

Boundary conditions and Dirac fields on AdS_n

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We study Dirac fields on AdS_n in both global and Poincaré charts and, for each mass window, we classify the boundary conditions at conformal infinity that ensure the existence of advanced and retarded propagators. We distinguish the well-known MIT–bag class from a generalized family, thereby extending to arbitrary dimensions the procedure initiated in [D. S. Blanco, *J. Math. Phys. (N.Y.)* **64**, 032301 (2023)]. As in the scalar case, we show that suitable generalized boundary data can support bound states. In four dimensions, we work out two explicit examples: (i) the MIT case, for which we construct the advanced/retarded propagators and the two-point function of the associated ground state and (ii) a representative generalized boundary condition, for which we construct the propagators and exhibit a normalizable bound state.

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I. INTRODUCTION

Quantum field theory (QFT) on curved spacetimes is by now a mature and lively area of mathematical and theoretical physics, which relies a lot on the algebraic approach; see Refs. [1–3] for complementary reviews. Algebraic QFT (AQFT) represents a very versatile and successful framework which allows us to study in depth several concrete free and interacting models and to answer relevant structural questions, ranging from the formulation of a covariant and local renormalization scheme to the implementation of local gauge invariance and the characterization of physically admissible states, just to quote a few notable examples.

In many analyses of the structural aspects of quantum field theory, a common hypothesis is that of global hyperbolicity of the underlying background. Without entering into the technical aspects of the definition of this class of spacetimes (see, e.g., Ref. [4] Ch. 8), it guarantees that solutions to symmetric hyperbolic equations, such as the Klein-Gordon, the Dirac or the Proca, can be constructed from initial data assigned on a Cauchy hypersurface. As a byproduct, one can establish the existence of a manifestly covariant quantization scheme for free field theories; see, e.g., Ref. [5]. In a few words, this is based on a two-step procedure. In the first, one constructs a suitable $*$ -algebra of

observables $\mathcal{A}$ which encodes structural properties such as dynamics and the canonical commutation or anticommutation relations. The second input is the choice of a state ω , which can be realized abstractly as a positive and normalized linear functional on $\mathcal{A}$. Subsequently, by means of the GNS theorem, one can associate to the pair $(\mathcal{A}, \omega)$ a Hilbert space representation, hence recovering the standard probabilistic interpretation proper of quantum theories. In all concrete models, fixing a state ω is tantamount to assigning the n -point correlation functions ω_n and, under the physically motivated assumption that the state is quasifree/Gaussian, these can all be determined starting from ω_2 ; see, e.g., Ref. [3], Ch. 3 & 5] advances AQFT.

Yet, among the plethora of quasifree/Gaussian states, not all are sensible. In the literature, a set of physically motivated conditions have been established. Among them, the most notable being the finiteness of the quantum fluctuations of all observables, the existence of a local and covariant regularization scheme to define Wick polynomials and the guarantee that the UV behavior of the state is the same as that of the Poincaré vacuum [6]. From a technical viewpoint, this translates in the now widely accepted Hadamard condition, which establishes a constraint on the singular structure of the two-point correlation function ω_2 . On a globally hyperbolic background, the existence of Hadamard states is always guaranteed, thanks to a deformation argument which relies on the property that, on static spacetimes, ground states automatically abide by the Hadamard condition [7].

The picture changes drastically when the assumption of global hyperbolicity is dropped. Already at the classical level, an initial value formulation is not sufficient to establish the existence of a unique solution for the

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underlying equation of motion, even at the level of non-interacting models. The temptation of dismissing such settings as physically irrelevant is unwarranted: well-known, experimentally established phenomena, such as the Casimir effect, are naturally modeled by field theories on manifolds with boundary; see, e.g., Ref. [8]. In particular, among the many lessons learned by investigating these models, noteworthy is that, at a geometric level, the lack of global hyperbolicity descends from the existence in the underlying manifold M of a nonempty boundary ∂M , consisting of one or more connected components. Furthermore, the metric induced thereon is still Lorentzian, thus endowing ∂M with a lower dimensional spacetime structure. Most notably, a central example of this class of backgrounds, whose relevance in high-energy and gravitational physics cannot be underestimated, is anti-de Sitter spacetime AdS_n , of dimension $n \geq 2$. This is a maximally symmetric solution of the vacuum Einstein equations with negative cosmological constant and, up to a conformal rescaling, it possesses a timelike boundary.

On the one hand, AdS_n is of paramount relevance due to its key role in several models, such as the AdS/CFT correspondence and, on the other hand, being maximally symmetric, it plays the same role of Minkowski spacetime when the cosmological constant is negative. For this reason, investigating the role of quantum field theories thereon is a very important and timely question, although the first works in this direction are from the 1970s [9]. Already at the level of scalar-free field theories, one can appreciate that the paradigm at the heart of AQFT does not need to be modified, although structural differences are present. First and foremost, the construction of an algebra of observables encoding the dynamics and the canonical commutation relations rests on the existence of an advanced and retarded propagator for the Klein-Gordon operator. Yet, the presence of a boundary entails that this is not unique unless suitable boundary conditions are assigned to ∂M . In Refs. [10,11], using the theory of Sturm-Liouville second-order differential equations, it has been shown that working in Fourier space and within a certain range of admissible values of the mass parameter, one can individuate a one-parameter family of admissible boundary conditions, among which one can find the Dirichlet ones, which are commonly used in the literature. To each of these, one can associate advanced/retarded fundamental solutions as well as the two-point correlation function of an underlying ground state, yielding nonequivalent Hilbert space representations. Such a procedure has been extended to cover an even larger class of boundary conditions in Ref. [12], proving ultimately the richness of free field theories on AdS spacetimes. It is important to emphasize that, in dropping the assumption that the underlying background is globally hyperbolic, one needs to verify that the quantization scheme is consistent with the principle of *F-locality* as first formulated in

Ref. [13]. This dictates that the restriction of the algebra of observables to any globally hyperbolic subregion of the underlying spacetime should be $*$ -isomorphic to the one constructed intrinsically in such a region using the usual procedure. At the level of free fields, this can be enforced already at the level of the advanced-minus-retarded fundamental solution, which implements the canonical commutation relations. More precisely, the one built starting from an admissible boundary condition, once restricted to a globally hyperbolic subregion, must differ only by a smooth term from the one which is intrinsically associated to it.

At first glance, one might think that the procedure which works so efficiently for a scalar field could be adapted to higher-spin systems. Yet, this turns out to be false if one has to deal with first-order rather than second-order partial differential equations, e.g., when working with Dirac fields. In this case, simple choices such as Dirichlet boundary conditions are known not to lead to well-defined, mixed initial-boundary value problems. Hence, one has to resort to different options, among which the most famous is the MIT-bag boundary condition [14]. Yet a systematic classification of the admissible boundary conditions and an analysis of the associated quantum field theory is to the best of our knowledge missing.

In view of these premises, in this work we study massive Dirac fields on AdS_n , both in global coordinates and in the Poincaré patch, denoted by PAdS_n . We address two open questions: first, for a given mass, we classify all boundary conditions at conformal infinity that make the dynamics well posed and entails the existence of advanced and retarded Green's fundamental solutions, meant as distributional inverses of the Dirac operator, which are respectively past and future propagating. We organize them into two classes: the familiar MIT-bag boundary conditions, mentioned above, and a generalized family that we identify and thoroughly characterize. This classification starts from and extends to arbitrary spacetime dimensions the procedure outlined in Ref. [15] for $n = 2$ and it is complementary to the analysis in Ref. [16] on the Cauchy problem with MIT-type boundary conditions.

Secondly, on the quantum side, we investigate which choices lead to physically acceptable states and whether bound states may appear. In close analogy with the scalar case, we show that within the generalized family, certain parameter ranges support normalizable bound modes, while the MIT case behaves regularly. We highlight that, in this work, we focus our attention on the construction of the fundamental solutions and of two-point correlation functions, seen as a bidistribution on PAdS_n whose singular structure is of global Hadamard form. Although these are the building blocks necessary to construct the algebra of observables associated to a Dirac field with prescribed boundary conditions, we shall not delve into the analysis, leaving it for future work. As a matter of fact, while its definition is a straightforward adaptation to the case in hand

of the procedure outlined in Ref. [17] on globally hyperbolic spacetimes, the investigation of the physical properties is a rather subtle task. On the one hand, already when considering a real scalar field, it is not obvious whether the associated $*$ -algebra of observables is causal, since this is a byproduct of the support properties of the fundamental solutions. Establishing them in presence of boundary conditions is by no means straightforward, since all standard techniques cannot be applied and, already in the simpler geometric setting of half-Minkowski spacetime, this has been proven for a massive real scalar field endowed with Robin boundary conditions in Ref. [18]. On the other hand, one might wish to explore the connection between boundary conditions and holography. However, working in the spirit of algebraic holography [19], an answer is not immediate. As a matter of fact, in Ref. [19], one considers field theories constructed on the global patch of AdS_n such that, on the associated $*$ -algebra of observables, the underlying group of isometries acts covariantly.

Hence, in order to apply our results, particularly those of Secs. IV and V, first of all, one should realize which among the fundamental solutions and the propagators in PAdS_n can be extended to the whole GAdS_n . The same problem investigated for a real scalar field suggests that one should consider only special values of the mass; see, e.g., Ref. [20]. Furthermore, covariance under the action of the full AdS_n isometry group might require a restriction of the class of admissible boundary conditions as it happens in the scalar case; see, e.g., Ref. [21]. From a physical viewpoint, one can interpret this feature as the signature that while all the boundary conditions considered in this work guarantee that PAdS_n behaves as an isolated system, not all of them entail total reflection at the boundary. In many instances, one must account also for the possibility of propagation along the boundary, but a proof of this fact requires us to write both the fundamental solutions and the two-point correlation functions in position space. Already in the case of a scalar field, this is not an easy task, and it requires an investigation of its own; see Ref. [18].

We complement our analysis by discussing in detail two examples in PAdS_4 . For the MIT-bag boundary condition, we construct the advanced and retarded propagators and a two-point function for a ground state, compatible with the PAdS_4 isometry group and with the analysis outlined in Ref. [22]. For a representative generalized boundary condition, we again construct the propagators explicitly and exhibit a normalizable bound state. This dichotomy illustrates in concrete terms that the MIT class yields a well-behaved quantum theory, whereas the generalized class can trigger qualitatively new features through boundary-supported modes.

The paper is organized as follows: in Sec. II, we review the geometry of global AdS_n and of the Poincaré patch, setting notation and recalling the relevant spin structure and Clifford conventions. In Sec. II A, we formulate the Dirac

equation as a first-order system on a manifold with a timelike boundary, reduce the dynamics to Fourier modes, and analyze self-adjointness of the ensuing spatial operator. In Sec. III, we deduce our mass-dependent classification of boundary conditions, highlighting the distinction between the MIT-bag class and its generalized counterpart. In Sec. IV, we establish a prescription to construct the associated advanced and retarded fundamental solutions. In Sec. V, we work out two explicit four-dimensional examples: MIT and a generalized boundary condition with a normalizable bound mode. We complement our analysis with two short appendixes. In the first, we elaborate on the Dirac equation and on our choice of γ matrices, while in the second, we outline a succinct overview of the construction of the algebra of fields in the scenario under investigation.

