcal{M}_J(\theta) \dot{y}_J}{\sqrt{r^2/l^2 - \dot{y}_J^2}} = - \frac{\mathcal{E}_J l^2}{r^2} \dot{y}_J. \quad (4.20)$$

Note that *there is no separate $P_\phi \partial \phi / \partial \dot{y}_J$ term and no explicit $l^2 \sin^2 \theta \dot{\phi}^2$ in $\dot{y}_J^2$* . The Wess-Zumino coupling has not been lost: it shifts the fixed charge through $Q(\theta) = J + \mu \nu \cos^4 \theta$ inside $\mathcal{M}_J(\theta)$. This is precisely the compatibility required between the fixed-charge dynamics and the complexity prescription.

Using Eq. (1.3), we identify

$$\dot{\mathcal{C}}_J(t) = \kappa |P_y^{(J)}(t)|. \quad (4.21)$$

For the reduced rest data (4.16), the equations of motion give

$$\begin{aligned} \dot{r}(t) &= - \frac{r_{UV}^3}{l^4} t + \mathcal{O}(t^3), \\ \dot{\theta}(t) &= - \frac{r_{UV}^2}{l^4} \frac{\mathcal{M}_J'(\theta_0)}{\mathcal{M}_J(\theta_0)} t + \mathcal{O}(t^3), \\ |P_y^{(J)}(t)| &= \frac{\mathcal{M}_J(\theta_0) r_{UV}}{l^2} \sqrt{1 + \left[\frac{\mathcal{M}_J'(\theta_0)}{\mathcal{M}_J(\theta_0)} \right]^2} |t| + \mathcal{O}(|t|^3). \end{aligned} \quad (4.22)$$

Consequently $\dot{\mathcal{C}}_J(0) = 0$ and $\mathcal{C}_J(t) - \mathcal{C}_J(0) = \mathcal{O}(t^2)$. On a constant- θ branch the coefficient reduces to $\mathcal{E}_J/l$. Appendix C supplies the details of this expansion.

The fixed-charge late-time behavior can also be read analytically. Provided $\mathcal{M}_J(\theta(t))$ remains finite along the asymptotic trajectory, Eqs. (4.13) and (4.20) imply

$$\begin{aligned} r(t) &= \frac{l^2}{|t|} + \mathcal{O}(|t|^{-3}), & \dot{y}_J &= \frac{l}{|t|} + \mathcal{O}(|t|^{-3}), \\ |P_y^{(J)}| &= \frac{\mathcal{E}_J}{l} |t| + \mathcal{O}(|t|^{-1}). \end{aligned} \quad (4.23)$$

Thus the quadratic leading growth follows from the reduced proper momentum without reintroducing the cyclic motion. The analytic expressions above are the quantities relevant to a definite- J sector.

Let us now study the falling dynamics of an extended probe, proposed as a collective model for the complexity of a nonlocal excitation.

V. SPREAD COMPLEXITY FOR A NONLOCAL (EXTENDED) OPERATOR

The fixed-charge particle of Sec. II, the baryon vertex of Sec. III, and the giant graviton of Sec. IV share a linear late-time proper-momentum rate within the rigid Poincaré-AdS configurations analyzed here. This common behavior is controlled by collective radial infall, while charge and brane structure enter the coefficient and subleading terms. This behavior should not be read as a theorem that every boundary-local probe has $\mathcal{C} \sim t^2$ for large times. In fact, the examples mentioned share a highly symmetric, zero-temperature conformal background and a collective-coordinate truncation.

Instead, in this section, we explore a different aspect of the above proposal, namely we consider the spread complexity of an *nonlocal (extended) operator*. We choose a string that falls in AdS under gravity and is extended along one spatial direction. This is the holographic version of a line operator. We also *adopt* the proposal that the rate of change of the Krylov complexity $\dot{\mathcal{C}}(t)$ is proportional to the proper momentum [16,22,23].

