

Modularity, 4d mirror symmetry, and vertex operator algebra modules of 4D $\mathcal{N}=2$ SCFTs with $a=c$

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The infinite series of 4D $\mathcal{N}=2$ superconformal field theories with central charge relation $a_{4D} = c_{4D}$ are closely related to the $\mathcal{N}=4$ super Yang-Mills theory. In this paper, we study the modular properties of their associated vertex operator algebras $\mathbb{V}[\mathcal{T}_{p,N}]$, where $\mathcal{T}_{p,N}$ are those $a=c$ theories with an $SU(N)$ gauge group. We exploit the closed-form formula for the Schur index of the $\mathcal{N}=4$ $SU(N)$ theories $\mathcal{T}_{SU(N)}$ to derive the space of characters of the vertex operator algebra $\mathbb{V}[\mathcal{T}_{p,N}]$ and the S, T matrices, and find the (nonmonic) modular linear differential equations that constrain the module characters when possible. We investigate the geometric interpretation of some of these modular data from the viewpoint of 4D mirror symmetry. Using insights from the flavored modular differential equation and defect index, we investigate a map between module characters of $\mathbb{V}[\mathcal{T}_{SU(N)}]$ and those of $\mathbb{V}[\mathcal{T}_{p,N}]$.

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I. INTRODUCTION

4D $\mathcal{N}=2$ superconformal field theories (SCFTs) have been a central object of study in theoretical physics and mathematics due to their richness and fascinating relations with many areas in mathematics and physics. Thanks to the amount of supersymmetry, there are many protected quantities that can be computed exactly. Any 4D $\mathcal{N}=2$ SCFT $\mathcal{T}$ has a protected subsector of local operators, known as Schur operators, that furnish an associated vertex operator algebra (VOA) $\mathbb{V}[\mathcal{T}]$ [1–8]. This fact is often referred to as the SCFT/VOA correspondence.

The vertex operator algebra $\mathbb{V}[\mathcal{T}]$ is nonunitary as long as the 4D theory $\mathcal{T}$ is unitary, where a Virasoro subalgebra of central charge $c_{2D} = -12c_{4D}$ is guaranteed to exist. The Schur index, given as the $t \rightarrow q$ limit of the full superconformal index $\mathcal{I}(b; p, q, t)$ of the 4D theory $\mathcal{T}$ [9], is identified with the vacuum character of $\mathbb{V}[\mathcal{T}]$,

$$\mathcal{I}_{\mathcal{T}}(b, q) = \text{tr}_{\mathcal{H}} q^{E - \mathcal{R} + \frac{c_{4D}}{2}} b^f = \text{tr}_{\mathbb{V}[\mathcal{T}]} q^{L_0 - \frac{c_{2D}}{24}} b^f = \text{ch}_0[\mathbb{V}[\mathcal{T}]]. \quad (1)$$

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Here b denotes flavor fugacity if the 4D theory enjoys a continuous flavor symmetry with Cartan f . Nonvacuum modules of $\mathbb{V}[\mathcal{T}]$ are expected to correspond to insertion of Bogomol'nyi-Prasad-Sommerfield (BPS) defects in the 4D theory [10–13]. The associated variety $X_{\mathbb{V}[\mathcal{T}]}$ of $\mathbb{V}[\mathcal{T}]$ is conjectured to be identified with the Higgs branch of vacua of $\mathcal{T}$, which suggests that the VOA is of the type quasilisse [14,15], a generalization of rational VOAs. In particular, a nontrivial Higgs branch implies that the VOA is nonrational. Nonrationality sometimes implies a more complicated representation theory, such as the presence of nonordinary modules and logarithmic modules.

In [15], based on the identification of the associated variety $X_{\mathbb{V}[\mathcal{T}]}$ with the Higgs branch of vacua of $\mathcal{T}$, it was proposed that the unflavored Schur index $\mathcal{I}_{\mathcal{T}}(q)$ and the other unflavored $\mathbb{V}[\mathcal{T}]$ characters must satisfy a modular linear differential equation (MLDE) [14,16]. An equation of this nature also separately emerged earlier [17,18] and was recently reinvigorated for classifying rational conformal field theories [19–29]. The existence of such an MLDE suggests that a certain level of modularity is retained in the representation theory of $\mathbb{V}[\mathcal{T}]$, at least for the ordinary characters, namely, they must have nonsingular unflavored characters. For modules that are not ordinary, their characterization requires flavor refinement of the standard MLDE. The recent work [30–33] initiated the analysis of these equations and used them to probe the representation theory of $\mathbb{V}[\mathcal{T}]$ and more generally quasilisse VOAs. Nonetheless, a general understanding of the structure of the representation theory is still lacking. It will be

very beneficial to examine simple examples whose associated VOAs are known in explicit form to gain valuable insights.

The 4D $\mathcal{N} = 4$ super-Yang-Mills (SYM) theories $\mathcal{T}_G$ with gauge group G are one such infinite series where the associated VOAs are relatively well understood, given by a free field realization associated with complex reflection groups [8,34], though the representation theory of these VOAs remains underdeveloped. Interestingly, another closely related series of 4D $\mathcal{N} = 2$ SCFTs $\hat{\Gamma}(G)$ was proposed in [35] that share some important features with the $\mathcal{N} = 4$ SYM theories. In particular, a subclass of these theories, which are referred to as $\mathcal{T}_{p,N}$ in this paper, are constructed as an $SU(N)$ gauge theory with Argyres-Douglas matter and satisfy the central charge relation $c_{4D} = a_{4D}$. The Schur index $\mathcal{I}_{\mathcal{T}_{p,N}}(q)$ is related to that of $\mathcal{N} = 4$ $SU(N)$ SYM theory by a simple specialization of the parameters. This relation inspires an isomorphism (as a vector space) between the associated VOA $\mathbb{V}[\mathcal{T}_{SU(2)}]$ and $\mathbb{V}[\mathcal{T}_{3,2}]$ [36].

The recent work [37] began the investigation of the modularity of $\mathbb{V}[\mathcal{T}_{p,N}]$, where MLDEs were found for a series of examples. In this paper, we continue this study by combining the Fermi gas method [38–40] and elliptic function integration techniques [41] to derive exact formulas for the Schur index of $\mathcal{T}_{p,N}$, which are just polynomials of the Eisenstein series, allowing us to determine their modular properties explicitly. For the infinite series $\mathcal{T}_{2,2\ell+1}$, we argue that the space $\mathcal{V}_0$ of $\mathbb{V}[\mathcal{T}_{2,2\ell+1}]$ characters has dimension

$$1 + 3\ell(2 + \ell). \quad (2)$$

We provide a basis for the space of characters and compute the S, T matrices. We further analyze more intricate cases, $\mathcal{T}_{3,2}$, $\mathcal{T}_{3,4}$, and $\mathcal{T}_{4,3}$, and construct the space of characters. We approximate the nilpotency index $\mathfrak{n}$ [42] with the dimension $\dim \mathcal{V}_0$ and verify that it is compatible with the bound $\text{rank} \leq \mathfrak{n} - 1$ in all known examples.

We also explore geometric aspects of modularity. In particular, the fixed locus of the $U(1)_r$ action on the Coulomb branch has been observed to encode modules of the associated VOA [43,44], which is extended to and proven for Argyres-Douglas theories [45,46]; evidence of a possible generalization to A_1 class- $\mathcal{S}$ theories are also found [47,48]. Certain $\mathcal{T}_{p,N}$ theories admit class- $\mathcal{S}$ realization, such as

$$\mathcal{T}_{3,2} = (A_2, D_4), \quad \mathcal{T}_{4,3} = (A_3, E_6), \quad \mathcal{T}_{6,5} = (A_5, E_8). \quad (3)$$

This allows us to study the fixed varieties using the affine Springer fiber techniques [45], and we find that the number of fixed varieties matches the number of nonlogarithmic

characters given in terms of Jordan block sizes of the T matrix. However, the role of logarithmic characters in this geometric picture remains an open question.

We further examine the relation between the characters of $\mathbb{V}[\mathcal{T}_{SU(N)}]$ and those of $\mathbb{V}[\mathcal{T}_{p,N}]$, generalizing the identification between their vacuum character. To obtain the $\mathbb{V}[\mathcal{T}_{SU(N)}]$ characters, we employ insights into the structure of the BPS defect index in four dimensions and the property of flavored MLDEs. In particular, we define the difference operator

$$\Delta_{(N)} \text{ch}(b, q) := b^{-(N^2-1)} q^{-\frac{N^2-1}{2}} \text{ch}(bq, q) - \text{ch}(b, q), \quad (4)$$

which acts on the $\mathbb{V}[\mathcal{T}_{SU(N)}]$ -vacuum characters and generates potential nonvacuum characters $\text{ch}^{SU(N)}(b, q)$. Under the map

$$\text{ch}^{SU(N)}(b, q) \rightarrow q^{-\frac{c_{2d}[\mathcal{T}_{p,N}] - pc_{2d}[\mathcal{T}^{SU(N)}]}{24}} \text{ch}^{SU(N)}(q^{\frac{p}{2}-1}, q^p), \quad (5)$$

$\text{ch}^{SU(N)}(b, q)$ are mapped to linear combinations of characters of the $\mathbb{V}[\mathcal{T}_{p,N}]$.

This paper is organized as follows. In Sec. II we review the basics of the SCFT/VOA correspondence and the exact computation of the Schur index of $\mathcal{N} = 4$ $SU(N)$ SYM theory. We also comment on observations from computing the defect index and flavored MLDEs. In Sec. III we reorganize the analytic expressions for the Schur indices of $\mathcal{T}_{p,N}$ theories as polynomials of the Eisenstein series and study their modular properties exactly. We also compute the number of fixed varieties using the affine Springer fiber techniques and match with corresponding nonlogarithmic VOA modules. Finally, we investigate the map from $\mathbb{V}[\mathcal{T}_{SU(N)}]$ characters to those of $\mathbb{V}[\mathcal{T}_{p,N}]$.

II. VERTEX OPERATOR ALGEBRA

A. Vertex operator algebra and Schur index

In any 4D $\mathcal{N} = 2$ SCFT $\mathcal{T}$, there is a subsector of local operators, known as Schur operators, annihilated by the four supercharges $Q_-^1, S_1^-, \tilde{Q}_{2\pm}, \tilde{S}^{2\pm}$. These operators, restricted on the $z = x_3 + ix_4$ plane, furnish an associated VOA $\mathbb{V}[\mathcal{T}]$ [1]. This VOA is always nontrivial and is in general nonunitary, due to the relation between the 4D and 2D central charges

$$c_{2D} = -12c_{4D}. \quad (6)$$

This relation between 4D $\mathcal{N} = 2$ SCFTs and 2D VOA is often referred to as the 4D/2D correspondence, or SCFT/VOA correspondence.

For any $\mathcal{T}$, the associated VOA $\mathbb{V}[\mathcal{T}]$ is believed to be quasirational, a generalization of the notion of a rational VOA. It was shown in [15] that the Higgs branch of $\mathcal{T}$ can be identified with the associated variety of the VOA $\mathbb{V}[\mathcal{T}]$.

This fact implies that for theories with a nontrivial Higgs branch, the associated VOA will be nonrational and carries a more complicated representation theory. It is an interesting and challenging problem to explore the representation theory of the associated VOA $\mathbb{V}[\mathcal{T}]$.

To tackle this problem using the tools from physics, we focus on the Schur index $\mathcal{I}(b, q)$ of $\mathcal{T}$. The Schur index is a special limit $t \rightarrow q$ of the 4D $\mathcal{N} = 2$ superconformal index $\mathcal{I}(b, p, q, t)$, which counts the Schur operators with weight, and is given by the supertrace

$$\mathcal{I}(b, q) = q^{\frac{c_{4D}}{2}} \text{tr}_{\mathcal{H}}(-1)^F q^{E-R} b^f. \quad (7)$$

Here $\mathcal{H}$ denotes the Hilbert space of $\mathcal{T}$, b denotes collectively fugacities associated with flavor symmetries, and f denotes the Cartan generators of the flavor group. This 4D quantity coincides with the vacuum character of the corresponding VOA,

$$\mathcal{I}(b, q) = \text{ch}_{\mathbb{V}[\mathcal{T}]}(-1)^F q^{L - \frac{c_{4D}}{24}} b^f = \text{ch}_0(b, q). \quad (8)$$

As a special limit of the superconformal index, $\mathcal{I}(b, q)$ is invariant under exactly marginal deformations of the theory and is thus a superconformal invariant, which is very useful in understanding 4D $\mathcal{N} = 2$ dualities.

In a 2D rational VOA, there are finitely many irreducible modules, whose characters all admit nonsingular unflavoring limit $b \rightarrow 1$. The unflavored characters $\text{ch}_i(\tau)$, where $\text{ch}_0(\tau)$ denotes the vacuum character, form a vector-valued modular form with respect to the modular group $SL(2, \mathbb{Z})$ (or a congruence subgroup), for example,

$$\text{ch}_i\left(-\frac{1}{\tau}\right) = \sum_j S_{ij} \text{ch}_j(\tau). \quad (9)$$

As a standard notation, we also use $q := e^{2\pi i \tau}$ to denote the modular parameter. The modular property also gives rise to a fusion algebra through the Verlinde formula,

$$N_{ijk} = \sum_{\ell} \frac{S_{i\ell} S_{j\ell} \overline{S_{\ell k}}}{S_{0\ell}}. \quad (10)$$

In a sense, the modular property is one of the central structures of a VOA.

To analyze the modular property of $\mathbb{V}[\mathcal{T}]$, it is most convenient to start with a closed-form expression of the vacuum character, namely, the Schur index $\mathcal{I}(b, q)$. There are many ways of computing the Schur index. For gauge theories, using the independence of the gauge coupling, the Schur index can be computed at the zero-coupling limit or using the framework of supersymmetric localization [49–51]. For class- $\mathcal{S}$ theories from 6D compactified on a Riemann surface Σ , the Schur index can be computed from the 2D q -deformed SYM theory on Σ [52]. Recently,

significant progress has been made in computing the Schur index exactly and analytically [38–41, 53–55], where the resulting expressions are often written in terms of theta functions, Eisenstein series, or Weierstrass elliptic functions (or other generalizations), making the modular properties of the index more manifest.

In this paper, we focus on a class of $\mathcal{N} = 2$ superconformal field theories referred to as the $\mathcal{T}_{p,N}$ theories, which satisfy the central charge relation $a_{4D} = c_{4D}$, and would like to explore the modular property of the VOA $\mathbb{V}[\mathcal{T}_{p,N}]$. These theories can be constructed via conformal gauging of the diagonal flavor symmetry G [56] shared among several copies of Argyres-Douglas $D_{p_i}(G)$ theories [57, 58]. A $D_p(G)$ theory is constructed from the compactification of the 6D (2,0) theory of type G on a Riemann sphere with one irregular puncture $G^{h^\vee}[p - h^\vee]$ and a full regular puncture [59]. The flavor symmetry of $D_p(G)$ is at least G coming from the full regular puncture and sometimes can enhance to a larger group if the irregular puncture carries additional flavor symmetries. Requiring the absence of additional flavor symmetry puts constraints on p depending on G , summarized as follows for reference:

$SU(N)$	$(p, N) = 1$	(11)
$SO(2N)$	$p \notin 2\mathbb{Z}_{>0}$	
E_6	$p \notin 3\mathbb{Z}_{>0}$	
E_7	$p \notin 2\mathbb{Z}_{>0}$	
E_8	$p \notin 30\mathbb{Z}_{>0}$	

After gauging the diagonal of the G -flavor symmetry of n copies of $D_{p_i}(G)$, the resulting theory is in general not conformal. For its β function to vanish, the flavor central charges k_i of the components $D_{p_i}(G)$ must satisfy the condition

$$\sum_{i=1}^n k_i = 4h_G^\vee, \quad k_i = \frac{2(p_i - 1)}{p_i} h^\vee. \quad (12)$$

This condition places strong restrictions on the parameters p_i [35]: the number n of copies of $D_{p_i}(G)$ must be less than or equal to 4, and

$$(p_1, p_2, p_3, p_4) = (2, 2, 2, 2), (1, 3, 3, 3), (1, 2, 4, 4), (1, 2, 3, 6). \quad (13)$$

The resulting theories are denoted as $\hat{\Gamma}(G)$, where $\Gamma = D_4, E_6, E_7, E_8$. They are depicted in Fig. 1.

The physical properties of $\hat{\Gamma}(G)$ theories were studied in detail in [35], including the rank of the flavor group, the scaling dimensions of the Coulomb branch operators, and the dimension of the Coulomb branch moduli space, and finally the a_{4D}, c_{4D} central charges. In particular, the

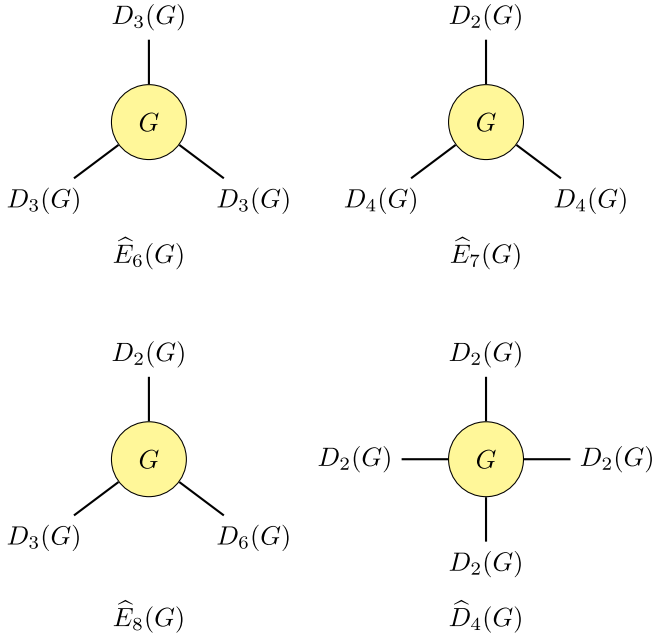


FIG. 1. The four possible series of conformal gauging of $D_p(G)$ theories.

condition for $a_{4D} = c_{4D}$ is satisfied by a selected subset of $\hat{\Gamma}(G)$ theories,

$$\begin{aligned} \hat{D}_4(SU(2\ell+1)), & \quad \hat{E}_6(SU(3\ell\pm 1)), \\ \hat{E}_7(SU(4\ell\pm 1)), & \quad \hat{E}_8(SU(6\ell\pm 1)) \end{aligned} \quad (14)$$

$$\hat{E}_6(SO(6\ell)), \quad \hat{E}_6(SO(6\ell+4)), \quad (15)$$

where ℓ is a positive integer. In this paper, we focus on $G = SU(N)$, and therefore only those in the first line. Note also that these theories have no residual flavor symmetry [35]. We follow the notation in [37] to call them $\mathcal{T}_{2,N}$, $\mathcal{T}_{3,N}$, $\mathcal{T}_{4,N}$, and $\mathcal{T}_{6,N}$, respectively [60]. Explicitly, their a_{4D} , c_{4D} central charges are given by the simple formula [37]

$$a_{4D} = c_{4D} = \left(1 - \frac{1}{p}\right)(N^2 - 1). \quad (16)$$

The Coulomb branch operators of $D_p(G)$ theory can be obtained from the deformations of the Seiberg-Witten curve [57]

$$\begin{aligned} \mathcal{C}(p, G) = \left\{ j - \frac{h_G^\vee}{p} s \mid j - \frac{h_G^\vee}{p} s > 1, j \in \text{Cas}(G), \right. \\ \left. s = 1, \dots, p-1 \right\}. \end{aligned} \quad (17)$$

The full Coulomb branch operators of the $\hat{\Gamma}(G)$ theory include those from each $D_{p_i}(G)$ component as well as the vector multiplet, which contributes the degree of

Casimir elements of the gauge group. The rank of the $\hat{\Gamma}(G)$ is given by

$$\text{rank}(\hat{\Gamma}(G)) = \text{rank} G \left(1 + \frac{\text{rank} \Gamma}{2}\right). \quad (18)$$

Explicitly,

$$\text{rank} \mathcal{T}_{2,2\ell+1} = 6\ell, \quad (19)$$

$$\text{rank} \mathcal{T}_{3,3\ell+1} = 12\ell, \quad \text{rank} \mathcal{T}_{3,3\ell-1} = 12\ell - 8, \quad (20)$$

$$\text{rank} \mathcal{T}_{4,4\ell+1} = 18\ell, \quad \text{rank} \mathcal{T}_{4,4\ell-1} = 18\ell - 9 \quad (21)$$

$$\text{rank} \mathcal{T}_{6,6\ell+1} = 30\ell, \quad \text{rank} \mathcal{T}_{6,6\ell-1} = 30\ell - 10. \quad (22)$$

Among all of the $\mathcal{T}_{p,N}$, one notices that three of them are identified again with known Argyres-Douglas theories [35],

$$\begin{aligned} \mathcal{T}_{(3,2)} &= (A_2, D_4) = D_4^6[3] = D_4^4[2], \\ \mathcal{T}_{(4,3)} &= (A_3, E_6) = E_6^{12}[4], \\ \mathcal{T}_{(6,5)} &= (A_5, E_8) = E_8^{30}[6]. \end{aligned} \quad (23)$$

The Schur index of the $\mathcal{T}_{p,N}$ theories is

$$\mathcal{I}_{\mathcal{T}_{p,N}}(q) = \oint [dx] \prod_{j=1}^{\ell} \mathcal{I}_{D_{p_j}(SU(N))}(\mathbf{x}, q) \times \mathcal{I}_{\text{vec}}(\mathbf{x}, q), \quad (24)$$

where the Schur index of $D_p(SU(N))$ theory is [5,6,62]

$$\mathcal{I}_{D_p(SU(N))}(\mathbf{x}, q) = \text{PE} \left[\frac{q - q^p}{(1-q)(1-q^p)} \chi_{\text{adj}}^{SU(N)}(\mathbf{x}) \right], \quad (25)$$

and

$$\mathcal{I}_{\text{vec}}(\mathbf{x}, q) = \text{PE} \left[-\frac{2q}{1-q} \chi_{\text{adj}}^{SU(N)}(\mathbf{x}) \right]. \quad (26)$$

The integration Haar measure is defined as

$$[dx] = \prod_{i=1}^{N-1} \frac{dx_i}{2\pi i x_i} \prod_{\substack{j,k=1 \\ j \neq k}}^N \left(1 - \frac{x_j}{x_k}\right), \quad (27)$$

subject to the constraint $x_1 x_2 \cdots x_N = 1$. Comparing this with the Schur index of $\mathcal{N} = 4$ SYM theory

$$I_G^{\mathcal{N}=4}(b, q) = \int [d\vec{z}] \text{PE} \left[\left(-\frac{2q}{1-q} + \frac{q^{\frac{1}{2}}}{1-q} (b + b^{-1}) \right) \chi_{\text{adj}}^G(\vec{z}) \right], \quad (28)$$

one finds the remarkable relation (here we have inserted the suitable power of q factor)

$$\mathcal{I}_{\mathcal{T}_{p,N}}(q) = q^{-\frac{c_{2D}[\mathcal{T}_{p,N}] - p c_{2D}[\mathcal{T}_{SU(N)}]}{24}} I_{SU(N)}^{\mathcal{N}=4}(b = q^{p/2-1}, q^p). \quad (29)$$

An exact expression for the Schur index of $\mathcal{N} = 4$ $SU(N)$ SYM theory then gives the exact expression for the $\mathcal{T}_{p,N}$ Schur index, a perfect starting point for modularity analysis. In the next subsection, we review the computation of the $\mathcal{N} = 4$ $SU(N)$ Schur index using the fermi-gas approach.

B. 4D $\mathcal{N} = 4$ Schur index

To analyze the modularity of the $\mathcal{T}_{p,N}$ Schur index, it is most convenient to start with its closed form in terms of Jacobi theta functions or Eisenstein series with well-known modular properties. Fortunately, several works have computed the closed-form expression of the flavored Schur index of $\mathcal{N} = 4$ SYM theory [39–41, 53, 54, 63], which then supplies the closed form of $\mathcal{I}_{\mathcal{T}_{p,N}}$. One particular powerful approach is the fermi-gas formalism proposed in [38, 39], and later generalized in [40] to obtain the flavored $\mathcal{N} = 4$ $SU(N)$ Schur index in closed form. In the following, we review the computation in [40], where we reorganize the results in terms of Eisenstein series using the integration formula developed in [41].

