

Bulk viscosity and n -component fluids

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Understanding the hydrodynamics of out-of-equilibrium dense viscous fluids is of key importance to accurate descriptions of physical systems such as compact stars, particularly their mergers. We consider a near-equilibrium relativistic fluid with n independent and small chemical potentials restoring the system back toward equilibrium. By diagonalizing the evolution equation for the out-of-equilibrium chemical potentials, we construct an explicit evolution equation for the bulk scalar of the system in second-order hydrodynamics in terms of equilibrium quantities. We find expressions for the $2n$ transport quantities and show that, in the rest frame of the fluid, the system admits a Green's function corresponding to an n -component fluid.

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I. INTRODUCTION

There are many open questions in the field of transport phenomena in dense relativistic fluids, with many existing studies focusing on a specific equation of state for the fluid. Such systems are found in nature, with a prominent example being neutron star mergers observed through their gravitational wave signal [1,2]. Neutron star mergers provide an environment where a dense near-equilibrium fluid is being brought toward equilibrium by chemical reactions. Accurate descriptions of the underlying phenomena are one of the key steps toward properly understanding dense, strongly interacting systems and especially transport in such matter [3].

In this manuscript, we are particularly interested in the case where a repeated compression-decompression cycle causes energy dissipation due to a finite bulk viscosity, which consequently has an effect on the observed merger signal [4–8]. The qualitative behavior of the bulk viscosity is known to be different for different types of matter present in neutron stars, namely, for hadronic matter and quark matter [9–13]—the transport properties depend on the equation of state of the system as well as the rates of the relevant chemical reactions.

With a simple system involving few particle species, the relevant equations derived using relativistic hydrodynamics are largely manageable, but quickly become unwieldy for a

more complicated system if treated directly [9,10,14]. A large number of particle species appears, in particular, in the hadronic phase, where the number of potentially active composite particle types can be significant.

In this manuscript, we will be rather general: Instead of selecting a specific equation of state—or even necessarily one associated with neutron stars—we consider a general dense fluid with n independent chemical potentials associated with reactions bringing the system toward equilibrium. By doing so, we provide a general framework for analyzing relativistic bulk viscous fluids in the linearized regime irrespective of the number of particle species. We derive the relevant evolution equation for such systems and show that under certain general conditions they factorize into n subsystems. We also find expressions for the associated transport coefficients in terms of equilibrium quantities. The derivation as well as the equations themselves are useful not only for a better grasp of the general behavior of relativistic dense viscous fluids, but also for describing complicated systems where n is larger than 1 or 2, important for a more nuanced understanding of neutron star merger dynamics.

The paper is structured as follows: In Sec. II we first define precisely what characterizes our near-equilibrium system. This includes writing down a system of evolution equations for the set of chemical potentials associated with the chemical reactions bringing the system toward equilibrium. Next, in Sec. III we diagonalize the set of equations to bring it into a solvable form. In Sec. IV, we write down the equation for the bulk scalar, finding the associated Green's function as well as the relevant transport coefficients in terms of equilibrium quantities. As the notation used can be somewhat abstract, in each of the Secs. II–IV we complement the general definitions and derivations with a corresponding example for $n = 2$, associated with (unpaired)

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quark matter (QM) in the absence of neutrinos. We will be rather superficial when it comes to the phenomenology of this particular system, mainly focusing on how the general formulas used and derived in this work appear in it, and encourage interested readers to refer to [15], which the example discussion is based on. We emphasize that this is purely for illustrative purposes, and the treatment, in general, is applicable for a much wider class of systems. Finally, in Sec. V we discuss the results.

II. LINEARIZED n -COMPONENT FLUID

In this section, we outline a general formulation for the relevant physical system. We adapt a version of the notation of Carter [16] in order to provide general expressions and give explicit expressions in the QM case. We use the summation convention between upper and lower indices. Consider a system of $k > 1$ particle species $\{p^a\}_{a \in K}$, $|K| = k$, with associated chemical potentials $\{\mu^a\}_{a \in K}$ whose chemical equilibrium has been disturbed and is being restored by a set of chemical reactions $\mathcal{C}$. The disturbance of equilibrium manifests as the chemical potentials becoming dependent on a spacetime point. For some $C \in \mathcal{C}$ the chemical reactions take the form $x_a^C p^a \rightarrow \tilde{x}_a^C p^a$ (they also come with a corresponding set of reverse reactions $C' \in \mathcal{C}'$ of the form $x_a^{C'} p^a \leftarrow \tilde{x}_b^{C'} p^b$; in introducing this notation, we have implicitly fixed what is the “forward” and the “backward” reaction, fixing a sign ambiguity in the chemical potentials.) for some integer (“stoichiometric”) coefficients¹ $x_a^C, \tilde{x}_a^C$ ($x_a^{C'}, \tilde{x}_a^{C'}$), which we set to zero for particle species not appearing in the reaction. We use uppercase Latin indices to refer to the various chemical reactions ($A, B, C, \dots$), while lowercase Latin indices ($a, b, c, \dots$) enumerate particle species, with contravariant indices corresponding to chemical potentials and, later on, covariant indices corresponding to their conjugate particle densities.

QM example. $K = \{u, d, s, e(\nu)\}$ and the relevant reactions are

$$\begin{aligned} C_1: u + s &\longleftrightarrow u + d, & C_2: u + e &\longrightarrow d + (\nu), \\ C_3: u + e &\longrightarrow s + (\nu), \end{aligned} \quad (1)$$

where, for simplicity of presentation, we consider a scenario where the effect of the neutrinos (ν) is absent,² so that $k = 4$. Thus, abbreviating C_i as i in the reaction indices, we find the following coefficients:

$$\begin{aligned} \{x_u^1, x_d^1, x_s^1, x_e^1\} &= \{1, 0, 1, 0\}, \\ \{\tilde{x}_u^1, \tilde{x}_d^1, \tilde{x}_s^1, \tilde{x}_e^1\} &= \{1, 1, 0, 0\}; \\ \{x_u^2, x_d^2, x_s^2, x_e^2\} &= \{1, 0, 0, 1\}, \\ \{\tilde{x}_u^2, \tilde{x}_d^2, \tilde{x}_s^2, \tilde{x}_e^2\} &= \{0, 1, 0, 0\}; \\ \{x_u^3, x_d^3, x_s^3, x_e^3\} &= \{1, 0, 0, 1\}, \\ \{\tilde{x}_u^3, \tilde{x}_d^3, \tilde{x}_s^3, \tilde{x}_e^3\} &= \{0, 0, 1, 0\}. \end{aligned} \quad (2)$$

The system becomes simple to treat if we assume it to be near equilibrium. That is to say, we assume that the reaction rates $\Gamma^C - \Gamma^{C'}$ can be linearized in some small (in a sense to be established momentarily) linear combinations of the chemical potentials μ^a . Note that specifying the inverse reaction C' here determines the sign of the chemical potential. The linearized expansion is most conveniently rewritten using small reaction chemical potentials³ $\tilde{\mu}^A$ describing deviation from chemical equilibrium associated with a specific reaction. Then, using the corresponding linearized reaction rates λ associated with a specific chemical reaction C and its inverse C' , one may write

$$\begin{aligned} \tilde{\mu}^C &\equiv (x_a^C - \tilde{x}_a^C)\mu^a, \\ \Gamma^C - \Gamma^{C'} &= \lambda_A^C \tilde{\mu}^A = \lambda_A^C (x_a^A - \tilde{x}_a^A)\mu^a. \end{aligned} \quad (3)$$

