

Fermion scattering in a Bose-Einstein condensate

César E. Echevarría^{*,†}, José F. Nieves[†], and Francisco Orbe[‡]

*Laboratory of Theoretical Physics, Department of Physics, University of Puerto Rico,
Río Piedras, Puerto Rico 00936*

Sarira Sahu[§]

*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,
Circuito Exterior, C. U., A. Postal 70-543, 04510 México DF, México*



(Received 8 April 2026; accepted 27 August 2026; published 7 October 2026)

In a recent work, we considered the propagation of fermions in the background of a scalar Bose-Einstein (BE) condensate. Using some illustrative Yukawa-type coupling models between the fermions and the scalar fields, we determined the dispersion relations of the fermions and the scalar modes in various models. To complement that work, here we consider the corresponding fermion spinors and propagators, which are required for the calculation of rates of processes involving the fermions, as well as the thermal, and/or higher order, corrections to such rates. We obtain and present here concise formulas which are useful for those applications. As an application and for illustrative purposes we consider the scattering rate for a generic fermion (which we denote by χ) with the fermion modes (f) in the BE condensate, using commonly used models for the $f\chi$ interactions. Due to the fact that the fermion modes dispersion relations are helicity dependent, the kinematics have some unique and nonstandard features. For example, at a particular value of the momentum one of the fermion modes has zero group velocity, which leads to effects of the type of the Van Hove singularity in the density of states of some condensed matter systems. The results of the framework presented here, besides their merit in their own right, can be useful in specific contexts and applications, such as cosmic-ray electron cooling through dark matter-electron scattering, and similar ones involving neutrino and/or electron propagation in a dark matter background.

DOI: [10.1103/3lmw-djxy](https://doi.org/10.1103/3lmw-djxy)

I. INTRODUCTION AND SUMMARY

It is well known that the dispersion relations of neutrinos that propagate in a medium are modified in important ways due to the interactions of the neutrinos with the particles in the background. Besides the effects due to the standard weak interactions when the neutrinos propagate through normal matter, many works have considered the effects due to nonstandard neutrino interactions and/or backgrounds in a variety of physical contexts. A significant effort along those lines involve extensions of the standard electroweak model in which the neutrinos interact with a scalar (ϕ) and fermion (f) by a Yukawa-type interaction of the form

$\bar{f}_R \nu_L \phi$ [1,2]. The effects when the neutrinos propagate in a medium that contains a background of those particles have been considered in a variety of physical contexts, including collective oscillations in supernova [3–5], the hot plasma of the early universe [6], cosmological observations such as cosmic microwave background [7], big bang nucleosynthesis data [8,9], and in particular dark matter-neutrino interactions [10–14].

Motivated by these developments, we have considered in previous works the calculations of various effects, including the damping and decoherence effects, on the neutrino propagation in such scalar and fermion backgrounds [2]. Of particular relevance to the present work, in Ref. [15] we presented a method to determine the effective potential and dispersion relation of a fermion in the background of the BE condensate in a general way and not tied to any specific application. The idea that the dark matter (DM) can be self-interacting is intriguing, and a DM background of scalar particles is a candidate for such environments [16–19]. In that context, our interest is the application to the case of a neutrino propagating in such backgrounds.

Besides the modification of the dispersion relation, the interaction also modifies the wave function associated with

*Contact author: cesar.echevarria@upr.edu

†Contact author: nieves@ltp.uprrp.edu

‡Contact author: francisco.orbe@upr.edu

§Contact author: sarira@nucleares.unam.mx

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

the propagating mode. The determination of the wave function is required to calculate in a consistent way the rates of processes involving the propagating particle. For example, in the case of fermions, the correct spinor to use is the solution to the field equation corresponding to the dispersion relation of the propagating mode [20].

In this work we complement Ref. [15], by considering the determination of the fermion spinors in the models considered there. To be specific, here we obtain and present concise formulas for the spinor projection matrices which are useful for calculations of process rates involving a fermion propagating in a BE condensate. As an application and for illustrative purposes we specifically consider the rate for the scattering process of a generic fermion (which we denote by χ) with the fermions in the BE condensate. The results presented here, besides their merit in their own right and serving as a guide for similar models, can be useful in specific contexts and applications that involve neutrino, electron, or nucleon interactions with dark matter particles [21–24].

For guidance and reference, in Sec. II we borrow material from Ref. [15] to briefly review the method we use for treating fermions f propagating in a Bose-Einstein (BE) condensate, in particular the fermion dispersion relations in the model (Model-I of that reference) that we use for the example calculations in the present work. In Sec. III we determine the spinors that follow from the field equations, corresponding to the dispersion relations. More importantly for our purposes here, we determine and present the corresponding formulas for the spinor projection matrices, in a convenient form suitable for the type of calculations we are considering, namely transition rates involving the fermion modes. In Sec. IV we obtain the formulas for the corresponding propagators, including their thermal counterpart. Although the formulas are not used directly in the present work, they are nevertheless required in order to determine the proper normalization factors of the spinors and the projection matrices obtained in Sec. III.

As example application of the fermion spinors and projection matrices, in Sec. V we consider the calculation of rates for processes involving the interaction of a fermion χ with a fermion mode f . For definiteness, we consider in detail the situation in which χ is a heavy fermion, which can be considered at rest. As simple as this case may seem at first sight, the analysis and results illustrate several technicalities that must be dealt with in this type of calculation involving the f fermions, that are due to the fact that the dispersion relations of the f modes are not the standard vacuum dispersion relations but in fact are helicity dependent, and therefore the kinematics differ from the case in which the fermions propagate in vacuum. For example, among other things, a particular consequence is the fact that there is a kinematical range in which the group velocity of one of the f modes decreases as the

momentum increases, and in fact vanishes at a particular point, reminiscent of the Van Hove singularity in condensed matter systems [25,26].

We give our conclusions and outlook in Sec. VI, and in the Appendix we give a compendium of the relevant formulas for the fermion dispersion relations, spinors, projection operators, and propagators that are useful for reference in future works.

II. PRELIMINARIES

For reference, and to state clearly the context of our work, we present in this section the model we use to treat the fermions and the BE condensate and summarize the main results obtained in Ref. [15] that we need in the present work.

A. Model

We consider the model labeled as “Model-I” in Ref. [15]. It consists of a complex scalar field ϕ , and two chiral fermions f_L and f_R , with an interaction Lagrangian

$$L_{\text{int}} = -\lambda \phi \bar{f}_R f_L + \text{H.c.} \quad (2.1)$$

Apart from the interaction term given above, the Lagrangian has the fermion terms

$$L_f = i \bar{f}_L \not{\partial} f_L + i \bar{f}_R \not{\partial} f_R, \quad (2.2)$$

and a standard ϕ^4 Lagrangian

$$L^{(\phi)} = (\partial^\mu \phi)^* (\partial_\mu \phi) - V_0, \quad (2.3)$$

with

$$V_0(\phi) = m_\phi^2 \phi^* \phi + \lambda_\phi (\phi^* \phi)^2 \quad (\lambda_\phi > 0). \quad (2.4)$$

There are two conserved charges, labeled as $Q_{1,2}$, with

$$\begin{aligned} Q_1(f_L) = Q_1(f_R) = 1, & \quad Q_1(\phi) = 0, \\ Q_2(\phi) = Q_2(f_R) = 1, & \quad Q_2(f_L) = 0, \end{aligned} \quad (2.5)$$

which satisfy

$$Q_A(\phi) + Q_A(f_L) - Q_A(f_R) = 0. \quad (2.6)$$

The thermal expectation value of any operator O is given by

$$\langle O \rangle = \frac{\text{Tr} Z O}{\text{Tr} Z}, \quad (2.7)$$

where Z is the partition function operator

$$Z = e^{-\beta H + \sum_A \alpha_A Q_A}. \quad (2.8)$$

The assignments in Eq. (2.6) imply that the chemical potentials satisfy

$$\mu_\phi + \mu_L - \mu_R = 0, \quad (2.9)$$

where we are denoting by μ_L and μ_R the chemical potential of f_L and f_R , respectively.

The method used in Ref. [15] consists in rewriting the Lagrangian in terms of the fields ϕ', f'_L, f'_R , defined by

$$\begin{aligned} \phi &= e^{-i\mu_\phi t} \phi', \\ f_L &= e^{-i\mu_L t} f'_L, \\ f_R &= e^{-i\mu_R t} f'_R. \end{aligned} \quad (2.10)$$

As explained in Ref. [15], we can then use the partition function operator without the chemical potential terms $\sum_A \alpha_A Q_A$, provided we use the Lagrangian written in terms of the prime fields ϕ', f'_L, f'_R . This idea is motivated by the results of previous works by other authors [27] that consider the BE condensation of scalar fields. However, we emphasize that the proof given in Appendix A of Ref. [15] is a proof at the operator level not involving Lagrangian dynamics arguments, which uses only the equation of motion and the commutators of the charges with the field; it does not assume that it is a free field, and it is applicable to any field (e.g., a fermion field). Therefore the result holds for any field, not just a scalar field, whether it is interacting or not.

For our current purposes the upshot is that the prime fields are the appropriate ones that describe the modes in the BE condensate, in particular f'_R and f'_L for the fermion modes. For practical purposes, the recipe is to substitute the transformations given in Eq. (2.10) into the Lagrangian to obtain the Lagrangian in terms of the prime fields. The interaction term in Eq. (2.1) does not change,

$$L_{\text{int}} = -\lambda \phi' \bar{f}'_R f'_L + \text{H.c.} \quad (2.11)$$

Regarding the scalar field, the result of substituting $\phi = e^{-i\mu_\phi t} \phi'$ in $L^{(\phi)}$, which we denote by $L^{(\phi')}$, is

$$L^{(\phi')} = K - U, \quad (2.12)$$

where

$$U = -(\mu_\phi^2 - m_\phi^2) \phi'^* \phi' + \lambda_\phi (\phi'^* \phi')^2, \quad (2.13)$$

and K contains the terms with derivatives of ϕ' . We do not write them explicitly because we will not need them in what follows, but they are of course important to determine the properties (e.g., dispersion relations) of the scalar modes, and are discussed at length in Ref. [15].