The Schur index of the $\mathcal{N} = 4$ $SU(N)$ SYM $\mathcal{T}_{SU(N)}$ can be written as a multivariate contour integral involving ratios of Jacobi theta functions,

$$\mathcal{I}_{SU(N)}(b, q) = \frac{1}{N!} \vartheta_4(\mathbf{b}) \eta(\tau)^{3(N-1)} \times \oint \frac{d\mathbf{a}}{2\pi i \mathbf{a}} \frac{\prod_{i \neq j} \vartheta_1(\mathbf{a}_i - \mathbf{a}_j)}{\prod_{i,j} \vartheta_4(\mathbf{a}_i - \mathbf{a}_j + \mathbf{b})}. \quad (30)$$

The integration contours are the unit circles $|a_i| = 1$. The contour integral appearing in the expression for the index can be rewritten using the Cauchy determinant formula

$$\frac{\prod_{i \neq j} \vartheta_1(\mathbf{a}_i - \mathbf{a}_j)}{\prod_{i,j} \vartheta_4(\mathbf{a}_i - \mathbf{a}_j + \mathbf{b})} = i^{N^2} \frac{u^{-N/2} q^{-\frac{N^2}{8}} b^{-\frac{N^2}{2}}}{\vartheta_1(\mathbf{u} + N(\mathbf{b} + \frac{\tau}{2})) \vartheta_1(\mathbf{u})^{N-1}} \times \det_{ij} \frac{\vartheta_4(\mathbf{a}_i - \mathbf{a}_j + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_i - \mathbf{a}_j + \mathbf{b})}, \quad (31)$$

where $u = e^{2\pi i \mathbf{u}}$ is an auxiliary parameter, which can be set to an arbitrary value at the end of the computation. The determinant can be expanded using the Laplace formula

$$\det_{ij} \frac{\vartheta_4(\mathbf{a}_i - \mathbf{a}_j + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_i - \mathbf{a}_j + \mathbf{b})} = \sum_{\sigma} (-1)^{\sigma} \prod_{i=1}^N \frac{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{\sigma(i)} + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{\sigma(i)} + \mathbf{b})}. \quad (32)$$

As a result, the integral reduces to the following sum of contour integrals (up to a prefactor):

$$\sum_{\sigma} (-1)^{\sigma} \oint \frac{d\mathbf{a}}{2\pi i \mathbf{a}} \prod_{i=1}^N \frac{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{\sigma(i)} + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{\sigma(i)} + \mathbf{b})}. \quad (33)$$

An important observation is that the integrand is elliptic [64] with respect to each $\mathbf{a}_i$. Moreover, for any two permutations $\sigma \sim \sigma'$ belonging to the same conjugacy class in S_N , we have

$$\begin{aligned} & \oint \frac{d\mathbf{a}}{2\pi i \mathbf{a}} \prod_{i=1}^N \frac{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{\sigma(i)} + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{\sigma(i)} + \mathbf{b})} \\ &= \oint \frac{d\mathbf{a}}{2\pi i \mathbf{a}} \prod_{i=1}^N \frac{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{\sigma'(i)} + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{\sigma'(i)} + \mathbf{b})}. \end{aligned} \quad (34)$$

Any permutation $\sigma \in S_N$ can be uniquely decomposed into a product of disjoint cycles

$$\sigma = (c_{11} \cdots c_{1\ell_1})(c_{21} \cdots c_{2\ell_2}) \cdots. \quad (35)$$

For a given cycle $c = (c_1 c_2 \cdots c_{\ell})$, one can isolate a product of terms in the integrand of the form

$$\prod_{i=1}^{\ell} \frac{\vartheta_4(\mathbf{a}_{c_i} - \mathbf{a}_{c_{i+1}} + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_{c_i} - \mathbf{a}_{c_{i+1}} + \mathbf{b})} \subset \prod_{i=1}^N \frac{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{\sigma(i)} + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{\sigma(i)} + \mathbf{b})}, \quad \mathbf{a}_{\ell+1} = \mathbf{a}_1. \quad (36)$$

As a result, the full integral can be decomposed into a product of contour integrals over each cycle,

$$\begin{aligned} & \oint \frac{d\mathbf{a}}{2\pi i \mathbf{a}} \prod_{i=1}^N \frac{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{\sigma(i)} + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{\sigma(i)} + \mathbf{b})} \\ &= \oint \frac{d\mathbf{a}_c}{2\pi i \mathbf{a}_c} \prod_{i=1}^{\ell} \frac{\vartheta_4(\mathbf{a}_{c_i} - \mathbf{a}_{c_{i+1}} + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_{c_i} - \mathbf{a}_{c_{i+1}} + \mathbf{b})} \times \oint \dots \end{aligned} \quad (37)$$

We define the integral corresponding to a cycle of length ℓ as

$$\mathcal{Z}_{\ell} := \oint \frac{d\mathbf{a}_c}{2\pi i \mathbf{a}_c} \prod_{i=1}^{\ell} \frac{\vartheta_4(\mathbf{a}_{c_i} - \mathbf{a}_{c_{i+1}} + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_{c_i} - \mathbf{a}_{c_{i+1}} + \mathbf{b})}. \quad (38)$$

The Schur index can then be expressed in terms of polynomials of $\mathcal{Z}_{\ell}$,

$$\begin{aligned} \mathcal{I}_{SU(N)} &= \frac{\vartheta_4(\mathbf{b}) \eta(\tau)^{3(N-1)}}{N!} \frac{i^{N^2} u^{-N/2} q^{-\frac{N^2}{8}} b^{-\frac{N^2}{2}}}{\vartheta_1(\mathbf{u} + N(\mathbf{b} + \frac{\tau}{2})) \vartheta_1(\mathbf{u})^{N-1}} \\ &\times \sum_{\vec{\ell}} (-1)^{C_{\vec{\ell}}} |C_{\vec{\ell}}| \mathcal{Z}_{\ell_1} \cdots \mathcal{Z}_{\ell_N}, \end{aligned} \quad (39)$$

where the sum is over all conjugacy classes $C_{\vec{\ell}}$ of S_N , labeled by partitions $\vec{\ell}$ of N ,

$$\vec{\ell} = \{\ell_1, \ell_2, \dots\}, \quad \sum_i \ell_i = N, \quad \ell_1 \geq \ell_2 \geq \dots \geq 0. \quad (40)$$

Alternatively, a conjugacy class $C_{\vec{\ell}}$ can also be represented by the multiplicities $\vec{n}$ of all cycle lengths $\ell = 1, \dots, N$, denoted $[1^{n_1} 2^{n_2} \dots N^{n_N}]$, with

$$|C_{\vec{n}}| = \frac{N!}{\prod_{\ell=1}^N \ell^{n_\ell} n_\ell!}, \quad \sum_{i=1}^N n_\ell \ell = N, \quad C_{\vec{\ell}} = C_{\vec{n}}. \quad (41)$$

The factor $(-1)^{C_{\vec{n}}}$ in (39) denotes the signature of the corresponding conjugacy class. Putting everything together, we arrive at the final expression

$$\mathcal{I}_{SU(N)} = \frac{\vartheta_4(\mathbf{b}) \eta(\tau)^{3(N-1)}}{N!} \frac{i^{N^2} u^{-N/2} q^{-\frac{N^2}{8}} b^{-\frac{N^2}{2}}}{\vartheta_1(\mathbf{u} + N(\mathbf{b} + \frac{\tau}{2})) \vartheta_1(\mathbf{u})^{N-1}} \times \sum_{\vec{n}} \frac{(-1)^{C_{\vec{n}}} N!}{\prod_{\ell=1}^N \ell^{n_\ell} n_\ell!} \prod_{\ell=1}^N Z_{\ell}^{n_\ell}. \quad (42)$$

Using the ellipticity of the integrand of $\mathcal{Z}_\ell$, we proceed to compute the components $\mathcal{Z}_\ell$ using the integration formula developed in [41] for contour integrals involving elliptic functions and Eisenstein series.

$\ell = 1$

This case is trivial,

$$\mathcal{Z}_1 = \oint \frac{da}{2\pi i a} \frac{\vartheta_4(\mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{b})} = \frac{\vartheta_4(\mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{b})}. \quad (43)$$

$\ell = 2$

In this case, we consider the integral

$$\mathcal{Z}_2 = \oint \frac{da_1}{2\pi i a_1} \frac{da_2}{2\pi i a_2} \frac{\vartheta_4(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{b})} \times \frac{\vartheta_4(\mathbf{a}_2 - \mathbf{a}_1 + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_2 - \mathbf{a}_1 + \mathbf{b})}. \quad (44)$$

The integrand has two imaginary poles

$$\mathbf{a}_1 = \mathbf{a}_2 - \mathbf{b} + \frac{\tau}{2}, \quad \mathbf{a}_1 = \mathbf{a}_2 + \mathbf{b} + \frac{\tau}{2}, \quad (45)$$

with the corresponding residues

$$\frac{i\vartheta_1(\mathbf{u})\vartheta_1(2\mathbf{b} + \mathbf{u})}{\eta(\tau)^3 \vartheta_1(2\mathbf{b})}, \quad -\frac{i\vartheta_1(\mathbf{u})\vartheta_1(2\mathbf{b} + \mathbf{u})}{\eta(\tau)^3 \vartheta_1(2\mathbf{b})}. \quad (46)$$

Choosing $a_0 = a_2 b u q^{1/2}$, the integral reduces to

$$\mathcal{Z}_2 = \sum_{\mathbf{a}_i \text{ real/Im}} R_j E_1 \left[\frac{-1}{a_j/a_0 q^{\pm 1/2}} \right] = \frac{i\vartheta_1(\mathbf{u})\vartheta_1(2\mathbf{b} + \mathbf{u})}{\eta(\tau)^3 \vartheta_1(2\mathbf{b})} \left(E_1 \left[\frac{+1}{u} \right] - E_1 \left[\frac{+1}{b^2 u} \right] \right). \quad (47)$$

$\ell = 3$

$$\prod_{i=1}^3 \oint \frac{d\mathbf{a}}{2\pi i \mathbf{a}} \prod_{i=1}^3 \frac{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{i+1} + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_i - \mathbf{a}_{i+1} + \mathbf{b})}. \quad (48)$$

We evaluate this in two steps. First, the integral over a_1 has poles

$$\mathbf{a}_1 = \mathbf{a}_2 - \mathbf{b} + \frac{\tau}{2}, \quad \mathbf{a}_1 = \mathbf{a}_3 + \mathbf{b} + \frac{\tau}{2}, \quad (49)$$

yielding

$$\begin{aligned} & \frac{i\vartheta_1(\mathbf{u})}{\eta(\tau)^3} \frac{\vartheta_4(\mathbf{a}_2 - \mathbf{a}_3 + \mathbf{b} + \mathbf{u}) \vartheta_1(-\mathbf{a}_2 + \mathbf{a}_3 + 2\mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_2 - \mathbf{a}_3 + \mathbf{b}) \vartheta_1(\mathbf{a}_3 - \mathbf{a}_2 + 2\mathbf{b})} \\ & \times \left(E_1 \left[\frac{1}{u} \right] + E_1 \left[\frac{1}{\frac{a_2}{a_3 b^2 u}} \right] \right), \\ & = \frac{i\vartheta_1(\mathbf{u})}{\eta(\tau)^3} \frac{\vartheta_4(\mathbf{a}_2 - \mathbf{a}_3 + \mathbf{b} + \mathbf{u}) \vartheta_1(-\mathbf{a}_2 + \mathbf{a}_3 + 2\mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_2 - \mathbf{a}_3 + \mathbf{b}) \vartheta_1(\mathbf{a}_3 - \mathbf{a}_2 + 2\mathbf{b})} \\ & \times \left(E_1 \left[\frac{1}{u} \right] - E_1 \left[\frac{1}{u b^2 \frac{a_3}{a_2}} \right] \right). \end{aligned} \quad (50)$$

The subsequent integral over a_2 has poles at

$$\mathbf{a}_2 = \mathbf{a}_3 - \mathbf{b} + \frac{\tau}{2}, \quad \mathbf{a}_2 = \mathbf{a}_3 + 2\mathbf{b} \quad (51)$$

with residues

$$-\frac{\vartheta_1(\mathbf{u})^2 \vartheta_4(3\mathbf{b} + \mathbf{u})}{\eta(\tau)^6 \vartheta_4(3\mathbf{b})}, \quad \frac{\vartheta_1(\mathbf{u})^2 \vartheta_4(3\mathbf{b} + \mathbf{u})}{\eta(\tau)^6 \vartheta_4(3\mathbf{b})}. \quad (52)$$

The resulting expression is

$$\begin{aligned} \mathcal{I} = & -\frac{1}{2} \frac{\vartheta_1(\mathbf{u})^2 \vartheta_4(3\mathbf{b} + \mathbf{u})}{\eta(\tau)^6 \vartheta_4(3\mathbf{b})} \\ & \times \left(1 + 8E_1 \left[\frac{-1}{b^3 u} \right] E_1 \left[\frac{+1}{u} \right] - 8E_1 \left[\frac{+1}{u} \right]^2 \right. \\ & \left. + 8E_2 \left[\frac{-1}{b^3 u} \right] - 8E_2 \left[\frac{+1}{u} \right] \right). \end{aligned} \quad (53)$$

From these examples, the result is proportional to the factor

$$\mathcal{Z}_\ell \propto \frac{\vartheta_1(\mathbf{u})^{\ell-1} \vartheta_{4 \text{ or } 1}(\ell \mathbf{b} + \mathbf{u})}{(\ell-1)! \vartheta_{4 \text{ or } 1}(\ell \mathbf{b})}, \quad (54)$$

where ϑ_4 appears for odd ℓ and ϑ_1 for even ℓ . Moreover, $\mathcal{Z}_\ell$ satisfies the recursion relation

$$\mathcal{Z}_{\ell+1} = \oint \frac{da_\ell}{2\pi i a_\ell} \frac{\vartheta_4(\mathbf{a}_\ell + \mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{a}_\ell + \mathbf{b})} [\mathcal{Z}_\ell]_{b^\ell \rightarrow b^\ell/a_\ell, \ell \mathbf{b} \rightarrow \ell \mathbf{b} - \mathbf{a}_\ell}, \quad (55)$$

and the twisted Eisenstein series appear in $\mathcal{Z}_\ell$ linearly: the contour integral can always be performed easily.

Alternatively, it was noted in [40] that

$$\frac{iu^{-1/2}\eta(\tau)^3}{\vartheta_1(\mathbf{u})} \mathcal{Z}_1 = \frac{iu^{-1/2}\eta(\tau)^3}{\vartheta_1(\mathbf{u})} \frac{\vartheta_4(\mathbf{b} + \mathbf{u})}{\vartheta_4(\mathbf{b})} = \sum_{p \in \mathbb{Z}} \frac{b^p q^{\frac{p}{2}}}{1 - uq^p}, \quad (56)$$

and

$$\left(\frac{iu^{-1/2}\eta(\tau)^3}{\vartheta_1(\mathbf{u})} \right)^\ell \mathcal{Z}_\ell(b, q) = \frac{1}{(\ell-1)!} \partial_u^{\ell-1} \times \left(\frac{iu^{-1/2}\eta(\tau)^3}{\vartheta_1(\mathbf{u})} \mathcal{Z}_1(b \rightarrow b^\ell q^{-\frac{\ell}{2}}) \right). \quad (57)$$

Derivatives of Jacobi theta functions and Eisenstein series can be reorganized into polynomials of Eisenstein series, using (A18), and

$$0 = u \partial_u E_k \begin{bmatrix} 1 \\ u \end{bmatrix} + (k+1) E_{k+1} \begin{bmatrix} 1 \\ u \end{bmatrix} + E_{k+1} + E_1 \begin{bmatrix} 1 \\ u \end{bmatrix} E_k \begin{bmatrix} 1 \\ u \end{bmatrix} - \sum_{\ell=2}^{k-1} E_\ell E_{k+1-\ell} \begin{bmatrix} 1 \\ u \end{bmatrix}. \quad (58)$$

We do not spell out the details here.

Although the above computation carries the variable u , the final result is independent of u . One particularly useful choice is $u = 1$, or $\mathbf{u} = 0$ [65]. For example, when $N = 2$, the above formula yields

$$\mathcal{I}_{SU(2)} = -\frac{i\vartheta_4(\mathbf{b})}{2\vartheta_4(2\mathbf{b})} \left(E_1 \begin{bmatrix} 1 \\ u \end{bmatrix} - E_1 \begin{bmatrix} 1 \\ b^2 u \end{bmatrix} \right) + \frac{\eta(\tau)^3 \vartheta_4(\mathbf{b} + \mathbf{u})^2}{2\vartheta_1(\mathbf{u}) \vartheta_1(2\mathbf{b} + \mathbf{u}) \vartheta_4(\mathbf{b})}. \quad (59)$$

Sending $\mathbf{u} \rightarrow 0$ by carefully implementing l'Hôpital's rule [notice that $E_1 \begin{bmatrix} 1 \\ u \end{bmatrix} = \frac{\vartheta_1'(\mathbf{u})}{2\pi i \vartheta_1(\mathbf{u})}$], the above reduces to a more compact expression,

$$\mathcal{I}_{SU(2)} = \frac{i\vartheta_4(\mathbf{b})}{\vartheta_4(2\mathbf{b})} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}, \quad (60)$$

which equals the index $\mathcal{I}_{SU(2)}$ in [41] identically.

Next, we consider $N = 3$, and the above computation yields

$$\mathcal{I}_{SU(3)} = \frac{\vartheta_4(\mathbf{b})}{24\vartheta_4(3\mathbf{b})} \left(1 + 8E_1 \begin{bmatrix} -1 \\ b^3 u \end{bmatrix} E_1 \begin{bmatrix} 1 \\ u \end{bmatrix} - 8E_1 \begin{bmatrix} 1 \\ u \end{bmatrix}^2 + 8E_2 \begin{bmatrix} -1 \\ b^3 u \end{bmatrix} - 8E_2 \begin{bmatrix} 1 \\ u \end{bmatrix} \right) + \frac{\eta(\tau)^3 \vartheta_4(\mathbf{b} + \mathbf{u})}{6\vartheta_1(\mathbf{u})^2 \vartheta_4(3\mathbf{b} + \mathbf{u})} \left(\frac{\eta(\tau)^3 \vartheta_4(\mathbf{b} + \mathbf{u})^2}{\vartheta_4(\mathbf{b})^2} - \frac{3i\vartheta_1(\mathbf{u}) \vartheta_1(2\mathbf{b} + \mathbf{u})}{\vartheta_1(2\mathbf{b})} \left(E_1 \begin{bmatrix} 1 \\ u \end{bmatrix} - E_1 \begin{bmatrix} 1 \\ b^2 u \end{bmatrix} \right) \right). \quad (61)$$

Sending $\mathbf{u} \rightarrow 0$ by implementing l'Hôpital's rule, the above reduces to a more compact expression,

$$\mathcal{I}_{SU(3)} = -\frac{1}{8} \frac{\vartheta_4(\mathbf{b})}{\vartheta_4(3\mathbf{b})} \left(-\frac{1}{3} + 4E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^2 - 4E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} \right). \quad (62)$$

This reproduces exactly the same $\mathcal{I}_{SU(3)}$ as in [41].

For reference, we list a few more examples of the Schur index $\mathcal{I}_{SU(N)}$ for $N = 2, \dots, 7$ in Appendix B.

To summarize, the Schur index $\mathcal{I}_{SU(N)}$ takes the form of

$$\mathcal{I}_{SU(N)}(b, q) = \begin{cases} \frac{\vartheta_4(\mathbf{b}|\tau)}{\vartheta_4(N\mathbf{b}|\tau)} \mathbb{E}_N(b, q), & N = 2\ell + 1, \\ \frac{i\vartheta_4(\mathbf{b}|\tau)}{\vartheta_1(N\mathbf{b}|\tau)} \mathbb{E}_N(b, q), & N = 2\ell, \end{cases} \quad (63)$$

where $\mathbb{E}_N(b, q)$ denotes some polynomial of Eisenstein series involving $E_k \begin{bmatrix} (-1)^f \\ b^f \end{bmatrix}$, $k = 0, 1, \dots, N-1$, $f = k, k+1, \dots, N-1$.

C. Defects, nonvacuum modules, and modular differential equations

In any 4D $\mathcal{N} = 2$ SCFT one can insert nonlocal BPS operators that preserve a certain amount of supersymmetry. For the purpose of studying the associated VOA $\mathbb{V}[T]$, one can consider nonlocal operators, sometimes referred to as defects, which preserve $Q_-^1, S_-^1, \tilde{Q}_{2-}, \tilde{S}^{2-}$. These operators were studied in [10,11,13,66]. In particular, one can consider surface defects spanning the space perpendicular to the z plane where the Schur operators reside, preserving a 2D $\mathcal{N} = (2, 2)$ superconformal algebra. Similarly, one can consider full or half Wilson line operators perpendicular to the z plane [11]. The Schur index in the presence of these

nonlocal operators can be computed [11,63,67,68]. These BPS defects are believed to correspond to nonvacuum modules of $\mathbb{V}[T]$.

Concretely, surface defects in the Argyres-Douglas theories (A_{N-1}, A_{M-1}) using the Higgsing procedure are identified with nonvacuum modules of $\mathcal{W}_N$ minimal models and $\mathcal{B}(k)_{A_{N-1}}$ algebra [10,69]. For A_1 -type class- $\mathcal{S}$ theories $\mathcal{T}_{g,n}$, it was shown that the Schur index of surface defects from Higgsing can be written as modular transformations of the original Schur index [70], and hence they are likely module characters of the associated VOA $\mathbb{V}[\mathcal{T}_{g,n}]$.

Line operators in the Argyres-Douglas theories were also studied in [11], whose Schur indices were shown to be linear combinations of nonvacuum characters of the associated VOA. For example, a line operator L in (A_1, A_2) has the Schur index

$$\mathcal{I}_L^{(A_1, A_2)} = q^{-\frac{1}{2}} \text{ch}_0 - q^{-\frac{1}{2}} \text{ch}_1, \quad (64)$$

where $\text{ch}_{0,1}$ are the irreducible characters of the $(2, 5)$ Virasoro minimal model, the associated VOA of (A_1, A_2) . Note that the coefficients depend on q . For $\mathcal{N} = 4$ $SU(2)$ SYM theory, the Schur index of a Wilson line operator W_j in the spin- j representation of $SU(2)$ is given by

$$\mathcal{I}_{W_j} = \delta_{j \in \mathbb{Z}} \mathcal{I}_{SU(2)} - \frac{i\vartheta_4(\mathfrak{b})}{2\vartheta_4(2\mathfrak{b})} \sum_{\substack{m=-j \\ m \neq 0}}^{+j} \frac{b^m - b^{-m}}{q^{m/2} - q^{-m/2}}. \quad (65)$$

We notice that the Wilson line index is a linear combination of $\mathcal{I}_{SU(2)}$ and $\frac{i\vartheta_4(\mathfrak{b})}{\vartheta_4(2\mathfrak{b})}$ with coefficients that are rational functions of b and $q^{1/2}$. For $SU(2)$ supersymmetric QCD with four flavors whose associated VOA is $\widehat{\mathfrak{so}}(8)_{-2}$, the same phenomenon also occurs: the Wilson line index

$$\begin{aligned} \mathcal{I}_{W_j}(M, q) = & \left(\delta_{j \in \mathbb{Z}} - \frac{1}{2} \mathcal{M}_{1j} - \frac{1}{2} \mathcal{M}_{2j} \right) \text{ch}_0 \\ & + \frac{1}{2} (\mathcal{M}_{1j} - \mathcal{M}_{2j}) \text{ch}_1 + \frac{1}{2} (\mathcal{M}_{3j} - \mathcal{M}_{2j}) \text{ch}_2 \\ & + \frac{1}{2} (\mathcal{M}_{3j} + \mathcal{M}_{4j}) \text{ch}_3 + \frac{1}{2} (\mathcal{M}_{3j} - \mathcal{M}_{4j}) \text{ch}_4, \end{aligned} \quad (66)$$

where $\text{ch}_{0,1,\dots,4}$ denotes the vacuum and four nonvacuum highest weight module characters of the $\widehat{\mathfrak{so}}(8)_{-2}$ [71,72],

$$\mathcal{M}_{ij} := \sum_{\substack{m=-j \\ m \neq 0}}^{+j} \frac{M_i^{2m} - M_i^{-2m}}{q^m - q^{-m}}. \quad (67)$$

The index is again a linear combination of characters with coefficients that are rational functions of the flavor

fugacities M and $q^{1/2}$. This structure may be universal, where the line operator index is a linear combination of the characters of the underlying associated VOA with coefficients being rational functions of the flavor fugacities and $q^{1/2}$. Reversing the logic, we may be able to extract nonvacuum characters from line operator indices by identifying a basis. This point of view helps us explore the relation between the representation theory of the associated VOAs of $\mathcal{N} = 4$ $SU(N)$ theories and $\mathcal{T}_{p,N}$ theories.