Note that the matrix λ_A^C is diagonal, whereas the same expression written using the chemical potentials μ^a would generally involve a nondiagonal matrix. For equations involving chemical potentials and related linearized quantities (notably particle densities), we always drop the remainder terms going beyond the linear approximation for notational simplicity; everything is derived assuming linearized chemical potentials. To make this more precise, the reaction chemical potentials are small in the sense that we require⁴ that $\tilde{\mu}^C/\mu^a \ll 1 \forall C \in \mathcal{C}, a \in K$. This also implies that the chemical potentials $\{\mu^a\}_{a \in K}$ can be expanded around an equilibrium value. That is to say, $\forall a \in K, \mu^a = \mu_0^a + \delta\mu^a$, where μ_0^a corresponds to a value obtained by fixing the constraint $\tilde{\mu}^C = 0 \forall C \in \mathcal{C}$, so that, in fact, in Eq. (3) one may replace each μ^a with $\delta\mu^a$ and take $\delta\mu^a/\mu_0^a \ll 1 \forall a \in K$ (no summation convention). In what follows, decorated ($\tilde{\cdot}, \hat{\cdot}$) chemical potentials are always taken to be small, as in, expanded to linear order around a vanishing equilibrium value, as are the deviations $\delta\mu$ that, in practice, appear only rarely. The importance of the near-equilibrium expansion is that the equilibrium chemical potentials $\{\mu_0^a\}_{a \in K}$ carry no dependence on the space-time point.

¹Often, these coefficients are all equal to either 1 or 0.

²In the physical system in question, this is justified at sufficiently low temperatures, below $O(10 \text{ MeV})$ or so, in the so-called neutrino-transparent regime. For more details, please see Ref. [15].

³Also known as “reaction affinities.”

⁴It is often possible to relax this condition by requiring that the reaction chemical potentials $\{\tilde{\mu}^C\}_{C \in \mathcal{C}}$ are smaller than only some of the chemical potentials associated with the particle species $\{\mu^a\}_{a \in K}$.

QM example. The linearized reaction chemical potentials are

$$\tilde{\mu}^1 \equiv \mu^u + \mu^s - \mu^u - \mu^d = \mu^s - \mu^d, \quad (4)$$

$$\tilde{\mu}^2 \equiv \mu^u + \mu^e - \mu^d, \quad (5)$$

$$\tilde{\mu}^3 \equiv \mu^u + \mu^e - \mu^s, \quad (6)$$

where each of the μ^a could be replaced with a $\delta\mu^a$.

At this point, there is still remaining degeneracy, as the reaction chemical potentials $\tilde{\mu}^A$ are generally not independent. Suppose now that there are $n \leq |C| = |C'|$ independent chemical potentials, in the sense that any possibly remaining $n - |C|$ rates can be written as linear combinations of the n independent chemical potentials using some real-valued coefficients c_α^A , determined by the form of the chemical reactions. While, in practice, it is convenient to choose this basis of n chemical potentials to be maximally coincident with the reaction chemical potentials $\tilde{\mu}^A$, for example, by fixing $c_\alpha^A = \delta_\alpha^A$ for $A \leq n$, to be general we allow for generic choices of the c_α^A , which can then be collected together to form a $|C| \times n$ matrix, determined by the form of the chemical reactions. We write, finally,

$$\Gamma^C - \Gamma^{C'} \equiv \lambda_A^C c_\alpha^A \hat{\mu}^\alpha, \quad (7)$$

where the basis elements $\hat{\mu}^\alpha$ that span the n -dimensional space can be collected together into an n -dimensional vector $\hat{\mu} = (\hat{\mu}^1, \dots, \hat{\mu}^n)$ of independent reaction chemical potentials. These are the degrees of freedom we use to measure deviation from equilibrium going forward, following the elimination of the $|C| - n$ superfluous reaction chemical potentials. To distinguish the independent degrees of freedom $\hat{\mu}^\alpha$ from the reaction chemical potentials $\tilde{\mu}^A$ labeled by uppercase Latin indices, and we label them by lowercase Greek indices from the start of the alphabet ($\alpha, \beta, \gamma, \dots$). The independent chemical potentials $\hat{\mu}^\alpha$ will appear in the hydrodynamic equations below and will be a central quantity, and by the end of this section, we will obtain a hydrodynamic evolution equation for it. In practice, once we move to the hydrodynamic equations, it will become convenient to use vector notation (e.g., $\hat{\mu}$) for n -dimensional quantities instead.

QM example. Noting that $\tilde{\mu}^3 = \tilde{\mu}^2 - \tilde{\mu}^1$, we take the linearly independent chemical potentials to be $\hat{\mu} = (\hat{\mu}^1, \hat{\mu}^2) = (\tilde{\mu}^1, \tilde{\mu}^2)$. We emphasize that this choice is arbitrary. At this point, we have accounted for the fact that the third chemical potential is linearly dependent and can use a simpler notation for the two independent chemical potentials. We indeed have $n = 2$, that is to say, two independent chemical potentials, and we find $c_\alpha^C = \delta_\alpha^C$ for $\alpha \in \{1, 2\}$, as well as $c_1^3 = -1, c_2^3 = +1$.

To make contact with hydrodynamics, consider a fluid with a velocity field u , and write $\vartheta = \text{div} u = \nabla_\nu u^\nu$,

$D_u = u^\nu \nabla_\nu$, where the $\{\nabla_\nu\}_{\nu=0}^d$ are spacetime derivatives.⁵ To fix conventions, we use the mostly minus metric, although the signature only enters into expressions obtained in fixed frames, and use the summation convention for spacetime indices. In second-order hydrodynamics, ϑ is formally treated as an independent variable in the evolution equations. Nevertheless, it is associated with the conserved Noether current j_a of particle numbers and connected to them via a continuity equation. We have

$$\begin{aligned} j_a &\equiv \text{div} n_a^{\text{noneq.}} u \equiv u^\nu \nabla_\nu n_a^{\text{noneq.}} + n_a^{\text{noneq.}} \nabla_\nu u^\nu \\ &\equiv D_u n_a^{\text{noneq.}} + n_a^{\text{noneq.}} \vartheta \quad \forall a \in K, \end{aligned} \quad (8)$$

which we will use in the derivations below (and where $\{n_a^{\text{noneq.}}\}_{a \in K}$ are nonequilibrium particle densities conjugate to the chemical potentials $\{\mu^a\}_{a \in K}$ and defined explicitly shortly). In the absence of rates, $j_a \equiv 0$, but the presence of rates will introduce a source term, as we shall see below.

