

Eigenstate thermalization in the two-site SYK and SYK chain models

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A study of Rényi entanglement entropy in the Sachdev-Ye-Kitaev (SYK) chain of Majorana fermions suggested that the model does not rapidly thermalize, despite being maximally chaotic. In this work, I examine the eigenstate thermalization hypothesis (ETH) for both the SYK chain and the two-site SYK models using exact diagonalization. I focus on two physically motivated few-body observables: the one-body particle number operator and the two-body hopping operator. These operators provide insight into local occupation and, in certain contexts, may reflect aspects of transport. Both operators have been employed in previous studies of the ETH in related SYK models. I show that single realizations of both models approximately satisfy ETH conditions for these operators, while ensemble averages strictly satisfy the ETH. Therefore, I conclude that the finite-size SYK chain and two-site SYK models can rapidly thermalize with respect to these observables through the ETH mechanism. This suggests that the subthermal behavior observed in previous studies of Rényi entanglement entropy does not manifest in finite-size systems and that these systems can thermalize rapidly via their light modes. It also indicates that the proposed gravitational dual may undergo rapid thermalization.

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I. INTRODUCTION

Chaotic quantum many-body systems are generally expected to thermalize rapidly, even when isolated from their environments. The eigenstate thermalization hypothesis (ETH) provides a theoretical foundation for this phenomenon [1–3]. The ETH asserts that a closed quantum system initialized in a pure state far from equilibrium can thermalize with respect to a set of observables, provided those observables satisfy certain conditions. In such cases, the expectation values of the observables evolve toward thermal values and remain close to them for most of the time, with only exponentially small fluctuations.

However, investigations of Rényi entanglement entropy in maximally chaotic systems have raised questions about the universality and speed of thermalization. In particular, evidence for slow thermalization has been reported in a one-dimensional generalization of the Sachdev-Ye-Kitaev (SYK) model, known as the SYK chain model, including its special case with two sites [4,5].

In this work, I study thermalization in the two-site SYK and SYK chain models from the perspective of the ETH. The SYK chain is a one-dimensional lattice model with quenched disorder, where each site hosts N Majorana

fermions with quartic random interactions [5]. Neighboring sites interact via 2-2 couplings between Majorana fermions. This model retains several key features of the original zero-dimensional SYK model [6,7], such as local criticality, extensive zero-temperature entropy, and maximal chaos. It also exhibits additional properties, including diffusive energy transport and butterfly velocity for the propagation of chaos in space [5]. From a condensed matter perspective, the SYK chain is a rare example of a solvable, strongly correlated, chaotic lattice model. It provides a valuable platform for studying thermalization, entanglement propagation, many-body localization, quantum phase transitions, and dissipative transport.

From the perspective of holographic duality, the SYK chain is conjectured to be dual to an incoherent black hole [4]. Since the isolated SYK chain evolves unitarily, understanding its thermalization dynamics may shed light on the formation and evaporation of such a black hole and contribute to ongoing discussions surrounding the black hole information paradox [8].¹

In this paper, I examine the validity of the ETH in the two-site SYK and SYK chain models by studying two conventional, generic, simple, few-body operators. I analyze both individual disorder realizations and ensemble-averaged data. Through exact diagonalization, I find that individual realizations exhibit what I term “approximate ETH behavior,” while ensemble-averaged results exhibit

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¹For progress on resolving the information paradox and related challenges, see [9–17].

ETH behavior more precisely. These findings suggest that the SYK chain and two-site SYK, and by extension its putative holographic duals, can indeed thermalize rapidly via the ETH mechanism. I discuss the apparent discrepancy with the conclusions of the Rényi entropy study [4] in the final section of the paper.

Previous studies have demonstrated the ETH in various zero-dimensional SYK-like models, including the original SYK model [18], the complex SYK model [19], and the supersymmetric SYK model [18]. These models feature all-to-all interactions without spatial structure. The influence of spatial locality and the absence of all-to-all couplings on ETH behavior has been noted as an open question [18]. The results presented in this work suggest that, within the analytically tractable regime considered, the presence of spatial locality and the absence of all-to-all interactions do not obstruct the emergence of ETH-like thermalization in the SYK chain and two-site SYK.

Furthermore, by analyzing the off-diagonal matrix elements of the operators, I demonstrate that the SYK chain and two-site SYK exhibit behavior consistent with random matrix theory (RMT) up to a characteristic energy scale, beyond which deviations from RMT emerge. Drawing an analogy with disordered conductors, I identify this scale as the Thouless energy E_T and show that it increases with the intersite coupling strength J_1 . In particular, stronger coupling between adjacent SYK sites extends the energy window over which the system displays RMT-like behavior.²

The remainder of the paper is organized as follows. In Sec. II, I review the ETH and the two-site SYK and SYK chain models. I introduce the concept of *approximate ETH behavior* and discuss the Thouless energy and diffusion constant of the SYK chain model. In Sec. III, I describe the computational setup, including the numerical construction of the two-site SYK and SYK chain models, the exploration of various coupling regimes, the construction of the few-body operators analyzed in this work, the justification for their selection, and the determination of their thermal values along with a representative entropy for ETH comparisons. In Sec. IV, I numerically verify the ETH for single realizations of the two-site SYK and SYK chain models and demonstrate that the ETH is approximately satisfied. In Sec. V, I verify the ETH for the ensemble-averaged models, showing that they satisfy the ETH precisely. I also identify the Thouless energy and examine the relation between the diffusion constant and the coupling strengths. Finally, in Sec. VI, I conclude with a summary of the results, a discussion of their implications and future directions, and a comparison with the study of Rényi entanglement entropy.

²For discussions of RMT behavior and the Thouless energy in the SYK and supersymmetric SYK models, see [20,21].

II. OVERVIEW

A. Eigenstate thermalization hypothesis

The ETH explains how isolated quantum systems in pure states can exhibit thermal behavior despite evolving unitarily [1–3]. In this subsection, I review the basic structure of the hypothesis.

Consider an isolated quantum system with finitely many degrees of freedom, $\infty > \mathcal{N} \gg 1$, and a nondegenerate, discrete energy spectrum $\{E_n\}$. Let H denote the Hamiltonian of the system, with energy eigenstates $|n\rangle$. For any pure state $|\psi\rangle$ of the system, the quantum average (i.e., expectation value) of energy is given by

$$\langle H \rangle_Q = \langle \psi | H | \psi \rangle, \quad (1)$$

and the quantum energy uncertainty is defined as

$$\Delta H_Q = [\langle \psi | H^2 | \psi \rangle - \langle \psi | H | \psi \rangle^2]^{1/2}. \quad (2)$$

The ETH is applicable to pure states for which the quantum energy uncertainty is much smaller than the average energy:

$$\Delta H_Q \ll \langle H \rangle_Q. \quad (3)$$

It is sufficient for the energy uncertainty to scale as $\Delta H_Q \sim \langle H \rangle_Q / \mathcal{N}^\nu$ for some $\nu > 0$.

For later use, we associate to each pure state $|\psi\rangle$ an artificial temperature $1/\beta_\psi$ such that the expectation value of the Hamiltonian in pure state $|\psi\rangle$ and in thermal state $e^{-\beta_\psi H}$ are equal:

$$\langle H \rangle_Q = \langle \psi | H | \psi \rangle = \frac{\text{Tr} e^{-\beta_\psi H} H}{\text{Tr} e^{-\beta_\psi H}} = \langle H \rangle_{\text{th}}. \quad (4)$$

The equation above implicitly defines the parameter β_ψ , such that the average energy of the pure state $|\psi\rangle$ matches that of the corresponding thermal state $e^{-\beta_\psi H}$.

Next, consider a finite collection of generic observables $\{A^\alpha\}_{\alpha=1}^\Gamma$, where the number of observables Γ is much smaller than the dimension of the Hilbert space, i.e., $\Gamma \ll \dim(\mathcal{H})$. These observables are considered generic in the sense that they correspond either to physically measurable quantities or to nonextensive operators acting on only a few degrees of freedom—for instance, the single-site number operator or two-site hopping operator.

For any state $|\psi\rangle$, the set $\{A^\alpha\}_{\alpha=1}^\Gamma$ is chosen such that the quantum averages of the observables

$$\langle A^\alpha \rangle_Q = \langle \psi | A^\alpha | \psi \rangle \quad (5)$$

are far from their thermal averages at temperature $1/\beta_\psi$:

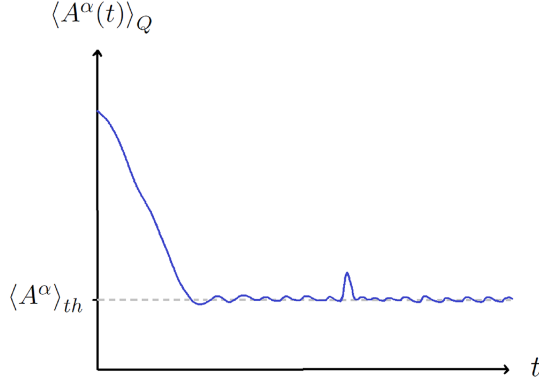


FIG. 1. Eigenstate thermalization hypothesis. If the ETH holds for a set of observables $\{A^\alpha\}_{\alpha=1}^\Gamma$, then those observables thermalize under unitary evolution. They are chosen such that their quantum expectation values in the initial pure state lie far from equilibrium. The ETH asserts that these expectation values $\langle A^\alpha(t) \rangle_Q$ relax to their thermal values $\langle A^\alpha \rangle_{\text{th}}$ and remain close to them for most of the time. Fluctuations around equilibrium are exponentially suppressed, though rare large deviations may occur.

$$\langle A^\alpha \rangle_{\text{th}} = \frac{\text{Tr} e^{-\beta_\psi H} A^\alpha}{\text{Tr} e^{-\beta_\psi H}}. \quad (6)$$

Thus, as the system starts from state $|\psi\rangle$, the observables start from nonequilibrium initial expectation values $\langle A^\alpha(0) \rangle_Q \neq \langle A^\alpha \rangle_{\text{th}}$ (see Fig. 1).

Expanding the pure state in terms of the energy eigenkets,

$$|\psi\rangle = \sum_n c_n |n\rangle, \quad (7)$$

the time-evolved state becomes

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |n\rangle. \quad (8)$$

Accordingly, the expectation values of the observables evolve as

$$\langle A^\alpha(t) \rangle_Q = \langle \psi(t) | A^\alpha | \psi(t) \rangle = \sum_{m,n} c_m^* c_n A_{mn}^\alpha e^{i(E_m - E_n)t/\hbar}, \quad (9)$$

where

$$A_{mn}^\alpha = \langle m | A^\alpha | n \rangle \quad (10)$$

denotes the matrix element of the observable A^α in the energy eigenbasis.

Having established the framework, we now state the ETH.

The ETH asserts that, under the assumptions and setup above, if the observables $\{A^\alpha\}_{\alpha=1}^\Gamma$ satisfy the following two

conditions, their time-evolved expectation values $\langle A^\alpha(t) \rangle_Q$ will relax to thermal equilibrium values $\langle A^\alpha \rangle_{\text{th}}$ and remain close to them for most of the time (Fig. 1).

The two conditions of ETH are as follows: For all n, m , and $1 \leq \alpha \leq \Gamma$, ETH requires

$$A_{nn}^\alpha = \mathcal{A}^\alpha(E_n), \quad (11)$$

$$|A_{nm}^\alpha| \sim e^{-S/2}, \quad (12)$$

where $\mathcal{A}^\alpha$ is a smooth function of energy and S is the entropy of the system, which scales extensively with the number of degrees of freedom, $S \sim \mathcal{N}$. In words, the ETH requires that the diagonal matrix elements A_{nn}^α follow a smooth energy-dependent curve and the off-diagonal elements be exponentially suppressed.

When these conditions hold, the ETH implies thermalization with respect to the observables $\{A^\alpha\}_{\alpha=1}^\Gamma$: Each observable relaxes to its thermal value and fluctuates only slightly around it for most of the time. The fluctuations are exponentially suppressed, though rare large deviations can occur.³

To demonstrate thermalization, ETH shows that the infinite time averages of the observables

$$\overline{A^\alpha} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle A^\alpha(t) \rangle_Q dt \quad (14)$$

are approximately equal to the smooth function $\mathcal{A}^\alpha(E_n)$, given that the diagonal matrix elements satisfy Eq. (11). This leads to the result that the infinite time average values are close to the thermal and microcanonical average values of the observables:

$$\overline{A^\alpha} \simeq \mathcal{A}^\alpha(E_n) = A_{nn}^\alpha \simeq \langle A^\alpha \rangle_{\text{th}} \simeq \langle A^\alpha \rangle_{\text{micro}}. \quad (15)$$

The approximations are valid up to $\mathcal{O}(1/S)$ and $\mathcal{O}(\Delta H^2)$ [2,3].