Therefore, if the conditions are such that

$$\mu_\phi^2 > m_\phi^2, \quad (2.14)$$

the minimum of the potential U is not at $\phi' = 0$, and therefore ϕ' develops a nonzero expectation value. The $U(1)$ symmetry corresponding Q_2 is broken, which is associated with BE transition. Explicitly,

$$\langle \phi' \rangle \equiv \frac{1}{\sqrt{2}} \phi_0, \quad (2.15)$$

$$\phi_0^2 = \frac{\mu_\phi^2 - m_\phi^2}{\lambda_\phi}. \quad (2.16)$$

As in Ref. [15], we assume that this condition is satisfied.

Regarding the fermion fields, as a result of making the substitution given in Eq. (2.10), together with the symmetry breaking effect just described, the bilinear fermion terms in the Lagrangian are

$$\begin{aligned} L_0 &= \bar{f}'_L i \not{\partial} f'_L + \bar{f}'_R i \not{\partial} f'_R + \mu_L \bar{f}'_L \not{t} f'_L + \mu_R \bar{f}'_R \not{t} f'_R \\ &\quad - (m \bar{f}'_R f'_L + \text{H.c.}), \end{aligned} \quad (2.17)$$

where

$$\begin{aligned} m &= \frac{\lambda \phi_0}{\sqrt{2}}, \\ &= \frac{\lambda}{\sqrt{2}} \left(\frac{\mu_\phi^2 - m_\phi^2}{\lambda_\phi} \right)^{1/2}, \end{aligned} \quad (2.18)$$

and u^μ is the four-vector

$$u^\mu = (1, \vec{0}). \quad (2.19)$$

The complete Lagrangian, in terms of the prime fields, is

$$L = L_0 + L^{(\phi')} + L'_{\text{int}}, \quad (2.20)$$

where $L^{(\phi')}$ is given in Eq. (2.12), while L'_{int} is the interaction term between the fermion and the scalar fields, which we will not need in what follows. Defining

$$f = f'_L + f'_R, \quad (2.21)$$

in momentum space L_0 is given by

$$L_0(k) = \bar{f}(k) (\not{K} - \Sigma(k)) f(k), \quad (2.22)$$

where

$$\Sigma = mL + m^* R - \mu_L \not{t} L - \mu_R \not{t} R. \quad (2.23)$$

In this model, the two chiral fermions form a Dirac particle, in which the left and right components have different dispersion relations.

Despite the similarity with the standard Higgs mechanism, in the type of model we are considering the fermion mass term is induced by the condensate, which in turn is induced by the chemical potentials. In particular, if the chemical potentials are zero, then Eq. (2.14) is not satisfied, and from Eqs. (2.16) and (2.18), ϕ_0 and m remain zero. Moreover, the chemical potentials also generate the terms proportional to $\not{\mu}$ in Eq. (2.23), reminiscent of the self-energy of a fermion in a thermal background (e.g., a neutrino in a matter background).

B. Dispersion relations

The field equation in momentum space is

$$(K - \Sigma)f = 0. \quad (2.24)$$

We use systematically the Weyl (*chiral*) representation of the gamma matrices,

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \gamma^i &= \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \\ \gamma^5 &\equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (2.25)$$

For completeness, we also mention that we use the convention

$$\epsilon^{0123} = +1. \quad (2.26)$$

Using the Weyl representation of the gamma matrices we put

$$\begin{aligned} f'_L &= \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \\ f'_R &= \begin{pmatrix} \xi \\ 0 \end{pmatrix}. \end{aligned} \quad (2.27)$$

Writing

$$k^\mu = (\omega, \vec{k}), \quad (2.28)$$

the equations to be solved then become

$$\begin{aligned} (\omega + \mu_L + \vec{\sigma} \cdot \vec{k})\eta - m^*\xi &= 0, \\ (\omega + \mu_R - \vec{\sigma} \cdot \vec{k})\xi - m\eta &= 0. \end{aligned} \quad (2.29)$$

In the one-generation case we are considering the phase of m is irrelevant, since it can be absorbed by a field redefinition, so that we could take $m^* = m$. However,

since in more general cases such field redefinitions cannot be done independently, we keep m arbitrary.

In general, leaving out the case that $\mu_R = \mu_L$ (i.e., assuming $\mu_\phi \neq 0$), these equations have nontrivial solutions only if ξ and η are proportional to the same eigenvector of $\vec{\sigma} \cdot \vec{k}$; and in any case we can choose the spinors ξ and η to be such. Therefore, we write the solution in the form (note: our convention for x, y here are opposite to the one in Ref. [15])

$$\begin{aligned} \xi &= x\chi_s(\vec{k}), \\ \eta &= y\chi_s(\vec{k}), \end{aligned} \quad (2.30)$$

where $\chi_s(\vec{k})$ is the spinor with definite helicity, defined by

$$(\vec{\sigma} \cdot \hat{k})\chi_s(\vec{k}) = s\chi_s(\vec{k}), \quad (2.31)$$

with $s = \pm 1$. For a given helicity s , the equations for x and y are

$$\begin{aligned} (\omega + s\kappa + \mu_L)y - m^*x &= 0, \\ (\omega - s\kappa + \mu_R)x - my &= 0, \end{aligned} \quad (2.32)$$

which imply that ω must satisfy

$$(\omega - s\kappa + \mu_R)(\omega + s\kappa + \mu_L) - |m|^2 = 0. \quad (2.33)$$

Expressing μ_R and μ_L in terms of their sum and their difference,

$$\mu_\pm \equiv \frac{1}{2}(\mu_R \pm \mu_L), \quad (2.34)$$

this equation can be written in the form

$$(\omega + \mu_+)^2 - (\kappa - s\mu_-)^2 - |m|^2 = 0. \quad (2.35)$$

For each s , we have two solutions, one with positive ω and another with a negative ω . They correspond to the positive and negative helicity states of the Dirac particle and its antiparticle, which are associated with the unbroken Q_1 . We label the two solutions for each s as $\omega_s^{(\pm)}$. With this notation the solutions are

$$\omega_s^{(\pm)}(\vec{k}) = \pm\omega_{0s} - \mu_+, \quad (2.36)$$

where

$$\omega_{0s} = [(\kappa - s\mu_-)^2 + |m|^2]^{1/2}. \quad (2.37)$$

Denoting the particle and antiparticle dispersion relations by ω_s and $\bar{\omega}_s$, respectively, they are to be identified according to

$$\begin{aligned}
\omega_s(\vec{k}) &= \omega_s^{(+)}(\vec{k}) \\
&= \omega_{0s} - \mu_+, \\
\bar{\omega}_s(\vec{k}) &= -\omega_s^{(-)}(-\vec{k}) \\
&= \omega_{0s} + \mu_+.
\end{aligned} \tag{2.38}$$

For future reference we record here the formula for the group velocity,

$$v_{gs} = \frac{\kappa - s\mu_-}{\omega_{0s}}. \tag{2.39}$$

The dispersion relations were obtained in Ref. [15]. Our main purpose in the present work is to find the corresponding spinor wave functions and propagator of the fermion modes, which are required to calculate rates for processes involving the fermions as well as thermal corrections to the dispersion relations.

Before entering into the details we would like to mention the following. As we have mentioned in the discussion of Eq. (2.10), the chemical potentials do not appear explicitly in the partition function operator as a result of the field transformations. However, the chemical potentials enter in the calculation of any physical quantity through the dispersion relations, spinors, and propagators. Moreover, if we denote by n_s the momentum distribution functions of the fermion modes at some temperature $T = 1/\beta$, and similarly for the antiparticles,

$$\begin{aligned}
n_s &= \frac{1}{e^{\beta\omega_s} + 1}, \\
\bar{n}_s &= \frac{1}{e^{\beta\bar{\omega}_s} + 1},
\end{aligned} \tag{2.40}$$

the chemical potentials are present through ω_s and $\bar{\omega}_s$. In the case we are considering, we can take $\mu_{L,R}$ to be the independent chemical potentials, with μ_ϕ given by Eq. (2.9). Then Eq. (2.16), together with Eq. (2.40), fixes the background composition of the system. In particular at $T \rightarrow 0$, some of the distribution functions are filled up to a certain value κ_F of the momentum, while others are zero, depending on the relative value and sign of $\mu_{L,R}$. Which ones are zero, and what is κ_F for those that are not zero, can be determined easily from the formulas for the dispersion relations given in Eq. (2.38). The ones that are not zero determine the fermion background composition of the zero temperature system. Therefore, by means of the field transformation, the effects of the chemical potentials are taken into account through the dispersion relations we have discussed above, and the wave functions and propagators that we consider next.

III. SPINORS

A. Spinors for $\omega_s^{(+)}$: The u_s spinors

The x , y coefficients corresponding to the $\omega_s^{(+)}$ dispersion relation must satisfy Eq. (2.32), for $\omega = \omega_s^{(+)}$. Remember that we are identifying $\omega_s^{(+)}$ with the particle dispersion relation ω_s , that is

$$\begin{aligned}
D_{Ls}y_s - m^*x_s &= 0, \\
D_{Rs}x_s - my_s &= 0,
\end{aligned} \tag{3.1}$$

where we define

$$\begin{aligned}
D_{Ls}(\vec{k}) &= \omega_s + s\kappa + \mu_L \\
&= \omega_{0s} + s\kappa - \mu_-, \\
D_{Rs}(\vec{k}) &= \omega_s - s\kappa + \mu_R \\
&= \omega_{0s} - s\kappa + \mu_-.
\end{aligned} \tag{3.2}$$

Of course, D_{Ls}, D_{Rs} satisfy

$$D_{Ls}D_{Rs} = |m|^2, \tag{3.3}$$

by virtue of the dispersion relation $\omega_s^{(+)}$. It follows that

$$\begin{aligned}
x_s &= z_s \sqrt{D_{Ls}}, \\
y_s &= z_s \alpha^* \sqrt{D_{Rs}},
\end{aligned} \tag{3.4}$$

where z_s is an arbitrary normalization factor, and α is the phase factor of m , that is $\alpha = m/|m|$.