On the other hand, another central quantity, the pressure of the system in equilibrium p , depends on the equilibrium chemical potentials $\{\mu_0^a\}_{a \in K}$, the equilibrium temperature T , and possible other variables that are independent of the spacetime point, such as particle masses. The bulk scalar of the system $\Pi = p_{\text{noneq.}} - p$ measures deviation from equilibrium in terms of the nonequilibrium pressure $p_{\text{noneq.}}$ and must take the form $\Pi = \Pi_\alpha \hat{\mu}^\alpha \equiv \langle \Pi, \hat{\mu} \rangle$ in the near-equilibrium linearized approximation, where Π_α are coefficients depending on equilibrium quantities exclusively. The equilibrium densities n_a conjugate to the equilibrium chemical potentials μ_0^a and the (generally nondiagonal but symmetric) susceptibilities, all in equilibrium, are given by $n_a \equiv \partial_a p, \chi_{ab} \equiv \partial_a \partial_b p$, respectively. Here $\partial_a \equiv \partial/\partial \mu_0^a$ are partial derivatives taken with respect to the equilibrium particle chemical potentials.

Importantly, in the definition of Π one may fully incorporate other constraints such as conservation laws that may exist in the system besides those given by the chemical reactions and the continuity equation. This is vital for treating physical systems, but does not affect the derivation presented here, so the exact form of these constraints and the Π is irrelevant for our present purposes. Nevertheless, it is the evolution equation of Π we are after.

QM example. Writing $\delta\mu^a$ for the deviations of the particle chemical potentials from their equilibrium values, the additional constraints fulfilled in QM can be written in the linearized regime as

$$(\chi_{ea} - \chi_{ua})\delta\mu^a = 0, \quad (\chi_{da} + \chi_{sa} - \chi_{ea})\delta\mu^a = 0. \quad (9)$$

⁵We work in flat spacetime, but use the notation ∇_ν to distinguish the spacetime derivatives from those taken with respect to the chemical potentials. To further eliminate the chance for confusion, we strive to use index-free notation when dealing with the spacetime derivatives.

These correspond to “charge neutrality” and “ β equilibrium,” respectively, with their equilibrium counterparts being $\mu_0^d = \mu_0^s = \mu_0^u + \mu_0^e$ and $2n_u - n_d - n_s - 3n_e = 0$. We note that in this special case these conditions imply that $\mu^u \approx \mu^d \approx \mu^s \gg \mu^e$. In QM, the two components Π_1 , Π_2 of the bulk scalar as well as the equilibrium susceptibilities and densities Π depends on can be expressed uniquely in terms of a single equilibrium chemical potential (and the temperature T)—for explicit expressions in QM, see the Appendix of [15]. Such a representation is not always possible, but it should not be taken as a problem as such; it simply means that the system is characterized by more than one free chemical potential. To reiterate, while the constraints and especially their practical implementation can be somewhat subtle, they are not relevant for the derivation presented in this work and are only mentioned for completeness for applying the results to specific physical systems.

Following [9,15], we next write down an evolution equation for the reaction chemical potentials. Generally, the near-equilibrium chemical potentials as well as the velocity u itself (or, specifically, its divergence ϑ) are taken to be functions $\mathbb{M}^{d+1} \rightarrow \mathbb{R}$. To begin, we note that the linearized rates of the chemical reaction disturb equilibrium by adding a source term to the continuity equation of the nonequilibrium particle densities $n_a^{\text{noneq.}}$. They are defined as the derivatives of the pressure with respect to the nonequilibrium chemical potentials $\partial p_{\text{noneq.}}/\partial \mu^a$. That is, we must have

$$j_a = \text{div} n_a^{\text{noneq.}} u = -(x_a^C - \tilde{x}_a^C) \delta_{CB} \lambda_A^B c_a^A \hat{\mu}^\alpha \equiv (\hat{\lambda} \hat{\mu})_a, \quad (10)$$

where we have defined the new (real-valued) quantity $\hat{\lambda}$, a $k \times n$ matrix of coefficients⁶ consisting of linearized production rates. Next, we note that in the linear regime the expansions

$$p^{\text{noneq.}} - p = n_a \delta \mu^a, \quad n_a^{\text{noneq.}} - n_a = \chi_{ab} \delta \mu^b \quad (11)$$

hold $\forall a, b \in K$ (recall that we always omit remainder terms). The third step is to connect the continuity equation (10) to the linearized expansions and write

$$\begin{aligned} \text{div} n_a^{\text{noneq.}} u &= D_u (\chi_{ab} \delta \mu^b) + \vartheta (n_a - \chi_{ab} \delta \mu^b) \\ &= \chi_{ab} D_u \mu^b + \vartheta n_a = (\hat{\lambda} \hat{\mu})_a. \end{aligned} \quad (12)$$

Here, the remainder term proportional to $\vartheta \delta \mu$ was discarded and, on the other hand, under the derivative we could replace $D_u \delta \mu^b$ by $D_u \mu^b$. We are almost done, but in order to get a convenient form for our purposes, we isolate the

⁶The overall sign is largely conventional and chosen to agree with [15] and to counteract the overall sign of Ω .

$\hat{\mu}$ dependence. To do so, we find a (symmetric) $k \times k$ matrix⁷ M such that $M^{ca} \chi_{ab} \mu^b$ is a vector of linear combinations of particle chemical potentials μ^a whose first $n \leq k$ components coincide with the values of the $\hat{\mu}^\alpha$. In this process, we find a rotated basis of k linearly independent chemical potentials whose first n components form $\hat{\mu}$. In principle, this construction can be realized in many ways, so long as the resulting basis is complete, but to be explicit we outline a specific method. Having chosen $c_a^A = \delta_a^A$ for $A \leq n$ for demonstrative purposes, we use its trivial inverse to define $M^{ab} = c_c^a (x_c^C - \tilde{x}_c^C) (\chi^{-1})^{cb}$ for the first n rows, $\alpha \leq n$, and simply set $M^{ab} = (\chi^{-1})^{ab}$ for the remaining $k \geq a > n$.

By multiplying the resulting expression by M and moving the θ -term to the right-hand side, we obtain

$$D_u M^{ab} \chi_{bc} \mu^c = M^{ab} \hat{\lambda}_{ba} \hat{\mu}^\alpha - \vartheta M^{ab} n_b. \quad (13)$$

In practice, we are only interested in the first n components of Eq. (13), as those are the ones appearing in Π , and by construction the left-hand side then restricts to $D_u \hat{\mu}$. Writing $\hat{M} = (M^{aa})$ for the restriction of M to its first n rows (so that $\hat{M}$ is an $n \times k$ matrix), we define an $n \times n$ matrix Ω via $\Omega \equiv -\hat{M} \hat{\lambda}$ ($\Omega_\beta^\alpha = -\hat{M}^{aa} \hat{\lambda}_{a\beta}$) and an auxiliary n -dimensional vector⁸ $z = (z^\alpha)$ by $z^\alpha \equiv (\Omega^{-1})^\alpha_\beta \hat{M}^{\beta a} n_a$.

Connecting these quantities to physics, the matrix Ω has units of $[\text{time}]^{-1}$ and its eigenvalues will be associated with the (inverse) relaxation times appearing in the fluid description, whereas z has units of $[\text{chemical potential}]^2$, and will be associated with the bulk viscous transport coefficients. Note that, despite the minus sign appearing in the definition of Ω , λ is constructed such that for the relevant physical systems the eigenvalues of Ω are positive.

