

Momentum space entanglement of four fermion field theory

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Momentum space entanglement of four fermion field theory is calculated from the Wilsonian effective action perturbatively using replica trick, local terms in low energy effective action are proved to be nonrelevant perturbatively, and nonlocal terms are the only source of entanglement between different momentum modes. The final result can be represented by a set of basketball Feynman diagrams with new Feynman rules proposed to interpret them.

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I. INTRODUCTION

Entanglement entropy, a fundamental measure of quantum correlations, has emerged as a powerful conceptual tool across several frontiers of theoretical physics. While its significance in quantum information science and quantum computing is well established [1], recent decades have witnessed remarkable extensions of its applicability. This measure now provides critical insights, ranging from black hole thermodynamics [2–4] and holographic principles [5] to condensed matter systems [6,7] and quantum field theory itself [8–11].

Traditional studies of entanglement entropy have predominantly focused on real-space bipartitions, motivated by the local nature of quantum field theory (QFT) [12–14]. However, a growing body of work has recently highlighted the profound implications of momentum-space entanglement entropy [8,15–20]. This momentum-space perspective naturally connects with Wilsonian renormalization group concepts, since the reduced density matrix for low-energy modes directly corresponds to the Wilsonian effective action, as first emphasized in Ref. [15].

In Ref. [15], the authors propose two kinds of methods for the perturbative calculation of several types of entanglement measures and employ the time-independent perturbation theory of ordinary quantum mechanics to evaluate momentum space entanglement of ϕ^n model. They uncover an intriguing connection between renormalizability and the

divergence properties of the mutual information, thereby providing key insights into the implications of momentum-space entanglement for renormalization group flows. This approach was subsequently employed in Ref. [18] to analyze the momentum space entanglement property of interacting fermions at finite density in one spatial dimension. There, divergences in single-mode entanglement were identified as the momentum approached the Fermi surface, and several renormalization group (RG) predictions were verified using mutual information.

In Ref. [17] it was demonstrated that the Wilsonian formulation of the renormalization group defines a quantum channel acting on the momentum-space density matrices of a quantum field theory. This result implies that the vacua of theories at a fixed point contain no entanglement between momentum scales. Such findings naturally raise the question of to what extent specific RG flows are reflected in momentum-space entanglement entropy. Motivated by these developments, in this work we focus on momentum-space entanglement and explore systematic methods for calculating such entanglement measures.

Another seminal formulation proposed in Ref. [15] establishes a connection between RG and reduced density matrix through the explicit representation of the momentum-space reduced density matrix,

$$\langle \varphi_{\mathbf{k}} | \rho_{\mu} | \tilde{\varphi}_{\mathbf{k}} \rangle = \lim_{\beta \rightarrow \infty} \frac{1}{Z(\beta)} \int_{\phi_{\mathbf{k}}(0)=\varphi_{\mathbf{k}}}^{\phi_{\mathbf{k}}(\beta)=\tilde{\varphi}_{\mathbf{k}}} \mathcal{D}\phi_{\mathbf{k}}(\tau) e^{-S_{\mu}^{\beta}}. \quad (1)$$

However, this formalism was not explicitly used in Ref. [15], subsequent work by Ref. [16] developed a novel replica-trick approach for computing entanglement and Rényi entropies directly from the effective action based

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on this representation, revealing intriguing diagrammatic interpretations that challenge conventional perturbative understanding. Although this method was originally developed for bosonic fields and successfully reproduces results obtained through conventional density matrix diagonalization in [15], its extension to fermionic systems, which is the topic of this paper, requires careful reconsideration. The Grassmannian nature of fermionic fields introduces significant complexities, rendering the path integral representation of the reduced density matrix substantially more intricate than in bosonic cases.

While momentum-space entanglement in fermionic systems has been extensively studied in condensed matter physics [18,21–26], our primary interest lies in relativistic quantum field theories. In this work, we extend the framework in Ref. [16] to fermionic systems, investigating whether analogous Feynman diagrammatic structures emerge in the computation of momentum-space entanglement entropy. Our analysis demonstrates affirmative results with unexpected new features.

The remainder of this paper is organized as follows: Sec. II presents a generalized formulation adapted for fermionic path integrals. Section III applies this formalism to a four-fermion interaction model previously studied in Ref. [15], while Sec. IV develops modified Feynman rules that capture the distinctive characteristics of fermionic entanglement entropy. Notably, we identify novel diagrammatic contributions whose physical origin requires further investigation. Technical details of the Grassmannian functional calculations are provided in the Appendix.

II. DENSITY MATRIX AND REPLICA TRICK FOR FERMIONS

The decomposition of physical degrees of freedom into high- and low-energy modes forms a cornerstone of the Wilsonian renormalization framework [27,28]. Within this paradigm, systematically integrating out high-momentum degrees of freedom generates RG flows for the coupling constants that govern low-energy effective interactions. This fundamental connection motivates the formal representation of reduced density matrices for low-momentum modes [Eq. (1)], from which entanglement entropies can be systematically computed. In this work, we extend this formalism to fermionic systems.

Conventional Wilsonian RG implementations typically employ Wick-rotated Euclidean four-momentum integrals to derive manifestly Lorentz-invariant effective actions. By contrast, our density-matrix formulation requires a different approach. Following Ref. [15], we construct the vacuum state using a Euclidean path integral in which spatial high-momentum modes are integrated out. Crucially, this procedure generates nonlocal contributions to the effective action—features shown to be essential for encoding entanglement signatures in bosonic systems as in Ref. [16].

As we demonstrate below, this key characteristic also persists in fermionic systems.

A. Reduced density matrix for low-momentum degrees of freedom for fermions

The path-integral representation of the vacuum state has been widely employed to compute Rényi entropies via the replica trick [13]. This construction extends naturally to fermionic systems, though with crucial distinctions arising from the Grassmannian nature of the path integrals [9].

To set the stage, consider a bare action S_Λ of a Dirac fermion field $\hat{\psi}(\vec{x})$ whose Fourier components are $\hat{\psi}_{\mathbf{k}}$. We impose an ultraviolet cutoff Λ on the spatial momentum such that $|\mathbf{k}| < \Lambda$ (In this paper bold letter $\mathbf{k}$ denotes spatial momentum). The ground state $|\Psi_\Lambda^0\rangle$ can be represented by the density matrix $\rho_\Lambda^0 = |\Psi_\Lambda^0\rangle\langle\Psi_\Lambda^0|$. Choose a basis formed by eigenvectors of the field operator, $\hat{\psi}_{\mathbf{k}}|\psi_{\mathbf{k}}\rangle = \psi_{\mathbf{k}}|\psi_{\mathbf{k}}\rangle$, where $\psi_{\mathbf{k}}$ is any Grassmann valued function. Then the matrix element of the vacuum density operator in the momentum space is given by Ref. [9],

$$\langle\theta_{\mathbf{k}}\rho_\Lambda|\tilde{\theta}_{\mathbf{k}}\rangle = \lim_{\beta\rightarrow\infty} \frac{1}{Z(\beta)} \int_{\psi_{\mathbf{k}}(0)=\theta_{\mathbf{k}}}^{\psi_{\mathbf{k}}(\beta)=\tilde{\theta}_{\mathbf{k}}} \mathcal{D}\tilde{\psi}_{\mathbf{k}}(\tau) \mathcal{D}\psi_{\mathbf{k}}(\tau) e^{-S_E(\psi_{\mathbf{k}},\tilde{\psi}_{\mathbf{k}})}. \quad (2)$$

where S_E is the Euclidean action and $Z(\beta)$ is the partition function that normalizes the path integral.

It was first pointed out in [15] that the density matrix ρ_μ for the degrees of freedom with momentum $\mathbf{k} < \mu$ is naturally associated with the Wilsonian effective action S_μ [see Eq. (1)]. The latter is obtained from the bare action S_Λ by integrating out all field modes with momentum $\mathbf{k}$ such that $\mathbf{k} \geq \mu$, where μ defines a momentum scale in the range $[0, \Lambda]$ that separates the momentum modes into two disjoint subsets. Accordingly, we decompose the fields in momentum space into low- and high-momentum modes,

$$\psi_{\mathbf{k}} = \psi_{0,\mathbf{k}} + \psi_{1,\mathbf{k}}, \quad (3)$$

with

$$\begin{aligned} \psi_{0,\mathbf{k}} &= \psi_{\mathbf{k}} & \text{if } |\mathbf{k}| \leq \mu, & \text{otherwise } 0, \\ \psi_{1,\mathbf{k}} &= \psi_{\mathbf{k}} & \text{if } |\mathbf{k}| > \mu, & \text{otherwise } 0, \end{aligned} \quad (4)$$

and similarly for $\tilde{\psi}_{\mathbf{k}}$. Then the reduced density matrix of low-momentum modes is obtained by tracing out high-momentum modes in Eq. (2),

$$\begin{aligned} \langle\theta_{0,\mathbf{k}}|\rho_\mu|\tilde{\theta}_{0,\mathbf{k}}\rangle &= \lim_{\beta\rightarrow\infty} \frac{1}{Z(\beta)} \int_{\psi_{0,\mathbf{k}}(0)=\theta_{0,\mathbf{k}}}^{\psi_{0,\mathbf{k}}(\beta)=\tilde{\theta}_{0,\mathbf{k}}} \mathcal{D}\tilde{\psi}_{0,\mathbf{k}}(\tau) \mathcal{D}\psi_{0,\mathbf{k}}(\tau) e^{-S_\mu(\psi_{0,\mathbf{k}},\tilde{\psi}_{0,\mathbf{k}})}, \end{aligned} \quad (5)$$

where the low energy effective action at scale μ is defined by,

$$e^{-S_\mu(\psi_{0,\mathbf{k}},\bar{\psi}_{0,\mathbf{k}})} = \int_{a.p.} \mathcal{D}\bar{\psi}_{1,\mathbf{k}} \mathcal{D}\psi_{1,\mathbf{k}} e^{-S_E(\psi_{\mathbf{k}},\bar{\psi}_{\mathbf{k}})}. \quad (6)$$

Here the subscript ‘‘a.p.’’ indicates that the high-momentum modes of the fermion field satisfy antiperiodic boundary conditions, as is standard in the evaluation of fermionic thermal partition functions. This representation allows the low-energy effective action of a fermionic system to be calculated perturbatively.

In contrast to the conventional approach of Ref. [9], where antiperiodicity is imposed on the low-momentum sector, we instead assign the minus sign arising from the Grassmann trace to the high-momentum modes. This choice enforces antiperiodic boundary conditions exclusively on the high-momentum fields. Consequently, the path integral is restricted to antiperiodic configurations of the high-momentum modes. After performing this momentum-shell integration, the high-momentum degrees of freedom are systematically eliminated, leaving a reduced system in which only the low-momentum modes remain as fundamental dynamical variables. This construction thereby establishes an effective theory for the infrared regime.

It is worth emphasizing that the low-energy effective action is a highly valuable object. On the one hand, its dependence on the scale μ provides key insights into the insensitivity of low-energy physics to the details of the ultraviolet description. On the other hand, it captures the complete information about the entanglement between high- and low-momentum modes. Thus, the Wilsonian effective action serves as a crucial bridge between renormalization and momentum-space entanglement—a connection that is intriguing yet far from fully understood.

B. Entanglement measures from the effective action

Having established the low-energy effective action, we now proceed to compute the Rényi entropy using the replica technique, which fundamentally reduces to evaluating $\text{Tr}\rho_\mu^n$. Our methodology follows the prescription outlined in Ref. [16], extended here to accommodate fermionic systems. The key step involves constructing a modified partition function, defined as (see Ref. [16] for a detailed description),

$$Z_n(\mu, \beta) \equiv \int_{n\beta} \mathcal{D}\bar{\psi}_{0,\mathbf{k}}(\tau) \mathcal{D}\psi_{0,\mathbf{k}}(\tau) e^{-\sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau L_\mu^\beta}, \quad (7)$$

where the effective Lagrangian density L_μ^β integrates to the action $S_\mu^\beta = \int_0^\beta d\tau L_\mu^\beta$. This construction arises from a two-step geometric manipulation of the Euclidean time domain: first, shifting the Euclidean time range for each replica of the reduced density matrix; and second, cyclically gluing these n copies across adjacent temporal boundaries.