II. THE GEOMETRY OF AN ADS SPACETIME

In this section, we outline the basic geometric data at the heart of our paper. In particular, we consider the n -dimensional anti-de Sitter AdS_n , $n \geq 2$, which is a maximally symmetric solution to the Einstein equations with a cosmological constant $\Lambda < 0$. If $n > 2$, it can be realized as an homogeneous, embedded submanifold of $\mathbb{R}^{n+1}$ endowed with the line element

$$ds^2 = -(dX^0)^2 - (dX^1)^2 + (dX^2)^2 + \dots + (dX^n)^2, \quad (1)$$

where $\{X^i\}_{i=0,\dots,n}$ are standard Cartesian coordinates. Herein, the n -dimensional anti-de Sitter spacetime AdS_n is the hyperboloid

$$-(X^0)^2 - (X^1)^2 + (X^2)^2 + \dots + (X^n)^2 = -\ell^2, \quad (2)$$

where $\ell^2 = -\frac{(n-1)(n-2)}{\Lambda}$ is the AdS_n radius of curvature. Henceforth, for simplicity we shall set $\ell = 1$. In the following, we shall consider both the *AdS global patch* and the *Poincaré fundamental domain*, which we discuss succinctly. The first one descends from the following parametrization of Eq. (2):

$$\begin{aligned} X^0 &= \sec \rho \cos \tau, \\ X^1 &= \sec \rho \sin \tau, \\ X^i &= y^i \tan \rho, \quad i = 2, \dots, n, \end{aligned} \quad (3)$$

where $\tau \in [-\pi, \pi)$, $\rho \in (0, \pi/2)$, while $\sum_{i=1}^{n-1} (y^i)^2 = 1$. These new coordinates cover the full AdS manifold and, inserting them into Eq. (1), one obtains

$$ds^2 = \sec^2 \rho (-d\tau^2 + d\rho^2 + \sin^2 \rho ds_{\mathbb{S}_{n-2}}^2), \quad i = 1, \dots, n-2, \quad (4)$$

where $ds_{\mathbb{S}_{n-2}}^2$ is the standard line element of the unit $(n-2)$ -sphere. In order to distinguish between the different

coordinate charts used in this work, we shall denote by GAdS_n the manifold when decorated with the global coordinate patch as per Eq. (3). Two comments are in due course. On the one hand, we observe that GAdS_n , endowed with the line element as per Eq. (4), is conformal to $\tilde{\mathcal{M}}_n \equiv \mathbb{R} \times \mathbb{S}_+^{n-1}$, an open subset of the n -dimensional Einstein static universe ESU_n . Here, $\mathbb{S}_+^{n-1}$ denotes half of the unit $(n-1)$ -sphere whose metric is parametrized by $d\rho^2 + \sin^2 \rho ds_{\mathbb{S}^{n-2}}^2$ with $\rho \in (0, \pi/2)$. On the other hand, in Eq. (4) we are making close contact with the conventions used in Ref. [15], since in the literature, the coordinate ρ is often replaced by $r \in (0, +\infty)$, implicitly defined by $\tan \rho = \sinh r$. In addition, one can notice starting from Eq. (3) that the coordinate τ is periodic and, in order to avoid any potential causal pathology, we shall always extend the domain of τ so that $\tau \in \mathbb{R}$. In other words, we consider the universal cover of the n -dimensional anti-de Sitter spacetime which will still be denoted by GAdS_n with a slight abuse of notation.

As mentioned above, we will also be considering the Poincaré patch/domain which is defined in terms of the coordinates

$$\begin{aligned} X^0 &= \frac{t}{z}, \\ X^i &= \frac{x^i}{z}, \quad i = 1, \dots, n-2, \\ X^{n-1} &= \frac{1}{2z}(1 + z^2 + \delta_{ij}x^i x^j - t^2), \\ X^n &= \frac{1}{2z}(1 - z^2 - \delta_{ij}x^i x^j + t^2). \end{aligned} \quad (5)$$

where δ_{ij} is the Kronecker δ , $t, x^i \in \mathbb{R}$, for all $i = 1, \dots, n-2$, while $z \in (0, +\infty)$. Since $X^{n-1} + X^n = \frac{1}{z} > 0$, the chart (5) covers only half of AdS_n and, from Eq. (1), it descends that the associated line element reads

$$ds^2 = \frac{1}{z^2}(-dt^2 + dz^2 + \delta_{ij}dx^i dx^j), \quad i = 1, \dots, n-2. \quad (6)$$

We shall denote this by PAdS_n , the Poincaré patch of AdS_n and, since $z > 0$, it is conformal to the upper half-space $\mathbb{H}^n := \{(t, z, x^1, \dots, x^{n-2}) | z > 0\}$, endowed with the standard Minkowski metric.

A. The Dirac equation on AdS_n and its boundary conditions

On top of both GAdS_n and PAdS_n we wish to consider a Dirac field, following the ideas outlined in Appendix A. Hence, focusing for definiteness on the Poincaré patch, we consider a spinor $\psi: \text{PAdS}_n \rightarrow \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$ which abides by the massive Dirac equation

$$\tilde{P}\psi := (D - m)\psi = 0, \quad (7)$$

where D is as per Eq. (A5) while $m \geq 0$ is a mass parameter. As mentioned in the Introduction, neither PAdS_n nor GAdS_n is a globally hyperbolic spacetime since they possess a timelike, conformal boundary. Hence, solutions of Eq. (7) cannot be uniquely obtained only by assigning initial data on a Cauchy surface, but also suitable boundary conditions must be supplemented. In the following, we classify the admissible ones extending the results of Ref. [15] to generic spacetime dimensions. To this end, we must start from an analysis of the structural aspects of the solutions of the Dirac equation both on PAdS_n and on GAdS_n ; see Sec. III A.

B. On the Dirac equation on PAdS_n

In this subsection, we consider the Poincaré patch of the n -dimensional anti-de Sitter spacetime and, for computational convenience, we consider n even and the following distinguished representation of the γ -matrices denoted by the symbol Υ , as follows:

$$\begin{aligned} \Upsilon^0 &= i \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \Upsilon^a = \begin{pmatrix} -\hat{\gamma}^{a-1} & 0 \\ 0 & \hat{\gamma}^{a-1} \end{pmatrix}, \\ a &= 1, \dots, n-1, \end{aligned} \quad (8)$$

where the matrices $\hat{\gamma}^{a-1}$ are defined in Appendix A. Accordingly, Eq. (7) reads

$$\tilde{P}\psi = (\Upsilon^\mu \nabla_\mu - m)\psi = 0. \quad (9)$$

In addition, given any pair of spinors on PAdS_n , say $\phi, \psi: \text{PAdS}_n \rightarrow \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$ abiding by Eq. (9), we can consider a canonical inner product subordinated to the Dirac current $J^\mu[\phi, \psi] = \bar{\phi} \Upsilon^\mu \psi$, as follows:

$$\langle \phi, \psi \rangle_{\text{PAdS}_n} \doteq \int_0^\infty \frac{dz}{z^{n-1}} \int_{\mathbb{R}^{n-2}} d\underline{x} \bar{\phi} \Upsilon^0 \psi, \quad (10)$$

where $d\underline{x} = \prod_{i=1}^{n-2} dx^i$, while $\bar{\phi} = \phi^\dagger \Upsilon^0$; see Eq. (A7). Both ϕ and ψ are here evaluated at an arbitrary but fixed value of the time coordinate t . This is legit since the Dirac current is covariantly conserved and therefore the rhs of Eq. (10) is actually time independent. Starting from this remark and observing also that constant time hypersurfaces on PAdS_n are isometric to $\mathbb{R}^+ \times \mathbb{R}^{n-2}$ endowed with the Riemannian line element $ds^2 = \frac{1}{z^2}(dz^2 + \delta_{ij}dx^i dx^j)$, $i, j = 1, \dots, n-2$, we can introduce

$$L^2(\mathbb{R}^+ \times \mathbb{R}^{n-2}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}) = \overline{(C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-2}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}), \langle \cdot, \cdot \rangle_{\text{PAdS}_n})}. \quad (11)$$

This is the Hilbert space of square integrable spinors obtained as the completion of $C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-2}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})$ with respect to the inner product (10). We highlight that the choice of n even is only due to the fact that when n is odd, the Υ -matrices are the same as in the $(n-1)$ -dimensional scenario; see Appendix A. The following analysis should therefore be slightly modified to account for this fact, but the final result of this section applies slavishly also to n odd and, therefore, for the sake of conciseness, we opted to focus only on one among the two possible scenarios. Since Eq. (6) entails that PAdS_n is conformal to $(\mathbb{H}^n, \eta)$, it is convenient to consider a conformally rescaled spinor $\Psi = z^{\frac{1-n}{2}} \psi: \mathbb{H}^n \rightarrow \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$. In addition, if ψ abides by Eq. (9), then

$$(\Upsilon^\mu \nabla_\mu - m)\psi = 0 \Leftrightarrow P\Psi = \left(\Upsilon^\mu \partial_\mu - \frac{m}{z} \right) \Psi = 0, \quad (12)$$

where the rhs is the conformally related Dirac equation on $\mathbb{H}^n$. Furthermore, the conformal rescaling entails that, given two solutions ϕ, ψ of the Dirac equation on PAdS_n , the inner product as per Eq. (10) becomes

$$\langle \phi, \psi \rangle_{\text{PAdS}_n} = \int_0^\infty dz \int_{\mathbb{R}^{n-2}} d\mathbf{x} \bar{\Phi} \Upsilon^0 \Psi = \langle \Phi, \Psi \rangle_{\mathbb{H}^n}, \quad (13)$$

where $\Phi = z^{\frac{1-n}{2}} \phi$, while the integral representation coincides with the natural inner product between spinors in $\mathbb{H}^n$. Notice that, on $\mathbb{H}^n$, the covariant Υ -matrices coincide with the Minkowski ones. Following the same rationale used in Eq. (11), it is convenient to realize that the constant time hypersurfaces of $\mathbb{H}^n$ are isometric to $\mathbb{H}_\delta^{n-1}$ endowed with the standard $(n-1)$ -dimensional Euclidean metric. Accordingly, we denote by

$$L^2(\mathbb{H}_\delta^{n-1}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}) = \overline{(C_0^\infty(\mathbb{H}_\delta^{n-1}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}), \langle \cdot, \cdot \rangle_{\mathbb{H}^{n-1}})}, \quad (14)$$

the Hilbert space of square integrable spinors obtained as the completion of $C_0^\infty(\mathbb{H}_\delta^{n-1}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})$ with respect to the inner product (13).