To calculate its spread complexity, we consider an F1 string parametrized by the (σ, τ) coordinates that moves according to

$$t = \tau, \quad x = x(\sigma), \quad r = r(\tau). \quad (5.1)$$

Strictly speaking, one should begin with a generic embedding

$$t = t(\sigma, \tau), \quad r = r(\sigma, \tau), \quad x = x(\sigma, \tau).$$

Then, one studies the corresponding equations of motion, and shows that the embedding in Eq. (5.1) is a consistent truncation. On the embedding (5.1), the action of the massive probe reads

$$S = \int dt L; \quad L = -m \sqrt{\frac{r^4}{l^4} - \dot{r}^2}.$$

We defined $m = \frac{L_x}{2\pi\alpha'}$ and $\int d\sigma x'(\sigma) = L_x$. (5.2)

The equation of motion and conserved Hamiltonian are

$$-\frac{d}{dt} \left[\frac{\dot{r}}{\sqrt{\frac{r^4}{l^4} - \dot{r}^2}} \right] = \frac{2r^3}{l^4 \sqrt{\frac{r^4}{l^4} - \dot{r}^2}}, \quad (5.3)$$

$$H = m \frac{r^4}{l^4 \sqrt{\frac{r^4}{l^4} - \dot{r}^2}}, \quad \Rightarrow \dot{r}(t) = -\frac{r^2}{H l^4} \sqrt{H^2 l^4 - m^2 r^4}. \quad (5.4)$$

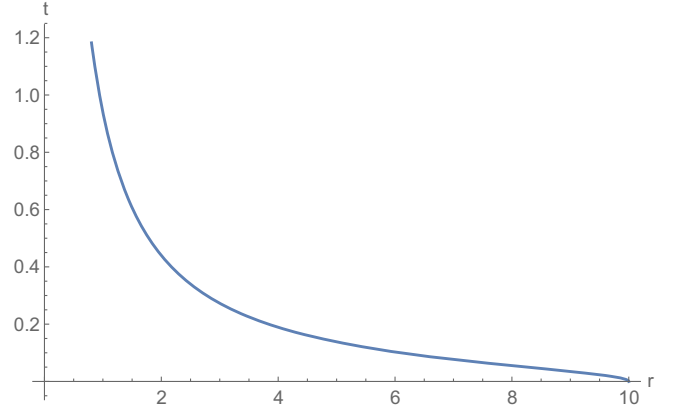


FIG. 1. We plot t vs r , where we set $l = 1$ and $r_{UV} = 10$.

Finding $\ddot{r}$ by derivative of $\dot{r}$ and writing everything in terms of $r(t)$ and $\dot{r}(t)$, the equation of motion in (5.3) is satisfied by the first-order equation in (5.4). Furthermore, we choose a string that starts falling from a position r_{UV} with zero initial velocity. This condition, together with Eq. (5.4), implies that

$$r_{UV}^4 = \frac{H^2 l^4}{m^2}. \quad (5.5)$$

Integrating Eq. (5.4), we find

$$t(r) = \frac{l^2}{r} \left[{}_2F_1 \left(-\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{r^4}{r_{UV}^4} \right) - \frac{r}{r_{UV}} \sqrt{\pi} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right]. \quad (5.6)$$

The integration constant was chosen to satisfy $t(r_{UV}) = 0$ or conversely $r(0) = r_{UV}$.

In Fig. 1 we give the plot of $t(r)$. This shows that the string takes infinite time to reach $r = 0$.

Expanding the expression (5.6) close to $r = r_{UV}$ gives

$$\frac{r_{UV}}{l^2} t \approx \sqrt{1 - \frac{r}{r_{UV}}} + \mathcal{O} \left(\left(1 - \frac{r}{r_{UV}} \right)^{\frac{3}{2}} \right). \quad (5.7)$$

Inverting this yields for short times

$$r(t) \approx r_{UV} \left[1 - \frac{r_{UV}^2}{l^4} t^2 \right], \quad \text{for } t \rightarrow 0. \quad (5.8)$$

This satisfies the conditions that the string starts at r_{UV} for $t = 0$ and has a zero initial velocity (along the radial direction), $\dot{r}|_{t=0}$.

On the other hand, expanding Eq. (5.6) near $\frac{r}{r_{UV}} \sim 0^+$ gives

$$\frac{r_{\text{UV}}}{l^2}(t+t_0) \approx \frac{r_{\text{UV}}}{r} - \frac{1}{6} \left(\frac{r}{r_{\text{UV}}} \right)^3 - \frac{3}{56} \left(\frac{r}{r_{\text{UV}}} \right)^7 + \dots,$$

$$\text{where } t_0 = \frac{l^2}{r_{\text{UV}}} \frac{\sqrt{\pi} \Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}. \quad (5.9)$$

This is inverted to obtain the result that for large times $r(t)$ behaves as

$$r(t) = \frac{l^2}{(t+t_0)} - \frac{l^{10}}{6r_{\text{UV}}^4(t+t_0)^5} + \mathcal{O}\left(\frac{1}{(t+t_0)^9}\right); \quad t \gg t_0. \quad (5.10)$$