In general, modules of $\mathbb{V}[T]$ are more complicated than those in a rational VOA. The simplest modules are referred to as ordinary modules [73], which have finite-dimensional weight spaces, namely, at any given conformal weight h , the corresponding subspace in the module is a finite-dimensional representation of the flavor symmetry group. Ordinary module characters have nonsingular unflavoring limits, similar to those in a rational VOA. Nonordinary modules may also exist, which have singular unflavoring limits. For example, the nonvacuum character $i\vartheta_4(\mathfrak{b})/\vartheta_1(2\mathfrak{b})$ in the $\mathcal{N} = 4$ $SU(2)$ theory has a singular unflavoring limit, since the weight space at the lowest conformal weight h is the infinite-dimensional irreducible representation of $SU(2)$ with spin $j = -1/2$. Similarly, all the nonvacuum modules of $\widehat{\mathfrak{so}}(8)_{-2}$ are nonordinary, since the highest weight state involves negative Dynkin labels of $\widehat{\mathfrak{so}}(8)$. There may also be modules with both logarithmic and nonordinary features, where L_0 does not act semi-simply and the character is singular in the unflavoring limit.

To understand the space of characters, one can borrow analytic tools from the study of rational VOAs, namely, the holomorphic modular bootstrap and (unflavored) MLDEs [16,17], which take the form

$$\left[D_q^{(N)} + \sum_{r=1}^N \phi_{2N-2r} D_q^{(r)} \right] \text{ch} = 0, \quad (68)$$

where ϕ_{2k} denotes a modular form [74] of weight $2k$, and $D_q^{(k)} := \partial_{(2k-2)} \circ \dots \circ \partial_{(2)} \circ \partial_{(0)}$, with $\partial_{(k)} := q\partial_q + kE_2(\tau)$. $D_q^{(k)}$ carries an effective modular weight $2k$, and (68) is of modular weight $2N$. In [14], unflavored characters of any quasilisse VOA were shown to satisfy an unflavored MLDE. For $\mathbb{V}[T]$ associated with a 4D $\mathcal{N} = 2$ SCFT $\mathcal{T}$, such an equation is argued to arise from the null state $|\mathcal{N}_T\rangle$ associated with the nilpotency of the stress tensor,

$$L_{-2}^n |0\rangle = |c_2\rangle + |\mathcal{N}_T\rangle, \quad |c_2\rangle \in C_2(\mathbb{V}[T]) = \text{span}\{a_{-h_a-1}b\}, \quad (69)$$

based on the identification between the Higgs branch of $\mathcal{T}$ and the associated variety of the VOA $\mathbb{V}[T]$ [15,16]. This ordinary differential equation can be used to constrain the space of ordinary module characters. However, for

nonordinary modules, flavor refinement becomes necessary. In [30–32], flavored MLDEs were constructed from null states for a simple set of $\mathbb{V}[\mathcal{T}]$ and affine Kac-Moody algebras, which lead to strong constraints on the full space of characters including nonordinary and logarithmic ones. These equations are partial differential equations with products of Eisenstein series $E_k[\frac{(-1)^f}{b^a}]$ as coefficients. Unlike the usual unflavored MLDE, a flavored MLDE is not covariant under the modular group due to the quasi-Jacobi nature of the Eisenstein series. Instead, they exhibit “quasimodularity,” namely,

$$\text{eq}^S \tau^{\text{wt}} \text{eq} + \tau^{\text{wt}-1} \mathfrak{b}_i \text{eq}_i + \tau^{\text{wt}-2} \mathfrak{b}_i \mathfrak{b}_j \text{eq}_{ij} + \dots, \quad (70)$$

where additional equations $\text{eq}_{ij\dots}$ are generated in the modular S transformation. In the cases we have studied explicitly, the set of partial differential equations $\text{eq}_{ij\dots}$ also annihilate all of the characters. It turns out that all of the flavored MLDEs that appear in the modular transformations come from null states, and there is an effective action of the modular group on the null states, e.g.,

$$|\mathcal{N}\rangle \xrightarrow{S} \sum_{\ell \geq 0} \sum_{i_1, \dots, i_\ell} \frac{1}{\ell!} \tau^{h[\mathcal{N}] - \ell} \mathfrak{b}_{i_1} \dots \mathfrak{b}_{i_\ell} h_1^{i_1} \dots h_1^{i_\ell} |\mathcal{N}\rangle, \quad (71)$$

where h^i denotes the Cartan generators of the flavor symmetry group and h_n^i denotes the generators of the corresponding affine Kac-Moody algebra.

Besides modular transformations, one can also consider the spectral-flow operation, which shifts the flavor fugacities [31,33]

$$\sigma: \mathfrak{b}_i \rightarrow \mathfrak{b}_i + n_i \tau, \quad (72)$$

where n_i are fractional numbers satisfying a certain integrality condition. Under the action of σ , the flavored MLDEs transform similar to the above quasimodularity property, which reads schematically as

$$\text{eq} \xrightarrow{\sigma} \sum_{\ell \geq 0} \sum_{i_1 \leq \dots \leq i_\ell} \frac{(-1)^\ell}{\ell!} n_{i_1} \dots n_{i_\ell} \text{eq}_{i_1 \dots i_\ell}. \quad (73)$$

The equations $\text{eq}_{i_1 \dots i_\ell}$ are the same as those appearing in the modular transformations. The σ operation also induces an effective action on the null states [33],

$$|\mathcal{N}\rangle \xrightarrow{\sigma} \sum_{\ell \geq 0} \sum_{i_1 \leq \dots \leq i_\ell} \frac{(-1)^\ell}{\ell!} n_{i_1} \dots n_{i_\ell} h_1^{i_1} \dots h_1^{i_\ell} |\mathcal{N}\rangle. \quad (74)$$

These transformation properties of the flavored MLDE indicate that the space of characters is invariant under the modular and spectral-flow transformations. In particular, starting from the vacuum character, the spectral-flow operation may generate nonvacuum characters that are

otherwise hard to access from first principles. We use this point of view to explore the relation between the representation theory of the associated VOAs of $\mathcal{N} = 4$ $SU(N)$ theories and $\mathcal{T}_{p,N}$ theories.

Before closing this section, let us make a few remarks about the unflavored MLDE that is crucial in analyzing the ordinary module characters. The nilpotency condition $L_{-2}^n |0\rangle = |c_2\rangle + |\mathcal{N}_T\rangle$ of the stress tensor gives rise to an interesting number $\mathfrak{n}$, which is referred to as the “nilpotency index,” as a measure of “complexity” of the theory $\mathcal{T}$ [42]. In favorable situations, this condition leads to a monic unflavored MLDE of order n . However, obstruction may arise where the monic MLDE is of higher order N in differentiation with respect to q ,

$$\left[D_q^{(N)} + \sum_{r=1}^N \phi_{2N-2r}(\tau) D_q^{(r)} \right] \text{ch} = 0, \quad (75)$$

where $\phi_{2N-2r}(\tau)$ have no pole in the fundamental domain of τ . Naturally, this suggests that

$$N \geq \mathfrak{n}. \quad (76)$$

If we relax the requirement of monicity, sometimes it is possible to find a nonmonic MLDE with lower order in q differentiation, as we will see in the next section in various examples, and also in previous works [15,48,70,75]. The presence of this nonmonic MLDE with lower differentiation order is often suggested by the presence of irrational or even complex indicial roots in the monic MLDE (68). Let us denote this number as $n_{\min}$ [76]. Obviously, both $\mathfrak{n}, n_{\min} \leq N$.

The unflavored vacuum character ch_0 is always a solution to any of these unflavored MLDEs. By construction and the property of the modular forms, the entire modular orbit of ch_0 must also be a solution. Let us define the span of the orbit as $\mathcal{V}_0$, where the subscript $_0$ signals its origin from the vacuum character. We define $n_0 := \dim \mathcal{V}_0$ and we expect

$$n_0 \leq n_{\min}. \quad (77)$$

Finally, we denote n_{ord} as the number of ordinary modules and, obviously, $n_{\text{ord}} \leq n_{\min}$.

To summarize, for any $\mathbb{V}[\mathcal{T}]$, there is a set of interesting quantities satisfying natural inequalities,

$$n_0, n_{\text{ord}} \leq n_{\min}, \quad n_{\min}, \mathfrak{n} \leq N. \quad (78)$$

In many cases where n_0, n_{ord} , and $n_{\min}$ are known explicitly, they are equal. These examples are 4D $\mathcal{N} = 4$ $SU(2)$ theory, $SU(2)$ supersymmetric QCD with four flavors together with a few other A_1 class- $\mathcal{S}$ theories, (A_1, A_{2n}) and (A_1, D_{2n+1}) Argyres-Douglas theories, as well as some more examples in the following section. In cases where

both $n_{\min}$ and the nilpotency index $\mathfrak{n}$ are known, they are also equal. Examples are $\mathcal{N} = 4$ $SU(2)$ theory, $SU(2)$ supersymmetric QCD with four flavors, and (A_1, A_{2n}) and (A_1, D_{2n+1}) Argyres-Douglas theories, $\mathcal{T}_{3,2}$. It would be interesting to verify if these equalities or inequalities are universal, or find out the criteria for them to hold.

III. MODULARITY OF $\mathbb{V}[\mathcal{T}_{p,N}]$

In this section, we study the modularity of $\mathbb{V}[\mathcal{T}_{p,N}]$ representation theory by carefully examining a few cases. Since $\mathbb{V}[\mathcal{T}_{p,N}]$ has no residual flavor symmetry, we expect only ordinary modules. The closed-form formula of $\mathcal{I}_{SU(N)}$ that we reviewed in the previous section can be used to derive and simplify that of $\mathcal{I}_{\mathcal{T}_{p,N}}$. From the closed-form formula, we derive the $SL(2, \mathbb{Z})$ orbit of $\mathcal{I}_{\mathcal{T}_{p,N}}$, from which the dimension n_0 of its span $\mathcal{V}_0 := \text{span}(SL(2, \mathbb{Z}) \cdot \mathcal{I}_{\mathcal{T}_{p,N}})$ is determined.

Since there is no residual flavor symmetry left in $\mathcal{T}_{p,N}$, all module characters are unflavored and they are expected to satisfy some unflavored MLDE. As remarked earlier, such an MLDE is sometimes not unique. When possible, we will find the nonmonic MLDE of the minimal order $n_{\min}$ in differentiation in q , and verify that $n_0 = n_{\min}$. With the modular orbit and a basis for its span, S, T matrices can be computed, from which we read off the Jordan normal form of T^2 . For simple examples, this data can then be matched with the Coulomb branch geometry using the affine Springer fiber [45]. Finally, we explore the relation between the potential module characters of $\mathbb{V}[\mathcal{T}_{SU(N)}]$ and characters of $\mathbb{V}[\mathcal{T}_{p,N}]$.

A. $\mathcal{T}_{p,N}$ Schur index

The Schur index of $\mathcal{T}_{p,N}$ is given by the simple formula [35]

$$\mathcal{I}_{\mathcal{T}_{p,N}}(q) = q^{-\frac{c_{2d}[\mathcal{T}_{p,N}] - p c_{2d}[\mathcal{T}_{SU(N)}]}{24}} \mathcal{I}_{SU(N)}(b \rightarrow q^{\frac{p}{2}-1}, q \rightarrow q^p). \quad (79)$$

Here $\mathcal{I}_{SU(N)}(b, q)$ denotes the Schur index of the $\mathcal{N} = 4$ $SU(N)$ theory, which can be written in the closed form explained in the previous section,

$$\mathcal{I}_{SU(N)}(b, q) = \begin{cases} \frac{\vartheta_4(\mathfrak{b}|\tau)}{\vartheta_4(N\mathfrak{b}|\tau)} \mathbb{E}_N(b, q), & N = 2\ell + 1, \\ \frac{\vartheta_4(\mathfrak{b}|\tau)}{\vartheta_1(N\mathfrak{b}|\tau)} \mathbb{E}_N(b, q), & N = 2\ell, \end{cases} \quad (80)$$

where $\mathbb{E}_N$ schematically denotes a polynomial of Eisenstein series. This closed form is particularly suited for studying the $\mathcal{T}_{p,N}$ index analytically due to an interesting limit of the prefactor f_N [77],

$$f_N(\mathfrak{b}, \tau) := \begin{cases} \frac{\vartheta_4(\mathfrak{b}|\tau)}{\vartheta_4(N\mathfrak{b}|\tau)}, & N = 2\ell + 1, \\ \frac{\vartheta_4(\mathfrak{b}|\tau)}{i^{N^2-1} \vartheta_1(N\mathfrak{b}|\tau)}, & N = 2\ell. \end{cases} \quad (81)$$

Let us begin with the simplest case with $p = 2$, where N takes values in all odd positive integers. Obviously,

$$f_N(\mathfrak{b}, \tau) \xrightarrow{\mathfrak{b} \rightarrow \left(\frac{\ell-1}{2}\right)\tau, \tau \rightarrow p\tau} \frac{\vartheta_4(0|2\tau)}{\vartheta_4(0|2\tau)} = 1. \quad (82)$$

Next, we consider $p = 3$ and a positive integer $N = 1, 2 \bmod 3$. Then, depending on the parity of N , we have

$$\begin{aligned} \text{odd } N: & \frac{\vartheta_4(\mathfrak{b}|\tau)}{\vartheta_4(N\mathfrak{b}|\tau)} \xrightarrow{\mathfrak{b} \rightarrow \left(\frac{\ell-1}{2}\right)\tau, \tau \rightarrow 3\tau} \frac{\vartheta_4\left(\frac{1}{2}\tau|3\tau\right)}{\vartheta_4\left(N\frac{1}{2}\tau|3\tau\right)}, \\ \text{even } N: & \frac{\vartheta_4(\mathfrak{b}|\tau)}{\vartheta_1(N\mathfrak{b}|\tau)} \xrightarrow{\mathfrak{b} \rightarrow \left(\frac{\ell-1}{2}\right)\tau, \tau \rightarrow 3\tau} \frac{\vartheta_4\left(\frac{1}{2}\tau|3\tau\right)}{\vartheta_1\left(N\frac{1}{2}\tau|3\tau\right)}. \end{aligned} \quad (83)$$

Using the shift property of the Jacobi theta function

$$\vartheta_4\left(\mathfrak{z} + N\frac{\tau}{2} \middle| \tau\right) = -(-1)^{\frac{N}{2}-1} b^{-\frac{N}{2}} q^{-\frac{N^2}{8}} \vartheta_1(\mathfrak{z}|\tau), \quad (84)$$

we deduce

$$f_N(\mathfrak{b}, \tau) \xrightarrow{\mathfrak{b} \rightarrow \left(\frac{\ell-1}{2}\right)\tau, \tau \rightarrow 3\tau} -(-1)^{\left\lfloor \frac{n_N-2}{4} \right\rfloor} q^{\frac{N^2-1}{24}}, \quad (85)$$

where n_N denotes that N is the n th smallest positive integer coprime to 3, for example, $n_2 = 2$, $n_4 = 3$, $n_5 = 4$.

Similarly, for $p = 4$ and $p = 6$, the factor simplifies, respectively, to

$$f_N(\mathfrak{b}, \tau) \xrightarrow{\mathfrak{b} \rightarrow \left(\frac{\ell-1}{2}\right)\tau, \tau \rightarrow p\tau} -(-1)^{\left\lfloor \frac{n_N-3}{2} \right\rfloor} q^{\frac{N^2-1}{8}}, \quad (86)$$

and

$$f_N(\mathfrak{b}, \tau) \xrightarrow{\mathfrak{b} \rightarrow \left(\frac{\ell-1}{2}\right)\tau, \tau \rightarrow p\tau} q^{\frac{N^2-1}{3}}. \quad (87)$$

These factors of q powers account for the difference of central charges in (79) (up to a sign, which will not be important),

$$f_N\left(\left(\frac{p}{2}-1\right)\tau, p\tau\right) = q^{\frac{c_{2d}[\mathcal{T}_{p,N}] - p c_{2d}[\mathcal{T}_{SU(N)}]}{24}}. \quad (88)$$

As a result, (79) can be simplified to a polynomial $\mathbb{E}$ in Eisenstein series,

$$\mathcal{I}_{\mathcal{T}_{p,N}} = \mathbb{E}_N \left(b \rightarrow q^{\left(\frac{p-1}{2}\right)}, q \rightarrow q^p \right), \quad (89)$$

where we have not been careful about the overall sign. This property of f_N can also be interpreted as a map from a special $\mathbb{V}[\mathcal{T}_{SU(N)}]$ -module character to a $\mathbb{V}[\mathcal{T}_{p,N}]$ module character; see the discussions in Sec. V.

Here we list a few explicit examples of $\mathcal{I}_{\mathcal{T}_{p,N}}$ with small p, N :

$$\mathcal{I}_{3,2} = E_1 \left[\frac{-1}{\sqrt{q}} \right] (3\tau), \quad (90)$$

$$\mathcal{I}_{3,4} = -\frac{1}{72} - \frac{1}{6} E_2(3\tau) + \frac{1}{24} E_1 \left[\frac{-1}{\sqrt{q}} \right] (3\tau) \quad (91)$$

$$- \frac{1}{6} E_1 \left[\frac{-1}{\sqrt{q}} \right] (3\tau)^3 - \frac{1}{2} E_1 \left[\frac{-1}{\sqrt{q}} \right] (3\tau) E_2 \left[\frac{-1}{q} \right] (3\tau), \quad (92)$$

$$\mathcal{I}_{2,3} = \frac{1}{24} + \frac{1}{2} E_2(2\tau), \quad \mathcal{I}_{2,5} = \frac{3}{640} + \frac{1}{16} E_2(2\tau) + \frac{1}{8} E_2(2\tau)^2 - \frac{1}{4} E_4(2\tau), \quad (93)$$

$$\mathcal{I}_{2,7} = \frac{5}{7168} + \frac{37}{3840} E_2(2\tau) + \frac{5}{192} E_2(2\tau)^2 + \frac{1}{48} E_2(2\tau)^3 \quad (94)$$

$$- \frac{5}{96} E_4(2\tau) - \frac{1}{8} E_2(2\tau) E_4(2\tau) + \frac{1}{6} E_6(2\tau), \quad (95)$$

and

$$\begin{aligned} \mathcal{I}_{4,3} &= -\frac{1}{24} + \frac{1}{2} E_1 \left[\frac{-1}{q} \right] (4\tau)^2 - \frac{1}{2} E_2 \left[\frac{1}{q^2} \right] (4\tau), \\ \mathcal{I}_{6,5} &= \frac{49}{7680} + \frac{1}{72} E_1 \left[\frac{-1}{q^2} \right] (6\tau) + \frac{1}{192} E_1 \left[\frac{-1}{q} \right] (6\tau) \\ &\quad - \frac{1}{96} E_1 \left[\frac{-1}{q^2} \right] (6\tau)^2 - \frac{1}{32} E_1 \left[\frac{-1}{q} \right] (6\tau)^2 + \frac{1}{12} E_2 \left[\frac{1}{q^2} \right] (6\tau) + \frac{1}{96} E_2 \left[\frac{-1}{q} \right] (6\tau) \\ &\quad - \frac{1}{8} E_1 \left[\frac{-1}{q} \right] (6\tau) E_2 \left[\frac{-1}{q} \right] (6\tau) + \frac{1}{8} E_1 \left[\frac{-1}{q} \right] (6\tau) E_1 \left[\frac{-1}{q^2} \right] (6\tau)^2 \\ &\quad - \frac{1}{3} E_2 \left[\frac{-1}{1} \right] (6\tau) E_1 \left[\frac{-1}{q^2} \right] (6\tau) + \frac{1}{4} E_2 \left[\frac{-1}{q} \right] (6\tau) E_1 \left[\frac{-1}{q^2} \right] (6\tau)^2 \\ &\quad + \frac{1}{4} E_3 \left[\frac{1}{q^2} \right] (6\tau) + \frac{1}{4} E_4 \left[\frac{1}{q^2} \right] (6\tau) - \frac{1}{8} E_2 \left[\frac{-1}{q} \right] (6\tau)^2 - \frac{1}{24} E_1 \left[\frac{-1}{q^2} \right] (6\tau)^4. \end{aligned} \quad (96)$$

B. Modularity of $\mathcal{T}_{2,2\ell+1}$

For odd $N = 2\ell + 1$, the Schur index of the $\mathcal{T}_{2,N}$ theory is given by

$$\mathcal{I}_{\mathcal{T}_{2,N}}(q) = q^{-\frac{c_{2D}[\mathcal{T}_{2,N}] - 2c_{2D}[\mathcal{T}_{SU(N)}]}{24}} I_{SU(N)}^{\mathcal{N}=4}(b=1, q^2). \quad (97)$$

Note that

$$c_{2D}[\mathcal{T}_{2,N}] = -12 \times \frac{N^2 - 1}{2}, \quad c_{2D}[\mathcal{T}_{SU(N)}] = -12 \times \frac{N^2 - 1}{4}. \quad (98)$$

Hence,

$$q^{\frac{c_{2D}[\mathcal{T}_{2,N}] - 2c_{2D}[\mathcal{T}_{SU(N)}]}{24}} = 1, \quad (99)$$

and the Schur index of $\mathcal{I}_{\mathcal{T}_{2,N}}(q)$ is obtained by simply replacing q with q^2 in the unflavored $\mathcal{N} = 4$ SYM index. As demonstrated in [41], the Schur index of unflavored SYM theory admits an elegant expression,

$$\mathcal{I}_{SU(2\ell+1)}(q) = (-1)^\ell \sum_{k=0}^{\ell} \frac{\tilde{\lambda}_{2k+2}^{(2\ell+3)}(2)}{\max(2k, 1)} \tilde{\mathbb{E}}_{2k}(\tau), \quad (100)$$

where we define [78]

$$\tilde{\mathbb{E}}_0(\tau) = 1, \quad \tilde{\mathbb{E}}_{2k} := 2k \oint \frac{dy}{2\pi i y} y^{-2k} \exp \left[- \sum_{n=1}^{\infty} \frac{1}{2n} E_{2n}(\tau) y^{2n} \right]. \quad (101)$$

For reference, we list some unflavored Schur indices of $\mathcal{N} = 4$ SYM theory in Appendix B.