QM example. The rotated basis is $\{\hat{\mu}^1, \hat{\mu}^2, \mu^s, \mu^e\}$, and $n_a = (n_u, n_d, n_s, n_e)$. The matrix M extracts the $\hat{\mu}$ dependence by connecting the bases $\{\mu^u, \mu^d, \mu^s, \mu^e\}$ and $\{\hat{\mu}^1, \hat{\mu}^2, \mu^s, \mu^e\}$ and reads for $a \in \{u, d, s, e\}$,

⁷We abuse notation slightly here, and use lowercase Latin indices to index M , even though the resulting quantity will mix reaction chemical potentials and particle chemical potentials in its covariant indices. However, the n independent reaction chemical potentials $\hat{\mu}^\alpha$ naturally embed to span a subspace of the larger space spanned by the k particle chemical potentials μ^a . More rigorously, one could take M to be a map to (a subspace of) the Cartesian product of the two spaces and introduce a fifth index type, but we opt against this as it would further clutter the notation, and we are ultimately only interested in one of the factors of this product space.

⁸We do not include ϑ in the definition of z , as in the case relevant for us, we will have $D_u z = 0$ but $D_u \vartheta \neq 0$. Note also that $\hat{M}, \lambda$ are generally nonsquare matrices, so that their individual inverses are not defined, and only the inverse of Ω is, so that the expression for z cannot generally be simplified further.

TABLE I. Summary of the most important notation used in Sec. II.

Symbol	Meaning
k	Number of particle species enumerated by $\{p_a\}_{a \in K}$
$\mathcal{C}$ ($\mathcal{C}'$)	Set of (inverse) chemical reactions responsible for equilibration
x_a^C ($\tilde{x}_b^{C'}$)	Weight of particle species a (b) in (inverse) chemical reaction $C \in \mathcal{C}$ ($C' \in \mathcal{C}'$)
n	Number of independent chemical potential degrees of freedom
μ^a	Chemical potentials of individual particle species split into equilibrium and nonequilibrium values by $\mu^a = \mu_0^a + \delta\mu^a$
$\tilde{\mu}^C$	Reaction chemical potential measuring deviation from equilibrium for reaction C expressed as a linear combination of the μ^a
$\hat{\mu} = (\hat{\mu}^\alpha)$	Vector of linearly independent reaction chemical potentials indexed by $\alpha \in \{1 \dots n\}$
c_a^C	Constants for expressing $\tilde{\mu}^C$ using $\hat{\mu}^\alpha$
λ_C	Linearized reaction rate associated with the reaction $C \in \mathcal{C}$ and the corresponding inverse reaction $C' \in \mathcal{C}'$
u, D_u, ϑ	Four-velocity of the fluid, convective derivative along it, and its divergence
Π	Bulk scalar quantifying the deviation of pressure $p_{\text{noneq.}}$ from equilibrium p
n_a, χ_{ab}	Particle densities and susceptibilities in equilibrium
M, λ	Coefficient matrices of inverse susceptibilities and reaction rates
Ω, z	Combinations of equilibrium quantities appearing in the evolution equation

$$M = \begin{pmatrix} (\chi^{-1})^{1u} & (\chi^{-1})^{1d} & (\chi^{-1})^{1s} & (\chi^{-1})^{1e} \\ (\chi^{-1})^{2u} & (\chi^{-1})^{2d} & (\chi^{-1})^{2s} & (\chi^{-1})^{2e} \\ (\chi^{-1})^{su} & (\chi^{-1})^{sd} & (\chi^{-1})^{ss} & (\chi^{-1})^{se} \\ (\chi^{-1})^{eu} & (\chi^{-1})^{ed} & (\chi^{-1})^{es} & (\chi^{-1})^{ee} \end{pmatrix},$$

$$(\chi^{-1})^{1a} \equiv (\chi^{-1})^{sa} - (\chi^{-1})^{da},$$

$$(\chi^{-1})^{2a} \equiv (\chi^{-1})^{ua} + (\chi^{-1})^{ea} - (\chi^{-1})^{da}. \quad (14)$$

Finally, the matrix of rates is

$$\hat{\lambda} = \begin{pmatrix} \lambda_3 & -\lambda_2 - \lambda_3 \\ \lambda_1 & \lambda_2 \\ -\lambda_1 - \lambda_3 & \lambda_3 \\ \lambda_3 & -\lambda_2 - \lambda_3 \end{pmatrix}. \quad (15)$$

Here λ_i are the elements of the diagonal matrix λ_A^B , and each correspond to the linearized rates of the reactions C_i , $i \in \{1, 2, 3\}$, defined in earlier examples. The quantities Ω, z obtained by matrix multiplication do not simplify in a meaningful way in the special case of QM.

Considering the first n components, Eq. (13) becomes

$$(\Omega^{-1} D_u + \text{id}_n) \hat{\mu} = -z \vartheta \Leftrightarrow \Omega \hat{\mu} = -\Omega z \vartheta - D_u \hat{\mu}, \quad (16)$$

where both forms will prove useful. From now on, we use vector notation almost exclusively, as we have exclusively moved to n -component quantities and will suspend the summation convention, as we encounter more nontensorial super- and subscripts. Here (already in the definition of z) and from now on, we have implicitly assumed that the matrix Ω is invertible. If this is the case, one can factorize the system of equations for $\hat{\mu}$. In practice, this is true as long

as the rates λ are all nonvanishing, unless the pressure takes a particularly degenerate form. For a physical system, the degeneracy of Ω also essentially implies that one is considering extraneous degrees of freedom: If Ω is r -fold degenerate, one is, in fact, dealing with only $n - (r - 1)$ independent chemical potentials.

To finish this section, we summarize the main notation used in the manuscript in Table I.

III. DIAGONALIZING THE DIFFERENTIAL EQUATION

We are now ready to manipulate the system of differential equations. Note that what follows makes no assumptions on the specific form or physical meaning of the quantities considered, other than that the square matrices are non-degenerate and the relevant functions otherwise sufficiently well behaved (that everything is well defined in the relevant regime of temperatures and chemical potentials and at all spacetime points, in particular, that ϑ is C^n smooth, and eventually that D_u commutes with the coefficient matrix). Thus, the approach is valid for any system of differential equations of the above form.

To start, we consider the characteristic polynomial of an invertible real-valued $n \times n$ matrix Ω . To remind the reader, it is given by

$$p_\Omega: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}[x]: \Omega \mapsto p_\Omega(x) = \sum_{i=0}^n c_i[\Omega] x^k$$

$$= \prod_{\omega \in \Sigma[\Omega]} (x - \omega), \quad (17)$$

where $\Sigma[\Omega]$ is the set of eigenvalues of Ω and the $n + 1$ coefficients $\{c_i[\Omega]\}_{i=0}^n$ are formed of invariants of Ω and its

powers. For example, $c_n[\Omega] \equiv 1$, $c_0[\Omega] \equiv (-1)^n \text{Det}\Omega$. While there exists no simple formula for the c_i valid for all i , their exact form is not of particular interest until we fix n , only that they are not degenerate. We denote $\tilde{p}_\Omega(\Omega)\Omega = p_\Omega(\Omega) - c_0[\Omega]\text{id}_n$. That is, $\tilde{p}_\Omega(\Omega) = \sum_{i=1}^n c_i[\Omega]\Omega^{i-1}$. Then, by Cayley-Hamilton

$$\tilde{p}_\Omega(\Omega)\Omega = -c_0[\Omega]\text{id}_n \Leftrightarrow \Omega^{-1} = -c_0^{-1}[\Omega]\tilde{p}_\Omega(\Omega). \quad (18)$$