³In some literature, the conditions of ETH are expressed in terms of the following equation:

$$A_{mn}^\alpha = \mathcal{A}^\alpha(E_n) \delta_{mn} + e^{-S/2} f^\alpha(\bar{E}, \omega) R_{mn}, \quad (13)$$

where $\bar{E} = (E_n + E_m)/2$, $\omega = E_m - E_n$, f^α is a smooth function, and R_{mn} is a Gaussian random variable with zero mean and unit variance. The key difference between conditions (11) and (12), as originally introduced by the founders of the ETH [1–3], and the form in Eq. (13) is the inclusion of the random variable R_{mn} in the off-diagonal matrix elements. However, it is not generally necessary for the off-diagonal elements to be randomly distributed for the ETH to hold and for the closed system to reach thermal equilibrium. While quantized chaotic systems satisfy Eq. (13), the specific distribution (e.g., Gaussian) of the off-diagonal matrix elements is not required for thermalization. Therefore, we do not consider the distribution of off-diagonal elements in this work.

Furthermore, the ETH shows that the infinite time average of the fluctuations around the thermal values are suppressed by the off-diagonal matrix elements as follows:

$$|\langle A^\alpha(t) \rangle_Q - \overline{A^\alpha}|^2 = \sum_{m \neq n} |c_n|^2 |c_m|^2 |A_{nm}^\alpha|^2. \quad (16)$$

If the off-diagonal matrix elements satisfy Eq. (12), the right-hand side of the above equation will be exponentially small:

$$|\langle A^\alpha(t) \rangle_Q - \overline{A^\alpha}|^2 \sim e^{-S}. \quad (17)$$

This occurs because $|c_n|^2$ and $|c_m|^2$ sum to unity.

Equation (15) demonstrates that the infinite time average values of the observables coincide with their thermal averages, while Eq. (17) guarantees that fluctuations around these thermal average values are exponentially small for most of the time.

This is the ETH in its most rigorous form. However, in certain cases, not all the conditions of the ETH are simultaneously satisfied. Nevertheless, the general conclusions of the ETH regarding thermalization of observables still hold. For instance, in systems such as the single realization of the SYK chain model, as discussed in Sec. IV, Eq. (12) may not be satisfied, and the off-diagonal matrix elements are only subexponentially small rather than exponentially small. Nonetheless, since their maximum absolute values are still small, one can use Eq. (16) to obtain

$$|\langle A^\alpha(t) \rangle_Q - \overline{A^\alpha}|^2 \leq \max_{n \neq m} |A_{nm}^\alpha|^2 \quad (18)$$

and conclude that the fluctuations remain small, though not exponentially small.

In such cases, we say that the ETH is *approximately* satisfied. As long as the off-diagonal matrix elements are one or more orders of magnitude smaller than the diagonal elements, significant deviations from the thermal average values do not occur during typical times, allowing us to assume that thermalization is achieved in the system.

Another point concerning the conditions of the ETH is inequality (3), which requires the quantum fluctuation of energy to be much smaller than the quantum average energy. This inequality is used in the proof of the ETH to demonstrate Eq. (15) for the diagonal matrix elements. If one can directly check the satisfaction of Eq. (15), the validity of inequality (3) does not need to be verified. This is what I do in Secs. IV and V. I directly check Eq. (15) both for single realizations of the SYK chain and two-site SYK models and for the ensemble-averaged theories and present the results there.

B. SYK chain model

The SYK chain model [5] is a generalization of the original SYK model [6,7]. In this subsection, I first provide

a brief overview of the original SYK model and then describe the SYK chain model.

The original SYK model is an ensemble-averaged theory [6,7]. Each single realization of the ensemble consists of N Majorana fermions with quartic random all-to-all interactions. The Hamiltonian for each realization is given by

$$H = \sum_{1 \leq i < j < k < l \leq N} J_{ijkl} \chi_i \chi_j \chi_k \chi_l, \quad (19)$$

where the Majorana fermion operators satisfy the Clifford algebra

$$\{\chi_i, \chi_j\} = \delta_{ij} \quad (20)$$

and the couplings $\{J_{ijkl}\}$ are independent random real numbers drawn from a Gaussian distribution with zero mean

$$\overline{J_{ijkl}} = 0 \quad (21)$$

and variance

$$\overline{J_{ijkl}^2} = \frac{3!J^2}{N^3}. \quad (22)$$

Here, J sets the average strength of the couplings.

The SYK model is solvable in the large- N limit. It exhibits many interesting properties at strong coupling $N \gg \beta J \gg 1$, such as extensive zero-temperature entropy, local criticality (with power law correlations in time), and maximal chaos. In this limit, the averaged theory also exhibits holographic behavior and demonstrates connections to nearly AdS_2 gravity. However, the model is $(0+1)$ -dimensional and lacks spatial locality. Consequently, some expected properties of higher-dimensional systems, such as diffusion and the butterfly effect, cannot be realized in the original model.

A higher-dimensional lattice generalization of the SYK model with spatial locality was proposed by Gu, Qi, and Stanford [5]. In $(1+1)$ dimensions, the model consists of a chain of SYK sites (also known as links), referred to as the SYK chain model, as illustrated in Fig. 2.

In this paper, I focus on the SYK chain model and its special case with two sites. Like the original SYK model, the SYK chain model is an ensemble-averaged theory.

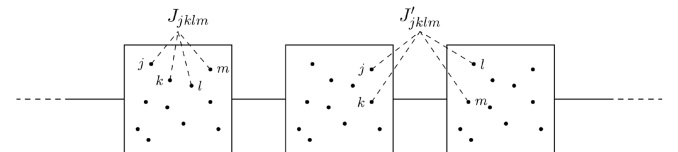


FIG. 2. SYK chain model. The model consists of M sites with periodic boundary conditions. Each site contains N Majorana fermions with SYK interactions. Nearest-neighboring sites interact via four-fermion interactions with two fermions from each site. (Graphic adapted from [5]).

The Hamiltonian for a single realization of the model is given by

$$H = \sum_{x=1}^M \sum_{1 \leq i < j < k < l \leq N} J_{ijkl,x} \chi_{i,x} \chi_{j,x} \chi_{k,x} \chi_{l,x} + \sum_{x=1}^M \sum_{\substack{1 \leq i < j \leq N \\ 1 \leq k < l \leq N}} J'_{ijkl,x} \chi_{i,x} \chi_{j,x} \chi_{k,x+1} \chi_{l,x+1}, \quad (23)$$

where $\{J_{ijkl,x}\}$ and $\{J'_{ijkl,x}\}$ are independent zero-mean Gaussian random couplings with variances

$$\overline{J_{ijkl,x}^2} = \frac{3!J_0^2}{N^3}, \quad \overline{J'_{ijkl,x}^2} = \frac{J_1^2}{N^3}. \quad (24)$$

Each site of the lattice, labeled by $x = 1, \dots, M$, contains N Majorana fermions with on-site SYK interactions. Nearest-neighbor SYK sites also interact via 2-2 fermion interactions, as shown in the second summation in the Hamiltonian of Eq. (23). The Majorana fermions satisfy the anticommutation relation

$$\{\chi_{i,x}, \chi_{j,y}\} = \delta_{xy} \delta_{ij} \quad (25)$$

and periodic boundary conditions: $\chi_{i,0} \equiv \chi_{i,M}$. The model's local properties can be described in terms of an effective coupling constant

$$J = \sqrt{J_0^2 + J_1^2}. \quad (26)$$

The SYK chain model is solvable in the large- N limit. At strong coupling ($N \gg \beta J \gg 1$), the model retains the interesting properties of the original SYK model, such as maximal chaos, extensive zero-temperature entropy, and local criticality. In addition, due to its spatial locality, the model exhibits new features, such as energy diffusion and the butterfly effect.

The model at strong coupling describes a strongly correlated diffusive metal. Also, the relation $D = v_B^2/2\pi T$ between its temperature-independent diffusion constant D and its butterfly speed v_B is consistent with proposals of incoherent metals. The SYK chain model, at strong coupling, provides a rare example of strongly interacting chaotic lattice models that are analytically tractable. As noted in Ref. [5], “it provides an interesting platform for studying various properties of strongly correlated systems, such as thermalization, entanglement propagation, dissipative transport, etc.”

Reference [4] studied the spread of Rényi entropy in the SYK chain model and discussed its thermalization properties. However, the ETH for the model has not yet been directly investigated. In the following sections, I investigate the eigenstate thermalization of the SYK chain model.

The diffusion constant D of the model is a function of its couplings strengths:

$$D = \frac{2\pi J_1^2}{3\sqrt{2}J\alpha_K}, \quad (27)$$

where $\alpha_K \approx 2.852$ [5]. The diffusion of energy in the system, indicated by the constant D , arises due to the dynamics of the collective mode of the time reparametrization field, which is identified as light degrees of freedom in [4]. This collective mode is the most important low-energy, long-wavelength mode in the system. Gu, Qi, and Stanford [5] find that single Majorana fermions do not propagate between sites; only collective modes made up of pairs of fermions exhibit spatial dynamics.

In study of off-diagonal matrix elements of observables in the next section, I identify the Thouless energy

$$E_T = \frac{\hbar D}{L^2}, \quad (28)$$

where L is the size of the system, and investigate whether the Thouless energy agrees with the expression in Eq. (27).

III. COMPUTATIONAL SETUP

I construct the SYK chain model with M sites, each containing N Majorana fermions. Thus, the total number of Majorana fermions in the system is NM .

Reference [4] could explicitly evaluate Rényi entanglement entropy and infer slow thermalization behavior, in the case of two-site SYK with $M = 2$. They resorted to the weak intersite coupling limit, defined as

$$\frac{J_1^2}{J^2} \ll \frac{1}{\beta J} \ll 1, \quad (29)$$

to obtain many of their results.

Here, I study the two-site SYK model with $M = 2$ in the weak intersite coupling limit. Additionally, I explore the SYK chain model with $M = 3$ sites in the weak coupling regime and with $M = 4$ sites in three distinct coupling regimes.

Throughout this work, I set $J = \sqrt{5}$. For $M = 2, 3$, I use

$$J_0 = \sqrt{4.99}, \quad J_1 = 0.1, \quad (30)$$

which satisfies the weak coupling condition in Eq. (29).

For $M = 4$, I examine the following three parameter sets:

- (a) $J_0 = 2, J_1 = 1$ (stronger intrasite coupling);
- (b) $J_0 = J_1 = \sqrt{5/2}$ (equal inter- and intrasite coupling); and
- (c) $J_0 = 1, J_1 = 2$ (stronger intersite coupling).

Note that even in case (b), where inter- and intrasite couplings are equal, the model differs from the original SYK model, because the couplings in the SYK chain are

not all to all. The SYK chain includes both on-site interactions (parametrized by J_0) and nearest-neighbor intersite couplings (parametrized by J_1). Therefore, one should not expect this model to directly reproduce ETH results from the original SYK model.

To construct the matrix representation of the Hamiltonian, I first relabel the Majorana fermions by a single index:

$$\chi_{i,x} \rightarrow \chi_\alpha, \quad (31)$$

with

$$\alpha = i + (x-1)N, \quad (32)$$

where $1 \leq i \leq N$ and $1 \leq x \leq M$, so that α ranges from 1 to NM . The χ_α obey the Clifford algebra:

$$\{\chi_\alpha, \chi_\beta\} = \delta_{\alpha\beta}. \quad (33)$$

Because Majorana fermions are Hermitian ($\chi_\alpha^\dagger = \chi_\alpha$), I define complex fermion operators as

$$c_\alpha = \chi_{2\alpha} - i\chi_{2\alpha-1}, \quad (34)$$

$$c_\alpha^\dagger = \chi_{2\alpha} + i\chi_{2\alpha-1}, \quad \alpha = 1, \dots, NM/2, \quad (35)$$

which satisfy the canonical fermionic anticommutation relations:

$$\{c_\alpha, c_\beta\} = \{c_\alpha^\dagger, c_\beta^\dagger\} = 0, \quad \{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}. \quad (36)$$

I define a vacuum state $|0\rangle$ such that

$$c_\alpha|0\rangle = 0 \quad (37)$$

and construct the Hilbert space basis using

$$(c_1^\dagger)^{n_1} \dots (c_L^\dagger)^{n_L} |0\rangle, \quad n_\alpha = 0, 1, \quad (38)$$

where $L = NM/2$. This yields a $2^{NM/2}$ -dimensional Hilbert space.