A. The holographic Krylov complexity

To calculate the complexity, we adopt the working hypothesis that this is proportional to the proper momentum of the falling string, extending the proposal of [16,22,23]. We change to the proper coordinate as proposed in [16,29–31],

$$ds^2 = dy^2 = \frac{l^2}{r^2} dr^2 \rightarrow y = \int \frac{l}{r} dr = l \log r, \quad (5.11)$$

$$\dot{C} = \kappa |P_y| = \kappa \left| P_r \frac{dr}{dy} \right| = \kappa \frac{ml|\dot{r}|}{r\sqrt{1-l^4\dot{r}^2/r^4}}. \quad (5.12)$$

For *short times*, the rate of change of the complexity is linear in time, as is conventionally observed in conformal field theories [16]. Indeed, using (5.8), the early time rate of change of the complexity is

$$\dot{C}(t) \approx \kappa \left[\frac{2mr_{\text{UV}}^2}{l^3} t + \frac{6mr_{\text{UV}}^4}{l^7} t^3 + \mathcal{O}(t^5) \right], \quad t \rightarrow 0. \quad (5.13)$$

On the other hand, for *large times*, we use the expansion in Eq. (5.10) and find

$$\dot{C}(t) \approx \kappa \left[\frac{mr_{\text{UV}}^2}{l^3} (t+t_0) - \frac{ml^5}{3r_{\text{UV}}^2(t+t_0)^3} + \mathcal{O}(t^{-4}) \right], \quad t \rightarrow \infty. \quad (5.14)$$

Hence, for large times, the rate of change of the complexity is also linear at leading order. In summary, we find that for both short and large times, the rate of change of the complexity is to leading order, linear in time. This is the same behavior as observed for pointlike, uncharged and structureless probes of mass m (dual to local operators of conformal dimension m) [16,22,23]

$$\dot{C}_{\text{particle}}(t) = \kappa \frac{|H|}{l} t = \kappa \frac{mr_{\text{UV}}}{l^2} t, \quad t > 0. \quad (5.15)$$

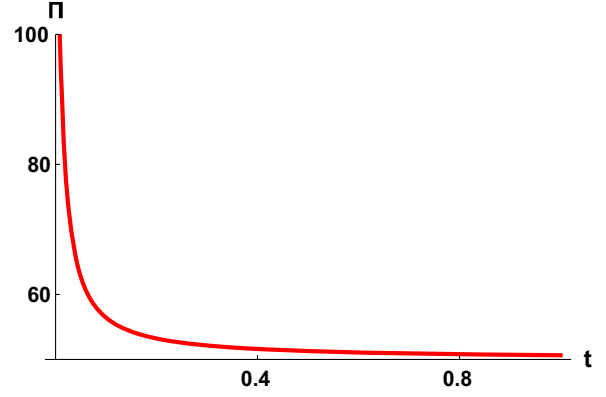


FIG. 2. Π vs t plot. We set $m = 0.001$, $l = 1$, $r_{\text{UV}} = 50$.

What is new and interesting for the nonlocal operators (dual to extended probes in gravity) are the subleading terms and the intermediate regimes; details that are absent in the case of a massive pointlike probe of Eq. (5.15). In Fig. 2, we plot the quotient

$$\begin{aligned} \Pi &= \frac{\text{rate of change for complexity of string}}{\text{rate of change for complexity of particle}} \\ &= \frac{\dot{C}_{\text{string}}}{\dot{C}_{\text{particle}}} = \frac{m\sqrt{r_{\text{UV}}^4 - r(t)^4} \frac{1}{t}}{Hr(t)}. \end{aligned} \quad (5.16)$$

Consider a particle and a string that fall from the same position r_{UV} and have the same mass m . Using the early-time expansion for $\dot{C}_{\text{string}}$ in (5.13), one finds

$$\Pi|_{t \sim 0} = \frac{2r_{\text{UV}}}{l} \quad (5.17)$$

which is the (large) constant depicted in Fig. 2. On the other hand, using the late-time expansion in (5.14), one finds

$$\Pi|_{t \gg 1} = \frac{r_{\text{UV}}}{l}, \quad (5.18)$$

also reflected in Fig. 2. To refine the comparison, we define the quotient

$$\begin{aligned} \Psi &= \frac{\text{second derivative of complexity of string}}{\text{second derivative of complexity of particle}} \\ &= \frac{\ddot{C}_{\text{string}}}{\ddot{C}_{\text{particle}}} = \frac{r_{\text{UV}}^4 + r^4(t)}{lr_{\text{UV}}^3}. \end{aligned} \quad (5.19)$$

Close to the UV cutoff in the geometry this yields $\Psi = \frac{2r_{\text{UV}}}{l}$, which is same as in (5.17).

