

Krylov complexity has it all

Wolfgang Mück 

Dipartimento di Fisica “Ettore Pancini,” Università degli Studi di Napoli Federico II Via Cintia, 80126 Napoli, Italy and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli Via Cintia, 80126 Napoli, Italy



(Received 4 June 2026; accepted 9 September 2026; published 28 September 2026)

This paper establishes that Krylov complexity contains the entire information about the dynamics of a quantum operator, extending the list of equivalent quantities that can serve this purpose, such as the Lanczos coefficients, the return amplitude, and the spectral density. To demonstrate this equivalence, an explicit recursive algorithm is constructed to calculate Lanczos coefficients from the Taylor expansion of the Krylov complexity around $t = 0$. Furthermore, the paper discusses the distinction between Krylov and spread complexity, clarifying why a similar recursive algorithm cannot exist for the latter without additional dynamical input. These results provide a “proof of principle” for using Krylov complexity as a complete characterization of operator evolution in quantum systems.

DOI: [10.1103/3g6c-jx8t](https://doi.org/10.1103/3g6c-jx8t)

Introduction. The definition of Krylov complexity [1] and the proposal of its use as a measure to characterize chaos in operator dynamics [1–5] has revived interest in the recursion method [6] and related Krylov subspace techniques based on the Lanczos algorithm [7]. Krylov complexity, in simple terms, is a time-dependent measure for the effective size (dimension) of the subspace of operators that a given observable explores during its time evolution. A similar quantity for quantum states is spread complexity, which measures the spreading of states in Hilbert space during their time evolution [8–10]. The excellent reviews [11–13] provide an overview of these recent developments, with applications in fields that range from quantum computing to quantum gravity.

It has been known for some time [6] that the dynamics of an operator $\mathcal{O}(t)$ can be represented in a variety of equivalent ways and that a number of quantities independently contain the entire information about this dynamics. This derives from the following chain of reasoning: In the Heisenberg picture, an operator $\mathcal{O}$ time-evolves as $\mathcal{O}(t) = e^{it\mathcal{L}}\mathcal{O}$, where $\mathcal{L} = [H, \cdot]$ is the Liouvillian. Given an operator inner product $(\mathcal{O}|\mathcal{O}')$, the Lanczos algorithm diagonalizes the set of operators $\mathcal{L}^n\mathcal{O}$, $n = 0, 1, 2, \dots$, providing an ordered basis of operators $\mathcal{O}_n$ known as the Krylov basis, which satisfy the recurrence relations $\mathcal{L}\mathcal{O}_n = b_{n+1}\mathcal{O}_{n+1} + b_n\mathcal{O}_{n-1}$. The (non-negative) real numbers b_n are known as Lanczos coefficients and represent the

Liouvillian acting on the Krylov basis. Moreover, defining a chain of wave functions $\phi_n(t) \sim (\mathcal{O}_n|\mathcal{O}(t))$, the operator dynamics maps onto a hopping model on this (semi-infinite) chain. Furthermore, the recurrence relations define a set of real orthogonal polynomials with an associated orthogonality measure. Therefore, the following quantities contain, independently, the entire information about the dynamics of $\mathcal{O}(t)$: (i) the set of Lanczos coefficients, (ii) the return amplitude $\phi_0(t)$ or its Laplace transform, (iii) the spectral density associated with the orthogonal polynomials, and (iv) the moments of the spectral density, because they also appear as coefficients in the Taylor expansion of the return amplitude.

In this paper, I will establish that the above list can be extended by (v) Krylov complexity $K(t)$. In other words, knowing the Krylov complexity of an operator $\mathcal{O}(t)$ is equivalent to knowing its dynamics. For this purpose, I will explicitly construct an algorithm that recursively calculates the Lanczos coefficients from the coefficients of the Taylor expansion of $K(t)$ around $t = 0$. This algorithm also can serve as a test whether a given (even) function $K(t)$ is compatible with a Krylov complexity. I will also discuss why this does not apply to spread complexity and provide a hint on what would be needed in this case.