The matrix representation of the χ_α operators can then be constructed recursively [22]:

$$\chi_\beta^{(K)} = \chi_\beta^{(K-1)} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta = 1, \dots, 2K-2, \quad (39)$$

$$\chi_{2K-1}^{(K)} = I_{2^{K-1}} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (40)$$

$$\chi_{2K}^{(K)} = I_{2^{K-1}} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (41)$$

where I_d denotes the $d \times d$ identity matrix and K runs from 1 to $NM/2$. I initialize the recursion with

$$\chi_1^{(1)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y, \quad \chi_2^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X. \quad (42)$$

This generates the full set of $2^{NM/2} \times 2^{NM/2}$ matrices $\{\chi_\alpha\}$.

Once the matrices $\{\chi_\alpha\}$ are constructed, one can recover the original indexing:

$$i = \alpha \bmod N, \quad x = \frac{\alpha - i}{N} + 1. \quad (43)$$

This allows us to construct the Hamiltonian of the SYK chain model as given in Eq. (23).

What we did above was to map every two Majorana fermions (labeled by Latin indices) onto one complex fermion (labeled by Greek indices). In the language of complex fermions, we can say that there are M sites, and on each site, there can be $N/2$ complex fermions (where N is even here and throughout the paper). Therefore, the indices of complex fermions $\alpha = 1, \dots, N/2$ belong to site $x = 1$; the indices $\alpha = N/2 + 1, \dots, N$ belong to site 2; the indices $\alpha = N + 1, \dots, 3N/2$ belong to site 3; and so on. Finally, the indices of complex fermions $\alpha = \frac{N(M-1)}{2} + 1, \dots, \frac{NM}{2}$ belong to site $x = M$. It is more convenient to express operators in terms of complex fermion operators, as I do below.⁴

In the following sections, I examine the ETH for two types of nonextensive operators: the particle number operator and the hopping operator, which act on one and two fermionic modes, respectively. These operators are natural probes of thermalization in fermionic systems, as they can encode important information about local occupation and, in some cases, transport properties. They have also been employed in ETH studies of the original SYK model [18], the complex SYK model [19], and the supersymmetric SYK model [18]. Moreover, due to the all-to-all random interactions and permutation symmetry among fermion modes within each SYK site, as well as translational symmetry in the SYK chain, the behavior of ETH indicators is expected to be largely independent of the specific choice of fermion index. Consequently, analyzing representative instances of these operators—such as a single number operator $\hat{n}_j$ and a single hopping term $\hat{h}_{ij}$ —captures the essential features of thermalization dynamics in these models.

The particle number operator is constructed from the annihilation and creation operators as

$$\hat{n}_1 = c_1^\dagger c_1, \quad (44)$$

⁴In another picture, one can map the complex fermions to qubits and regard fermion operators as gate operators that act on qubits.

where I use Greek indices to label complex fermion operators c_α and $c_\beta^\dagger$. Physically, this corresponds to mapping the first two Majorana fermions on the first SYK site, $\chi_{1,1}$ and $\chi_{2,1}$, onto a single complex fermion. The number operator $\hat{n}_1$ then counts the occupation number of this fermion, which can be either 0 or 1. As defined, $\hat{n}_1$ is a local operator, influenced only by the first two Majorana fermions of the first site.

In contrast to the number operator, the hopping operator can be entirely nonlocal, acting between two distinct SYK sites. Using complex fermion indices, I define the hopping operator between sites 1 and 2 as

$$\hat{h}_{12} = c_{\frac{N}{2}+1}^\dagger c_1 + c_1^\dagger c_{\frac{N}{2}+1} \quad (45)$$

and the operator between sites 1 and 3 as

$$\hat{h}_{13} = c_{N+1}^\dagger c_1 + c_1^\dagger c_{N+1}. \quad (46)$$

In the former case, the index $\alpha = 1$ corresponds to a complex fermion on site $x = 1$, while $\alpha = \frac{N}{2} + 1$ corresponds to one on site $x = 2$. In the latter, $\alpha = N + 1$ corresponds to a complex fermion on site $x = 3$.

Expressed in terms of Majorana fermions, $\hat{h}_{12}$ involves the first two Majorana modes of sites 1 and 2, namely, $\chi_{1,1}, \chi_{2,1}$ and $\chi_{1,2}, \chi_{2,2}$, respectively. Similarly, $\hat{h}_{13}$ involves the first two Majorana modes of sites 1 and 3: $\chi_{1,1}, \chi_{2,1}$ and $\chi_{1,3}, \chi_{2,3}$.

Despite their nonlocality, the hopping operators remain nonextensive, as they act on only a few Majorana modes.

In the following sections, I present results for the number operator $\hat{n}_1$ and the hopping operators defined above. I examine $\hat{h}_{12}$ in the context of the two-site SYK model and $\hat{h}_{13}$ for SYK chains with more than two sites.

The microcanonical average of $\hat{n}_1$ is

$$\langle \hat{n}_1 \rangle_{\text{micro}} = \frac{1}{2}, \quad (47)$$

and, according to the ETH—as expressed in Eq. (15)—the diagonal matrix elements of $\hat{n}_1$ in the energy eigenbasis are expected to be approximately equal to this value.

In contrast, the diagonal matrix elements of $\hat{h}_{12}$ and $\hat{h}_{13}$ vanish exactly by symmetry:

$$\langle n | \hat{h}_{12} | n \rangle = \langle n | \hat{h}_{13} | n \rangle = 0. \quad (48)$$

This follows directly from the fermionic structure of the model and the Hermiticity of the operators. Consequently, the microcanonical averages also vanish:

$$\langle \hat{h}_{12} \rangle_{\text{micro}} = \langle \hat{h}_{13} \rangle_{\text{micro}} = 0. \quad (49)$$

In accordance with the ETH, the diagonal matrix elements of $\hat{h}_{12}$ and $\hat{h}_{13}$ in the energy eigenbasis are, therefore, expected to be approximately zero.

In the case of off-diagonal matrix elements, the strict version of the ETH, as given in Eq. (12), predicts that all such elements should be of the order of $e^{-S/2}$, where S is the thermodynamic entropy. In contrast, the weaker or approximate formulation of the ETH, discussed in Sec. II A, requires only that the maximum absolute value of these off-diagonal elements be sufficiently small.

The entropy of a quantum system depends on its state: It is zero for pure states and maximal for the maximally mixed state, where the maximum value is $\ln(\dim(\mathcal{H}))$. For all other states, including thermal states, the entropy lies between these extremes. For the purpose of comparison in our numerical analysis, I use the mean value

$$\bar{S} = \frac{1}{2} \ln(\dim(\mathcal{H})) \quad (50)$$

to assess how small the off-diagonal matrix elements are relative to ETH expectations. For the two-site SYK and SYK chain models, the Hilbert space dimension is $\dim(\mathcal{H}) = 2^{NM/2}$, which gives the mean entropy as

$$\bar{S} = NM \ln 2/2. \quad (51)$$

IV. THERMALIZATION OF SINGLE REALIZATIONS OF TWO-SITE SYK AND SYK CHAIN MODELS

The two-site SYK and SYK chain models are formulated as ensemble-averaged theories. However, the ETH applies to individual quantum systems, not ensemble averages. Therefore, it is crucial to verify whether the ETH holds for single realizations of these models before considering the ensemble-averaged versions.

If single realizations of the disorder do not satisfy the ETH, drawing any conclusions from the satisfaction of the ETH in the disorder-averaged theories becomes problematic. This is because the proof of thermalization for generic observables within the ETH framework relies on the assumption that the system of interest is an individual system with a fixed Hamiltonian.

In this section, I numerically examine the validity of the eigenstate thermalization hypothesis for single realizations of the two-site SYK and SYK chain models.

The numerical results indicate that, for the operators examined in this study, the ETH is *approximately* satisfied in individual realizations of the disorder (see Sec. II A for the definition of *approximate satisfaction of the ETH*). Consequently, the analysis of the ETH in the ensemble-averaged formulations of the two-site SYK and SYK chain models, as pursued in the subsequent section, remains well justified.

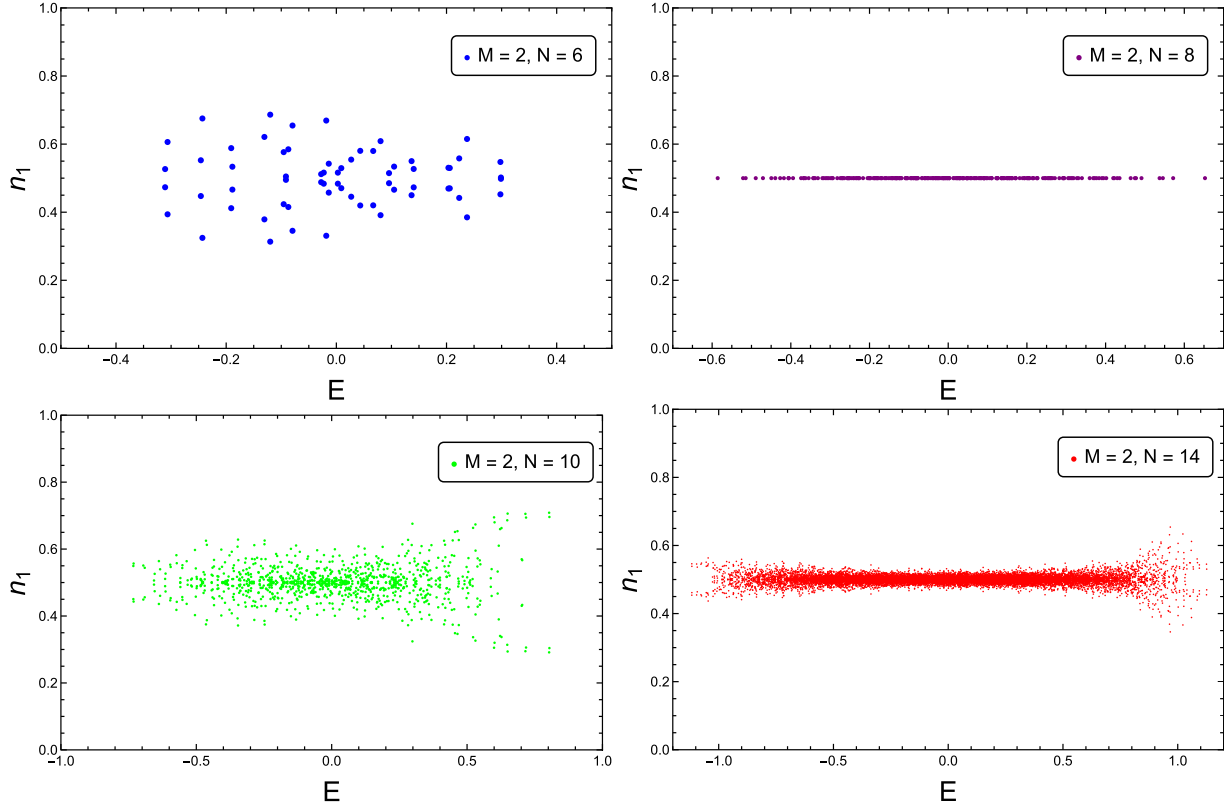


FIG. 3. Absolute value of the diagonal matrix elements of the particle number operator n_1 plotted against the energy eigenvalues E for a single realization of the two-site SYK model in the weak-link limit. Results are shown for $MN = 12, 16, 20, 28$ Majorana fermions with $M = 2$. In all cases, the diagonal elements fluctuate around $1/2$ except for the case $N = 8$, where they are exactly $1/2$ due to an additional symmetry.

A. Thermalization in single realizations of the two-site SYK model

In the weak coupling regime, introduced in Sec. III, I examined single realizations of the two-site SYK model with $M = 2$ and $N = 6, 8, 10, 14$, using coupling strengths $J_0 = \sqrt{4.99}$ and $J_1 = 0.1$ for intersite and intra-site interactions, respectively. For each case, I computed the absolute values of matrix elements of the particle number operator $\hat{n}_1$ and the hopping operator $\hat{h}_{12}$ in the energy eigenbasis, i.e., $|\langle m|\hat{n}_1|n\rangle|$ and $|\langle m|\hat{h}_{12}|n\rangle|$.