Therefore the complete Dirac spinor corresponding to $\omega_s^{(+)}$ is

$$u_s = z_s \begin{pmatrix} \sqrt{D_{Ls}} \\ \alpha^* \sqrt{D_{Rs}} \end{pmatrix} \chi_s, \tag{3.5}$$

and in particular

$$u_s \bar{u}_s = z_s^2 \begin{pmatrix} m & D_{Ls} \\ D_{Rs} & m^* \end{pmatrix} P_s, \tag{3.6}$$

where

$$P_s(\vec{k}) \equiv \frac{1}{2}(1 + s\vec{\sigma} \cdot \hat{k}). \tag{3.7}$$

Using the standard convention for the one-particle state with momentum $k_s^\mu = (\omega_s, \vec{k})$ and helicity s , which we denote by $|f(\vec{k}, s)\rangle$, we have

$$\langle 0|f(x)|f(\vec{k}, s)\rangle = u_s(\vec{k})e^{-ik_s \cdot x}. \tag{3.8}$$

The corresponding one-particle contribution to the propagator

$$iS_{F0}(x-y) = \langle 0|Tf(x)\bar{f}(y)|0\rangle, \quad (3.9)$$

in momentum space [with $k^\mu \equiv (\omega, \vec{k})$], is

$$(S_{F0})_{1P}(k) = \frac{u_s(\vec{k})\bar{u}_s(\vec{k})}{2\omega_s(\omega - \omega_s + i\epsilon)}. \quad (3.10)$$

The normalization factor z_s is determined by the requirement that the one-particle contribution to the propagator given in Eq. (3.10), using Eq. (3.6), coincides with the exact propagator near the $\omega = \omega_s$ pole. As we show in Sec. IV [see Eq. (4.14)], this condition gives

$$z_s = \sqrt{\frac{\omega_s}{\omega_{0s}}}. \quad (3.11)$$

We can write an expression for $u_s\bar{u}_s$ in a form that may be more convenient for some calculations as follows. Remembering that we are using the Weyl (chiral) representation of the gamma matrices, we have

$$-\gamma^0\gamma^5\vec{\gamma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad (3.12)$$

and therefore

$$P_s \equiv \frac{1}{2} \begin{pmatrix} 1 + s\vec{\sigma} \cdot \hat{k} & 0 \\ 0 & s\vec{\sigma} \cdot \hat{k} \end{pmatrix} = \frac{1}{2}(1 - s\gamma^0\gamma^5\vec{\gamma} \cdot \hat{k}). \quad (3.13)$$

Introducing the vectors

$$\begin{aligned} u^\mu &= (1, \vec{0}), \\ n^\mu &= (0, \hat{k}), \end{aligned} \quad (3.14)$$

we can then write

$$P_s = \frac{1}{2}(1 + s\gamma^5\not{n}) = \frac{1}{2}(1 - s\gamma^5\not{u}). \quad (3.15)$$

On the other hand, we can also write

$$\begin{aligned} \begin{pmatrix} m & D_{Ls} \\ D_{Rs} & m^* \end{pmatrix} &= mR + m^*L + D_{Ls}\gamma^0L + D_{Rs}\gamma^0R \\ &= mR + m^*L + D_{Ls}\not{n}L + D_{Rs}\not{n}R, \end{aligned} \quad (3.16)$$

and with these we then have

$$u_s\bar{u}_s = \frac{z_s^2}{2}(mR + m^*L + D_{Ls}\not{n}L + D_{Rs}\not{n}R)(1 - s\gamma^5\not{u}). \quad (3.17)$$

B. Spinors for $\omega_s^{(-)}$: The v_s spinors

We consider now the spinors corresponding to the $\omega_s^{(-)}$ dispersion relation. The x, y coefficients satisfy

$$\begin{aligned} (\omega^{(-)} + s\kappa + \mu_L)y_s - m^*x_s &= 0, \\ (\omega^{(-)} - s\kappa + \mu_R)x_s - my_s &= 0, \end{aligned} \quad (3.18)$$

and putting $\omega_s^{(-)} = -\bar{\omega}_s$ [remember Eq. (2.38)],

$$\begin{aligned} (\bar{\omega}_s - s\kappa - \mu_L)y_s + m^*x_s &= 0, \\ (\bar{\omega}_s + s\kappa - \mu_R)x_s + my_s &= 0. \end{aligned} \quad (3.19)$$

Defining

$$\begin{aligned} \bar{D}_{Ls}(\vec{k}) &= \bar{\omega}_s - s\kappa - \mu_L \\ &= \omega_{0s} - s\kappa + \mu_-, \\ \bar{D}_{Rs}(\vec{k}) &= \bar{\omega}_s + s\kappa - \mu_R \\ &= \omega_{0s} + s\kappa - \mu_-, \end{aligned} \quad (3.20)$$

the equations are written in the form

$$\begin{aligned} \bar{D}_{Ls}y_s &= -m^*x_s, \\ \bar{D}_{Rs}x_s &= -my_s. \end{aligned} \quad (3.21)$$

Notice that, since $\omega_s^{(-)}$ satisfies Eq. (2.33), $\bar{\omega}_s$ satisfies

$$(\bar{\omega}_s - s\kappa - \mu_L)(\bar{\omega}_s + s\kappa - \mu_R) - |m|^2 = 0, \quad (3.22)$$

which in turn imply

$$\bar{D}_{Rs}\bar{D}_{Ls} = |m|^2. \quad (3.23)$$

Therefore, the solution to Eq. (3.21) can be written as

$$\begin{aligned} x_s &= \bar{z}_s \sqrt{\bar{D}_{Ls}}, \\ y_s &= -\bar{z}_s \alpha^* \sqrt{\bar{D}_{Rs}}, \end{aligned} \quad (3.24)$$

where $\alpha = m/|m|$ as before. The complete Dirac spinor corresponding to $\omega_s^{(-)}$, which we denote by $u^{(-)}$, is therefore

$$u_s^{(-)}(\vec{k}) = \bar{z}_s \begin{pmatrix} \sqrt{\bar{D}_{Ls}} \\ -\alpha^* \sqrt{\bar{D}_{Rs}} \end{pmatrix} \chi_s(\vec{k}), \quad (3.25)$$

and in particular

$$u_s^{(-)}(\vec{k})\bar{u}_s^{(-)}(\vec{k}) = \bar{z}_s^2 \begin{pmatrix} -m & \bar{D}_{Ls} \\ \bar{D}_{Rs} & -m^* \end{pmatrix} P_s(\vec{k}). \quad (3.26)$$

The v spinor that appears in the plane wave expansion of the field is defined as

$$v_s(\vec{k}) \equiv u_s^{(-)}(-\vec{k}) = \bar{z}_s \begin{pmatrix} \sqrt{\bar{D}_{Ls}} \\ -\alpha^* \sqrt{\bar{D}_{Rs}} \end{pmatrix} \chi_s(-\vec{k}), \quad (3.27)$$

and therefore

$$v_s(\vec{k}) \bar{v}_s(\vec{k}) = \bar{z}_s^2 \begin{pmatrix} -m & \bar{D}_{Ls} \\ \bar{D}_{Rs} & -m^* \end{pmatrix} P_s(-\vec{k}). \quad (3.28)$$

To determine the normalization factor, notice that

$$v_s(-\vec{k}) \bar{v}_s(-\vec{k}) = u_s^{(-)}(\vec{k}) \bar{u}_s^{(-)}(\vec{k}) = \bar{z}_s^2 \begin{pmatrix} -m & \bar{D}_{Ls} \\ \bar{D}_{Rs} & -m^* \end{pmatrix} P_s(\vec{k}). \quad (3.29)$$

Using the standard convention for the one-(anti)particle state with momentum $k_s^\mu = (\bar{\omega}_s, \vec{k})$ and helicity s , which we denote by $|\bar{f}(\vec{k}, s)\rangle$, we have

$$\langle 0 | \bar{f}(x) | f(\vec{k}, s) \rangle = \bar{v}_s(\vec{k}) e^{-ik_s \cdot x}. \quad (3.30)$$

The corresponding one-particle contribution to the propagator, in momentum space [with $k^\mu \equiv (\omega, \vec{k})$], is

$$(S_{F0})_{1\bar{p}}(k) = \frac{v_s(\vec{k}) \bar{v}_s(\vec{k})}{2\bar{\omega}_s(\omega + \bar{\omega}_s - i\epsilon)} \Big|_{\vec{k} \rightarrow -\vec{k}}. \quad (3.31)$$

The normalization factor $\bar{z}_s$ is determined by the requirement that the one-particle contribution to the propagator given in Eq. (3.31), using Eq. (3.29), coincides with the exact propagator near the $\omega = \omega_s^{(-)}$ pole. As we show in Sec. IV [see Eq. (4.14)], this condition gives

$$\bar{z}_s = \sqrt{\frac{\bar{\omega}_s}{\omega_{0s}}}. \quad (3.32)$$

Similar to what we did for the $u_s \bar{u}_s$ projection operator, it is easy to see that in this case the expression corresponding to Eq. (3.17) is

$$v_s \bar{v}_s = \frac{\bar{z}_s^2}{2} (-mR - m^*L + \bar{D}_{Ls} \not{L} + \bar{D}_{Rs} \not{R})(1 + s\gamma^5 \not{H}). \quad (3.33)$$

C. Limit $\mu_{RL} = 0$ and $m^* = m$

1. $u\bar{u}$

In this limit $z_s = 1$, and

$$\begin{aligned} 2u_s \bar{u}_s &= (m + \omega \not{H} - s\kappa \not{H} \gamma^5)(1 - s\gamma^5 \not{H} \not{H}) \\ &= (m + \omega \not{H} - s\kappa \not{H} \gamma^5) + [-sm\gamma^5 \not{H} \not{H} + s\omega\gamma^5 \not{H} + \kappa \not{H}] \\ &= m + \omega \not{H} + \kappa \not{H} + s\kappa\gamma^5 \not{H} + s\omega\gamma^5 \not{H} - sm\gamma^5 \not{H} \not{H} \\ &= m + \mathcal{K} + sm\gamma^5 \mathcal{L} - sm\gamma^5 \not{H} \not{H}, \end{aligned} \quad (3.34)$$

where

$$\mathcal{L}^\mu = \frac{1}{m}(\kappa u^\mu + \omega n^\mu) = \frac{1}{m}(\kappa, \omega \hat{k}). \quad (3.35)$$

On the other hand, the standard expression in vacuum is

$$\begin{aligned} 2u_s \bar{u}_s &= (\mathcal{K} + m)(1 + s\gamma^5 \mathcal{L}) \\ &= \mathcal{K} + m + sm\gamma^5 \mathcal{L} - s\gamma^5 \mathcal{K} \mathcal{L}. \end{aligned} \quad (3.36)$$

Substituting in $\mathcal{K}\mathcal{L}$ the relations

$$\begin{aligned} k^\mu &= \omega u^\mu + \kappa \vec{n}, \\ \mathcal{L}^\mu &= \frac{1}{m}(\kappa u^\mu + \omega n^\mu), \end{aligned} \quad (3.37)$$

and using the identities

$$u^2 = 1, \quad n^2 = -1, \quad u \cdot n = 0, \quad (3.38)$$

it follows that

$$\mathcal{K}\mathcal{L} = m \not{H} \not{H}, \quad (3.39)$$

which proves that Eq. (3.34) coincides with the standard expression in vacuum.