Lanczos coefficients from Krylov complexity. The starting point is the hopping model on the one-dimensional chain associated with the time evolution of $\mathcal{O}(t)$ in the Krylov basis [1,6],¹

¹We follow the convention of [14] absorbing factors of i into the wave functions $\phi_n(t)$ such that all of them are real functions.

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$$\partial_t \phi_n(t) = -b_{n+1} \phi_{n+1}(t) + b_n \phi_{n-1}(t), \quad \phi_n(0) = \delta_{n,0}. \quad (1)$$

Note that $b_n \geq 0$ is ensured by the Lanczos algorithm and $b_n = 0$ for $n = N$ terminates it, leading to an N -dimensional Krylov space (and a chain of length N). Define also

$$\Delta_n = b_n^2, \quad h_n = \prod_{k=1}^n \Delta_k, \quad h_0 = 1, \quad (2)$$

which naturally appear in the monic version of the Lanczos algorithm [14] as the Lanczos coefficients (Δ_n) and the norm of the n th Krylov state (h_n). Then, it is clear that the solutions $\phi_n(t)$ of (1) have Taylor expansions of the form

$$\phi_n(t) = \sqrt{h_n} \sum_{m=0}^{\infty} C_{n+m,m} \frac{t^{n+2m}}{(n+2m)!}, \quad (3)$$

where the prefactor has been inserted for convenience. The hopping equation (1) translates into the following recurrence formula for the matrix elements $C_{k,l}$:

$$C_{k,l} = -\Delta_{k-l+1} C_{k,l-1} + C_{k-1,l}. \quad (4)$$

The initial condition in (1) implies $C_{0,0} = 1$, and then (4) immediately gives $C_{k,0} = 1 \forall k$. The matrix $C_{k,l}$ is lower triangular. An important point to remember is that the row C_p only contains the Lanczos coefficients Δ_q with $q \leq p$.

Consider the Krylov complexity

$$K(t) = \sum_{n=0}^{\infty} n \phi_n(t)^2. \quad (5)$$

After substituting (3) and performing a number of summation reorderings, one obtains

$$K(t) = \sum_{p=0}^{\infty} B_p \frac{t^{2p}}{(2p)!} \quad (6)$$

with the coefficients²

$$B_p = \sum_{n=0}^p \sum_{k=0}^n \binom{2p}{p-n+2k} (p-n) h_{p-n} C_{p-n+k,k} C_{p-k,n-k}. \quad (7)$$

The sum on the right-hand side of (7) is symmetric under $k \leftrightarrow n-k$. In other words, if one considers the (n, k)

plane, on which the sums in (7) are represented by the triangle $0 \leq n \leq p$, $0 \leq k \leq n$, the sum over the points above the line $n = 2k$ is equal to the sum over the points below it. Therefore, (7) can be expressed as

$$B_p = \sum_{k=0}^{\lfloor p/2 \rfloor} \left[2 \sum_{n=0}^{p-2k-1} \binom{2p}{p-n} (p-n-2k) \times h_{p-n-2k} C_{p-n-k,k} C_{p-k,n+k} - \binom{2p}{p} (p-2k) h_{p-2k} C_{p-k,k} C_{p-k,k} \right]. \quad (8)$$

The term on the second line compensates the double summation over the points on the line $n = 2k$.

By simple inspection, one finds that B_p contains only the coefficients Δ_q with $q \leq p$, and that the coefficient Δ_p only appears in the terms with $k = 0$. Thus, (8) can be split into

$$B_p = B_{p,1} + 2 \sum_{n=0}^{p-1} \binom{2p}{p-n} (p-n) h_{p-n} C_{p,n} - p \binom{2p}{p} h_p \quad (9)$$

with

$$B_{p,1} = \sum_{k=1}^{\lfloor p/2 \rfloor} \left[2 \sum_{n=0}^{p-2k-1} \binom{2p}{p-n} (p-n-2k) \times h_{p-n-2k} C_{p-n-k,k} C_{p-k,n+k} - \binom{2p}{p} (p-2k) h_{p-2k} C_{p-k,k} C_{p-k,k} \right]. \quad (10)$$