On the other hand, at late times, it is easy to see that $\Psi = \frac{r_{\text{UV}}}{l}$; see Fig. 3.

With this, we hope to have made the point that extended operators display a different complexity than pointlike

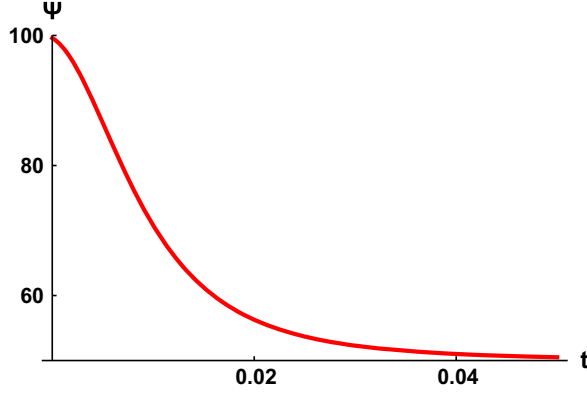


FIG. 3. Ψ vs t plot. We set $m = 0.001$, $l = 1$, $r_{UV} = 50$.

operators do. These differences are apparent in the short- and long-times subleading behavior. It would be interesting to study the effect of string fluctuations on the complexity. Of course, having a purely field theoretical calculation showing similar behavior would be very valuable and we leave this for future investigations. It might be possible to study our string in the context of [50]. Let us present some conclusions and future directions.

VI. CONCLUSIONS AND FUTURE DIRECTIONS

In this work we tested the proper-momentum proposal for holographic spread complexity using probes with conserved charge, composite brane structure, and genuine spatial extension. The analysis exposes an important distinction. If an internal coordinate is cyclic, its Noether charge must be fixed before the proper momentum is defined. The corresponding Routhian retains the physical charge dependence while removing the spurious constant initial rate produced by counting the conjugate velocity as a Krylov-spreading direction. Both the R -charged particle and the giant graviton then satisfy the universal quadratic short-time onset of unitary spread complexity. The unreduced momentum remains a useful diagnostic of the complete bulk trajectory, but it is not the complexity rate in a sector of definite charge.

The baryon vertex and the spatially stretched fundamental string probe different kinds of structure. In the former, wrapped-brane dynamics and attached strings modify the effective energy of an object that remains pointlike to the boundary observer. In the latter, the nonlocal extension changes the radial effective action and produces distinct subleading and intermediate-time behavior. Across the rigid Poincaré-AdS configurations studied here, the fixed-charge proper momentum is linear at late times. This common scaling is evidence for a robust pattern in the tested class, but the present examples do not establish a universality theorem. In particular, no general identification can be made between the gauge-group rank and the number of sites of a Krylov chain: the Krylov dimension depends

on the seed, the symmetry sector, degeneracies, and the effective Hilbert space.

The most direct next step is a boundary calculation. One should project a concrete charged state or operator onto a definite symmetry sector, determine its survival amplitude and sectoral Lanczos coefficients, and compare the resulting spread complexity with the bulk Routhian [33,34]. More precisely, the open problem is to identify a controlled large- N , fixed-charge regime of the boundary theory in which this comparison can be carried out. It would also be valuable to include worldsheet and worldvolume fluctuations, thereby testing the limits of the rigid collective-coordinate approximation.

Finally, backgrounds that vary the spacetime dimension, flux realization, and infrared geometry provide targeted tests of the interpretation. $\text{AdS}_3 \times S^3 \times \text{CY}_2$ is closer to the original $\text{AdS}_3/\text{CFT}_2$ original derivation. $\text{AdS}_7 \times S^4$ replaces DBI/F1 physics by an M5/M2 construction. Confining, thermal, and black-hole backgrounds can test whether an infrared scale or horizon changes the separation between collective radial growth and probe-dependent data. The families in [51–60] offer concrete laboratories. These extensions will determine which features follow from AdS kinematics and which retain detailed information about charge, compositeness, or spatial extension.

