

$\pi^0 \rightarrow 2\gamma$ decay under strong magnetic fields in the NJL modelMáximo Coppola¹, Daniel Gomez Dumm², and Norberto N. Scoccola^{1,3}¹*Departamento de Física Teórica, Comisión Nacional de Energía Atómica,
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We study the anomalous $\pi^0 \rightarrow \gamma\gamma$ decay under an external uniform magnetic field in the framework of a two-flavor Nambu-Jona-Lasinio model. It is seen that the full decay width gets strongly reduced with the external field, and that the differential width is almost independent of the direction of the outgoing photons. We also find that the result for the total width can be very well approximated by a simple expression obtained at the lowest order in the chiral expansion, which is just a direct extension to finite magnetic fields of the well-known $B = 0$ result that follows from the anomalous Wess-Zumino-Witten action. As a result, the magnetic suppression of the width can be understood in terms of the evolution of both the neutral pion mass and decay constant with the magnetic field.

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I. INTRODUCTION

In the past few decades, there has been an increasing interest in the effect of intense magnetic fields on the physics of strong interactions [1–4]. To a great extent, this is due to the impact that such large magnetic fields might have in the study of the early Universe [5,6], in the analysis of high-energy noncentral heavy ion collisions [7,8], and in the description of compact stellar objects like magnetars [9,10]. In addition, magnetic fields have been shown to induce several interesting phenomena in quantum chromodynamics (QCD), such as the chiral magnetic effect [11–13], the enhancement of the QCD quark-antiquark condensate (magnetic catalysis) [14], the decrease in critical temperatures for chiral restoration and deconfinement QCD transitions [inverse magnetic catalysis (IMC)] [15], etc.

The study of strong interactions in the presence of external magnetic fields is also useful to probe QCD dynamics, through, for instance, the analysis of the effects of these fields on the properties of light hadrons. The corresponding theoretical calculations usually represent a nontrivial task, since they require to deal in general with QCD in a nonperturbative regime. In view of this difficulty, magnetic effects have been studied considering various effective theoretical approaches to strong interaction

dynamics. For example, the effect of intense external magnetic fields on light pseudoscalar meson masses has been studied in the framework of Nambu-Jona-Lasinio (NJL)-like models [16–34], quark-meson models [35–38], chiral perturbation theory (ChPT) [39–41], path integral Hamiltonians [42,43], effective chiral confinement Lagrangians [44,45], and QCD sum rules [46]. In addition, several results for the spectrum of these mesons in the presence of background magnetic fields have been obtained from lattice QCD (LQCD) calculations [15,47–52].

The effect of a magnetic field on the decay widths of light pseudoscalar mesons is a related topic that has also been considered in the literature. However, most of the existing works have been devoted to the analysis of the leptonic decays of charged pions [53–58]. The main conclusion of Refs. [55,58], where actual predictions for these processes are given, is that the presence of the external field leads to a great enhancement of the corresponding decay widths. On the other hand, the influence of a magnetic field on the anomalous decay $\pi^0 \rightarrow \gamma\gamma$ has received far less attention. In the past few years, this problem has been studied in Refs. [59,60] in the framework of ChPT, where the validity of the analysis is restricted to values of eB up to $\sim m_\pi^2$. At this point, we should emphasize the importance of the $\pi^0 \rightarrow \gamma\gamma$ decay in the context of chiral theories. In fact, it was through a careful analysis of the corresponding decay amplitude (in vacuum) that anomalies were first discovered [61,62], and this led to the realization that the $U(1)_A$ symmetry is not conserved in quantum field theories like QCD. Having this in mind, it is clear that the understanding of how the $\pi^0 \rightarrow \gamma\gamma$ decay width gets

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modified by the presence of an external magnetic field appears to be a quite relevant issue.

The aim of the present work is to study the influence of a strong magnetic field $\vec{B}$ on the π^0 anomalous decay using a two-flavor NJL model. We recall that the NJL model is an effective chiral quark model of QCD in which gluon degrees of freedom are integrated out in favor of some local quark-antiquark interactions that lead to the dynamical breaking of chiral symmetry. The model is nonrenormalizable and, therefore, a regularization procedure has to be adopted in order to deal with divergent quantities. The choice of this procedure may be considered as part of the definition of the model itself. In this work, we adopt a magnetic-field-independent regularization (MFIR) method [63–65], which has been shown to be convenient when dealing with magnetized systems [66]. While in the NJL model quark condensates emerge from the usual mean field approximation, mesons are treated as excitations whose masses are obtained by considering second-order corrections in the corresponding bosonized action. The quark propagator in the presence of a uniform magnetic field, considered in this work in the Schwinger form [2,3], is used to evaluate the π^0 polarization function and its decay amplitude to two photons. As an additional ingredient, we consider the case of B -dependent effective quark-quark coupling constants. This possibility has been previously explored in effective models [67–69] in order to reproduce the IMC effect observed at finite temperature in LQCD calculations.

This paper is organized as follows. In Sec. II, we present the theoretical formalism, introducing the magnetized two-flavor NJL-like Lagrangian to be used in our calculations. Expanding the bosonized action in powers of fluctuations around mean field values up to quadratic order, we derive expressions for certain relevant quantities associated with the $\pi^0 \rightarrow \gamma\gamma$ decay width, such as quark masses, quark propagators, the quark-pion coupling strength, and the neutral pion mass. In Sec. III, we calculate the magnetized $\pi^0 \rightarrow \gamma\gamma$ decay width from the third-order expansion of the bosonized action. We also show that in the chiral limit, the resulting coupling constant $g_{\pi^0\gamma\gamma}$ has a simple expression that turns out to be a direct extension to finite magnetic field of the well-known $B = 0$ result that follows from the gauged Wess-Zumino-Witten action. In Sec. IV, we present and discuss our numerical results, while in Sec. V we provide a summary of our work, together with our main conclusions. We also include Appendices A, B, and C to provide some technical details of our calculations.

II. NJL MODEL UNDER STRONG MAGNETIC FIELDS: MEAN FIELD PROPERTIES, π^0 MASS, AND COUPLING TO QUARKS

The Lagrangian density of the NJL two-flavor model, including couplings with an electromagnetic field, is given by

$$\mathcal{L} = \bar{\psi}(i\not{D} - m_0)\psi + G[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (1)$$

where $\psi = (ud)^T$, τ_i are Pauli matrices, and m_0 is the current quark mass, which is assumed to be equal for u and d quarks. The interaction between fermions and the electromagnetic field $\mathcal{A}_\mu$ is driven by the covariant derivative

$$D_\mu = \partial_\mu + i\hat{Q}\mathcal{A}_\mu, \quad (2)$$

where $\hat{Q} = \text{diag}(q_u, q_d)$, with $q_u = 2e/3$ and $q_d = -e/3$, and $e > 0$ being the proton electric charge. In addition, Eq. (1) involves the electromagnetic tensor $F^{\mu\nu} = \partial^\mu\mathcal{A}^\nu - \partial^\nu\mathcal{A}^\mu$. The gauge field $\mathcal{A}^\mu = A^\mu + a^\mu$ involves both the external classical field A^μ and the photon degrees of freedom, given by the dynamical quantum field a^μ . We consider the particular case in which the external electromagnetic field is given by an homogeneous stationary magnetic field $\vec{B}$ orientated along the positive 3-axis.

Since we are interested in studying a meson decay, it is convenient to bosonize the fermionic theory, introducing a scalar field $\sigma(x)$ and pseudoscalar fields $\vec{\pi}(x)$ while integrating out fermion fields. The bosonized action can be written as

$$S_{\text{bos}} = -i \times \ln \det(i\mathcal{D}) - \frac{1}{4G} \int d^4x [\sigma(x)\sigma(x) + \vec{\pi}(x) \cdot \vec{\pi}(x)] + S_{\text{EM}}, \quad (3)$$

with

$$i\mathcal{D}_{x,y} = [i\not{D}_x - m_0 - \sigma(x) - i\gamma_5\vec{\tau} \cdot \vec{\pi}(x)]\delta^{(4)}(x - y), \quad (4)$$

where a direct product to an identity matrix in color space is understood. The pure electromagnetic field contribution to the effective action is given by

$$S_{\text{EM}} = -\frac{1}{4} \int d^4x F_{\mu\nu}F^{\mu\nu} = -\frac{B^2}{2}V^{(4)} + \frac{1}{2} \int d^4x a_\mu(x)(\eta^{\mu\nu}\partial_\alpha\partial^\alpha - \partial^\mu\partial^\nu)a_\nu(x), \quad (5)$$

where the first and second terms on the rhs correspond to the classical and propagating contributions, respectively. For the Minkowski metric, the convention $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is adopted. Since we consider a static and uniform external magnetic field, in what follows the classical contribution to S_{EM} will be dropped. Notice, however, that this contribution could be relevant in some other contexts, such as the study of charged pion superfluidity [70].