For the number operator, the diagonal matrix elements were found to fluctuate around the microcanonical average value of $1/2$, consistent with Eq. (47), while the off-diagonal elements remained significantly smaller than this thermal value.

For the hopping operator, the diagonal matrix elements vanished exactly due to symmetry, as noted in Eq. (48), and the off-diagonal elements were small compared to unity.

Figure 3 displays the diagonal matrix elements of the number operator as a function of energy eigenvalues. With the exception of $N = 8$, these elements fluctuate about $1/2$, and their deviation from this thermal value diminishes as N increases.

For $N = 8$, the diagonal elements are exactly equal to $1/2$, a feature attributable to symmetry, as discussed in the context of the original SYK model (see Ref. [18]).

Figure 4 presents matrix plots of $|\langle m|\hat{n}_1|n\rangle|$, clearly showing dark blue diagonal elements (corresponding to values near 0.5) and light cyan off-diagonal elements (corresponding to values close to zero). In many cases, the off-diagonal elements vanish to machine precision, which is consistent with the block-diagonal structure of the Hamiltonian, similar to the original SYK model. Nevertheless, most nonzero off-diagonal elements are not as small as $e^{-\tilde{S}/2}$, indicating a failure of the strict ETH.

Even so, the average magnitude of the off-diagonal elements decreases with increasing N , as does the maximum observed value. These trends are summarized in Table I, where they are compared with the ETH prediction $e^{-\tilde{S}/2} = e^{-NM \ln 2/4}$.

As the data in Table I illustrate, the strict formulation of the ETH is not fulfilled, as $\max_{n \neq m} |A_{nm}| \gg e^{-\tilde{S}/2}$ even for large N . However, the approximate form of the ETH is satisfied: The off-diagonal matrix elements remain small compared to the thermal value, and their magnitude decreases with increasing N .

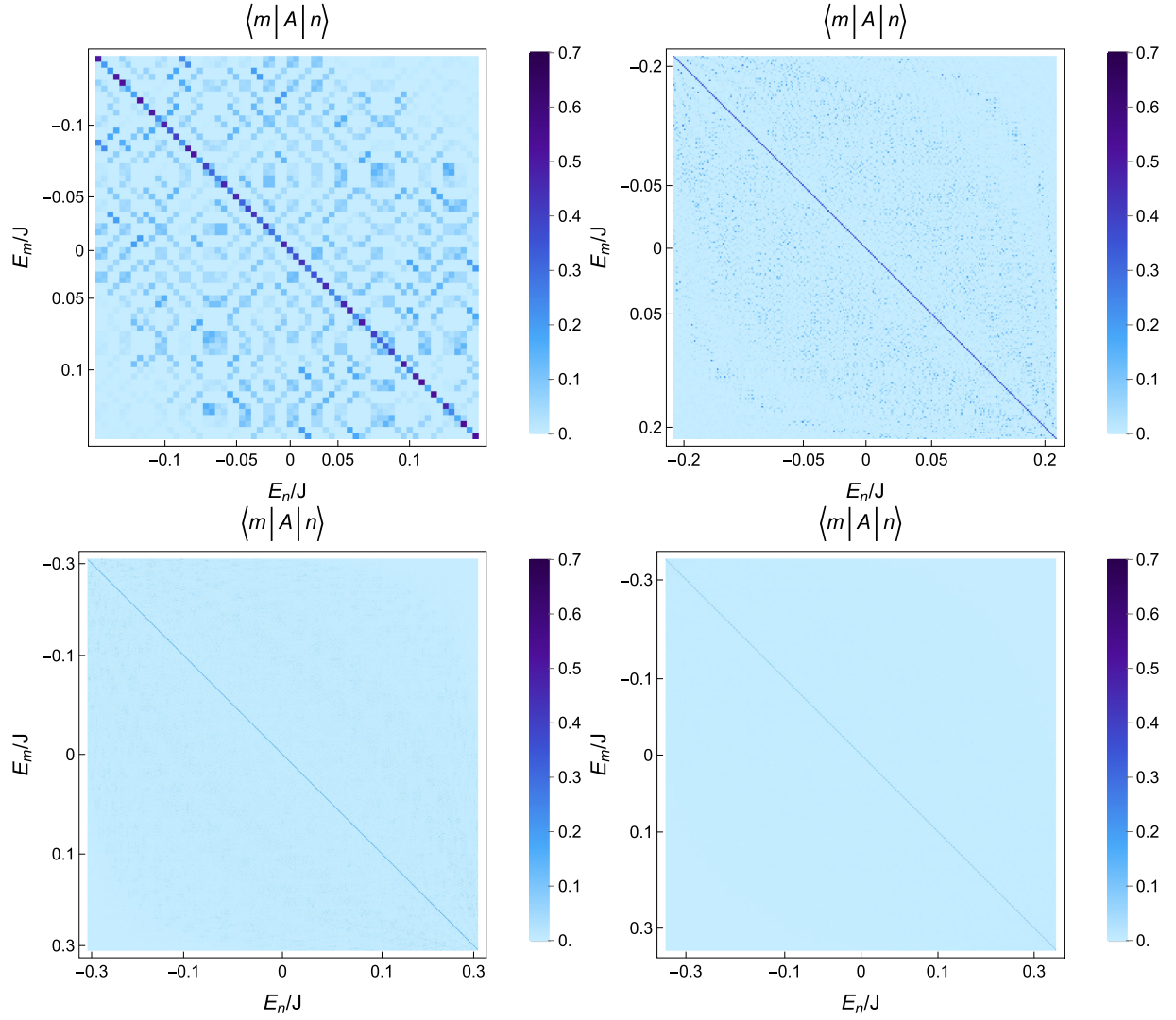


FIG. 4. Matrix plots of $|A_{mn}| = |\langle m|A|n\rangle|$ for the particle number operator $A = \hat{n}_1$ in the weak intersite coupling regime, $J_0 = \sqrt{4.99}$, $J_1 = 0.1$, of the two-site SYK model, $M = 2$. I choose single realizations of the two-site SYK model with $M = 2$ sites and $N = 6, 8, 10, 14$ Majorana fermions on each site. The horizontal and vertical axes show the energy eigenvalues, E_m/J , E_n/J , corresponding to the matrix elements $|O_{mn}|$. All four plots exhibit agreement with the approximate formulation of the ETH. The diagonal elements of the number operator in the energy eigenbasis fluctuate around the microcanonical ensemble average value of $1/2$, while the off-diagonal elements are subexponentially small. This suggests that the ETH is not satisfied in its strict form but holds approximately. Top left panel, $N = 6$; top right panel, $N = 8$; bottom left panel, $N = 10$; bottom right panel, $N = 14$; the data for the latter are downsampled to 2048×2048 for the purpose of presentation.

TABLE I. Maximum and mean values of the off-diagonal matrix elements of the particle number operator $A = \hat{n}_1$ in the energy eigenbasis for typical single realizations of the two-site SYK model ($M = 2$), with different numbers of Majorana fermions N per site.

M	N	$\max_{n \neq m} A_{nm} $	$\text{mean}_{n \neq m} A_{nm} $	$e^{-\tilde{S}/2}$
2	6	0.40	0.023	0.125
2	8	0.37	0.010	0.063
2	10	0.24	0.007	0.031
2	14	0.15	0.002	0.008

TABLE II. Maximum and mean values of the off-diagonal matrix elements of the hopping operator $O = \hat{h}_{12}$ in the energy eigenbasis for typical single realizations of the two-site SYK model ($M = 2$), with different numbers of Majorana fermions N per site.

M	N	$\max_{n \neq m} O_{nm} $	$\text{mean}_{n \neq m} O_{nm} $	$e^{-\tilde{S}/2}$
2	6	0.37	0.042	0.125
2	8	0.33	0.017	0.063
2	10	0.22	0.012	0.031
2	14	0.10	0.003	0.008

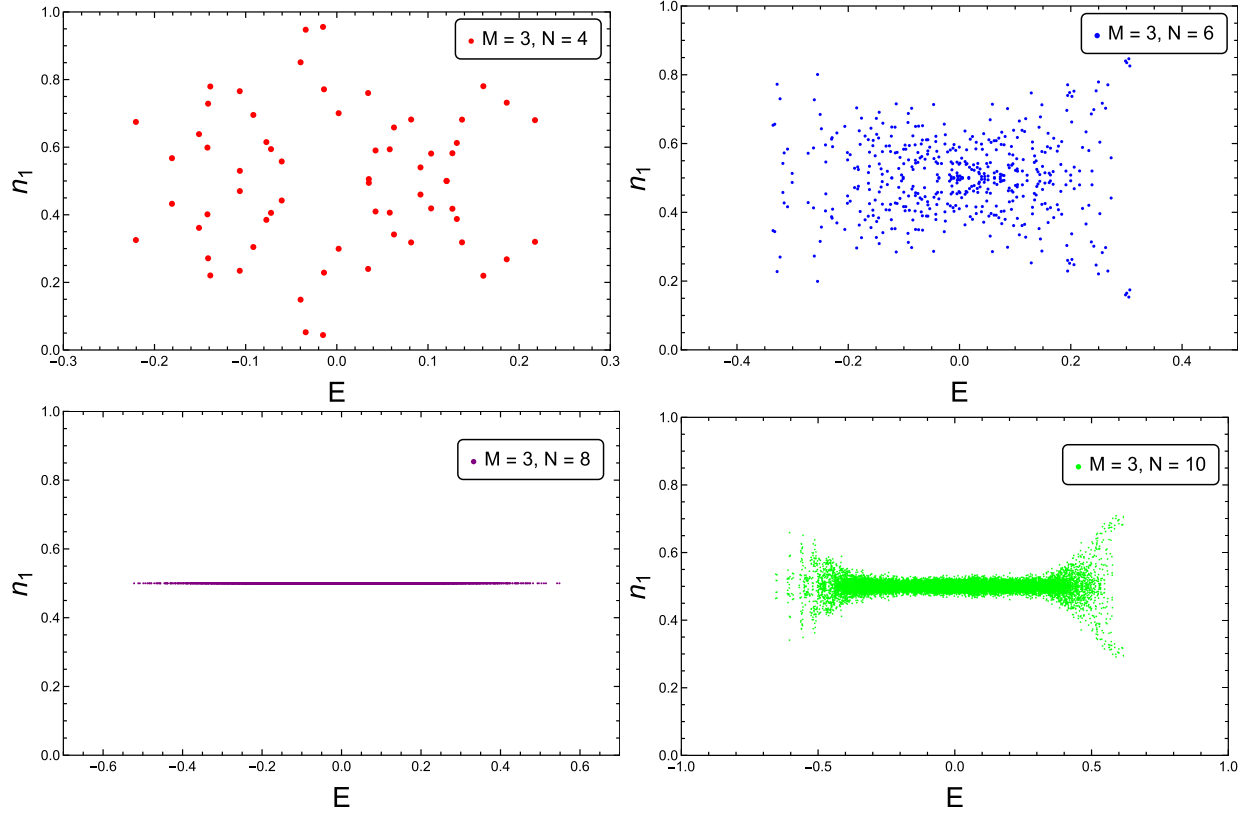


FIG. 5. The absolute value of the on-diagonal elements of the particle number operator n_1 as a function of energy eigenvalues E for single realizations of SYK chain model with three sites in the weak-link regime. We considered $MN = 12, 18, 24, 30$ Majorana fermions, with $M = 3$, and observed the diagonal terms fluctuating around $1/2$.

Moreover, the mean values of the off-diagonal elements approach or fall below the ETH expected value, $\text{mean}_{n \neq m}[|A_{nm}|] \lesssim \exp(-\tilde{S}/2)$, suggesting that the ETH is satisfied in the ensemble-averaged framework.

A similar analysis was performed for the hopping operator $\mathcal{O} = \hat{h}_{12}$. Table II reports the maximum and mean values of its off-diagonal matrix elements. These trends mirror those observed for the number operator, supporting the conclusion that the ETH holds approximately in single disorder realizations for both local and nonlocal observables.

B. Thermalization in single realizations of the three-site SYK chain model

For the three-site SYK chain ($M = 3$), I studied single realizations with $N = 4, 6, 8, 10$ Majorana fermions per site, again focusing on the weak coupling regime ($J_0 = \sqrt{4.99}, J_1 = 0.1$). In this case, I computed matrix elements of both the particle number operator $\hat{n}_1$ and the hopping operator $\hat{h}_{13}$ in the energy eigenbasis.