2. $v\bar{v}$

We proceed in a similar way for $v\bar{v}$. In this limit $\bar{\omega} = \omega$, and therefore

$$\begin{aligned} 2v_s \bar{v}_s &= (-m + \omega \not{H} + s\kappa \not{H} \gamma^5)(1 + s\gamma^5 \not{H} \not{H}) \\ &= -m + \omega \not{H} + s\kappa \not{H} \gamma^5 - sm\gamma^5 \not{H} \not{H} - s\omega\gamma^5 \not{H} + \kappa \not{H} \\ &= -m + \mathcal{K} - sm\gamma^5 \mathcal{L} - sm\gamma^5 \not{H} \not{H}, \end{aligned} \quad (3.40)$$

while the standard expression in vacuum is

$$\begin{aligned} 2v_s \bar{v}_s &= (\mathcal{K} - m)(1 + s\gamma^5 \mathcal{L}) \\ &= \mathcal{K} - m - sm\gamma^5 \mathcal{L} - s\gamma^5 \mathcal{K} \mathcal{L}. \end{aligned} \quad (3.41)$$

Using Eq. (3.39) once more verifies the equivalence.

IV. PROPAGATOR

The expression or the propagator of the f modes is required in order to determine the normalization factor of the spinors. Here we give the details of the formulas that we have quoted in Sec. III for that purpose. The formulas we present here are also appropriate for calculating amplitudes of processes involving the f modes in intermediate states, including possible thermal effects when required.

Going back to Eqs. (2.22) and (2.23), the propagator is the inverse of $\mathcal{K} - \Sigma$, or

$$S_{F0}^{-1}(k) = \mathcal{K} + \mu_L \not{L} + \mu_R \not{R} - mL - m^* R, \quad (4.1)$$

which we write explicitly in the Weyl representation,

$$S_{F0}^{-1}(k) = \begin{pmatrix} -m^* & \omega + \mu_L + \vec{\sigma} \cdot \vec{k} \\ \omega + \mu_R - \vec{\sigma} \cdot \vec{k} & -m \end{pmatrix}. \quad (4.2)$$

Using

$$\begin{aligned} P_+ + P_- &= 1, \\ P_+ - P_- &= \vec{\sigma} \cdot \hat{k}, \end{aligned} \quad (4.3)$$

where P_s are the matrices defined in Eq. (3.7), we can write the propagator in the form

$$S_{F0}^{-1}(k) = \sum_s S_s^{-1} P_s, \quad (4.4)$$

where

$$S_s^{-1} = \begin{pmatrix} -m^* & \omega + \mu_L + s\kappa \\ \omega + \mu_R - s\kappa & -m \end{pmatrix}. \quad (4.5)$$

It then follows immediately that

$$S_{F0} = \sum_s S_s P_s, \quad (4.6)$$

where

$$S_s \equiv \frac{N_s}{d_s}, \quad (4.7)$$

with

$$\begin{aligned} d_s(\omega) &= (\omega + \mu_L + s\kappa)(\omega + \mu_R - s\kappa) - |m|^2, \\ N_s(\omega) &= \begin{pmatrix} m & \omega + \mu_L + s\kappa \\ \omega + \mu_R - s\kappa & m^* \end{pmatrix}. \end{aligned} \quad (4.8)$$

It follows [see Eq. (2.33)] that the dispersion relations are the solutions to

$$d_s = 0, \quad (4.9)$$

and indeed

$$d_s = (\omega - \omega_s^{(+)})(\omega - \omega_s^{(-)}). \quad (4.10)$$

Therefore,

$$\begin{aligned} S_s &= \frac{N_s(\omega)}{(\omega - \omega_s^{(+)})(\omega - \omega_s^{(-)})} \\ &= \frac{1}{\omega_s^{(+)} - \omega_s^{(-)}} \left[\frac{N_s(\omega)}{\omega - \omega_s^{(+)}} - \frac{N_s(\omega)}{\omega - \omega_s^{(-)}} \right]. \end{aligned} \quad (4.11)$$

Following the standard (Feynman) prescription, the poles of the propagator are to be shifted by replacing¹

$$\omega_s^{(\pm)} \rightarrow \omega_s^{(\pm)} \mp i\epsilon. \quad (4.12)$$

Equivalently, in terms of the particle and antiparticle dispersion relations ω_s and $\bar{\omega}_s$,

$$\begin{aligned} \omega_s &\rightarrow \omega_s - i\epsilon, \\ \bar{\omega}_s &\rightarrow \bar{\omega}_s - i\epsilon. \end{aligned} \quad (4.13)$$

Thus, for ω near one of the $\omega_s^{(+)}$ poles, the propagator S_{F0} given in Eq. (4.6) reduces to

$$S_{F0}|_{\omega \rightarrow \omega_s^{(+)}} = \frac{N_s(\omega_s^{(+)})}{(\omega_s^{(+)} - \omega_s^{(-)})(\omega - \omega_s^{(+)} + i\epsilon)} P_s. \quad (4.14)$$

The wave function normalization factor z_s in Eq. (3.6) is determined by the condition that this coincides with the expression for $(S_{F0})_{1p}$ given in Eq. (3.10). Using Eqs. (3.2) and (3.6) we arrive at Eq. (3.11). Similarly, for the v spinors, near the $\omega_s^{(-)}$ pole,

$$\begin{aligned} S_{F0}|_{\omega \rightarrow \omega_s^{(-)}} &= -\frac{N_s(\omega_s^{(-)})}{(\omega_s^{(+)} - \omega_s^{(-)})(\omega - \omega_s^{(-)} - i\epsilon)} P_s \\ &= -\frac{N_s(-\bar{\omega}_s)}{(\omega_s^{(+)} - \omega_s^{(-)})(\omega + \bar{\omega}_s - i\epsilon)} P_s. \end{aligned} \quad (4.15)$$

This must be compared with Eq. (3.31). Using Eqs. (3.28) and (4.8) we arrive at Eq. (3.32).

The propagator can be written in covariant form, analogous to the $u\bar{u}$ and $v\bar{v}$ projection operators, which is more convenient for practical calculations. Firstly, introducing the off-shell version of $D_{L,R}$,

$$\begin{aligned} E_{Ls}(\omega, \vec{k}) &= \omega + s\kappa + \mu_L, \\ E_{Rs}(\omega, \vec{k}) &= \omega - s\kappa + \mu_R, \end{aligned} \quad (4.16)$$

¹This is required for consistency with the one-particle approximation term as well.

we can write

$$N_s = (mR + m^*L + E_{Ls}\not{u}L + E_{Rs}\not{u}R). \quad (4.17)$$

Notice that

$$\begin{aligned} D_{L,R}(\vec{k}) &= E_{L,R}(\omega_s, \vec{k}), \\ \bar{D}_{L,R}(\vec{k}) &= -E_{L,R}(-\bar{\omega}_s, \vec{k}). \end{aligned} \quad (4.18)$$

Thus,

$$S_{F0} = \sum_s \frac{N_s P_s}{(\omega - \omega_s^{(+)} + i\epsilon)(\omega - \omega_s^{(-)} - i\epsilon)}, \quad (4.19)$$

or in terms of the particle and antiparticle dispersion relations ω_s and $\bar{\omega}_s$,

$$\begin{aligned} S_{F0} &= \sum_s \frac{N_s P_s}{(\omega - \omega_s + i\epsilon)(\omega + \bar{\omega}_s - i\epsilon)} \\ &= \sum_s \frac{1}{\omega_s + \bar{\omega}_s} \left[\frac{N_s P_s}{\omega - \omega_s + i\epsilon} - \frac{N_s P_s}{\omega + \bar{\omega}_s - i\epsilon} \right], \end{aligned} \quad (4.20)$$

with P_s and N_s given in Eqs. (3.15) and (4.17), respectively.

The thermal propagator can be constructed by the usual rules. Denoting the propagator of the f fermion given in Eq. (4.20) by $S_{F0}^{(f)}(k)$, we have, for example,

$$S_{F11}^{(f)}(q) = S_{F0}^{(f)}(q) - \left(S_{F0}^{(f)}(q) - \bar{S}_{F0}^{(f)}(q) \right) \eta_f(q), \quad (4.21)$$

$$S_{F12}^{(f)}(q) = -\left(S_{F0}^{(f)}(q) - \bar{S}_{F0}^{(f)}(q) \right) [\eta_f(q) - \theta(-q \cdot u)], \quad (4.22)$$

and so on. The factor η_f is the usual one, but the chemical potentials do not appear in the distribution functions, that is

$$\eta_f(q) = \theta(q \cdot u) n_F(x_f) + \theta(-q \cdot u) n_F(-x_f), \quad (4.23)$$

where

$$n_F(x) = \frac{1}{e^x + 1}, \quad (4.24)$$

with

$$x_f = \beta q \cdot u. \quad (4.25)$$

Although the chemical potentials do not appear explicitly in Eqs. (4.24) and (4.25), their presence is taken into account through the dispersion relations, wave functions, and propagators of the modes. Calculations of rates of processes, including the fermion modes, would involve the

use of the propagators defined above. Thus, a calculation of the transition probability for any process involving the fermion modes would depend on the fermion distribution functions (e.g., the Pauli blocking factors of the fermion modes).

V. EXAMPLES

As examples that illustrate the applicability of the formulas we have obtained and their usefulness, we will consider the calculation of the rates for processes of the type

$$f + \chi \rightarrow f + \chi, \quad (5.1)$$

and we will present the results and details specifically for the case that χ is a heavy fermion, in the sense that $\kappa \ll m_\chi$, and essentially at rest. Due to the fact that the momentum dependence of the dispersion relations and spinors of the f fermions are not standard, the kinematics involved in the calculations of the rates for these processes also differ from the standard case of a free f . In principle, a complete calculation of such rates would involve their thermal averages, taking into account the fermion distribution functions. In this sense, the calculations we present below are useful for considering possible application of the results to situations of phenomenological interest (for example, in the context of DM physics), and the results can serve as benchmark guides for further exploration of the type of model we are considering, in specific realistic settings.