We proceed by expanding the bosonized action in powers of fluctuations of the bosonic fields around their mean field (MF) values. We assume that the scalar field

$\sigma(x)$ has a nontrivial translational invariant MF value $\bar{\sigma}$, while vacuum expectation values of pion fields $\pi_i(x)$ are $\bar{\pi}_i = 0$ so as to respect the parity symmetry of vacuum. Thus, we write

$$\mathcal{D}_{x,y} = \text{diag}(\mathcal{D}_{x,y}^{\text{MF},u}, \mathcal{D}_{x,y}^{\text{MF},d}) + \delta\mathcal{D}_{x,y}. \quad (6)$$

The MF piece, which is diagonal in flavor space, includes the interaction of quark fields with the classical electromagnetic field A^μ . One has

$$\mathcal{D}_{x,y}^{\text{MF},f} = -i[i\cancel{D}_x - Q_f A(x) - M]\delta^{(4)}(x-y), \quad (7)$$

where $M = m_0 + \bar{\sigma}$ is the quark effective mass and $f = u, d$. On the other hand, $\delta\mathcal{D}_{x,y}$ reads

$$\delta\mathcal{D}_{x,y} = i[\hat{Q}\cancel{d}(x) + \delta\sigma(x) + i\gamma_5 \vec{\tau} \cdot \delta\vec{\pi}(x)]\delta^{(4)}(x-y). \quad (8)$$

Replacing in the bosonized effective action and expanding in powers of meson fluctuations around the MF values, we get

$$S_{\text{bos}} = S_{\text{bos}}^{\text{MF}} + S_{\text{bos}}^{\text{quad}} + S_{\text{bos}}^{\text{cub}} + \dots \quad (9)$$

Here, the MF action per unit volume is given by

$$\frac{S_{\text{bos}}^{\text{MF}}}{V^{(4)}} = -\frac{\bar{\sigma}^2}{4G} - \frac{iN_c}{V^{(4)}} \sum_{f=u,d} \text{Tr} \ln(i\mathcal{D}^{\text{MF},f}), \quad (10)$$

where N_c is the number of colors, and the trace is understood to be taken in both coordinate and Dirac spaces. Next, we introduce the MF quark propagator in the presence of the classical magnetic field, $\mathcal{S}^{\text{MF},f} = (i\mathcal{D}^{\text{MF},f})^{-1}$. We choose to use for this propagator the Schwinger form, given by

$$\mathcal{S}_{x,y}^{\text{MF},f} = e^{i\Phi_f(x,y)} \int_p e^{-ip(x-y)} \tilde{\mathcal{S}}_f(p), \quad (11)$$

where

$$\begin{aligned} \tilde{\mathcal{S}}_f(p) = & -i \int_0^\infty d\rho \exp \left[-i\rho \left(M^2 - p_\parallel^2 + \vec{p}_\perp^2 \frac{\tan(\rho B_f)}{\rho B_f} - i\epsilon \right) \right] \\ & \times \left[(p_\parallel \cdot \gamma_\parallel + M)(1 - s_f \gamma^1 \gamma^2 \tan(\rho B_f)) - \frac{\vec{p}_\perp \cdot \vec{\gamma}_\perp}{\cos^2(\rho B_f)} \right], \end{aligned} \quad (12)$$

with $B_f = |q_f B|$ and $s_f = \text{sign}(q_f B)$. Here, we have defined “parallel” and “perpendicular” four-vectors

$$p_\parallel^\mu \equiv (p^0, 0, 0, p^3), \quad p_\perp^\mu \equiv (0, p^1, p^2, 0), \quad (13)$$

and we have introduced a shorthand notation for the momentum integral

$$\int_p \equiv \int \frac{d^4 p}{(2\pi)^4}. \quad (14)$$

We have also used the notations $p_\parallel \cdot \gamma_\parallel = p^0 \gamma^0 - p^3 \gamma^3$ and $\vec{p}_\perp \cdot \vec{\gamma}_\perp = p^1 \gamma^1 + p^2 \gamma^2$. The gauge dependence of the propagator is encoded in the so-called Schwinger phase $\Phi_f(x, y)$ associated to the classical external field

$$\Phi_f(x, y) = q_f \int_x^y d\xi_\mu \left[A^\mu(\xi) + \frac{1}{2} F_A^{\mu\nu} (\xi_\nu - y_\nu) \right], \quad (15)$$

where $F_A^{\mu\nu}$ is the corresponding electromagnetic strength tensor.

As usual, from the minimization condition $\partial S_{\text{bos}}^{\text{MF}} / \partial \bar{\sigma} = 0$, one obtains a gap equation, which can be written as

$$M = m_0 + 4GMN_c I, \quad (16)$$

with

$$I = \frac{i}{2M} \sum_{f=u,d} \int_p \text{Tr}_D \tilde{\mathcal{S}}_f(p), \quad (17)$$

where Tr_D stands for trace in Dirac space. Notice that Eq. (16) is a gauge-independent expression, as it should be. Performing the usual deformation of the proper time integration paths (which effectively reduces to the substitution $\rho \rightarrow -i\tau$) [71], one gets

$$I = \frac{1}{8\pi^2} \sum_{f=u,d} \int_0^\infty \frac{d\tau}{\tau^2} e^{-\tau M^2} \tau B_f \coth(\tau B_f). \quad (18)$$

This is a divergent integral and has to be properly regularized. As stated in Sec. I, we use the MFIR scheme: for a given unregularized quantity, the corresponding (divergent) $B \rightarrow 0$ limit is subtracted and then added in a regularized form. Thus, each quantity can be separated into a “ $B = 0$ ” term and a “magnetic” contribution. The regularized integral is given by [29]

$$I^{\text{reg}} = I^{0,\text{reg}} + I^{\text{mag}}, \quad (19)$$

where

$$\begin{aligned} I^{\text{mag}} = & \frac{1}{8\pi^2} \sum_{f=u,d} \int_0^\infty \frac{d\tau}{\tau^2} e^{-\tau M^2} [\tau B_f \coth(\tau B_f) - 1] \\ = & \frac{M^2}{8\pi^2} \sum_{f=u,d} \left[\frac{\ln \Gamma(x_f)}{x_f} - \frac{\ln(2\pi)}{2x_f} + 1 - \left(1 - \frac{1}{2x_f} \right) \ln x_f \right], \end{aligned} \quad (20)$$

with $x_f = M^2 / (2B_f)$. In turn, the divergent “ $B = 0$ ” piece can be regularized by introducing a 3D momentum cutoff Λ . One has [72]

$$I^{0,\text{reg}} = \frac{1}{2\pi^2} \left[\Lambda \sqrt{\Lambda^2 + M^2} + M^2 \ln \left(\frac{M}{\Lambda + \sqrt{\Lambda^2 + M^2}} \right) \right]. \quad (21)$$

It is worth noticing that, in general, the “ $B = 0$ ” term still depends implicitly on B (e.g., through the values of the dressed quark masses M), hence it should not be confused with the value of the studied quantity at vanishing external field.

Next, we consider quadratic terms in the expansion of S_{bos} in powers of $\delta\mathcal{D}_{x,y}$. We omit the description of charged pion contributions, since charged pions cannot mix neither with the photon nor the π^0 . In contrast, neutral pions are in general expected to mix with photons [59]. Here, we consider only lowest-order dynamical electromagnetic contributions to the $\pi^0 \rightarrow \gamma\gamma$ decay. Thus, we dismiss possible $\pi^0 - \gamma$ mixing effects. As usual, the π^0 effective propagator can be obtained from a resummation of quark loop chains, using the random phase approximation. Since we are dealing with a neutral particle, the contributions of Schwinger phases associated with quark propagators [see Eq. (11)] cancel out, leading to gauge invariant polarization functions that depend only on the difference $x - y$ (i.e., they are translationally invariant). If one performs a Fourier transformation, the conservation of momentum implies that the polarization function is diagonal in the momentum basis. Thus, in this basis we have

$$S_{\pi^0} = -\frac{1}{2} \int_p \delta\pi^0(-q) \left[\frac{1}{2G} - \Pi_{\pi^0}(q_{\parallel}^2, q_{\perp}^2) \right] \delta\pi^0(q), \quad (22)$$

where

$$\Pi_{\pi^0}(q_{\parallel}^2, q_{\perp}^2) = -iN_c \sum_f \text{Tr}_D [\bar{i}\tilde{S}_f(p^+) i\gamma_5 \tilde{S}_f(p^-) i\gamma_5]. \quad (23)$$