Crucially, for fermionic systems, the antiperiodicity condition must now be imposed on the low-momentum modes over the extended interval $[0, n\beta]$, in contrast to conventional bosonic treatments. With this framework established, we can write,

$$\text{Tr}\rho_\mu^n = \lim_{\beta \rightarrow \infty} \frac{Z_n(\mu, \beta)}{[Z(\mu, \beta)]^n}, \quad (8)$$

where the partition function is redefined by discarding terms that are at zeroth order in the low-energy effective action,

$$Z(\mu, \beta) \equiv \int_\beta \mathcal{D}\bar{\psi}_{0,\mathbf{k}}(\tau) \mathcal{D}\psi_{0,\mathbf{k}}(\tau) e^{-S_\mu^\beta(\psi_{0,\mathbf{k}},\bar{\psi}_{0,\mathbf{k}})}. \quad (9)$$

The central object of interest is the ‘‘glued’’ effective action, defined as

$$S_{\mu,r}^\beta \equiv \sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau L_\mu^\beta.$$

At second order, this glued action decomposes into two contributions: one corresponding to the original effective action evaluated at temperature $1/n\beta$, and the other a nonlocal component that contributes directly to the Rényi entropy. Accordingly, we may write

$$S_{\mu,r}^\beta \equiv S_\mu^{n\beta} + S_{\mu,\text{nl}}^\beta. \quad (10)$$

Furthermore, to facilitate a perturbative expansion, we define the unperturbed partition function of the low-momentum modes as

$$Z_0(\mu, \beta) \equiv \int_\beta \mathcal{D}\bar{\psi}_{0,\mathbf{k}}(\tau) \mathcal{D}\psi_{0,\mathbf{k}}(\tau) e^{S_\mu^{\beta,(0)}(\bar{\psi}_{0,\mathbf{k}},\psi_{0,\mathbf{k}})}, \quad (11)$$

where $S_\mu^{\beta,(0)}$ represent the free part of the effective action.

Then we can write Eq. (8) as,

$$\begin{aligned} \text{Tr}\rho_\mu^n &= \lim_{\beta \rightarrow \infty} \frac{Z_n(\mu, \beta)}{[Z(\mu, \beta)]^n} \\ &= \lim_{\beta \rightarrow \infty} \frac{Z_n(\mu, \beta)}{Z(\mu, n\beta)} \\ &= \lim_{\beta \rightarrow \infty} \frac{Z_n(\mu, \beta)/Z_0(\mu, n\beta)}{Z(\mu, n\beta)/Z_0(\mu, n\beta)}. \end{aligned} \quad (12)$$

Finally the Rényi entropies can be expressed as

$$\begin{aligned} H_n(\mu) &= \frac{1}{1-n} \log \text{Tr}\rho_\mu^n \\ &= \frac{1}{n-1} \lim_{\beta \rightarrow \infty} \left(\log \frac{Z(\mu, n\beta)}{Z_0(\mu, n\beta)} - \log \frac{Z_n(\mu, \beta)}{Z_0(\mu, n\beta)} \right). \end{aligned} \quad (13)$$

Using the decomposition of Eq. (10), the perturbative expansion to leading order yields

$$H_n(\mu) = \frac{1}{n-1} \lim_{\beta \rightarrow \infty} \langle S_{\mu, \text{nl}}^{n\beta} \rangle. \quad (14)$$

In Ref. [16], the authors established a general theorem demonstrating that if the effective action consists solely of local terms in Euclidean time, the Rényi entropies vanish identically at the nonperturbative level. Our findings extend this conclusion by showing that even when the effective action contains both local and nonlocal temporal contributions, the entanglement structure is still entirely governed by the nonlocal sector within perturbation theory. Specifically, nonlocal contributions persist as the exclusive perturbative source of quantum entanglement, while local terms alone fail to generate nontrivial entropy corrections. This distinction highlights the intrinsic link between temporal nonlocality in the effective action and the emergence of quantum correlations quantified by Rényi entropies.

In practice, exact expressions for the effective action are generally inaccessible. As emphasized in Ref. [16], the appearance of nontrivial entanglement between distinct momentum modes requires identifying the lowest-order effective action that incorporates nonlocal temporal terms. In our case, such contributions first arise at the second-order expansion of the effective action.

A subtle technical point arises in deriving Eq. (14): the perturbative procedure is applied at two distinct stages—first, in the derivation of the effective action itself, and second, in the evaluation of the composite path integral for the replica system. This dual perturbative treatment underscores the importance of consistently truncating approximations while retaining entanglement-sensitive terms across both computational steps.

III. APPLICATION TO $(\bar{\psi}\psi)^2$ THEORY

In this section, we apply the preceding formalism to a four-fermion interaction model. While detailed derivations are provided in the Appendix, we summarize the key results at each computational stage below.

This interaction model has been widely studied in various contexts. For instance, Ref. [15] computed the entanglement entropy via time-independent perturbation theory, yielding leading-order results. The same framework was later extended to finite-density fermionic systems in Ref. [18], but such approaches rely on explicit density-matrix diagonalization and are not suitable for higher-order corrections. A related study, Ref. [21], employed the replica trick to derive a systematic expansion of the Rényi entropies $S_n(A)$ for momentum-space subsets near the Fermi surface. Although conceptually similar to our approach, their diagrammatic representation (“pants diagrams”) is not analytically tractable, whereas our method provides a calculable framework.

We consider the Euclidean action,

$$S_\Lambda^\beta[\psi_{\mathbf{k}}] = \sum_{K,Q} \bar{\psi}_K D_{K,Q} \Psi_Q - g \frac{\beta}{V} \sum_{K,K',Q,Q'} \bar{\psi}_K \psi_Q \bar{\psi}_{K'} \psi_{Q'} \delta_{K+K'-Q-Q',0}, \quad (15)$$

with Euclidean momentum $K^\mu \equiv (-i\omega_j, \mathbf{k})$ where $\omega_j = (2j+1)\pi/\beta$ are the fermion Matsubara frequencies. The spacetime dimension d is left arbitrary. This Euclidean action can be derived from the Lagrangian,

$$\mathcal{L} = \bar{\psi}(i\cancel{\partial} - m)\psi + g(\bar{\psi}\psi)^2, \quad (16)$$

by transforming the fields to momentum space via the standard Fourier transforms in finite-temperature field theory,

$$\begin{aligned} \psi(X) &= \frac{1}{\sqrt{V}} \sum_K e^{-iK \cdot X} \psi_K, \\ \bar{\psi}(X) &= \frac{1}{\sqrt{V}} \sum_K e^{iK \cdot X} \bar{\psi}_K, \end{aligned} \quad (17)$$

with the Euclidean coordinate defined as $X^\mu \equiv (-i\tau, \mathbf{x})$; then we find the Euclidean action above with $D_{K,Q} \equiv \frac{S_0^{-1}(K)}{T} \delta_{K,Q}$, where $S_0^{-1}(K) \equiv m - \cancel{K}$ is the thermal fermion propagator.

Given the Euclidean action, the first step is to calculate the low energy effective action. To order g^2 , there are four Feynman diagrams as in Figs. 1–4 that contribute.

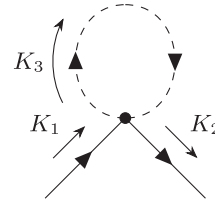


FIG. 1. Order g contribution to the low-energy effective action: momentum conservation prevents entanglement between high- and low-momentum modes, so this term is omitted.

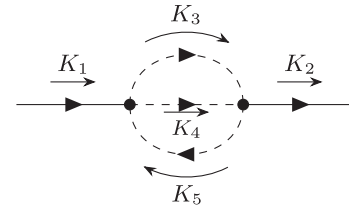


FIG. 2. Order g^2 two-point contribution to the low-energy effective action. In fermionic systems, two types of contractions of the legs are possible, as shown in Fig. 5.

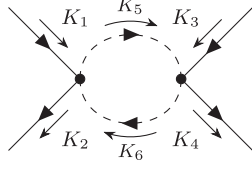


FIG. 3. Order g^2 four-point contribution to the low-energy effective action, with three possible fermionic leg contractions (see Fig. 6).

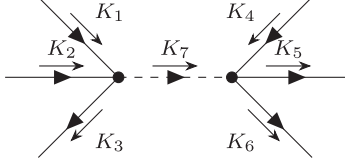


FIG. 4. Order g^2 six-point contribution to the low-energy effective action, denoted by $S_\mu^{\beta,(6)}$.

Note that due to the Dirac indices, there are several types of contractions for the two-point and four-point effective

actions (see the Appendix). In particular, the two-point diagram can be contracted as illustrated in Fig. 5.

And the four-point diagram has three types of contractions as in Fig. 6.

From these diagram we can write down the effective action as

$$S_\mu^\beta = S_{0,E}^\beta + S_\mu^{\beta,(2,1)} + S_\mu^{\beta,(2,2)} + S_\mu^{\beta,(4,1)} + S_\mu^{\beta,(4,2)} + S_\mu^{\beta,(4,3)} + S_\mu^{\beta,(6)}, \quad (18)$$

and the “glued” effective action as

$$S_{\mu,r}^\beta = S_{0,E}^{n\beta} + S_{\mu,r}^{\beta,(2,1)} + S_{\mu,r}^{\beta,(2,2)} + S_{\mu,r}^{\beta,(4,1)} + S_{\mu,r}^{\beta,(4,2)} + S_{\mu,r}^{\beta,(4,3)} + S_{\mu,r}^{\beta,(6)}, \quad (19)$$

see the Appendix for explicit expressions.

Then according to Eq. (13), we can write the Rényi entropy as Eq. (20).

$$H_n(\mu) = \frac{1}{n-1} \lim_{\beta \rightarrow \infty} (\langle S_{\mu,r}^{n\beta,(2,1)} \rangle + \langle S_{\mu,r}^{n\beta,(2,2)} \rangle + \langle S_{\mu,r}^{n\beta,(4,1)} \rangle + \langle S_{\mu,r}^{n\beta,(4,2)} \rangle + \langle S_{\mu,r}^{n\beta,(4,3)} \rangle + \langle S_{\mu,r}^{n\beta,(6)} \rangle - \langle S_\mu^{n\beta,(2,1)} \rangle - \langle S_\mu^{n\beta,(2,2)} \rangle - \langle S_\mu^{n\beta,(4,1)} \rangle - \langle S_\mu^{n\beta,(4,2)} \rangle - \langle S_\mu^{n\beta,(4,3)} \rangle - \langle S_\mu^{n\beta,(6)} \rangle). \quad (20)$$

As analytically shown in the Appendix and anticipated in Eq. (10), the composite (“glued”) effective action admits a systematic decomposition as the sum of the original effective action at temperature $n\beta$ and an additional nonlocal term.

This property holds for each individual Feynman diagram and establishes that nonlocal interactions are not merely necessary for generating entanglement, but constitute the sole source of momentum-mode entanglement.

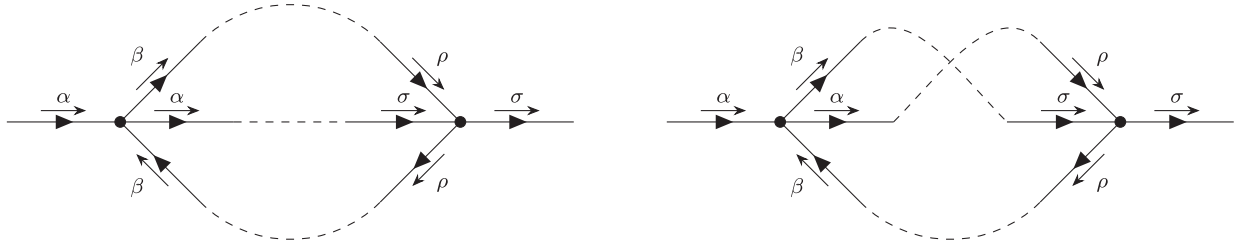


FIG. 5. Two contractions of the order g^2 two-point contribution to the low-energy effective action, distinguished by Dirac indices and denoted $S_\mu^{\beta,(2,1)}$ and $S_\mu^{\beta,(2,2)}$, respectively.