After the $q \rightarrow q^2$ replacement, we obtain the Schur index of $\mathcal{T}_{2,N}$ theories,

$$\mathcal{I}_{\mathcal{T}_{2,N}}(q) = (-1)^\ell \sum_{k=0}^{\ell} \frac{\tilde{\lambda}_{2k+2}^{(2\ell+3)}(2)}{\max(2k, 1)} \tilde{\mathbb{E}}_{2k}(2\tau). \quad (102)$$

The argument 2τ complicates the modular property. For example,

$$\begin{aligned} E_4(2\tau) &\xrightarrow{S} E_4\left(-\frac{2}{\tau}\right) = \left(\frac{\tau}{2}\right)^4 E_4\left(\frac{\tau}{2}\right), \\ E_6(2\tau) &\xrightarrow{S} E_6\left(-\frac{2}{\tau}\right) = \left(\frac{\tau}{2}\right)^6 E_6\left(\frac{\tau}{2}\right). \end{aligned} \quad (103)$$

To make the analysis slightly easier, we use the property of the Eisenstein series

$$\sum_{\pm} E_k \left[\begin{smallmatrix} \phi \\ \pm z \end{smallmatrix} \right] (\tau) = 2E_k \left[\begin{smallmatrix} \phi \\ z^2 \end{smallmatrix} \right] (2\tau), \quad (104)$$

which trades Eisenstein series at 2τ to the those at τ , paying the price of more terms in the expression. Now we have

$$\mathcal{I}_{\mathcal{T}_{2,N}}(q) = (-1)^\ell \sum_{k=0}^{\ell} \frac{\tilde{\lambda}_{2k+2}^{(2\ell+3)}(2)}{\max(2k, 1)} \tilde{\mathbb{E}}_{2k}(\tau) |_{E_k(2\tau) \rightarrow \frac{1}{2}(E_k[\frac{1}{+1}] + E_k[\frac{1}{-1}])}. \quad (105)$$

1. $\mathcal{T}_{2,3}$ theory

We start by analyzing the modular orbit of the simplest example: the $\mathcal{T}_{2,3}$ theory. The Schur index of the $\mathcal{T}_{2,3}$ theory is

$$\mathcal{I}_{\mathcal{T}_{2,3}}(q) = \frac{1}{24} + \frac{1}{4} \left(E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right). \quad (106)$$

Under the modular S transformation,

$$\mathcal{I}_{\mathcal{T}_{2,3}}(q) \xrightarrow{S} S\mathcal{I}_{\mathcal{T}_{2,3}} = \frac{1}{24} + \frac{1}{4} \tau^2 \left(E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) + \frac{i\tau}{8\pi}. \quad (107)$$

After successive application of T , one obtains

$$\begin{aligned} T^n S\mathcal{I}_{\mathcal{T}_{2,3}}(q) &= \frac{1}{24} + \frac{1}{4} (\tau + n)^2 \left(E_2 \begin{bmatrix} -1 \\ (-1)^n \end{bmatrix} + E_2 \begin{bmatrix} +1 \\ +1 \end{bmatrix} \right) \\ &\quad + \frac{(\tau + n)i}{8\pi}. \end{aligned} \quad (108)$$

Clearly, most of the expressions are linearly dependent. A finite linearly independent basis can be easily obtained by considering $\mathbf{T}^{(\ell)} f := T^\ell f - f$. Concretely, we can define recursively

$$\begin{aligned} \text{ch}_0 &:= \mathcal{I}_{\mathcal{T}_{2,3}}, \\ \text{ch}_1, \text{ch}_2, \dots, \quad \text{ch}_6 &:= S\text{ch}_0, \quad T\text{ch}_1, \mathbf{T}^{(2)}\text{ch}_2, \quad T\text{ch}_3, \\ &\quad \mathbf{T}^{(2)}\text{ch}_4, \quad T\text{ch}_5, \\ \text{ch}_7, \text{ch}_8, \text{ch}_9 &:= S\text{ch}_5, \quad \mathbf{T}^{(1)}\text{ch}_7, \quad \mathbf{T}^{(1)}\text{ch}_8. \end{aligned} \quad (109)$$

By inspection, the second line manifests six linearly independent expressions,

$$\tau^{\ell=2,1,0} \left(E_2 \begin{bmatrix} -1 \\ \pm 1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right). \quad (110)$$

The third line manifests three independent expressions,

$$\tau^{\ell=0,1,2} \left(E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right). \quad (111)$$

And, finally, the original Schur index ch_0 supplies the constant term as a final linearly independent expression, giving a basis of ten linearly independent expressions spanning the modular orbit of $\mathcal{I}_{\mathcal{T}_{2,3}}$. One can also extract these ten linearly independent expressions by inspecting the original Schur index ch_0 since it contains a weight-zero term and a weight-two term, where the weight-two term generates three sets of linearly independent objects

$$\begin{aligned}
\left(E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) &\xrightarrow{SL(2, \mathbb{Z})} \tau^{\ell=0,1,2} \left(E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + E_2 \begin{bmatrix} +1 \\ +1 \end{bmatrix}\right), \\
&\tau^{\ell=0,1,2} \left(E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} +1 \\ +1 \end{bmatrix}\right), \quad \text{and} \\
&\tau^{\ell=0,1,2} \left(E_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} +1 \\ +1 \end{bmatrix}\right),
\end{aligned} \tag{112}$$

while the weight-zero term (a constant) remains invariant. As a result, we obtain the dimension of the modular orbit

$$\dim \mathcal{V}_{T_{2,3}} = 1 + 3 \times 3 = 10. \tag{113}$$

In [37], an order-12 unflavored MLDE was proposed, which annihilates the Schur index,

$$\begin{aligned}
0 = & [D_q^{(12)} - 1510E_4D_q^{(10)} - 55440E_6D_q^{(9)} - 233400E_4^2D_q^{(8)} + 2364600E_4E_6D_q^{(7)} \\
& + 2000(31228E_4^3 - 41013E_6^2)D_q^{(6)} + 1422624000E_4^2E_6D_q^{(5)} \\
& + (3925360000E_4^4 + 40438916000E_4E_6^2)D_q^{(4)} \\
& + (420470400000E_4^3E_6 + 344509200000E_6^3)D_q^{(3)} \\
& + (1168824000000E_4^5 + 7510426000000E_4^2E_6^2)D_q^{(2)} \\
& + (23905224000000E_4^4E_6 + 31682361200000E_4E_6^3)D_q^{(1)}] \text{ch}_0.
\end{aligned} \tag{114}$$

This equation has ten rational indicial roots and two irrational indicial roots. The two irrational roots actually suggest that this is not the unflavored MLDE of the lowest order in the q derivative. Note also that there is no $D_q^{(0)}$ term in the equation, implying that a constant is a solution to the equation. By direct computation, we find at modular weight 36 an equation of order 10,

$$\begin{aligned}
0 = & \left[\left(\frac{55}{196} E_4^4 + E_4 E_6^2 \right) D_q^{(10)} - \frac{405}{14} E_4^3 E_6 D_q^{(9)} + \left(-\frac{11775}{49} E_4^5 - 525 E_4^2 E_6^2 \right) D_q^{(8)} \right. \\
& \times \left(\frac{94125}{7} E_4^4 E_6 - 29400 E_4 E_6^3 \right) D_q^{(7)} + \left(-\frac{2364000}{49} E_4^6 + 126300 E_4^3 E_6^2 \right) D_q^{(6)} \\
& + \left(\frac{14796000}{7} E_4^5 E_6 - 4498200 E_4^2 E_6^3 \right) D_q^{(5)} \\
& + \left(\frac{19430000}{7} E_4^7 + 5253500 E_4^4 E_6^2 - 5586000 E_4 E_6^4 \right) D_q^{(4)} \\
& + \left(-\frac{1033500000}{7} E_4^6 E_6 + 639555000 E_4^3 E_6^3 \right) D_q^{(3)} \\
& \times \left(\frac{10959000000}{49} E_4^8 + 575450000 E_4^5 E_6^2 + 2830730000 E_4^2 E_6^4 \right) D_q^{(2)} \\
& \left. \times \left(\frac{1683000000}{7} E_4^7 E_6 + 14264950000 E_4^2 E_6^3 + 14543200000 E_4 E_6^5 \right) D_q^{(1)} \right] \text{ch}_0.
\end{aligned} \tag{115}$$

This equation share exactly the same ten rational indicial roots with (114) without the irrational roots. Again, there is no $D_q^{(0)}$ term in the equation, implying that a constant is a solution to the equation. The existence of this equation implies that the modular orbit of $\text{ch}_0^{T_{2,3}}$ spans the full space of module characters.

With the above basis $\text{ch}_0, \dots, \text{ch}_9$, we are able to compute the S, T matrices for $T_{2,3}$:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ -8 & 0 & 8 & 0 & -5 & 0 & 4 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -8 & 0 & 8 & 0 & -4 & 0 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -8 & 8 & 0 & -4 & 0 & 2 & 0 & 0 & 0 & \frac{1}{2} \\ -16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \end{pmatrix}. \quad (116)$$

The Jordan decomposition of T contains blocks of sizes $[3, 3, 3, 1]$,

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \end{pmatrix}. \quad (117)$$

These sizes count precisely the number of independent objects (112) at different modular weights that appear in the modular orbit, and can also be read off directly from the Schur index $\mathcal{I}_{\mathcal{T}_{2,3}}$ (106). Roughly speaking, each block corresponds to one nonlogarithmic solution,

$$1, \quad E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ +1 \end{bmatrix}, \quad E_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ +1 \end{bmatrix}, \quad E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ +1 \end{bmatrix}. \quad (118)$$

2. $\mathcal{T}_{2,5}$ theory

Let us consider the next example in the series: the $\mathcal{T}_{2,5}$ theory. The Schur index of the $\mathcal{T}_{2,5}$ theory is

$$\begin{aligned} \mathcal{I}_{\mathcal{T}_{2,5}} = & \frac{1}{640} \left[3 + 20 \left(E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right. \\ & \left. + 20 \left(E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2 - 80 \left(E_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right]. \end{aligned} \quad (119)$$

In $\mathcal{I}_{\mathcal{T}_{2,5}}$, there is one weight-zero, one weight-two, and one weight-four term. Under the modular transformation, terms of different weight decouple, while terms of the same weight transform as a collection, leading to a set of linearly independent expressions,

$$\begin{aligned} & 1 \\ & \tau^{\ell=0,1,2} \left(E_2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \Big|_{(\alpha,\beta)=(1,-1),(-1,1),(-1,-1)}, \\ & \tau^{\ell=0,1,2,3,4} \left[\left(E_2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2 - 4 \left(E_4 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + E_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right] \Big|_{(\alpha,\beta)=(1,-1),(-1,1),(-1,-1)}. \end{aligned} \quad (120)$$

Therefore, the dimension of the $SL(2, \mathbb{Z})$ orbit of ch_0 is

$$\dim \mathcal{V}_{T_{2,5}} = 1 + 3 \times 3 + 3 \times 5 = 25. \quad (121)$$

This indicates that the minimal order of derivatives in the unflavored MLDE should be at least 25. By direct computation, we find an unflavored at modular weight 74 of the form

$$[D_q^{(37)} + \lambda E_6 D_q^{(34)} + \dots] \text{ch}_0 = 0, \quad (122)$$

where λ is a fraction with numerator and denominator

$$\begin{aligned} \text{numerator} = & 421282402952490486445992287338190685460922751300553151483 \\ & 241822067901749047988562616579113550476031711391007616264 \\ & 950318480787195182424277942115734662906185298984911599590 \\ & 79110107872217894568437469668581051521077474520, \end{aligned} \quad (123)$$

$$\begin{aligned} \text{denominator} = & 198743721653470831065556134904415296204844638429340239369 \\ & 695204038832081520624654129118171344687480575165865484484 \\ & 953211885868906433778828082693618682833770608875281132559 \\ & 8979100523693841443176524289415862155427. \end{aligned} \quad (124)$$

However, this is not the MLDE with the lowest order of derivation. We further find another equation at weight 254, which starts with

$$\begin{aligned} & [733377661001283792156623707366556995116638016839211654400000000000000000E_4^{51} \\ & + 282690869316323266546364828331938074747341836739646939747840000000000000000E_6^2 E_4^{48} \\ & - 788488239387148139158480815011365352662900146444998542130714400000000000000000E_6^4 E_4^{45} \\ & + 936477446565668332715258488848552171843527574494029013614332140000000000000000E_6^6 E_4^{42} \\ & - 56249995376002643544568653897696847585154162228403884602600683880000000000000000E_6^8 E_4^{39} \\ & + 123776120446998765213233804570322354284093941230721564683716185710000000000000000E_6^{10} E_4^{36} \\ & + 6000090031464451799668871835615184439713164025098402431439406513868500000000000000E_6^{12} E_4^{33} \\ & - 60501492602890104728207162664524254851454513928274275105761754068139975000000000000E_6^{14} E_4^{30} \\ & + 2535743475516345489116903895129723725497482304856216630957579751764253337500000000E_6^{16} E_4^{27} \\ & - 674937213267060398825325560565542490748053015409023766724262837052317705125000000E_6^{18} E_4^{24} \\ & + 1271222347065419612392286229650454325173285620228112223945671203806321222934687500E_6^{20} E_4^{21} \\ & - 1796410685751471391288649628690748883914914584246863279966572708957721408722703125E_6^{22} E_4^{18} \\ & + 1971197053860851321044537423579541440657171136270930092419202405217776367136340625E_6^{24} E_4^{15} \\ & - 1655501119018411519887498805059736262157521610608157585689094805749542231614393750E_6^{26} E_4^{12} \\ & + 961307884181455524086753029316646694951039128891591369755961770416467062921587500E_6^{28} E_4^9 \\ & - 302702480531330081358260891093741687255683069618196709811824107244611560172418800E_6^{30} E_4^6 \\ & + 23013695122573891150533825839037706228944742721941092539531308051951642856858320E_6^{32} E_4^3 \\ & + 604669704120972379283681439557999913127508129846543569232279348643581825075328E_6^{34}] D_q^{(25)} + \dots \end{aligned} \quad (125)$$

We may also consider a different basis for the span of the modular orbit, for example,

$$\begin{aligned}
 \text{ch}_1 &= S\text{ch}_1, \\
 \text{ch}_{2\ell}, \text{ch}_{2\ell+1} &= T\text{ch}_{2\ell-1}, (T^2 - 1)\text{ch}_{2\ell-1}, \quad \ell = 1, \dots, 4, \\
 \text{ch}_{10} &= T\text{ch}_9, \\
 \text{ch}_{11} &= S\text{ch}_9, \\
 \text{ch}_\ell &= (T - 1)\text{ch}_{\ell-1}, \quad \ell = 12, \dots, 15, \\
 \text{ch}_{16} &= S\text{ch}_{15}, \\
 \text{ch}_{2\ell+1}, \text{ch}_{2\ell+2} &= T\text{ch}_{2\ell}, (T^2 - 1)\text{ch}_{2\ell}, \quad \ell = 8, \dots, 10, \\
 \text{ch}_{21} &= T\text{ch}_{20}, \\
 \text{ch}_{22} &= S\text{ch}_{20}, \\
 \text{ch}_{23}, \text{ch}_{24} &= (T - 1)\text{ch}_{22}, (T - 1)\text{ch}_{23}.
 \end{aligned} \tag{126}$$

This basis leads to the following S and T matrices:

$$S = \begin{pmatrix}
 0 & 1 & 0 \\
 1 & 0 \\
 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 104 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & -\frac{1}{48} & \frac{1}{64} & -\frac{5}{384} & 104 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7}{8} & \frac{3}{8} & -\frac{1}{8} \\
 0 & 0 & 80 & 0 & -53 & 0 & 44 & 0 & -39 & 0 & \frac{107}{3} & 0 & 0 & 0 & 0 & 8 & 0 & 72 & 0 & -48 & 0 & 40 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
 376 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{12} & -\frac{1}{8} & \frac{7}{96} & -\frac{5}{96} & 376 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
 0 & 0 & 464 & 0 & -268 & 0 & 211 & 0 & -181 & 0 & \frac{485}{3} & 0 & 0 & 0 & 0 & 8 & 0 & 456 & 0 & -264 & 0 & 208 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
 288 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & -\frac{1}{8} & \frac{1}{16} & -\frac{1}{24} & 288 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 768 & 0 & -408 & 0 & 312 & 0 & -263 & 0 & 232 & 0 & 0 & 0 & 0 & 0 & 0 & 768 & 0 & -408 & 0 & 312 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 384 & 0 & -192 & 0 & 144 & 0 & -120 & 0 & 105 & 0 & 0 & 0 & 0 & 0 & 0 & 384 & 0 & -192 & 0 & 144 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 384 & 0 & -192 & 0 & 144 & 0 & -120 & 0 & 104 & 0 & 0 & 0 & 0 & 0 & 0 & 384 & 0 & -192 & 0 & 144 & 0 & 0 & 0 & 0 \\
 0 & 5376 & 0 & -1152 & 0 & 672 & 0 & -480 & 0 & 376 & 0 & 0 & 0 & 0 & 0 & 0 & 5376 & 0 & -1152 & 0 & 672 & 0 & 0 & 0 & 0 \\
 0 & 13824 & 0 & -1152 & 0 & 576 & 0 & -384 & 0 & 288 & 0 & 0 & 0 & 0 & 0 & 0 & 13824 & 0 & -1152 & 0 & 576 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
 0 & \frac{7}{8} & -\frac{3}{8} & \frac{1}{8} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 8 & 0 & -5 & 0 & 4 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 8 & 0 & -4 & 0 & 3 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 8 & 0 & -4 & 0 & 3 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -24 & 24 & 0 & -4 & 0 & 2 & 0 & \frac{3}{2} & -\frac{3}{2} & \frac{3}{2}
 \end{pmatrix},$$

and

[illegible]

We can compute the Jordan type of the T matrix, showing block sizes $[5, 5, 5, 3, 3, 3, 1]$. Obviously, these sizes count precisely the modular weights of the objects (121) from performing $SL(2, \mathbb{Z})$, which can be easily read off from inspecting the original Schur index $\mathcal{I}_{\mathcal{T}_{2,5}}$. The seven blocks corresponds to seven nonlogarithmic solutions to the MLDE, given explicitly by

$$1, \quad \left(E_2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \Big|_{(\alpha, \beta) = (1, -1), (-1, 1), (-1, -1)} \left[\left(E_2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2 - 4 \left(E_4 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + E_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right] \Big|_{(\alpha, \beta) = (1, -1), (-1, 1), (-1, -1)}. \quad (128)$$

3. $\mathcal{T}_{2.7}$ theory

As the last example in the $\mathcal{T}_{2,\text{odd}}$ series, we consider the $\mathcal{T}_{2,7}$ theory, where a basis of length $1 + 3 \times 3 + 3 \times 5 + 3 \times 7 = 46$ can be constructed in complete similarity with the $\mathcal{T}_{2,3}, \mathcal{T}_{2,5}$ cases. We compute the S, T matrix using the basis, indicating that it is a complete basis for the modular orbit of $\text{ch}_0^{\vee[\mathcal{T}_{2,7}]}$. In particular, the T matrix has Jordan block sizes $[7, 7, 7, 5, 5, 5, 3, 3, 3, 1]$, as expected. Unfortunately, we are unable to explicitly construct an order-46 MLDE due to a constraint on computational resources.

4. $\mathcal{T}_{2,2\ell+1}$ theory

Now the modular structure for $\mathcal{T}_{2,2\ell+1}$ becomes almost transparent, and we make some predictions for all $\mathcal{T}_{2,2\ell+1}$ cases. From the equation (105), the index contains components of different weight decoupled in modular transformations, and we generate individually the following linearly independent expressions:

$$1, \quad \tau^{0,1,\dots,2k} \tilde{\mathbb{E}}_{2k}(2\tau) \Big|_{E_k(2\tau) \rightarrow \frac{1}{2} \left(E_k \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)},$$

$$\tau^{0,1,\dots,2k} \tilde{\mathbb{E}}_{2k}(2\tau) \Big|_{E_k(2\tau) \rightarrow \frac{1}{2} \left(E_k \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)},$$

$$\tau^{0,1,\dots,2k} \tilde{\mathbb{E}}_{2k}(2\tau) \Big|_{E_k(2\tau) \rightarrow \frac{1}{2} \left(E_k \begin{bmatrix} -1 \\ -1 \end{bmatrix} + E_k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)}, \quad (129)$$

where the 1 comes from the weight-zero term, and $k = 1, \dots, \ell$. In other words, besides the weight-zero term, each term in $\mathcal{T}_{2,2\ell+1}$ of weight $2k$ gives rise to $3 \times (2k + 1)$ objects. In total, there are

$$1 + 3 \times 3 + 3 \times 5 + 3 \times 7 + \dots + 3 \times (2\ell + 1) = 1 + 3\ell(2 + \ell) \quad (130)$$

linearly independent expressions in the span of the modular orbit $\text{ch}_0^{\mathcal{T}_{2,2\ell+1}}$. We conjecture that this span is the space of $\mathbb{V}[\mathcal{T}_{2,2\ell+1}]$ characters. The T matrix will have $1 + 3\ell$ Jordan blocks, with sizes $[2\ell + 1, 2\ell + 1, 2\ell + 1, \dots, 5, 5, 5, 3, 3, 3, 1]$. Each block corresponds to a non-logarithmic solution to the MLDE, given simply by the τ^0 objects in (129). The fact that a constant is always in the span of the modular orbit implies that all of the unflavored MLDEs for $\mathcal{T}_{2,2\ell+1}$ do not have the $D_q^{(0)}$ term.

As reviewed in Sec. II C, [42] defined the nilpotency index $\mathfrak{n}$, which is used to organize 4D $\mathcal{N} = 2$ SCFTs. Recall that the index is given by the power $\mathfrak{n}$ in the nilpotency relation

$$L_{-2}^{\mathfrak{n}}|0\rangle = |c_2\rangle + |\mathcal{N}_T\rangle, \quad |c_2\rangle \in C_2(\mathbb{V}[\mathcal{T}]) := \text{span}\{a_{-h_{a-1}}b\}. \quad (131)$$

This quantity is conjectured to be larger than the rank of the 4D theory [42],

$$\mathfrak{n} - 1 \geq \text{rank}\mathcal{T}. \quad (132)$$

As mentioned previously, many simple cases indicate $\mathfrak{n} = n_0 = n_{\min}$. Hence, since we do not know the value of $\mathfrak{n}$ for $\mathcal{T}_{2,2\ell+1}$, we could try to use n_0 or $n_{\min}$ to approximate the nilpotency index $\mathfrak{n}$. We can immediately check that this assumption is consistent with the inequality

$$n_0 - 1 = 6\ell + 3\ell^2 > \text{rank}\mathcal{T}_{2,2\ell+1} = 6\ell. \quad (133)$$

C. Modularity of $\mathcal{T}_{3,N}$

The modular properties of the $\mathcal{T}_{p \neq 2, N}$ theory are significantly more intricate than those of the $\mathcal{T}_{2, N}$ theory, due to the complex web of relations between the twisted Eisenstein series. To simplify the analysis of modular transformation, it is convenient to reexpress twisted Eisenstein series evaluated at $p\tau$ in terms of those at τ using the identities

$$E_k \begin{bmatrix} -1 \\ b^p \end{bmatrix} (p\tau) = \frac{1}{p} \sum_{p=0}^{l-1} E_k \begin{bmatrix} 1 \\ e^{2\pi i l/p} b \end{bmatrix},$$

$$E_k \begin{bmatrix} 1 \\ b^p \end{bmatrix} (p\tau) = \frac{1}{p} \sum_{p=0}^{l-1} E_k \begin{bmatrix} 1 \\ e^{2\pi i l/p} b \end{bmatrix}, \quad (134)$$

where we have omitted the (τ) on the right for brevity.

1. $\mathcal{T}_{3,2}$ theory

We begin with the simplest nontrivial example: the $\mathcal{T}_{3,2}$ theory. This case was discussed in [37]. For completeness, we rephrase their results using the Eisenstein series. The Schur index is

$$\mathcal{I}_{3,2} = \frac{1}{3} \left(E_1 \begin{bmatrix} -1 \\ q^{1/6} \end{bmatrix} + E_1 \begin{bmatrix} -1 \\ e^{2\pi i/3} q^{1/6} \end{bmatrix} + E_1 \begin{bmatrix} -1 \\ e^{4\pi i/3} q^{1/6} \end{bmatrix} \right). \quad (135)$$

Under the T transformation, the Eisenstein series transforms as

$$E_1 \begin{bmatrix} -1 \\ aq^{1/6} \end{bmatrix} \rightarrow E_1 \begin{bmatrix} -1 \\ e^{4\pi i/3} aq^{1/6} \end{bmatrix}, \quad (136)$$

which leaves the full Schur index invariant after summing over all three terms. The first nontrivial modular action appears under the S transformation. We construct the following set of independent expressions:

$$\text{ch}_1 = \text{Sch}_0 = -\frac{1}{6} + \frac{\tau}{3} \left(E_1 \begin{bmatrix} 1 \\ e^{\frac{2\pi i}{3}} \end{bmatrix} + E_1 \begin{bmatrix} 1 \\ e^{\frac{4\pi i}{3}} q^{-1/3} \end{bmatrix} + E_1 \begin{bmatrix} 1 \\ e^{\frac{2\pi i}{3}} q^{1/3} \end{bmatrix} \right), \quad (137)$$

$$\text{ch}_2 = T\text{ch}_1 = -\frac{1}{6} + \frac{1}{3} \left(E_1 \begin{bmatrix} 1 \\ e^{\frac{2\pi i}{3}} \end{bmatrix} + E_1 \begin{bmatrix} 1 \\ q^{-1/3} \end{bmatrix} + E_1 \begin{bmatrix} 1 \\ e^{\frac{4\pi i}{3}} q^{1/3} \end{bmatrix} \right) + \frac{\tau}{3} \left(E_1 \begin{bmatrix} 1 \\ e^{\frac{2\pi i}{3}} \end{bmatrix} + E_1 \begin{bmatrix} 1 \\ q^{-1/3} \end{bmatrix} + E_1 \begin{bmatrix} 1 \\ e^{\frac{4\pi i}{3}} q^{1/3} \end{bmatrix} \right), \quad (138)$$

$$\left(e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}} q^{-\frac{1}{3}}, e^{\frac{2\pi i}{3}} q^{\frac{1}{3}}\right) \xrightarrow{T} \left(e^{\frac{2\pi i}{3}}, q^{-\frac{1}{3}}, e^{\frac{4\pi i}{3}} q^{\frac{1}{3}}\right) \xrightarrow{T} \left(e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}} q^{-\frac{1}{3}}, q^{\frac{1}{3}}\right) \xrightarrow{T} \left(e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}} q^{-\frac{1}{3}}, e^{\frac{2\pi i}{3}} q^{\frac{1}{3}}\right)$$

FIG. 2. A periodic structure under the T transformation.

$$\begin{aligned} \text{ch}_3 &= T\text{ch}_2 \\ &= -\frac{1}{6} + \frac{2}{3} \left(E_1 \left[\frac{1}{e^{\frac{2\pi i}{3}}} \right] + E_1 \left[\frac{1}{e^{\frac{4\pi i}{3}} q^{-1/3}} \right] + E_1 \left[\frac{1}{q^{1/3}} \right] \right) \\ &\quad + \frac{\tau}{3} \left(E_1 \left[\frac{1}{e^{\frac{2\pi i}{3}}} \right] + E_1 \left[\frac{1}{e^{\frac{4\pi i}{3}} q^{-1/3}} \right] + E_1 \left[\frac{1}{q^{1/3}} \right] \right), \end{aligned} \quad (139)$$

$$\begin{aligned} \text{ch}_4 &= T\text{ch}_3 - \text{ch}_1 = E_1 \left[\frac{1}{e^{\frac{2\pi i}{3}}} \right] + E_1 \left[\frac{1}{e^{\frac{2\pi i}{3}} q^{-1/3}} \right] \\ &\quad + E_1 \left[\frac{1}{e^{\frac{2\pi i}{3}} q^{1/3}} \right]. \end{aligned} \quad (140)$$

In the last line, the subtraction of ch_1 from $T\text{ch}_3$ simplifies the final expression by removing the constant and the τ terms. Note that the exponents $1/3$ indicate some periodicity structure when applying T , as shown in Fig. 2.