Here, $p^{\pm} = p \pm q/2$, while $\tilde{S}_f(p)$ is given in Eq. (12). Once again, employing the MFIR scheme we regularize the above polarization function by separating $\Pi_{\pi^0}^{\text{reg}} = \Pi_{\pi^0}^{\text{mag}} + \Pi_{\pi^0}^{0,\text{reg}}$. The corresponding expressions are given in Appendix A. Choosing the frame in which the π^0 meson is at rest, its mass can be obtained by solving the equation

$$\frac{1}{2G} - \Pi_{\pi^0}^{\text{reg}}(m_{\pi^0}^2, 0) = 0. \quad (24)$$

As a final remark, we recall that the pion field wave function has to be renormalized. Notice that the external magnetic field introduces a spatial anisotropy reflected in the momentum dependence $\Pi_{\pi^0}(q_{\parallel}^2, q_{\perp}^2)$, which distinguishes parallel from perpendicular components. We introduce a (parallel) “wave function renormalization constant” $Z_{\parallel}^{1/2}$, defined by fixing the residue of the two-point function at the pion pole. This constant, and the related quark-pion coupling constant $g_{\pi^0 qq}$, are given by [29]

$$Z_{\parallel}^{-1} = g_{\pi^0 qq}^{-2} = \frac{\partial \Pi_{\pi^0}}{\partial q_{\parallel}^2} \Big|_{q_{\parallel}^2 = m_{\pi^0}^2, q_{\perp}^2 = 0}. \quad (25)$$

III. $\pi^0 \rightarrow \gamma\gamma$ DECAY WIDTH

A. Contributions to the decay amplitude

For a pion of momentum q decaying into two photons of momenta p and k , the decay width is given by

$$\Gamma_{\pi^0 \gamma\gamma}(B) = \frac{1}{2E_{\pi}} \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3 2p^0} \frac{d^3 k}{(2\pi)^3 2k^0} \sum_{\lambda_p, \lambda_k = \pm} \frac{|iT|^2}{VT}, \quad (26)$$

where λ_p and λ_k are the helicities of the outgoing photons, while V and T are the volume and time interval in which the interaction is active. At the end of the calculation, the limit $V, T \rightarrow \infty$ will be taken. Note that a factor $1/2$ has been included to account for the identical nature of the outgoing photons. The transition amplitude is given by the relevant $\hat{S}$ -matrix element between the initial and final states,

$$iT = \langle \gamma(p, \lambda_p) \gamma(k, \lambda_k) | (\hat{S} - 1) | \pi^0(q) \rangle. \quad (27)$$

By properly expanding the fields in terms of creation and annihilation operators, we get

$$iT = \epsilon_{\mu}(p, \lambda_p)^* \epsilon_{\nu}(k, \lambda_k)^* \times \int d^4 x d^4 y d^4 z e^{ipz} e^{iky} e^{-iqx} \langle 0 | T [J^{\mu}(z) J^{\nu}(y) J_5(x)] | 0 \rangle, \quad (28)$$

where $\epsilon_{\mu}(p, \lambda_p)$ and $\epsilon_{\nu}(k, \lambda_k)$ are the photon polarization vectors, T is the time-ordering operator, and the quark currents are given by

$$J^{\mu}(x) = \bar{\psi}(x) (-\hat{Q}\gamma^{\mu}) \psi(x), \\ J_5(x) = \bar{\psi}(x) (-g_{\pi^0 qq} i\gamma_5 \tau_3) \psi(x). \quad (29)$$

The connected three-point correlation function in Eq. (28) can be obtained through Wick contractions. After taking trace over color, flavor, and coordinate spaces, we get

$$iT(q, p, k) = \epsilon_{\mu}(p, \lambda_p)^* \epsilon_{\nu}(k, \lambda_k)^* \int d^4 x d^4 y d^4 z e^{ipz} e^{iky} e^{-iqx} \times \sum_{f=u,d} \left[i\mathcal{G}_f^{\mu\nu}(x, y, z) + i\mathcal{G}_f^{\nu\mu}(x, z, y) \right], \quad (30)$$

where

$$i\mathcal{G}_f^{\mu\nu}(x, y, z) = iN_c \sum_{f=u,d} \text{Tr}_D \left[(-g_{\pi^0 qq} \kappa_f i\gamma_5) i\mathcal{S}_{x,z}^{\text{MF},f} \times (-q_f \gamma^{\mu}) i\mathcal{S}_{z,y}^{\text{MF},f} (-q_f \gamma^{\nu}) i\mathcal{S}_{y,x}^{\text{MF},f} \right]. \quad (31)$$

Here, the factors $\kappa_u = -\kappa_d = 1$ arise from the Pauli matrix τ_3 in the quark axial current $J_5(x)$. The contributions $\mathcal{G}_f^{\mu\nu}(x, y, z)$ and $\mathcal{G}_f^{\nu\mu}(x, z, y)$ to the sum in Eq. (30) correspond to the Feynman diagrams depicted in Fig. 1.

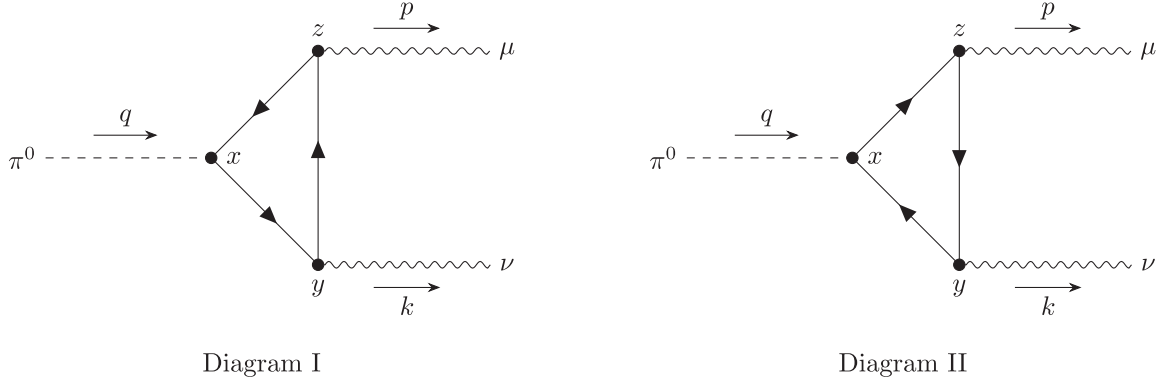


FIG. 1. One-loop Feynman diagrams that contribute to the $\pi^0 \rightarrow \gamma\gamma$ decay. Diagrams I and II correspond to the amplitudes $i\mathcal{G}_f^{\mu\nu}(x, y, z)$ and $i\mathcal{G}_f^{\nu\mu}(x, z, y)$, respectively. Internal (external) arrows indicate the charge (momentum) flux. The interaction points in configuration space are indicated with labels x , y , and z .

Taking into account the form of the quark propagators given by Eq. (11), we obtain

$$\begin{aligned} \mathcal{G}_f^{\mu\nu}(x, y, z) = & -g_{\pi^0 qq} N_c \kappa_f q_f^2 \exp[i(\Phi_f(x, z) + \Phi_f(z, y) + \Phi_f(y, x))] \\ & \times \int_{rst} e^{-is(x-z)} e^{-ir(z-y)} e^{-it(y-x)} \text{Tr}_D[i\gamma_5 i\bar{\mathcal{S}}_f(s) \gamma^\mu i\bar{\mathcal{S}}_f(r) \gamma^\nu i\bar{\mathcal{S}}_f(t)]. \end{aligned} \quad (32)$$

It is easy to see that the sums of Schwinger phases appearing in the amplitude are such that gauge-dependent pieces cancel out. Recalling that we consider an homogeneous magnetic field $\vec{B}$ orientated along the 3-axis, we obtain the gauge-independent result

$$\begin{aligned} \Phi_f(x, z) + \Phi_f(z, y) + \Phi_f(y, x) \\ = q_f(x_\mu - y_\mu) F^{\mu\nu}(z_\nu - x_\nu)/2 \\ = s_f B_f \epsilon_{ij3}(z_i - x_i)(x_j - y_j)/2. \end{aligned} \quad (33)$$

At this point, we note that the two amplitudes in Eq. (30) are related by a charge conjugation transformation $\hat{\mathcal{C}}$, namely,

$$\hat{\mathcal{C}} \mathcal{G}_f^{\nu\mu}(x, z, y) \hat{\mathcal{C}}^{-1} = \mathcal{G}_f^{\mu\nu}(x, y, z). \quad (34)$$

Therefore, we can write

$$\begin{aligned} iT(q, p, k) = & -ig_{\pi^0 qq} N_c \epsilon_\mu^*(p, \lambda_p) \epsilon_\nu^*(k, \lambda_k) \\ & \times \sum_{f=u,d} \kappa_f q_f^2 \sum_{\epsilon=\pm 1} \int_{rst} G^{(\epsilon s_f)}(r, s, t) \\ & \times \text{Tr}_D[i\gamma^5 i\bar{\mathcal{S}}_{\epsilon f}(t) \gamma^\nu i\bar{\mathcal{S}}_{\epsilon f}(r) \gamma^\mu i\bar{\mathcal{S}}_{\epsilon f}(s)], \end{aligned} \quad (35)$$

where the subscript ϵf indicates that s_f has to be replaced by ϵs_f in the corresponding propagator, as dictated by charge conjugation. The function $G^{(\epsilon s_f)}(s, r, t)$ is defined by

$$G^{(\epsilon s_f)}(r, s, t) = \int d^4x d^4y d^4z e^{i(pz + ky - qx)} e^{-is(x-z) - ir(z-y) - it(y-x)} \exp\left[i \frac{\epsilon s_f B_f}{2} \epsilon_{ij3}(z_i - x_i)(x_j - y_j)\right]. \quad (36)$$