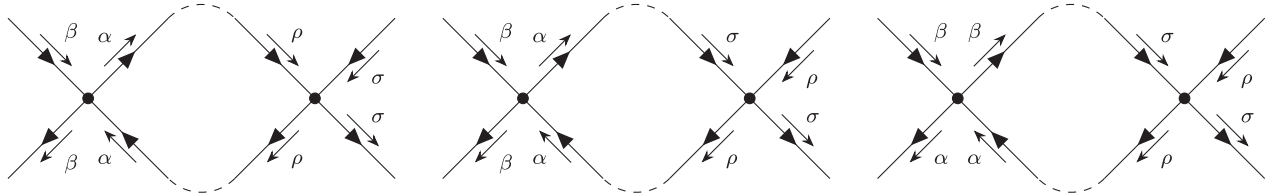


FIG. 6. Three contractions of the order g^2 four-point contribution to the low-energy effective action, distinguished by Dirac indices and denoted $S_\mu^{\beta,(4,1)}$, $S_\mu^{\beta,(4,2)}$ and $S_\mu^{\beta,(4,3)}$, respectively.

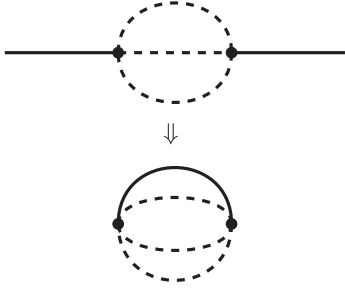


FIG. 7. Schematic contractions of the two-point effective action.

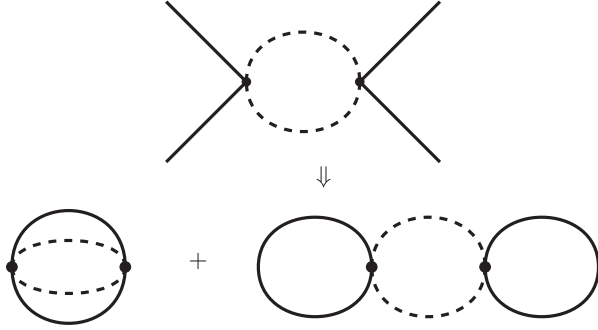


FIG. 8. Schematic contractions of the four-point effective action.

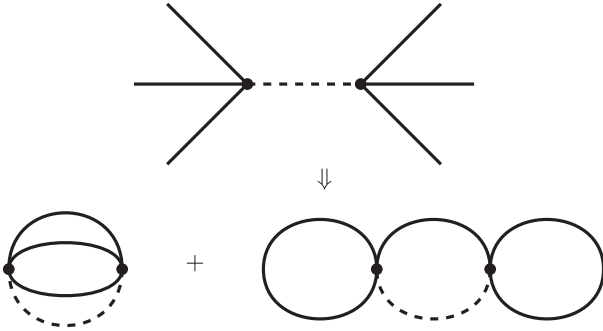


FIG. 9. Schematic contractions of the six-point effective action.

The evaluation of the expectation values in Eq. (20) corresponds to contracting the remaining low-momentum external legs in Figs. 1–4, following a procedure analogous to the contraction scheme described in Ref. [16], as shown in Figs. 7–9. It should be emphasized, however, that these diagrams serve as schematic representations; the actual computations involve the full complexity of fermionic propagators.

The four-point effective action allows two possible contractions. It turns out that the contraction shown on the right in Fig. 8 vanishes even at finite temperature, while the corresponding contraction in Fig. 9 is identically zero due to momentum conservation at the vertex.

Including all Dirac indices, the Rényi entropy can once again be expressed in terms of a set of basketball diagrams, as illustrated in Fig. 10.

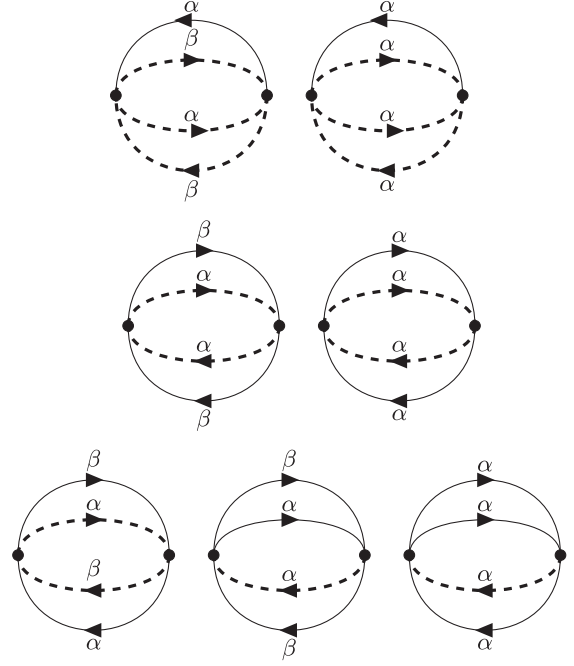


FIG. 10. Feynman diagram representation of Rényi entropies.

So far for the four fermion theory the Rényi entropy have been calculated and a similar patten as in Ref. [16] has been found, for von Neumann entropy it is explained also in Ref. [16] that the replacement of $\frac{n}{n-1}$ by $\log \frac{1}{g}$ gives the leading order contribution to von Neumann entropies.

IV. CONCLUSIONS

The methodology established in 16 provides a systematic framework for deriving entanglement entropies directly from the Wilsonian effective action. This approach offers two major advantages: (1) inherent extensibility to fermionic systems, as demonstrated in our work, and (2) systematic accessibility to higher-order corrections, particularly for Rényi entropies. We emphasize that the computational framework involves two distinct perturbative expansions: (1) Effective action derivation—Higher-order terms can be systematically incorporated through standard perturbative techniques. (2) Modified partition function evaluation—Current calculations employ first-order perturbation theory, leaving room for future higher-order extensions. This dual perturbative structure identifies two critical directions for advancing the formalism.

A key result emerges in the diagrammatic representation of Rényi entropies (Fig. 10), where “basketball” diagrams encode all possible Dirac index contractions and momentum configurations.

More specifically, the two point effective action allows two kinds of contraction of external low-momentum legs. Their respective contributions to Rényi entropy are denoted by $H_n^{2,1}(\mu)$ and $H_n^{2,2}(\mu)$, where

$$\mathcal{T}_{\eta_3\eta_5}(\mathbf{k}_3, \mathbf{k}_5) \equiv \text{tr}(\Lambda_{\mathbf{k}_3}^{\eta_3} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0)$$

and

$$\mathcal{T}_{\eta_1\eta_2\eta_3\eta_4}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \equiv \text{tr}(\Lambda_{\mathbf{k}_1}^{\eta_1} \gamma^0 \Lambda_{\mathbf{k}_2}^{\eta_2} \gamma^0 \Lambda_{\mathbf{k}_3}^{\eta_3} \gamma^0 \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0)$$

etc. $[\Lambda_{\mathbf{k}}^{\eta}]$ are projection operators defined in Eq. (A8):

$$H_n^{2,1}(\mu) = \frac{n}{n-1} 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5} \delta_{\mathbf{k}_1+\mathbf{k}_5, \mathbf{k}_3+\mathbf{k}_4} \frac{\mathcal{T}_{+-}(\mathbf{k}_4, \mathbf{k}_1) \mathcal{T}_{+-}(\mathbf{k}_3, \mathbf{k}_5) + \mathcal{T}_{-+}(\mathbf{k}_4, \mathbf{k}_1) \mathcal{T}_{-+}(\mathbf{k}_3, \mathbf{k}_5)}{(M_1 + M_3 + M_4 + M_5)^2}, \quad (21)$$

$$H_n^{2,2}(\mu) = -\frac{n}{n-1} 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5} \delta_{\mathbf{k}_1+\mathbf{k}_5, \mathbf{k}_3+\mathbf{k}_4} \frac{\mathcal{T}_{-++}(\mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_3, \mathbf{k}_1) + \mathcal{T}_{+-+}(\mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_3, \mathbf{k}_1)}{(M_1 + M_3 + M_4 + M_5)^2}, \quad (22)$$

the Feynman diagrams for these two terms are shown in Figs. 11 and 12 (1_α represent $\mathbf{k}_{1,\alpha}$ etc.).

Note that the only distinction between these two diagrams lies in the structure of the Dirac indices of the propagators; there are no other possible arrangements of these indices. A similar pattern also appears in the contributions from the four-point and six-point effective actions.

Likewise, the four-point effective action yields three distinct contributions to the Rényi entropy, denoted as $H_\mu^{4,1}(\mu)$, $H_\mu^{4,1}(\mu)$, and $H_\mu^{4,1}(\mu)$, respectively,

$$H_n^{4,1}(\mu) = \frac{n}{n-1} 2g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1\mathbf{k}_3\mathbf{k}_5\mathbf{k}_6} \delta_{\mathbf{k}_1+\mathbf{k}_6, \mathbf{k}_3+\mathbf{k}_5} \frac{\mathcal{T}_{-+}(\mathbf{k}_1, \mathbf{k}_3) \mathcal{T}_{-+}(\mathbf{k}_5, \mathbf{k}_6) + \mathcal{T}_{+-}(\mathbf{k}_1, \mathbf{k}_3) \mathcal{T}_{+-}(\mathbf{k}_5, \mathbf{k}_6)}{(M_1 + M_3 + M_5 + M_6)^2}. \quad (23)$$

The corresponding diagram is shown in Fig. 13.

Similarly, we have

$$H_n^{4,2}(\mu) = -\frac{n}{n-1} 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1\mathbf{k}_3\mathbf{k}_5\mathbf{k}_6} \delta_{\mathbf{k}_1+\mathbf{k}_6, \mathbf{k}_3+\mathbf{k}_5} \frac{\mathcal{T}_{-++}(\mathbf{k}_3, \mathbf{k}_6, \mathbf{k}_5, \mathbf{k}_1) + \mathcal{T}_{+-+}(\mathbf{k}_3, \mathbf{k}_6, \mathbf{k}_5, \mathbf{k}_1)}{(M_1 + M_3 + M_5 + M_6)^2}, \quad (24)$$

the corresponding diagram is shown in Fig. 14.

The third contribution reads:

$$H_n^{4,3}(\mu) = \frac{n}{n-1} 2g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1\mathbf{k}_3\mathbf{k}_5\mathbf{k}_6} \delta_{\mathbf{k}_1+\mathbf{k}_6, \mathbf{k}_3+\mathbf{k}_5} \frac{\mathcal{T}_{+-}(\mathbf{k}_5, \mathbf{k}_1) \mathcal{T}_{+-}(\mathbf{k}_6, \mathbf{k}_3) + \mathcal{T}_{-+}(\mathbf{k}_5, \mathbf{k}_1) \mathcal{T}_{-+}(\mathbf{k}_6, \mathbf{k}_3)}{(M_1 + M_3 + M_5 + M_6)^2}, \quad (25)$$

and the corresponding diagram is shown in Fig. 15.

For six-point effective action, we have,

$$\begin{aligned} H_n^6(\mu) &= \frac{n}{n-1} 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_7} \delta_{\mathbf{k}_1+\mathbf{k}_2, \mathbf{k}_3+\mathbf{k}_7} \frac{1}{(M_1 + M_2 + M_3 + M_7)^2} \\ &\quad \times \{ \mathcal{T}_{+-}(\mathbf{k}_3, \mathbf{k}_1) \mathcal{T}_{+-}(\mathbf{k}_7, \mathbf{k}_2) - \mathcal{T}_{-++}(\mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_7, \mathbf{k}_1) \\ &\quad + \mathcal{T}_{-+}(\mathbf{k}_3, \mathbf{k}_1) \mathcal{T}_{-+}(\mathbf{k}_7, \mathbf{k}_2) - \mathcal{T}_{-++}(\mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_7, \mathbf{k}_1) \} \end{aligned} \quad (26)$$

and its diagram is shown in Fig. 16.

Note that although we have specifically partitioned the momentum space into two subsets—above and below an arbitrary scale μ —our method and the resulting expressions are valid for any partition of the momentum space. Moreover, compared to the application of this method to the bosonic case in Ref. [16], we refine the calculation by

taking the zero-temperature limit at the final step, as detailed in the Appendix, rather than at an intermediate step as done in Ref. [16]. This procedure renders our final results more reliable and facilitates their generalization to higher-order calculations.