A simple possible type of $f\chi$ interaction that we can consider is (a vectorlike interaction)

$$L_{f\chi} = g_{LL} (\bar{f}_L \gamma_\mu f_L) (\bar{\chi}_L \gamma^\mu \chi_L), \quad (5.2)$$

which in terms of the primed f field becomes

$$L_{f\chi} = g_{LL} (\bar{f}'_L \gamma_\mu f'_L) (\bar{\chi}_L \gamma^\mu \chi_L), \quad (5.3)$$

or variations of this; for example,

$$L_{f\chi} = g_{RL} (\bar{f}'_L \gamma_\mu f'_L) (\bar{\chi}_R \gamma^\mu \chi_R), \quad (5.4)$$

and similar ones with $(L \leftrightarrow R)$.

As a side note we mention that although it could be thought that the simplest interaction we can consider are scalar-type interactions of the form

$$L_{f\chi} = g_L (\bar{f}_R f_L) (\bar{\chi}_R \chi_L) + \text{H.c.}, \quad (5.5)$$

this is actually not consistent. This interaction does not respect the global symmetries of the model we are considering (Model-I), corresponding to the global charges

$$\begin{aligned} Q_1(f_L) = Q_1(f_R) = 1, \quad Q_1(\phi) = 0, \\ Q_2(\phi) = Q_2(f_R) = 1, \quad Q_2(f_L) = 0. \end{aligned} \quad (5.6)$$

In other words, in terms of the primed fields this interaction term would become

$$L_{f\chi} = g_L e^{-i(\mu_L - \mu_R)t} (\tilde{f}'_R f'_L) (\tilde{\chi}_R \chi_L) + \text{H.c.}, \quad (5.7)$$

which introduces a time-dependent coupling. Therefore we stick to one of the vector interactions, which are consistent with the symmetries of Model-I.

For definiteness we consider the g_{LL} interaction. Dropping the prime superscript on f ,

$$L_{f\chi} = g_{LL} (\tilde{f}_L \gamma_\mu f_L) (\tilde{\chi}_L \gamma^\mu \chi_L), \quad (5.8)$$

with the understanding that f now stands for the fermion field of the fermion modes in the BE condensate. We want to calculate the rates for

$$f(k, s) + \chi(p, \lambda) \rightarrow f(k', s') + \chi(p', \lambda'), \quad (5.9)$$

using obvious labels for the kinematic variables. We write k^μ and p^μ in the form

$$\begin{aligned} k^\mu &= (\omega_s, \vec{k}), \\ p^\mu &= (E, \vec{P}), \end{aligned} \quad (5.10)$$

and similarly

$$\begin{aligned} k'^\mu &= (\omega_{s'}, \vec{k}'), \\ p'^\mu &= (E', \vec{P}'). \end{aligned} \quad (5.11)$$

The amplitude is

$$M = g_{LL} (\bar{u}_f(\vec{k}', s') \gamma_\mu L u_f(\vec{k}, s)) (\bar{u}_\chi(\vec{P}', \lambda') \gamma^\mu L u_\chi(\vec{P}, \lambda)). \quad (5.12)$$

To calculate the probability, we average over λ (unpolarized χ) sum over λ' , but keep s and s' since in the end we have to multiply by the f distribution functions, which depend on the helicity of f . Thus,

$$|\overline{M}|^2 = g_{LL}^2 \left(\frac{1}{2} \left(\text{Tr} \gamma_\mu L \rho_f \gamma_\nu L \rho'_f \right) \text{Tr} \gamma^\mu L \rho_\chi \gamma^\nu L \rho'_\chi \right). \quad (5.13)$$

For χ ,

$$\rho_\chi = \not{p} + m_\chi, \quad (5.14)$$

and similarly for ρ'_χ . For f ,

$$\rho_f = u_f(\vec{k}, s) \bar{u}_f(\vec{k}, s), \quad (5.15)$$

and similarly for ρ'_f , where $u_f(k, s)$ is the spinor for the fermions in the BE Model-I. Here we will use the expression we have obtained for the projection matrices $\bar{u}_f u_f$, given in Eq. (3.17) and summarized in Appendix, namely,

$$\rho_f = \frac{z_s^2}{2} (mR + m^* L + D_{Ls} \not{L} + D_{Rs} \not{R}) (1 - s \gamma^5 \not{L}). \quad (5.16)$$

The formulas for ρ'_f are obtained by replacing $\vec{k} \rightarrow \vec{k}'$ and $s \rightarrow s'$ in the corresponding formulas.

A. Evaluation of the traces

For χ , following standard procedure,

$$\begin{aligned} L^{\mu\nu} &\equiv \frac{1}{2} \text{Tr} \gamma^\mu L \rho_\chi \gamma^\nu L \rho'_\chi \\ &= [p^\mu p'^\nu + p'^\mu p^\nu - p \cdot p' g^{\mu\nu} + i \epsilon^{\mu\nu\alpha\beta} p_\alpha p'_\beta]. \end{aligned} \quad (5.17)$$

1. f part

For the f part, we can simplify the calculation by noticing that

$$\text{Tr} \gamma_\mu L \rho_f \gamma_\nu L \rho'_f = \text{Tr} \gamma_\mu (L \rho_f R) \gamma_\nu (L \rho'_f R), \quad (5.18)$$

and from Eq. (5.16)²

$$L \rho_f R = \frac{z_s^2}{2} D_{Rs} L \not{L} (1 - s \not{L} \not{R}) R = \frac{z_s^2}{2} D_{Rs} L (\not{L} - s \not{R}), \quad (5.19)$$

and similarly for ρ'_f ,

$$L \rho'_f R = \frac{z_{s'}^2}{2} D'_{Rs'} L (\not{L} - s' \not{R}'). \quad (5.20)$$

Therefore,

$$\text{Tr} \gamma_\mu L \rho_f \gamma_\nu L \rho'_f = \frac{z_s^2}{2} \frac{z_{s'}^2}{2} D_{Rs} D'_{Rs'} (2F_{\mu\nu}), \quad (5.21)$$

²It is instructive to notice that in the standard case,

$$L(\mathcal{K} + m)(1 + s \gamma^5 \not{\mathcal{L}}) R = L(\mathcal{K} - sm \not{\mathcal{L}}) = L(\omega - s\kappa)(\not{u} - s \not{n}),$$

which agrees with what we get if we take the $\mu_{L,R} \rightarrow 0$ limit (and $m = m^*$) in the following expression for $L \rho_f R$. In the second equality we have used $k^\mu = \omega u^\mu + \kappa n^\mu$ and $m \not{\mathcal{L}}^\mu = \kappa u^\mu + \omega n^\mu$, and therefore

$$\begin{aligned} k^\mu - sm \not{\mathcal{L}}^\mu &= (\omega - s\kappa) u^\mu + (\kappa - s\omega) n^\mu \\ &= (\omega - s\kappa) u^\mu + (s\kappa - \omega) s n^\mu = (\omega - s\kappa)(u^\mu - s n^\mu). \end{aligned}$$

where

$$F_{\mu\nu} \equiv \frac{1}{2} \text{Tr} \left\{ R \gamma_\mu (\not{p} - s \not{n}) \gamma_\nu (\not{p} - s' \not{n}') \right\}, \quad (5.22)$$

which can be evaluated by using the same formulas for the traces that we have used in the χ part. The result then is the same that we have obtained above for $L_{\mu\nu}$, with the replacements,

$$\begin{aligned} p_\mu &\rightarrow q_\mu \equiv u_\mu - s n_\mu, \\ p'_\mu &\rightarrow q'_\mu \equiv u_\mu - s' n'_\mu. \end{aligned} \quad (5.23)$$

Thus,

$$F_{\mu\nu} = [q_\mu q'_\nu + q'_\mu q_\nu - q \cdot q' g_{\mu\nu} + i \epsilon_{\mu\nu\alpha\beta} q^\alpha q'^\beta], \quad (5.24)$$

with q and q' defined above.

The amplitude squared is then given by

$$|\overline{M}|^2 = \frac{1}{2} g_{LL}^2 z_s^2 z_{s'}^2 D_{Rs} D'_{Rs'} L^{\mu\nu} F_{\mu\nu}, \quad (5.25)$$

with $L^{\mu\nu}$ and $F^{\mu\nu}$ given in Eqs. (5.17) and (5.24). By straightforward algebra, it follows that

$$L^{\mu\nu} F_{\mu\nu} = 4(p \cdot q)(p' \cdot q'), \quad (5.26)$$

with

$$\begin{aligned} p \cdot q &= E + s \vec{P} \cdot \hat{\kappa}, \\ p' \cdot q' &= E' + s' \vec{P}' \cdot \hat{\kappa}'. \end{aligned} \quad (5.27)$$

B. $f + \chi \rightarrow f + \chi$ with χ at rest (heavy χ)

We now consider in detail the case that χ is initially at rest, and heavy, i.e., with a mass m_χ such that

$$\kappa \ll m_\chi. \quad (5.28)$$

In a physical setting, χ can stand for a nucleon, with the process taking place with κ much less than the nucleon mass, or some other particle in the medium, but which is not part of the BE condensate background. Because the f mode is an excitation of the BE condensate, rather than a particle that travels from far away and scatters from a target, the quantity of interest is the process rate, typically denoted by $v\sigma$. Denoting by n_χ the number density of χ particles, the total transition probability per second is given by $n_\chi v\sigma$. Our starting point to determine $v\sigma$ is

$$v\sigma = \frac{1}{4E\omega_s} |\overline{M}|^2 dI, \quad (5.29)$$

where dI is the phase space integral

$$dI \equiv (2\pi)^4 \delta^{(4)}(p' + k' - p - k) \frac{d^3 P'}{(2\pi)^3 2E'} \frac{d^3 \kappa'}{(2\pi)^3 2\omega_{s'}}. \quad (5.30)$$

As can be seen from Eq. (2.38), the dispersion relations depend on the helicity and have a nonstandard momentum dependence. A particular consequence is the fact that there is a region of κ in which the dispersion relation of one of the modes has negative group velocity. Specifically this occurs for $s = +1$ if $\mu_- > 0$, or $s = -1$ if $\mu_- < 0$. Considering for definiteness the case $\mu_- > 0$, then ω_+ has negative group velocity for $\kappa < \mu_-$, and in fact becomes zero at $\kappa = \mu_-$. This is reminiscent of the Van Hove singularity of the density of states in condensed matter systems [25,26]. For definiteness, in the remainder of this work we adopt the convention that

$$\mu_- > 0. \quad (5.31)$$

The relevant point for us is that

$$\left. \frac{d\omega_+}{d\kappa} \right|_{\kappa=\mu_-} = 0. \quad (5.32)$$

The opposite convention $\mu_- < 0$ can be treated similarly with corresponding interchange of the s helicity states.