QM example. For $n = 2$, we have $c_0[\Omega] = \text{Det}\Omega^{-1}$, $c_1[\Omega] = -\text{Tr}\Omega$, $c_2[\Omega] = 1$, so that the statement of the Cayley-Hamilton theorem becomes

$$\begin{aligned} \Omega^2 - \text{Tr}\Omega\Omega + \text{Det}\Omega &= 0 \\ \Leftrightarrow \Omega^{-1} &= -\frac{1}{\text{Det}\Omega}(\Omega - \text{Tr}\Omega\text{id}_2) = \text{Tr}\Omega^{-1}\text{id}_2 - \text{Det}\Omega^{-1}\Omega. \end{aligned} \quad (19)$$

Using the representation of Eq. (18), we can write the set of differential equations of the form shown in (16) as an n th-order set of differential equations diagonalized for the $\hat{\mu}$, assuming that $[D_u, \Omega] = 0$, which for the physical case holds since Ω contains only equilibrium quantities. First, we note that for any $i \geq 1$

$$D_u\Omega^i\hat{\mu} = \Omega^{i-1}D_u\Omega\hat{\mu} = -D_u\Omega^i\mathbf{z}\vartheta - D_u^2\Omega^{i-1}\hat{\mu}. \quad (20)$$

This recursion gives

$$D_u\Omega^i\hat{\mu} = (-1)^i D_u^{i+1}\hat{\mu} + \sum_{j=0}^{i-1} (-1)^{j+1} D_u^{j+1}\Omega^{i-j}\mathbf{z}\vartheta. \quad (21)$$

Substituting this to Eq. (16) by using Eq. (18) and extracting the $i = 1$ -term out of the characteristic polynomial we have

than Eq. (23), reading immediately

$$\begin{aligned} c_2[\Omega]D_u^2\hat{\mu} - c_1[\Omega]D_u\hat{\mu} + c_0[\Omega]\hat{\mu} &= -c_0[\Omega]\mathbf{z}\vartheta - c_2[\Omega]\Omega\mathbf{z}D_u\vartheta \\ \Leftrightarrow D_u^2\hat{\mu} + \text{Tr}\Omega D_u\hat{\mu} + \text{Det}\Omega\hat{\mu} &= -\text{Det}\Omega\mathbf{z}\vartheta - \Omega\mathbf{z}D_u\vartheta. \end{aligned} \quad (24)$$

IV. BULK SCALAR EVOLUTION AND TRANSPORT COEFFICIENTS

To understand the underlying physical system, we are looking for an evolution equation for the bulk scalar. We contract the vector-valued Eq. (23) with the vector Π . For reasons of convention and to get the appropriate units, we rescale the equation by multiplying it with $\text{Det}\Omega^{-1}$; this, in particular, sets the coefficient of Π without derivatives to unity for any n . The result is the following differential equation for the bulk scalar (again, we emphasize that it is only the $\{\hat{\mu}^\alpha\}_{\alpha=1}^n$ and ϑ that are acted on nontrivially by the derivatives; the components of Π appearing in the scalar products are given in terms of equilibrium quantities and commute with the convective derivative D_u):

$$\begin{aligned} \text{Det}\Omega^{-1}p_\Omega(-D_u)\Pi &= -\text{Det}\Omega^{-1}\langle\Pi, (\Omega + D_u\text{id}_n)^{-1}\Omega\mathbf{z}\rangle[p_\Omega(-D_u)\vartheta] \\ \Rightarrow \sum_{\alpha=1}^n T_\alpha D_u^\alpha \Pi + \Pi &= -\sum_{\alpha=1}^n Z_\alpha D_u^{\alpha-1}\vartheta. \end{aligned} \quad (25)$$

$$\begin{aligned} \sum_{i=2}^n c_i[\Omega](-1)^i D_u^i\hat{\mu} - c_1[\Omega]D_u\hat{\mu} + c_0[\Omega]\hat{\mu} \\ = -c_0[\Omega]\mathbf{z}\vartheta - \sum_{i=2}^n c_i[\Omega] \sum_{j=0}^{i-2} (-1)^j D_u^{j+1}\Omega^{i-j-1}\mathbf{z}\vartheta. \end{aligned} \quad (22)$$

Equation (22) is already of the desired form. However, we may make it more visually appealing by explicitly extracting the characteristic polynomial. We formally write $\sum_{j=0}^{i-2} (-1)^j \Omega^{i-j-1} D_u^{j+1} = (\Omega + D_u\text{id}_n)^{-1}[\Omega^i D_u + (-1)^i \Omega D_u^i]$, which equals zero for $i = 1$ and the identity for $i = 0$, to obtain

$$\begin{aligned} 0 &= \sum_{i=0}^n c_i[\Omega](-D_u)^i\hat{\mu} + \frac{1}{D_u\text{id}_n + \Omega} \\ &\times \left(\sum_{i=0}^n c_i[\Omega](-D_u)^i\Omega + D_u \sum_{i=0}^n c_i[\Omega]\Omega^i \right) \mathbf{z}\vartheta \\ &= p_\Omega(-D_u)[\hat{\mu} + (D_u\text{id}_n + \Omega)^{-1}\Omega\mathbf{z}\vartheta]. \end{aligned} \quad (23)$$

Above, $p_\Omega(-D_u)$ is to be understood as an n th-order differential operator whose coefficients are the coefficients of the characteristic polynomial with an alternating sign, and importantly the inverse operator appearing in the equation is nothing nonlocal, but simply a finite (double) sum of the powers of Ω and D_u acting on $\mathbf{z}\vartheta$. The system of equations is now diagonalized for the $\hat{\mu}$, and easy to treat with standard methods.

QM example. When $n = 2$, the general formula for the derivatives is not needed, only that $D_u\Omega\hat{\mu} = -\Omega\mathbf{z}D_u\vartheta - D_u^2\hat{\mu}$. In addition, Eq. (22) is more convenient

Above, we have implicitly defined $2n$ transport coefficients as T_α , the n combinations of relaxation times and Z_α , the n bulk viscous coefficients. They are all given in terms of equilibrium quantities. While the evolution equation (25) is already of a usable form for, e.g., numerical simulations of viscous systems, we can obtain further qualitative understanding of the behavior of the corresponding physical systems. Because of the structure of the $\{c_\alpha(\Omega)\}_{\alpha=1}^n$, the form of the constant terms and the highest derivatives on both sides of the equation is always fixed: 1 and $\text{Det}\Omega^{-1}$ on the left-hand side, $\langle \Pi, z \rangle$ and $\text{Det}\Omega^{-1} \langle \Pi, \Omega z \rangle$ on the right-hand side, respectively. More generally, at any fixed $n \geq 1$ and $i \in \{1, \dots, n\}$ the coefficient of $D_u^i \Pi$ ($D_u^i \vartheta$) equals the coefficient of $D_u^{i+1} \Pi$ ($D_u^{i+1} \vartheta$) as $n \mapsto n+1$ so that only the coefficient of the first derivative is new at a given order.