Finally, to interpret these structures, we propose modified Feynman rules (Figs. 17–18) incorporating a novel

$$H_n^{2,1}(\mu) = \frac{n}{n-1} \text{ (Feynman diagram of } H_n^{2,1}(\mu) \text{)}$$

FIG. 11. Feynman diagram of $H_n^{2,1}(\mu)$.

$$H_n^{2,2}(\mu) = \frac{n}{n-1} \text{ (Feynman diagram of } H_n^{2,2}(\mu) \text{)}$$

FIG. 12. Feynman diagram of $H_n^{2,2}(\mu)$.

$$H_n^{4,1}(\mu) = \frac{n}{n-1} \text{ (Feynman diagram of } H_n^{4,1}(\mu) \text{)}$$

FIG. 13. Feynman diagram of $H_n^{4,1}(\mu)$.

$$H_n^{4,2}(\mu) = \frac{n}{n-1} \text{ (Feynman diagram of } H_n^{4,2}(\mu) \text{)}$$

FIG. 14. Feynman diagram of $H_n^{4,2}(\mu)$.

$$H_n^{4,3}(\mu) = \frac{n}{n-1} \text{ (Feynman diagram of } H_n^{4,3}(\mu) \text{)}$$

FIG. 15. Feynman diagram of $H_n^{4,3}(\mu)$.

sign alternation rule for adjacent projection operators—a distinctive feature rooted in the Grassmannian nature of fermionic path integrals (see the Appendix). While this sign alternation mechanism lacks a complete theoretical interpretation, this empirical consistency across both bosonic

$$H_n^6(\mu) = \frac{n}{n-1} \text{ (Feynman diagram of } H_n^6(\mu) \text{)}$$

FIG. 16. Feynman diagram of $H_n^6(\mu)$.

$$\text{Feynman diagram of } H_n^6(\mu) = \frac{1}{V} \Lambda_{\mathbf{k}}^{\eta} \gamma^0$$

FIG. 17. Fermion propagator.

$$\text{Feynman diagram of } H_n^6(\mu) = gV \delta_{\mathbf{k}_1+\mathbf{k}_2, \mathbf{k}_3+\mathbf{k}_4} \frac{1}{M_1+M_2+M_3+M_4}$$

FIG. 18. Four-fermion vertex.

and fermionic systems suggests a universal characteristic of momentum-space entanglement. The phenomenological success of these diagrammatic rules raises fundamental questions: Are their structural similarities between scalar and fermionic theories merely coincidental, or do they reflect deeper principles governing momentum-mode entanglement? Resolving this ambiguity requires investigating higher-order corrections to test whether the observed patterns persist or not.

These open questions underscore the need for a unified theoretical framework capable of explaining the microscopic origin of entanglement signatures across diverse quantum field theories.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: MOMENTUM-SPACE ENTROPY IN $(\bar{\psi}\psi)^2$ THEORY

We first give explicitly the Feynman rules read from the Euclidean actions as follows,

- (1) enumerate all topologically equivalent diagrams;
- (2) dashed lines represent high momentum modes and solid lines represent low-momentum modes;

- (3) fermion propagators are given by $\frac{1}{\beta}S_0(K)$;
- (4) the four fermion vertex is given by $-g\frac{\beta}{V}$;
- (5) momentum conservation $\beta V\delta_{K_i, K_f}$ at each vertex where K_i is the sum of incoming momentum and K_f is the sum of outgoing momentum;
- (6) external legs are $\psi_{0,\mathbf{k}}$ and $\bar{\psi}_{0,\mathbf{k}}$;
- (7) momentum integral for high-momentum modes in the range $[\mu, \Lambda]$ and momentum integral for low-momentum modes in the range $[0, \mu]$;
- (8) extra minus sign due to fermion loop.

Figure 1 is a local contribution that does not contribute so we will not consider this term. We define the two kinds of two-point effective action as $S_\mu^{\beta,(2,1)}$ and $S_\mu^{\beta,(2,2)}$, define the three kinds of four-point effective action as $S_\mu^{\beta,(4,1)}$, $S_\mu^{\beta,(4,2)}$, $S_\mu^{\beta,(4,3)}$, and the six-point effective action as $S_\mu^{\beta,(6)}$, then from the Feynman rules above we can write down the expressions explicitly,

$$S_\mu^{\beta,(2,1)} = -\frac{1}{2} \times 2 \times 4 \times \left(g\frac{\beta}{V}\right)^2 \sum_{K_1 K_2 K_3 K_4 K_5} (-1) \text{tr} \left(\frac{S_0(K_3)}{\beta} \frac{S_0(K_5)}{\beta} \right) \bar{\psi}_{0,K_1} \frac{S_0(K_4)}{\beta} \psi_{0,K_2} \times \delta_{K_1+K_5-K_3-K_4,0} \cdot \delta_{K_3+K_4-K_2-K_5,0}, \quad (\text{A1})$$

$$S_\mu^{\beta,(2,2)} = -\frac{1}{2} \times 2 \times 4 \times \left(g\frac{\beta}{V}\right)^2 \sum_{K_1 K_2 K_3 K_4 K_5} \bar{\psi}_{0,K_1} \frac{S_0(K_4)}{\beta} \frac{S_0(K_5)}{\beta} \frac{S_0(K_3)}{\beta} \psi_{0,K_2} \times \delta_{K_1+K_5-K_3-K_4,0} \cdot \delta_{K_3+K_4-K_2-K_5,0}, \quad (\text{A2})$$

$$S_\mu^{\beta,(4,1)} = -\frac{2 \times 2}{2} \times \left(g\frac{\beta}{V}\right)^2 \sum_{K_1 K_2 K_3 K_4 K_5 K_6} \bar{\psi}_{0,K_1} \psi_{0,K_2} (-1) \text{tr} \left(\frac{S_0(K_5)}{\beta} \frac{S_0(K_6)}{\beta} \right) \bar{\psi}_{0,K_3} \psi_{0,K_4} \times \delta_{K_1+K_6-K_2-K_5,0} \cdot \delta_{K_3+K_5-K_6-K_4,0}, \quad (\text{A3})$$

$$S_\mu^{\beta,(4,2)} = -\frac{2 \times 4}{2} \times \left(g\frac{\beta}{V}\right)^2 \sum_{K_1 K_2 K_3 K_4 K_5 K_6} \bar{\psi}_{0,K_1} \psi_{0,K_2} \bar{\psi}_{0,K_3} \left(\frac{S_0(K_6)}{\beta} \frac{S_0(K_5)}{\beta} \right) \psi_{0,K_4} \times \delta_{K_1+K_6-K_2-K_5,0} \cdot \delta_{K_3+K_5-K_6-K_4,0}, \quad (\text{A4})$$

$$S_\mu^{\beta,(4,3)} = -\frac{2 \times 2}{2} \times \left(g\frac{\beta}{V}\right)^2 \sum_{K_1 K_2 K_3 K_4 K_5 K_6} \bar{\psi}_{0,K_1} \frac{S_0(K_5)}{\beta} \psi_{0,K_4} \bar{\psi}_{0,K_3} \frac{S_0(K_6)}{\beta} \psi_{0,K_2} \times \delta_{K_1+K_6-K_2-K_5,0} \cdot \delta_{K_3+K_5-K_6-K_4,0}, \quad (\text{A5})$$

$$S_\mu^{\beta,(6)} = -\frac{1}{2} \times 2 \times 4 \times \left(g\frac{\beta}{V}\right)^2 \sum_{K_1 K_2 K_3 K_4 K_5 K_6 K_7} \bar{\psi}_{0,K_1} \psi_{0,K_3} \bar{\psi}_{0,K_4} \psi_{0,K_6} \bar{\psi}_{0,K_2} \frac{S_0(K_7)}{\beta} \psi_{0,K_5} \times \delta_{K_1+K_2-K_3-K_7,0} \cdot \delta_{K_7+K_4-K_5-K_6,0}. \quad (\text{A6})$$

The above expressions for effective actions are in momentum space, in order to find the “glued” effective action, we had better transform the above expressions in Euclidean time formalism in which the mixed representation of propagators can be used to finish Matsubara summations.

For that purpose we employ the following transformations of low-momentum modes,

$$\begin{aligned} \psi_{0,K} &= T \int_0^\beta d\tau e^{-i\omega_j \tau} \psi_{0,\mathbf{k}}(\tau), & \psi_{0,\mathbf{k}}(\tau) &= \sum_j e^{i\omega_j \tau} \psi_{0,K}, \\ \bar{\psi}_{0,K} &= T \int_0^\beta d\tau e^{i\omega_j \tau} \bar{\psi}_{0,\mathbf{k}}(\tau), & \bar{\psi}_{0,\mathbf{k}}(\tau) &= \sum_j e^{-i\omega_j \tau} \bar{\psi}_{0,K}. \end{aligned} \quad (\text{A7})$$

Besides, let us review the mixed representation of fermions propagators; define projection operators as follows,

$$\Lambda_{\mathbf{k}}^{\pm} \equiv \frac{M \pm \gamma^0(\boldsymbol{\gamma} \cdot \mathbf{k} + m)}{2M} \quad (\text{A8})$$

Then we can define the mixed representation of fermion propagator as

then the thermal fermion propagator can be written as

$$\begin{aligned} \mathcal{S}_0(K) &= \frac{k_0 \gamma^0 - \boldsymbol{\gamma} \cdot \mathbf{k} + m}{(M - k_0)(M + k_0)} \\ &= -\sum_{\eta=\pm} \frac{1}{k_0 - \eta M} \Lambda_{\mathbf{k}}^{\eta} \gamma^0 \\ &= \sum_{\eta=\pm} \mathcal{S}_0^{\eta}(K) \Lambda_{\mathbf{k}}^{\eta} \gamma^0. \end{aligned} \quad (\text{A9})$$

where

$$\begin{aligned} \bar{\mathcal{S}}_0^{\eta}(\tau, M) &= T \sum_j e^{i\omega_j \tau} \mathcal{S}_0^{\eta}(K), \\ \mathcal{S}_0^{\eta}(K) &= \int_0^{\beta} d\tau e^{-i\omega_j \tau} \bar{\mathcal{S}}_0^{\eta}(\tau, M) \end{aligned} \quad (\text{A10})$$

$$\bar{\mathcal{S}}_0^{\eta}(\tau, M) = e^{-\eta M \tau} [1 - n_F(\eta M)]. \quad (\text{A11})$$

Substituting the mixed representation into effective action $S_{\mu}^{\beta, (2,1)}$, we get

$$\begin{aligned} S_{\mu}^{\beta, (2,1)} &= 4g^2 \frac{1}{\beta V^2} \sum_{K_1 \rightarrow K_5} \delta_{K_1+K_5-K_3-K_4,0} \cdot \delta_{K_3+K_4-K_2-K_5,0} \sum_{\eta_3 \eta_4 \eta_5} \mathcal{S}_0^{\eta_3}(K_3) \mathcal{S}_0^{\eta_4}(K_4) \mathcal{S}_0^{\eta_5}(K_5) \\ &\quad \times \text{tr}(\Lambda_{\mathbf{k}_3}^{\eta_3} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0) \bar{\psi}_{0,K_1} \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \psi_{0,K_2}. \end{aligned} \quad (\text{A12})$$

Define $\mathcal{T}_{\eta_3 \eta_5}(\mathbf{k}_3, \mathbf{k}_5) \equiv \text{tr}(\Lambda_{\mathbf{k}_3}^{\eta_3} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0)$, then substitute the transformation Eq. (A7) in, we have

$$\begin{aligned} S_{\mu}^{\beta, (2,1)} &= 4g^2 \frac{1}{\beta^3 V^2} \sum_{K_1 \rightarrow K_5} \delta_{K_1+K_5-K_3-K_4,0} \cdot \delta_{K_3+K_4-K_2-K_5,0} \sum_{\eta_3 \eta_4 \eta_5} \int_0^{\beta} d\tau_1 \cdots d\tau_5 \\ &\quad \times \bar{\mathcal{S}}_0^{\eta_3}(\tau_3, M_3) \bar{\mathcal{S}}_0^{\eta_4}(\tau_4, M_4) \bar{\mathcal{S}}_0^{\eta_5}(\tau_5, M_5) \mathcal{T}_{\eta_3 \eta_5}(\mathbf{k}_3, \mathbf{k}_5) \bar{\psi}_{0,\mathbf{k}_1}(\tau_1) \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \psi_{0,\mathbf{k}_2}(\tau_2) \\ &\quad \times e^{i\omega_{j_1} \tau_1 - i\omega_{j_2} \tau_2 - i\omega_{j_3} \tau_3 - i\omega_{j_4} \tau_4 - i\omega_{j_5} \tau_5}. \end{aligned}$$