In order to study Eq. (12), it is convenient to split $\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$ as $\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor-1}} \oplus \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor-1}}$ and consequently $\Psi = \Psi^{(1)} \oplus \Psi^{(2)}$, where $\Psi^{(1,2)}$ will be referred to as the upper- and lower-half components of the spinor Ψ . As a consequence, one can exploit Eq. (8) to rewrite the Dirac equation as the coupled system

$$\left(\hat{\gamma}^0 \partial_z + \hat{\gamma}^j \partial_j - \frac{m}{z} \right) \Psi^{(2)} = -i \partial_t \Psi^{(1)}, \quad (15a)$$

$$\left(\hat{\gamma}^0 \partial_z + \hat{\gamma}^j \partial_j + \frac{m}{z} \right) \Psi^{(1)} = i \partial_t \Psi^{(2)}, \quad (15b)$$

where $j = 1, \dots, n-2$ labels the spatial directions orthogonal to z ; see Eq. (5). Deriving with respect to the time variable Eq. (15a), and employing Eq. (15b), yields

$$\left(\eta^{\mu\nu} \partial_\mu \partial_\nu - \hat{\gamma}^0 \frac{m}{z^2} - \frac{m^2}{z^2} \right) \Psi^{(1)} = 0, \quad (16)$$

where we exploited that the γ -matrices are time independent. Reversing the role of $\Psi^{(1)}$ and $\Psi^{(2)}$ yields

$$\left(\eta^{\mu\nu} \partial_\mu \partial_\nu + \hat{\gamma}^0 \frac{m}{z^2} - \frac{m^2}{z^2} \right) \Psi^{(2)} = 0. \quad (17)$$

Since the translations along all directions tangent to the boundary of $\mathbb{H}^n$ are isometries of Eq. (6), it is convenient to assume that $\Psi^{(i)} \in \mathcal{S}'(\mathbb{H}^n)$ and $i = 1, 2$, so that, working at the level of integral kernel, we can take the Fourier transform along all coordinates barring z , as follows:

$$\Psi^{(1)}(t, z, x^i) = \int_{\mathbb{R}^{n-1}} \frac{d\omega d\mathbf{k}}{(2\pi)^{\frac{n-1}{2}}} e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})} \Psi_{\omega, \mathbf{k}}^{(1)}(z), \quad (18)$$

where $\mathbf{x} = (x^1, \dots, x^{n-2}) \in \mathbb{R}^{n-2}$, $\omega \in \mathbb{R}$, while $\mathbf{k} = (k^1, \dots, k^{n-2}) \in \mathbb{R}^{n-2}$. The expression $\mathbf{k} \cdot \mathbf{x}$ denotes the standard Euclidean inner product. Plugging Eq. (18) inside Eq. (16) yields the following second-order, vector-valued ODE:

$$\begin{aligned} & \left(\frac{d^2}{dz^2} - \hat{\gamma}^0 \frac{m}{z^2} - \frac{m^2}{z^2} + \omega^2 - |\mathbf{k}|^2 \right) \Psi_{\omega, \mathbf{k}}^{(1)}(z) \\ &= \left(\frac{d^2}{dz^2} - \hat{\gamma}^0 \frac{\nu - \frac{1}{2}}{z^2} - \frac{(\nu - \frac{1}{2})^2}{z^2} + \omega^2 - |\mathbf{k}|^2 \right) \Psi_{\omega, \mathbf{k}}^{(1)}(z) = 0, \end{aligned} \quad (19)$$

where $z \in (0, +\infty)$ and $\nu \doteq m + \frac{1}{2}$, while $|\mathbf{k}|^2$ denotes the standard Euclidean norm of $\mathbf{k} \in \mathbb{R}^{n-2}$. At last, given $\hat{\gamma}^0$ as per Appendix A, Eq. (19) is a system of $\frac{N}{2} = 2^{\lfloor \frac{n}{2} \rfloor - 1}$ decoupled, second-order ordinary differential equations of Sturm-Liouville type; see Ref. [23]. The collection of linearly independent solutions of Eq. (19) reads

$$\begin{aligned} \mathbf{J}_i^{(\nu)}(z) &= [\sqrt{z} J_\nu(qz)] e_i, & i = 1, \dots, \frac{N}{4}, \\ \mathbf{Y}_i^{(\nu)}(z) &= [\sqrt{z} Y_\nu(qz)] e_i, & i = 1, \dots, \frac{N}{4}, \\ \mathbf{J}_i^{(\nu-1)}(z) &= [\sqrt{z} J_{\nu-1}(qz)] e_i, & i = \frac{N}{4} + 1, \dots, \frac{N}{2}, \\ \mathbf{Y}_i^{(\nu-1)}(z) &= [\sqrt{z} Y_{\nu-1}(qz)] e_i, & i = \frac{N}{4} + 1, \dots, \frac{N}{2}, \end{aligned} \quad (20)$$

where $q^2 \doteq \omega^2 - |\mathbf{k}|^2$, $\{e_i\}_{i=1, \dots, \frac{N}{2}}$ is the canonical basis of $\mathbb{R}^{\frac{N}{2}}$, J_ν denotes the Bessel function of first kind of order ν , while Y_ν denotes that of second kind of the same order; see

Ref. [24]. It is worth observing that, from a mere mathematical viewpoint, contrary to what occurs when working with a real scalar field on PAdS_n , there is no reason to restrict the values of ν and therefore of m on account of Eq. (20). In other words, *a priori*, at the level of ODEs, there exists no counterpart of the Breitenlohner-Freedman bound [25,26]. Yet, as we can infer from direct inspection of Eq. (20), the set of the first $\frac{N}{2}$ ODEs is invariant under the exchange $\nu \rightarrow -\nu$, while the second half under $\nu \rightarrow 2 - \nu$. Taking into account both symmetries, we can limit ourselves to considering the case $\nu \geq 0$, as the remaining

values can be studied exploiting the mentioned symmetry. Henceforth, we shall always stick to this choice and, furthermore, we shall discard the extremal case $\nu = 0$ since it would require a separate analysis. Focusing instead on $\Psi^{(2)}$ and working once more at the level of Fourier transform, Eq. (15) yields

$$\Psi_{\omega, \mathbf{k}}^{(2)}(z) = -\frac{1}{\omega} \left(\hat{\gamma}^0 \frac{d}{dz} - i k_j \hat{\gamma}^j + \frac{m}{z} \right) \Psi_{\omega, \mathbf{k}}^{(1)}(z), \quad (21)$$

which entails that Eq. (20) gets transformed into

$$\begin{aligned} \tilde{\mathbf{J}}_i^{(\nu)}(z) &= -\frac{\sqrt{z}}{\omega} [q J_{\nu-1}(qz) \mathbf{1} - i J_{\nu}(qz) k_j \hat{\gamma}^j] e_i, & i = 1, \dots, \frac{N}{4}, \\ \tilde{\mathbf{Y}}_i^{(\nu)}(z) &= -\frac{\sqrt{z}}{\omega} [q Y_{\nu-1}(qz) \mathbf{1} - i Y_{\nu}(qz) k_j \hat{\gamma}^j] e_i, & i = 1, \dots, \frac{N}{4}, \\ \tilde{\mathbf{J}}_i^{(\nu-1)}(z) &= \frac{\sqrt{z}}{\omega} [q J_{\nu}(qz) \mathbf{1} + i J_{\nu-1}(qz) k_j \hat{\gamma}^j] e_i, & i = \frac{N}{4} + 1, \dots, \frac{N}{2}, \\ \tilde{\mathbf{Y}}_i^{(\nu-1)}(z) &= \frac{\sqrt{z}}{\omega} [q Y_{\nu}(qz) \mathbf{1} + i Y_{\nu-1}(qz) k_j \hat{\gamma}^j] e_i, & i = \frac{N}{4} + 1, \dots, \frac{N}{2}. \end{aligned} \quad (22)$$

Remark 2.1. From Eq. (21), it is clear that the above procedure is not valid for $\omega = 0$. When considering this particular value, one has to take into account that Eq. (15) reduces to a decoupled system of differential equations whose solutions can be expressed in terms of modified Bessel functions $K_{\alpha}(|\mathbf{k}|z)$, $I_{\alpha}(|\mathbf{k}|z)$, of order ν and $\nu - 1$. In Sec. V, we show how these solutions might contribute in the construction of the propagators of the Dirac equation.

C. Solutions of the Dirac equation on GAdS_n

We focus our attention on GAdS_n , whose associated line element is codified in Eq. (3) and, exactly as in the previous section, we set n even. Since Eq. (4) entails that GAdS_n is conformal to $\tilde{\mathcal{M}}_n \subset \text{ESU}_n$, we consider a conformally rescaled spinor $\Psi = (\sec \rho)^{\frac{1-n}{2}} \psi: \tilde{\mathcal{M}}_n \rightarrow \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$ so that, if ψ abides by Eq. (9) and if we work with the Υ -matrices as per Eq. (8), then

$$(\Upsilon^{\mu} \nabla_{\mu} - m \sec \rho) \Psi = 0, \quad (23)$$

where ∇_{μ} is the spin connection on $\tilde{\mathcal{M}}$ as per Eq. (A4) built out of the Υ -matrices. In addition, given any pair of spinors on GAdS_n , say $\phi, \psi: \text{GAdS}_n \rightarrow \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$, both solutions of the Dirac equation, we can consider a canonical inner product subordinated to the Dirac current $J^{\mu}[\phi, \psi] = \bar{\phi} \Upsilon^{\mu} \psi$, from which we can construct the following inner product:

$$\langle \phi, \psi \rangle_{\text{GAdS}_n} \doteq \int_0^{\frac{\pi}{2}} d\rho \sec^{n-1} \rho \int_{\mathbb{S}^{n-2}} dS^{n-2} \bar{\phi} \Upsilon^0 \psi, \quad (24)$$

where dS^{n-2} is the standard volume form on the unit $(n-2)$ -sphere, while $\bar{\phi}$ is the Dirac conjugate as per Eq. (A7). As in the previous section, both ϕ and ψ are here implicitly evaluated at an arbitrary but fixed value of the time coordinate t . Yet, since the Dirac current is covariantly conserved, this choice is irrelevant since the rhs of Eq. (24) does not depend on time. Furthermore the conformal rescaling entails that the inner product as per Eq. (24) becomes

$$\langle \phi, \psi \rangle_{\text{GAdS}_n} \doteq \int_0^{\frac{\pi}{2}} d\rho \int_{\mathbb{S}^{n-2}} dS^{n-2} \bar{\Phi} \Upsilon^0 \Psi = \langle \Phi, \Psi \rangle_{\tilde{\mathcal{M}}_n}, \quad (25)$$

where $\Phi = \sec^{\frac{1-n}{2}} \rho \phi$, while the integral representation coincides with the natural inner product between spinors in $\tilde{\mathcal{M}}_n$. As in the previous section, it is convenient to split $\Psi = \Psi^{(1)} \oplus \Psi^{(2)}$ and to write

$$\Upsilon^{\mu} \nabla_{\mu} = \begin{pmatrix} -D_{S^{n-1}} & i\partial_t \\ i\partial_t & D_{S^{n-1}} \end{pmatrix}, \quad (26)$$

where $D_{S^{n-1}}$ is the Dirac operator on S^{n-1} , the unit $(n-1)$ -sphere. This yields the system

$$\begin{aligned} (D_{S^{n-1}} + m \sec \rho) \Psi^{(1)} &= i\partial_t \Psi^{(2)} \\ (D_{S^{n-1}} - m \sec \rho) \Psi^{(2)} &= -i\partial_t \Psi^{(1)}. \end{aligned} \quad (27)$$

Since the Dirac operator on S^{n-1} is time independent, by applying to the second equation ∂_t and inserting the first one, it descends that

$$\left(-\frac{\partial^2}{\partial t^2} + D_{S^{n-1}}^2 + \hat{\gamma}^0 m \sec \rho \tan \rho - m^2 \sec^2 \rho\right) \Psi^{(1)} = 0. \quad (28)$$

As discussed thoroughly in Ref. [27], $D_{S^{n-1}}$ has a discrete, completely imaginary spectrum, $\lambda_k = i(k + \frac{n-1}{2})$, $k \in \mathbb{Z}$. Its eigenfunctions are the *spinor spherical harmonics* $\zeta_{k,m}(\rho, \Omega^i)$, where $m = -|k|, \dots, |k|$, while $\{\Omega^i\}_{i=1, \dots, n-2}$ denote the standard angular coordinates of S^{n-2} . These harmonics are orthogonal with respect to the $L^2(S^{n-1})$ inner product and form a complete basis for the space of square integrable spinors on S^{n-1} . Assuming that $\Psi^{(1)}$ is a tempered distribution along the time direction, we can take the Fourier transform and write

$$\Psi^{(1,\pm)}(t, \rho, \Omega^i) = \sum_{k \in \mathbb{Z}} \sum_{m=-k}^k \int_{\mathbb{R}} \frac{d\omega}{\sqrt{2\pi}} e^{i(\omega t \mp \lambda \rho)} \times \frac{1}{2} (1 \pm \hat{\gamma}^0) f_{\omega, \lambda}^{(1,\pm)}(\rho) \zeta_{\lambda}(\rho, \Omega^i). \quad (29)$$