Equation (25) is an n th-order linear differential equation with constant, and by assumption nondegenerate, coefficients. This already implies that the general solution will be a superposition of exponentials. Still, to find an explicit solution, we consider the rest frame, so that $D_u \rightarrow \nabla_t$ (recall that ∇ is used purely to avoid confusion with the thermodynamic derivatives, and in the typical flat-space scenario this can be thought of as an ordinary partial derivative), in the linearized approximation (but we treat ϑ as an independent variable in the spirit of second-order hydrodynamics). As long as the derivatives $\nabla_t^a \vartheta$ are all piecewise continuous on $\mathbb{R}$, the existence of a unique solution to the corresponding initial-value problem is guaranteed, and in particular, the system admits a Green's function analysis. As a linear ordinary differential equation with constant coefficients, the form of the Green's function is well known. The homogeneous equation admits n linearly independent exponential solutions, and the Green's function is their sum multiplied by the Heaviside function $\theta(t)$. For the physical system under consideration, we look for decaying solutions so that the general form is

$$G \equiv \sum_\alpha G_\alpha: \mathbb{R} \rightarrow \mathbb{R} \quad \text{such that} \\ t \mapsto G(t) \equiv \theta(t) \sum_{\alpha=1}^n \frac{\zeta_\alpha}{\tau_\alpha} e^{-t/\tau_\alpha}. \quad (26)$$

The physical implication of the form in Eq. (26) is that the system is in a superposition of n separate fluid components, each with a relaxation time τ_α and a bulk viscosity ζ_α , and each obeying the Israel-Stewart equation for bulk viscosity, the $n=1$ counterpart of the evolution equation (23) $\tau D_u \Pi + \Pi + \zeta D_u u = 0$, for $\tau = \tau_\alpha, \zeta = \zeta_\alpha$. Formally, for each component we then have $\tau = \chi'/\lambda'$ and $\zeta = n^2/\lambda'$ for some rate, density, and susceptibility λ', n', χ' , but these need not correspond to physical quantities.

The general form of (26) is implicit by the structure of the relevant equations. Indeed, just as the case $n=2$ corresponds to the so-called Burgers model in the study of viscoelastic materials, the case for general n corresponds

to a generalized Maxwell model [17]. In particular, the Green's function of (26) is of the form of a so-called Prony series. In the context of viscoelasticity, this is a commonly used fitting ansatz [18–20]. However, here we are able to solve the coefficients explicitly in terms of equilibrium quantities.

QM example. For $n=2$, writing τ_+, τ_- for the eigenvalues of Ω^{-1} , Eq. (25) becomes

$$\tau_+ \tau_- D_u^2 \Pi + (\tau_+ + \tau_-) D_u \Pi + \Pi \\ = -\langle \Pi, z \rangle \vartheta - \tau_+ \tau_- \langle \Pi, \Omega z \rangle. \quad (27)$$

The Green's function has the form

$$G(t) = \theta(t) \frac{\zeta_1}{\tau_1} e^{-t/\tau_1} + \theta(t) \frac{\zeta_2}{\tau_2} e^{-t/\tau_2}. \quad (28)$$

From the homogeneous equation, we find

$$\text{Det}\Omega^{-1} p_\Omega(\tau_\alpha^{-1}) G_\alpha(t) = 0 \quad \forall \alpha \in \{1 \dots n\}. \quad (29)$$

Given that Ω is nondegenerate by assumption, for G_α not identically zero, this dictates that the inverse relaxation times $\{\tau_\alpha^{-1}\}_{\alpha=1}^n$ are the eigenvalues of Ω ,

$$T_\alpha = (-1)^\alpha \text{Det}\Omega^{-1} c_\alpha[\Omega], \quad \{\tau_\alpha\}_{\alpha=1}^n = \Sigma[\Omega^{-1}]. \quad (30)$$

The latter relation motivates the notation $\Omega^{-1} = \mathcal{T}$ in [15]. For the solutions to be decaying, the relaxation times must be positive and, as was noted above, this is the case for physical systems, and we assume so going forward.

The coefficients $\{\zeta_\alpha\}_{\alpha=1}^n$ are “partial bulk viscosities.” To relate them to the coefficients $\{Z_\alpha\}_{\alpha=1}^n$, known in terms of equilibrium quantities, we generalize the approach of [21]. For demonstrative purposes, it suffices if the bulk scalar Π also splits into n distinct components; for instance, if ϑ varies slowly so that $\vartheta(t-t') \approx \vartheta(t)$ under the convolution integral appearing in the particular solution of the differential equation using the Green's function.⁹ The split obtained here is generally distinct from the earlier split of $\Pi = \langle \Pi, \hat{\mu} \rangle$, as, denoting $\Pi = \sum_{\alpha=1}^n \pi_\alpha$, each component fulfills the $n=1$ equation,

$$\tau_\alpha \nabla_t \pi_\alpha + \pi_\alpha + \zeta_\alpha \vartheta = 0 \quad \forall \alpha \in \{1, \dots, n\}. \quad (31)$$

This fixes the partial bulk viscosities $\{\zeta_\alpha\}_{\alpha=1}^n$. We consider

$$\sum_{\alpha=1}^n \prod_{\beta=1, \beta \neq \alpha}^n (\tau_\beta \nabla_t + 1) (\tau_\alpha \nabla_t \pi_\alpha + \pi_\alpha + \zeta_\alpha \vartheta) = 0 \quad (32)$$

and find that the π_α -dependent parts reduce to Eq. (25) automatically. To match the ϑ -dependent part, we make

⁹That is to say, we are in the Navier-Stokes regime.