For the number operator, the diagonal matrix elements typically fluctuate around $1/2$ for $N = 4, 6, 10$. For $N = 8$, all diagonal elements are exactly equal to $1/2$, reflecting

the same symmetry discussed previously. These results are presented in Fig. 5.

Figure 6 provides matrix plots of $|\langle m | \hat{n}_1 | n \rangle|$. As in the two-site case, the diagonal elements exhibit values near 0.5 , while most off-diagonal elements are close to zero or vanish to machine precision. This again reflects the block-diagonal structure of the Hamiltonian. However, the non-zero off-diagonal matrix elements generally exceed the ETH prediction $e^{-\tilde{S}/2}$.

Table III presents the mean and maximum values of the off-diagonal elements for various values of N , together with the corresponding ETH prediction.

The results in Table III confirm that the strict ETH is not satisfied, as $\max_{n \neq m} |A_{nm}| \gg e^{-\tilde{S}/2}$ for all cases. Nevertheless, the approximate form of the ETH holds: The off-diagonal elements are small compared to the thermal value and decrease with N .

Additionally, the mean values of the off-diagonal matrix elements approach or fall below the ETH expected value, $\text{mean}_{n \neq m}[|A_{nm}|] \lesssim \exp(-\tilde{S}/2)$, indicating that the ETH is satisfied for the ensemble-averaged SYK chain with three sites.

For the hopping operator $\hat{h}_{13}$, the diagonal matrix elements are strictly zero for all N , reflecting the system's

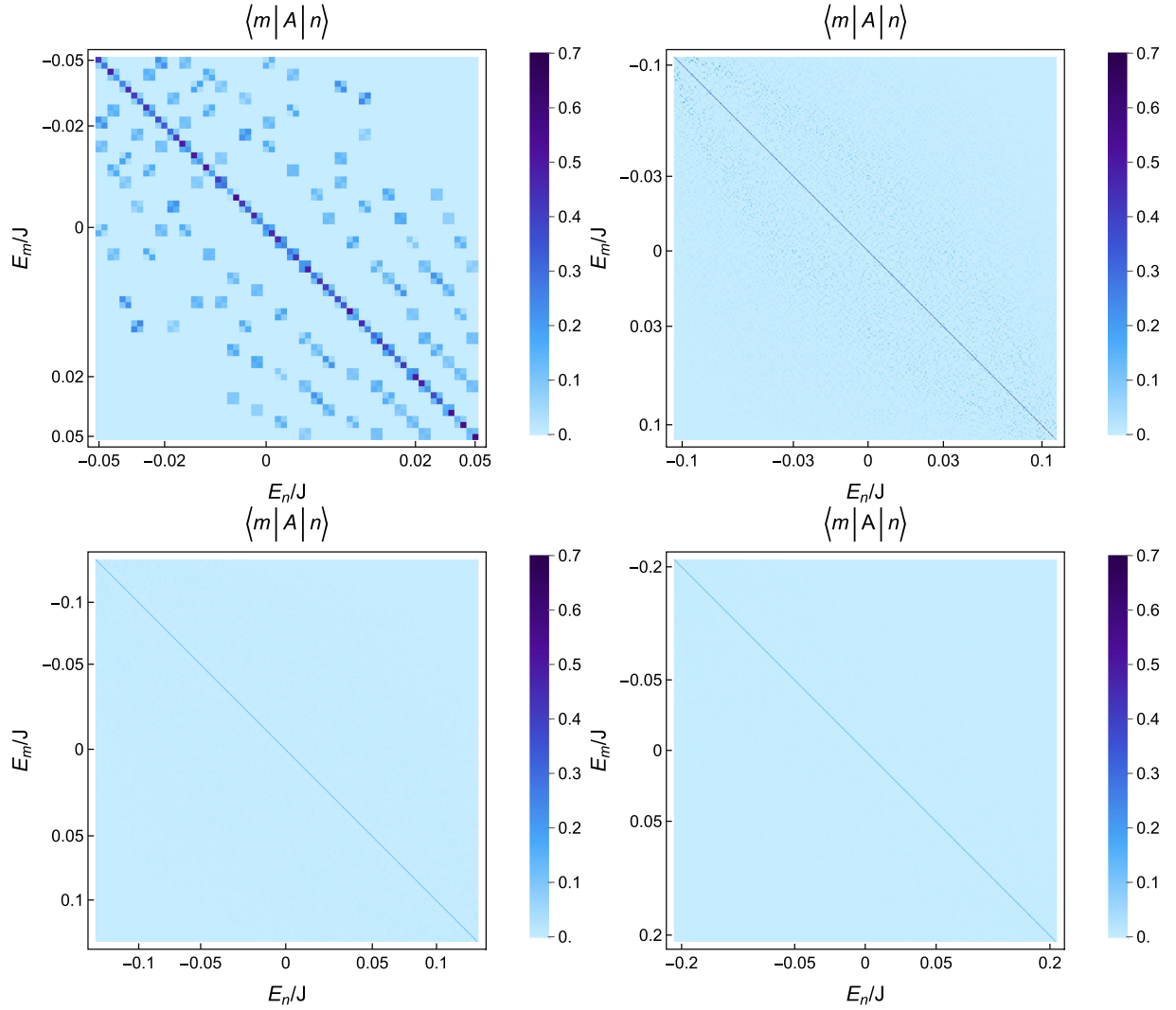


FIG. 6. Matrix plots of $|A_{mn}| = |\langle m|A|n\rangle|$ for the particle number operator $A = \hat{n}_1$ in the weak intersite coupling regime $J_0 = \sqrt{4.99}, J_1 = 0.1$, of the SYK chain model with $M = 3$ sites. We choose single realizations of the model with $N = 4, 6, 8, 10$ Majorana fermions on each site. The horizontal and vertical axes show the energy eigenvalues, $E_m/J, E_n/J$, corresponding to the matrix elements $|A_{mn}|$. All four plots illustrate approximate ETH behavior: the diagonal elements of the number operator in the energy eigenbasis fluctuate around the microcanonical average value of $1/2$, while the off-diagonal elements are subexponentially small. This indicates that ETH holds approximately, though not in its strict form. Top left panel, $N = 4$; top right panel, $N = 6$; bottom left panel, $N = 8$; bottom right panel, $N = 10$; the data for $N = 8, 10$ are downsampled to 1024×1024 for the purpose of presentation.

TABLE III. Maximum and mean values of the off-diagonal matrix elements of the particle number operator $A = \hat{n}_1$ in the energy eigenbasis for typical single realizations of the three-site SYK chain ($M = 3$), with different numbers of Majorana fermions N per site.

M	N	$\max_{n \neq m} A_{nm} $	$\text{mean}_{n \neq m} A_{nm} $	$e^{-\bar{S}/2}$
3	4	0.37	0.016	0.125
3	6	0.34	0.007	0.044
3	8	0.30	0.002	0.016
3	10	0.22	0.001	0.006

symmetries. The off-diagonal elements are small compared to unity, and their values also decrease with increasing N . Table IV reports the corresponding maximum and mean values, together with the ETH prediction.

Similar to the number operator, the off-diagonal elements decrease with increasing N , and their mean values approach or fall below the ETH prediction $e^{-\bar{S}/2}$. The maximum values, however, remain significantly larger than the ETH prediction, again indicating that the strict ETH is not satisfied.

In summary, the results for single realizations of the three-site SYK chain reinforce the findings from the

TABLE IV. Maximum and mean values of the off-diagonal matrix elements of the hopping operator $\mathcal{O} = \hat{h}_{13}$ in the energy eigenbasis for typical single realizations of the three-site SYK chain ($M = 3$), with different numbers of Majorana fermions N per site.

M	N	$\max_{n \neq m} \mathcal{O}_{nm} $	$\text{mean}_{n \neq m} \mathcal{O}_{nm} $	$e^{-\bar{S}/2}$
3	4	0.53	0.028	0.125
3	6	0.38	0.011	0.044
3	8	0.24	0.003	0.016
3	10	0.19	0.002	0.006

two-site case. The off-diagonal matrix elements of both the number operator and the hopping operator remain small, and their mean values decrease with increasing system size, approaching or falling below the ETH prediction. While the strict ETH is not fulfilled—particularly as the maximum off-diagonal elements consistently exceed the expected bound—the approximate form of the ETH holds.

C. Thermalization of single realizations of the SYK chain model with four sites

Turning to the four-site SYK chain ($M = 4$), I studied single realizations with $N = 6$ Majorana fermions per site

for the three types of couplings described in Sec. III: $J_0 = 2, J_1 = 1$; $J_0 = J_1 = \sqrt{5/2}$; and $J_0 = 1, J_1 = 2$. As in previous cases, I computed $|\langle m | \hat{n}_1 | n \rangle|$ and observed that the diagonal elements fluctuate around $1/2$, as shown in Fig. 7.

Figure 8 presents matrix plots of $|\langle m | \hat{n}_1 | n \rangle|$. The pattern remains consistent with that found for two- and three-site chains: Diagonal entries cluster around 0.5, while off-diagonal entries are mostly small or vanish at machine precision. This reflects the block-diagonal nature of the Hamiltonian. However, the nonzero off-diagonal matrix elements generally exceed the ETH prediction $e^{-\bar{S}/2}$.

A quantitative summary of the mean and maximum off-diagonal matrix elements of $A = \hat{n}_1$ is provided in Table V, alongside the corresponding ETH estimates. These results reveal that the strict ETH is violated, with $\max_{n \neq m} |A_{nm}| \gg e^{-\bar{S}/2}$ in all coupling scenarios. Nonetheless, an approximate version of the ETH remains valid: The off-diagonal elements are small compared to unity.

Moreover, the mean off-diagonal matrix elements tend to approach or fall below the ETH estimate, $\text{mean}_{n \neq m} [|A_{nm}|] \lesssim e^{-\bar{S}/2}$. This supports the view that the ETH holds in the ensemble-averaged sense for the four-site SYK chain. It is worth noting that, while the maximum

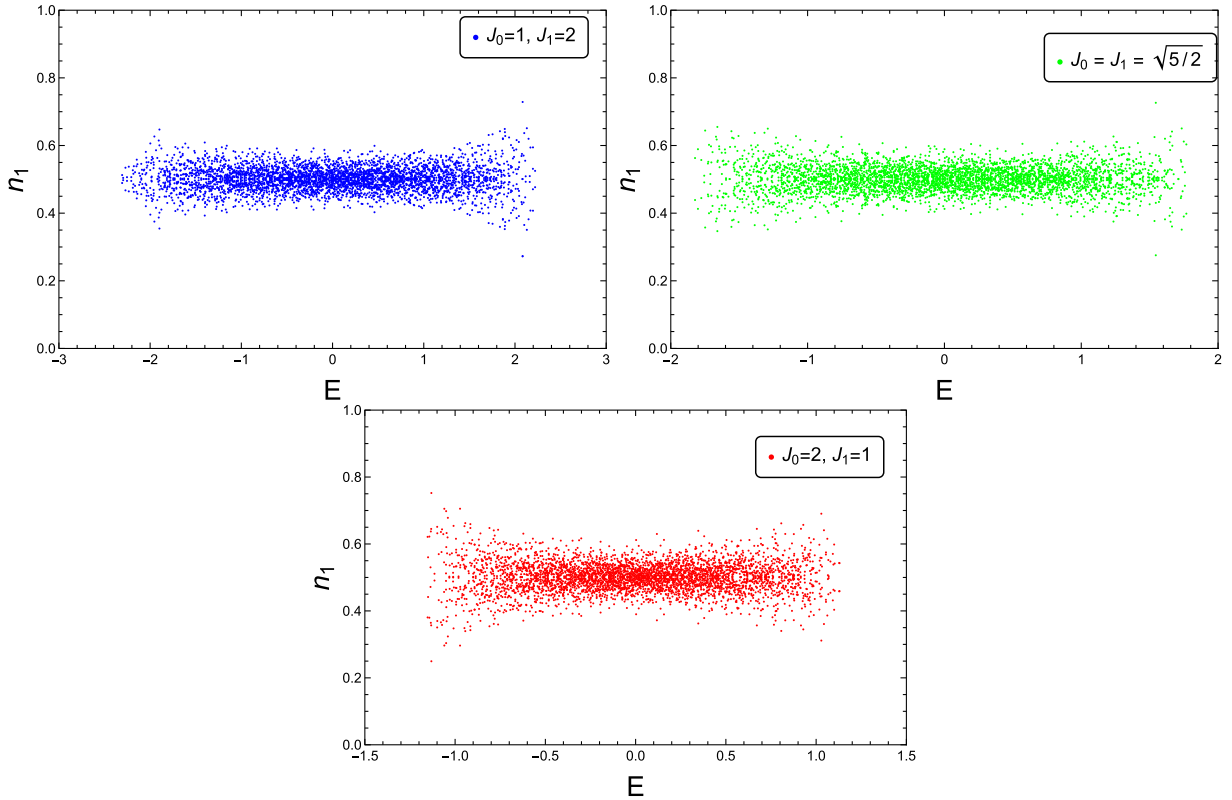


FIG. 7. The absolute value of the on-diagonal elements of the particle number operator n_1 as a function of energy eigenvalues E for single realization of SYK chain model with $M = 4$ sites and $N = 6$ fermions on each site. We consider three regimes of coupling strengths and observe that the diagonal terms fluctuate around the thermal value of $1/2$ in all cases. Top left panel, $J_0 = 1, J_1 = 2$, $J = \sqrt{5}$; top right panel, $J_0 = J_1 = \sqrt{5/2}$, $J = \sqrt{5}$; bottom panel, $J_0 = 2, J_1 = 1$, $J = \sqrt{5}$.