The superscripts $\pm$ indicate that we are splitting $\Psi^{(1)}$ further into two half-spinors. In fact, one can verify that $\frac{1}{2}(1 \pm \hat{\gamma}^0)$ is the projector on the upper-half or lower-half components for the half-spinors $\Psi^{(1,2)}$. The reader should be aware of the fact that, in general, the sum in Eq. (29) should be regarded as a multi-indexed sum over a countable set which takes into account also the multiplicities of the eigenvalues. Combining Eqs. (28) and (29), we end up with the following ordinary differential equation:

$$\left(\frac{d^2}{d\rho^2} \pm m \sec \rho \tan \rho - m^2 \sec^2 \rho + \omega^2\right) f_{\omega, \lambda}^{(1,\pm)}(\rho) = 0. \quad (30)$$

Generalizing the analysis in Ref. [15] for $n = 2$, the linearly independent solutions of Eq. (30) read

$$f_{\omega, \lambda}^{(1,\pm)}(\rho) = (2\nu_{\pm} + 1) \sigma(\rho)^{\pm m} F\left(\omega, -\omega; \nu_{\pm}; \frac{1 - \sin \rho}{2}\right),$$

$$h_{\omega, \lambda}^{(1,\pm)}(\rho) = \omega \cos \rho \sigma(\rho)^{\mp m} F\left(1 + \omega, 1 - \omega; 2 - \nu_{\pm}; \frac{1 - \sin \rho}{2}\right), \quad (31)$$

where $\nu_{\pm} \doteq \pm m + \frac{1}{2}$, $\sigma(\rho) \doteq (\frac{1 - \sin \rho}{1 + \sin \rho})^{\frac{1}{2}}$, while $F(a, b; c; z)$ denotes the Gaussian hypergeometric function; see Ref. [24]. Eventually, using Eq. (31), one determines $\Psi^{(2)}$ replacing in Eq. (29) $f_{\omega, \lambda}^{(1,\pm)}(\rho)$ with

$$f_{\omega, \lambda}^{(2,\pm)}(\rho) = -\frac{1}{\omega} \left(\pm \frac{d}{d\rho} + m \sec \rho\right) f_{\omega, \lambda}^{(1,\pm)}(\rho). \quad (32)$$

To conclude the section, we highlight that the preceding analysis can be repeated also for n odd, though taking into account the differences in the form of the Υ -matrices. Hence Eqs. (29) and (31) still apply. Observe also that, *mutatis mutandis*, the content of Remark 2.1 also applies to this scenario.

III. BOUNDARY CONDITIONS FOR THE DIRAC EQUATION

Having constructed the solutions of the Dirac equation both on PAdS_n and on GAdS_n , we investigate the class of admissible boundary conditions which can be assigned to the underlying dynamical problem in addition to initial data, so to single out a unique solution. In Ref. [10], when discussing the same issue for a real scalar field, one had to resort to using techniques proper of Sturm-Liouville problems; see Refs. [23, 28]. On the contrary, since the Dirac operator identifies a first-order partial differential equation, we can rely on the fact that both in PAdS_n and in GAdS_n , the Dirac equation can be written in a Schrödinger form, as follows:

$$i\partial_t \psi = H\psi, \quad (33)$$

where, having set for simplicity $\hbar = 1$, t is the time coordinate while H is a suitable differential operator which plays the role of a Hamiltonian. As it will emerge from the following analysis, we shall be interested in interpreting H as a symmetric operator on the space of square integrable spinors living on constant time hypersurfaces with respect to the inner product (10) on PAdS_n and (24) on GAdS_n . Most notably, H shall admit self-adjoint extensions, each of which will be associated to a boundary condition. From the operator theoretical viewpoint, this serves as a constraint on the domain of H itself, while from that of the Dirac equation (7), it will identify an admissible boundary condition yielding in particular a unique advanced and retarded fundamental solution. In the following, we will be employing the theory of deficiency indices, and we refer to Ref. [29] for an introduction to this topic.

A. PAdS_n case

In this section, we consider the Dirac equation on PAdS_n as per Eq. (9), and exploiting the defining properties of the Υ -matrices as per Eq. (A2), we recast it in a Schrödinger-like form, as follows:

$$i\partial_t \psi = i\Upsilon^0 (\Upsilon^k \nabla_k - m) \psi \doteq H\psi, \quad k = 1, \dots, n-1, \quad (34)$$

where $H \doteq i\Upsilon^0 (\Upsilon^k \nabla_k - m)$ is the Dirac Hamiltonian on PAdS_n . As in Sec. II B, we can exploit the conformal

rescaling $\psi \rightarrow \Psi \doteq z^{\frac{1-n}{2}}\psi$ to individuate an equivalent equation on $\mathbb{H}^n$, namely

$$i\partial_t \Psi = i\Upsilon^0 \left(\Upsilon^k \partial_k - \frac{m}{z} \right) \Psi \doteq H \Psi, \quad k = 1, \dots, n-1, \quad (35)$$

where $H \doteq i\Upsilon^0(\Upsilon^k \partial_k - \frac{m}{z})$ is the conformal Dirac Hamiltonian. Notice that, with a slight abuse of notation, we are denoting by the same symbols the covariant Υ -matrices on PAdS_n and on $\mathbb{H}^n$, though $\Upsilon_{\mathbb{H}^n}^\mu = \frac{1}{z} \Upsilon_{\text{PAdS}_n}^\mu$. Equations (34) and (35) suggest to read H (respectively, H) as an operator on $L^2(\mathbb{R}^+ \times \mathbb{R}^{n-2}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})$; see Eq. (11) [respectively, on $L^2(\mathbb{H}_\delta^{n-1}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor})}$; see Eq. (14)]. Observe that H is manifestly symmetric, namely

$$\langle \phi, H\psi \rangle_{\text{PAdS}_n} - \langle H\phi, \psi \rangle_{\text{PAdS}_n} = 0, \quad \forall \phi, \psi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-2}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}),$$

and a similar conclusion can be drawn for H . Therefore, it is meaningful to look for their self-adjoint extensions, hence extending the procedure used in Ref. [15] on $G\text{AdS}_2$ and, on account of Eq. (13), it is sufficient to discuss one among the two operators. Focusing on H , we start by using Eq. (8) to write it in the following matrix form:

$$H = i\Upsilon^0 \left(\Upsilon^k \partial_k - \frac{m}{z} \right) = \begin{pmatrix} 0 & -\hat{\gamma}^0 \partial_z - \hat{\gamma}^i \partial_i + \frac{m}{z} \\ \hat{\gamma}^0 \partial_z + \hat{\gamma}^i \partial_i + \frac{m}{z} & 0 \end{pmatrix},$$

where the index i runs over all coordinates barring t and z . In analogy to Eq. (18), since $D(H)$, the domain of H or of any of its self-adjoint extensions must be contained in $L^2(\mathbb{H}_\delta^{n-1}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})$; we can consider the Fourier-Plancherel transform along all directions tangent to the boundary. In other words, for any $\Psi \in D(H)$, it holds that

$$H\Psi = \int_{\mathbb{R}^{n-2}} \frac{d\mathbf{k}}{(2\pi)^{\frac{n-2}{2}}} e^{-i\mathbf{k} \cdot \mathbf{x}} \mathbb{D}_{\mathbf{k}} \Psi_{\mathbf{k}}(z),$$

$$\mathbb{D}_{\mathbf{k}} \doteq \begin{pmatrix} 0 & -\hat{\gamma}^0 \frac{d}{dz} + i\hat{\gamma}^i k_i + \frac{m}{z} \\ \hat{\gamma}^0 \frac{d}{dz} - i\hat{\gamma}^i k_i + \frac{m}{z} & 0 \end{pmatrix}, \quad (36)$$

where the subscript $\mathbf{k}$ highlights the dependence on the momenta. As a consequence, we can focus our attention on $\mathbb{D}_{\mathbf{k}}$, seen as an operator on $L^2((0, \infty); \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})$ where, here, we consider the standard Lebesgue measure. As a first step, we read $\mathbb{D}_{\mathbf{k}}$ as an operator with domain $\text{Dom}(\mathbb{D}_{\mathbf{k}}) \equiv C_0^\infty((0, \infty); \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})$ and, for every $\Psi_{\mathbf{k}}, \Phi_{\mathbf{k}} \in \text{Dom}(\mathbb{D}_{\mathbf{k}})$, it holds integrating by parts that

$$\langle \mathbb{D}_{\mathbf{k}} \Psi_{\mathbf{k}}, \Phi_{\mathbf{k}} \rangle - \langle \Psi_{\mathbf{k}}, \mathbb{D}_{\mathbf{k}} \Phi_{\mathbf{k}} \rangle = [\bar{\Psi}_{\mathbf{k}} \Upsilon^1 \Phi_{\mathbf{k}}]_{z=0} = 0, \quad (37)$$

where $\bar{\Psi}_{\mathbf{k}} \doteq \Psi_{\mathbf{k}}^\dagger \Upsilon^0$ while the Υ -matrices are defined in Eq. (8). This entails that $\mathbb{D}_{\mathbf{k}}$ is a symmetric operator, and we can look for its self-adjoint extensions by using the von Neumann theory of deficiency indices. To this end, we must start by identifying the domain of the adjoint operator, which reads

$$D(\mathbb{D}_{\mathbf{k}}^\dagger) = \{ \Psi_{\mathbf{k}} \in L^2((0, \infty); \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}) | \mathbb{D}_{\mathbf{k}} \Psi_{\mathbf{k}} \in L^2((0, \infty); \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}) \}.$$

Subsequently, we need to characterize the deficiency spaces $\mathcal{N}_\pm(\mathbb{D}_{\mathbf{k}}^\dagger) = \ker(\mathbb{D}_{\mathbf{k}}^\dagger \pm i\mathbb{I})$ and, to this end, we introduce the following notation to denote certain collections of distinguished elements of $\mathbb{C}^{\frac{N}{2}}$, with $N = 2^{\lfloor \frac{n}{2} \rfloor}$, as follows:

$$\begin{aligned} \mathbf{A} &= (a_1, \dots, a_{\frac{N}{4}}, 0, \dots, 0), \\ \mathbf{B} &= (b_1, \dots, b_{\frac{N}{4}}, 0, \dots, 0), \\ \mathbf{C} &= (0, \dots, 0, c_1, \dots, c_{\frac{N}{4}}), \\ \mathbf{D} &= (0, \dots, 0, d_1, \dots, d_{\frac{N}{4}}), \end{aligned} \quad (38)$$

where $\{a_i, b_i, c_i, d_i\}_{i=1, \dots, N/4}$ are arbitrary complex numbers. Consequently, splitting $\Psi = \Psi^{(1)} \oplus \Psi^{(2)}$ as per Sec. II C, Eqs. (19) and (22) entail that any element lying in $\mathcal{N}_\pm(\mathbb{D}_{\mathbf{k}}^\dagger)$ can be written as the following linear combinations:

$$\Psi_{\pm i, \mathbf{k}}^{(1)}(z) = \sqrt{z} [J_\nu(qz) \mathbf{A} + Y_\nu(qz) \mathbf{B} + J_{\nu-1}(qz) \mathbf{C} + Y_{\nu-1}(qz) \mathbf{D}], \quad (39a)$$

$$\begin{aligned} \Psi_{\pm i, \mathbf{k}}^{(2)}(z) &= \mp i \sqrt{z} \{ J_\nu(qz) [q \mathbf{C} + i k_j \hat{\gamma}^j \mathbf{A}] \\ &\quad + Y_\nu(qz) [q \mathbf{D} + i k_j \hat{\gamma}^j \mathbf{B}] \\ &\quad + J_{\nu-1}(qz) [q \mathbf{A} - i k_j \hat{\gamma}^j \mathbf{C}] \\ &\quad - Y_{\nu-1}(qz) [q \mathbf{B} - i k_j \hat{\gamma}^j \mathbf{D}] \}, \end{aligned} \quad (39b)$$

where we have used implicitly Eqs. (20) and (22). In order to deduce whether $\Psi_{\pm i, \mathbf{k}}^{(1)}(z)$ is square integrable, it suffices to control the asymptotic expansion as $z \rightarrow 0$ or $z \rightarrow +\infty$, setting $\omega = \pm i$. Starting from the former and using Ref. [[24], §10], it descends as follows:

$$|\Psi_{\pm i, \mathbf{k}}^{(1)}(z)|^{2z \rightarrow 0} \sim \left[\frac{\text{Im}(q)}{2} \right]^{-2\nu} \frac{\Gamma^2(\nu-1)}{\pi^2} \left\{ (\nu-1)^2 z^{1-2\nu} |\mathbf{B}|^2 + \left[\frac{\text{Im}(q)}{2} \right]^2 z^{3-2\nu} |\mathbf{D}|^2 \right\}, \quad (40)$$

where we have reported only the most singular terms. The symbols $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ respectively denote the real and the imaginary part, while $\Gamma(\cdot)$ is the Euler γ function. As one can infer, the rhs does not depend on $\omega = \pm i$, and we can distinguish three regimes:

- (1) if $\nu \geq 2$, then $\Psi_{\pm i, \mathbf{k}}^{(1)}(z)$ is never square integrable unless $\mathbf{B} = \mathbf{D} = 0$,
- (2) if $1 \leq \nu < 2$, then $\Psi_{\pm i, \mathbf{k}}^{(1)}(z)$ is square integrable only if $\mathbf{B} = 0$, and
- (3) if $0 < \nu < 1$, then $\Psi_{\pm i, \mathbf{k}}^{(1)}(z)$ is always square integrable.

Accordingly, we can study these scenarios separately as $z \rightarrow \infty$. More precisely,

- (1) if $\nu \geq 2$, then

$$\Psi_{\pm i, \mathbf{k}}^{(1)}(z) = \sqrt{z}[J_\nu(qz)\mathbf{A} + J_{\nu-1}(qz)\mathbf{C}].$$

Since $\sqrt{z}J_\nu(qz) \stackrel{z \rightarrow \infty}{\sim} \cos(qz - \frac{i\pi}{2} - \frac{\pi}{4})$ and $q^2 = -1 - |\mathbf{k}|^2$, using Eq. (38), it descends that $\Psi_{\pm i, \mathbf{k}}^{(1)}(z)$ is square integrable if and only if $\mathbf{A} = \mathbf{C} = 0$.

- (2) If $1 \leq \nu < 2$, then

$$\Psi_{\pm i, \mathbf{k}}^{(1)}(z) = \sqrt{z}[J_\nu(qz)\mathbf{A} + J_{\nu-1}(qz)\mathbf{C} + Y_{\nu-1}(qz)\mathbf{D}].$$

Using the same asymptotic expansion as in the previous point and Eq. (38), we can infer that $\mathbf{A}$ must vanish. Accordingly, we are left only with a linear combination between the Bessel functions of first and second kind of order $\nu - 1$. Provided $\text{Im}(q) > 0$, their lone square integrable linear combination is $\sqrt{z}H_{\nu-1}^{(1)}(qz) \stackrel{z \rightarrow \infty}{\sim} \exp[iqz - \frac{i\pi}{2} - \frac{\pi}{4}]$, where $H_{\nu-1}^{(1)}(qz)$ denotes the first Hankel function. On the contrary, if $\text{Im}(q) < 0$, we can draw the same conclusion provided that we replace the first with the second Hankel function $H_{\nu-1}^{(2)}(qz)$. As a consequence, it turns out that $\mathbf{D} = \pm i\mathbf{C}$ depending on the sign of the imaginary part of q .

- (3) If $0 < \nu < 1$, then no constraint to Eq. (39a) needs to be imposed due to the behavior close to $z = 0$. On the contrary, close to infinity, using the asymptotic expansion of the Bessel functions and observing that Eq. (38) entails that the vectors $\mathbf{A}$ and $\mathbf{B}$ are orthogonal to $\mathbf{C}$ and to $\mathbf{D}$, we can infer that $\Psi_{\pm i, \mathbf{k}}^{(1)}(z)$ is square integrable provided that $\mathbf{D} = \pm i\mathbf{C}$ and $\mathbf{B} = \pm i\mathbf{A}$, the sign being dependent on that of the imaginary part of q as explained in the previous point.

We focus our attention on $\Psi_{\pm i, \mathbf{k}}^{(2)}(z)$ taking into account Eq. (39b) and the constraints on the coefficients $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ deduced from the analysis of $\Psi_{\pm i, \mathbf{k}}^{(1)}(z)$:

TABLE I. Summary of Von-Neumann theory for the operator $\mathbb{D}$.

ν	$n_\pm(\mathbb{D}_{\mathbf{k}}^\dagger)$	Expected boundary conditions
$(0, 1)$	$\frac{N}{2}$	Parametrized by elements of $U(\frac{N}{2})$
$[1, +\infty)$	0	No conditions

- (1) If $\nu \geq 2$, we already know that $\mathbf{A} = \mathbf{B} = \mathbf{C} = \mathbf{D} = 0$ and therefore, $\Psi_{\pm i, \mathbf{k}}^{(2)}(z) = 0$. In other words, $n_\pm(\mathbb{D}_{\mathbf{k}}^\dagger) = \dim(\mathcal{N}_\pm(\mathbb{D}_{\mathbf{k}}^\dagger)) = 0$, which is tantamount to saying that $\mathbb{D}_{\mathbf{k}}$ is essentially self-adjoint.
- (2) if $1 \leq \nu < 2$, the preceding analysis yields $\mathbf{A} = \mathbf{B} = 0$ and $\mathbf{C} = \pm i\mathbf{D}$ depending on the sign of $\text{Im}(q)$. Focusing on $\text{Im}(q) > 0$, it turns out that

$$\Psi_{\pm i, \mathbf{k}}^{(2)}(z) = \mp i\sqrt{z}[qH_\nu^{(1)}(qz) + iH_{\nu-1}^{(1)}(qz)\hat{\gamma}^j k_j]\mathbf{C}.$$

Taking into account that as $z \rightarrow 0$, $H_\nu^{(1)}(qz) \propto (qz)^{-\nu}$, we need to impose $\mathbf{C} = 0$ in order for $\Psi_{\pm i, \mathbf{k}}^{(2)}(z)$ to be square integrable. Hence $\mathbb{D}_{\mathbf{k}}$ is essentially self-adjoint.

- (3) If $0 < \nu < 1$, the preceding analysis yields $\mathbf{A} = \pm i\mathbf{B}$ and $\mathbf{C} = \pm i\mathbf{D}$ depending on the sign of $\text{Im}(q)$. Considering for definiteness $\text{Im}(q) > 0$, this entails that

$$\Psi_{\mathbf{k}}^{(2)}(z) = \frac{\sqrt{z}}{\omega} \left\{ H_\nu^{(1)}(qz)[q\mathbf{C} + i\hat{\gamma}^j k_j \mathbf{A}] - H_{\nu-1}^{(1)}(qz)[q\mathbf{A} - i\hat{\gamma}^j k_j \mathbf{C}] \right\},$$

which is square integrable both at the origin and at infinity. To summarize, in this case, $n_\pm(\mathbb{D}_{\mathbf{k}}^\dagger) = \dim(\mathcal{N}_\pm(\mathbb{D}_{\mathbf{k}}^\dagger)) = \frac{N}{2}$ and, hence, on account of von Neumann theory of deficiency indices, the number of self-adjoint extensions is in one-to-one correspondence with the elements of the unitary group $U(\frac{N}{2})$.

To conclude, we summarize in Table I the results obtained for $\Psi_{\pm i, \mathbf{k}}^{(1,2)}(z)$.

We conclude this section by showing how to associate specific boundary conditions to each self-adjoint extension of the operator $\mathbb{D}_{\mathbf{k}}$. For definiteness, we shall focus on the mass range $\nu \in (0, 1)$. As a guiding example we consider first the case $\nu = \frac{1}{2}$, namely $m = 0$. This is a special scenario since all singular terms in Eq. (19) vanish, and this entails that all solutions of Eq. (19) are regular at $z = 0$. Therefore, one can equivalently write Eq. (37) as

$$\begin{aligned} & [\Phi_{\omega,\mathbf{k}}^{(1)}(0) + i\hat{\gamma}^0\Phi_{\omega,\mathbf{k}}^{(2)}(0)]^\dagger [\Psi_{\omega,\mathbf{k}}^{(1)}(0) + i\hat{\gamma}^0\Psi_{\omega,\mathbf{k}}^{(2)}(0)] \\ &= [\Phi_{\omega,\mathbf{k}}^{(1)}(0) - i\hat{\gamma}^0\Phi_{\omega,\mathbf{k}}^{(2)}(0)]^\dagger [\Psi_{\omega,\mathbf{k}}^{(1)}(0) - i\hat{\gamma}^0\Psi_{\omega,\mathbf{k}}^{(2)}(0)], \end{aligned}$$

where $\hat{\gamma}^0 = i\gamma^0$; see Appendix A. If we assume that the boundary values $\Phi_{\omega,\mathbf{k}}^{(1)}(0) \pm i\hat{\gamma}^0\Phi_{\omega,\mathbf{k}}^{(2)}(0)$ are related by a linear transformation, which is indeed the case since the Dirac equation is linear, then it descends that

$$\begin{aligned} \Phi_{\omega,\mathbf{k}}^{(1)}(0) + i\hat{\gamma}^0\Phi_{\omega,\mathbf{k}}^{(2)}(0) &= U[\Phi_{\omega,\mathbf{k}}^{(1)}(0) \\ &\quad - i\hat{\gamma}^0\Phi_{\omega,\mathbf{k}}^{(2)}(0)], \quad U \in U\left(\frac{N}{2}\right), \end{aligned}$$

and the same applies for $\Psi_{\omega,\mathbf{k}}^{(1)}(0) \pm i\hat{\gamma}^0\Psi_{\omega,\mathbf{k}}^{(2)}(0)$. Thus, the boundary conditions for $m = 0$ are

$$(\mathbf{1} - U)\Psi_{\omega,\mathbf{k}}^{(1)}(0) + i(\mathbf{1} + U)\hat{\gamma}^0\Psi_{\omega,\mathbf{k}}^{(2)}(0) = 0, \quad (41)$$

where $U \in U(\frac{N}{2})$. Let us now assume that $\Phi_{\omega,\mathbf{k}}(z)$, $\Psi_{\omega,\mathbf{k}}(z) \in L^2((0, \infty); \mathbb{C}^{2[\frac{N}{2}]})$ and that $\Phi_{\omega,\mathbf{k}}(z)$ also abides by Eq. (41). Bearing in mind Eq. (37), it turns out that

$$\begin{aligned} \langle \mathbb{D}_{\mathbf{k}} \Phi_{\omega,\mathbf{k}}, \Psi_{\omega,\mathbf{k}} \rangle - \langle \Phi_{\omega,\mathbf{k}}, \mathbb{D}_{\mathbf{k}} \Psi_{\omega,\mathbf{k}} \rangle &= \bar{\Phi}_{\omega,\mathbf{k}}(0) \Upsilon^1 \Psi_{\omega,\mathbf{k}}(0) \\ &= \Phi_{\omega,\mathbf{k}}^{(1)\dagger}(0) \hat{\gamma}^0 \Psi_{\omega,\mathbf{k}}^{(2)}(0) - \Phi_{\omega,\mathbf{k}}^{(2)\dagger}(0) \hat{\gamma}^0 \Psi_{\omega,\mathbf{k}}^{(1)}(0) \\ &= \frac{1}{2i} [\Phi_{\omega,\mathbf{k}}^{(1)}(0) - i\hat{\gamma}^0\Phi_{\omega,\mathbf{k}}^{(2)}(0)]^\dagger \left\{ U^\dagger [\Psi_{\omega,\mathbf{k}}^{(1)}(0) + i\hat{\gamma}^0\Psi_{\omega,\mathbf{k}}^{(2)}(0)] - [\Psi_{\omega,\mathbf{k}}^{(1)}(0) - i\hat{\gamma}^0\Psi_{\omega,\mathbf{k}}^{(2)}(0)] \right\}, \end{aligned}$$

where we have exploited that we are working in the regular case and that Eq. (41) holds for $\Phi_{\omega,\mathbf{k}}$. The rhs vanishes if

$$(\mathbf{1} - U)\Psi_{\omega,\mathbf{k}}^{(1)}(0) + i(\mathbf{1} + U)\hat{\gamma}^0\Psi_{\omega,\mathbf{k}}^{(2)}(0) = 0.$$