The $k = 0$ terms in (9) can be rewritten by recursively applying (4) and using identities and sums for the binomial coefficients as

$$\begin{aligned} & 2 \sum_{n=0}^{p-1} \binom{2p}{p-n} (p-n) h_{p-n} C_{p,n} - p \binom{2p}{p} h_p \\ &= 2 \binom{2p-2}{p-1} h_p + B_{p,2} \end{aligned} \quad (11)$$

with

$$B_{p,2} = 4p \sum_{n=0}^{p-2} \binom{2p-2}{p-n-2} h_{p-n-1} C_{p-1,n+1}, \quad (12)$$

which contains only Δ_q with $q < p$. Thereby, the term with h_p has been explicitly isolated in B_p .

Using the above formulas, the algorithm given in Table I recursively calculates the Lanczos coefficients Δ_p from

²If, instead of the Krylov complexity, one considers the norm $\sum_{n=0}^{\infty} \phi_n(t)^2 = 1$, one obtains similar coefficients A_p , but without the factor $(p-n)$. It is a nice exercise to show that (4) ensures $A_p = \delta_{p,0}$.

TABLE I. Krylov complexity to Lanczos coefficients algorithm.

Input	Coefficients B_p ($p = 1, 2, \dots, N$) in the Taylor series (6)
Start	$h_0 = 1, C_{0,0} = 1$
$p = 1$	$h_1 = \frac{1}{2}B_1, \Delta_1 = \frac{h_1}{h_0}$, stop, if $\Delta_1 \leq 0$. Prepare the data needed for the next step: $C_{1,0} = 1, C_{1,1} = -\Delta_1$.
$2 \leq p \leq N$	Calculate $B_{p,1}$ from (10) and $B_{p,2}$ from (12). $h_p = \frac{B_p - B_{p,1} - B_{p,2}}{2^{\binom{p-2}{p-1}}}, \Delta_p = \frac{h_p}{h_{p-1}}$, stop, if $\Delta_p \leq 0$. Prepare the data needed for the next step: $C_{p,0} = 1, C_{p,q} = -\Delta_{p-q+1}C_{p,q-1} + C_{p-1,q}$, $q = 1, \dots, p$.
Result	Lanczos coefficients Δ_p ($p = 1, 2, \dots, N$) or incompatibility of the input if the algorithm stops at $p < N$

the Taylor series coefficients B_p of Krylov complexity. Because the number of input parameters (N) is, in principle, arbitrary, the Krylov complexity $K(t)$ contains the full information about the entire set of Lanczos coefficients.

This does not mean that given any set of coefficients B_p obtained from the Taylor series of some even function $F(t^2)$ with $F(0) = 0$, one can find a set of Lanczos coefficients that produces the Krylov complexity $K(t) = F(t^2)$. In fact, the algorithm also serves as a test of whether the input coefficients B_p are compatible with a Krylov complexity. The caveat is that the construction involves the condition $\Delta_p \geq 0 \forall p$. Therefore, as soon as the algorithm returns a

single $\Delta_p < 0$, it is clear that the input coefficients cannot represent a Krylov complexity. In the case in which the algorithm stops at some $p = P$ because $\Delta_P = 0$, one may be tempted to say that the Krylov space has dimension P , but then the remaining input coefficients B_p for $p > P$ must uniquely depend on the Lanczos coefficients Δ_q with $q < P$ if they are to represent a Krylov complexity. Of course, this cannot be guaranteed either, and in most cases they simply will not. Finally, the algorithm is a proof of principle. Practical issues such as numerical stability are to be investigated.