Now the Matsubara frequencies summations can be easily performed, and we arrive at a formula that is suitable for replica tricks as follows,

$$S_{\mu}^{\beta, (2,1)} = 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_5} \delta_{\mathbf{k}_1+\mathbf{k}_5, \mathbf{k}_3+\mathbf{k}_4} \cdot \delta_{\mathbf{k}_1, \mathbf{k}_2} \sum_{\eta_3 \eta_4 \eta_5} (I_1^{(2,1)} + I_2^{(2,1)})$$

where

$$\begin{aligned} I_1^{(2,1)} &\equiv -\int_0^{\beta} d\tau_1 \int_0^{\tau_1} d\tau_2 \bar{\mathcal{S}}_0^{\eta_3}(\tau_1 - \tau_2, M_3) \bar{\mathcal{S}}_0^{\eta_4}(\tau_1 - \tau_2, M_4) \bar{\mathcal{S}}_0^{\eta_5}(\tau_1 - \tau_2, M_5) \mathcal{T}_{\eta_3 \eta_5}(\mathbf{k}_3, \mathbf{k}_5) \\ &\quad \times \bar{\psi}_{0,\mathbf{k}_1}(\tau_1) \cdot \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \cdot \psi_{0,\mathbf{k}_2}(\tau_2), \\ I_2^{(2,1)} &\equiv \int_0^{\beta} d\tau_1 \int_{\tau_1}^{\beta} d\tau_2 \bar{\mathcal{S}}_0^{\eta_3}(\tau_2 - \tau_1, M_3) \bar{\mathcal{S}}_0^{\eta_4}(\tau_2 - \tau_1, M_4) \bar{\mathcal{S}}_0^{\eta_5}(\tau_2 - \tau_1, M_5) \mathcal{T}_{\eta_3 \eta_5}(\mathbf{k}_3, \mathbf{k}_5) \\ &\quad \times \bar{\psi}_{0,\mathbf{k}_1}(\tau_1) \cdot \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \cdot \psi_{0,\mathbf{k}_2}(\tau_2). \end{aligned}$$

In the same vein we have the expressions for other effective actions as follows,

$$S_{\mu}^{\beta, (2,2)} = -4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_5} \delta_{\mathbf{k}_1+\mathbf{k}_5, \mathbf{k}_3+\mathbf{k}_4} \cdot \delta_{\mathbf{k}_1, \mathbf{k}_2} \sum_{\eta_3 \eta_4 \eta_5} \{I_1^{(2,2)} + I_2^{(2,2)}\} \quad (\text{A13})$$

where

$$\begin{aligned} I_1^{(2,2)} &\equiv -\int_0^{\beta} d\tau_1 \int_0^{\tau_1} d\tau_2 \bar{\psi}_{0,\mathbf{k}_1}(\tau_1) \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \Lambda_{\mathbf{k}_3}^{\eta_3} \gamma^0 \psi_{0,\mathbf{k}_2}(\tau_2) \\ &\quad \times \bar{\mathcal{S}}_0^{\eta_3}(\tau_1 - \tau_2, M_3) \bar{\mathcal{S}}_0^{\eta_4}(\tau_1 - \tau_2, M_4) \bar{\mathcal{S}}_0^{\eta_5}(\tau_1 - \tau_2, M_5), \\ I_2^{(2,2)} &\equiv \int_0^{\beta} d\tau_1 \int_{\tau_1}^{\beta} d\tau_2 \bar{\psi}_{0,\mathbf{k}_1}(\tau_1) \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \Lambda_{\mathbf{k}_3}^{\eta_3} \gamma^0 \psi_{0,\mathbf{k}_2}(\tau_2) \\ &\quad \times \bar{\mathcal{S}}_0^{\eta_3}(\tau_2 - \tau_1, M_3) \bar{\mathcal{S}}_0^{\eta_4}(\tau_2 - \tau_1, M_4) \bar{\mathcal{S}}_0^{\eta_5}(\tau_2 - \tau_1, M_5). \end{aligned} \quad (\text{A14})$$

The expression for $S_\mu^{\beta,(4,1)}$ reads

$$S_\mu^{\beta,(4,1)} = 2 \frac{g^2}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_6} \delta_{\mathbf{k}_1 + \mathbf{k}_6, \mathbf{k}_2 + \mathbf{k}_5} \delta_{\mathbf{k}_3 + \mathbf{k}_5, \mathbf{k}_4 + \mathbf{k}_6} \sum_{\eta_5 \eta_6} \{I_1^{(4,1)} + I_2^{(4,1)}\} \quad (\text{A15})$$

where

$$\begin{aligned} I_1^{(4,1)} &\equiv - \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_4 \bar{\psi}_{0,\mathbf{k}_1}(\tau_1) \psi_{0,\mathbf{k}_2}(\tau_1) \bar{\psi}_{0,\mathbf{k}_3}(\tau_4) \psi_{0,\mathbf{k}_4}(\tau_4) \mathcal{T}_{\eta_5 \eta_6}(\mathbf{k}_5, \mathbf{k}_6) \\ &\quad \times \bar{\mathcal{S}}_0^{\eta_5}(\tau_1 - \tau_4, M_5) \bar{\mathcal{S}}_0^{-\eta_6}(\tau_1 - \tau_4, M_6), \\ I_2^{(4,1)} &\equiv - \int_0^\beta d\tau_1 \int_{\tau_1}^\beta d\tau_4 \bar{\psi}_{0,\mathbf{k}_1}(\tau_1) \psi_{0,\mathbf{k}_2}(\tau_1) \bar{\psi}_{0,\mathbf{k}_3}(\tau_4) \psi_{0,\mathbf{k}_4}(\tau_4) \mathcal{T}_{\eta_5 \eta_6}(\mathbf{k}_5, \mathbf{k}_6) \\ &\quad \times \bar{\mathcal{S}}_0^{-\eta_5}(\tau_4 - \tau_1, M_5) \bar{\mathcal{S}}_0^{\eta_6}(\tau_4 - \tau_1, M_6). \end{aligned} \quad (\text{A16})$$

The expression for $S_\mu^{\beta,(4,2)}$ reads

$$S_\mu^{\beta,(4,2)} = 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_6} \delta_{\mathbf{k}_1 + \mathbf{k}_6, \mathbf{k}_2 + \mathbf{k}_5} \delta_{\mathbf{k}_3 + \mathbf{k}_5, \mathbf{k}_4 + \mathbf{k}_6} \sum_{\eta_5 \eta_6} \{I_1^{(4,2)} + I_2^{(4,2)}\}, \quad (\text{A17})$$

where

$$\begin{aligned} I_1^{(4,2)} &= \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_4 \bar{\psi}_{0,\mathbf{k}_1}(\tau_1) \psi_{0,\mathbf{k}_2}(\tau_1) \bar{\psi}_{0,\mathbf{k}_3}(\tau_4) \Lambda_{\mathbf{k}_6}^{\eta_6} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \psi_{0,\mathbf{k}_4}(\tau_4) \\ &\quad \times \bar{\mathcal{S}}_0^{-\eta_6}(\tau_1 - \tau_4, M_6) \bar{\mathcal{S}}_0^{\eta_5}(\tau_1 - \tau_4, M_5), \\ I_2^{(4,2)} &= \int_0^\beta d\tau_1 \int_{\tau_1}^\beta d\tau_4 \bar{\psi}_{0,\mathbf{k}_1}(\tau_1) \psi_{0,\mathbf{k}_2}(\tau_1) \bar{\psi}_{0,\mathbf{k}_3}(\tau_4) \Lambda_{\mathbf{k}_6}^{\eta_6} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \psi_{0,\mathbf{k}_4}(\tau_4) \\ &\quad \times \bar{\mathcal{S}}_0^{\eta_6}(\tau_4 - \tau_1, M_6) \bar{\mathcal{S}}_0^{-\eta_5}(\tau_4 - \tau_1, M_5). \end{aligned} \quad (\text{A18})$$

The expression for $S_\mu^{\beta,(4,3)}$ reads

$$S_\mu^{\beta,(4,3)} = 2g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_6} \delta_{\mathbf{k}_1 + \mathbf{k}_6, \mathbf{k}_2 + \mathbf{k}_5} \delta_{\mathbf{k}_3 + \mathbf{k}_5, \mathbf{k}_4 + \mathbf{k}_6} \sum_{\eta_5 \eta_6} \{I_1^{(4,3)} + I_2^{(4,3)}\} \quad (\text{A19})$$

where

$$\begin{aligned} I_1^{(4,3)} &\equiv \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_4 \bar{\psi}_{0,\mathbf{k}_1}(\tau_1) \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \psi_{0,\mathbf{k}_4}(\tau_4) \bar{\psi}_{0,\mathbf{k}_3}(\tau_4) \Lambda_{\mathbf{k}_6}^{\eta_6} \gamma^0 \psi_{0,\mathbf{k}_2}(\tau_1) \\ &\quad \times \bar{\mathcal{S}}_0^{-\eta_6}(\tau_1 - \tau_4, M_6) \bar{\mathcal{S}}_0^{\eta_5}(\tau_1 - \tau_4, M_5), \\ I_2^{(4,3)} &\equiv \int_0^\beta d\tau_1 \int_{\tau_1}^\beta d\tau_4 \bar{\psi}_{0,\mathbf{k}_1}(\tau_1) \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \psi_{0,\mathbf{k}_4}(\tau_4) \bar{\psi}_{0,\mathbf{k}_3}(\tau_4) \Lambda_{\mathbf{k}_6}^{\eta_6} \gamma^0 \psi_{0,\mathbf{k}_2}(\tau_1) \\ &\quad \times \bar{\mathcal{S}}_0^{\eta_6}(\tau_4 - \tau_1, M_6) \bar{\mathcal{S}}_0^{-\eta_5}(\tau_4 - \tau_1, M_5) \end{aligned} \quad (\text{A20})$$

The expression for $S_\mu^{\beta,(6)}$ reads

$$S_\mu^{\beta,(6)} = -4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_7} \delta_{\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_3 + \mathbf{k}_7} \cdot \delta_{\mathbf{k}_7 + \mathbf{k}_4, \mathbf{k}_5 + \mathbf{k}_6} \sum_{\eta_7} \{I_1^{(6)} + I_2^{(6)}\} \quad (\text{A21})$$

where

$$\begin{aligned}
I_1^{(6)} &\equiv \int_0^\beta d\tau_2 \int_0^{\tau_2} d\tau_5 \bar{\psi}_{0,\mathbf{k}_1}(\tau_2) \psi_{0,\mathbf{k}_3}(\tau_2) \bar{\psi}_{0,\mathbf{k}_4}(\tau_5) \psi_{0,\mathbf{k}_6}(\tau_5) \bar{\psi}_{0,\mathbf{k}_2}(\tau_2) \Lambda_{\mathbf{k}_7}^{\eta_7} \gamma^0 \psi_{0,\mathbf{k}_5}(\tau_5) \\
&\quad \times \tilde{S}_0^{\eta_7}(\tau_2 - \tau_5, \mathbf{k}_7), \\
I_2^{(6)} &\equiv - \int_0^\beta d\tau_2 \int_{\tau_2}^\beta d\tau_5 \bar{\psi}_{0,\mathbf{k}_1}(\tau_2) \psi_{0,\mathbf{k}_3}(\tau_2) \bar{\psi}_{0,\mathbf{k}_4}(\tau_5) \psi_{0,\mathbf{k}_6}(\tau_5) \bar{\psi}_{0,\mathbf{k}_2}(\tau_2) \Lambda_{\mathbf{k}_7}^{\eta_7} \gamma^0 \psi_{0,\mathbf{k}_5}(\tau_5) \\
&\quad \times \tilde{S}_0^{-\eta_7}(\tau_5 - \tau_2, \mathbf{k}_7).
\end{aligned} \tag{A22}$$

Then we can proceed to calculate the “glued” effective action, the essential part is the following sum of integrations,

$$\begin{aligned}
A_1(\omega_j, \omega_{j'}, M) &= \frac{1}{n\beta} \sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau \int_{u\beta}^\tau d\tau' e^{i\omega_j \tau + i\omega_{j'} \tau'} e^{-M(\tau - \tau')} \\
&= \delta_{j+j'}^0 \frac{1}{M + i\omega_{j'}} + \frac{1}{n\beta} \cdot n \sum_\nu \delta_{j+j'}^{\nu\nu} \frac{e^{(i\omega_j - M)\beta} - 1}{(M + i\omega_{j'})(M - i\omega_j)} \\
&\equiv B_1(\omega_j, \omega_{j'}, M) + C_1(\omega_j, \omega_{j'}, M), \\
A_2(\omega_j, \omega_{j'}, M) &= \frac{1}{n\beta} \sum_{u=0}^{n-1} \int_{u\beta}^{(u+1)\beta} d\tau \int_\tau^{(u+1)\beta} d\tau' e^{i\omega_j \tau + i\omega_{j'} \tau'} e^{-M(\tau' - \tau)} \\
&= \delta_{j+j'}^0 \frac{1}{M - i\omega_{j'}} + \frac{1}{n\beta} \cdot n \sum_\nu \delta_{j+j'}^{\nu\nu} \frac{e^{(i\omega_{j'} - M)\beta} - 1}{(M + i\omega_j)(M - i\omega_{j'})} \\
&\equiv B_2(\omega_j, \omega_{j'}, M) + C_2(\omega_j, \omega_{j'}, M).
\end{aligned} \tag{A23}$$

In the above formula we have defined the local part as B and nonlocal part as C and we will see that the part B in “glued” effective action coincides with the original effective action at temperature $n\beta$.