In general, p^μ is given as in Eq. (5.10), here we consider the case that χ is

(1) initially at rest,

$$p^\mu = (m_\chi, \vec{0}) \Rightarrow p \cdot q = m_\chi, \quad (5.33)$$

(2) heavy,

$$\kappa, \omega_s(\kappa) \ll m_\chi. \quad (5.34)$$

Energy-momentum conservation implies that

$$E' \sim m_\chi, \quad (5.35)$$

so that we can set

$$p' \cdot q' \rightarrow m_\chi. \quad (5.36)$$

Therefore, in the case we are considering,

$$L^{\mu\nu} F_{\mu\nu} = 4m_\chi^2, \quad (5.37)$$

and whence

$$|\overline{M}|^2 = 2m_\chi^2 g_{LL}^2 z_s^2 z_{s'}^2 D_{Rs} D'_{Rs'}. \quad (5.38)$$

Regarding the phase-space integral, doing the integral over P' ,

$$dI = \frac{1}{4\pi^2} \frac{1}{2E'} \delta(E' + \omega'_{s'} - m_\chi - \omega_s) \frac{d^3\kappa'}{2\omega'_{s'}}, \quad (5.39)$$

where

$$E' = \sqrt{|\vec{k} - \vec{k}'|^2 + m_\chi^2} \simeq \frac{|\vec{k} - \vec{k}'|^2}{2m_\chi} + m_\chi. \quad (5.40)$$

Therefore,

$$dI = \frac{1}{4\pi^2} \frac{1}{2E'} \frac{\kappa'^2}{2\omega'_{s'}} \frac{1}{\left| \frac{dE'}{d\kappa'} + \frac{d\omega'_{s'}}{d\kappa'} \right|} d\Omega', \quad (5.41)$$

where κ' must satisfy

$$\omega'_{s'} + E' = \omega_s + m_\chi, \quad (5.42)$$

or explicitly,

$$[(\kappa' - s'\mu_-)^2 + |m|^2]^{1/2} + \frac{|\vec{k} - \vec{k}'|^2}{2m_\chi} = [(\kappa - s\mu_-)^2 + |m|^2]^{1/2}. \quad (5.43)$$

Defining

$$x = \cos \theta' \equiv \hat{k} \cdot \hat{k}', \quad (5.44)$$

we then have

$$dI = \frac{1}{8\pi} \frac{\kappa'^2}{E' \omega'_{s'}} \frac{1}{\left| \frac{\kappa' - \kappa x}{m_\chi} + \frac{d\omega'_{s'}}{d\kappa'} \right|} dx. \quad (5.45)$$

The expression for $v\sigma$ is then

$$v d\sigma = \frac{1}{4m_\chi \omega_s} |\overline{M}|^2 dI, \quad (5.46)$$

where $|\overline{M}|^2$ is given in Eq. (5.38). Given κ , s , and s' , then κ' is given by the solution of Eq. (5.43). If there are several solutions, then there is a rate for each such κ' . The rate has a particular κ dependence. In particular, Eq. (5.45) reveals that if the solution for κ' is near the point $\kappa' = \mu_-$, at which $\frac{d\omega'_{s'}}{d\kappa'}$ is zero, then the term $\frac{\kappa' - \kappa x}{m_\chi}$ must be taken into account, and the recoil energy of the χ in Eq. (5.43) cannot be dropped when determining κ' . But in the rest of the factors that appear in the expression for $v\sigma$ we can use the large m_χ limit and set $E' \rightarrow m_\chi$ and $\omega'_{s'} = \omega_s$. We then obtain

$$dI = \frac{1}{8\pi} \frac{\kappa'^2}{m_\chi \omega_s} \frac{1}{\left| \frac{\kappa' - \kappa x}{m_\chi} + \frac{d\omega'_{s'}}{d\kappa'} \right|} dx, \quad (5.47)$$

and in $|M|^2$ we can put

$$\begin{aligned} z'_{s'} &= z_s, \\ D'_{Rs'} &= \omega_s - s'\kappa' + \mu_R, \\ &= \omega_{0s} - s'\kappa' + \mu_-. \end{aligned} \quad (5.48)$$

Putting it all together in Eq. (5.46),

$$\begin{aligned} v d\sigma &= \frac{g_{LL}^2 \kappa'^2}{16\pi \omega_{0s}^2} (\omega_{0s} - s\kappa + \mu_-)(\omega_{0s} - s'\kappa' + \mu_-) \\ &\times \frac{1}{\left| \frac{\kappa' - \kappa x}{m_\chi} + \frac{\kappa' - s'\mu_-}{\omega'_{0s'}} \right|} dx, \end{aligned} \quad (5.49)$$

where $\omega'_{0s'}$ is given by the expression in Eq. (2.37), with the replacements $\kappa, s \rightarrow \kappa', s'$.

We now consider the solutions of Eq. (5.43), separately one by one.

(1) *Case I:* Consider $s = -$. Ignoring the χ recoil energy term in Eq. (5.43), the equation for κ' becomes

$$|\kappa' - s'\mu_-| = |\kappa + \mu_-|, \quad (5.50)$$

whose only solution is

$$\kappa' = \kappa + \mu_- + s'\mu_-. \quad (5.51)$$

In either case, it follows that

$$\begin{aligned} \kappa' - s'\mu_- &= \kappa + \mu_-, \\ \omega'_{0s'} &= \omega_{0-}, \end{aligned} \quad (5.52)$$

and therefore

$$\frac{\kappa' - s'\mu_-}{\omega'_{0s'}} = \frac{\kappa + \mu_-}{\omega_{0-}}. \quad (5.53)$$

This implies that keeping the term $(\kappa' - \kappa x)/m_\chi$ in the denominator of Eq. (5.49), as well as the χ recoil energy in Eq. (5.43) would give insignificant corrections to the solution given in Eq. (5.51). By straightforward algebra, the following explicit expressions for $v\sigma$ follow from Eq. (5.49):

$$\begin{aligned}
v\sigma &= \frac{g_{LL}^2 (\kappa + \mu_- + s' \mu_-)^2}{8\pi (\kappa + \mu_-) \omega_{0-}} [\omega_{0-} + \kappa + \mu_-] [\omega_{0-} - s'(\kappa + \mu_-)] \\
&= \frac{g_{LL}^2}{8\pi} \left\{ \frac{1}{(\kappa + \mu_-) \omega_{0-}} \right\} \begin{cases} (\kappa + 2\mu_-)^2 |m|^2 & (s' = +) \\ \kappa^2 \{ \sqrt{(\kappa + \mu_-)^2 + |m|^2} + \mu_- + \kappa \}^2 & (s' = -) \end{cases} \quad (5.54)
\end{aligned}$$

Sample plots of $v\sigma$ given in Eq. (5.54) are given in Fig. 1. For comparison and reference, Fig. 2 shows the plot of $v\sigma$ calculated by setting $\mu_{\pm} = 0$.

- (2) *Case 2:* Consider $s = +$, $s' = -$. Again ignoring the χ recoil energy in Eq. (5.43), the equation for κ' becomes

$$\kappa' = -\mu_- \pm |\kappa - \mu_-|. \quad (5.55)$$

Only the solution with the positive sign is possible. Then a solution exists (the process is possible) provided

$$\kappa > 2\mu_-, \quad (5.56)$$

so that

$$\kappa' = \kappa - 2\mu_- \geq 0. \quad (5.57)$$

Furthermore,

$$\frac{\kappa' - s' \mu_-}{\omega'_{0s'}} = \frac{\kappa - \mu_-}{\omega_{0+}} \geq \frac{\mu_-}{\omega_{0+}}, \quad (5.58)$$

which implies, as in Case 1 above, that keeping the term $(\kappa' - \kappa x)/m_\chi$ in the denominator of Eq. (5.49), as well as the χ recoil energy in Eq. (5.43) would give insignificant corrections to the solution given in

Eq. (5.57). In this case the following explicit form for $v\sigma$ follows from Eq. (5.49),

$$v\sigma = \frac{g_{LL}^2 (\kappa - 2\mu_-)^2}{8\pi (\kappa - \mu_-) \omega_{0+}} |m|^2 \quad (\kappa > 2\mu_-). \quad (5.59)$$

Due to the kinematical restriction $\kappa > 2\mu_-$, the apparent singularity at $\kappa = \mu_-$ actually lies outside the physical region. Sample plot of $v\sigma$ in this case is shown in Fig. 3.

- (3) *Case 3:* Consider $s = +$, $s' = +$. In this case it is necessary to include the term $(\kappa' - \kappa x)/m_\chi$ in Eq. (5.49) and take into account the χ recoil energy in Eq. (5.43). To see this, let us pretend that we can neglect them and proceed as in the previous Case 1 and Case 2 above. Then Eq. (5.43) gives

$$\kappa' = \mu_- \pm |\kappa - \mu_-|, \quad (5.60)$$

which can have two solutions for κ' . As a complement to understanding the origin of the two solutions for κ' , the dispersion relations are plotted in Fig. 4. For ω_+ around its minimum value, the equation $\omega_+(\kappa) = \omega$ gives two values of κ . Thus, if $\kappa < \mu_-$, both values of κ' are allowed,

$$\kappa' = \begin{cases} 2\mu_- - \kappa & (+) \\ \kappa & (-) \end{cases}. \quad (5.61)$$

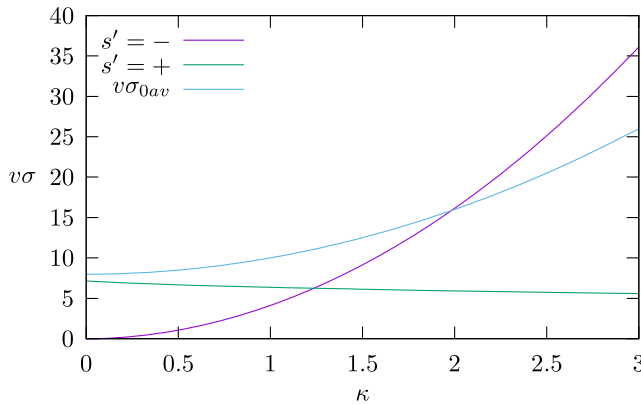


FIG. 1. Plot of $v\sigma$ for $s = -$ and s' as indicated (Case-1), for $\mu_- = 1$, $\mu_+ = 1.5$, $|m| = 2$. For reference, also plotted is $v\sigma_{0av}$, which is calculated by setting $\mu_{\pm} = 0$, and then summing over s' and averaging over s .