ACKNOWLEDGMENTS

For discussions, for comments on the manuscript, and for sharing their ideas with us, we wish to thank: Pawel Caputa, Dimitrios Chatzis, Ali Fatemiabhari, Madison Hammond, Yolanda Lozano, Alfonso Ramallo, Diego Rodriguez-Gomez, and Ricardo Santamaria. C. N. is supported by STFC's Grants No. ST/Y509644-1, No. ST/X000648/1, and No. ST/T000813/1. D. R. would like to acknowledge the Mathematical Research Impact Centric Support (MATRICS) Grant No. MTR/2023/000005 received from ANRF, India. The work of H. N. is supported in part by CNPq Grant No. 304583/2023-5 and FAPESP Grant No. 2019/21281-4. H. N. would also like to thank the ICTP-SAIFR for their support through FAPESP Grant No. 2021/14335-0, and to CEA-Saclay for their hospitality during a part of this project.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: THE BARYONIC VERTEX IN CASE OF $\text{AdS}_3 \times S^3 \times \text{CY}_2$

We study the complexity of the D1–D5 two-dimensional SCFT measured by a composite operator of the baryon-vertex kind. We probe the spacetime with a D7-brane that extends along $[t, S^3, \text{CY}_2]$ and falls along a geodesic

parametrized by $r(t)$. The metric of the spacetime is given by

$$ds^2 = \frac{r^2}{l^2}(-dt^2 + d\vec{x}^2) + \frac{l^2 dr^2}{r^2} + \mu^2 d\Omega_3^2 + \nu^2 ds_{CY_2}^2. \quad (A1)$$

The parameters (μ, ν) indicate the radii of S^3 and the Calabi-Yau manifold. These are related to the numbers (N_1, N_5) of D1- and D5-branes via quantisation of fluxes. A pure-gauge one-form is switched on the probe D7-brane. We consider the electric F_3 in this configuration or the magnetic F_7 that indicates the presence of D1-branes. The induced metric on the D7 probe is

$$ds_{\text{ind}, D7}^2 = \frac{r^2}{l^2} \left(-1 + \frac{l^4 \dot{r}^2}{r^4} \right) dt^2 + \mu^2 d\Omega_3^2 + \nu^2 ds_{CY_2}^2. \quad (A2)$$

The Born-Infeld part of the action of this D7-brane is

$$S_{BI, D7} = T_{D7} \mu^3 \nu^4 \int dt \sqrt{\frac{r^2}{l^2} \left(1 - \frac{l^4 \dot{r}^2}{r^4} \right)}. \quad (A3)$$

The Wess-Zumino term on the D7 consists only of the contribution from the C_6 background field (with $F_7 = dC_6$). In the absence of other background fluxes, and switching on a gauge field on the D7 probe, we find that for this configuration

$$S_{WZ, D7} = -T_{D7} \int_{t \times S^3 \times CY_2} C_6 \wedge f_2 = T_{F1} N_{D1} \int dt a_t, \quad (A4)$$

$$N_{D1} = \int_{S^3 \times CY_2} F_7.$$

The Wess-Zumino term is canceled by the presence of N_{D1} F1 strings that extend from the D7 probe and a D-brane fixed at r_{UV} . As above, these strings are modeled here as being “rigid” having no kinetic term, but contributing to the action of the composite object with a potential term. As is characteristic of strings, we model this contribution as being proportional to the length of the string. This gives

$$S_{F1} = N_{D1} T_{F1} \int dt [r_{UV} - r(t)]. \quad (A5)$$

As a result the full Lagrangian for the system is of the same form as in Eq. (3.9), with the coefficients $\alpha = T_{D7} \mu^3 \nu^4 \text{Vol}_{S^3 \times CY_2}$ and $\beta = T_{F1} N_{D1}$. This dynamic is studied in Sec. III C. One can make a Routhian analysis, like the one described in Appendix B below.