After integration over space coordinates, one gets

$$\begin{aligned} G^{(\epsilon s_f)}(r, s, t) = & \frac{4}{B_f^2} (2\pi)^{10} \delta^{(4)}(q - p - k) \delta^{(2)}(p_\parallel + s_\parallel - r_\parallel) \delta^{(2)}(k_\parallel + r_\parallel - t_\parallel) \\ & \times \exp\left\{i \frac{2\epsilon s_f}{B_f} [(p^1 + s^1 - r^1)(k^2 + r^2 - t^2) - (k^1 + r^1 - t^1)(p^2 + s^2 - r^2)]\right\}. \end{aligned} \quad (37)$$

As can be seen, while parallel momentum is conserved at each vertex, the conservation of perpendicular momenta gets broken owing to Schwinger phases, which mix the interaction points. Nevertheless, momentum is still conserved for the overall $\pi^0 \rightarrow \gamma\gamma$ process (as indicated by the first Dirac delta), allowing us to define the decay amplitude $\mathcal{M}(q, p, k)$ as

$$iT(q, p, k) = (2\pi)^4 \delta^{(4)}(q - p - k) i\mathcal{M}(q, p, k). \quad (38)$$

It is convenient to write $\mathcal{M}(q, p, k)$ by indicating explicitly the contraction with photon polarization vectors. We obtain

$$i\mathcal{M}(q, p, k) = \epsilon_\mu(p, \lambda_p)^* \epsilon_\nu(k, \lambda_k)^* (R_u^{\mu\nu} - R_d^{\mu\nu}), \quad (39)$$

where $R_f^{\mu\nu}$ is given by

$$\begin{aligned} R_f^{\mu\nu} = & -i16\pi^2 N_c g_{\pi^0 qq} \frac{q_f^2}{B_f^2} \int_{rst} (2\pi)^4 \delta^{(2)}(p_\parallel + s_\parallel - r_\parallel) \delta^{(2)}(k_\parallel + r_\parallel - t_\parallel) \\ & \times \sum_{\varepsilon=\pm 1} \exp \left\{ i \frac{2\varepsilon s_f}{B_f} [(p^1 + s^1 - r^1)(k^2 + r^2 - t^2) - (k^1 + r^1 - t^1)(p^2 + s^2 - r^2)] \right\} \\ & \times \text{Tr}_D [i\gamma^5 i\tilde{\mathcal{S}}_{ef}(s) \gamma^\mu i\tilde{\mathcal{S}}_{ef}(r) \gamma^\nu i\tilde{\mathcal{S}}_{ef}(t)]. \end{aligned} \quad (40)$$

B. Decay width calculation

Taking into account Eq. (26) and the definitions in Eqs. (38) and (39), the $\pi^0 \rightarrow \gamma\gamma$ decay width is given by

$$\Gamma_{\pi^0 \gamma\gamma}(B) = \frac{1}{2E_\pi} \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3 2p^0} \frac{d^3 k}{(2\pi)^3 2k^0} (2\pi)^4 \delta^{(4)}(q - p - k) \sum_{\lambda_p, \lambda_k=\pm} |\epsilon_\mu(p, \lambda_p)^* \epsilon_\nu(k, \lambda_k)^* (R_u^{\mu\nu} - R_d^{\mu\nu})|^2. \quad (41)$$

This expression can be conveniently worked out in the pion rest frame, i.e., taking $q^\mu = (m_{\pi^0}, \vec{0})$. Momentum conservation leads then to $\vec{p} = -\vec{k}$ and $p^0 + k^0 = m_{\pi^0}$. We assume that outgoing photons can be taken at tree level in Quantum Electrodynamics (QED), neglecting higher-order corrections that could arise from the presence of the external magnetic field. Hence, we consider the usual on-shell conditions $p^2 = k^2 = 0$.

The calculation of $R_f^{\mu\nu}$ in the pion rest frame is outlined in Appendix B. After some work, we arrive at the result

$$R_f^{\mu\nu} = i \frac{g_{\pi^0 qq} e^2 N_c M}{\pi^2} p^0 \epsilon^{0\mu\nu\lambda} p_\lambda \mathcal{I}_f, \quad (42)$$

where we have used the convention $\epsilon^{0123} = +1$. The proper-time integral $\mathcal{I}_f$ is given by

$$\begin{aligned} \mathcal{I}_f = & \frac{q_f^2 B_f}{e^2} \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_0^\infty d\tau_3 \frac{1 + t_1 t_3 - (t_1 + t_3)t_2}{(\tau_1 + \tau_2 + \tau_3) T_f} \exp[-(\tau_1 + \tau_2 + \tau_3) M^2] \\ & \times \exp \left[\frac{4\tau_1 \tau_3 (p^0)^2 + (\tau_1 + \tau_3) \tau_2 p_\parallel^2}{\tau_1 + \tau_2 + \tau_3} - \frac{(t_1 + t_3) t_2 \vec{p}_\perp^2}{T_f B_f} \right], \end{aligned} \quad (43)$$

where we have introduced the definitions $t_i = \tanh(\tau_i B_f)$ and $T_f = t_1 + t_2 + t_3 + t_1 t_2 t_3$.

The contraction with photon polarization vectors can be carried out taking into account the expression for $\epsilon_\mu(p, \lambda_p)$ in Appendix C. One has

$$\begin{aligned} \epsilon_\mu(p, \pm)^* \epsilon_\nu(k, \pm)^* \epsilon^{0\mu\nu\lambda} p_\lambda &= \mp i e^{\mp 2i\phi} p^0, \\ \epsilon_\mu(p, +)^* \epsilon_\nu(k, -)^* \epsilon^{0\mu\nu\lambda} p_\lambda &= \epsilon_\mu(p, -)^* \epsilon_\nu(k, +)^* \epsilon^{0\mu\nu\lambda} p_\lambda = 0, \end{aligned} \quad (44)$$

where ϕ is an arbitrary phase. Therefore, from Eq. (42), we get

$$\begin{aligned} & \sum_{\lambda_p, \lambda_k=\pm} |\epsilon_\mu(p, \lambda_p)^* \epsilon_\nu(k, \lambda_k)^* (R_u^{\mu\nu} - R_d^{\mu\nu})|^2 \\ &= 2m_{\pi^0}^4 \alpha^2 \left(\frac{g_{\pi^0 qq} N_c M}{\pi} \right)^2 (\mathcal{I}_u - \mathcal{I}_d)^2, \end{aligned} \quad (45)$$

where $\alpha = e^2/(4\pi)$ is the fine structure constant.

As it is usually done, we can express the decay width in terms of an effective pion-to-two-photon coupling $g_{\pi^0 \gamma\gamma}$, which will now depend on the external field. Writing

$$\Gamma_{\pi^0 \gamma\gamma}(B) = \frac{\pi}{4} \alpha^2 m_{\pi^0}^3 g_{\pi^0 \gamma\gamma}^2(B), \quad (46)$$

and integrating over phase space in Eq. (41), we get

$$g_{\pi^0\gamma\gamma}^2(B) = \left(\frac{g_{\pi^0 qq} N_c M}{2\pi^2} \right)^2 \int_0^1 d \cos \theta (\mathcal{I}_u - \mathcal{I}_d)^2, \quad (47)$$

where θ is the angle between the magnetic field $\vec{B}$ and the three-momentum $\vec{p}$ of one of the outgoing photons. One has then $p_{\parallel}^2 = \vec{p}_{\perp}^2 = m_{\pi^0}^2 \sin^2 \theta / 4$. Finally, to perform the integrals in Eq. (43), we find it convenient to introduce new variables x , y , and z , defined by

$$\tau_1 = xz, \quad \tau_2 = yz, \quad \tau_3 = (1 - x - y)z. \quad (48)$$

In this way, the proper-time integrals can be rewritten as

$$\begin{aligned} \mathcal{I}_f &= \frac{q_f^2 B_f}{e^2} \int_0^\infty dz \int_0^1 dx \int_0^{1-x} dy \frac{z}{T_f} [1 + t_1 t_3 - (t_1 + t_3) t_2] \\ &\times \exp \left\{ -z [M^2 - x(1 - x - y) m_{\pi^0}^2] \right. \\ &\left. + \frac{m_{\pi^0}^2}{4} \sin^2 \theta \left[zy(1 - y) - \frac{(t_1 + t_3) t_2}{T_f B_f} \right] \right\}. \end{aligned} \quad (49)$$

As expected, the decay shows only an axially symmetric angular distribution, since full rotational symmetry is broken by the external magnetic field. The corresponding differential decay width reads

$$\frac{d\Gamma_{\pi^0\gamma\gamma}(B)}{d \cos \theta} = \frac{\pi}{4} \alpha^2 m_{\pi^0}^3 \left(\frac{g_{\pi^0 qq} N_c M}{2\pi^2} \right)^2 (\mathcal{I}_u - \mathcal{I}_d)^2, \quad (50)$$

where the dependence with θ is given by Eq. (49).