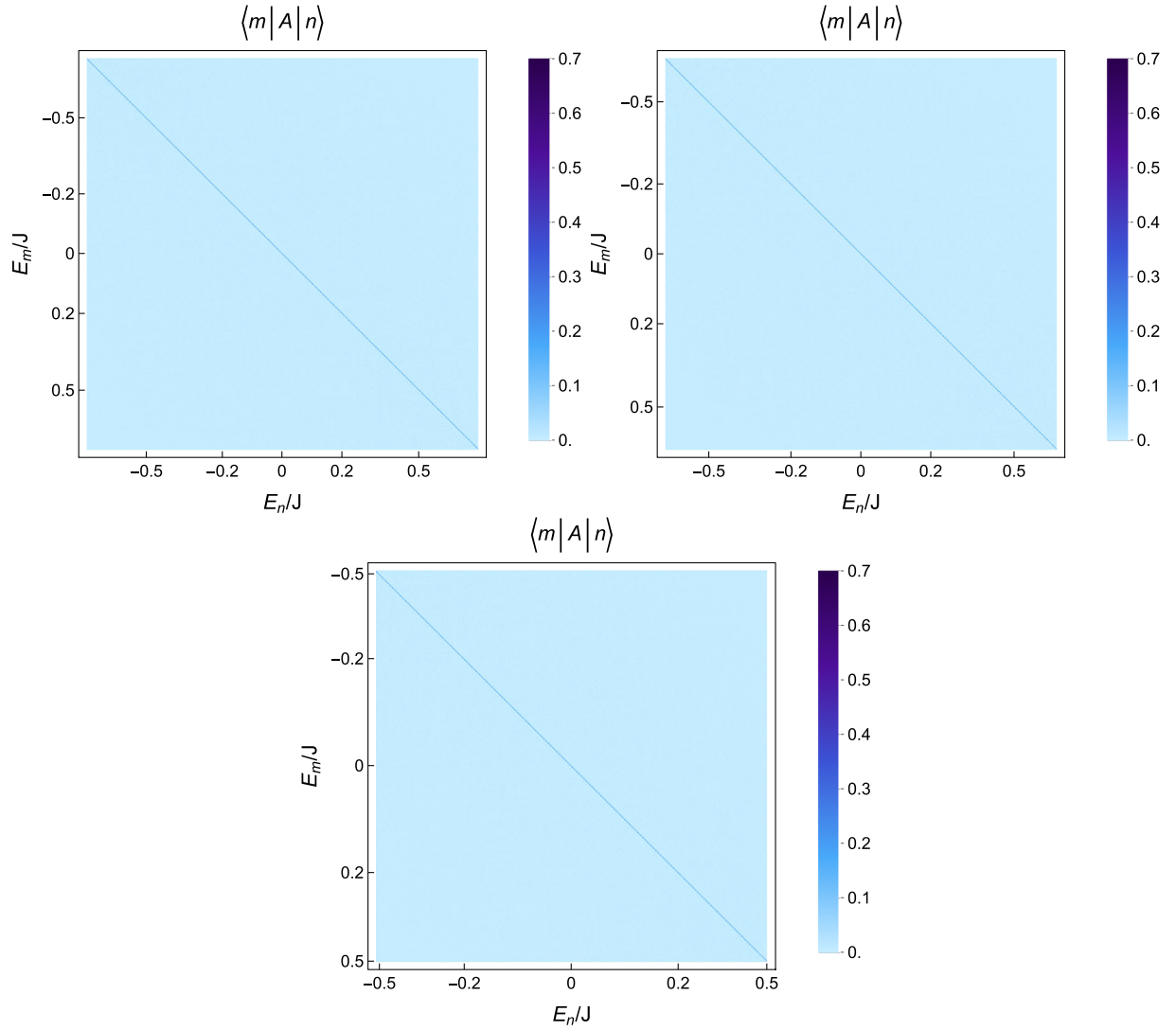


FIG. 8. Matrix plots of $|A_{mn}| = |\langle m|A|n \rangle|$ for the particle number operator $A = \hat{n}_1$ for three different values of coupling strengths J_0 and J_1 . We choose a single realization of the SYK chain model with $M = 4$ sites and $N = 6$ Majorana fermions on each site. The horizontal and vertical axes show the energy eigenvalues E_m/J and E_n/J , corresponding to the matrix elements $|A_{mn}|$. All three plots show behavior consistent with an approximate version of the ETH. The diagonal matrix elements of the number operator fluctuate about the microcanonical average value $1/2$, while the off-diagonal elements remain subexponentially suppressed. These results suggest that the ETH is fulfilled approximately rather than strictly. For the purpose of presentation, the data are downsampled to 1024×1024 . Top left panel, $J_0 = 1$, $J_1 = 2$; top right panel, $J = \sqrt{5}$, $J_0 = J_1 = \sqrt{5}/2$, $J = \sqrt{5}$; bottom panel, $J_0 = 2$, $J_1 = 1$, $J = \sqrt{5}$.

TABLE V. Maximum and mean values of the off-diagonal matrix elements of the particle number operator $A = \hat{n}_1$ in the energy eigenbasis for representative single realizations of the SYK chain model with $M = 4$ sites and $N = 6$ Majorana fermions per site at three different coupling strengths.

J_0	J_1	$\max_{n \neq m} A_{nm} $	$\text{mean}_{n \neq m} A_{nm} $	$e^{-\bar{S}/2}$
2	1	0.16	0.002	0.016
$\sqrt{5/2}$	$\sqrt{5/2}$	0.17	0.002	0.016
1	2	0.20	0.002	0.016

off-diagonal element varies with the coupling strength and realization, the mean value remains largely unaffected by the choice among the three coupling types considered.

Turning next to the hopping operator $\hat{h}_{13}$, a similar analysis yields consistent findings. Numerically, I confirmed the strict satisfaction of Eq. (48) for all three coupling cases. Given that the microcanonical average is also zero [Eq. (49)], the first ETH condition—the agreement between diagonal matrix elements and the microcanonical average—is exactly satisfied for $\hat{h}_{13}$.

TABLE VI. Maximum and mean values of the off-diagonal matrix elements of the hopping operator $\mathcal{O} = \hat{h}_{13}$ in the energy eigenbasis for typical single realizations of the SYK chain model with $M = 4$ sites and $N = 6$ Majorana fermions on each site for three different coupling strengths.

J_0	J_1	$\max_{n \neq m} \mathcal{O}_{nm} $	$\text{mean}_{n \neq m} \mathcal{O}_{nm} $	$e^{-\bar{S}/2}$
2	1	0.20	0.003	0.016
$\sqrt{5/2}$	$\sqrt{5/2}$	0.18	0.003	0.016
1	2	0.23	0.003	0.016

For the off-diagonal elements, Table VI compiles the mean and maximum values of $|\mathcal{O}_{nm}|$ for $\mathcal{O} = \hat{h}_{13}$ across the different coupling scenarios, together with the ETH estimates. As before, the strict ETH is not satisfied, since $\max_{n \neq m} |\mathcal{O}_{nm}| \gg e^{-\bar{S}/2}$ in every case. Nevertheless, the off-diagonal elements remain small compared to unity, consistent with an approximate ETH.

Again, the mean off-diagonal values approach or fall below $e^{-\bar{S}/2}$, reinforcing the conclusion that the ETH holds in the ensemble-averaged four-site SYK chain. It should be emphasized that, although the maximum off-diagonal element exhibits sensitivity to the coupling and realization, the mean value remains remarkably stable across the different coupling cases.

V. THERMALIZATION OF ENSEMBLE-AVERAGED THEORIES OF THE TWO-SITE SYK AND SYK CHAIN MODELS

In the preceding section, I examined the ETH conditions for individual realizations of the two-site SYK and SYK chain models and found them to be approximately satisfied. These results suggest that single realizations of these models thermalize rapidly with respect to the observables considered, with subexponentially small fluctuations of their expectation values around the corresponding thermal values.

I now turn to the ensemble-averaged forms of the two-site SYK and SYK chain models, which exhibit additional interesting properties. A natural question arises: What can be said about the thermalization of these ensemble-averaged theories? Unfortunately, the ETH is formulated only for individual quantum systems, and no developed extension currently exists that can be directly applied to ensemble-averaged models.

Nevertheless, it is common practice to perform ETH-inspired analyses in such settings. In what follows, I adopt the procedure employed in Refs. [18,19] for studying the original SYK, supersymmetric SYK, and complex SYK models.

The analysis of diagonal matrix elements proceeds as follows. First, I partition the energy spectrum into small intervals. For each realization of the disorder, I compute the

diagonal matrix elements of a chosen operator $\mathcal{O}$ in the energy eigenbasis, $\mathcal{O}_{nn} = \langle n | \mathcal{O} | n \rangle$. I then average the $\mathcal{O}_{nn}$ corresponding to eigenstates whose energies E_n lie within the same interval. Finally, I average over disorder realizations and report the resulting quantity as the disorder-averaged diagonal matrix element of $\mathcal{O}$ as a function of energy. I compare this result to the expectation from the ETH.

For off-diagonal matrix elements, the ETH predicts that their typical magnitudes should scale as $e^{-S/2}$:

$$g_{\mathcal{O}}(\bar{E}, \omega) := \langle m | \mathcal{O} | n \rangle \sim e^{-S/2}. \quad (52)$$

Here, $g_{\mathcal{O}}$ denotes the off-diagonal matrix elements of $\mathcal{O}$, classified as a function of the average energy $\bar{E} = (E_m + E_n)/2$ and the energy difference $\omega = E_m - E_n$.

The analysis of off-diagonal matrix elements proceeds in a similar manner. First, I partition the $\bar{E}$ and ω domains into small intervals of widths $\delta\bar{E}$ and $\delta\omega$, respectively. For each realization of the disorder, I compute the off-diagonal matrix elements $g_{\mathcal{O}}(\bar{E}, \omega)$. I then group all matrix elements $g_{\mathcal{O}}(\bar{E}, \omega)$ whose corresponding $(\bar{E}, \omega)$ values fall within the same intervals $(\bar{E}, \bar{E} + \delta\bar{E})$ and $(\omega, \omega + \delta\omega)$ and compute their average. Finally, I average the results over disorder realizations and report the resulting quantity $\bar{g}_{\mathcal{O}}(\bar{E}, \omega)$ as the disorder-averaged off-diagonal matrix elements of $\mathcal{O}$. I then compare these results to the ETH expectation.

In Eq. (52), I expressed the off-diagonal matrix elements of the operator $\mathcal{O}$ as a function of the average energy $\bar{E}$ and the energy difference ω . This parametrization enables a systematic study of the chaotic and random matrix theory (RMT)-like features of the off-diagonal elements. In the RMT, at fixed $\bar{E}$, $g_{\mathcal{O}}(\bar{E}, \omega)$ is expected to be approximately constant as a function of ω . Thus, to compare the behavior of the SYK chain model with that of a random matrix theory, I fix $\bar{E}$ and plot $g_{\mathcal{O}}(\bar{E}, \omega)$ as a function of ω .

The portion of $g_{\mathcal{O}}(\bar{E}, \omega)$ that remains nearly constant with respect to ω exhibits RMT-like behavior, while deviations from constancy indicate non-RMT behavior. The crossover between these two regimes occurs at a characteristic energy scale denoted E_T . In local models with spatial structure, this energy scale is often associated with the diffusive Thouless energy. Since the SYK chain model possesses one-dimensional spatial structure and exhibits diffusion of collective modes [5], E_T can be identified with the Thouless energy of the model. A similar identification has even been proposed for the complex, original, and supersymmetric SYK models, despite their zero-dimensional nature [18,19].