Hence, we have individuated an extension of $\mathbb{D}_{\mathbf{k}}$ which we denote by $\mathbb{D}_{\mathbf{k}}^U$ such that $\mathbb{D}_{\mathbf{k}}^U = (D^U)_{\mathbf{k}}^\dagger$ and that

$$\begin{aligned} \text{Dom}(\mathbb{D}_{\mathbf{k}}^U) &= \{ \Psi_{\mathbf{k}} \in L^2((0, \infty); \mathbb{C}^{2[\frac{N}{2}]}) | (\mathbf{1} - U)\Psi_{\omega,\mathbf{k}}^{(1)}(0) \\ &\quad + i(\mathbf{1} + U)\hat{\gamma}^0\Psi_{\omega,\mathbf{k}}^{(2)}(0) = 0 \}. \end{aligned}$$

Keeping Eq. (41) as an inspiration, we extend our conclusion to the case when $\Psi_{\omega,\mathbf{k}}(z)$ is singular at $z = 0$. To this end, we introduce the $\frac{N}{2} \times \frac{N}{2}$ rescaling matrix $R(z) = \text{diag}(z^{\nu-\frac{1}{2}}\mathbf{1}, z^{-\nu-\frac{1}{2}}\mathbf{1})$ and we define the rescaled spinors as

$${}_R\Psi_{\omega,\mathbf{k}}^{(1)}(z) = R(z)\Psi_{\omega,\mathbf{k}}^{(1)}(z), \quad {}_R\Psi_{\omega,\mathbf{k}}^{(2)}(z) = R^{-1}(z)\Psi_{\omega,\mathbf{k}}^{(2)}(z), \quad (42)$$

where $\Psi_{\omega,\mathbf{k}}^{(1,2)}(z)$ are as per Eq. (20). Observe that the rescaled spinor components are regular at $z = 0$ since, exploiting Eq. (39) for $\nu \in (\frac{1}{2}, 1)$, we obtain

$$\begin{aligned} \lim_{z \rightarrow 0} [{}_R\Psi_{\omega,\mathbf{k}}^{(1)}(z)] &= \begin{pmatrix} -\frac{2^\nu \Gamma(\nu)}{\pi} \mathbf{B} \\ \frac{2^{1-\nu}}{\Gamma(\nu)} \mathbf{C} \end{pmatrix}, \\ \lim_{z \rightarrow 0} [{}_R\Psi_{\omega,\mathbf{k}}^{(2)}(z)] &= \begin{pmatrix} -\frac{2^\nu \Gamma(\nu)}{\pi} [q\mathbf{D} + ik_j \hat{\gamma}^j \mathbf{B}] \\ \frac{2^{1-\nu}}{\Gamma(\nu)} [q\mathbf{A} - ik_j \hat{\gamma}^j \mathbf{C}] \end{pmatrix}, \quad (43) \end{aligned}$$

where we have used the asymptotic expansion of the Bessel functions as $z \rightarrow 0^+$. The case $\nu \in (0, \frac{1}{2})$ is obtained similarly. We remind the reader that for $\nu \in (0, 1)$, an element of Eq. (39) lies in the domain of a self-adjoint extension of $\mathbb{D}_{\mathbf{k}}$ once we fix one between the parameters $(\mathbf{A}, \mathbf{B})$ and one between those denoted by $(\mathbf{C}, \mathbf{D})$. Thus, the boundary values of the rescaled spinors encode all data of the solution. We show how Eq. (41) can be rewritten in terms of rescaled spinors. For every $\Psi_{\omega,\mathbf{k}}, \Phi_{\omega,\mathbf{k}} \in L^2((0, \infty); \mathbb{C}^{2[\frac{N}{2}]})$, it holds that

$$\begin{aligned} &\lim_{z \rightarrow 0} \{ \bar{\Phi}_{\omega,\mathbf{k}}(z) \Upsilon^1 \Psi_{\omega,\mathbf{k}}(z) \} \\ &= \lim_{z \rightarrow 0} \{ \Phi_{\omega,\mathbf{k}}^{(1)\dagger}(z) \hat{\gamma}^0 \Psi_{\omega,\mathbf{k}}^{(2)}(z) - \Phi_{\omega,\mathbf{k}}^{(2)\dagger}(z) \hat{\gamma}^0 \Psi_{\omega,\mathbf{k}}^{(1)}(z) \} \\ &= \lim_{z \rightarrow 0} \{ {}_R\Phi_{\omega,\mathbf{k}}^{(1)\dagger}(z) \hat{\gamma}^0 {}_R\Psi_{\omega,\mathbf{k}}^{(2)}(z) - {}_R\Phi_{\omega,\mathbf{k}}^{(2)\dagger}(z) \hat{\gamma}^0 {}_R\Psi_{\omega,\mathbf{k}}^{(1)}(z) \} \\ &= {}_R\Phi_{\omega,\mathbf{k}}^{(1)\dagger}(0) \hat{\gamma}^0 {}_R\Psi_{\omega,\mathbf{k}}^{(2)}(0) - {}_R\Phi_{\omega,\mathbf{k}}^{(2)\dagger}(0) \hat{\gamma}^0 {}_R\Psi_{\omega,\mathbf{k}}^{(1)}(0). \end{aligned}$$

On account of the rescaling, the evaluation at $z = 0$ is well defined and, therefore, we can repeat the same analysis as for $\nu = \frac{1}{2}$, drawing exactly the same conclusions. Hence, the most general boundary condition for $\nu \in (0, 1)$ which identifies $\mathbb{D}_{\mathbf{k}}^U$, a self-adjoint extension of $\mathbb{D}_{\mathbf{k}}$, reads

$$(\mathbf{1} - U)_R\Psi_{\omega,\mathbf{k}}^{(1)}(0) + i(\mathbf{1} + U)\hat{\gamma}^0{}_R\Psi_{\omega,\mathbf{k}}^{(2)}(0) = 0, \quad (44)$$

where $U \in U(\frac{N}{2})$ while $\hat{\gamma}^0 = i\gamma^0$ is as per Appendix A and $\Psi_{\omega,\mathbf{k}}^{(1,2)} \in L^2((0, \infty); \mathbb{C}^{2[\frac{N}{2}]})$.

Remark 3.1. In 1975, while studying hadron models, K. Johnson identified a set of self-adjoint boundary conditions

for the Dirac equation on a subset of Minkowski spacetime; see Ref. [14]. These are known as MIT-bag boundary conditions and can be expressed as $n_\mu \Upsilon^\mu \Psi|_{\partial\mathcal{M}} = \pm \Psi|_{\partial\mathcal{M}}$, where $\mathcal{M}$ denotes the underlying manifold; $\partial\mathcal{M}$ its boundary; while n_μ is an outward pointing unit vector, orthogonal to $\partial\mathcal{M}$. In our case, the MIT-bag boundary conditions can be written as $\Upsilon^1_R \Psi(0) = \pm_R \Psi(0)$, where ${}_R \Psi(0) = {}_R \Psi^{(1)}(0) \oplus {}_R \Psi^{(2)}(0)$ as per Eq. (42).

Notice that, starting from Eq. (43), we can recover these distinguished boundary conditions for suitable unitary matrices U , though we postpone the discussion to Sec. V. Henceforth, we refer to Eq. (44) as the generalized MIT-bag boundary condition.

Physically, the MIT-bag boundary conditions were introduced as an effective description of quark confinement. Practically speaking, in Ref. [14], Johnson models a hadron as a bag, i.e., a subregion of Minkowski spacetime inside which quark fields are nonidentically vanishing, and inside which one can find all the stress-energy content of the hadron. As a consequence, one has to require that the normal component of the Dirac current vanishes at the boundary of the bag, so that no energy-momentum flux can cross the wall. This boils down to choosing a self-adjoint realization of the Dirac Hamiltonian. In our setup, the conformal boundary plays the role of an idealized, perfectly reflecting surface. Imposing MIT-bag at conformal infinity therefore selects a self-adjoint realization of the Dirac Hamiltonian. It yields a unitary time evolution, while different unitary matrices U can be interpreted as more general reflective boundary couplings, potentially supporting boundary-localized modes, as observed in the bosonic case, see Ref. [18].

We conclude this section by collecting our finding on the admissible self-adjoint boundary conditions in Table II.

TABLE II. Boundary conditions associated to all self-adjoint extensions of $\mathbb{D}_k$.

ν	Boundary conditions
$\frac{1}{2}$	$(\mathbf{1} - U)\Psi^{(1)}(0) + i(\mathbf{1} + U)\hat{\gamma}^0\Psi^{(2)}(0) = 0$
$(0, 1) \setminus \{\frac{1}{2}\}$	$(\mathbf{1} - U)_R \Psi^{(1)}(0) + i(\mathbf{1} + U)_R \hat{\gamma}^0 \Psi^{(2)}(0) = 0$
$[1, +\infty)$	No conditions