APPENDIX B: BARYON VERTEX USING THE ROUTHIAN

Consider the D5-brane probe on $\text{AdS}_5 \times S^5$ in Sec. III B. The metric and Ramond 4-form potential are

$$ds^2 = \frac{r^2}{l^2}(-dt^2 + d\vec{x}_3^2) + \frac{l^2 dr^2}{r^2} + l^2 d\theta^2 + l^2 \sin^2 \theta d\Omega_4,$$

$$C_4 = 4l^4 \left(\frac{3\theta}{8} - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32} \right) \text{Vol}_{\Omega_4} = C_4(\theta) \text{Vol}_{\Omega_4},$$

$$F_4 = dC_4 = 4l^4 \sin^4 \theta d\theta \wedge \text{Vol}_{\Omega_4}. \quad (B1)$$

We probe this spacetime with a D5-brane extended along the five sphere and time. We switch on a gauge field on the brane $a_1 = a_t(\theta)dt$, with nonzero curvature $f_2 = a'_t(\theta)d\theta \wedge dt = f_{\theta t}d\theta \wedge dt$. As in the main body of the paper, we consider a brane embedding of the type

$$[t, \Omega_5], \quad r(t). \quad (B2)$$

The Born-Infeld and Wess-Zumino contributions to the probe brane action are

$$S_{BI} = l^5 T_{D5} \int dt d\theta d\Omega_4 \sin^4 \theta \sqrt{\frac{r^2}{l^2} \left(1 - \frac{l^4 \dot{r}^2}{r^4} \right) - f_{\theta t}^2},$$

$$= l^5 T_{D5} \text{Vol}_{\Omega_4} \int dt d\theta \sin^4 \theta \sqrt{\frac{r^2}{l^2} \left(1 - \frac{l^4 \dot{r}^2}{r^4} \right) - f_{\theta t}^2}, \quad (B3)$$

$$S_{WZ} = -T_{D5} \int_{\Omega_4, \theta, t} C_4 \wedge f_2 = -T_{D5} \text{Vol}_{\Omega_4} \int dt d\theta C_4(\theta) f_{\theta t},$$

$$S_{BIWZ} = T_{D5} \text{Vol}_{\Omega_4} \int dt d\theta \left[l^5 \sin^4 \theta \sqrt{\frac{r^2}{l^2} \left(1 - \frac{l^4 \dot{r}^2}{r^4} \right) - f_{\theta t}^2} - C_4(\theta) f_{\theta t} \right]. \quad (B4)$$

We denote $\mu = T_{D5} \text{Vol}_{\Omega_4}$ and define the conjugate momentum

$$\Pi_\theta = \frac{\delta S_{\text{BIWZ}}}{\delta f_{\theta t}} = \mu \left(-\frac{l^5 f_{\theta t} \sin^4 \theta}{\sqrt{\frac{r^2}{l^2} (1 - \frac{l^4 \dot{r}^2}{r^4}) - f_{\theta t}^2}} - C_4(\theta) \right). \quad (\text{B5})$$

We can find $f_{\theta t}$ in terms of Π_θ ,

$$f_{\theta t} = \pm (\Pi_\theta + \mu C_4) \sqrt{\frac{\frac{r^2}{l^2} (1 - \frac{l^4 \dot{r}^2}{r^4})}{(\Pi_\theta + \mu C_4)^2 + \mu^2 l^{10} \sin^8 \theta}}. \quad (\text{B6})$$

Then, we define the Routhian, a Legendre transform of the Lagrangian along the ‘‘cyclic coordinate’’ $f_{\theta t}$. We have

$$\begin{aligned} R &= -L + \Pi_\theta f_{\theta t}, \\ R &= \int d\theta \sqrt{(\Pi_\theta + \mu C_4(\theta))^2 + \mu^2 l^{10} \sin^8 \theta} \\ &\quad \times \int dt \sqrt{\frac{r^2}{l^2} \left(1 - \frac{l^4 \dot{r}^2}{r^4} \right)}, \\ R &= \alpha \times \int dt \sqrt{\frac{r^2}{l^2} \left(1 - \frac{l^4 \dot{r}^2}{r^4} \right)}. \end{aligned} \quad (\text{B7})$$

Using the Routhian, we find the same dynamics that we study in Eq. (3.9).

To obtain the contribution coming from Gauss’ law, we observe that the momentum Π_θ is piecewise constant away from the sources. In other words Π_θ plays the role of the electric displacement $\vec{D}$ -field. That is,

$$\vec{\nabla} \cdot \vec{D} = \partial_\theta \Pi_\theta = \sum_i q_i \delta(\theta - \theta_i). \quad (\text{B8})$$

Using this equation, we conclude that Gauss’ law on the compact θ -space requires F1 strings that carry away the induced flux. We must therefore add the F1 action in Eq. (3.8). This gives the same dynamical contribution as the treatment in the main text.