Our expression for the decay width can be evaluated in the limit of vanishing external field, where $t_i \rightarrow \tau_i B_f$ and $T_f \rightarrow z B_f$. In this case, the θ dependence in Eq. (49) vanishes, recovering spherical symmetry. We obtain

$$\begin{aligned} g_{\pi^0\gamma\gamma}(0) &= g_{\pi^0 qq} \frac{N_c M}{6\pi^2} \int_0^\infty dz \int_0^1 dx \\ &\times \int_0^{1-x} dy e^{-z [M^2 - x(1-x-y) m_{\pi^0}^2]} \\ &= g_{\pi^0 qq} \frac{N_c M}{3\pi^2} \left[\frac{1}{m_{\pi^0}} \arcsin \left(\frac{m_{\pi^0}}{2M} \right) \right]^2, \end{aligned} \quad (51)$$

in agreement with the result obtained in Ref. [73]. Notice that in that reference the usual Feynman form of quark propagators was considered.

C. Chiral expansion

As stated, it is also interesting to consider the anomalous π^0 decay at the lowest order in the current quark mass expansion. Taking the limit $m_{\pi^0} \rightarrow 0$ in Eq. (49) we have

$$\begin{aligned} \mathcal{I}_f^{\text{ch}} &= \frac{q_f^2 B_f}{e^2} \int_0^\infty dz \int_0^1 dx \\ &\times \int_0^{1-x} dy \frac{z}{T_f} [1 + t_1 t_3 - (t_1 + t_3) t_2] e^{-z (M^{\text{ch}})^2}, \end{aligned} \quad (52)$$

where M^{ch} is the (B -dependent) effective quark mass obtained in the chiral limit $m_0 = 0$. This integral can be analytically performed, leading to

$$\mathcal{I}_f^{\text{ch}} = \frac{q_f^2}{e^2} \frac{1}{2(M^{\text{ch}})^2}. \quad (53)$$

Replacing this expression into Eq. (47), and noting that in this case the integral over $\cos \theta$ can be trivially performed, we get

$$g_{\pi^0\gamma\gamma}^{\text{ch}}(B) = \frac{N_c}{12\pi^2 f_{\pi^0}^{\text{ch}}(B)}, \quad (54)$$

where we have made use of the generalized Goldberger-Treiman relation $f_{\pi^0}^{\text{ch}} g_{\pi^0 qq}^{\text{ch}} = M^{\text{ch}}$, which is shown to be valid at finite B (in the chiral limit) within the NJL model [29]. It is worth mentioning that, in fact, three (vector and axial-vector) pion-to-vacuum form factors arise for the π^0 in the presence of the external magnetic field [56]. Here, we denote by $f_{\pi^0}^{\text{ch}}$ the one that reduces to the usual decay constant f_π at $B = 0$. Moreover, Eq. (54) turns out to be a direct extension of the well-known $B = 0$ result, which follows from the anomalous gauged Wess-Zumino-Witten action introduced in Refs. [74,75], to a finite magnetic field.

In this way, the expression for the magnetized $\pi^0 \rightarrow \gamma\gamma$ decay width is found to be given by

$$\Gamma_{\pi^0\gamma\gamma}^{\text{LO}}(B) = \frac{\alpha^2}{\pi^3} \frac{m_{\pi^0}^3(B)}{[f_{\pi^0}^{\text{ch}}(B)]^2}, \quad (55)$$

where the kinematical factor $m_{\pi^0}^3$ is taken to be nonzero, keeping the lowest order in the chiral expansion. Clearly, for $B = 0$ this equation reduces to the well-known result obtained in ChPT [76].

IV. NUMERICAL RESULTS

In this section we provide numerical results arising from our calculation of the $\pi^0 \rightarrow \gamma\gamma$ decay width. The values of related quantities, such as m_{π^0} and $g_{\pi^0 qq}$, are estimated using the SU(2) version of the local NJL model. For the model parameters, we take the values $m_0 = 5.419$ MeV, $\Lambda = 6.395$ MeV, and $G\Lambda^2 = 2.136$, which (for vanishing external magnetic field) correspond to a quark effective mass $M = 350$ MeV and a quark-antiquark condensate $\langle \bar{f}f \rangle = (-243.3 \text{ MeV})^3$. This parametrization, denoted as S350, properly reproduces the empirical values of m_{π^0} and f_{π^0} in vacuum, namely, $m_{\pi^0} = 135$ MeV and

TABLE I. Model parameters and predictions for $g_{\pi^0 qq}$ and $\Gamma_{\pi^0 \gamma \gamma}$ at $B = 0$.

	$-\langle q\bar{q} \rangle^{1/3}$ MeV	m_0 MeV	$G\Lambda^2$	Λ MeV	M MeV	$g_{\pi^0 qq}$	$\Gamma_{\pi^0 \gamma \gamma}$ eV
S320	246.9	5.185	2.138	639.5	320	3.396	7.67
S350	243.3	5.418	2.252	613.4	350	3.720	7.65
S380	241.4	5.543	2.366	596.1	380	4.046	7.65

$f_\pi = 92.4$ MeV. To test the sensitivity of our results to the model parametrization, we have also considered two alternative parameter sets, denoted as S320 and S380, which also reproduce the phenomenological values of m_{π^0} and f_π in vacuum, and lead to quark effective masses $M = 320$ and 380 MeV, respectively. The corresponding model parameters are listed in Table I together with the predictions for $g_{\pi^0 qq}$ and $\Gamma_{\pi^0 \gamma \gamma}$ at $B = 0$. As one can see, the values of $\Gamma_{\pi^0 \gamma \gamma}$ are all compatible with the experimental range $\Gamma_{\pi^0 \gamma \gamma}^{\text{exp}} = (7.72 \pm 0.11)$ eV within less than 1% accuracy.

In addition, since local NJL models fail to describe the IMC effect observed in LQCD at finite temperature, we also consider a NJL Lagrangian with a magnetic-field-dependent four-quark interaction coupling $G(B)$, so as to

account for the effect of the magnetic field on sea quarks. For definiteness, we adopt for $G(B)$ the form proposed in Ref. [21], namely,

$$G(B) = G\mathcal{F}(B), \quad (56)$$

where

$$\mathcal{F}(B) = \kappa_1 + (1 - \kappa_1)e^{-\kappa_2(eB)^2}, \quad (57)$$

with $\kappa_1 = 0.321$ and $\kappa_2 = 1.31 \text{ GeV}^{-2}$. Even though $G(B)$ does not appear explicitly in our expressions for the decay width $\Gamma_{\pi^0 \gamma \gamma}$, the B dependence affects the values of M , m_{π^0} , and $g_{\pi^0 qq}$, which implicitly depend on the four-quark coupling strength.