B. GAdS_n case

In this subsection, we consider the global patch of the n -dimensional AdS spacetime, namely GAdS_n . Also in this case, we are interested in identifying the admissible boundary conditions which can be supplemented to the Dirac equation (7). Yet, the analysis follows closely the steps already highlighted when considering PAdS_n and the Poincaré patch, and, therefore, we limit ourselves to stating the main results and highlighting the minor difference with the preceding section. It is worth mentioning that in Ref. [15], a thorough analysis on GAdS_2 is available. The starting point consists of writing on GAdS_n the counterpart of the conformal Hamiltonian individuated in Eq. (35). It reads

$$\tilde{H} = -\Upsilon^0(\Upsilon^k \nabla_k - m \sec \rho), \quad k = 1, \dots, n-1,$$

where ∇ denotes the spinor covariant derivative specified for the conformally related manifold $\tilde{\mathcal{M}}_n$; see Eq. (A4) and Sec. II. Using Eq. (29), we introduce the counterpart of the operator $\mathbb{D}_k$ as

$$\tilde{H} \sum_{\lambda} \underline{f}_{\omega, \lambda}(\rho) \underline{\zeta}_{\lambda}(\rho, \Omega^i) = \sum_{\lambda} [\tilde{\mathbb{D}} \underline{f}_{\omega, \lambda}(\rho)] \underline{\zeta}_{\lambda}(\rho, \Omega^i),$$

$$\tilde{\mathbb{D}} \doteq \begin{pmatrix} 0 & 0 & -\frac{d}{d\rho} + m \sec \rho & 0 \\ 0 & 0 & 0 & \frac{d}{d\rho} + m \sec \rho \\ \frac{d}{d\rho} + m \sec \rho & 0 & 0 & 0 \\ 0 & -\frac{d}{d\rho} + m \sec \rho & 0 & 0 \end{pmatrix}, \quad (45)$$

where $\underline{f}_{\omega, \lambda}(\rho) \underline{\zeta}_{\lambda}(\rho, \Omega^i)$ denotes a vector whose components are split in two halves. The first takes the form $e^{\mp i \lambda \rho} f_{\omega, \lambda}^{(1, \pm)}(\rho) \zeta_{\lambda}(\rho, \Omega^i)$, while the second one takes the form $e^{\mp i \lambda \rho} f_{\omega, \lambda}^{(2, \pm)}(\rho) \zeta_{\lambda}(\rho, \Omega^i)$. Notice how Eq. (29) entails that $\tilde{\mathbb{D}}$ is a 4×4 matrix regardless of the dimension of GAdS_n , with $n > 3$. Once more, $\tilde{\mathbb{D}}$ as per Eq. (45) can be

regarded as a higher-dimensional generalization of the one discussed for $n = 2$ in Ref. [15], which in turn also applies to the scenario with $n = 3$. The induced inner product starting from Eqs. (24) and (25) reads

$$\langle f_{\omega, \lambda}, g_{\omega, \lambda} \rangle = \sum_{\lambda} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\rho f_{\omega, \lambda}^{\dagger}(\rho) g_{\omega, \lambda}(\rho),$$

where $f_{\omega,\lambda}, g_{\omega,\lambda} \in L^2((0, \frac{\pi}{2}); \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})$ have, respectively, components $f^{(1,\pm)}, f^{(2,\pm)}$, and $g^{(1,\pm)}, g^{(2,\pm)}$. As one could infer, $\tilde{\mathbb{D}}$ is symmetric with respect to this inner product only if it vanishes, as follows:

$$\langle \tilde{\mathbb{D}} f_{\omega,\lambda}, g_{\omega,\lambda} \rangle - \langle f_{\omega,\lambda}, \tilde{\mathbb{D}} g_{\omega,\lambda} \rangle \propto \sum_{\lambda} [\bar{f}_{\omega,\lambda} \Upsilon^1 g_{\omega,\lambda}]_0^{\frac{\pi}{2}}, \quad (46)$$

where $\bar{f} = f^\dagger \Upsilon^0$. It suffices to consider $C_0^\infty(0, \frac{\pi}{2})$ to ensure that this holds true and hence to conclude that $\tilde{\mathbb{D}}$ is a symmetric operator on $L^2(0, \frac{\pi}{2})$. In order to characterize the self-adjoint extensions of $\tilde{\mathbb{D}}$, one should proceed exactly as in the previous section. Since *mutatis mutandis* the procedure is identical, we list only the final result:

- (i) If $\nu \doteq m + \frac{1}{2} \in (0, 1)$ the deficiency indices associated to $\tilde{\mathbb{D}}$ and to its adjoint $\tilde{\mathbb{D}}^\dagger$ are $n_\pm(\tilde{\mathbb{D}}^\dagger) = 4$.
- (ii) If $\nu \geq 1$ $\tilde{\mathbb{D}}$ is essentially self-adjoint, thus $n_\pm(\tilde{\mathbb{D}}^\dagger) = 0$.

To conclude, we exploit once more the analogy with Sec. III A and Ref. [15] in order to write the sought boundary conditions on GAdS_n :

- (i) For $\nu \in (0, 1)$ we introduce the rescaling factor $R(\rho) = \sigma(\rho)^{|\nu - \frac{1}{2}|}$, where $\sigma(\rho)$ is as per Eq. (31). The boundary conditions are then given by

$$(\mathbf{1} - U) \begin{pmatrix} {}_R f^{(2)}(\frac{\pi}{2}) \\ {}_R f^{(2)}(0) \end{pmatrix} = i(\mathbf{1} + U) \begin{pmatrix} {}_R f^{(1)}(\frac{\pi}{2}) \\ {}_R f^{(1)}(0) \end{pmatrix}, \quad (47)$$

where $U \in U(4)$, while

$${}_R f^{(1)}(\rho) = R(\rho) f^{(1)}(\rho), \quad {}_R f^{(2)}(\rho) = R^{-1}(\rho) f^{(2)}(\rho). \quad (48)$$

- (ii) For $\nu \geq 1$, $\mathbb{D}$ is essentially self-adjoint. Thus, we do not need to impose any boundary condition on the square integrable spinors.

Also in this scenario, a suitable choice of U is tantamount to selecting boundary conditions as in the MIT-bag model [14].

IV. GREEN'S OPERATOR FOR THE DIRAC EQUATION

In this section, we derive an expression for the advanced G^- and retarded G^+ fundamental solutions for the Dirac equation on PAdS_n for each of the boundary conditions which have been individuated in Sec. II B. We focus on this scenario since on GAdS_n the computations are analogous; see Secs. III A and III B. We recall that a fundamental solution is a map $\tilde{\mathcal{G}}^\pm: C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-1}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}) \rightarrow C^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-1}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})$ such that

$$\tilde{P} \circ \tilde{\mathcal{G}}^\pm = \tilde{\mathcal{G}}^\pm \circ \tilde{P}|_{C_0^\infty} = \text{Id}_{C_0^\infty}, \quad (49)$$

where $\tilde{P}$ is the Dirac operator as per Eq. (9). Equivalently, we look for a bidistribution $\tilde{\mathcal{G}}^\pm \in \mathcal{D}'((\mathbb{R}^+ \times \mathbb{R}^{n-1}) \times (\mathbb{R}^+ \times \mathbb{R}^{n-1}); \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}} \otimes (\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})^*)$ such that, working at the level of integral kernels,

$$(\tilde{P} \otimes \mathbb{I}) \tilde{\mathcal{G}}^\pm(x, x') = (\mathbb{I} \otimes \tilde{P}) \tilde{\mathcal{G}}^\pm(x, x') = \delta(x - x') \mathbf{1}_{\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}}, \quad (50)$$

where $\delta(x - y)$ is the Dirac δ -distribution on $\mathbb{R}^+ \times \mathbb{R}^{n-1}$, $\tilde{P}$ is the Dirac operator as per Eq. (9), while $\tilde{\tilde{P}}$ is the anti-Dirac operator defined in Eq. (A8). Once more, it is convenient to consider a conformally equivalent problem on $\mathbb{H}^n$; see Sec. II B. To establish the correspondence, we recall that any $\psi: \mathbb{R}^+ \times \mathbb{R}^{n-1} \rightarrow \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$ is conformally related to $\Psi := z^{\frac{1-n}{2}} \psi: \mathbb{H}^n \rightarrow \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$ such that $P\psi = z^{\frac{n+1}{2}} \tilde{P}\Psi$ where P is the Dirac operator on PAdS_n while $\tilde{P}$ is defined in Eq. (12). Equation (49) entails that, denoting by $\mathcal{G}^\pm$ the retarded and advanced fundamental solutions associated to P , these are related to $\tilde{\mathcal{G}}^\pm$ via the identity

$$\mathcal{G}^\pm = z^{\frac{n-1}{2}} \circ \tilde{\mathcal{G}}^\pm \circ z^{-\frac{n+1}{2}}.$$

Hence, we can focus our attention on the construction of the retarded fundamental solutions directly on $\mathbb{H}^n$, so to avoid considering the test functions whose support intersects the boundary at $z = 0$. The most notable advantage is that the identity

$$\begin{aligned} S \doteq P \circ P' &= \left(\Upsilon^\nu \partial_\nu + \frac{m}{z} \mathbf{1} \right) \circ \left(\Upsilon^\mu \partial_\mu - \frac{m}{z} \mathbf{1} \right) \\ &= \eta^{\mu\nu} \partial_\mu \partial_\nu \mathbf{1} + \Upsilon^1 \frac{m}{z^2} - \frac{m^2}{z^2} \mathbf{1}, \end{aligned} \quad (51)$$

entails that we construct the advanced and retarded fundamental solution $\mathcal{K}^\pm$ for S , since

$$\mathcal{G}^\pm = \left(\Upsilon^\nu \partial_\nu + \frac{m}{z} \mathbf{1} \right) \mathcal{K}^\pm. \quad (52)$$

In turn, since $\Upsilon^1 = \text{diag}(-\mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{1})$ as per Eq. (8), we can infer that

$$\mathcal{K}^\pm = \begin{pmatrix} \tilde{K}_\nu^\pm \mathbf{1}_{\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor - 1}}} & 0 & 0 & 0 \\ 0 & \tilde{K}_{\nu-1}^\pm \mathbf{1}_{\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor - 1}}} & 0 & 0 \\ 0 & 0 & \tilde{K}_{\nu-1}^\pm \mathbf{1}_{\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor - 1}}} & 0 \\ 0 & 0 & 0 & \tilde{K}_\nu^\pm \mathbf{1}_{\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor - 1}}} \end{pmatrix}, \quad (53)$$

where $\tilde{K}_\nu^\pm$ are the advanced and retarded fundamental solution of the Klein-Gordon equation on $\mathbb{H}^n$ with mass $\nu = m + \frac{1}{2}$. We denote by $\mathcal{K}(x, x')$ the integral kernel of the advanced-minus-retarded Green's operator for S ; see Ref. [10], and by $G(x, x')$ its counterpart for P as per Eq. (52). The former can be determined once we fix any admissible boundary condition; that is, Eq. (53) ought to satisfy Eq. (44). In order to translate this statement in a condition that $\tilde{K}_\nu(x, x')$ has to abide by, we proceed as follows: taking once more Ref. [10] as an inspiration and using the notation introduced in Eq. (18), we write

$$G(x, x') = \int_{\mathbb{R}^{n-2}} \frac{d\mathbf{k}}{(2\pi)^{\frac{n-2}{2}}} e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} e^{i\omega(t-t')} \mathbf{G}_{\omega, \mathbf{k}}(z, z'), \quad (54)$$

where $G^- = \Theta(t' - t)G$, $G^+ = -\Theta(t' - t)G$ and

$$P_x G(x, x') = 0, \quad (55a)$$

$$G(x, x') \overline{P_{x'}} = 0, \quad (55b)$$

$$G(x, x')|_{t=t'} = \Upsilon^0 \delta(z - z') \delta(\mathbf{x} - \mathbf{x}'), \quad (55c)$$

in addition to a boundary condition as per Table II. Replacing Eq. (54) in Eq. (55c) yields

$$\int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \mathbf{G}_{\omega, \mathbf{k}}(z, z') = \Upsilon^0 \delta(z - z'). \quad (56)$$

As one can infer from Eqs. (54), (55a), and (55b), the columns of $\mathbf{G}_{\omega, \mathbf{k}}(z, z')$, regarded as functions of z , are eigenfunctions of $\mathbb{D}_{\mathbf{k}}$ associated to the eigenvalue ω , while its rows, regarded as functions of z' , are conjugated eigenfunctions associated to the same eigenvalue. In other words, Eq. (56) establishes a spectral representation of the δ -distribution in terms of eigenfunctions of $\mathbb{D}_{\mathbf{k}}$ subject to boundary conditions as per Table II.

Remark 4.1. We observe that, combining Eq. (55) with (a minor adaptation of) Refs. [30,31] entails that the singularities of $\mathcal{G}^\pm$ are supported both along light rays and along their reflection at the conformal boundary located at $z = 0$. As a byproduct, whenever one considers a globally hyperbolic region $\mathcal{O}$ of $\mathbb{R}^+ \times \mathbb{R}^{n-1}$, the restriction $\mathcal{G}^\pm|_{\mathcal{O} \times \mathcal{O}}$

differs from the propagators of the Dirac operator on $\mathcal{O}$ by a smooth term. This entails, in turn, that if one constructs the C^* -algebra of observables for the Dirac field with prescribed boundary conditions as per Eq. (55), this abides by the principle of F-locality as per Ref. [13]. Nonetheless, we highlight once more that this property alone does not guarantee *a priori* a physically sensible quantum field theory; e.g., it does not guarantee that the $*$ -algebra of observables is causal. In order to make that happen, one needs to prove that Eq. (55) entails

$$(x, x') \in \text{supp } G(x, x') \Rightarrow x \sim x',$$

where $x \sim x'$ means that $x, x' \in \text{PAdS}_n$ can be connected by a casual, possibly reflected, curve. For the time being, we postpone the investigation of this property, and thus the construction of a physically reliable $*$ -algebra, to future works.