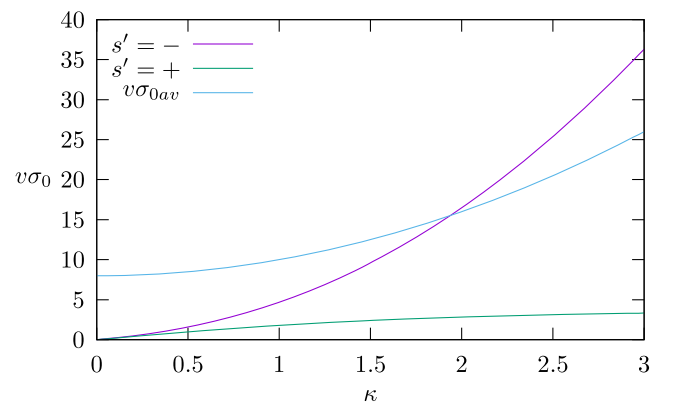


FIG. 2. Plot of $v\sigma$ for $s = -$ and s' as indicated, for $\mu_{\pm} = 0$, $|m| = 2$. For reference, also plotted is $v\sigma_{0av}$, which is calculated by setting $\mu_{\pm} = 0$ as above, and then summing over s' and averaging over s .

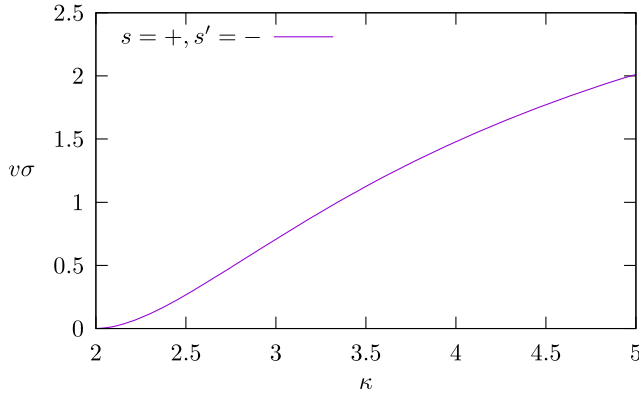


FIG. 3. Plot of $v\sigma$ for $s = +$ and $s' = -$, $\mu_- = 1, \mu_+ = 1.5, |m| = 2$. As discussed in the text (Case 2), the process can take place provided $\kappa > 2\mu_-$.

If $\mu_- < \kappa < 2\mu_-$, again both values of κ' are allowed,

$$\kappa' = \begin{cases} 2\mu_- - \kappa & (-) \\ \kappa & (+) \end{cases}. \quad (5.62)$$

If $\kappa > 2\mu_-$,

$$\kappa' = \kappa. \quad (5.63)$$

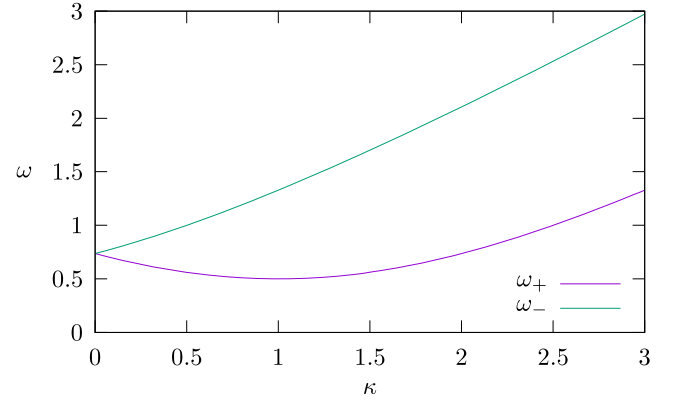


FIG. 4. Plot of dispersion relations, for the sample values $\mu_- = 1, \mu_+ = 1.5, |m| = 2$. The lower curve, ω_+ , has a minimum at $\kappa = \mu_-$. Around that point, there can be two values of κ for a given value of ω , which is the reason why there are two possible solutions for κ' discussed in Case 3 in the text.

In other words, in summary, the process $f_+(\vec{\kappa}) \rightarrow f_+(\vec{\kappa}')$ can occur with
(a) *Case 3a*:

$$\kappa' = \kappa \quad (\text{all } \kappa), \quad (5.64)$$

Eq. (5.49) can be written in the following explicit form,

$$v\sigma = \frac{g_{LL}^2}{8\pi} \frac{\kappa^2}{|\kappa - \mu_-|\omega_{0+}} \left(\sqrt{(\kappa - \mu_-)^2 + |m|^2} + \mu_- - \kappa \right)^2 \quad \text{all } \kappa, \quad (5.65)$$

which literally exposes the singularity at $\kappa = \mu_-$.

(b) *Case 3b*:

$$\kappa' = 2\mu_- - \kappa \quad (\kappa < 2\mu_-). \quad (5.66)$$

Similarly, the result in this case is

$$v\sigma = \frac{g_{LL}^2}{8\pi} \frac{(2\mu_- - \kappa)^2 |m|^2}{|\kappa - \mu_-|\omega_{0+}} \quad (\kappa' = 2\mu_- - \kappa \quad \text{with } \kappa < 2\mu_-) \quad (5.67)$$

again illustrating the singularity at $\kappa = \mu_-$.

The lesson is that the $(\kappa' - \kappa x)/m_\chi$ term cannot be neglected, and the χ recoil energy in Eq. (5.43) must be taken into account when solving for κ' , when κ is in the region $\kappa \simeq \mu_-$. The above formulas for $v\sigma$ in Case 3a and 3b can be used for κ away from the singularity $\kappa \simeq \mu_-$.

C. Discussion

It is encouraging that the final formulas for $v\sigma$ that we have obtained in the examples considered above, look

compact and simple. However, we should not miss the fact that altogether they describe some rather peculiar effects. First of all the kinematics differs from the one in the standard case (with the vacuum dispersion relations), which can be traced back to the fact that the dispersion relations depend on the helicity and have a nonstandard momentum dependence.

On the opposite side of the case of a heavy χ that we have considered above, is the case of an extremely

relativistic, or massless χ , with the initial f being at rest, in symbols,

$$\chi(\vec{P}) + f(\vec{k} = 0) \rightarrow \chi(\vec{P}') + f(\vec{k}'), \quad (5.68)$$

where the χ particle is extremely relativistic, or massless. Momentum conservation gives the recoil f momentum

$$\vec{k}' = \vec{P} - \vec{P}', \quad (5.69)$$

and the energy conservation condition requires

$$P' + \sqrt{(|\vec{P} - \vec{P}'| - s'\mu_-)^2 + |m|^2} = P + \sqrt{\mu_-^2 + |m|^2}. \quad (5.70)$$

We have carried out some preliminary work to calculate the cross section in this case, which also hints to peculiar and interesting features in its own right.

VI. SUMMARY AND OUTLOOK

In summary, in this work we have considered the determination of the fermion spinors, the corresponding projection matrices and propagators, in the models considered in Ref. [15] for the propagation of fermions interacting with a BE condensate. We have obtained concise formulas which are useful for calculations of process rates involving the fermion modes of the BE condensate. For convenience and future reference, we have summarized in Appendix the formulas for the fermion dispersion relations, spinors, projection operators, and propagators that are useful for reference in future works.

As an application we have considered the calculation of the scattering rate of the fermion mode f of the BE condensate, with a generic fermion which we denoted by χ , using commonly used models for the $f\chi$ interactions. For definiteness, we considered in detail the case in which χ is a heavy fermion, initially at rest. Even in this relatively simple situation the analysis and results illustrate several peculiarities that must be dealt with in this type of calculation involving the f fermion modes. They are a consequence of the fact that their dispersion relations are not the standard vacuum dispersion relations, but actually are helicity dependent, and therefore the kinematics differ from the case in which the fermions propagate in vacuum.

In general terms, the results we have presented here, besides their value in their own right and serving as a guide for treating similar models, can be useful in specific contexts and applications, for example involving neutrino,

electron, or nucleon propagation and interactions with dark matter particles.

ACKNOWLEDGMENTS

The work of S. S. is partially supported by DGAPA-UNAM (Mexico) PAPIIT Project No. IN105326.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: SUMMARY

For convenience we summarize here the main formulas for the fermion dispersion relations, spinors, and propagator.

1. Dispersion relations

$$\begin{aligned} \text{particle: } \omega_s(\vec{k}) &= [(\kappa - s\mu_-)^2 + |m|^2]^{1/2} - \mu_+, \\ \text{antiparticle: } \bar{\omega}_s(\vec{k}) &= [(\kappa - s\mu_-)^2 + |m|^2]^{1/2} + \mu_+, \end{aligned} \quad (A1)$$

with

$$\mu_{\pm} \equiv \frac{1}{2}(\mu_R \pm \mu_L). \quad (A2)$$

2. Spinors

$$\begin{aligned} \text{particle: } u_s(\vec{k}) &= z_s \begin{pmatrix} \sqrt{D_{Ls}} \\ \alpha^* \sqrt{D_{Rs}} \end{pmatrix} \chi_s(\vec{k}) \\ \text{antiparticle: } v_s(\vec{k}) &= \bar{z}_s \begin{pmatrix} \sqrt{\bar{D}_{Ls}} \\ -\alpha^* \sqrt{\bar{D}_{Rs}} \end{pmatrix} \chi_s(-\vec{k}), \end{aligned} \quad (A3)$$

where

$$(\vec{\sigma} \cdot \hat{k}) \chi_s(\vec{k}) = s \chi_s(\vec{k}) \quad (\text{helicity spinor}), \quad (A4)$$

$$\begin{aligned} D_{Ls}(\vec{k}) &= \omega_s + s\kappa + \mu_L, \\ D_{Rs}(\vec{k}) &= \omega_s - s\kappa + \mu_R, \end{aligned} \quad (A5)$$

$$\begin{aligned} \bar{D}_{Ls}(\vec{k}) &= \bar{\omega}_s - s\kappa - \mu_L, \\ \bar{D}_{Rs}(\vec{k}) &= \bar{\omega}_s + s\kappa - \mu_R, \end{aligned} \quad (A6)$$

$$\begin{aligned}
z_s(\vec{k}) &= \left\{ \frac{\omega_s(\vec{k})}{\sqrt{(\kappa - s\mu_-)^2 + |m|^2}} \right\}^{\frac{1}{2}}, \\
\bar{z}_s(\vec{k}) &= \left\{ \frac{\bar{\omega}_s(\vec{k})}{\sqrt{(\kappa - s\mu_-)^2 + |m|^2}} \right\}^{\frac{1}{2}}, \\
\alpha &= \frac{m}{|m|}.
\end{aligned} \tag{A7}$$