The SYK chain model is a one-dimensional system, and its diffusion constant has been theoretically derived, as shown in Eq. (27). To take the analysis further, I investigate whether the Thouless energy, which is proportional to the diffusion constant [see Eq. (28)], follows the dependence of the diffusion constant on the coupling strengths,

as expressed in Eq. (27). Specifically, I check if the relationship

$$E_T \propto D \quad (53)$$

holds, with the diffusion constant given by

$$D = \frac{2\pi J_1^2}{3\sqrt{2}J\alpha_K}. \quad (54)$$

A. Thermalization of the ensemble-averaged two-site SYK model

For $N = 6, 10$, and 14 the ETH conditions were examined in the ensemble-averaged theory of the two-site SYK model ($M = 2$) at the weak-link regime with $J_0 = \sqrt{4.99}$ and $J_1 = 0.1$. The procedure outlined previously was applied to the number operator $\hat{n}_1$ and the hopping operator $\hat{h}_{12}$, where $\hat{h}_{12}$ acts on the first and second sites in a similar fashion to $\hat{h}_{13}$, which acts on the first and third sites. Averages were taken over a sufficient number of disorder realizations to ensure that the resulting plots were smooth, with further increases in the number of realizations yielding no improvement in plot quality. Specifically, 40000 realizations were used for $N = 6$, 1000 for $N = 10$, and 25 for $N = 14$.

The results for the diagonal matrix elements of the number operator $\mathcal{O} = \hat{n}_1$ are shown in Fig. 9. To generate the plot, the absolute difference between the diagonal matrix elements of the number operator in the energy

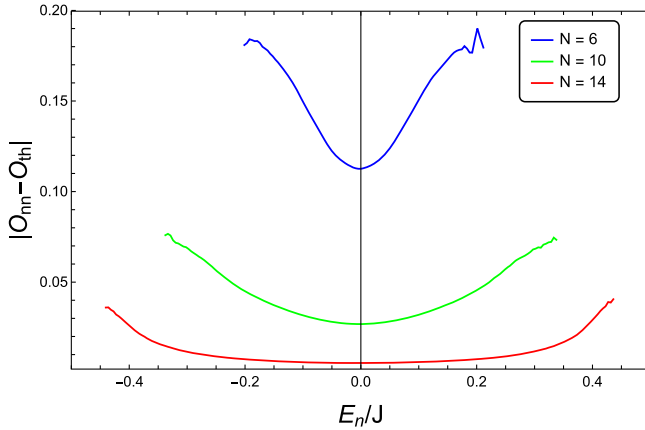


FIG. 9. Absolute difference between the diagonal elements $\mathcal{O}_{nn}$ of the particle number operator, $\mathcal{O} = \hat{n}_1$, and its thermal value, $\mathcal{O}_{th} = \text{Tr}[\rho_{mc}\mathcal{O}]/Z = 1/2$, plotted as a function of energy eigenvalues E_n/J for the ensemble-averaged two-site SYK model with $N = 6, 10$, and 14 Majorana fermions per site at the weak-link regime with $J_0 = \sqrt{4.99}$ and $J_1 = 0.1$. Ensemble averages were taken over 40000, 1000, and 25 disorder realizations, respectively. The smoothness of the curves and the small magnitude of the deviations demonstrate complete agreement with the ETH for all three values of N .

eigenbasis and its microcanonical average value, $\frac{1}{2}$, was calculated for each realization, i.e., $|\mathcal{O}_{nn} - \frac{1}{2}|$. Averages were then computed over all disorder realizations using the procedure described earlier.

As observed, the ETH condition, as stated in Eq. (11) or (15), is satisfied for the diagonal elements of the number operator. The deviations from the microcanonical average value are small, and they decrease further as N increases. Therefore, the ETH is strictly satisfied for the diagonal matrix elements of the number operator $\hat{n}_1$.

An interesting feature appears in Fig. 9. The deviation from the microcanonical average value is smaller at smaller absolute energies and becomes larger at larger absolute energies. This feature is also observed in the original SYK, supersymmetric SYK, and complex SYK models [18,19].

For the hopping operator $\hat{h}_{12}$, the diagonal matrix elements are strictly zero due to symmetry, just like $\hat{h}_{13}$, as discussed in Sec. IV C. Thus, their disorder averages are also zero and in strict agreement with their microcanonical value of zero.

For the off-diagonal matrix elements, however, there is no symmetry, and the values must be obtained numerically. The ensemble-averaged absolute values of the off-diagonal matrix elements of the number operator $\hat{n}_1$ and the hopping operator $\hat{h}_{12}$ are numerically computed and plotted in Figs. 10 and 11.

As observed, the ETH expectations are satisfied for the off-diagonal matrix elements of both $\hat{n}_1$ and $\hat{h}_{12}$. The ensemble-averaged absolute values of the off-diagonal matrix elements are smaller than $e^{-\tilde{S}/2}$ (see Table I for

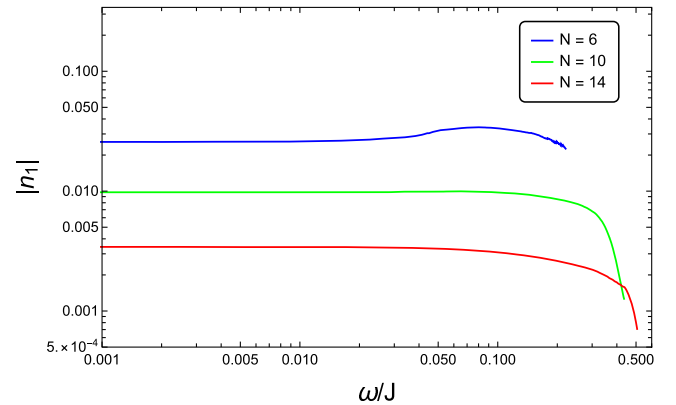


FIG. 10. Absolute value of the off-diagonal elements $\mathcal{O}_{mn}$ of the particle number operator, $\mathcal{O} = \hat{n}_1$, plotted as a function of the energy difference $\omega/J = (E_m - E_n)/J$ at fixed average eigenenergy $(E_m + E_n)/2$ for $N = 6, 10$, and 14 Majorana fermions per site in the ensemble-averaged two-site SYK model at the weak-link regime with $J_0 = \sqrt{4.99}$ and $J_1 = 0.1$. Ensemble averages were performed over 40000, 1000, and 25 disorder realizations, respectively. The exponentially small values of the off-diagonal elements demonstrate strict agreement with ETH for all three values of N .

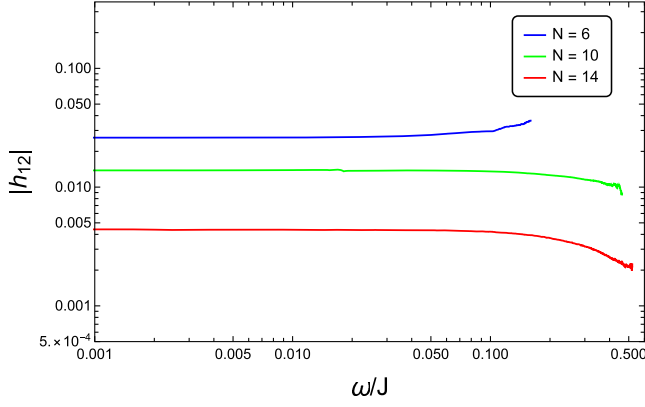


FIG. 11. Absolute value of the off-diagonal elements $\mathcal{O}_{mn}$ of the hopping operator, $\mathcal{O} = \hat{h}_{12}$, plotted as a function of the energy difference $\omega/J = (E_m - E_n)/J$ at fixed average eigenenergy $(E_m + E_n)/2$ for $N = 6, 10$, and 14 Majorana fermions per site in the ensemble-averaged two-site SYK model at the weak-link regime with $J_0 = \sqrt{4.99}$ and $J_1 = 0.1$. Ensemble averages were performed over 40000, 1000, and 25 disorder realizations, respectively. The exponentially small values of the off-diagonal elements demonstrate strict agreement with ETH for all three values of N .

the values of $e^{-\bar{s}/2}$). Thus, the ETH is strictly satisfied for both operators in the ensemble-averaged two-site SYK model.

Finally, the Thouless energy E_T can be identified in Figs. 10 and 11 as the energy scale where the behavior transitions from being independent of ω (RMT behavior) to being dependent on ω (non-RMT behavior). These results suggest that, for most of the energy domain, the two-site SYK model exhibits RMT-like behavior.

B. Thermalization in the ensemble-averaged three-site SYK chain

I investigate thermalization in the ensemble-averaged SYK chain model with $M = 3$ sites and $N = 8$ Majorana fermions per site, averaging over 1000 disorder realizations. As established in Sec. IV B, the diagonal matrix elements of the number operator $\hat{n}_1$ are identically equal to the microcanonical value $1/2$ for $N = 8$, and the diagonal elements of the hopping operator $\hat{h}_{13}$ are identically zero. Therefore, in this section, I focus exclusively on the off-diagonal matrix elements of $\hat{n}_1$ and $\hat{h}_{13}$ in the ensemble-averaged three-site SYK chain. As in the analysis of single realizations, I consider the weak-link regime, with coupling constants $J_0 = \sqrt{4.99}$ and $J_1 = 0.1$.

The ensemble-averaged absolute values of the off-diagonal matrix elements of $\hat{n}_1$ and $\hat{h}_{13}$ are computed numerically following the procedure outlined earlier and are presented in Fig. 12.

As shown in Fig. 12, the ensemble-averaged off-diagonal elements of both $\hat{n}_1$ and $\hat{h}_{13}$ satisfy the ETH expectations.

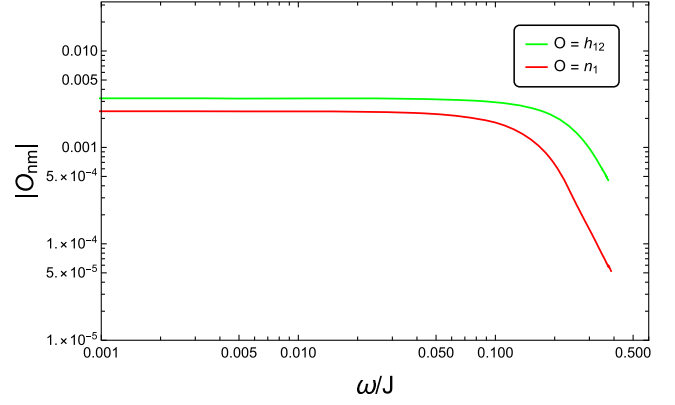


FIG. 12. The ensemble-averaged absolute value of the off-diagonal matrix elements of the number operator $\hat{n}_1$ and the hopping operator $\hat{h}_{13}$ as a function of energy difference ω/J for the SYK chain with $M = 3$ sites and $N = 8$ Majorana fermions per site at the weak-link regime with $J_0 = \sqrt{4.99}$ and $J_1 = 0.1$. The average is taken over 1000 disorder realizations. The moving average over neighboring matrix elements is performed at fixed average eigenenergy. The results demonstrate strict agreement with the ETH predictions for both of the operators.

In particular, their magnitudes are consistently smaller than $e^{-\bar{s}/2}$ (see Table III for the numerical value of $e^{-\bar{s}/2}$ for $M = 3$ and $N = 8$). Thus, the ETH holds strictly for both operators in the ensemble-averaged three-site SYK chain.

Moreover, Fig. 12 enables an estimation of the Thouless energy E_T , identifiable as the scale where the behavior of the off-diagonal elements transitions from a flat (RMT-like) dependence to a decay with increasing ω . Interestingly, the location of this crossover appears to be largely independent of the choice of operator, reinforcing the validity of the theoretical analysis of Thouless energy in the SYK chain model given in Ref. [5].

C. Thermalization in the ensemble-averaged four-site SYK chain

The ETH conditions are examined in the ensemble-averaged SYK chain with $M = 4$ sites and $N = 6$ Majorana fermions per site, across all three coupling regimes defined in Sec. III, namely, $J_0 = 2, J_1 = 1$; $J_0 = J_1 = \sqrt{5/2}$; and $J_0 = 1, J_1 = 2$. Following the procedure outlined earlier, the number operator $\hat{n}_1$ and the hopping operator $\hat{h}_{13}$ are analyzed, with averages taken over 1000 disorder realizations.