Recall that after replica, the low-momentum modes are antiperiodic in the Euclidean time range $[0, n\beta]$, so we can transform the low-momentum modes as,

$$\psi_{0,\mathbf{k}}(\tau) = \sum_j e^{i\omega_j \tau} \psi_{0,K}, \quad \bar{\psi}_{0,\mathbf{k}}(\tau) = \sum_j e^{-i\omega_j \tau} \bar{\psi}_{0,K}, \tag{A24}$$

now we have $\omega_j = \frac{(2j+1)\pi}{n\beta}$, using these tools, we arrive at the form of the “glued” effective action.

The expression for $S_{\mu,r}^{\beta,(2,1)}$ reads,

$$S_{\mu,r}^{\beta,(2,1)} \equiv 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_5} \delta_{\mathbf{k}_1 + \mathbf{k}_5, \mathbf{k}_3 + \mathbf{k}_4} \cdot \delta_{\mathbf{k}_1, \mathbf{k}_2} \sum_{\eta_3 \eta_4 \eta_5} (I_{1,r}^{(2,1)} + I_{2,r}^{(2,1)}) \tag{A25}$$

where

$$\begin{aligned}
I_{1,r}^{(2,1)} &= - \sum_{j_1, j_2} \bar{\psi}_{0,K_1} \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \psi_{0,K_2} \mathcal{T}_{\eta_3 \eta_5}(\mathbf{k}_3, \mathbf{k}_5) [1 - n_F(\eta_3 M_3)] [1 - n_F(\eta_4 M_4)] n_F(\eta_5 M_5) \\
&\quad \times (n\beta) A_1(-\omega_{j_1}, \omega_{j_2}, \eta_3 M_3 + \eta_4 M_4 - \eta_5 M_5), \\
I_{2,r}^{(2,1)} &= \sum_{j_1, j_2} \bar{\psi}_{0,K_1} \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \psi_{0,K_2} \mathcal{T}_{\eta_3 \eta_5}(\mathbf{k}_3, \mathbf{k}_5) n_F(\eta_3 M_3) n_F(\eta_4 M_4) [1 - n_F(\eta_5 M_5)] \\
&\quad \times (n\beta) A_2(-\omega_{j_1}, \omega_{j_2}, -\eta_3 M_3 - \eta_4 M_4 + \eta_5 M_5).
\end{aligned} \tag{A26}$$

The expression for $S_{\mu,r}^{\beta,(2,2)}$ reads,

$$S_{\mu,r}^{\beta,(2,2)} \equiv -4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_5} \delta_{\mathbf{k}_1 + \mathbf{k}_5, \mathbf{k}_3 + \mathbf{k}_4} \cdot \delta_{\mathbf{k}_1, \mathbf{k}_2} \sum_{\eta_3 \eta_4 \eta_5} (I_{1,r}^{(2,2)} + I_{2,r}^{(2,2)}) \tag{A27}$$

where

$$\begin{aligned}
 I_{1,r}^{(2,2)} &\equiv - \sum_{j_1 j_2} \bar{\psi}_{0,K_1} \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \Lambda_{\mathbf{k}_3}^{\eta_3} \gamma^0 \psi_{0,K_2} [1 - n_F(\eta_3 M_3)] [1 - n_F(\eta_4 M_4)] n_F(\eta_5 M_5) \\
 &\quad \times (n\beta) A_1(-\omega_{j_1}, \omega_{j_2}, \eta_3 M_3 + \eta_4 M_4 - \eta_5 M_5), \\
 I_{2,r}^{(2,2)} &\equiv \sum_{j_1 j_2} \bar{\psi}_{0,K_1} \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \Lambda_{\mathbf{k}_3}^{\eta_3} \gamma^0 \psi_{0,K_2} n_F(\eta_3 M_3) n_F(\eta_4 M_4) [1 - n_F(\eta_5 M_5)] \\
 &\quad \times (n\beta) A_2(-\omega_{j_1}, \omega_{j_2}, -\eta_3 M_3 - \eta_4 M_4 + \eta_5 M_5).
 \end{aligned} \tag{A28}$$

The expression for $S_{\mu,r}^{\beta,(4,1)}$ reads,

$$S_{\mu,r}^{\beta,(4,1)} = 2 \frac{g^2}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_6} \delta_{\mathbf{k}_1 + \mathbf{k}_6, \mathbf{k}_2 + \mathbf{k}_5} \delta_{\mathbf{k}_3 + \mathbf{k}_5, \mathbf{k}_4 + \mathbf{k}_6} \sum_{\eta_5 \eta_6} \{I_{1,r}^{(4,1)} + I_{2,r}^{(4,1)}\} \tag{A29}$$

where

$$\begin{aligned}
 I_{1,r}^{(4,1)} &\equiv - \sum_{j_1 j_2 j_3 j_4} \bar{\psi}_{0,K_1} \psi_{0,K_2} \bar{\psi}_{0,K_3} \psi_{0,K_4} \mathcal{T}_{\eta_5 \eta_6}(\mathbf{k}_5, \mathbf{k}_6) [1 - n_F(\eta_5 M_5)] n_F(\eta_6 M_6) \\
 &\quad \times (n\beta) A_1(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_5 M_5 - \eta_6 M_6), \\
 I_{2,r}^{(4,1)} &\equiv - \sum_{j_1 j_2 j_3 j_4} \bar{\psi}_{0,K_1} \psi_{0,K_2} \bar{\psi}_{0,K_3} \psi_{0,K_4} \mathcal{T}_{\eta_5 \eta_6}(\mathbf{k}_5, \mathbf{k}_6) n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] \\
 &\quad \times (n\beta) A_2(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, -\eta_5 M_5 + \eta_6 M_6).
 \end{aligned} \tag{A30}$$

The expression for $S_{\mu,r}^{\beta,(4,2)}$ read,

$$S_{\mu,r}^{\beta,(4,2)} = 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_6} \delta_{\mathbf{k}_1 + \mathbf{k}_6, \mathbf{k}_2 + \mathbf{k}_5} \delta_{\mathbf{k}_3 + \mathbf{k}_5, \mathbf{k}_4 + \mathbf{k}_6} \sum_{\eta_5 \eta_6} \{I_{1,r}^{(4,2)} + I_{2,r}^{(4,2)}\} \tag{A31}$$

where

$$\begin{aligned}
 I_{1,r}^{(4,2)} &= \sum_{j_1 \rightarrow j_4} \bar{\psi}_{0,K_1} \psi_{0,K_2} \bar{\psi}_{0,K_3} \Lambda_{\mathbf{k}_6}^{\eta_6} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \psi_{0,K_4} n_F(\eta_6 M_6) [1 - n_F(\eta_5 M_5)] \\
 &\quad \times (n\beta) A_1(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_5 M_5 - \eta_6 M_6), \\
 I_{2,r}^{(4,2)} &= \sum_{j_1 \rightarrow j_4} \bar{\psi}_{0,K_1} \psi_{0,K_2} \bar{\psi}_{0,K_3} \Lambda_{\mathbf{k}_6}^{\eta_6} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \psi_{0,K_4} n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] \\
 &\quad \times (n\beta) A_2(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_6 M_6 - \eta_5 M_5).
 \end{aligned} \tag{A32}$$

The expression for $S_{\mu,r}^{\beta,(4,3)}$ reads,

$$S_{\mu,r}^{\beta,(4,3)} = 2g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_6} \delta_{\mathbf{k}_1 + \mathbf{k}_6, \mathbf{k}_2 + \mathbf{k}_5} \delta_{\mathbf{k}_3 + \mathbf{k}_5, \mathbf{k}_4 + \mathbf{k}_6} \sum_{\eta_5 \eta_6} \{I_{1,r}^{(4,3)} + I_{2,r}^{(4,3)}\} \tag{A33}$$

where

$$\begin{aligned}
 I_{1,r}^{(4,3)} &\equiv \sum_{j_1 \rightarrow j_4} \bar{\psi}_{0,K_1} \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \psi_{0,K_4} \bar{\psi}_{0,K_3} \Lambda_{\mathbf{k}_6}^{\eta_6} \gamma^0 \psi_{0,K_2} n_F(\eta_6 M_6) [1 - n_F(\eta_5 M_5)] \\
 &\quad \times (n\beta) A_1(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_5 M_5 - \eta_6 M_6), \\
 I_{2,r}^{(4,3)} &\equiv \sum_{j_1 \rightarrow j_4} \bar{\psi}_{0,K_1} \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \psi_{0,K_4} \bar{\psi}_{0,K_3} \Lambda_{\mathbf{k}_6}^{\eta_6} \gamma^0 \psi_{0,K_2} n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] \\
 &\quad \times (n\beta) A_2(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_6 M_6 - \eta_5 M_5).
 \end{aligned} \tag{A34}$$

The expression for $S_{\mu,r}^{\beta,(6)}$ reads,

$$S_{\mu,r}^{\beta,(6)} = -4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_7} \delta_{\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_3 + \mathbf{k}_7} \cdot \delta_{\mathbf{k}_7 + \mathbf{k}_4, \mathbf{k}_5 + \mathbf{k}_6} \sum_{\eta_7} \{I_{1,r}^{(6)} + I_{2,r}^{(6)}\} \quad (\text{A35})$$

where

$$\begin{aligned} I_{1,r}^{(6)} &\equiv \sum_{j_1 \rightarrow j_6} \bar{\psi}_{0,K_1} \psi_{0,K_3} \bar{\psi}_{0,K_4} \psi_{0,K_6} \bar{\psi}_{0,K_2} \Lambda_{\mathbf{k}_7}^{\eta_7} \gamma^0 \psi_{0,K_5} [1 - n_F(\eta_7 M_7)] \\ &\quad \times (n\beta) A_1(-\omega_{j_1} - \omega_{j_2} + \omega_{j_3}, -\omega_{j_4} + \omega_{j_5} + \omega_{j_6}, \eta_7 M_7), \\ I_{2,r}^{(6)} &\equiv - \sum_{j_1 \rightarrow j_6} \bar{\psi}_{0,K_1} \psi_{0,K_3} \bar{\psi}_{0,K_4} \psi_{0,K_6} \bar{\psi}_{0,K_2} \Lambda_{\mathbf{k}_7}^{\eta_7} \gamma^0 \psi_{0,K_5} n_F(\eta_7 M_7) \\ &\quad \times (n\beta) A_2(-\omega_{j_1} - \omega_{j_2} + \omega_{j_3}, -\omega_{j_4} + \omega_{j_5} + \omega_{j_6}, -\eta_7 M_7). \end{aligned} \quad (\text{A36})$$

Recall that

$$\begin{aligned} H_n(\mu) &= \frac{1}{n-1} \lim_{\beta \rightarrow \infty} (\langle S_{\mu,r}^{n\beta,(2,1)} \rangle + \langle S_{\mu,r}^{n\beta,(2,2)} \rangle + \langle S_{\mu,r}^{n\beta,(4,1)} \rangle + \langle S_{\mu,r}^{n\beta,(4,2)} \rangle + \langle S_{\mu,r}^{n\beta,(4,3)} \rangle + \langle S_{\mu,r}^{n\beta,(6)} \rangle \\ &\quad - \langle S_{\mu}^{n\beta,(2,1)} \rangle - \langle S_{\mu}^{n\beta,(2,2)} \rangle - \langle S_{\mu}^{n\beta,(4,1)} \rangle - \langle S_{\mu}^{n\beta,(4,2)} \rangle - \langle S_{\mu}^{n\beta,(4,3)} \rangle - \langle S_{\mu}^{n\beta,(6)} \rangle); \end{aligned} \quad (\text{A37})$$

now we prove the claim that the local part in the “glued” effective action coincides with the original effective action at temperature $n\beta$. Comparing these two terms, we find that what we need to prove is actually the following identities.