APPENDIX C: FIXED-CHARGE EXPANSION FOR THE FALLING GIANT GRAVITON

This appendix gives the short-time expansion appropriate to the fixed- J prescription of Secs. IV B and IV C. The cyclic angle ϕ has already been eliminated, and the initial data are at rest in the reduced (r, θ) configuration space, as in Eq. (4.16). Consequently

$$K = l^2 \mathcal{M}_J^2(\theta_0), \quad \mathcal{E}_J = \frac{\mathcal{M}_J(\theta_0) r_{\text{UV}}}{l}. \quad (\text{C1})$$

Expanding the radial solution (4.13) gives

$$\begin{aligned} r(t) &= r_{\text{UV}} \left[1 - \frac{r_{\text{UV}}^2}{2l^4} t^2 + \frac{3r_{\text{UV}}^4}{8l^8} t^4 + \mathcal{O}(t^6) \right], \\ \dot{r}(t) &= -\frac{r_{\text{UV}}^3}{l^4} t + \mathcal{O}(t^3). \end{aligned} \quad (\text{C2})$$

The θ equation follows either from the reduced Euler-Lagrange equation or by expanding Eq. (4.15) about the endpoint at which $K = l^2 \mathcal{M}_J^2(\theta_0)$. One finds

$$\begin{aligned} \theta(t) &= \theta_0 - \frac{r_{\text{UV}}^2}{2l^4} \frac{\mathcal{M}_J'(\theta_0)}{\mathcal{M}_J(\theta_0)} t^2 + \mathcal{O}(t^4), \\ \dot{\theta}(t) &= -\frac{r_{\text{UV}}^2}{l^4} \frac{\mathcal{M}_J'(\theta_0)}{\mathcal{M}_J(\theta_0)} t + \mathcal{O}(t^3). \end{aligned} \quad (\text{C3})$$

It follows that the reduced proper velocity is

$$|\dot{y}_J(t)| = \frac{r_{\text{UV}}^2}{l^3} \sqrt{1 + \left[\frac{\mathcal{M}_J'(\theta_0)}{\mathcal{M}_J(\theta_0)} \right]^2} |t| + \mathcal{O}(t^3). \quad (\text{C4})$$

Substitution in Eq. (4.20) yields

$$|P_y^{(J)}(t)| = \frac{\mathcal{M}_J(\theta_0) r_{\text{UV}}}{l^2} \sqrt{1 + \left[\frac{\mathcal{M}_J'(\theta_0)}{\mathcal{M}_J(\theta_0)} \right]^2} |t| + \mathcal{O}(t^2). \quad (\text{C5})$$

Therefore $\dot{C}_J(0) = 0$ and the fixed-charge complexity starts quadratically, in agreement with Eq. (1.2). If $\mathcal{M}_J'(\theta_0) = 0$, the result reduces to the constant- θ branch $|P_y^{(J)}| = (\mathcal{E}_J/l)|t|$.

APPENDIX D: FIXED-CHARGE ROUTHIAN TREATMENT FOR CONSERVED CHARGES

When a probe moves both radially in AdS and along an internal direction, one may define an unreduced generalized momentum that includes every semiclassical velocity. A cyclic internal coordinate, however, is conjugate to a conserved charge and must not be treated as an independent spreading direction inside a fixed-charge sector. The appropriate reduced functional is the Routhian, obtained by Legendre transforming with respect to the cyclic velocity before defining the proper momentum. This appendix collects details used in Secs. II and IV, following the fixed-charge logic of [21].

Let us start with a generic one-dimensional radial Routhian of the form

$$\mathfrak{R}(r, \dot{r}; q) = \mathfrak{g}(r; q) \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2}}, \quad (\text{D1})$$

where q denotes one or several fixed charges. The variable associated with the radial proper distance is defined from

$$\dot{y} = \frac{l\dot{r}}{r}, \quad \mathfrak{R} = \mathfrak{g}(r; q) \sqrt{\frac{r^2}{l^2} - \dot{y}^2}. \quad (\text{D2})$$

The fixed-charge proper momentum is therefore

$$P_y^{(q)} = \frac{\partial \mathfrak{R}}{\partial \dot{y}} = -\frac{\mathfrak{g}(r; q) \dot{y}}{\sqrt{\frac{r^2}{l^2} - \dot{y}^2}}. \quad (\text{D3})$$

This is the quantity proposed to represent, up to the positive normalization κ , the complexity rate in a fixed-charge sector. A velocity along the cyclic variable no longer

appears in $\dot{y}$; its effect survives through the fixed parameter q in $\mathfrak{g}(r; q)$ and in the conserved energy. The function $\mathfrak{g}$ may be constant or may depend on the remaining reduced coordinates.