The results for the diagonal matrix elements of the number operator, $\mathcal{O} = \hat{n}_1$, are shown in Fig. 13.

As illustrated in Fig. 13, the ETH expectation is satisfied for the diagonal elements of the number operator. The deviations from the microcanonical average value are small, and Eq. (15) is consequently satisfied. Thus, the ETH holds for the diagonal matrix elements of $\hat{n}_1$ across all three coupling regimes.

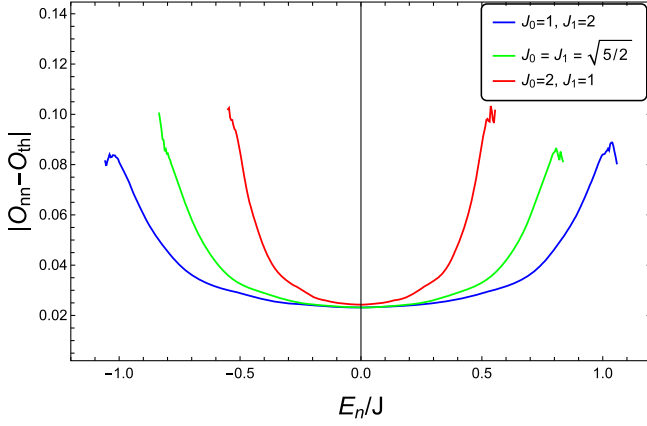


FIG. 13. The absolute difference between the diagonal elements $\mathcal{O}_{nn}$ of the particle number operator, $\mathcal{O} = \hat{n}_1$, and its thermal value $\mathcal{O}_{\text{th}} = \text{Tr}[\rho_{\text{mc}} \mathcal{O}]/Z = 1/2$, plotted as a function of energy eigenvalues E_n/J for three different coupling strengths J_0 and J_1 . The ensemble average is taken over 1000 disorder realizations of the SYK chain model with $M = 4$ sites and $N = 6$ Majorana fermions per site. The curves exhibit a visible dependence on the ratio J_1/J_0 , yet the smoothness of the curves and the smallness of the difference between the diagonal elements and the thermal value indicate complete agreement with the ETH for all three pairs of coupling strengths.

Several notable features emerge from Fig. 13. For all coupling cases, the deviation from the microcanonical value is smaller at lower absolute energies and grows larger at higher absolute energies. This behavior has also been observed in the original SYK model, the supersymmetric SYK model, and the complex SYK model [18,19]. The rate of growth of these deviations is faster in the case of stronger intrasite couplings ($J_0 = 2$, $J_1 = 1$) and becomes progressively slower as the intersite couplings become stronger ($J_0 = 1$, $J_1 = 2$). This difference reflects the distinct roles played by intrasite and intersite coupling terms in the SYK chain Hamiltonian [Eq. (23)]. However, since the deviations remain within the $\mathcal{O}(1/N)$ domain permitted by the ETH, they do not affect the overall validity of the ETH in this system.

For the hopping operator $\hat{h}_{13}$, the diagonal matrix elements are strictly zero due to symmetry, as discussed and reported for single realizations in Sec. IV C. Consequently, their disorder-averaged values are also zero and in exact agreement with the corresponding microcanonical value.

In contrast, the off-diagonal matrix elements are not constrained by symmetry and must be determined numerically. The ensemble-averaged absolute values of the off-diagonal matrix elements of the number operator $\hat{n}_1$ and the hopping operator $\hat{h}_{13}$ are computed and presented in Figs. 14 and 15, respectively.

As shown in these figures, the ETH predictions are satisfied for the off-diagonal matrix elements of both $\hat{n}_1$ and $\hat{h}_{13}$. The ensemble-averaged absolute values of the

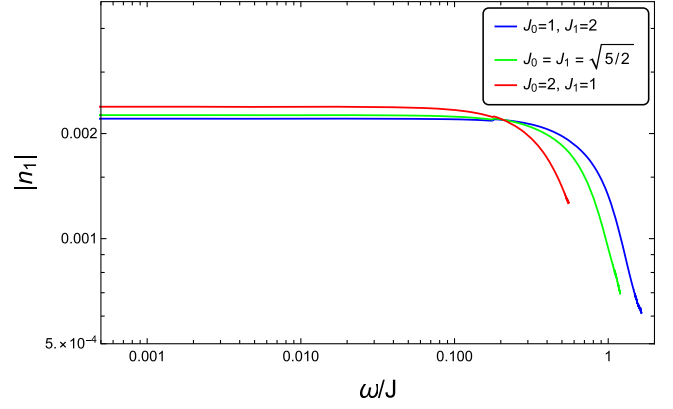


FIG. 14. The absolute value of the off-diagonal elements $\mathcal{O}_{mn}$ of the particle number operator, $\mathcal{O} = \hat{n}_1$, as a function of the energy eigenvalue difference $\omega/J = (E_m - E_n)/J$, at a fixed average eigenenergy $(E_m + E_n)/2$ for three different coupling strengths J_0 and J_1 . The ensemble average is taken over 1000 disorder realizations of the SYK chain model with $M = 4$ sites and $N = 6$ Majorana fermions per site. The average energy is fixed at $(E_m + E_n)/2NM = 0.02 \pm 0.001$, and a moving average in ω over the 800 nearest neighbors of the matrix elements is performed. The exponentially small values of the off-diagonal elements are in agreement with the ETH. Additionally, the curves show a dependence on the ratio J_1/J for larger ω/J , indicating that the Thouless energy and the range of RMT-like behavior increase as J_1/J increases, consistent with theoretical results for the diffusion constant of the model.

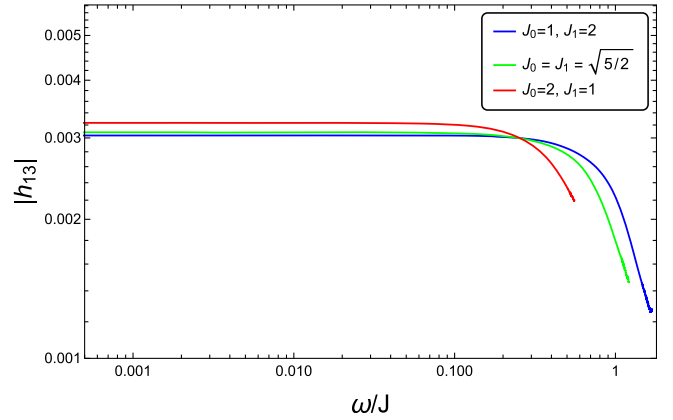


FIG. 15. The absolute value of the off-diagonal elements $\mathcal{O}_{mn}$ of the hopping operator between sites 1 and 3, $\mathcal{O} = \hat{h}_{13}$, as a function of the energy eigenvalue difference $\omega/J = (E_m - E_n)/J$, at a fixed average eigenenergy $(E_m + E_n)/2$, for three different coupling strengths J_0 and J_1 . The ensemble average is taken over 1000 disorder realizations of the SYK chain model with $M = 4$ sites and $N = 6$ Majorana fermions per site. The average energy is fixed at $(E_m + E_n)/2NM = 0.02 \pm 0.001$, and a moving average is performed in ω over 800 nearest neighbors of the matrix elements. The exponentially small values of the off-diagonal elements are consistent with the ETH. Additionally, the curves exhibit a dependence on the ratio J_1/J for larger ω/J , indicating that the Thouless energy and the range of RMT-like behavior increase as J_1/J increases, in accordance with theoretical results for the diffusion constant of the model.

off-diagonal matrix elements are smaller than $e^{-\tilde{S}/2}$ (see Tables V and VI for the values of $e^{-\tilde{S}/2}$). Thus, the ETH is satisfied for all three coupling regimes considered.

Moreover, the behavior of the Thouless energy E_T , determined by the crossover from the ω -independent regime (characteristic of RMT) to the ω -dependent regime (non-RMT behavior), reveals an interesting trend. Specifically, as J_1 increases from 1 (red curves) to $\sqrt{5/2}$ (green curves) and to 2 (blue curves), while keeping $J = \sqrt{5}$ fixed, the value of E_T increases. These findings are consistent with the analytic results for the diffusion constant of the SYK chain model reported in Ref. [5], Eq. (54), and the corresponding definitions of the Thouless energy, Eqs. (28) and (53).

Therefore, I conclude that, as the intersite coupling J_1 increases, the range of ω over which the system exhibits RMT-like chaotic behavior becomes larger.

VI. DISCUSSION AND CONCLUSION

In this work, I have investigated thermalization via the ETH in the strongly correlated, nonlocal $(1+1)$ -dimensional SYK chain model and its special case, the two-site SYK model. Using exact diagonalization, I examined two representative few-body operators: the local particle number operator acting on a single site and the nonlocal hopping operator acting between two sites. Both single realizations and ensemble averages over many disorder realizations were analyzed. The main goal was to verify whether the ETH conditions hold, both approximately and strictly, and to understand the mechanisms underlying thermalization in these systems.

The key findings are as follows. For individual disorder realizations, the diagonal elements of the observables in the energy eigenbasis fluctuate around the corresponding thermal values, and the off-diagonal elements are small but not exponentially suppressed. Consequently, the ETH is satisfied approximately in single realizations, allowing for thermalization with slightly larger fluctuations around equilibrium. In contrast, the ensemble-averaged theory satisfies the ETH conditions more precisely: The diagonal elements follow smooth energy-dependent curves, and the off-diagonal elements are exponentially small, fully consistent with the ETH ansatz.

An important conclusion is that thermalization occurs much faster than might have been inferred from earlier studies based on Rényi entropies [4]. The faster thermalization revealed by ETH analysis may have several explanations. One possibility is as follows. In the SYK chain model at strong coupling ($N \gg \beta J \gg 1$), the dominant light degrees of freedom are collective modes associated with the time-reparametrization field. These soft modes can lead to rapid equilibration of few-body observables, even when the system is spatially local and lacks all-to-all interactions. The heavy modes that Ref. [4] identified

through Rényi entropy studies, which were claimed to induce slow thermalization, do not appear crucial for the ETH mechanism of thermalization.

Additional insight comes from Ref. [23], which appeared after the first version of this work was posted on arXiv and cited it. In that study, the authors show that thermalization in the SYK chain is state dependent: States with sufficiently high effective temperature thermalize rapidly, even with respect to Rényi entropies. Moreover, their analysis provides evidence that the previously reported subthermal behavior [4] is a large- N artifact rather than an intrinsic feature of the model at finite N . Their results are consistent with and further support the conclusions drawn in the present work regarding the relatively fast thermalization observed through ETH analysis.

Additionally, I found that the Thouless energy E_T , which marks the crossover from RMT-like behavior to non-RMT behavior, increases with the strength of the intersite coupling J_1 . This observation matches theoretical expectations based on diffusion of energy in the SYK chain, where the diffusion constant grows with increasing J_1 [5]. As the intersite coupling strengthens, energy spreads more efficiently between sites, leading to an enlarged energy window over which RMT-like chaotic behavior persists.

These findings have broader implications. First, the spatial locality of the SYK chain model does not obstruct thermalization via the ETH, suggesting that the ETH holds beyond zero-dimensional models with all-to-all couplings. Second, from a holographic perspective, since the SYK chain is conjectured to be dual to an incoherent black hole, the validity of the ETH for few-body operators indicates that black holes in the dual description also exhibit rapid thermalization.

Comparing with previous ETH studies in zero-dimensional SYK-like models [18,19], this work extends the understanding to $(1+1)$ -dimensional systems with spatial locality, demonstrating that thermalization mechanisms remain robust even when spatial structure is introduced. The results suggest that the dominant physics responsible for the ETH and rapid relaxation is tied to collective low-energy dynamics.

Several future directions arise naturally from this work. While I focused on two conventional operators, it would be valuable to systematically examine whether more generic few-body operators also display ETH behavior in the SYK chain and its generalizations. Additionally, understanding how thermalization in ensemble-averaged theories can be rigorously derived from the ETH at the level of individual realizations remains an open and important question. Finally, extensions of this analysis to higher-dimensional SYK models introduced in Refs. [5,24] would shed further light on the universality of the ETH across strongly interacting quantum chaotic systems.

Overall, this study demonstrates that, despite the absence of all-to-all interactions and despite earlier indications of

slow entanglement growth, the SYK chain and two-site SYK models thermalize rapidly through the ETH mechanism, confirming the robustness of eigenstate thermalization in strongly interacting lattice systems.

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