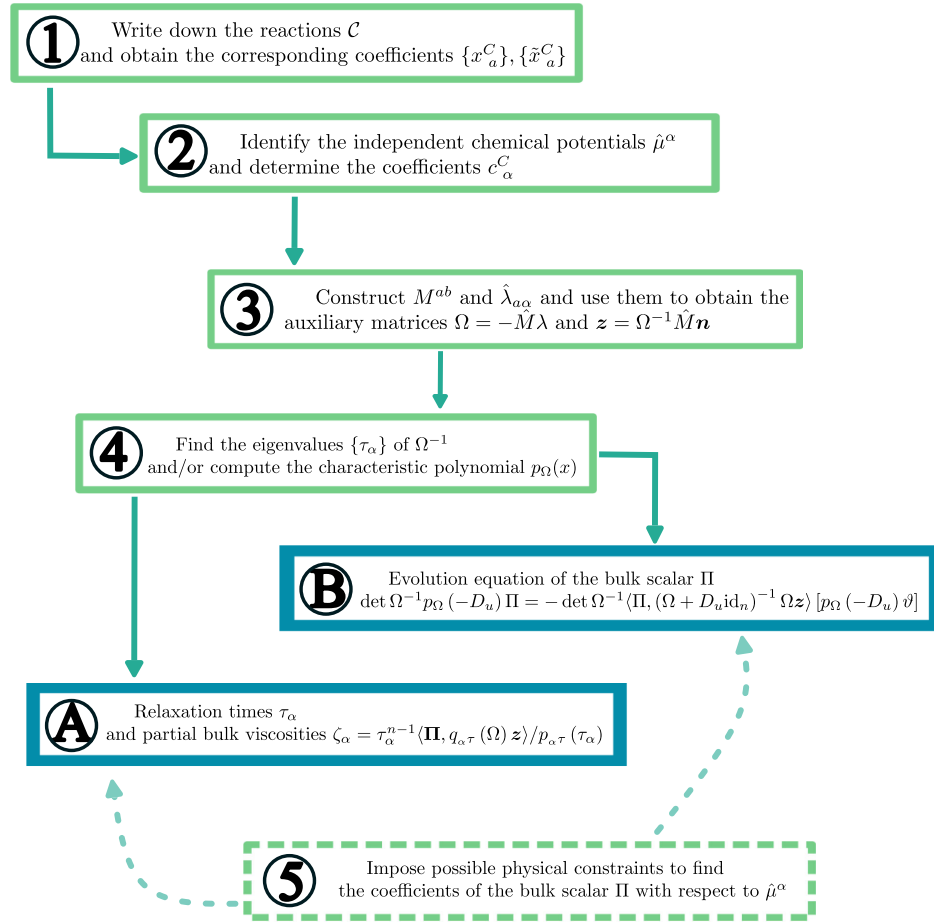


FIG. 1. A flowchart demonstrating the steps starting from identifying the appropriate physical system and leading to obtaining the transport coefficients and the evolution equation of the bulk scalar. The details of the fifth step, distinguished by dashed lines, depend on the specifics of the system under consideration.

some auxiliary definitions. We write for any n $\mathfrak{P}^\alpha: \mathbb{R}^n \rightarrow \mathbb{R}, \mathbf{y} \mapsto \mathfrak{P}^\alpha(\mathbf{y})$, defined for $\alpha \leq n$, for a function that evaluates to the sum of the products of $\alpha - 1$ combinations of the elements of $\mathbf{y}$. This quantity is related to terms seen in the characteristic polynomial written in terms of eigenvalues, but without oscillating signs. Indeed, we also define for $m \leq n$ a reciprocal characteristic polynomial as $q: \mathbb{R}^{m \times m} \rightarrow \mathbb{R}[x]$, $A \mapsto q_A(x) = \sum_{i=0}^m c_{m-i}[A]x^i$ and, finally, an $(n-1) \times (n-1)$ matrix ${}_\beta \mathcal{T} = \text{diag}(\tau_1, \dots, \tau_{\hat{\beta}}, \dots, \tau_n)$, i.e., with the β th entry removed, and ${}_\beta \mathcal{T}$ for the corresponding $n-1$ vector. Then, we find that [in the second step substituting the rhs of the first line of Eq. (25)]

$$Z_\alpha = \sum_{\beta=1}^n \mathfrak{P}^\alpha({}_\beta \mathcal{T}) \zeta_\beta$$

$$\Rightarrow \zeta_\alpha = \frac{\tau_\alpha^{n-1}}{p_{\alpha T}(\tau_\alpha)} \langle \Pi, q_{\alpha T}(\Omega) \mathbf{z} \rangle \quad \forall \alpha \in \{1 \dots n\}. \quad (33)$$

QM example. We have $T_1 = \tau_+ \tau_-$, $T_2 = \tau_+ + \tau_-$. The eigenvalues indeed match the relaxation times, $\tau_1 = \tau_+$, $\tau_2 = \tau_-$, and the bulk viscous transport coefficients are

$$Z_1 = \langle \Pi, \mathbf{z} \rangle, \quad Z_2 = \text{Det} \Omega^{-1} \langle \Pi, \Omega \mathbf{z} \rangle$$

$$\Leftrightarrow \zeta_1 = \tau_1 \frac{\langle \Pi, \mathbf{z} \rangle - \tau_2 \langle \Pi, \Omega \mathbf{z} \rangle}{\tau_1 - \tau_2},$$

$$\zeta_2 = \tau_2 \frac{\langle \Pi, \mathbf{z} \rangle - \tau_1 \langle \Pi, \Omega \mathbf{z} \rangle}{\tau_2 - \tau_1}. \quad (34)$$

Figure 1 shows a flowchart describing the general process of obtaining the evolution equation and/or the transport coefficients. It is important to highlight that, for a fixed n , the approach of this work gives both the evolution equation as well as the transport coefficients in an explicit form composed of matrix invariants. Explicit examples in terms of matrix invariants can be found in the Appendix for small n (including, for completeness, a reiterated $n=2$ without specifying the particle and reaction content).

V. DISCUSSION AND OUTLOOK

We have analyzed the transport properties of a general dense system with n independent chemical potentials in second-order hydrodynamics in the linearized regime.

The main results of this work are the evolution equation (25) of the bulk scalar as well as the form of the relevant transport coefficients (relaxation times and bulk viscosities) that were found in two different representations, both expressed in terms of equilibrium rates, densities, and susceptibilities. In our analysis, we found that the system factorizes into an n -component fluid, with each component evolving according to the Israel-Stewart equation.

In particular, the generally convoluted expressions for the bulk viscosities for the subsystems are given by simple expressions (as long as the bulk scalar splits into n components, e.g., in the Navier-Stokes regime) and by far the most complex part of their determination is essentially reduced to determining the eigenvalues of an $n \times n$ matrix Ω^{-1} .

The form obtained for the transport coefficients is readily implemented for a given physical system. This will be especially useful for, e.g., numerical purposes in viscous hydrodynamical simulations in comparison to directly writing out the expressions in terms of susceptibilities, rates, and densities, which already for $n = 2$ results in remarkably unwieldy expressions.

The results found will be useful when studying the transport properties of increasingly complicated systems involving multiple particle species, which arise naturally in the multifaceted setting of neutron star physics. As the results and techniques used have been rather general, they improve the general understanding of viscous systems and will hopefully prove helpful for the study of viscosity in systems beyond neutron stars, as well.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: EXPLICIT RESULTS FOR SMALL n

For a fixed n , the coefficients $c_\alpha[\Omega]$ appearing in the characteristic polynomial can be written down in terms of matrix invariants, e.g., with the Faddeev-LeVerrier algorithm [22]. Here, we reproduce such forms of the evolution equation and form of the coefficients of (25) for $n = 2$, $n = 3$ which have been studied in the context of compact stars and also display $n = 4$. Regardless of the applications cited here, we emphasize that the method is general for systems following differential equations of this form. For $n = 1, 2$, we reproduce results such as [9,10,15,21] and for $n = 3$ we find a considerable simplification to the forms presented in [14]. As was already seen in [15] (see the Appendix), the simplification using linear algebra is considerable over working with the explicit coefficients of the matrices. This process is feasible for any n , even when explicitly solving the eigenvalues is generally not possible.