The five expressions $\text{ch}_0, \dots, \text{ch}_4$ span the full modular orbit of ch_0 . The resulting S and T matrices are

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 3e^{\frac{\pi i}{3}} & -3 & 3e^{-\frac{\pi i}{3}} & 2e^{\frac{\pi i}{3}} \end{pmatrix}, \quad (141)$$

and

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2}(1-i\sqrt{3}) & \frac{3}{2}(1+i\sqrt{3}) & -2 & \frac{1}{2}(1-i\sqrt{3}) \\ 3 & \frac{3}{2}(3+i\sqrt{3}) & -\frac{13}{2} + \frac{3i\sqrt{3}}{2} & -3i\sqrt{3} & \frac{1}{2}(5+i\sqrt{3}) \\ \frac{3(\sqrt{3}+3i)}{\sqrt{3}-i} & -\frac{3(\sqrt{3}-3i)}{\sqrt{3}-i} & -\frac{6(\sqrt{3}+2i)}{\sqrt{3}-i} & 6 & \frac{\sqrt{3}+5i}{\sqrt{3}-i} \end{pmatrix}. \quad (142)$$

Their Jordan forms are

$$\begin{aligned} S &\sim \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 \end{pmatrix}, \\ T &\sim \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{2i\pi}{3}} & 1 \\ 0 & 0 & 0 & 0 & e^{\frac{2i\pi}{3}} \end{pmatrix}. \end{aligned} \quad (143)$$

There are three Jordan blocks in the Jordan decomposition of T , with block sizes $[2,2,1]$. The three blocks correspond to three nonlogarithmic solutions to the MLDE,

$$\text{ch}_0, \quad \text{ch}_1 + e^{\frac{2\pi i}{3}} \text{ch}_2 - e^{\frac{\pi i}{3}} \text{ch}_3, \quad \text{ch}_4. \quad (144)$$

As shown in [37], the fifth-order MLDE does not contain a $D_q^{(0)}$ term, so a constant is a solution. Indeed, the constant 1 can be written as a linear combination of these nonlogarithmic solutions,

$$1 = -6\text{ch}_0 - 6(\text{ch}_1 + e^{\frac{2\pi i}{3}} \text{ch}_2 - e^{\frac{\pi i}{3}} \text{ch}_3) + 2\sqrt{3}e^{\frac{5\pi i}{6}} \text{ch}_4. \quad (145)$$

The equality comes from the nontrivial identities

$$0 = E_1 \left[\frac{1}{q^{1/3}} \right] - e^{\frac{\pi i}{3}} E_1 \left[\frac{1}{e^{-\frac{2\pi i}{3}} q^{1/3}} \right] + e^{\frac{2\pi i}{3}} E_1 \left[\frac{1}{e^{\frac{2\pi i}{3}} q^{1/3}} \right], \quad (146)$$

and

$$\begin{aligned} 1 &= -2E_1 \left[\frac{-1}{q^{1/6}} \right] - 2E_1 \left[\frac{-1}{e^{-\frac{2\pi i}{3}} q^{1/6}} \right] \\ &\quad - 2E_1 \left[\frac{-1}{e^{\frac{2\pi i}{3}} q^{1/6}} \right] + 2i\sqrt{3}E_1 \left[\frac{1}{e^{\frac{2\pi i}{3}}} \right] \\ &\quad + (1 + 3\sqrt{3}i)E_1 \left[\frac{1}{q^{1/3}} \right] + (4 - 2i\sqrt{3})E_1 \left[\frac{1}{e^{-\frac{2\pi i}{3}} q^{1/3}} \right] \\ &\quad + (-5 - i\sqrt{3})E_1 \left[\frac{1}{e^{\frac{2\pi i}{3}} q^{1/3}} \right]. \end{aligned} \quad (147)$$

The $\mathcal{T}_{3,2}$ theory has a known and explicit associated VOA given by the $\mathcal{A}(6)$ algebra [36,79,80]. The corresponding null state $|\mathcal{N}_T\rangle$ was constructed in [37], which shows the nilpotency index $\mathbf{n} = 5$, which is equal to the dimension n_0 of the modular orbit span $\mathcal{V}_0$. In this case, the nilpotency index precisely saturates the inequality

$$n_0 - 1 = \mathbf{n} - 1 = 4 = \text{rank } \mathcal{T}_{3,2}. \quad (148)$$

2. $\mathcal{T}_{3,4}$ theory

The next nontrivial example is the $\mathcal{T}_{3,4}$ theory, whose Schur index is

$$\begin{aligned} \mathcal{I}_{\mathcal{T}_{3,4}}(q) = & -\frac{1}{72} + \frac{1}{72} \left(E_1 \begin{bmatrix} -1 \\ q^{1/6} \end{bmatrix} + E_1 \begin{bmatrix} -1 \\ e^{-\frac{2\pi i}{3}} q^{1/6} \end{bmatrix} + E_1 \begin{bmatrix} -1 \\ e^{\frac{2\pi i}{3}} q^{1/6} \end{bmatrix} \right) \\ & + \frac{1}{6} \left(-\frac{1}{3} E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{3} E_2 \begin{bmatrix} 1 \\ e^{-\frac{2\pi i}{3}} \end{bmatrix} - \frac{1}{3} E_2 \begin{bmatrix} 1 \\ e^{\frac{2\pi i}{3}} \end{bmatrix} \right) \\ & + \frac{1}{18} \left(E_1 \begin{bmatrix} -1 \\ q^{1/6} \end{bmatrix} + E_1 \begin{bmatrix} -1 \\ e^{-\frac{2\pi i}{3}} q^{1/6} \end{bmatrix} + E_1 \begin{bmatrix} -1 \\ e^{\frac{2\pi i}{3}} q^{1/6} \end{bmatrix} \right) \\ & \times \left(E_2 \begin{bmatrix} 1 \\ q^{1/3} \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ e^{-\frac{2\pi i}{3}} q^{1/3} \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ e^{\frac{2\pi i}{3}} q^{1/3} \end{bmatrix} \right) \\ & - \frac{1}{162} \left(E_1 \begin{bmatrix} -1 \\ q^{1/6} \end{bmatrix} + E_1 \begin{bmatrix} -1 \\ e^{-\frac{2\pi i}{3}} q^{1/6} \end{bmatrix} + E_1 \begin{bmatrix} -1 \\ e^{\frac{2\pi i}{3}} q^{1/6} \end{bmatrix} \right)^3. \end{aligned} \quad (149)$$

As in the case of the $\mathcal{T}_{3,2}$ theory, the Schur index of the $\mathcal{T}_{3,4}$ theory is invariant under the T transformation. The S transformation yields schematically

$$\frac{-5}{324} + \frac{1}{108} \tau(\cdots) + \frac{1}{108} \tau^2(\cdots) + \frac{1}{162} \tau^3(\cdots), \quad (150)$$

where the ellipses represent some polynomials of twisted Eisenstein series, of modular weight 1,2,3 with characteristics, for example, $(e^{\frac{2\pi i}{3}}, q^{-\frac{1}{3}}, e^{\frac{4\pi i}{3}} q^{\frac{1}{3}})$. The exponents $\frac{1}{3}$ indicate the periodicity of the T transformation, as shown in Fig. 2.

Now we are ready to compute the modular orbit of $\text{ch}_0^{\mathcal{T}_{3,4}} = \mathcal{I}_{\mathcal{T}_{3,4}}$. We construct a basis in a recursive manner,

$$\text{ch}_1, \text{ch}_2, \text{ch}_3 = \text{Sch}_0, T\text{ch}_1, T^2\text{ch}_2, \quad (151)$$

$$\text{ch}_4, \text{ch}_5, \text{ch}_6 = T\text{ch}_3 - \text{ch}_1, T\text{ch}_4, T^2\text{ch}_5, \quad (152)$$

$$\text{ch}_7, \text{ch}_8, \text{ch}_9 = T\text{ch}_6 - \text{ch}_4, T\text{ch}_7, T^2\text{ch}_8, \quad (153)$$

$$\text{ch}_{10}, \text{ch}_{11}, \text{ch}_{12} = T\text{ch}_9 - \text{ch}_7, T\text{ch}_{10}, T^2\text{ch}_{11}. \quad (154)$$

The element ch_{12} marks the end of the successive T transformations, since $\text{ch}_{10}, \text{ch}_{11}, \text{ch}_{12}$ forms a periodic sequence of length 3 under the T transformation,

$$T\text{ch}_{12} = \text{ch}_{10}. \quad (155)$$

At the beginning of each line, we subtract ch_{i-2} from $T\text{ch}_i$ to cancel the τ^3 , τ^2 , and τ terms, using the periodicity of the T transformation, as shown in Fig. 2.

Moving forward, we consider the following elements:

$$\text{ch}_{13} = \text{Sch}_{10}, \quad (156)$$

$$\begin{aligned} \text{ch}_{14}, \text{ch}_{15}, \text{ch}_{16} = & (T - id)\text{ch}_{13}, (T - id)\text{ch}_{14}, \\ & \times (T - id)\text{ch}_{15} + 972\text{ch}_0. \end{aligned} \quad (157)$$

In the last entry we subtracted 972ch_0 to cancel the weight-three terms in $(T - id)\text{ch}_{15}$, which simplifies the subsequent modular transformations, where such a simplification pattern repeats once again. Also, $T\text{ch}_{16} = 0$. To proceed, we enter the remaining sector:

$$\text{ch}_{17}, \text{ch}_{18}, \text{ch}_{19} = \text{Sch}_{16}, T\text{ch}_{17}, T\text{ch}_{18}, \quad (158)$$

$$\text{ch}_{20}, \text{ch}_{21}, \text{ch}_{22} = T\text{ch}_{19} - \text{ch}_{17}, T\text{ch}_{20}, T\text{ch}_{21}, \quad (159)$$

$$\text{ch}_{23}, \text{ch}_{24}, \text{ch}_{25} = T\text{ch}_{22} - \text{ch}_{20}, T\text{ch}_{23}, T\text{ch}_{24} \quad (160)$$

$$\text{ch}_{26}, \text{ch}_{27}, \text{ch}_{28} = \text{Sch}_{23}, (T - id)\text{ch}_{26}, (T - id)\text{ch}_{27} - 36\text{ch}_{16}, \quad (161)$$

$$\text{ch}_{29}, \text{ch}_{30}, \text{ch}_{31} = \text{Sch}_{28}, T\text{ch}_{29}, T\text{ch}_{30}, \quad (162)$$

$$\text{ch}_{32} = T\text{ch}_{31} - \text{ch}_{29}. \quad (163)$$

In the above, $T\text{ch}_{25} = \text{ch}_{23}$, and the subtraction $(T - id)\text{ch}_{27} - 36\text{ch}_{16}$ removes the weight-two terms in $(T - id)\text{ch}_{27}$. At the very end, the construction terminated at ch_{32} , since further action of T does not yield new linearly independent elements, for example,

$$0 = -3e^{\frac{i\pi}{6}}\text{ch}_{29} + 3\text{ch}_{30} + 3e^{\frac{i\pi}{6}}i\text{ch}_{31} - 2e^{\frac{i\pi}{3}}\text{ch}_{32} + \text{ch}_{33}, \quad (164)$$

which is due to an identity between Eisenstein series,

$$E_1 \begin{bmatrix} 1 \\ q^{1/3} \end{bmatrix} + e^{\frac{4\pi i}{3}} E_1 \begin{bmatrix} 1 \\ e^{\frac{4\pi i}{3}} q^{1/3} \end{bmatrix} + e^{\frac{2\pi i}{3}} E_1 \begin{bmatrix} 1 \\ e^{-\frac{2\pi i}{3}} q^{1/3} \end{bmatrix} = 0. \quad (165)$$

The resulting modular orbit thus spans a 33-dimensional space $\mathcal{V}_0$. Unfortunately, we have not constructed an order-33 MLDE to verify $n_{\min} = n_0$. The rank-33 S and T matrices can be computed directly with the basis $\text{ch}_{0,\dots,32}$, and we omit their explicit form here. The sizes of the Jordan blocks for the T and S matrices are given by $\{2, 1, 3, 3, 4, 4, 3, 4, 2, 3, 4\}$ and $\{1, \dots, 1\}$, respectively. Each block in T corresponds to a nonlogarithmic solution of the MLDE, given by the following expressions:

$$\text{ch}_0, \text{ch}_{10}, \text{ch}_{12}, \text{ch}_{13}, \text{ch}_{17}, \text{ch}_{24}, \text{ch}_{25}, \text{ch}_{26}, \text{ch}_{29}, \quad (166)$$

$$\text{ch}_{29} + e^{\frac{2\pi i}{3}}\text{ch}_{30} - e^{\frac{\pi i}{3}}\text{ch}_{31}. \quad (167)$$

In particular, the constant solution is a solution to the MLDE and can be written in terms of the nonlogarithmic solutions,

$$1 = \frac{18}{5}\text{ch}_{28} + \frac{18}{5}\text{ch}_9 + \frac{18e^{2\pi i/3}}{5}\text{ch}_{30} - \frac{18}{5}e^{\pi i/3}\text{ch}_{31} + \frac{6}{5}\sqrt{3}e^{-\pi i/6}\text{ch}_{32}. \quad (168)$$

With $\dim \mathcal{V}_0 = 33$, we again approximate the nilpotency index $\mathbf{n} = 33$. We can check against the conjecture in [42] and verify that

$$\dim \mathcal{V}_0 - 1 = \mathbf{n} - 1 = 32 > 12 = \text{rank } \mathcal{T}_{3,4}. \quad (169)$$

D. Modularity of $\mathcal{T}_{4,3}$

The $\mathcal{T}_{4,3}$ theory has a Schur index given by

$$\mathcal{I}_{\mathcal{T}_{4,3}} = -\frac{1}{48} \left(-1 + 12E_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} (4\tau) - 24E_1 \begin{bmatrix} -1 \\ q \end{bmatrix} (4\tau)^2 + 24E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} (4\tau) \right), \quad (170)$$

where we have identified the prefactor $\frac{i\partial_4(\tau|4\tau)}{\partial_1(\tau|4\tau)} = 1$. Using

$$E_k \begin{bmatrix} -1 \\ b^p \end{bmatrix} (p\tau) = \frac{1}{p} \sum_{\ell=0}^{p-1} E_k \begin{bmatrix} -1 \\ e^{2\pi i \ell / p} b \end{bmatrix}, \quad (171)$$

we are ready to compute the modular orbit of $\text{ch}_0^{\mathcal{T}_{4,3}} = \mathcal{I}_{\mathcal{T}_{4,3}}$. A complication arises from identities between Eisenstein series, for example,

$$0 = E_1 \begin{bmatrix} 1 \\ -q^{1/4} \end{bmatrix} - E_1 \begin{bmatrix} 1 \\ q^{1/4} \end{bmatrix} + iE_1 \begin{bmatrix} 1 \\ -iq^{1/4} \end{bmatrix} - iE_1 \begin{bmatrix} 1 \\ iq^{1/4} \end{bmatrix}, \quad (172)$$

$$0 = E_2 \begin{bmatrix} 1 \\ -q^{1/4} \end{bmatrix} - E_2 \begin{bmatrix} 1 \\ q^{1/4} \end{bmatrix} - \frac{1}{4}E_1 \begin{bmatrix} 1 \\ -q^{1/4} \end{bmatrix} + \frac{1}{4}E_1 \begin{bmatrix} 1 \\ q^{1/4} \end{bmatrix} + \frac{1}{2}E_1 \begin{bmatrix} -1 \\ i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ -iq^{1/4} \end{bmatrix} + \frac{1}{2}E_1 \begin{bmatrix} 1 \\ i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ -iq^{1/4} \end{bmatrix} - \frac{1}{2}E_1 \begin{bmatrix} -1 \\ i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ iq^{1/4} \end{bmatrix} - \frac{1}{2}E_1 \begin{bmatrix} 1 \\ i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ iq^{1/4} \end{bmatrix}. \quad (173)$$

Similar to the discussions of $\mathcal{T}_{3,4}$, it leads to some unexpected linear relations between elements in the modular orbit of $\text{ch}_0^{\mathcal{T}_{4,3}}$. For example, the τ^2 terms in $T^2\text{Sch}_0$ and Sch_0 are actually equal,

$$4(T^2\text{Sch}_0 - \text{Sch}_0)|_{\tau^2} = E_2 \begin{bmatrix} 1 \\ -q^{1/4} \end{bmatrix} - E_2 \begin{bmatrix} 1 \\ q^{1/4} \end{bmatrix} - \frac{1}{4}E_1 \begin{bmatrix} 1 \\ -q^{1/4} \end{bmatrix} + \frac{1}{4}E_1 \begin{bmatrix} 1 \\ q^{1/4} \end{bmatrix} + \frac{1}{2}E_1 \begin{bmatrix} -1 \\ i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ -iq^{1/4} \end{bmatrix} + \frac{1}{2}E_1 \begin{bmatrix} 1 \\ i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ -iq^{1/4} \end{bmatrix} - \frac{1}{2}E_1 \begin{bmatrix} -1 \\ i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ iq^{1/4} \end{bmatrix} - \frac{1}{2}E_1 \begin{bmatrix} 1 \\ i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ iq^{1/4} \end{bmatrix} = 0. \quad (174)$$

Otherwise, one would expect such an equality appearing between $T^4\text{Sch}_0$ and Sch_0 to cancel the leading τ^2 .

Taking into account all of these relations, we construct the following basis of $\mathcal{V}_0$ in a recursive manner:

$$\begin{aligned} \text{ch}_1 &= \text{Sch}_0, \\ \text{ch}_2, \text{ch}_3, \text{ch}_4, \text{ch}_5 &= T\text{ch}_1, T\text{ch}_2 - \text{ch}_1, T^\ell \text{ch}_3, \ell = 1, 2 \\ \text{ch}_6, \text{ch}_7, \text{ch}_8 &= T\text{ch}_5 + i\text{ch}_3 - \text{ch}_4 - i\text{ch}_5, T^\ell \text{ch}_6, \ell = 1, 2 \\ \text{ch}_9 &= \text{Sch}_8, \\ \text{ch}_{10}, \text{ch}_{11}, \text{ch}_{12} &= T\text{ch}_9, T\text{ch}_{10} - \text{ch}_9, T\text{ch}_{11}. \end{aligned} \quad (175)$$

These expressions span a 13-dimensional space. In [37], a 17th-order MLDE was constructed, with 13 rational indicial roots and four irrational indicial roots. This suggests the existence of a 13th-order MLDE. Indeed, we find such an MLDE at modular weight 72,

$$\begin{aligned}
0 = & [(574745600E_4^9 - 1076920320E_6^2E_4^6 + 1096671156E_6^4E_4^3 + 1424635751E_6^6)E_6E_4D_q^{(13)} \\
& - 432(124899200E_6^2E_4^9 - 122363780E_6^4E_4^6 + 276220301E_6^6E_4^3)D_q^{(12)} \\
& + \left(\frac{1}{7}(-9326814848000)E_6E_4^{11} + 4224114652800E_6^3E_4^8 - 2783340033180E_6^5E_4^5 - 1838877565469E_6^7E_4^2\right)D_q^{(11)} \\
& + 8(7442481904000E_4^9 - 3342340125600E_6^2E_4^6 + 19895117706660E_6^4E_4^3 - 20234101571453E_6^6)E_6^2E_4D_q^{(10)} \\
& - \frac{1}{7}6600(107830112000E_4^9 - 464290614480E_6^2E_4^6 + 3764819611896E_6^4E_4^3 - 5674341339373E_6^6)E_6E_4^3D_q^{(9)} \\
& + \frac{900}{7}(125131002656000E_4^9 - 533992662885360E_6^2E_4^6 + 135615545950920E_6^4E_4^3 + 277277681006249E_6^6)E_6^2E_4^2D_q^{(8)} \\
& + \frac{25}{49}(643463494355968000E_4^{12} - 2731343006933587200E_6^2E_4^9 \\
& + 3659914196088200400E_6^4E_4^6 - 4407695411247251480E_6^6E_4^3 + 2872041231540420093E_6^8)E_6E_4D_q^{(7)} \\
& - 21000(195522293753600E_4^9 - 227017724188080E_6^2E_4^6 - 287151556728792E_6^4E_4^3 - 652446141959227E_6^6)E_6^2E_4^3D_q^{(6)} \\
& + 16000(672100320320000E_4^{12} + 1275146989916800E_6^2E_4^9 - 5549761806160320E_6^4E_4^6 \\
& - 5608956608000116E_6^6E_4^3 + 4426262074119193E_6^8)E_6E_4^2D_q^{(5)} \\
& + 1000(186244985483392000E_4^{12} - 1504510704852769600E_6^2E_4^9 \\
& + 3377616482254971600E_6^4E_4^6 - 2490552991649009420E_6^6E_4^3 + 1680134794564701167E_6^8)E_6^2E_4D_q^{(4)} \\
& + \frac{40000}{7}(45809333636352000E_4^{12} - 106878284832457600E_6^2E_4^9 - 346593764375468400E_6^4E_4^6 \\
& + 830194932722473920E_6^6E_4^3 + 90356200990717277E_6^8)E_6E_4^3D_q^{(3)} \\
& + \frac{60000}{7}(1077666799527296000E_4^{12} - 4957511831704644800E_6^2E_4^9 \\
& + 6663480952508528400E_6^4E_4^6 - 4603907186041058980E_6^6E_4^3 + 6994092915923002351E_6^8)E_6^2E_4^2D_q^{(2)} \\
& + \frac{10000}{49}(43422946454384640000E_4^{15} - 225296315114048640000E_6^2E_4^{12} \\
& + 680500256154933166400E_6^4E_4^9 - 1372309846461167300400E_6^6E_4^6 \\
& + 1494610562792323885020E_6^8E_4^3 - 523092474786506400853E_6^{10})E_6E_4D_q^{(1)}] \text{ch}_0^{\mathcal{T}_{4,3}}. \tag{176}
\end{aligned}$$

This equation shares the same 13 rational indicial roots as the 17th-order MLDE constructed in [37]. Since the order of the MLDE equals the dimension n_0 of the span $\mathcal{V}_0$ of the modular orbit, $\mathcal{V}_0$ is expected to be the entire space of module characters. Using the above basis $\text{ch}_0, \dots, \text{ch}_{12}$, we can compute the S , T matrices. In particular, the T matrix reads

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & -2i & 4 & 6i & -6 & -6i & 4 & 2i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -32i & 53 & 96i & -110 & -96i & 61 & 32i & -4 & -1 & 0 & 2 & 0
\end{pmatrix}. \tag{177}$$

The Jordan decomposition leads to the following block structure:

$$T \rightarrow \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \end{pmatrix}, \quad (178)$$

featuring six blocks with block sizes $[3,3,2,2,2,1]$. Each block corresponds to a nonlogarithmic solution to the MLDE, given by the following four expressions:

$$\text{ch}_0, \text{ch}_6, \text{ch}_7, \text{ch}_8, \quad (179)$$

$$\begin{aligned} \text{ch}_{11} - 4\text{ch}_3 - 8(1+i)\text{ch}_4 + (4-8i)\text{ch}_5, \\ \text{ch}_{12} - 4(2+i)\text{ch}_3 - 8i\text{ch}_4 - 4i\text{ch}_5. \end{aligned} \quad (180)$$

Finally, with $n_0 = \dim \mathcal{V}_0 = 13$, we may use it to approximate the nilpotency index $\mathfrak{n} = 13$. We can check against the conjecture in [42] and verify that

$$n_0 - 1 = 12 > 9 = \text{rank} \mathcal{T}_{4,3}. \quad (181)$$

IV. MODULARITY AND 4D MIRROR SYMMETRY

The previous discussions establish the modularity properties of the $\mathcal{T}_{p,N}$ theory, providing the full space of simple and logarithmic modules characters of the associated VOA $\mathbb{V}[\mathcal{T}_{p,N}]$. In particular, the simple module characters are given by the nonlogarithmic solutions to the MLDE.