For the point particle of Sec. II,

$$L_p = -m \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2} - l^2 \dot{\psi}^2}, \quad J = \frac{\partial L_p}{\partial \dot{\psi}}. \quad (\text{D4})$$

Solving the first relation for $\dot{\psi}(r, \dot{r}, J)$ and Legendre transforming gives

$$R_J = J\dot{\psi} - L_p = \frac{\sqrt{J^2 + m^2 l^2}}{l} \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2}} \equiv \hat{m} \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2}}, \quad \hat{m}^2 = \frac{J^2 + m^2 l^2}{l^2}. \quad (\text{D5})$$

The charge J has become part of the “effective mass” $\hat{m}$. The Routhian-derived Hamiltonian is

$$H_J = P_r \dot{r} - R_J = -\frac{\hat{m} r^2 / l^2}{\sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2}}} \quad (\text{D6})$$

up to the same sign convention used in the main text. Hence

$$\dot{r} = \pm \frac{r^2}{H_J l^4} \sqrt{H_J^2 l^4 - \hat{m}^2 l^2 r^2}, \quad r(t) = \frac{|H_J| l^2}{\sqrt{H_J^2 l^2 + \hat{m}^2 l^2}}. \quad (\text{D7})$$

Using $\dot{y} = l\dot{r}/r$ in Eq. (D3), one obtains

$$P_y^{(J)}(t) = -\frac{H_J}{l} t, \quad \dot{C}_J(t) \propto |P_y^{(J)}(t)| = \frac{|H_J|}{l} |t|. \quad (\text{D8})$$

In particular $P_y^{(J)}(0) = 0$. This is the desired fixed-charge short-time behavior: the complexity itself starts as $C_J(t) - C_J(0) \sim t^2$ [20]. The constant term in Eq. (2.10) is therefore not a fixed- J Krylov growth rate; it is the contribution of the nonzero internal velocity $\dot{\psi}(0)$ in the unreduced configuration space.

We now apply the same logic to the giant graviton of Sec. IV. Its Born-Infeld plus Wess-Zumino Lagrangian is

$$L_{\text{giant}} = \nu \left[\cos^3 \theta \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2} - l^2 \dot{\theta}^2 - l^2 \sin^2 \theta \dot{\phi}^2} - \mu \cos^4 \theta \dot{\phi} \right]. \quad (\text{D9})$$

The unsquared relation $J = \partial L_{\text{giant}} / \partial \dot{\phi}$ fixes the physical solution (4.7). With $Q(\theta) = J + \mu \nu \cos^4 \theta$, the partial Legendre transform is

$$R_J = J\dot{\phi} - L_{\text{giant}} = -\mathcal{M}_J(\theta) \sqrt{\frac{r^2}{l^2} - \frac{l^2 \dot{r}^2}{r^2} - l^2 \dot{\theta}^2},$$

$$\mathcal{M}_J(\theta) = \frac{\sqrt{Q^2(\theta) + \nu^2 l^2 \sin^2 \theta \cos^6 \theta}}{l \sin \theta}. \quad (\text{D10})$$

Using $\tilde{R}_J = -R_J$ leaves the reduced equations unchanged and gives a positive effective mass. The Wess-Zumino coupling is encoded in the shifted charge $Q(\theta)$.

The reduced proper coordinate and momentum are

$$\dot{y}_J^2 = \frac{l^2 \dot{r}^2}{r^2} + l^2 \dot{\theta}^2, \quad \tilde{R}_J = \mathcal{M}_J(\theta) \sqrt{\frac{r^2}{l^2} - \dot{y}_J^2}, \quad (\text{D11})$$

$$P_y^{(J)} = -\frac{\mathcal{M}_J(\theta) \dot{y}_J}{\sqrt{r^2/l^2 - \dot{y}_J^2}} = -\frac{\mathcal{E}_J l^2}{r^2} \dot{y}_J. \quad (\text{D12})$$

No ϕ velocity is reintroduced. For the rest data (4.16), Appendix C proves that $P_y^{(J)}(0) = 0$ and $\dot{C}_J \sim t$. A trajectory with a prescribed $\dot{\theta}(0) \neq 0$ is a different bulk preparation and lies outside the standard Krylov identification about that reference time; it is not used to evade the universal condition $\dot{C}_K(0) = 0$.

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