3. Projection matrices

$$\begin{aligned}
u_s \bar{u}_s &= z_s^2 \begin{pmatrix} m & D_{Ls} \\ D_{Rs} & m^* \end{pmatrix} P_s(\vec{k}), \\
v_s \bar{v}_s &= \bar{z}_s^2 \begin{pmatrix} -m & \bar{D}_{Ls} \\ \bar{D}_{Rs} & -m^* \end{pmatrix} P_s(-\vec{k}),
\end{aligned} \tag{A8}$$

where

$$P_s(\vec{k}) \equiv \frac{1}{2}(1 + s\vec{\sigma} \cdot \hat{k}). \tag{A9}$$

An alternative form that may be convenient for some calculations is

$$\begin{aligned}
u_s \bar{u}_s &= \frac{z_s^2}{2} (mR + m^*L + D_{Ls}\not{L} + D_{Rs}\not{R})(1 - s\gamma^5\not{n}), \\
v_s \bar{v}_s &= \frac{\bar{z}_s^2}{2} (-mR - m^*L + \bar{D}_{Ls}\not{L} + \bar{D}_{Rs}\not{R})(1 + s\gamma^5\not{n}),
\end{aligned} \tag{A10}$$

where

$$\begin{aligned}
u^\mu &= (1, \vec{0}), \\
n^\mu &= (0, \hat{k}).
\end{aligned} \tag{A11}$$

In the limit $\mu_{R,L} = 0$ and $m^* = m$ these formulas reduce to the standard vacuum formulas, as it should be,

$$\begin{aligned}
u_s \bar{u}_s &= \frac{1}{2}(\not{k} + m)(1 + s\gamma^5\not{\ell}), \\
v_s \bar{v}_s &= \frac{1}{2}(\not{k} - m)(1 + s\gamma^5\not{\ell}),
\end{aligned} \tag{A12}$$

where

$$\not{\ell}^\mu = \frac{1}{m}(\kappa u^\mu + \omega n^\mu) = \frac{1}{m}(\kappa, \omega \hat{k}). \tag{A13}$$

4. Propagator

$$S_{F0} = \sum_s S_s P_s(\vec{k}), \tag{A14}$$

where,

$$\begin{aligned}
S_s &= \frac{N_s}{(\omega - \omega_s + i\epsilon)(\omega + \bar{\omega}_s - i\epsilon)} \\
&= \frac{1}{\omega_s + \bar{\omega}_s} \left[\frac{N_s}{\omega - \omega_s + i\epsilon} - \frac{N_s}{\omega + \bar{\omega}_s - i\epsilon} \right],
\end{aligned} \tag{A15}$$

$$N_s = \begin{pmatrix} m & \omega + \mu_L + s\kappa \\ \omega + \mu_R - s\kappa & m^* \end{pmatrix}. \tag{A16}$$

An alternative form analogous to $u\bar{u}$ and $v\bar{v}$ above, which is more convenient for practical calculations in particular those involving the thermal propagators, is to use

$$P_s = \frac{1}{2}(1 - s\gamma^5\not{n}), \tag{A17}$$

and

$$N_s = (mR + m^*L + E_{Ls}\not{L} + E_{Rs}\not{R}), \tag{A18}$$

where $E_{L,R}$ are the off-shell version of $D_{L,R}$,

$$\begin{aligned}
E_{Ls}(\omega, \vec{k}) &= \omega + s\kappa + \mu_L, \\
E_{Rs}(\omega, \vec{k}) &= \omega - s\kappa + \mu_R.
\end{aligned} \tag{A19}$$

Notice that

$$\begin{aligned}
D_{L,R} &= E_{L,R}(\omega_s, \vec{k}) \\
\bar{D}_{L,R} &= -E_{L,R}(-\bar{\omega}_s, \vec{k}).
\end{aligned} \tag{A20}$$

In summary,

$$\begin{aligned}
S_{F0} &= \sum_s \frac{N_s P_s}{(\omega - \omega_s + i\epsilon)(\omega + \bar{\omega}_s - i\epsilon)} \\
&= \sum_s \frac{1}{\omega_s + \bar{\omega}_s} \left[\frac{N_s P_s}{\omega - \omega_s + i\epsilon} - \frac{N_s P_s}{\omega + \bar{\omega}_s - i\epsilon} \right],
\end{aligned} \tag{A21}$$

with P_s and N_s given in Eqs. (A17) and (A18).

- [1] K. S. Babu, Garv Chauhan, and P. S. Bhupal Dev, Neutrino nonstandard interactions via light scalars in the earth, sun, supernovae and the early universe, *Phys. Rev. D* **101**, 095029 (2020).
- [2] Jose F Nieves and Sarira Sahu, Neutrino effective potential in a fermion and scalar background in the resonance region, *Phys. Rev. D* **105**, 095022 (2022).
- [3] See for example, H. Duan, G. M. Fuller, and Y. Z. Qian, Collective neutrino oscillations, *Annu. Rev. Nucl. Part. Sci.* **60**, 569 (2010).
- [4] S. Chakraborty, R. Hansen, I. Izaguirre, and G. Raffelt, Collective neutrino flavor conversion: Recent developments, *Nucl. Phys.* **B908**, 366 (2016).
- [5] Y. Y. Y. Wong, Analytical treatment of neutrino asymmetry equilibration from flavor oscillations in the early universe, *Phys. Rev. D* **66**, 025015 (2002).
- [6] C. M. Ho, D. Boyanovsky, and H. J. de Vega, Neutrino oscillations in the early universe: A real-time formulation, *Phys. Rev. D* **72**, 085016 (2005).
- [7] Elena Sellentin and Ruth Durrer, Detecting the cosmological neutrino background in the CMB, *Phys. Rev. D* **92**, 063012 (2015).
- [8] Gary Steigman, Neutrinos and big bang nucleosynthesis, *Adv. High Energy Phys.* **2012**, 268321 (2012).
- [9] G. Mangano, G. Miele, S. Pastor, T. Pinto, O. Pisanti, and P. D. Serpico, Effects of nonstandard neutrino-electron interactions on relic neutrino decoupling, *Nucl. Phys.* **B756**, 100 (2006).
- [10] G. Mangano, A. Melchiorri, P. Serra, A. Cooray, and M. Kamionkowski, Cosmological bounds on dark matter-neutrino interactions, *Phys. Rev. D* **74**, 043517 (2006).
- [11] T. Binder, L. Covi, A. Kamada, H. Murayama, T. Takahashi, and N. Yoshida, Matter power spectrum in hidden neutrino interacting dark matter models: A closer look at the collision term, *J. Cosmol. Astropart. Phys.* **11** (2016) 043.
- [12] R. Primulando and P. Uttayarat, Dark matter-neutrino interaction in light of collider and neutrino telescope data, *J. High Energy Phys.* **06** (2018) 026.
- [13] A. Olivares-Del Campo, C. Boehm, S. Palomares-Ruiz, and S. Pascoli, Dark matter-neutrino interactions through the lens of their cosmological implications, *Phys. Rev. D* **97**, 075039 (2018).
- [14] S. Pandey, S. Karmakar, and S. Rakshit, Interactions of astrophysical neutrinos with dark matter: A model building perspective, *J. High Energy Phys.* **01** (2019) 095; **11** (2021) 215(E).
- [15] Jose F. Nieves and Sarira Sahu, Model for the propagation of fermions in a Bose-Einstein condensate, *Phys. Rev. D* **107**, 116012 (2023).
- [16] Raghuvver Garani, Michel H. G. Tytgat, and Jérôme Vandecasteele, Condensed dark matter with a Yukawa interaction, *Phys. Rev. D* **106**, 116003 (2022).
- [17] Kay Kirkpatrick, Anthony E. Mirasola, and Chanda Prescod-Weinstein, Analysis of Bose-Einstein condensation times for self-interacting scalar dark matter, *Phys. Rev. D* **106**, 043512 (2022).
- [18] C. G. Boehmer and T. Harko, Can dark matter be a Bose-Einstein condensate?, *J. Cosmol. Astropart. Phys.* **06** (2007) 025.
- [19] Maria Crăciun and Tiberiu Harko, Testing Bose-Einstein condensate dark matter models with the SPARC galactic rotation curves data, *Eur. Phys. J. C* **80**, 735 (2020).
- [20] Juan Carlos D'Olivo and Jose F Nieves, Damping rate of a fermion in a medium, *Phys. Rev. D* **52**, 2987 (1995).
- [21] Dilip Kumar Ghosh, Tushar Gupta, Matti Heikinheimo, Katri Huitu, and Sk Jeesun, Boosted dark matter driven by cosmic rays and diffuse supernova neutrinos, *Phys. Rev. D* **111**, 063019 (2025).
- [22] Atanu Guha and Jong-Chul Park, Constraints on cosmic-ray boosted dark matter with realistic cross section, *J. Cosmol. Astropart. Phys.* **07** (2024) 074.
- [23] Christopher V. Cappiello, Qinrui Liu, Gopolang Mohlabeng, and Aaron C. Vincent, Cosmic ray-boosted dark matter at IceCube, *Phys. Rev. D* **110**, 095031 (2024).
- [24] Deep Ghosh and Anirban Das, Probing freeze-in dark matter using Bose-Einstein condensate in neutron star, *arXiv:2605.22245*.
- [25] L. Van Hove, The occurrence of singularities in the elastic frequency distribution of a crystal, *Phys. Rev.* **89**, 1189 (1953).
- [26] See also, Wenbin Wu *et al.*, The discovery of three-dimensional van Hove singularity, *Nat. Commun.* **15**, 2313 (2024).
- [27] See, for example, Refs. [28–30].
- [28] H. Arthur Weldon, Chemical potentials in real-time thermal field theory, *Phys. Rev. D* **76**, 125029 (2007).
- [29] Joseph I. Kapusta, Bose-Einstein condensation, spontaneous symmetry breaking, and gauge theories, *Phys. Rev. D* **24**, 426 (1981).
- [30] Howard E. Haber and H. Arthur Weldon, Finite-temperature symmetry breaking as Bose-Einstein condensation, *Phys. Rev. D* **25**, 502 (1982).