4D mirror symmetry is a mysterious correspondence between the associated VOA and the Coulomb branch geometry of the circle-compactified 4D theory $\mathcal{T}$. In [43,44], it was observed and proposed that there is a correspondence between the Coulomb branch geometry and the representation theory of VOAs. This relation was subsequently extended to a large class of Argyres-Douglas theories [45,46] and to class- $\mathcal{S}$ theories of type A_1 [47,48]. In this section, we explore the connection between the simple modules of the associated VOA and the fixed loci under $U(1)_r$ in the Coulomb branch geometry for $\mathcal{T}_{p,N}$

theories that admit a class $\mathcal{S}$ construction. We see in these cases that the number of simple modules matches the number of fixed points in the Coulomb branch geometry.

Recall that an Argyres-Douglas theory can be realized as a compactification of the 6D $\mathcal{N} = (2,0)$ theory on a Riemann surface with one irregular singularity (at $z = \infty$) labeled by $J^b[k]$ and at most one regular singularity (at $z = 0$). Here, k, b are both integers and $k > -b$; the allowed values of b are restricted by the group J [59,81–84]. The Coulomb branch geometry is determined by the boundary condition of the adjoint-valued Higgs field, which has the following asymptotics at the irregular singularity:

$$\Phi(z) \stackrel{z \rightarrow \infty}{\sim} (T_k z^{\frac{k}{b}} + \dots) dz. \quad (182)$$

The irregular singularity can be classified by requiring that the Higgs field is well defined upon analytic continuation around the singularity $z \rightarrow ze^{2\pi i}$ [85]. That is, there exists a gauge transformation g that ensures the single-valuedness of the Higgs field

$$\Phi(e^{2\pi i} z) = g\Phi(z)g^{-1} \Rightarrow gT_k g^{-1} = e^{2\pi i k/b} T_k. \quad (183)$$

This gauge transformation g induces a grading on the Lie algebra

$$\mathfrak{j} = \bigoplus_{\ell \in \mathbb{Z}/b\mathbb{Z}} \mathfrak{j}_{\ell/b}, \quad (184)$$

where $\mathfrak{j}_{\ell/b}$ denotes the subspace on which g acts with eigenvalue $e^{2\pi i \ell/b}$. Here, the leading coefficient T_k of the Higgs field expansion is a regular semisimple element in $\mathfrak{j}_{k/b} \subset \mathfrak{j}$.

Redefining the Higgs field via

$$\Phi(z) = \frac{\Phi'(z)}{z}, \quad (185)$$

the boundary condition near the irregular singularity becomes

$$\Phi'^{z \rightarrow \infty} (T_k z^\nu + \dots) dz., \quad \nu := \frac{k}{b} + 1 > 0. \quad (186)$$

Near the regular singularity, Φ has the boundary behavior

$$\Phi(z)^{z \rightarrow \infty} \left(\frac{f^\vee}{z} + \dots \right) dz, \quad (187)$$

where f is a regular nilpotent element in a Levi subalgebra $\mathfrak{l} \subset \mathfrak{j}$ [86], and $f^\vee$ is valued in the closure $\bar{\mathcal{O}}_{f^\vee}$, where $\mathcal{O}_{f^\vee}$ denotes the nilpotent orbit in $\mathfrak{j}^\vee$ that is the Spaltenstein dual of $\mathcal{O}_f$.

On the Hitchin side, the data is encoded in a parabolic subalgebra $\mathfrak{p}^\vee = \mathfrak{l}^\vee + \mathfrak{n}^\vee \subset \mathfrak{j}^\vee$ corresponding to the parabolic subgroup $P^\vee \subset J^\vee$. Here $\mathfrak{n}^\vee$ denotes the positive root space that is not contained in $\mathfrak{l}^\vee$, and $\mathfrak{l}^\vee$ is the dual of the Levi subalgebra $\mathfrak{l}$. In the massless limit, the redefined Higgs field behaves as

$$\Phi' \sim (\beta + \dots) dz, \quad \beta \in \mathfrak{n}^\vee. \quad (188)$$

This asymptotics is related to (187) by gauge transformations $j_0 + j_1 z + j_2 z^2 + \dots$, $j_0 \in P^\vee$, $j_i \in J^\vee$, due to the fact that $\bar{\mathcal{O}}_{f^\vee} = G^\vee \cdot \mathfrak{n}^\vee$, where $G^\vee$ is the connected and simply connected Lie group with Lie algebra $\mathfrak{j}^\vee$.

The Hitchin moduli space admits a $\mathbb{C}^*$ action that rescales the coordinates (x, z) as

$$x \rightarrow \lambda^\alpha x, \quad z \rightarrow \lambda^\beta z. \quad (189)$$

Given an elliptic element γ , for which the associated spectral curve is $\mathbb{C}^*$ invariant under the identification $\Phi(z)' = \gamma \gamma^{-1} dz$, the affine Springer fiber is defined as

$$Sp_{\gamma, \mathfrak{p}^\vee} := \{g \in \mathbf{P}^\vee \backslash \mathbf{J}^\vee | g \gamma g^{-1} \in \tilde{\mathfrak{n}}^\vee\}, \quad (190)$$

where $\tilde{\mathfrak{n}}^\vee$ is the subalgebra with root system $\Delta_{\tilde{\mathfrak{n}}^\vee} = \hat{\Delta}_+^\vee \backslash \Delta_{\mathfrak{l}^\vee}^\vee$, and $\mathbf{P}^\vee$ is the parahoric group of group $\mathbf{J}^\vee$ of gauge transformations, with root system $\hat{\Delta}_+^\vee \cup \Delta_{\mathfrak{l}^\vee}^\vee$.

The fixed locus of the affine Springer fiber under the torus action is described using the sets

$$L_\nu = \{\alpha + l\delta \in \hat{\Delta}^\vee | \nu(\alpha(\rho)) + l = 0\}, \quad (191)$$

and

$$S_\nu = \{\alpha + l\delta \in \hat{\Delta}^\vee | \nu(\alpha(\rho)) + l = \nu\}, \quad (192)$$

and the fixed varieties are given by [87,88]

$$Sp_{\gamma, \mathfrak{p}^\vee}^T = \sqcup H_{\tilde{w}}, \{\tilde{w} \in W_{\mathbf{P}^\vee} \backslash W_{\text{aff}} / W_\nu | \text{Ad}(\tilde{w})\gamma \in \tilde{\mathfrak{n}}^\vee\}. \quad (193)$$

Here, W_ν and $W_{\mathbf{P}^\vee}$ denote the Weyl groups of L_ν and $\mathbf{P}^\vee$, respectively. The dimension of each component is

$$\dim H_{\tilde{w}} = |\tilde{w}L_\nu \backslash \Delta_{\tilde{\mathfrak{n}}^\vee}| - |\tilde{w}S_\nu \backslash \Delta_{\tilde{\mathfrak{n}}^\vee}|. \quad (194)$$

We would like to apply this formalism to the three simplest $\mathcal{T}_{p,N}$ theories that can be identified as Argyres-Douglas theories,

$$\begin{aligned} \mathcal{T}_{3,2} &= (A_2, D_4), & \mathcal{T}_{4,3} &= (A_3, E_6), \\ \mathcal{T}_{6,5} &= (A_5, E_8). \end{aligned} \quad (195)$$

Let us begin with the $\mathcal{T}_{3,2}$ theory. The relevant VOA for the (A_2, D_4) theory is

$$W_{-h^\vee + \frac{b}{b+k}}(\mathfrak{j}, f), \quad (196)$$

with $h^\vee = 6$, $b = 6$, $k = 3$, $\mathfrak{j} = \mathfrak{d}_4$, and $f = [7, 1]$ corresponding to the principal nilpotent orbit in $\mathfrak{d}_4$ (namely, the absence of a regular puncture). Alternatively, $b = 4$, $k = 2$ also yields a consistent realization. In both cases, the level is

$$-6 + \frac{2}{3}. \quad (197)$$

This level is inadmissible, and the corresponding value of ν is

$$\nu = \frac{k}{b} + 1 = \frac{3}{2} = \frac{u}{m}, \quad u = 3, \quad m = 2. \quad (198)$$

We observe that there is no twist, and since D_4 is a simply laced, we have the affine roots $\hat{\Delta} = \hat{\Delta}^\vee$. Here, $f = [7, 1]$ implies that the standard Levi subalgebra satisfies $\Delta_{\mathfrak{l}}^\vee = \Delta_+^\vee$. Consequently, we obtain

$$\Delta_{\tilde{\mathfrak{n}}^\vee} = \hat{\Delta}_+ \backslash \Delta_+ = \{\alpha + l\delta | l > 0, \alpha \in \Delta\}, \quad (199)$$

and the Weyl group $W_{\mathbf{P}^\vee}$ is the finite Weyl group of $\mathfrak{d}_4$, since $\mathbf{P}^\vee$ is a parahoric subgroup with the root system $\hat{\Delta}_+^\vee \cup \Delta_+^\vee = \hat{\Delta}_+$ [89]. The elliptic element can be chosen as

$$\gamma = \sum_{\alpha + l\delta \in S_\nu, \alpha + l\delta \in \hat{\Delta}_+} e_\alpha z^l. \quad (200)$$

in terms of the Chevalley basis.

To determine the fixed varieties of the affine Springer fiber, we follow the algorithm in [87,88] and solve the defining equations for two sets L_ν and S_ν of affine roots,

$$L_\nu = \{\alpha + l\delta \in \hat{\Delta}^\vee | \nu\alpha(\rho) + l = 0\}, \quad (201)$$

and

$$S_\nu = \{\alpha + l\delta \in \hat{\Delta}^\vee | \nu\alpha(\rho) + l = \nu\}. \quad (202)$$

Solving these equations yields the sets

$$\begin{aligned} L_\nu &= \{6\delta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, 3\delta - \alpha_2 - \alpha_4, 3\delta - \alpha_1 - \alpha_2, 3\delta - \alpha_2 - \alpha_3, \\ &\quad -3\delta + \alpha_1 + \alpha_2, -3\delta + \alpha_2 + \alpha_3, -6\delta + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, -3\delta + \alpha_2 + \alpha_4\} \\ &= \{\pm(6\delta - e_1 - e_3), \pm(3\delta - e_2 - e_4), \pm(3\delta - e_1 + e_3), \pm(3\delta - e_2 + e_4)\} \end{aligned} \quad (203)$$

and

$$S_\nu = \left\{ \begin{array}{l} 9\delta - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4, 6\delta - \alpha_1 - \alpha_2 - \alpha_4, 6\delta - \alpha_2 - \alpha_3 - \alpha_4, 3\delta - \alpha_1, \\ 3\delta - \alpha_2, \alpha_1, 3\delta - \alpha_3, -3\delta + \alpha_1 + \alpha_2 + \alpha_3, \alpha_3, -6\delta + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \\ -3\delta + \alpha_2 + \alpha_3 + \alpha_4, \alpha_4 \end{array} \right\}. \quad (204)$$

The set L_ν generates a Weyl group W_ν , which can be used to describe the fixed varieties of the affine Springer fiber,

$$\{\tilde{w} \in W_{\mathfrak{d}_4^\vee} \setminus W_{\text{aff}}/W_\nu | \text{Ad}(\tilde{w})\gamma \in \alpha + l\delta, l > 0\}. \quad (205)$$

Explicitly, W_ν is generated by the reflections $s_{\hat{\beta}_i}$, where

$$\begin{aligned} \hat{\beta}_1 &= 6\delta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \\ \hat{\beta}_2 &= 3\delta - \alpha_2 - \alpha_4, \\ \hat{\beta}_3 &= 3\delta - \alpha_1 - \alpha_2, \\ \hat{\beta}_4 &= 3\delta - \alpha_2 - \alpha_3. \end{aligned} \quad (206)$$

One can verify that $(\hat{\beta}_i, \hat{\beta}_j) = 0$ for $i \neq j$. The action of the affine Weyl group on any affine weight is

$$s_{\hat{\alpha}}\hat{\lambda} = \hat{\lambda} - (\hat{\lambda}, \hat{\alpha}^\vee)\hat{\alpha}. \quad (207)$$

From this expression, it follows that $s_{\hat{\beta}_i}s_{\hat{\beta}_j}\hat{\lambda} = s_{\hat{\beta}_j}s_{\hat{\beta}_i}\hat{\lambda}$, $s_{\hat{\beta}_i}s_{\hat{\beta}_i}\hat{\lambda} = \hat{\lambda}$. Thus, W_ν is an Abelian group. It is known that the affine Weyl group can be decomposed into a semidirect product of the Weyl group and the coroot lattice,

$$W_{\text{aff}} \cong W_{\mathfrak{d}_4^\vee} \ltimes Q^\vee \Rightarrow W_{\mathfrak{d}_4^\vee} \setminus W_{\text{aff}} \simeq Q^\vee. \quad (208)$$

Recall that a generic translation is generated by

$$t_{\sum_{i=1}^4 m_i \alpha_i}, \quad m_i \in \mathbb{Z}, \quad (209)$$

where the action of translation on an affine root is

$$t_{\hat{\beta}}(\hat{\alpha}) = \hat{\alpha} - (\alpha, \beta)\delta. \quad (210)$$

Thus, the problem reduces to the following steps.

(1) First, denote the set in which γ takes values as S_γ (200). Then, determine the translation that satisfies

$$t_{\sum_{i=1}^4 m_i \alpha_i}(S_\gamma) \in \alpha + l\delta, \quad l > 0. \quad (211)$$

(2) Identify the translations that belong to different orbits of W_ν .

The generic elements of W_ν shift the finite part of the affine root, while translations only affect the δ component. For elements of the translation group that lie in the orbit of W_ν the only possibility is the identity element in W_ν . Hence, the simple modules should correspond to the allowed translations. Solving the constraint in the first step, we obtain three solutions that corresponding to the simple module:

$$\begin{array}{cccc} m_1 = -5, & m_2 = -8, & m_3 = -5, & m_4 = -5, \\ m_1 = -4, & m_2 = -7, & m_3 = -4, & m_4 = -4, \\ m_1 = -3, & m_2 = -5, & m_3 = -3, & m_4 = -3, \end{array} \quad (212)$$

with respective fixed point dimensions 7,7,0 if we apply (194) directly. Unfortunately, these dimensions do not match with the Jordan block structure of the T matrix [90].

The analysis for $\mathcal{T}_{4,3}$ and $\mathcal{T}_{6,5}$ proceeds in a similar fashion. The presence of a trivial puncture simplifies the computation, reducing the $\Delta_{\mathfrak{h}^\vee}$ to $\hat{\Delta}_+^\vee \setminus \Delta_+^\vee$, and hence leads to translation. Since W_ν is finite, the generic element in W_ν will shift the finite part, and the only element that does not do so is $\mathbb{I}$; hence, the coset of W_ν is trivial. For the $\mathcal{T}_{4,3}$ theory, the relevant L_ν and S_ν are

$$\begin{aligned}
L_\nu = \pm \{ & -4\delta + \alpha_2 + \alpha_4 + \alpha_5, \quad -4\delta + \alpha_2 + \alpha_3 + \alpha_4, \quad -8\delta + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\
& -4\delta + \alpha_3 + \alpha_4 + \alpha_5, \quad -4\delta + \alpha_4 + \alpha_5 + \alpha_6, \quad -8\delta + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \\
& -8\delta + \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \quad -12\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \\
& -4\delta + \alpha_1 + \alpha_3 + \alpha_4 \}
\end{aligned} \tag{213}$$

and

$$\begin{aligned}
S_\nu = \{ & \alpha_2, \quad -4\delta + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \quad -4\delta + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \\
& -8\delta + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \quad \alpha_3, \quad -4\delta + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \\
& \alpha_4, \quad \alpha_5, \quad \alpha_6, \quad -4\delta + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\
& -8\delta + \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \quad \alpha_1, \quad -8\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5, \\
& -12\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \quad -4\delta + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \\
& 4\delta - \alpha_2 - \alpha_4, \quad 8\delta - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6, \quad 8\delta - \alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5, \\
& 4\delta - \alpha_3 - \alpha_4, \quad 4\delta - \alpha_4 - \alpha_5, \quad 4\delta - \alpha_5 - \alpha_6, \\
& 16\delta - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6, \quad 8\delta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \\
& 12\delta - \alpha_1 - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6, \quad 12\delta - \alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6, \\
& 4\delta - \alpha_1 - \alpha_3, \quad 8\delta - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 \}.
\end{aligned} \tag{214}$$

The allowed translation for $t_{\sum m_i \alpha_i}$ is

$$\begin{aligned}
m_1 = -11m_2 = -15m_3 = -21m_4 = -29m_5 = -21m_6 = -11, \\
m_1 = -11m_2 = -15m_3 = -20m_4 = -28m_5 = -20m_6 = -11, \\
m_1 = -10m_2 = -14m_3 = -19m_4 = -27m_5 = -19m_6 = -10, \\
m_1 = -10m_2 = -13m_3 = -18m_4 = -25m_5 = -18m_6 = -10, \\
m_1 = -9m_2 = -13m_3 = -17m_4 = -24m_5 = -17m_6 = -9, \\
m_1 = -8m_2 = -11m_3 = -15m_4 = -21m_5 = -15m_6 = -8.
\end{aligned} \tag{215}$$

This matches with the modularity, implying that the theory has six simple modules. The dimension from the naive application of (194) for each fixed variety is $[16, 16, 16, 8, 5, 0]$.

For the $\mathcal{T}_{6,5}$ theory, the L_ν and S_ν are

$$\begin{aligned}
L_\nu = \pm \{ & \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 - 6\delta, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 - 6\delta, \\
& 2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 - 6\delta, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 - 30\delta, \\
& \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 - 12\delta, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 - 6\delta, \\
& \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 - 6\delta, \\
& 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 - 24\delta, \\
& \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8 - 18\delta, \\
& \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 - 18\delta, \\
& \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 - 6\delta, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 - 18\delta, \\
& \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 - 12\delta, \\
& \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 - 12\delta,
\end{aligned}$$

$$\begin{aligned}
& \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 - 12\delta, \\
& \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 - 24\delta, \\
& \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 - 12\delta, \\
& \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 - 12\delta, \\
& \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8 - 18\delta, \\
& \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 - 6\delta\}
\end{aligned} \tag{216}$$

and

$$\begin{aligned}
S_\nu = \{ & \alpha_2, -6\delta + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, -6\delta + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \\
& -6\delta + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\
& -30\delta + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \\
& -12\delta + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_3, \\
& -6\delta + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_4, \\
& \alpha_5, -24\delta + 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \alpha_6, \alpha_7, \alpha_8, \\
& -12\delta + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \\
& -18\delta + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8, \\
& -18\delta + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \\
& -6\delta + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, -6\delta + \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \\
& -12\delta + \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8, \alpha_1, \\
& -18\delta + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \\
& -12\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \\
& -12\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \\
& -24\delta + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \\
& -12\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \\
& -18\delta + \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \\
& -6\delta + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, 6\delta - \alpha_2 - \alpha_4 - \alpha_5 - \alpha_6, 6\delta - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \\
& 30\delta - 2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 5\alpha_4 - 4\alpha_5 - 3\alpha_6 - 2\alpha_7 - \alpha_8, \\
& 12\delta - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7, \\
& 12\delta - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7, \\
& 36\delta - 2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 6\alpha_4 - 5\alpha_5 - 4\alpha_6 - 3\alpha_7 - 2\alpha_8, \\
& 6\delta - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6, 6\delta - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7, 6\delta - \alpha_5 - \alpha_6 - \alpha_7 - \alpha_8, \\
& 24\delta - 2\alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 2\alpha_6 - 2\alpha_7 - \alpha_8, \\
& 18\delta - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7 - \alpha_8, 6\delta - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \\
& 18\delta - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7, \\
& 12\delta - \alpha_1 - \alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6 - \alpha_7 - \alpha_8, \\
& 24\delta - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 4\alpha_4 - 4\alpha_5 - 3\alpha_6 - 2\alpha_7 - \alpha_8,
\end{aligned}$$

$$\begin{aligned}
&12\delta - \alpha_1 - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6 - \alpha_7, \\
&24\delta - \alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 3\alpha_6 - 2\alpha_7 - \alpha_8, \\
&12\delta - \alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6 - \alpha_7, \\
&12\delta - \alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6, 18\delta - \alpha_1 - \alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - 2\alpha_6 - 2\alpha_7 - \alpha_8, \\
&18\delta - \alpha_1 - \alpha_2 - 2\alpha_3 - 3\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7 - \alpha_8, 6\delta - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5\}.
\end{aligned} \tag{217}$$

The fixed variety can be described by the following translations $t_{\sum_i m_i \alpha_i}$:

$$\begin{aligned}
&m_1 = -56, \quad m_2 = -83, \quad m_3 = -111, \quad m_4 = -165, \quad m_5 = -134, \quad m_6 = -102, \quad m_7 = -69, \quad m_8 = -35 \\
&m_1 = -56, \quad m_2 = -82, \quad m_3 = -110, \quad m_4 = -163, \quad m_5 = -133, \quad m_6 = -102, \quad m_7 = -69, \quad m_8 = -35 \\
&m_1 = -55, \quad m_2 = -82, \quad m_3 = -109, \quad m_4 = -162, \quad m_5 = -132, \quad m_6 = -101, \quad m_7 = -69, \quad m_8 = -35 \\
&m_1 = -55, \quad m_2 = -81, \quad m_3 = -109, \quad m_4 = -161, \quad m_5 = -131, \quad m_6 = -100, \quad m_7 = -68, \quad m_8 = -35 \\
&m_1 = -54, \quad m_2 = -80, \quad m_3 = -107, \quad m_4 = -159, \quad m_5 = -130, \quad m_6 = -99, \quad m_7 = -67, \quad m_8 = -34 \\
&m_1 = -54, \quad m_2 = -79, \quad m_3 = -106, \quad m_4 = -157, \quad m_5 = -128, \quad m_6 = -98, \quad m_7 = -67, \quad m_8 = -34 \\
&m_1 = -53, \quad m_2 = -79, \quad m_3 = -105, \quad m_4 = -156, \quad m_5 = -127, \quad m_6 = -97, \quad m_7 = -66, \quad m_8 = -34 \\
&m_1 = -53, \quad m_2 = -78, \quad m_3 = -105, \quad m_4 = -155, \quad m_5 = -126, \quad m_6 = -96, \quad m_7 = -65, \quad m_8 = -33 \\
&m_1 = -52, \quad m_2 = -77, \quad m_3 = -103, \quad m_4 = -153, \quad m_5 = -125, \quad m_6 = -96, \quad m_7 = -65, \quad m_8 = -33 \\
&m_1 = -52, \quad m_2 = -76, \quad m_3 = -102, \quad m_4 = -151, \quad m_5 = -123, \quad m_6 = -94, \quad m_7 = -64, \quad m_8 = -33 \\
&m_1 = -51, \quad m_2 = -76, \quad m_3 = -101, \quad m_4 = -150, \quad m_5 = -122, \quad m_6 = -93, \quad m_7 = -63, \quad m_8 = -32 \\
&m_1 = -50, \quad m_2 = -74, \quad m_3 = -99, \quad m_4 = -147, \quad m_5 = -120, \quad m_6 = -92, \quad m_7 = -63, \quad m_8 = -32 \\
&m_1 = -50, \quad m_2 = -73, \quad m_3 = -98, \quad m_4 = -145, \quad m_5 = -118, \quad m_6 = -90, \quad m_7 = -61, \quad m_8 = -31 \\
&m_1 = -48, \quad m_2 = -71, \quad m_3 = -95, \quad m_4 = -141, \quad m_5 = -115, \quad m_6 = -88, \quad m_7 = -60, \quad m_8 = -31 \\
&m_1 = -46, \quad m_2 = -68, \quad m_3 = -91, \quad m_4 = -135, \quad m_5 = -110, \quad m_6 = -84, \quad m_7 = -57, \quad m_8 = -29.
\end{aligned} \tag{218}$$

The dimension is [36,36,36,36,36,26,22,19,17,12,10,8,5,3,0] following (194). However, the modular orbit of $\mathcal{T}_{6.5}$ is too large and we do not have results to compare it with the Coulomb branch geometry, though we expect the result to be different from this naive computation of dimension.

V. $\mathbb{V}[\mathcal{T}_{p,N}]$ MODULE CHARACTERS FROM $\mathcal{N}=4$ $SU(N)$ THEORY

The relation between the Schur index $\mathcal{I}_{SU(N)}$ of $\mathcal{N}=4$ $SU(N)$ theory and $\mathcal{I}_{\mathcal{T}_{p,N}}$ of $\mathcal{T}_{p,N}$ theory suggests some deep relation between the associated VOAs. In [36] an isomorphism between the two VOAs as graded vector spaces was proposed for $p=3$, $N=2$, where the conformal weight structure is significantly altered under the isomorphism.