$$\begin{aligned} &\sum_{j_7} \delta_{j_1+j_2, j_3+j_7} \delta_{j_4+j_7, j_5+j_6} \mathcal{S}_0(K_7) \\ &= \sum_{\eta_7} \{ [1 - n_F(\eta_7 M_7)] B_1(-\omega_{j_1} - \omega_{j_2} + \omega_{j_3}, -\omega_{j_4} + \omega_{j_5} + \omega_{j_6}, \eta_7 M_7) \\ &\quad + n_F(\eta_7 M_7) B_2(-\omega_{j_1} - \omega_{j_2} + \omega_{j_3}, -\omega_{j_4} + \omega_{j_5} + \omega_{j_6}, -\eta_7 M_7) \} \Lambda_{\mathbf{k}_7}^{\eta_7} \gamma^0, \\ &\quad - \sum_{j_5, j_6} \delta_{j_1+j_6, j_2+j_5} \delta_{j_3+j_5, j_4+j_6} \mathcal{S}_0^{\eta_6}(K_6) \mathcal{S}_0^{\eta_5}(K_5) \\ &= (n\beta) \{ n_F(\eta_6 M_6) [1 - n_F(\eta_5 M_5)] B_1(\omega_{j_2} - \omega_{j_1}, \omega_{j_4} - \omega_{j_3}, \eta_5 M_5 - \eta_6 M_6) \\ &\quad + n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] B_2(\omega_{j_2} - \omega_{j_1}, \omega_{j_4} - \omega_{j_3}, \eta_6 M_6 - \eta_5 M_5) \}, \\ &\quad \frac{1}{(n\beta)^2} \sum_{j_3 j_4 j_5} \delta_{j_1+j_5, j_3+j_4} \delta_{j_1, j_2} \mathcal{S}_0^{\eta_3}(K_3) \mathcal{S}_0^{\eta_4}(K_4) \mathcal{S}_0^{\eta_5}(K_5) \\ &= -[1 - n_F(\eta_3 M_3)] [1 - n_F(\eta_4 M_4)] n_F(\eta_5 M_5) \times B_1(-\omega_{j_1}, \omega_{j_2}, \eta_3 M_3 + \eta_4 M_4 - \eta_5 M_5) \\ &\quad + n_F(\eta_3 M_3) n_F(\eta_4 M_4) [1 - n_F(\eta_5 M_5)] \times B_2(-\omega_{j_1}, \omega_{j_2}, -\eta_3 M_3 - \eta_4 M_4 + \eta_5 M_5). \end{aligned} \quad (\text{A38})$$

These identities can be proved easily using the mixed representation of propagators.

Now we can write Rényi entropies as

$$H_n(\mu) \equiv H_n^{2,1}(\mu) + H_n^{2,2}(\mu) + H_n^{4,1}(\mu) + H_n^{4,2}(\mu) + H_n^{4,3}(\mu) + H_n^6(\mu) \quad (\text{A39})$$

where

$$\begin{aligned} H_n^{2,1}(\mu) &= \frac{1}{n-1} \lim_{\beta \rightarrow \infty} 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_5} \delta_{\mathbf{k}_1 + \mathbf{k}_5, \mathbf{k}_3 + \mathbf{k}_4} \delta_{\mathbf{k}_1, \mathbf{k}_2} \sum_{\eta_3 \eta_4 \eta_5} \sum_{j_1 j_2} \langle \bar{\psi}_{0,K_1} \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \psi_{0,K_2} \rangle \mathcal{T}_{\eta_3 \eta_5}(\mathbf{k}_3, \mathbf{k}_5) \\ &\quad \times (n\beta) \{ n_F(\eta_3 M_3) n_F(\eta_4 M_4) [1 - n_F(\eta_5 M_5)] C_2(-\omega_{j_1}, \omega_{j_2}, -\eta_3 M_3 - \eta_4 M_4 + \eta_5 M_5) \\ &\quad - [1 - n_F(\eta_3 M_3)] [1 - n_F(\eta_4 M_4)] n_F(\eta_5 M_5) C_1(-\omega_{j_1}, \omega_{j_2}, \eta_3 M_3 + \eta_4 M_4 - \eta_5 M_5) \}, \end{aligned} \quad (\text{A40})$$

$$\begin{aligned}
 H_n^{2,2}(\mu) = & -\frac{1}{n-1} \lim_{\beta \rightarrow \infty} 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_5} \delta_{\mathbf{k}_1+\mathbf{k}_5, \mathbf{k}_3+\mathbf{k}_4} \delta_{\mathbf{k}_1, \mathbf{k}_2} \sum_{\eta_3 \eta_4 \eta_5} \sum_{j_1 j_2} \langle \bar{\psi}_{0,K_1} \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \Lambda_{\mathbf{k}_3}^{\eta_3} \gamma^0 \psi_{0,K_2} \rangle \\
 & \times (n\beta) \{ n_F(\eta_3 M_3) n_F(\eta_4 M_4) [1 - n_F(\eta_5 M_5)] C_2(-\omega_{j_1}, \omega_{j_2}, -\eta_3 M_3 - \eta_4 M_4 + \eta_5 M_5) \\
 & - [1 - n_F(\eta_3 M_3)] [1 - n_F(\eta_4 M_4)] n_F(\eta_5 M_5) C_1(-\omega_{j_1}, \omega_{j_2}, \eta_3 M_3 + \eta_4 M_4 - \eta_5 M_5) \}, \quad (A41)
 \end{aligned}$$

$$\begin{aligned}
 H_n^{4,1}(\mu) = & \frac{1}{n-1} \lim_{\beta \rightarrow \infty} 2g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_6} \delta_{\mathbf{k}_1+\mathbf{k}_6, \mathbf{k}_2+\mathbf{k}_5} \delta_{\mathbf{k}_3+\mathbf{k}_5, \mathbf{k}_4+\mathbf{k}_6} \sum_{\eta_5 \eta_6} \sum_{j_1 \rightarrow j_4} \langle \bar{\psi}_{0,K_1} \psi_{0,K_2} \bar{\psi}_{0,K_3} \psi_{0,K_4} \rangle \\
 & \times \{ [1 - n_F(\eta_5 M_5)] n_F(\eta_6 M_6) C_1(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_5 M_5 - \eta_6 M_6) \\
 & + n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] C_2(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, -\eta_5 M_5 + \eta_6 M_6) \} \\
 & \times (-n\beta) \mathcal{T}_{\eta_5 \eta_6}(\mathbf{k}_5, \mathbf{k}_6), \quad (A42)
 \end{aligned}$$

$$\begin{aligned}
 H_n^{4,2}(\mu) = & \frac{1}{n-1} \lim_{\beta \rightarrow \infty} 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_6} \delta_{\mathbf{k}_1+\mathbf{k}_6, \mathbf{k}_2+\mathbf{k}_5} \delta_{\mathbf{k}_3+\mathbf{k}_5, \mathbf{k}_4+\mathbf{k}_6} \\
 & \times \sum_{\eta_5 \eta_6} \sum_{j_1 \rightarrow j_4} \langle \bar{\psi}_{0,K_1} \psi_{0,K_2} \bar{\psi}_{0,K_3} \Lambda_{\mathbf{k}_6}^{\eta_6} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \psi_{0,K_4} \rangle (n\beta) \\
 & \times \{ n_F(\eta_6 M_6) [1 - n_F(\eta_5 M_5)] C_1(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_5 M_5 - \eta_6 M_6) \\
 & + n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] C_2(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_6 M_6 - \eta_5 M_5) \}, \quad (A43)
 \end{aligned}$$

$$\begin{aligned}
 H_n^{4,3}(\mu) = & \frac{1}{n-1} \lim_{\beta \rightarrow \infty} 2g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_6} \delta_{\mathbf{k}_1+\mathbf{k}_6, \mathbf{k}_2+\mathbf{k}_5} \delta_{\mathbf{k}_3+\mathbf{k}_5, \mathbf{k}_4+\mathbf{k}_6} \\
 & \times \sum_{\eta_5 \eta_6} \sum_{j_1 \rightarrow j_4} \langle \bar{\psi}_{0,K_1} \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \psi_{0,K_4} \bar{\psi}_{0,K_3} \Lambda_{\mathbf{k}_6}^{\eta_6} \gamma^0 \psi_{0,K_2} \rangle (n\beta) \\
 & \times \{ n_F(\eta_6 M_6) [1 - n_F(\eta_5 M_5)] C_1(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_5 M_5 - \eta_6 M_6) \\
 & + n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] C_2(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_5 M_5 - \eta_6 M_6) \}, \quad (A44)
 \end{aligned}$$

$$\begin{aligned}
 H_n^6(\mu) = & \frac{1}{n-1} \lim_{\beta \rightarrow \infty} (-1) 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \rightarrow \mathbf{k}_7} \delta_{\mathbf{k}_1+\mathbf{k}_2, \mathbf{k}_3+\mathbf{k}_7} \cdot \delta_{\mathbf{k}_7+\mathbf{k}_4, \mathbf{k}_5+\mathbf{k}_6} \sum_{j_1 \rightarrow j_6} \sum_{\eta_7} \\
 & \times \langle \bar{\psi}_{0,K_1} \psi_{0,K_3} \bar{\psi}_{0,K_4} \psi_{0,K_6} \bar{\psi}_{0,K_2} \Lambda_{\mathbf{k}_7}^{\eta_7} \gamma^0 \psi_{0,K_5} \rangle (n\beta) \\
 & \times \{ [1 - n_F(\eta_7 M_7)] \cdot C_1(-\omega_{j_1} - \omega_{j_2} + \omega_{j_3}, -\omega_{j_4} + \omega_{j_5} + \omega_{j_6}, \eta_7 M_7) \\
 & - n_F(\eta_7 M_7) C_2(-\omega_{j_1} - \omega_{j_2} + \omega_{j_3}, -\omega_{j_4} + \omega_{j_5} + \omega_{j_6}, -\eta_7 M_7) \}. \quad (A45)
 \end{aligned}$$

For the remaining calculation, we need the following result,

$$\begin{aligned}
 & \sum_{j_1, j_2} \delta_{j_1, j_2} \frac{1}{M_1 + i\omega_{j_1}} C_1(-\omega_{j_1}, \omega_{j_2}, M) \\
 & = n \cdot \left\{ \frac{1 - e^{(M_1-M)\beta}}{(M-M_1)^2} \frac{1}{e^{n\beta M_1} + 1} - \frac{\beta}{M-M_1} \frac{1}{e^{n\beta M} + 1} \right\}, \quad (A46)
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{j_1, j_2} \delta_{j_1, j_2} \frac{1}{M_1 + i\omega_{j_1}} C_2(-\omega_{j_1}, \omega_{j_2}, M) \\
 & = n \cdot \left\{ \frac{e^{-(M_1+M)\beta} - 1}{(M+M_1)^2} \frac{e^{n\beta M_1}}{e^{n\beta M_1} + 1} + \frac{\beta}{M+M_1} \frac{1}{e^{n\beta M} + 1} \right\}. \quad (A47)
 \end{aligned}$$

For $H_n^{2,1}$, we have

$$\langle \bar{\psi}_{0,K_1} \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \psi_{0,K_2} \rangle = -\frac{1}{n\beta} \sum_{\eta_1} \mathcal{T}_{\eta_4 \eta_1}(\mathbf{k}_4, \mathbf{k}_1) \mathcal{S}_0^{\eta_1}(K_1) \delta_{K_1, K_2}$$

then we need to calculate

$$\begin{aligned} \sum_{j_1 j_2} \mathcal{S}_0^{\eta_1}(K_1) \delta_{j_1, j_2} \times \{ & -[1 - n_F(\eta_3 M_3)][1 - n_F(\eta_4 M_4)] n_F(\eta_5 M_5) \cdot C_1(-\omega_{j_1}, \omega_{j_2}, \eta_3 M_3 + \eta_4 M_4 - \eta_5 M_5) \\ & + n_F(\eta_3 M_3) n_F(\eta_4 M_4) [1 - n_F(\eta_5 M_5)] \cdot C_2(-\omega_{j_1}, \omega_{j_2}, -\eta_3 M_3 - \eta_4 M_4 + \eta_5 M_5) \}. \end{aligned} \quad (\text{A48})$$