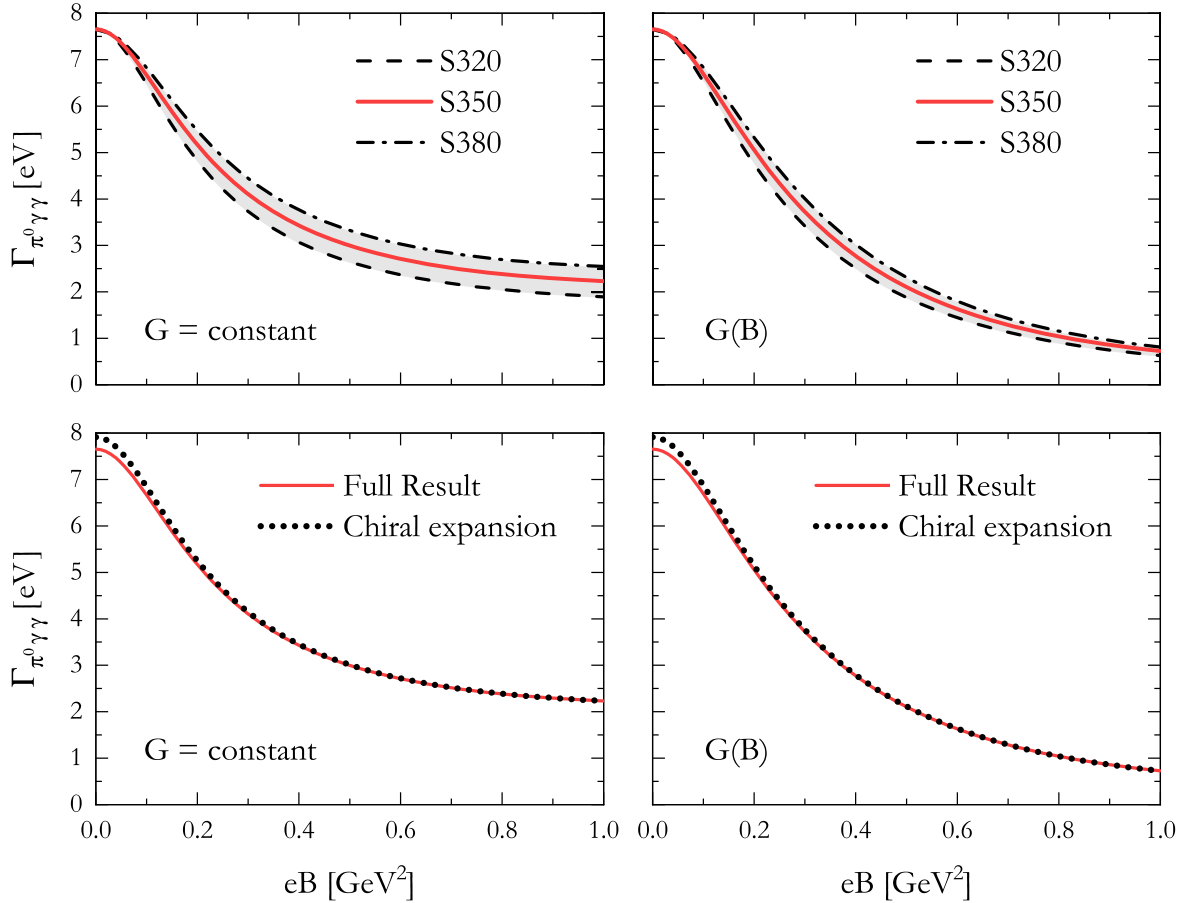


FIG. 2. $\pi^0 \rightarrow \gamma\gamma$ decay width as a function of eB , for $G = \text{constant}$ (left) and $G(B)$ (right). Solid red lines correspond to parameter set S350. Band limits in upper panels correspond to S320 (dashed lines) and S380 (dash-dotted lines). Dotted lines in lower panels correspond to the lowest-order chiral expansion in Eq. (55).

Our numerical results for the $\pi^0 \rightarrow \gamma\gamma$ decay width in the presence of the external magnetic field are collected in Fig. 2. In the upper panels, we show our results for $\Gamma_{\pi^0\gamma\gamma}$ for the three parameter sets of Table I, considering both a constant coupling G (left) and a magnetic-field-dependent coupling $G(B)$ (right). As seen, the NJL model predicts a rather strong suppression of the width, which is even more striking in the case of $G(B)$. We can also observe that this feature is basically independent of the model parametrization. Before discussing the possible origin of this suppression, it is convenient to compare the above results with the predictions arising from Eq. (55), obtained at the lowest order in the current quark mass expansion. This comparison is shown in the lower panels of Fig. 2 for the parameter set S350. As seen from the graphs, our calculations show that for both $G = \text{constant}$ (left) and $G(B)$ (right), the result in Eq. (55) is actually a good approximation to the full width. Indeed, the accuracy of this approximation is found to be less than 4% at $B = 0$, becoming even smaller as the magnetic field increases.

The similarity between the numerical results for $\Gamma_{\pi^0\gamma\gamma}(B)$ and $\Gamma_{\pi^0\gamma\gamma}^{\text{LO}}(B)$ can be used as a hint to trace the origin of the width suppression with the external magnetic field. In this sense, we can take into account the fact that at the lowest order in the chiral expansion, there is a simple relation between the $\pi^0 \rightarrow \gamma\gamma$ width, the pion mass, and the pion decay constant. The behavior of these three quantities as functions of B is shown in Fig. 3 for set S350. Looking at

these curves, and considering Eq. (55), it becomes clear that the width suppression originates from the joint effect of the decrease of the neutral pion mass and the increase of $f_{\pi^0}^{\text{ch}}$ with the magnetic field. In fact, it is seen that both effects turn out to be numerically important. We note in this sense that in Ref. [60], where the width calculation was performed within the framework of ChPT, only the magnetic behavior of the pion mass was taken into account. It is interesting to observe that the magnetic behavior of the pion decay constant can be qualitatively understood taking into account the generalized Gell-Mann-Oakes-Renner relation [29] $(m_{\pi^0} f_{\pi^0}^{\text{ch}})^2 = -m_0 \langle \bar{u}u + \bar{d}d \rangle^{\text{ch}}/2$ (valid at lowest order in the current quark mass m_0), which explains the enhancement of $f_{\pi^0}^{\text{ch}}$ with B in terms of both the magnetic catalysis effect and the decrease in m_{π^0} as B increases. Regarding the use of $G(B)$, it is seen from Fig. 3 that magnetic effects on m_{π^0} and $f_{\pi^0}^{\text{ch}}$ are more pronounced when considering $G(B)$ in comparison with the case of a constant G . This leads to a stronger suppression of the width for $G(B)$ that also holds away from the chiral limit, as shown in Fig. 2.

Finally, the angular dependence of the decaying photon rate can be analyzed from Eq. (50). Though in principle the differential width $d\Gamma(B)/d\cos\theta$ depends on the polar angle θ between the magnetic field and one of the photon three-momenta, it is seen that the θ -dependent contribution is suppressed by powers of the ratio $(m_{\pi^0}/M)^2$. Indeed, from our numerical calculations we find that for values of

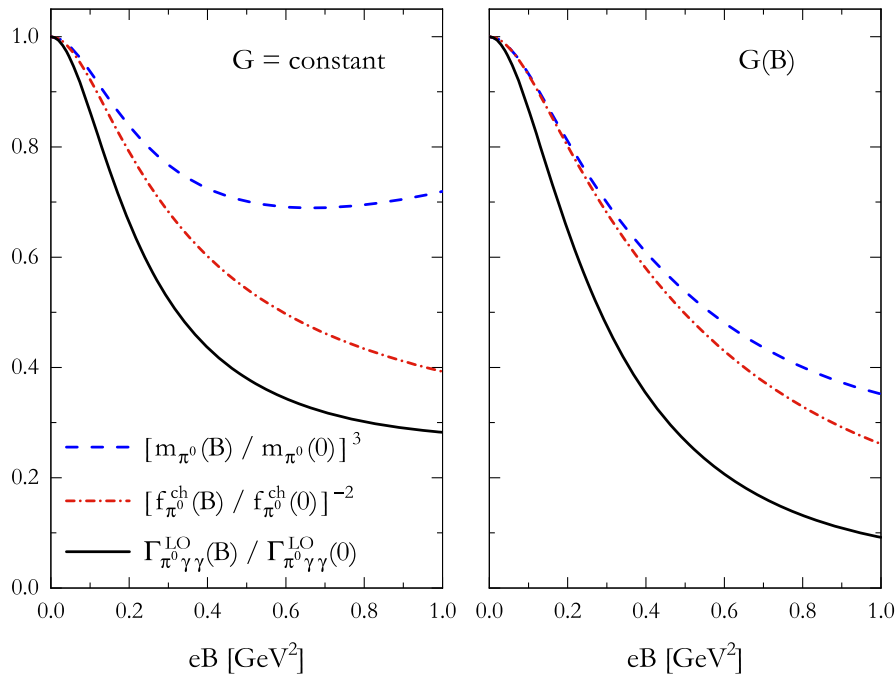


FIG. 3. Cubed π^0 mass (dashed lines), inverse squared pion decay constant (dashed-dotted lines), and $\pi^0 \rightarrow \gamma\gamma$ decay width (solid lines) at lowest order in ChPT as functions of eB . All three quantities are normalized with respect to their values at $B = 0$. The cases of a constant coupling G and a B -dependent coupling $G(B)$ are shown in the left and right panels, respectively. All values correspond to the parameter set S350.

eB up to 1 GeV^2 , the differential width varies less than 0.1% within the full range of values of θ .

V. SUMMARY AND CONCLUSIONS

In this work we have studied the anomalous $\pi^0 \rightarrow \gamma\gamma$ decay in the presence of a static uniform external magnetic field. Our analysis has been performed in the framework of a two-flavor Nambu-Jona-Lasinio effective model for low-energy QCD dynamics, in which pions are treated as quantum fluctuations in the random phase approximation. The ultraviolet divergences associated with the nonrenormalizability of the model have been regularized using the magnetic-field-independent regularization method, which has been shown to be free from unphysical oscillations and to reduce the dependence of the results on the model parameters [66]. Additionally, we have explored the possibility of using a magnetic-field-dependent coupling constant $G(B)$ to account for the effect of the magnetic field on sea quarks.

Our calculations indicate that the $\pi^0 \rightarrow \gamma\gamma$ decay width gets strongly reduced with the external field, a fact that can be more easily understood by considering the current quark mass expansion. At leading order in m_{π^0} , we find that the result for the full width reduces to a simple expression, which turns out to be just a direct extension of the well-known $B = 0$ result that follows from the anomalous gauged Wess-Zumino-Witten action to a finite magnetic field. That expression linearly relates $\Gamma_{\pi^0\gamma\gamma}^{\text{LO}}$ with the cube of the pion mass and the inverse of the squared pion decay constant, $(f_{\pi^0}^{\text{ch}})^{-2}$, see Eq. (55). Since m_{π^0} decreases while $f_{\pi^0}^{\text{ch}}$ increases with B , the joint effect implies a strong suppression of $\Gamma_{\pi^0\gamma\gamma}^{\text{LO}}$ with the external field. Our analysis also shows that the effects of both the behavior of m_{π^0} and $f_{\pi^0}^{\text{ch}}$ with B contribute significantly to the width reduction. Moreover, it is seen that this qualitative analysis can be extrapolated away from the chiral limit: from our numerical results, it is found that the calculation of $\Gamma_{\pi^0\gamma\gamma}$ at the lowest order in the chiral expansion is actually a good approximation to the full calculation, with the differences lying below 4% for values of eB up to 1 GeV^2 . The full analysis is found to be basically independent of the model parametrization, within phenomenologically acceptable parameter ranges. Regarding the use of a B -dependent four-quark coupling, we observe that magnetic effects on m_{π^0} and f_{π^0}

are more pronounced when considering $G(B)$ with respect to the case of constant G . This leads to an even stronger suppression of the $\pi^0 \rightarrow \gamma\gamma$ width.