A VOA is the vacuum module of itself. It is therefore natural to ask whether there is any relation between the nonvacuum modules of $\mathbb{V}[\mathcal{T}_{SU(N)}]$ and $\mathbb{V}[\mathcal{T}_{p,N}]$.

Concretely, given a module character $\text{ch}(b, q)$ of the $\mathbb{V}[\mathcal{T}_{SU(N)}]$, we would like to understand whether the map, extrapolated from (79),

$$\text{ch}^{SU(N)}(b, q) \rightarrow q^{-\frac{c_{2D}[\mathcal{T}_{p,N}] - pc_{2D}[\mathcal{T}_{SU(N)}]}{24}} \text{ch}^{SU(N)}(q^{\frac{p}{2}-1}, q^p) \tag{219}$$

provides a module character of the $\mathbb{V}[\mathcal{T}_{p,N}]$ or a solution to the corresponding MLDE.

In the above discussions, we have gained some insight into the module characters of the $\mathbb{V}[\mathcal{T}_{p,N}]$ through the study of MLDEs and modular orbits. Unfortunately, the representation theory of the VOA $\mathbb{V}[\mathcal{T}_{SU(N)}]$ remains largely unexplored except for the $SU(2)$ case [91]. For higher N , we rely on the discussions in Sec. II C to speculate on how such a relation may look like at the level of characters. It would be very interesting to understand the modules of $\mathbb{V}[\mathcal{T}_{SU(N)}]$ more rigorously.

We start with the relatively trivial case with $p = 3$, $N = 2$. The VOA $\mathbb{V}[\mathcal{T}_{SU(2)}]$ has two nonlogarithmic irreducible modules [91]. Additionally, at the level of character, there is a logarithmic character whose corresponding module remains to be constructed. We list the three $\mathbb{V}[\mathcal{T}_{SU(2)}]$ characters in closed form [41],

$$\text{ch}_0 = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}, \quad \text{ch}_M = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} \left(1 - E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} \right), \quad (220)$$

$$\text{ch}_{\log} = \text{Sch}_0 = \frac{i\tau\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} + \frac{i\mathfrak{b}\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}. \quad (221)$$

Following the map (219), the vacuum character ch_0 gives the vacuum character for $\mathbb{V}[\mathcal{T}_{3,2}]$. The second character ch_M contains two terms, one being $-\text{ch}_0$, and the second $i\vartheta_4(\mathfrak{b})/\vartheta_1(2\mathfrak{b})$. Under (219),

$$\frac{\vartheta_4(i\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} \rightarrow \frac{i\vartheta_4(\frac{\tau}{2}|3\tau)}{\vartheta_1(\frac{\tau}{2}|3\tau)} = 1. \quad (222)$$

A constant is obviously a solution to the fifth-order MLDE. Hence, both of the nonlogarithmic characters are mapped to module characters of $\mathbb{V}[\mathcal{T}_{3,2}]$.

The logarithmic character $\text{ch}_{\log}$ under the map gives (up to an overall constant)

$$\begin{aligned} & \frac{2\pi\tau\vartheta_4(\frac{\tau}{2}|3\tau)}{q^{1/8}\vartheta_1(\tau|3\tau)} \left(1 + E_6 \begin{bmatrix} -1 \\ q^{1/2} \end{bmatrix} (3\tau) \right) \\ &= \ln q (1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + 6q^{12} \\ & \quad + 12q^{13} + 6q^{19} + 12q^{21} + 6q^{25} + \dots). \end{aligned} \quad (223)$$

It is straightforward to verify that this is indeed a solution to the fifth-order MLDE. Hence, all of the $\mathbb{V}[\mathcal{T}_{SU(2)}]$ characters are mapped to those of $\mathbb{V}[\mathcal{T}_{3,2}]$.

Next, we consider a universal module character for all N . As discussed in [30], the residue of a special pole in the integrand that computes the Schur index is given by

$$\text{Res} = q^{\frac{\dim \mathfrak{g}}{8}} \prod_{i=1}^r \frac{\left(b^{d_i-1} q^{\frac{d_i+1}{2}}; q \right) \left(b^{-d_i+1} q^{\frac{1-d_i}{2}}; q \right)}{\left(b^{d_i} q^{\frac{d_i}{2}}; q \right) \left(b^{-d_i} q^{1-\frac{d_i}{2}}; q \right)}. \quad (224)$$

Here d_i denotes the degrees of invariants of the Lie algebra $\mathfrak{su}(N)$. The residue is related to some Gukov-Witten-type surface defects, where the $SU(N)$ gauge field develops a certain singularity along a codimension-two surface [92]. Up to some numerical constant, the residue equals the vacuum character of a system of $bc\beta\gamma$ ghosts that provides the free field realization for the simple VOA $\mathbb{V}[\mathcal{T}_{SU(N)}]$ [8,34],

$$\text{Res} = \text{ch}_0(\mathbb{V}_{bc\beta\gamma}^{SU(N)}). \quad (225)$$

In particular, the system of $bc\beta\gamma$ ghosts constitute a reducible but indecomposable module of $\mathbb{V}[\mathcal{T}_{SU(N)}]$, and hence $\text{ch}_0(\mathbb{V}_{bc\beta\gamma}^{SU(N)})$ is a module character of $\mathbb{V}[\mathcal{T}_{SU(N)}]$. Interestingly, up to a sign,

$$\text{ch}_0(\mathbb{V}_{bc\beta\gamma}^{SU(N)}) = \begin{cases} \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(N\mathfrak{b})}, & N = \text{odd}, \\ \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(N\mathfrak{b})}, & N = \text{even}, \end{cases} \quad (226)$$

precisely the prefactor f_N of $\mathcal{I}_{SU(N)}$ in (63). As discussed before,

$$q^{-\frac{c_{2D}[\mathcal{T}_{p,N}] - p c_{2D}[\mathcal{T}_{SU(N)}]}{24}} \text{ch}_0(\mathbb{V}_{bc\beta\gamma}^{SU(N)}) \xrightarrow{b \rightarrow q^{\frac{p}{2}-1}, q \rightarrow q^p} 1. \quad (227)$$

As noted in explicit examples above, a constant is always inside the modular orbit of $\mathcal{I}_{\mathcal{T}_{p,N}}$ and hence must always be a solution to the MLDE.

Next, we consider the theory $\mathcal{T}_{2,3}$. We start by postulating candidates for the module character of $\mathbb{V}[\mathcal{T}_{SU(3)}]$. The Wilson line index of the $SU(3)$ theory was computed in [68]. For example, the Wilson line associated with the Young diagram [2, 2] has an index given by

$$\begin{aligned} \langle W_{[2,2]} \rangle_{SU(3)} &= \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})} \left[\frac{\sqrt{q}(b^3 q^{\frac{1}{2}} - 1)(-b^5 q - 2b^4 q^{\frac{1}{2}}(q+1) - b^3(q(q+4) + 2))}{b^4(q^2 - 1)^2} \right] \\ & \quad + \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})} \left[\frac{\sqrt{q}((b^2(2q(q+2) + 1)q^{\frac{1}{2}} + 2b(q+1)q + q^{3/2}))}{b^4(q^2 - 1)^2} \right] \\ & \quad + \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})} \frac{\sqrt{q}(b^2 - 1)[(b^2 + 1)\sqrt{q} + 2bq + 2b]}{b^2(q^2 - 1)} \left(E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} + E_1 \begin{bmatrix} -1 \\ b^2 \sqrt{q} \end{bmatrix} \right). \end{aligned} \quad (228)$$

Here we observe two interesting combinations:

$$\text{ch}_1^{SU(3)} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})} \left(E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} + E_1 \begin{bmatrix} -1 \\ b^2 \sqrt{q} \end{bmatrix} \right), \quad \text{ch}_2^{SU(3)} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}. \quad (229)$$

These two expressions, together with the vacuum character ch_0 , span all of the $SU(3)$ Wilson line index. It is therefore natural to speculate that $\frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}(E_1\left[\begin{smallmatrix} -1 \\ b \end{smallmatrix}\right] + E_1\left[\begin{smallmatrix} -1 \\ b^2\sqrt{q} \end{smallmatrix}\right])$ constitutes a module character of $\mathbb{V}[\mathcal{T}_{SU(3)}]$, and we can investigate its image under the map (219). Interestingly, the expression has a pole as $b \rightarrow q^{1/2}$, $q \rightarrow q^2$ with a constant residue and a constant b -independent term. Both of these values are trivial solutions to the MLDE of $\mathcal{T}_{2,3}$.

Admittedly, the $\mathcal{T}_{2,3}$ case is somewhat trivial. However, there is a similar but nontrivial example: $\mathcal{T}_{p=4, N=3}$. Again, we consider

$$\begin{aligned} & q^{-\frac{c_{2D}[\mathcal{T}_{4,3}] - 4c_{2D}[\mathcal{T}_{SU(3)}]}{24}} \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})} \\ & \times \left(E_1\left[\begin{smallmatrix} -1 \\ b \end{smallmatrix}\right] + E_1\left[\begin{smallmatrix} -1 \\ b^2\sqrt{q} \end{smallmatrix}\right] \right) \Big|_{b \rightarrow q^{\frac{p}{2}-1}, q \rightarrow q^p} \\ & = \frac{1}{q} \frac{\vartheta_4(\tau|4\tau)}{\vartheta_4(3\tau|4\tau)} \left(E_1\left[\begin{smallmatrix} -1 \\ q \end{smallmatrix}\right](4\tau) + E_1\left[\begin{smallmatrix} +1 \\ q^2 \end{smallmatrix}\right](4\tau) \right) \\ & = \frac{1}{2} - q - q^2 - q^4 - 2q^5 - q^8 - q^9 - 2q^{10} + \dots \quad (230) \end{aligned}$$

It is straightforward to verify that this is indeed a solution to the 13th-order MLDE (176). In fact, this expression can be written in terms of modular transformations of the vacuum character ch_i of $\mathbb{V}[\mathcal{T}_{4,3}]$,

$$\begin{aligned} & = \frac{1}{50} e^{-i \arctan \frac{4}{3}} (720 e^{i \arctan \frac{4}{3}} \text{ch}_0 + 52 \sqrt{10} e^{-i \arctan 3} \text{ch}_3 \\ & + 104 \sqrt{5} e^{-i \arctan \frac{1}{2}} \text{ch}_4 + 50 \sqrt{10} e^{+i \arctan 3} \text{ch}_5 \\ & + (27 + 114i) \text{ch}_6 - (87 + 15i) \text{ch}_7 - (18 + 77i) \text{ch}_8 \\ & + 13i(\text{ch}_{11} + \text{ch}_{12})), \quad (231) \end{aligned}$$

and hence is a linear combination of $\mathbb{V}[\mathcal{T}_{4,3}]$ -module characters. Note that before taking the specialization

$b \rightarrow q^{\frac{p}{2}-1}$, $q \rightarrow q^p$, the expression sits outside of the modular orbit of $\mathcal{I}_{SU(3)}$, since $E_1\left[\begin{smallmatrix} -1 \\ b^2\sqrt{q} \end{smallmatrix}\right]$ has a simple pole at the unflavoring limit $b \rightarrow 1$, while $\mathcal{I}_{SU(3)}(b, q)$ and its orbit do not.

Another interesting series of examples is the $\mathcal{T}_{2,2\ell+1}$ theories. To obtain candidates for modules, as we have discussed in Sec. II C, we exploit the observational insight from the study of flavored modular differential equations, where the space of characters is closed under the action of spectral-flow transformations. Concretely, we define a difference operator $\Delta_{(N)} \text{ch}(b, q) := b^{-(N^2-1)} q^{-\frac{N^2-1}{2}} \text{ch}(bq, q) - \text{ch}(b, q)$.

Let us first revisit the known cases. For $N = 2$, we have

$$\Delta_{(2)} \mathcal{I}_{SU(2)}(b, q) = \frac{i \vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}, \quad (232)$$

precisely reproducing the character of the associated $bc\beta\gamma$ system, which also appears in the $SU(2)$ Wilson line index. Similarly, for $SU(3)$, we have

$$\begin{aligned} \Delta_{(3)} \mathcal{I}_{SU(3)}(b, q) & = -\frac{3 \vartheta_4(\mathfrak{b})}{2 \vartheta_4(3\mathfrak{b})} \\ & + \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})} \left(E_1\left[\begin{smallmatrix} -1 \\ b \end{smallmatrix}\right] + E_1\left[\begin{smallmatrix} 1 \\ b^2 \end{smallmatrix}\right] \right), \quad (233) \end{aligned}$$

$$\Delta_{(3)}^2 \mathcal{I}_{SU(3)}(b, q) = -\frac{3 \vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})}. \quad (234)$$

Again, the difference operator generates the expressions we have seen in the computation of the Wilson line index.

Now we try to construct $\mathbb{V}[\mathcal{T}_{2,5}]$ modules from the $\mathcal{N} = 4$ $SU(5)$ theory. Under the difference operator $\Delta_{(5)}$, the Schur index $\mathcal{I}_{SU(5)}^{SU(N)}$ gives

$$\begin{aligned} \Delta_{(5)} \mathcal{I}_{SU(5)} & = -\frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(5\mathfrak{b})} \left(-225 + 120 E_1\left[\begin{smallmatrix} 1 \\ b^4 \end{smallmatrix}\right] + 70 E_1\left[\begin{smallmatrix} -1 \\ b \end{smallmatrix}\right] \right. \\ & - 36 E_1\left[\begin{smallmatrix} -1 \\ b \end{smallmatrix}\right]^2 + 8 E_1\left[\begin{smallmatrix} -1 \\ b \end{smallmatrix}\right]^3 - 24 E_1\left[\begin{smallmatrix} -1 \\ b^2 \end{smallmatrix}\right]^2 \\ & + E_1\left[\begin{smallmatrix} -1 \\ b \end{smallmatrix}\right] \left(120 - 72 E_1\left[\begin{smallmatrix} -1 \\ b^3 \end{smallmatrix}\right] - 48 E_2\left[\begin{smallmatrix} -1 \\ b^3 \end{smallmatrix}\right] \right) + 48 E_2\left[\begin{smallmatrix} -1 \\ b^3 \end{smallmatrix}\right] \\ & + E_1\left[\begin{smallmatrix} 1 \\ b^2 \end{smallmatrix}\right] \left(70 - 48 E_1\left[\begin{smallmatrix} -1 \\ b \end{smallmatrix}\right] + 24 E_1\left[\begin{smallmatrix} -1 \\ b \end{smallmatrix}\right]^2 - 24 E_2\left[\begin{smallmatrix} -1 \\ b^2 \end{smallmatrix}\right] \right) \\ & \left. + E_2\left[\begin{smallmatrix} 1 \\ b^2 \end{smallmatrix}\right] \left(36 - 24 E_1\left[\begin{smallmatrix} -1 \\ b \end{smallmatrix}\right] \right) + 96 E_2\left[\begin{smallmatrix} 1 \\ b^4 \end{smallmatrix}\right] + 16 E_3\left[\begin{smallmatrix} -1 \\ b^3 \end{smallmatrix}\right] + 48 E_3\left[\begin{smallmatrix} 1 \\ b^4 \end{smallmatrix}\right] \right). \quad (235) \end{aligned}$$

Under the map (219), $\Delta_{(5)} \text{ch}_0^{SU(5)}(b, q)$ develops a singularity as $\mathfrak{b} \rightarrow 0$, $\tau \rightarrow 2\tau$. However we can Laurent expand $\Delta_{(5)} \text{ch}_0^{SU(5)}(b, q)|_{q \rightarrow q^2}$ around $\mathfrak{b} = 0$. It is straightforward to show that the $\mathfrak{b}^0$ term solves the 25th-order MLDE of $\mathbb{V}[\mathcal{T}_{2,5}]$.

Alternatively, we can show that the $\mathfrak{b}^0$ term sits inside the span of the modular orbit of $\text{ch}_0^{T_{2,5}}$,

$$\Delta_{(5)} \text{ch}_0^{SU(5)}(b, q^2)|_{\mathfrak{b}^0} = -1000\text{ch}_{15}^{\mathbb{V}[T_{2,5}]}(q) - \text{ch}_{20}^{\mathbb{V}[T_{2,5}]}(q) - 5\text{ch}_{21}^{\mathbb{V}[T_{2,5}]}(q) + 60\text{ch}_{24}^{\mathbb{V}[T_{2,5}]}(q).$$

Interestingly, the $\mathfrak{b}^{-1}$ term is also in the span,

$$\Delta_{(5)} \text{ch}_0^{SU(5)}(b, q^2)|_{\mathfrak{b}^{-1}} = -\frac{i}{18\pi} \left(2600\text{ch}_{15}^{\mathbb{V}[T_{2,5}]}(q) + 3\text{ch}_{20}^{\mathbb{V}[T_{2,5}]}(q) + 3\text{ch}_{21}^{\mathbb{V}[T_{2,5}]}(q) - 161\text{ch}_{24}^{\mathbb{V}[T_{2,5}]}(q) \right). \quad (236)$$

We can continue to act $\Delta_{(5)}$ on the above expression and find

$$\Delta_{(5)}^2 \text{ch}_0^{SU(5)}(b, q^2)|_{\mathfrak{b}^0} = -\frac{2}{3} \left(23000\text{ch}_{15}^{\mathbb{V}[T_{2,5}]}(q) + 3\text{ch}_{20}^{\mathbb{V}[T_{2,5}]}(q) + 15\text{ch}_{21}^{\mathbb{V}[T_{2,5}]}(q) - 1430\text{ch}_{24}^{\mathbb{V}[T_{2,5}]}(q) \right). \quad (237)$$

Note, however, that the $\mathfrak{b}^{-1}$ term in $\Delta_{(5)}^2 \text{ch}_0^{SU(5)}(b, q^2)$ is constant and therefore sits trivially in the span. Finally, $\Delta_{(5)}^3 \text{ch}(b \rightarrow 1, q^2)$ is also a constant.

A similar pattern appears in $T_{2,7}$. In this case, we are not able to obtain the 49th-order MLDE explicitly. However, we show that the difference operator $\Delta_{(7)}^\ell$ acting on the Schur index provides elements in the span of the modular orbit of $\mathbb{V}[T_{2,7}]$, which also guarantees that the map (219) produces a module character of $\mathbb{V}[T_{2,7}]$. Explicitly, we list a few examples below:

$$\begin{aligned} \Delta_{(7)} \text{ch}_0^{SU(7)}(b, q^2)|_{\mathfrak{b}^{-2}} &= \frac{665}{114\pi^2} \text{ch}_{36} + \frac{25}{7104\pi^2} \text{ch}_{41} + \frac{25}{7104\pi^2} \text{ch}_{42} - \frac{24455}{85248\pi^2} \text{ch}_{45}, \\ \Delta_{(7)}^2 \text{ch}_0^{SU(7)}(b, q^2)|_{\mathfrak{b}^{-2}} &= \frac{4865}{72\pi^2} \text{ch}_{36} + \frac{25}{3552\pi^2} \text{ch}_{41} + \frac{25}{3552\pi^2} \text{ch}_{42} - \frac{179855}{42624\pi^2} \text{ch}_{45}, \\ \Delta_{(7)}^2 \text{ch}_0^{SU(7)}(b, q^2)|_{\mathfrak{b}^{-1}} &= -\frac{53263i}{18\pi} \text{ch}_{36} - \frac{325i}{444\pi} \text{ch}_{41} - \frac{275i}{222\pi} \text{ch}_{42} + \frac{1964131i}{10656\pi} \text{ch}_{45}, \\ \Delta_{(7)}^3 \text{ch}_0^{SU(7)}(b, q^2)|_{\mathfrak{b}^{-2}} &= \frac{175}{\pi^2} \text{ch}_{36} + \frac{175}{16\pi^2} \text{ch}_{45}, \\ \Delta_{(7)}^3 \text{ch}_0^{SU(7)}(b, q^2)|_{\mathfrak{b}^{-1}} &= -\frac{271159i}{18\pi} \text{ch}_{36} - \frac{325i}{444\pi} \text{ch}_{41} - \frac{275i}{222\pi} \text{ch}_{42} + \frac{10026283i}{10656\pi} \text{ch}_{45}, \\ \Delta_{(7)}^3 \text{ch}_0^{SU(7)}(b, q^2)|_{\mathfrak{b}^0} &= -\frac{1394981}{3} \text{ch}_{36} + \frac{5775}{148} \text{ch}_{41} - \frac{29025}{148} \text{ch}_{42} + \frac{51440147}{1776} \text{ch}_{45}. \end{aligned}$$

Similar expressions can be obtained for $\Delta_{(7)}^{\ell=4,5,6} \text{ch}_0$, which are relatively trivial.

As a final example, we consider the case $T_{3,4}$. By direct computation, we find that

$$\Delta_{(4)} \text{ch}_0^{SU(4)} = -\frac{i\vartheta_4(\mathfrak{b})}{2\vartheta_1(4\mathfrak{b})} \left(5 - 3E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} - 3E_1 \begin{bmatrix} 1 \\ b^3 \end{bmatrix} - 2E_1 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} + E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^2 + 2E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} E_1 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} - 2E_2 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} - E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} \right), \quad (238)$$

$$\Delta_{(4)}^2 \text{ch}_0^{SU(4)} = -\frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(4\mathfrak{b})} \left(-16 + 3E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} + 3E_1 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} + 2E_1 \begin{bmatrix} +1 \\ b^2 \end{bmatrix} \right), \quad (239)$$

$$\Delta_{(4)}^3 \text{ch}_0^{SU(4)} = -\frac{16i\vartheta_4(\mathfrak{b})}{\vartheta_1(4\mathfrak{b})}. \quad (240)$$

We can check if these expressions are mapped by (219) to a module character of $\mathbb{V}[T_{3,4}]$. Although we are unable to find the 33rd-order unflavored MLDE that annihilates the vacuum character $\text{ch}_0^{\mathbb{V}[T_{3,4}]}(q)$ explicitly, we are able to check whether the image sits in the span of the modular orbit. Explicitly, we find that under (219)

$$\begin{aligned}
\Delta_{(4)}\text{ch}_0^{SU(4)} &\rightarrow -18\text{ch}_0^{\mathbb{V}[T_{3,4}]}(q) - \frac{1}{54}\text{ch}_{16}^{\mathbb{V}[T_{3,4}]} - \frac{134}{5}\text{ch}_{28}^{\mathbb{V}[T_{3,4}]} \\
&\quad - \frac{169}{5}\text{ch}_{29}^{\mathbb{V}[T_{3,4}]} - \frac{169}{5}e^{\frac{2\pi i}{3}}\text{ch}_{30}^{\mathbb{V}[T_{3,4}]} \\
&\quad - \frac{169}{10}e^{\frac{4\pi i}{3}}\text{ch}_{31}^{\mathbb{V}[T_{3,4}]} + \frac{169}{5\sqrt{3}}e^{\frac{5\pi i}{6}}\text{ch}_{32}^{\mathbb{V}[T_{3,4}]}, \\
\Delta_{(4)}^2\text{ch}_0^{SU(4)} &\rightarrow -\frac{873}{5}\text{ch}_{28} + \frac{933}{5}\left(-\text{ch}_{29} - e^{\frac{2\pi i}{3}}\text{ch}_{30} + e^{\frac{\pi i}{3}}\text{ch}_{31}\right) \\
&\quad + \frac{311}{5}\sqrt{3}ie^{\frac{\pi i}{3}}\text{ch}_{32}. \tag{241}
\end{aligned}$$

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: SPECIAL FUNCTIONS AND NOTATION

In the following and in the main text, we use the normal and Fraktur fonts to denote fugacities in characters and Schur indices,

$$a = e^{2\pi i a}, \quad b = e^{2\pi i b}, \quad \tilde{b} = e^{2\pi i \tilde{b}}, \dots, z = e^{2\pi i z}. \tag{A1}$$

The only exception is $q = e^{2\pi i \tau}$, which is the standard notation.