1. $n = 2$

The results for $n = 2$ can be collected from the examples in the main text, but we list them all below. From $p_\Omega(x) = x^2 - \text{Tr}\Omega x + \text{Det}\Omega$, the evolution equation (25) becomes

$$\text{Det}\Omega^{-1}D_\mu^2\Pi + \text{Det}\Omega^{-1}\text{Tr}\Omega D_\mu\Pi + \Pi = -\langle\Pi, z\rangle\vartheta - \text{Det}\Omega^{-1}\langle\Pi, \Omega z\rangle D_\mu\vartheta. \quad (\text{A1})$$

For any n , the inverse relaxation times $\{\tau_\alpha^{-1}\}_{\alpha=1}^n$ are the eigenvalues of Ω . The bulk viscous coefficients are

$$\begin{aligned} Z_1 &= \zeta_1 + \zeta_2 = \langle\Pi, z\rangle, \\ Z_2 &= \zeta_1\tau_2 + \zeta_2\tau_1 = \text{Det}\Omega^{-1}\langle\Pi, \Omega z\rangle \end{aligned} \quad (\text{A2})$$

or, alternatively,

$$\zeta_1 = \frac{\tau_1 Z_1 - Z_2}{\tau_1 - \tau_2} = \tau_1 \frac{\langle\Pi, z\rangle - \tau_2 \langle\Pi, \Omega z\rangle}{\tau_1 - \tau_2}, \quad \zeta_2 = \frac{\tau_2 Z_1 - Z_2}{\tau_2 - \tau_1} = \tau_2 \frac{\langle\Pi, z\rangle - \tau_1 \langle\Pi, \Omega z\rangle}{\tau_2 - \tau_1}. \quad (\text{A3})$$

2. $n = 3$

For $n = 3$ the equations are still very reasonable: The evolution equation is

$$\begin{aligned} \text{Det}\Omega^{-1}D_\mu^3\Pi + \text{Det}\Omega^{-1}\text{Tr}\Omega D_\mu^2\Pi + \frac{\text{Det}\Omega^{-1}}{2}(\text{Tr}^2\Omega - \text{Tr}\Omega^2)D_\mu\Pi + \Pi \\ = -\langle\Pi, z\rangle\vartheta - \text{Det}\Omega^{-1}(\text{Tr}\Omega\langle\Pi, \Omega z\rangle - \langle\Pi, \Omega^2 z\rangle)D_\mu\vartheta - \text{Det}\Omega^{-1}\langle\Pi, \Omega z\rangle D_\mu^2\vartheta \end{aligned} \quad (\text{A4})$$

and the bulk viscous coefficients are

$$\begin{aligned} Z_1 &= \zeta_1 + \zeta_2 + \zeta_3 = \langle \mathbf{\Pi}, \mathbf{z} \rangle, \\ Z_2 &= (\tau_2 + \tau_3)\zeta_1 + (\tau_1 + \tau_3)\zeta_2 + (\tau_1 + \tau_2)\zeta_3 = \text{Det}\Omega^{-1}(\text{Tr}\Omega\langle \mathbf{\Pi}, \Omega\mathbf{z} \rangle - \langle \mathbf{\Pi}, \Omega^2\mathbf{z} \rangle), \\ Z_3 &= \tau_2\tau_3\zeta_1 + \tau_1\tau_3\zeta_2 + \tau_1\tau_2\zeta_3 = \text{Det}\Omega^{-1}\langle \mathbf{\Pi}, \Omega\mathbf{z} \rangle. \end{aligned} \quad (\text{A5})$$

As mentioned in the main text, for any $n > 1$ all but one of the bulk viscous coefficients Z_α reduce to a form found in the $n - 1$ -component system. The partial bulk viscosities read

$$\begin{aligned} \zeta_1 &= \frac{\tau_1^2(\langle \mathbf{\Pi}, \mathbf{z} \rangle - (\tau_2 + \tau_3)\langle \mathbf{\Pi}, \Omega\mathbf{z} \rangle + \tau_2\tau_3\langle \mathbf{\Pi}, \Omega^2\mathbf{z} \rangle)}{(\tau_1 - \tau_2)(\tau_1 - \tau_3)}, & \zeta_2 &= \frac{\tau_2^2(\langle \mathbf{\Pi}, \mathbf{z} \rangle - (\tau_1 + \tau_3)\langle \mathbf{\Pi}, \Omega\mathbf{z} \rangle + \tau_1\tau_3\langle \mathbf{\Pi}, \Omega^2\mathbf{z} \rangle)}{(\tau_2 - \tau_1)(\tau_2 - \tau_3)}, \\ \zeta_3 &= \frac{\tau_3^2(\langle \mathbf{\Pi}, \mathbf{z} \rangle - (\tau_1 + \tau_2)\langle \mathbf{\Pi}, \Omega\mathbf{z} \rangle + \tau_1\tau_2\langle \mathbf{\Pi}, \Omega^2\mathbf{z} \rangle)}{(\tau_3 - \tau_1)(\tau_3 - \tau_2)}. \end{aligned} \quad (\text{A6})$$

3. $n = 4$

Even for $n = 4$, which to the best of our knowledge has not been considered in the context of dense relativistic hydrodynamics, even in specific simplifying cases, the results are simple enough to be written down in a relatively compact form, although, in practice, it would be less tedious to do it indirectly using the coefficients of the characteristic polynomial. The evolution equation is

$$\begin{aligned} &\text{Det}\Omega^{-1}D_u^4\Pi + \text{Det}\Omega^{-1}\text{Tr}\Omega D_u^3\Pi + \frac{\text{Det}\Omega^{-1}}{2}(\text{Tr}^2\Omega - \text{Tr}\Omega^2)D_u^2\Pi + \frac{\text{Det}\Omega^{-1}}{6}(\text{Tr}^3\Omega - 3\text{Tr}^2\Omega\text{Tr}\Omega + 2\text{Tr}\Omega^3) + \Pi \\ &= -\langle \mathbf{\Pi}, \mathbf{z} \rangle\vartheta - \frac{\text{Det}\Omega^{-1}}{2} \left[(\text{Tr}^2\Omega - \text{Tr}\Omega^2)\langle \mathbf{\Pi}, \Omega\mathbf{z} \rangle - 2\text{Tr}\Omega\langle \mathbf{\Pi}, \Omega^2\mathbf{z} \rangle + 2\langle \mathbf{\Pi}, \Omega^3\mathbf{z} \rangle \right] D_u\vartheta \\ &\quad - \text{Det}\Omega^{-1}(\text{Tr}\Omega\langle \mathbf{\Pi}, \Omega\mathbf{z} \rangle - \langle \mathbf{\Pi}, \Omega^2\mathbf{z} \rangle)D_u^2\vartheta - \text{Det}\Omega^{-1}\langle \mathbf{\Pi}, \Omega\mathbf{z} \rangle D_u^3\vartheta, \end{aligned} \quad (\text{A7})$$

with bulk viscous coefficients

$$\begin{aligned} Z_1 &= \langle \mathbf{\Pi}, \mathbf{z} \rangle, & Z_2 &= \frac{\text{Det}\Omega^{-1}}{2}[(\text{Tr}^2\Omega - \text{Tr}\Omega^2)\langle \mathbf{\Pi}, \Omega\mathbf{z} \rangle - 2\text{Tr}\Omega\langle \mathbf{\Pi}, \Omega^2\mathbf{z} \rangle + 2\langle \mathbf{\Pi}, \Omega^3\mathbf{z} \rangle], \\ Z_3 &= \text{Det}\Omega^{-1}(\text{Tr}\Omega\langle \mathbf{\Pi}, \Omega\mathbf{z} \rangle - \langle \mathbf{\Pi}, \Omega^2\mathbf{z} \rangle), & Z_4 &= \text{Det}\Omega^{-1}\langle \mathbf{\Pi}, \Omega\mathbf{z} \rangle. \end{aligned} \quad (\text{A8})$$

The partial bulk viscosities again follow the general pattern from Eq. (33).

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