The case of spread complexity. A similar algorithm, which calculates the Lanczos coefficients a_n and b_n [6] of the Hamiltonian dynamics of a certain state from its spread complexity, cannot exist. Spread complexity is defined by

$$K_s(t) = \sum_{n=0}^{\infty} n |\phi_n(t)|^2, \quad (13)$$

with the wave functions ϕ_n satisfying the chain of hopping equations³

$$\begin{aligned} \partial_t \phi_n(t) &= -b_{n+1} \phi_{n+1}(t) - i a_n \phi_n(t) + b_n \phi_{n-1}(t), \\ \phi_n(0) &= \delta_{n,0}, \end{aligned} \quad (14)$$

where both a_n and b_n are real, and one may take $b_n \geq 0$. Splitting the wave functions into their real and complex parts $\phi_n = \varphi_n + i\psi_n$, (14) becomes a coupled system for real functions (with real coefficients)

$$\begin{aligned} \partial_t \varphi_n(t) &= -b_{n+1} \varphi_{n+1}(t) + a_n \psi_n(t) + b_n \varphi_{n-1}(t), & \varphi_n(0) &= \delta_{n,0}, \\ \partial_t \psi_n(t) &= -b_{n+1} \psi_{n+1}(t) - a_n \varphi_n(t) + b_n \psi_{n-1}(t), & \psi_n(0) &= 0. \end{aligned} \quad (15)$$

The solutions to these can be Taylor expanded as

$$\begin{aligned} \varphi_n(t) &= \sqrt{h_n} \sum_{m=0}^{\infty} C_{n+2m,2m} \frac{t^{n+2m}}{(n+2m)!}, \\ \psi_n(t) &= \sqrt{h_n} \sum_{m=0}^{\infty} C_{n+2m+1,2m+1} \frac{t^{n+2m+1}}{(n+2m+1)!}, \end{aligned} \quad (16)$$

where the coefficients have been arranged into a single matrix, which again turns out to be lower triangular and can be generated by certain recurrence relations that follow from (15). In any case, as $K_s(t)$ in (13) depends on the norm $|\phi_n(t)|^2 = \varphi_n(t)^2 + \psi_n(t)^2$, its Taylor expansion only contains even powers

of t .⁴ Therefore, spread complexity $K_s(t)$ offers exactly the same amount of input information as Krylov complexity $K(t)$. Since the algorithm in Table I, if it does not fail, uniquely determines the Lanczos coefficients $\Delta_n = b_n^2$ (with $a_n = 0$), there cannot be an algorithm which determines both a_n and b_n from the same type of input.

One may wonder what kind of additional input, together with spread complexity, would be needed such that the entire set of Lanczos coefficients can be recursively

³Other forms of the hopping equation, e.g., those in [10,15], differ from this by multiplying ϕ_n by suitable powers of i .

⁴This fact is immediately relevant for checking the holographic Krylov-momentum proposal [15–34].

determined.⁵ For example, one may consider the second spread complexity,

$$K_2(t) = \sum_{n=0}^{\infty} n^2 |\phi_n(t)|^2. \quad (17)$$

Explicitly considering the first few Taylor coefficients of $K_s(t)$ and $K_2(t)$, one realizes that, at order t^{2p} they both contain combinations of Δ_q with $q \leq p$ and $(a_q - a_0)$ with $q < p$. More precisely, the t^2 coefficients of $K_s(t)$ and $K_2(t)$ coincide and are given by $2\Delta_1$, so that one can determine Δ_1 (and arbitrarily choose a_0). From the t^4

⁵A shift $a_n \rightarrow a_n + c$ corresponds to an overall shift of energy, which does not change the complexity functions. Therefore, the coefficient a_0 can always be arbitrarily chosen.

coefficients, one can calculate Δ_2 and $(a_1 - a_0)$, and so on. Therefore, a recursive algorithm to determine all Δ_p and $(a_p - a_0)$ seems possible, with the caveat that it terminates if $\Delta_p \leq 0$. The explicit construction of such an algorithm or some other compatibility test would be interesting and also contribute to our understanding of the holographic Krylov-momentum proposal.

Acknowledgments. I thank Ali Fatemiabhari and Carlos Nuñez for stimulating discussions. Partial support by the INFN under the research initiative ST&FI is gratefully acknowledged.

Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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