1. Dedekind eta function

The Dedekind eta function $\eta(\tau)$ is defined as

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{+\infty} (1 - q^n). \tag{A2}$$

Under modular transformations,

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}}\eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau). \tag{A3}$$

2. Jacobi theta functions

The standard Jacobi theta functions are defined using the q -Pochhammer symbol $(z; q) := \prod_{k=0}^{+\infty} (1 - zq^k)$,

$$\begin{aligned}
\vartheta_1(\mathfrak{z}|\tau) &= iq^{\frac{1}{8}}z^{-\frac{1}{2}}(q; q)(z; q)(z^{-1}q; q) \\
&= -iq^{\frac{1}{8}}z^{\frac{1}{2}}(q; q)(zq; q)(z^{-1}; q), \tag{A4}
\end{aligned}$$

$$\begin{aligned}
\vartheta_2(\mathfrak{z}|\tau) &= q^{\frac{1}{8}}z^{-\frac{1}{2}}(q; q)(-z; q)(-z^{-1}q; q) \\
&= q^{\frac{1}{8}}z^{\frac{1}{2}}(q; q)(-zq; q)(-z^{-1}; q), \tag{A5}
\end{aligned}$$

$$\vartheta_3(\mathfrak{z}|\tau) = (q; q)(-zq^{1/2}; q)(-z^{-1}q^{1/2}; q), \tag{A6}$$

$$\vartheta_4(\mathfrak{z}|\tau) = (q; q)(zq^{1/2}; q)(z^{-1}q^{1/2}; q). \tag{A7}$$

We often omit the τ from the notation and, in particular,

$$\vartheta_i(0) = \vartheta_i(\mathfrak{z} = 0|\tau), \quad \vartheta_i^{(k)}(0) = \partial_{\mathfrak{z}}^k|_{\mathfrak{z}=0}\vartheta_i(\mathfrak{z}|\tau). \tag{A8}$$

These functions can be rewritten in infinite sums,

$$\begin{aligned}
\vartheta_1(\mathfrak{z}|\tau) &:= -i \sum_{r \in \mathbb{Z} + \frac{1}{2}} (-1)^{r-\frac{1}{2}} e^{2\pi i r \mathfrak{z}} q^{\frac{r^2}{2}}, \\
\vartheta_2(\mathfrak{z}|\tau) &:= \sum_{r \in \mathbb{Z} + \frac{1}{2}} e^{2\pi i r \mathfrak{z}} q^{\frac{r^2}{2}}, \tag{A9}
\end{aligned}$$

$$\begin{aligned}
\vartheta_3(\mathfrak{z}|\tau) &:= \sum_{n \in \mathbb{Z}} e^{2\pi i n \mathfrak{z}} q^{\frac{n^2}{2}}, \quad \vartheta_4(\mathfrak{z}|\tau) := \sum_{n \in \mathbb{Z}} (-1)^n e^{2\pi i n \mathfrak{z}} q^{\frac{n^2}{2}}. \tag{A10}
\end{aligned}$$

The Jacobi theta functions are quasiperiodic functions of $\mathfrak{z}$, for example,

$$\vartheta_1(\mathfrak{z} + m\tau + n) = (-1)^{m+n} e^{-2\pi i m \mathfrak{z}} q^{-\frac{1}{2}m^2} \vartheta_1(\mathfrak{z}). \tag{A11}$$

When $\mathfrak{z}$ is shifted by half-periods, the four Jacobi theta functions transform into each other. We summarize the shift property in the following diagram:

$$\begin{array}{ccccc}
& \vartheta_1 & \xrightarrow{1} & \vartheta_2 & \xrightarrow{-1} & \vartheta_1 \\
& \nearrow \text{slanted} & & \nearrow \text{slanted} & & \nearrow \text{slanted} \\
& \vartheta_4 & \xrightarrow{1} & \vartheta_3 & \xrightarrow{1} & \vartheta_4 \\
& \nearrow \text{slanted} & & \nearrow \text{slanted} & & \nearrow \text{slanted} \\
& \vartheta_1 & \xrightarrow{1} & \vartheta_2 & \xrightarrow{-1} & \vartheta_1
\end{array}$$

where $\mu = e^{-\pi i \mathfrak{z}} e^{-\frac{\pi i}{4}}$, and $f \xrightarrow{a} g$ means

$$\text{either } f\left(\mathfrak{z} + \frac{1}{2}\right) = ag(\mathfrak{z}) \quad \text{or} \quad f\left(\mathfrak{z} + \frac{\tau}{2}\right) = ag(\mathfrak{z}), \tag{A12}$$

depending on whether the arrow is horizontal or (slanted) vertical, respectively.

The modularity of $\vartheta_i(\mathfrak{z}|\tau)$ is well known. Under the S and T transformations, which act on the nome and flavor fugacity as $(\frac{\mathfrak{z}}{\tau}, -\frac{1}{\tau}) \xleftarrow{S} (\mathfrak{z}, \tau) \xrightarrow{T} (\mathfrak{z}, \tau + 1)$,

$$\begin{array}{ccc} -i\alpha\vartheta_1 & \xleftarrow{S} & \vartheta_1 \xrightarrow{T} e^{\frac{\pi i}{4}}\vartheta_1 \\ & \swarrow \quad \searrow & \\ \alpha\vartheta_2 & & \vartheta_2 \xrightarrow{T} e^{\frac{\pi i}{4}}\vartheta_2 \\ & \swarrow \quad \searrow & \\ \alpha\vartheta_3 & \xleftarrow{S} & \vartheta_3 \xrightarrow{T} \vartheta_3 \\ & \swarrow \quad \searrow & \\ \alpha\vartheta_4 & & \vartheta_4 \end{array}$$

Here we define $\alpha := \sqrt{-i\tau}e^{\frac{\pi i}{4}\tau^2}$. These transformations can be used to deduce the modular transformation of the Eisenstein series that we now review.

3. Eisenstein series

The Eisenstein series [93] depend on two characteristics ϕ, θ , and are defined by the infinite sum

$$E_{k \geq 1} \left[\begin{array}{c} \phi \\ \theta \end{array} \right] := -\frac{B_k(\lambda)}{k!} + \frac{1}{(k-1)!} \sum_{r \geq 0} \frac{(r+\lambda)^{k-1} \theta^{-1} q^{r+\lambda}}{1 - \theta^{-1} q^{r+\lambda}} + \frac{(-1)^k}{(k-1)!} \sum_{r \geq 1} \frac{(r-\lambda)^{k-1} \theta q^{r-\lambda}}{1 - \theta q^{r-\lambda}}, \quad (\text{A13})$$

where $\phi := e^{2\pi i \lambda}$ with $0 \leq \lambda < 1$, $B_k(x)$ denotes the k th Bernoulli polynomial, and the prime in the sum indicates that the $r = 0$ should be omitted when $\phi = \theta = 1$. To make a certain formula more compact, we also define

$$E_0 \left[\begin{array}{c} \phi \\ \theta \end{array} \right] = -1. \quad (\text{A14})$$

We have suppressed the τ dependence in the notation. However, when the argument is rescaled to $n\tau$, we write it explicitly, like

$$E_2 \left[\begin{array}{c} -1 \\ z \end{array} \right] (3\tau), \quad E_3 \left[\begin{array}{c} 1 \\ q^{1/4}b \end{array} \right] (4\tau), \quad \text{etc.} \quad (\text{A15})$$

Eisenstein series with even $k = 2n$ are related to the usual Eisenstein series $E_{2n}(\tau)$ by sending the $\theta, \phi \rightarrow 1$. When k is odd, $\theta = \phi = 1$ is a vanishing limit, except for the case $k = 1$ where it is singular,

$$E_{2n} \left[\begin{array}{c} +1 \\ +1 \end{array} \right] = E_{2n}, \quad E_1 \left[\begin{array}{c} +1 \\ z \end{array} \right] = \frac{1}{2\pi i} \frac{\vartheta'_1(\mathfrak{z})}{\vartheta_1(\mathfrak{z})},$$

$$E_{2n+1 \geq 3} \left[\begin{array}{c} +1 \\ +1 \end{array} \right] = 0. \quad (\text{A16})$$

In fact, all $E_k[\frac{\pm 1}{z}]$ are regular as $z \rightarrow 1$, except for $E_1[\frac{1}{z}]$, which has a simple pole. We often omit (τ) in $E_n(\tau)$, unless the argument is $n\tau$, where we write $E_k(n\tau)$.

A closely related property is the symmetry of the Eisenstein series

$$E_k \left[\begin{array}{c} \pm 1 \\ z^{-1} \end{array} \right] = (-1)^k E_k \left[\begin{array}{c} \pm 1 \\ z \end{array} \right]. \quad (\text{A17})$$

To analyze properties of Eisenstein series, it is useful to relate them with the Jacobi theta functions,

$$E_k \left[\begin{array}{c} +1 \\ z \end{array} \right] = \sum_{\ell=0}^{\lfloor k/2 \rfloor} \frac{(-1)^{k+1}}{(k-2\ell)!} \left(\frac{1}{2\pi i} \right)^{k-2\ell} \mathbb{E}_{2\ell} \frac{\vartheta_1^{(k-2\ell)}(\mathfrak{z})}{\vartheta_1(\mathfrak{z})}. \quad (\text{A18})$$

Here $\vartheta_i^{(k)}(\mathfrak{z})$ is the k th derivative in $\mathfrak{z}$. The conversion from $E_k[\frac{-1}{z}]$ can be obtained by replacing ϑ_1 with $\vartheta_{2,3,4}$ appropriately. An immediate consequence of this relation is that the Eisenstein series satisfy the following shift property:

$$E_k \left[\begin{array}{c} \pm 1 \\ zq^{\frac{n}{2}} \end{array} \right] = \sum_{\ell=0}^k \binom{n}{2}^{\ell} \frac{1}{\ell!} E_{k-\ell} \left[\begin{array}{c} (-1)^n (\pm 1) \\ z \end{array} \right]. \quad (\text{A19})$$

With the help of the known modular property of ϑ_i , we deduce that under the $S: (\tau, \mathfrak{z}) \rightarrow (-\frac{1}{\tau}, \frac{\mathfrak{z}}{\tau})$, the Eisenstein series transform according to the following formula:

$$E_n \left[\begin{array}{c} +1 \\ +z \end{array} \right] \xrightarrow{S} E_n \left[\begin{array}{c} +1 \\ e^{\frac{2\pi i \mathfrak{z}}{\tau}} \end{array} \right] \left(-\frac{1}{\tau} \right) = \left(\frac{1}{2\pi i} \right)^n \sum_{\ell=0}^k \frac{(-\log z)^{n-\ell} (\log q)^{\ell}}{(n-\ell)!} E_{\ell} \left[\begin{array}{c} +1 \\ z \end{array} \right], \quad (\text{A20})$$

$$E_n \left[\begin{array}{c} -1 \\ +z \end{array} \right] \xrightarrow{S} E_n \left[\begin{array}{c} -1 \\ e^{\frac{2\pi i \mathfrak{z}}{\tau}} \end{array} \right] \left(-\frac{1}{\tau} \right) = \left(\frac{1}{2\pi i} \right)^n \sum_{\ell=0}^k \frac{(-\log z)^{n-\ell} (\log q)^{\ell}}{(n-\ell)!} E_{\ell} \left[\begin{array}{c} +1 \\ -z \end{array} \right], \quad (\text{A21})$$

$$E_n \left[\begin{array}{c} 1 \\ -z \end{array} \right] \xrightarrow{S} E_n \left[\begin{array}{c} 1 \\ -e^{\frac{2\pi i \mathfrak{z}}{\tau}} \end{array} \right] \left(-\frac{1}{\tau} \right) = \left(\frac{1}{2\pi i} \right)^n \sum_{\ell=0}^k \frac{(-\log z)^{n-\ell} (\log q)^{\ell}}{(n-\ell)!} E_{\ell} \left[\begin{array}{c} -1 \\ z \end{array} \right], \quad (\text{A22})$$

$$E_n \begin{bmatrix} -1 \\ -z \end{bmatrix} \xrightarrow{S} E_n \begin{bmatrix} -1 \\ -e^{\frac{2\pi i \mathfrak{z}}{\tau}} \end{bmatrix} \left(-\frac{1}{\tau} \right) = \left(\frac{1}{2\pi i} \right)^n \sum_{\ell=0}^k \frac{(-\log z)^{n-\ell} (\log q)^\ell}{(n-\ell)!} E_\ell \begin{bmatrix} -1 \\ -z \end{bmatrix}. \quad (\text{A23})$$

Under the T transformation, the Eisenstein series transform according to the following rules:

$$E_n \begin{bmatrix} +1 \\ +z \end{bmatrix} \xrightarrow{T} E_n \begin{bmatrix} +1 \\ +z \end{bmatrix}, \quad E_n \begin{bmatrix} -1 \\ +z \end{bmatrix} \xrightarrow{T} E_n \begin{bmatrix} -1 \\ -z \end{bmatrix} \quad (\text{A24})$$

$$E_n \begin{bmatrix} +1 \\ -z \end{bmatrix} \xrightarrow{T} E_n \begin{bmatrix} +1 \\ -z \end{bmatrix}, \quad E_n \begin{bmatrix} -1 \\ -z \end{bmatrix} \xrightarrow{T} E_n \begin{bmatrix} -1 \\ +z \end{bmatrix}. \quad (\text{A25})$$

For reference, we give a few explicit examples of modular transformations of Eisenstein series. Consider the special case $E_2(\tau) = E_2 \begin{bmatrix} +1 \\ +1 \end{bmatrix} = \lim_{z \rightarrow 1} E_2 \begin{bmatrix} +1 \\ z \end{bmatrix}$. Under the S transformation $(\tau, \mathfrak{z}) \rightarrow (-\frac{1}{\tau}, \frac{\mathfrak{z}}{\tau})$,

$$E_2 \begin{bmatrix} 1 \\ e^{2\pi i \mathfrak{z}} \end{bmatrix}(\tau) \rightarrow -\frac{\mathfrak{z}^2}{2} - \mathfrak{z} \tau E_1 \begin{bmatrix} 1 \\ e^{2\pi i \mathfrak{z}} \end{bmatrix} + \tau^2 E_2 \begin{bmatrix} 1 \\ e^{2\pi i \mathfrak{z}} \end{bmatrix}. \quad (\text{A26})$$

When taking the limit $\mathfrak{z} \rightarrow 0$, the second term does not vanish due the fact that $E_1 \begin{bmatrix} 1 \\ z \end{bmatrix}$ has a simple pole at $z = 1$,

$$\lim_{\mathfrak{z} \rightarrow 0} \mathfrak{z} E_1 \begin{bmatrix} 1 \\ e^{2\pi i \mathfrak{z}} \end{bmatrix} = -\frac{i}{2\pi}. \quad (\text{A27})$$

As a result, we recover the standard modular transformation of $E_2(\tau)$,

$$E_2(\tau) \xrightarrow{S} \tau^2 E_2(\tau) + \frac{i}{2\pi} \tau. \quad (\text{A28})$$

Besides $E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, no other $E_{k>2}(\tau)$ has this extra τ term, since $\lim_{\mathfrak{z} \rightarrow 0} \mathfrak{z}^{k-1} E_1 \begin{bmatrix} 1 \\ z \end{bmatrix} = 0$.

As a second example, we consider two different ways of computing the modular transformation of $E_2 \begin{bmatrix} +1 \\ -1 \end{bmatrix}$. First, according to the above rules,

$$E_2 \begin{bmatrix} +1 \\ -1 \end{bmatrix}(\tau) \xrightarrow{S} \left(\frac{1}{2\pi i} \right)^2 \sum_{\ell=0}^k \frac{(-\log z)^{2-\ell} (\log q)^\ell}{(2-\ell)!} E_\ell \begin{bmatrix} -1 \\ z \end{bmatrix}(\tau) \Big|_{z \rightarrow 1} = \tau^2 E_2 \begin{bmatrix} -1 \\ +1 \end{bmatrix}, \quad (\text{A29})$$

Alternatively, we can treat $-1 = e^{\frac{2\pi i \mathfrak{z}}{\tau}}$ with $\mathfrak{z} := \frac{\tau}{2}$, $z := e^{2\pi i \mathfrak{z}} = q^{\frac{1}{2}}$,

$$E_2 \begin{bmatrix} +1 \\ -1 \end{bmatrix}(\tau) \xrightarrow{S} E_2 \begin{bmatrix} +1 \\ e^{\frac{\pi i \tau}{\tau}} \end{bmatrix} \left(-\frac{1}{\tau} \right) = \left(\frac{1}{2\pi i} \right)^2 \sum_{\ell=0}^2 \frac{(-\pi i \tau)^{2-\ell} (2\pi i \tau)^\ell}{(2-\ell)!} E_\ell \begin{bmatrix} -1 \\ q^{\frac{1}{2}} \end{bmatrix}(\tau) = \tau^2 E_2 \begin{bmatrix} -1 \\ +1 \end{bmatrix}(\tau). \quad (\text{A30})$$

In going to the second line, we made use of the shift property (A19).

Combining the S , T transformations, we obtain

$$E_n \begin{bmatrix} -1 \\ z \end{bmatrix} \xrightarrow{STS} \left(\frac{1}{2\pi i} \right)^n \left[\left(\sum_{k \geq 0} \frac{1}{k!} (-\log z)^k y^k \right) \left(\sum_{\ell \geq 0} \left(\log q - 2\pi i \right)^\ell y^\ell E_\ell \begin{bmatrix} -1 \\ +z \end{bmatrix} \right) \right]_n.$$

There are some useful identities involving Eisenstein series and their products. For example, using the following elegant identity, we can trade the argument $p\tau$ for τ :

$$E_n \begin{bmatrix} \pm 1 \\ b^p \end{bmatrix}(p\tau) = \frac{1}{p} \sum_{\ell=0}^{p-1} E_n \begin{bmatrix} \pm 1 \\ b e^{2\pi i \ell / p} \end{bmatrix}(\tau), \quad p = 1, 2, 3, \dots \quad (\text{A31})$$

As we mentioned earlier, we often omit the (τ) on the right. There are infinitely many between products of Eisenstein series, for example,

$$0 = E_1 \begin{bmatrix} 1 \\ q^{1/4} \end{bmatrix} - E_1 \begin{bmatrix} 1 \\ -q^{1/4} \end{bmatrix} + i E_1 \begin{bmatrix} 1 \\ i q^{1/4} \end{bmatrix} - i E_1 \begin{bmatrix} 1 \\ -i q^{1/4} \end{bmatrix}, \quad (\text{A32})$$

$$\begin{aligned}
0 = & E_2 \begin{bmatrix} 1 \\ -q^{1/4} \end{bmatrix} - E_2 \begin{bmatrix} 1 \\ q^{1/4} \end{bmatrix} - \frac{1}{4} E_1 \begin{bmatrix} 1 \\ -q^{1/4} \end{bmatrix} + \frac{1}{4} E_1 \begin{bmatrix} 1 \\ q^{1/4} \end{bmatrix} + \frac{1}{2} E_1 \begin{bmatrix} -1 \\ i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ -iq^{1/4} \end{bmatrix} \\
& + \frac{1}{2} E_1 \begin{bmatrix} 1 \\ i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ -iq^{1/4} \end{bmatrix} - \frac{1}{2} E_1 \begin{bmatrix} -1 \\ i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ iq^{1/4} \end{bmatrix} - \frac{1}{2} E_1 \begin{bmatrix} 1 \\ i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ iq^{1/4} \end{bmatrix}.
\end{aligned} \tag{A33}$$

Many of these identities complicate the structure of the modular orbit of the Schur index.

APPENDIX B: $\mathcal{N}=4$ $SU(N)$ SCHUR INDEX

In this appendix we collect the Schur indices of $\mathcal{N} = 4$ $SU(N)$ theory with relatively low N .

$SU(2)$

The flavored Schur index is

$$\mathcal{I}_{SU(2)} = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}. \tag{B1}$$

$SU(3)$

The flavored Schur index is

$$\mathcal{I}_{SU(3)} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})} \left(\frac{1}{24} - \frac{1}{2} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^2 + \frac{1}{2} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} \right). \tag{B2}$$

$SU(4)$

The flavored Schur index is

$$\mathcal{I}_{SU(4)} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_1(4\mathfrak{b})} \left(\frac{i}{24} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} - \frac{i}{6} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^3 + \frac{i}{24} E_1 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} + \frac{i}{2} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} - \frac{i}{3} E_3 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} \right). \tag{B3}$$

$SU(5)$

The flavored Schur index is

$$\begin{aligned}
\mathcal{I}_{SU(5)} = & \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(5\mathfrak{b})} \left(\frac{3}{640} + \frac{1}{48} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} + \frac{1}{24} E_2 \begin{bmatrix} 1 \\ b^4 \end{bmatrix} - \frac{1}{48} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^2 - \frac{1}{24} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} \right. \\
& \left. - \frac{1}{4} E_4 \begin{bmatrix} 1 \\ b^4 \end{bmatrix} + \frac{1}{24} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^4 - \frac{1}{4} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^2 E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} + \frac{1}{8} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix}^2 + \frac{1}{3} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} E_3 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} \right).
\end{aligned} \tag{B4}$$

$SU(6)$

The flavored Schur index is

$$\begin{aligned}
\mathcal{I}_{SU(6)} = & \frac{i\vartheta_4(\mathfrak{b}, q)}{\vartheta_1(6\mathfrak{b}, q)} \left(-\frac{3}{640} E_1 \begin{bmatrix} -1 \\ b^5 \end{bmatrix} - \frac{3}{640} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} - \frac{1}{576} E_1 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} - \frac{1}{48} E_2 \begin{bmatrix} 1 \\ b^4 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} - \frac{1}{48} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} \right. \\
& - \frac{1}{48} E_1 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} + \frac{1}{24} E_3 \begin{bmatrix} -1 \\ b^5 \end{bmatrix} + \frac{1}{72} E_3 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} + \frac{1}{48} E_1 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^2 + \frac{1}{144} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^3 \\
& + \frac{1}{4} E_4 \begin{bmatrix} 1 \\ b^4 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} - \frac{1}{8} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix}^2 E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} - \frac{1}{6} E_3 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^2 \\
& \left. + \frac{1}{12} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^3 + \frac{1}{6} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} E_3 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} - \frac{1}{5} E_5 \begin{bmatrix} -1 \\ b^5 \end{bmatrix} - \frac{1}{120} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^5 \right).
\end{aligned} \tag{B5}$$

$SU(7)$

$$\begin{aligned} \mathcal{I}_{SU(7)} = & \frac{\partial_4(\mathbf{b})}{\partial_4(7\mathbf{b})} \left(\frac{5}{7168} - \frac{3}{640} E_1 \begin{bmatrix} -1 \\ b^5 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} - \frac{1}{576} E_1 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} - \frac{3}{1280} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^2 + \frac{3}{1280} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} + \frac{1}{576} E_2 \begin{bmatrix} 1 \\ b^4 \end{bmatrix} \right. \\ & + \frac{1}{180} E_2 \begin{bmatrix} 1 \\ b^6 \end{bmatrix} + \frac{1}{192} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix}^2 + \frac{1}{72} E_3 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} + \frac{1}{24} E_3 \begin{bmatrix} -1 \\ b^5 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} - \frac{1}{18} E_3 \begin{bmatrix} -1 \\ b^3 \end{bmatrix}^2 - \frac{1}{96} E_4 \begin{bmatrix} 1 \\ b^4 \end{bmatrix} - \frac{1}{24} E_4 \begin{bmatrix} 1 \\ b^6 \end{bmatrix} \\ & + \frac{1}{8} E_4 \begin{bmatrix} 1 \\ b^4 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^2 + \frac{1}{48} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} E_2 \begin{bmatrix} 1 \\ b^4 \end{bmatrix} - \frac{1}{5} E_5 \begin{bmatrix} -1 \\ b^5 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} + \frac{1}{6} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} E_3 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} + \frac{1}{6} E_6 \begin{bmatrix} 1 \\ b \end{bmatrix} \\ & + \frac{1}{48} E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix}^3 - \frac{1}{720} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^6 + \frac{1}{48} E_2^2 E_2 \begin{bmatrix} 1 \\ b \end{bmatrix} - \frac{1}{24} E_4 E_2 \begin{bmatrix} 1 \\ b \end{bmatrix} - \frac{1}{48} E_2^2 E_2 \begin{bmatrix} 1 \\ b^6 \end{bmatrix} + \frac{1}{24} E_4 E_2 \begin{bmatrix} 1 \\ b^6 \end{bmatrix} \\ & \left. - \frac{1}{12} E_2 E_4 \begin{bmatrix} 1 \\ b \end{bmatrix} + \frac{1}{12} E_2 E_4 \begin{bmatrix} 1 \\ b^6 \end{bmatrix} \right). \end{aligned} \quad (\text{B6})$$

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- [73] In rigorous mathematical language, “ordinary” requires semi-simplicity in the L_0 action. To simplify the narration in this paper, however, by “ordinary” we include those having logarithmic feature, but with nonsingular unflavored characters.
- [74] In general ϕ_{2k} can have poles in the fundamental domain of the complex moduli τ . In the absence of poles, the equation is called “monic.”
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- [77] This property is discussed in [37]. Here we present it using our notation which is more convenient for the subsequent modularity analysis.
- [78] The exponent in the integral is also known as Weierstrass’ elliptic function,
- $$P_2(y, \tau) := -\sum_{n=1}^{\infty} \frac{1}{2n} E_{2n}(\tau) y^{2n}. \quad (\text{B8})$$
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