As for the angular dependence of the decaying photons, an anisotropic angular distribution could be, in principle, expected, since rotational symmetry is broken by the magnetic field. Nevertheless, we find that angular-dependent contributions to the differential decay width $d\Gamma(B)/d\cos\theta$ get suppressed by powers of the ratio $(m_{\pi^0}/M)^2$, leading to nearly perfect isotropy: for magnetic fields as large as $eB = 1 \text{ GeV}^2$, $d\Gamma(B)/d\cos\theta$ varies less than 0.1% within the full range of values of θ .

In this work, we have considered only leading-order dynamical electromagnetic contributions, taking vacuum dispersion relations for the outgoing photons and dismissing possible effects arising from $\pi^0 - \gamma$ mixing. We leave the analysis of the effect of the corresponding corrections for future studies.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [77].

APPENDIX A: π^0 POLARIZATION FUNCTION

The regularized form of the π^0 polarization function, Eq. (23), in the MFIR scheme can be written as

$$\Pi_{\pi^0}^{\text{reg}}(q_{\parallel}^2, q_{\perp}^2) = \Pi_{\pi^0}^{0,\text{reg}}(q^2) + \Pi_{\pi^0}^{\text{mag}}(q_{\parallel}^2, q_{\perp}^2). \quad (\text{A1})$$

Here, the regularized “ $B = 0$ ” contribution, obtained using a 3D momentum cutoff Λ , can be written as

$$\Pi_{\pi^0}^{0,\text{reg}}(q^2) = 2N_c [I^{0,\text{reg}} - q^2 I_2^{0,\text{reg}}(q^2)], \quad (\text{A2})$$

where $I^{0,\text{reg}}$ is given by Eq. (21), while $I_2^{0,\text{reg}}$ reads [29,72]

$$I_2^{0,\text{reg}}(q^2) = \frac{1}{4\pi^2} \int_0^1 dy \left[\frac{\Lambda}{\sqrt{\Lambda^2 + M^2 - y(1-y)q^2}} + \ln \frac{\sqrt{M^2 - y(1-y)q^2}}{\Lambda + \sqrt{\Lambda^2 + M^2 - y(1-y)q^2}} \right]. \quad (\text{A3})$$

On the other hand, the magnetic contribution can be written as [29]

$$\begin{aligned} \Pi_{\pi^0}^{\text{mag}}(q_{\parallel}^2, q_{\perp}^2) = & \frac{N_c}{4\pi^2} \sum_{f=u,d} \int_0^\infty dz \int_0^1 dy e^{-z[M^2 - y(1-y)q_{\parallel}^2]} \left\{ e^{-\omega_f(y,z)\vec{q}_{\perp}^2} \left[\left(M^2 + \frac{1}{z} + y(1-y)q_{\parallel}^2 \right) \frac{B_f}{\tanh(zB_f)} \right. \right. \\ & \left. \left. + \frac{B_f^2}{\sinh^2(zB_f)} (1 - \omega_f(y,z)\vec{q}_{\perp}^2) \right] - \frac{e^{-zy(1-y)\vec{q}_{\perp}^2}}{z} \left[M^2 + \frac{2}{z} + y(1-y)q_{\parallel}^2 \right] \right\}, \end{aligned} \quad (\text{A4})$$

where

$$\omega_f(y, z) = \frac{\sinh(yzB_f) \sinh[(1-y)zB_f]}{B_f \sinh(zB_f)}. \quad (\text{A5})$$

For a pion at rest, this expression can be rewritten in terms of the digamma function, see Ref. [34] for details.

APPENDIX B: COMPUTATION OF THE $\pi^0 \rightarrow \gamma\gamma$ DECAY AMPLITUDE

To evaluate the decay amplitude we need to calculate the tensor $R_f^{\mu\nu}$ given by Eq. (40). Taking into account the Schwinger form of quark propagators in Eqs. (11) and (12), after integration over the “parallel” coordinates $s_{\parallel}$ and $t_{\parallel}$, we obtain

$$R_f^{\mu\nu} = i16\pi^2 N_c g_{\pi^0 qq} \frac{q_f^2}{B_f^2} \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_0^\infty d\tau_3 e^{-(\tau_1 + \tau_2 + \tau_3)M^2} \int \frac{d^2 r_{\parallel}}{(2\pi)^2} e^{\tau_1(r_{\parallel} - p_{\parallel})^2 + \tau_2 r_{\parallel}^2 + \tau_3(r_{\parallel} + k_{\parallel})^2} \sum_{\varepsilon=\pm 1} \mathcal{N}_{\varepsilon f}^{\mu\nu}, \quad (\text{B1})$$

where

$$\begin{aligned} \mathcal{N}_{\varepsilon f}^{\mu\nu} = & \int \frac{d^2 r_{\perp}}{(2\pi)^2} \frac{d^2 s_{\perp}}{(2\pi)^2} \frac{d^2 t_{\perp}}{(2\pi)^2} \exp \left\{ -\frac{1}{B_f} (t_1 \vec{s}_{\perp}^2 + t_2 \vec{r}_{\perp}^2 + t_3 \vec{t}_{\perp}^2) \right\} \\ & \times \exp \left\{ i \frac{2\varepsilon s_f}{B_f} [(p^1 + s^1 - r^1)(k^2 + r^2 - t^2) - (k^1 + r^1 - t^1)(p^2 + s^2 - r^2)] \right\} T_{\varepsilon f}^{\mu\nu}, \end{aligned} \quad (\text{B2})$$

with $t_i = \tanh(\tau_i B_f)$. Here, $T_{\varepsilon f}^{\mu\nu}$ stands for the Dirac trace

$$T_{\varepsilon f}^{\mu\nu} = \text{Tr}_D [i\gamma^5 iC_{\varepsilon f}(r_{\parallel} - p_{\parallel}, s_{\perp}, t_1) \gamma^{\mu} iC_{\varepsilon f}(r_{\parallel}, r_{\perp}, t_2) \gamma^{\nu} iC_{\varepsilon f}(r_{\parallel} + k_{\parallel}, t_{\perp}, t_3)], \quad (\text{B3})$$

where

$$C_{\varepsilon f}(u_{\parallel}, v_{\perp}, t_i) = (u_{\parallel} \cdot \gamma_{\parallel} + M)[1 + i\varepsilon s_f \gamma^1 \gamma^2 t_i] - \vec{v}_{\perp} \cdot \vec{\gamma}(1 - t_i^2). \quad (\text{B4})$$

From explicit computation of the traces in Eq. (B3), we obtain

$$\begin{aligned} T^{00} &= -8\varepsilon s_f M T_f p^0 p^3, \\ T^{11} &= T^{22} = 8\varepsilon s_f M [t_1 + t_3 - t_2(1 + t_1 t_3)] p^0 p^3, \\ T^{33} &= -8\varepsilon s_f M T_f p^0 (2r^3 - p^3), \\ T^{01} \pm iT^{02} &= \pm 4M f_1^{\pm} f_2^{\pm} f_3^{\mp} [f_1^{\mp}(s^1 \pm is^2) - f_3^{\pm}(t^1 \pm it^2)] p^3, \\ T^{10} \pm iT^{20} &= \mp 4M f_1^{\pm} f_2^{\mp} f_3^{\mp} [f_1^{\mp}(s^1 \pm is^2) - f_3^{\pm}(t^1 \pm it^2)] p^3, \\ T^{03} + T^{30} &= -16\varepsilon s_f M T_f r^0 p^0, \\ T^{03} - T^{30} &= 8M\varepsilon s_f \{ T_f [M^2 + p_{\parallel}^2 - r_{\parallel}^2 + 2p^3(p^3 + r^3)] + t_1(1 - t_2^2)(1 - t_3^2)(\vec{r}_{\perp} \cdot \vec{t}_{\perp}) \\ &\quad + t_2(1 - t_1^2)(1 - t_3^2)(\vec{s}_{\perp} \cdot \vec{t}_{\perp}) + t_3(1 - t_1^2)(1 - t_2^2)(\vec{r}_{\perp} \cdot \vec{s}_{\perp}) \} \\ &\quad + 8iM\varepsilon^{ij3} \{ r^i t^j (1 - t_2^2)(1 - t_3^2) + t^i s^j (1 - t_1^2)(1 - t_3^2) + s^i r^j (1 - t_1^2)(1 - t_2^2) \}, \\ T^{12} &= -T^{21} = 8iM [1 + t_1 t_3 - t_2(t_1 + t_3)] p^0 p^3, \\ T^{13} \pm iT^{23} &= \pm 4M f_1^{\pm} f_2^{\mp} \{ f_3^{\mp} [f_1^{\mp}(s^1 \pm is^2) + f_3^{\pm}(t^1 \pm it^2)] - 2f_2^{\pm} f_3^{\pm}(r^1 \pm ir^2) \} p^0, \\ T^{31} \pm iT^{32} &= \mp 4M f_2^{\pm} f_3^{\mp} \{ f_1^{\pm} [f_1^{\mp}(s^1 \pm is^2) + f_3^{\pm}(t^1 \pm it^2)] - 2f_2^{\mp} f_1^{\mp}(r^1 \pm ir^2) \} p^0, \end{aligned} \quad (\text{B5})$$

where we have defined $f_i^\pm = 1 \pm \varepsilon s_f t_i$ and $T_f = t_1 + t_2 + t_3 + t_1 t_2 t_3$. We have also considered the usual photon on-shell conditions $k^2 = p^2 = 0$ and taken into account the fact that in the rest frame of the pion, one has $k^0 = p^0$, $k^3 = -p^3$.