Note that the factor $n_F(\eta M)$ gives the constraint that $\eta = -1$ and $1 - n_F(\eta M)$ gives $\eta = 1$ in order to survive the zero temperature limit, substituting Eqs. (A46) and (A47) we finally arrive at

$$H_n^{2,1}(\mu) = \frac{n}{n-1} 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \mathbf{k}_3 \mathbf{k}_4 \mathbf{k}_5} \delta_{\mathbf{k}_1 + \mathbf{k}_5, \mathbf{k}_3 + \mathbf{k}_4} \frac{\mathcal{T}_{+-}(\mathbf{k}_4, \mathbf{k}_1) \mathcal{T}_{+-}(\mathbf{k}_3, \mathbf{k}_5) + \mathcal{T}_{-+}(\mathbf{k}_4, \mathbf{k}_1) \mathcal{T}_{-+}(\mathbf{k}_3, \mathbf{k}_5)}{(M_1 + M_3 + M_4 + M_5)^2}. \quad (\text{A49})$$

For $H_n^{2,2}$, we have

$$\langle \bar{\psi}_{0,K_1} \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \Lambda_{\mathbf{k}_3}^{\eta_3} \gamma^0 \psi_{0,K_2} \rangle = -\frac{1}{n\beta} \sum_{\eta_1} \text{tr}(\Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0 \Lambda_{\mathbf{k}_5}^{\eta_5} \gamma^0 \Lambda_{\mathbf{k}_3}^{\eta_3} \gamma^0 \Lambda_{\mathbf{k}_1}^{\eta_1} \gamma^0) \mathcal{S}_0^{\eta_1}(K_1) \delta_{K_1, K_2}. \quad (\text{A50})$$

For simplicity, define $\mathcal{T}_{\eta_1 \eta_2 \eta_3 \eta_4}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \equiv \text{tr}(\Lambda_{\mathbf{k}_1}^{\eta_1} \gamma^0 \Lambda_{\mathbf{k}_2}^{\eta_2} \gamma^0 \Lambda_{\mathbf{k}_3}^{\eta_3} \gamma^0 \Lambda_{\mathbf{k}_4}^{\eta_4} \gamma^0)$. According the result in Eq. (A48) we get

$$H_n^{2,2}(\mu) = -\frac{n}{n-1} 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \mathbf{k}_3 \mathbf{k}_4 \mathbf{k}_5} \delta_{\mathbf{k}_1 + \mathbf{k}_5, \mathbf{k}_3 + \mathbf{k}_4} \frac{\mathcal{T}_{-++-}(\mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_3, \mathbf{k}_1) + \mathcal{T}_{+-+-}(\mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_3, \mathbf{k}_1)}{(M_1 + M_3 + M_4 + M_5)^2}. \quad (\text{A51})$$

For $H_n^{4,1}$,

$$\begin{aligned} \langle \bar{\psi}_{0,K_1} \psi_{0,K_2} \bar{\psi}_{0,K_3} \psi_{0,K_4} \rangle &= \langle (\bar{\psi}_{0,K_1})_\alpha (\psi_{0,K_2})_\alpha (\bar{\psi}_{0,K_3})_\beta (\psi_{0,K_4})_\beta \rangle \\ &= \langle (\psi_{0,K_2})_\alpha (\bar{\psi}_{0,K_1})_\alpha (\psi_{0,K_4})_\beta (\bar{\psi}_{0,K_3})_\beta \rangle \\ &= \frac{\mathcal{S}_0(K_1)_{\alpha\alpha}}{n\beta} \delta_{K_1, K_2} \frac{\mathcal{S}_0(K_3)_{\beta\beta}}{n\beta} \delta_{K_3, K_4} - \frac{\mathcal{S}_0(K_3)_{\alpha\beta}}{n\beta} \delta_{K_2, K_3} \frac{\mathcal{S}_0(K_1)_{\beta\alpha}}{n\beta} \delta_{K_4, K_1}. \end{aligned} \quad (\text{A52})$$

In this contraction we have two terms corresponding to the two diagrams in Fig. 5. We now show that one term containing $\delta_{K_1, K_2} \delta_{K_3, K_4}$ vanishes even at finite temperature, this term is proportional to a factor as follows,

$$\begin{aligned} & \{ n_F(\eta_6 M_6) [1 - n_F(\eta_5 M_5)] C_1(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_5 M_5 - \eta_6 M_6) \\ & + n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] C_2(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_5 M_5 - \eta_6 M_6) \} \end{aligned} \quad (\text{A53})$$

Given $\delta_{K_1 K_2} \delta_{K_3, K_4}$, define $M \equiv \eta_5 M_5 - \eta_6 M_6$. Then we have

$$\begin{aligned} & n_F(\eta_6 M_6) [1 - n_F(\eta_5 M_5)] C_1(0, 0, M) + n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] C_2(0, 0, -M) \\ & \propto n_F(\eta_6 M_6) [1 - n_F(\eta_5 M_5)] \cdot \frac{e^{-M\beta} - 1}{M^2} + n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] \cdot \frac{e^{M\beta} - 1}{M^2} \\ & \propto n_F(\eta_6 M_6) [1 - n_F(\eta_5 M_5)] \left\{ \frac{n_F(\eta_5 M_5)}{1 - n_F(\eta_5 M_5)} \frac{1 - n_F(\eta_6 M_6)}{n_F(\eta_6 M_6)} - 1 \right\} \\ & \quad + n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] \cdot \left\{ \frac{1 - n_F(\eta_5 M_5)}{n_F(\eta_5 M_5)} \frac{n_F(\eta_6 M_6)}{1 - n_F(\eta_6 M_6)} - 1 \right\} \\ & = n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] - n_F(\eta_6 M_6) [1 - n_F(\eta_5 M_5)] \\ & \quad + n_F(\eta_6 M_6) [1 - n_F(\eta_5 M_5)] - n_F(\eta_5 M_5) [1 - n_F(\eta_6 M_6)] \\ & = 0. \end{aligned} \quad (\text{A54})$$

The other term gives a Matsubara summation as

$$\begin{aligned}
 & \sum_{j_1 \rightarrow j_4} \delta_{j_1, j_4} \delta_{j_2, j_3} \mathcal{S}_0^{\eta_1}(K_1) \mathcal{S}_0^{\eta_3}(K_3) \\
 & \times \{n_F(\eta_6 M_6)[1 - n_F(\eta_5 M_5)] \cdot C_1(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, \eta_5 M_5 - \eta_6 M_6) \\
 & + n_F(\eta_5 M_5)[1 - n_F(\eta_6 M_6)] \cdot C_2(-\omega_{j_1} + \omega_{j_2}, -\omega_{j_3} + \omega_{j_4}, -\eta_5 M_5 + \eta_6 M_6)\},
 \end{aligned} \quad (\text{A55})$$

this summation involves a double usage of Eqs. (A46) and (A47), finally

$$H_n^{4,1}(\mu) = \frac{n}{n-1} 2g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \mathbf{k}_3 \mathbf{k}_5 \mathbf{k}_6} \delta_{\mathbf{k}_1 + \mathbf{k}_6, \mathbf{k}_3 + \mathbf{k}_5} \frac{\mathcal{T}_{-+}(\mathbf{k}_1, \mathbf{k}_3) \mathcal{T}_{+-}(\mathbf{k}_5, \mathbf{k}_6) + \mathcal{T}_{+-}(\mathbf{k}_1, \mathbf{k}_3) \mathcal{T}_{-+}(\mathbf{k}_5, \mathbf{k}_6)}{(M_1 + M_3 + M_5 + M_6)^2}. \quad (\text{A56})$$

Similarly, we have

$$\begin{aligned}
 H_n^{4,2}(\mu) &= -\frac{n}{n-1} 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \mathbf{k}_3 \mathbf{k}_5 \mathbf{k}_6} \delta_{\mathbf{k}_1 + \mathbf{k}_6, \mathbf{k}_3 + \mathbf{k}_5} \frac{\mathcal{T}_{-++}(\mathbf{k}_3, \mathbf{k}_6, \mathbf{k}_5, \mathbf{k}_1) + \mathcal{T}_{+-+}(\mathbf{k}_3, \mathbf{k}_6, \mathbf{k}_5, \mathbf{k}_1)}{(M_1 + M_3 + M_5 + M_6)^2}, \\
 H_n^{4,3}(\mu) &= \frac{n}{n-1} 2g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \mathbf{k}_3 \mathbf{k}_5 \mathbf{k}_6} \delta_{\mathbf{k}_1 + \mathbf{k}_6, \mathbf{k}_3 + \mathbf{k}_5} \frac{\mathcal{T}_{+-}(\mathbf{k}_5, \mathbf{k}_1) \mathcal{T}_{+-}(\mathbf{k}_6, \mathbf{k}_3) + \mathcal{T}_{-+}(\mathbf{k}_5, \mathbf{k}_1) \mathcal{T}_{-+}(\mathbf{k}_6, \mathbf{k}_3)}{(M_1 + M_3 + M_5 + M_6)^2}.
 \end{aligned} \quad (\text{A57})$$

For H_n^6 , we have a six-point function, there are six ways of contraction, four of them cannot satisfy momentum conservation, only two of them are nonzero,

$$\begin{aligned}
 \langle \bar{\psi}_{0,K_1} \psi_{0,K_3} \bar{\psi}_{0,K_4} \psi_{0,K_6} \bar{\psi}_{0,K_2} \Lambda_{\mathbf{k}_7}^{\eta_7} \gamma^0 \psi_{0,K_5} \rangle &\sim -(\Lambda_{\mathbf{k}_7}^{\eta_7} \gamma^0)_{\rho\sigma} \left\{ -\frac{\mathcal{S}_0(K_4)_{\alpha\beta}}{n\beta} \delta_{K_3, K_4} \cdot \frac{\mathcal{S}_0(K_1)_{\beta\alpha}}{n\beta} \delta_{K_1, K_6} \cdot \frac{\mathcal{S}_0(K_2)_{\sigma\rho}}{n\beta} \delta_{K_5, K_2} \right. \\
 &\quad \left. + \frac{\mathcal{S}_0(K_4)_{\alpha\beta}}{n\beta} \delta_{K_3, K_4} \cdot \frac{\mathcal{S}_0(K_2)_{\beta\rho}}{n\beta} \delta_{K_2, K_6} \cdot \frac{\mathcal{S}_0(K_1)_{\sigma\alpha}}{n\beta} \delta_{K_5, K_1} \right\}.
 \end{aligned}$$

We are left with a summation as

$$\begin{aligned}
 & \sum_{j_1 \rightarrow j_6} \delta_{j_3, j_4} \delta_{j_1, j_6} \delta_{j_2, j_5} \mathcal{S}_0^{\eta_1}(K_1) \mathcal{S}_0^{\eta_2}(K_2) \mathcal{S}_0^{\eta_4}(K_4) \\
 & \times \{[1 - n_F(\eta_7 M_7)] \cdot C_1(-\omega_{j_1} - \omega_{j_2} + \omega_{j_3}, -\omega_{j_4} + \omega_{j_5} + \omega_{j_6}, \eta_7 M_7) \\
 & - n_F(\eta_7 M_7) C_2(-\omega_{j_1} - \omega_{j_2} + \omega_{j_3}, -\omega_{j_4} + \omega_{j_5} + \omega_{j_6}, -\eta_7 M_7)\}.
 \end{aligned} \quad (\text{A58})$$

This summation can also be calculated by a treble usage of Eqs. (A46) and (A47), then we arrive at

$$\begin{aligned}
 H_n^6(\mu) &= \frac{n}{n-1} 4g^2 \frac{1}{V^2} \sum_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_7} \delta_{\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_3 + \mathbf{k}_7} \frac{1}{(M_1 + M_2 + M_3 + M_7)^2} \\
 & \times \{\mathcal{T}_{+-}(\mathbf{k}_3, \mathbf{k}_1) \mathcal{T}_{+-}(\mathbf{k}_7, \mathbf{k}_2) - \mathcal{T}_{-++}(\mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_7, \mathbf{k}_1) \\
 & + \mathcal{T}_{-+}(\mathbf{k}_3, \mathbf{k}_1) \mathcal{T}_{-+}(\mathbf{k}_7, \mathbf{k}_2) - \mathcal{T}_{-+-}(\mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_7, \mathbf{k}_1)\}.
 \end{aligned} \quad (\text{A59})$$

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