To calculate the integrals over perpendicular momenta in Eq. (B2), notice that the traces include up to quadratic powers of $r_\perp$, $s_\perp$, and $t_\perp$ coordinates. In fact, we are left with Gaussian integrals that can be straightforwardly performed. Let us define

$$I_\perp(u_\pm^n) = \int \frac{d^2 r_\perp}{(2\pi)^2} \frac{d^2 s_\perp}{(2\pi)^2} \frac{d^2 t_\perp}{(2\pi)^2} (u^1 \pm i u^2)^n \times \exp\left\{-\frac{1}{B_f}(t_1 \vec{s}_\perp^2 + t_2 \vec{r}_\perp^2 + t_3 \vec{t}_\perp^2)\right\} \times \exp\left\{i \frac{2\varepsilon s_f}{B_f}[(p^1 + s^1 - r^1)(k^2 + r^2 - t^2) - (k^1 + r^1 - t^1)(p^2 + s^2 - r^2)]\right\}, \quad (\text{B6})$$

where $n = 0, 1$ and $u = r, s, t$. Taking $\vec{k}_\perp = -\vec{p}_\perp$, for $n = 0$ we obtain

$$I_\perp(1) = \frac{B_f^3}{64\pi^3 T_f} \exp\left[-\frac{t_2(t_1 + t_3)}{B_f T_f} \vec{p}_\perp^2\right], \quad (\text{B7})$$

while for $n = 1$ we have

$$I_\perp(r_\pm) = \frac{I_\perp(1)}{T_f} (t_1 + t_3)(p^1 \pm i p^2), \quad (\text{B8})$$

$$I_\perp(s_\pm) = -\frac{I_\perp(1)}{T_f} t_2 f_3^\pm (p^1 \pm i p^2), \quad (\text{B9})$$

$$I_\perp(t_\pm) = -\frac{I_\perp(1)}{T_f} t_2 f_1^\mp (p^1 \pm i p^2). \quad (\text{B10})$$

From the above expressions, it can be seen that $\mathcal{N}_{ef}^{\mu\nu} = 0$ for $\mu\nu = 01, 02, 10$, and 20 . On the other hand, notice that for $\mu = \nu$ the expressions of $\mathcal{N}_{ef}^{\mu\nu}$ depend linearly on ε . Thus, the corresponding sums over $\varepsilon = \pm 1$ also vanish. In the case of $\mathcal{N}_{ef}^{30}$ and $\mathcal{N}_{ef}^{03}$, one is faced with terms that are linear in ε and with integrals of terms that are quadratic in perpendicular coordinates. For the latter, in the integrals we can shift $r_\perp \rightarrow r'_\perp = r_\perp - p_\perp$ and choose the 1-axis in the direction of $\vec{p}_\perp$ (i.e., we can take $p^2 = 0$). Then, changing $r'^2 \rightarrow -r'^2$, $s^2 \rightarrow -s^2$, and $t^2 \rightarrow -t^2$, it is seen that the integrals change sign by changing $\varepsilon \rightarrow -\varepsilon$, which leads to $\sum_\varepsilon \mathcal{N}_{ef}^{30} = \sum_\varepsilon \mathcal{N}_{ef}^{03} = 0$. The only nonvanishing contributions to the amplitude can be obtained from the sums

$$\sum_{\varepsilon=\pm} (\mathcal{N}_{ef}^{13} \pm i \mathcal{N}_{ef}^{23}) = -\sum_{\varepsilon=\pm} (\mathcal{N}_{ef}^{31} \pm i \mathcal{N}_{ef}^{32}) = \mp 16 M I_\perp(1) [1 + t_1 t_3 - t_2(t_1 + t_3)] p^0 \times (p^1 \pm i p^2), \quad (\text{B11})$$

$$\sum_{\varepsilon=\pm} \mathcal{N}_{ef}^{12} = -\sum_{\varepsilon=\pm} \mathcal{N}_{ef}^{21} = 16 i M I_\perp(1) [1 + t_1 t_3 - t_2(t_1 + t_3)] p^0 p^3. \quad (\text{B12})$$

In fact, the above results for $\sum_\varepsilon \mathcal{N}_{ef}^{\mu\nu}$ can be more compactly written as

$$\sum_{\varepsilon=\pm} \mathcal{N}_{ef}^{\mu\nu} = 16 i M I_\perp(1) [1 + t_1 t_3 - t_2(t_1 + t_3)] p^0 \varepsilon^{0\mu\nu\lambda} p_\lambda. \quad (\text{B13})$$

Turning back to the expression of $R_f^{\mu\nu}$ in Eq. (B1), we are left with the integral over parallel components of r ,

$$I_\parallel = \int \frac{d^2 r_\parallel}{(2\pi)^2} e^{\tau_1(r_\parallel - p_\parallel)^2 + \tau_2 r_\parallel^2 + \tau_3(r_\parallel + k_\parallel)^2} \quad (\text{B14})$$

(notice that the result for $\sum_\varepsilon \mathcal{N}_{ef}^{\mu\nu}$ does not depend on $r_\parallel$). Performing a Wick rotation in the variable r^0 we get, in the pion rest frame,

$$I_\parallel = \frac{i}{4\pi(\tau_1 + \tau_2 + \tau_3)} \exp\left[\frac{4\tau_1\tau_3(p^0)^2 + (\tau_1 + \tau_3)\tau_2 p_\parallel^2}{\tau_1 + \tau_2 + \tau_3}\right]. \quad (\text{B15})$$

In this way, we end up with

$$R_f^{\mu\nu} = 256\pi^2 N_c g_{\pi^0 qq} M p^0 \varepsilon^{0\mu\nu\lambda} p_\lambda \frac{q_f^2}{B_f^2} \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \times \int_0^\infty d\tau_3 e^{-(\tau_1 + \tau_2 + \tau_3)M^2} I_\perp(1) I_\parallel [1 + t_1 t_3 - t_2(t_1 + t_3)], \quad (\text{B16})$$

which, after replacing the integrals $I_\perp(1)$ and $I_\parallel$ by the corresponding expressions in Eqs. (B7) and (B15), leads to the result in Eq. (42).

APPENDIX C: POLARIZATIONS OF OUTGOING PHOTONS

The polarization vectors $\epsilon^\mu(p, \pm 1)$ for an on-shell photon with momentum $\vec{p}$ are given by

$$\epsilon^0(p, \pm 1) = 0, \quad \vec{\epsilon}(p, \pm 1) = \frac{e^{\pm i\phi}}{\sqrt{2}|\hat{p}_\perp|} (i\hat{p}^2 \mp \hat{p}^1 \hat{p}^3, -i\hat{p}^1 \mp \hat{p}^2 \hat{p}^3, \pm |\hat{p}_\perp|^2), \quad (\text{C1})$$

where $\hat{p} = \vec{p}/|\vec{p}|$, $\hat{p}_\perp = \vec{p}_\perp/|\vec{p}|$, and ϕ is an arbitrary phase. They satisfy

$$\begin{aligned} \epsilon(p, \pm 1) \cdot \epsilon(p, \pm 1)^* &= -1, & \epsilon(p, +1) \cdot \epsilon(p, -1)^* &= 0, \\ \epsilon(p, \pm 1) \cdot p &= 0, & \vec{\epsilon}(p, \pm 1) \cdot \vec{p} &= 0, \end{aligned} \quad (C2)$$

as well as

$$\sum_{\lambda_p=\pm} \epsilon^\mu(p, \lambda_p) \epsilon^\nu(p, \lambda_p)^* = -\eta^{\mu\nu} + \frac{p^\mu \tilde{p}^\nu + \tilde{p}^\mu p^\nu}{p \cdot \tilde{p}}, \quad (C3)$$

where $\tilde{p}^\mu = (p^0, -\vec{p})$.

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