Constructing $\mathbf{G}_{\omega, \mathbf{k}}(z, z')$: Eq. (56) entails that, barring a factor Υ^0 , $\mathbf{G}_{\omega, \mathbf{k}}(z, z')$ can be constructed out of $\mathbb{D}_{\mathbf{k}}$ as per Eq. (36), solving

$$(\mathbb{D}_{\mathbf{k}} + \omega) G_{\mathbb{D}_{\mathbf{k}}}(z, z'; \omega) = \delta(z - z'), \quad (57)$$

together with one among the boundary conditions as per Table II. We recall that $\mathbb{D}_{\mathbf{k}}$ in Eq. (57) is acting on z and we write $G_{\mathbb{D}_{\mathbf{k}}}(z, z'; \omega)$ as

$$G_{\mathbb{D}_{\mathbf{k}}}(z, z'; \omega) = \begin{cases} \mathcal{F}_\omega(z) C_<(z'), & z < z' \\ \mathcal{F}_\omega(z) C_>(z'), & z > z', \end{cases} \quad (58)$$

where setting $N = 2^{\lfloor \frac{n}{2} \rfloor}$, $\mathcal{F}_\omega(z)$ is the $N \times N$ matrix whose columns are the solutions listed in Eqs. (20) and (21). On the contrary, $C_\geq(z')$ are $N \times N$ matrices to be determined from Eq. (57) and from the boundary conditions listed in Table II. In the following, we investigate more in detail the conditions that they have to fulfill, splitting the analysis between two scenarios depending on ν and, thus, on the mass of the field. We highlight that, in this section, we do not determine explicitly their expression; rather, we establish the algorithm to determine them. In Sec. V, we will apply it to two concrete cases.

Case $\nu \in (0, 1)$: for $0 < z < z' < +\infty$ and denoting by U a matrix which identifies an admissible boundary condition, $G_{\mathbb{D}_{\mathbf{k}}}(z, z'; \omega)$ has to fulfill

$$(\mathbf{1} - U)_R G_{\mathbb{D}_{\mathbf{k}}}^{(1)}(0, z'; \omega) + i(\mathbf{1} + U) \hat{\gamma}^0_R G_{\mathbb{D}_{\mathbf{k}}}^{(2)}(0, z'; \omega) = 0,$$

where ${}_R G_{\mathbb{D}_{\mathbf{k}}}^{(1,2)}(z, z'; \omega)$ denote the rescaled upper/lower half of the matrix $G_{\mathbb{D}_{\mathbf{k}}}(z, z'; \omega)$ as per Eq. (42). Equation (58) yields

$$\{(\mathbf{1} - U)[{}_R \mathcal{F}_\omega(0)]^{(1)} + i(\mathbf{1} + U) \hat{\gamma}^0 [{}_R \mathcal{F}_\omega(0)]^{(2)}\} C_<(z') = 0.$$

For $0 < z' < z < +\infty$, no boundary condition must be imposed and, following Ref. [28], we can infer that the columns of the matrix $\mathcal{F}_\omega(z)C_>(z')$ must be eigenfunctions of $\mathbb{D}_\mathbf{k}$ that are square integrable when $\text{Im}(\omega) \neq 0$. The analysis carried out on $\Psi_{\pm i}(z)$ (see Sec. II B), suggests that

$$[C_>(z')]^{(2)} = \pm i[C_>(z')]^{(1)},$$

where the sign is the same as that of $\text{Im}(\omega)$. At last, we impose a matching condition at $z = z'$ as follows. We consider Eq. (57) and we integrate over $z \in (z' - \varepsilon, z' + \varepsilon)$

for $\varepsilon > 0$. Given $\mathbb{D}_\mathbf{k}$ as per Eq. (36), we take the limit $\varepsilon \rightarrow 0$ so that it holds

$$\lim_{\varepsilon \rightarrow 0} [G(z' + \varepsilon, z'; \omega)|_{z > z'} - G(z' - \varepsilon, z'; \omega)|_{z < z'}] = -i\Upsilon^0\Upsilon^1.$$

This identity translates to

$$C_>(z') - C_<(z') = -i\mathcal{F}_\omega^{-1}(z')\Upsilon^0\Upsilon^1.$$

Gathering all the previous constraints above, one can conclude that the matrices $C_{\gtrless}(z')$ abide by

$$\begin{aligned} [C_>(z')]^{(2)} &= \pm i[C_>(z')]^{(1)}, \\ \{(\mathbf{1} - U)[{}_R\mathcal{F}_\omega(0)]^{(1)} + i(\mathbf{1} + U)\hat{\gamma}^0[{}_R\mathcal{F}_\omega(0)]^{(2)}\}[C_>(z') + i\mathcal{F}_\omega^{-1}(z')\Upsilon^0\Upsilon^1] &= 0, \\ C_<(z') &= C_>(z') + i\mathcal{F}_\omega^{-1}(z')\Upsilon^0\Upsilon^1. \end{aligned} \quad (59)$$

Case $\nu \in [1, +\infty)$: this case is simpler since we do not have to impose boundary conditions at $z = 0$. Hence, we need to consider only square integrable spinors, namely $\mathbf{B} = \mathbf{D} = 0$ as per Sec. III A, or, equivalently, $[C_<(z')]^{(2)} = 0$. Apart from this difference, the analysis is the same as in the previous case and we do not report it again.

Resolution of the Dirac δ : following Eq. (56), one must read $G_{\mathbb{D}_\mathbf{k}}(z, z', \omega)$ as a function of the spectral parameter ω . Yet Eq. (58) entails that $G_{\mathbb{D}_\mathbf{k}}$ depends on $\mathcal{F}_\omega(z)$ and on $C_{\gtrless}(z')$. Combining Eqs. (59) and (19), it descends that, rather than on ω , it is more convenient to work with q where $q^2 = \omega^2 - |\mathbf{k}|^2$. Repeating the same analysis outlined in Ref. [10], based in turn on the theory of Sturm-Liouville equations (see also Ref. [28]), it descends that

$$\delta(z - z') = -\frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \int_{\sigma(\mathbb{D}_\mathbf{k})} [G_{\mathbb{D}_\mathbf{k}}(z, z', \omega + i\varepsilon) - G_{\mathbb{D}_\mathbf{k}}^\dagger(z', z, \omega - i\varepsilon)] d\omega, \quad (60)$$

where $G_{\mathbb{D}_\mathbf{k}}(z, z', \omega \pm i\varepsilon)$ denote, respectively, the Green's functions of the operator $\mathbb{D}_\mathbf{k} - \omega \mp i\varepsilon$, while $\sigma(\mathbb{D}_\mathbf{k})$ denotes the spectrum of $\mathbb{D}_\mathbf{k}$. Following Ref. [28], in order to give a representation of Eq. (60), we look for $\{u_\omega(z)\}_{\omega \in \sigma(\mathbb{D}_\mathbf{k})}$, a complete set of generalized eigenfunctions for $\mathbb{D}_\mathbf{k}$, which satisfy the underlying chosen boundary condition as per Table II as well as the identity

$$\int_{\sigma(\mathbb{D}_\mathbf{k})} u_\omega(z) u_\omega^\dagger(z') d\omega = \delta(z - z').$$

A direct comparison of this identity with Eq. (56) allows us to determine $G_{\mathbb{D}_\mathbf{k}}$.

V. PROPAGATORS FOR THE DIRAC FIELD

In this section, we apply the algorithm developed in Sec. IV to a massive Dirac field and, for definiteness, we consider only PAdS₄ although the following analysis can be adapted to any spacetime dimension $n \geq 2$. In particular, we focus on two boundary conditions as per Table II, one being the MIT-bag boundary condition, namely $\Upsilon_R^1 \Psi(0) = \pm_R \Psi(0)$ as per Remark 3.1, and one being a generalized counterpart. In what follows, we assume for definiteness $m \in (0, \frac{1}{2})$, i.e., $\nu \in (\frac{1}{2}, 1)$.

Case 1: Upper-half generalized MIT-bag condition. We start our investigation by setting ${}_R\Psi^{(1)}(0) = 0$ which, according to Table II, corresponds to choosing $U = -\mathbf{1}$. We refer to it as the upper-half generalized MIT-bag condition. Recalling that in this setting, $\mathbf{k} = (k_1, k_2)$, and using Eqs. (19) and (15), we can infer that

$$\mathcal{F}_\omega(z) = \frac{\sqrt{z}}{\omega} \begin{pmatrix} \omega J_\nu(qz) & 0 & \omega Y_\nu(qz) & 0 \\ 0 & \omega J_{\nu-1}(qz) & 0 & \omega Y_{\nu-1}(qz) \\ -qJ_{\nu-1}(qz) & (ik_1 + k_2)J_{\nu-1}(qz) & -qY_{\nu-1}(qz) & (ik_1 + k_2)Y_{\nu-1}(qz) \\ (ik_1 - k_2)J_\nu(qz) & -qJ_\nu(qz) & (ik_1 - k_2)Y_\nu(qz) & -qY_\nu(qz) \end{pmatrix}. \quad (61)$$

Remark 5.1. The columns of $\mathcal{F}_\omega(z)$ are, by construction, the four linear independent solutions of the Dirac equation on PAdS_4 . Comparing with Eq. (43), one can infer that the first and the last columns are those implementing the chosen boundary condition. As a matter of fact, the first one satisfies $\Upsilon^1_R \Psi(0) = {}_R \Psi(0)$ while the last one satisfies $\Upsilon^1_R \Psi(0) = -{}_R \Psi(0)$. Hence, ${}_R \Psi^{(1)}(0) = 0$, and we stress that this is a boundary condition distinguished from those discussed in Ref. [14].

The next step consists of focusing on Eq. (59), which entails that the matrices $\mathcal{C}_\pm(z')$ solve

$$\begin{aligned} [\mathcal{C}_>(z')]^{(2)} &= \pm i[\mathcal{C}_>(z')]^{(1)}, \\ [{}_R \mathcal{F}_\omega(0)]^{(1)} [\mathcal{C}_>(z') + i\mathcal{F}_\omega^{-1}(z') \Upsilon^0 \Upsilon^1] &= 0, \\ \mathcal{C}_<(z') &= \mathcal{C}_>(z') + i\mathcal{F}_\omega^{-1}(z') \Upsilon^0 \Upsilon^1, \end{aligned} \quad (62)$$

where the sign in the first identity is determined by that of $\text{Im}(\omega)$, as per Sec. IV. A direct computation yields

$$\begin{aligned} \mathcal{C}_>^{(\pm)}(z') &= \frac{\pi\sqrt{z'}}{2} \begin{pmatrix} \mp i\omega J_\nu(qz) & 0 & \pm iqJ_{\nu-1}(qz) & \pm(ik_2 - k_1)J_\nu(qz) \\ 0 & -\omega Y_{\nu-1}(qz) & (ik_1 - k_2)Y_{\nu-1}(qz) & qY_\nu(qz) \\ \omega J_\nu(qz) & 0 & -qJ_{\nu-1}(qz) & -(ik_1 + k_2)J_\nu(qz) \\ 0 & \mp i\omega Y_{\nu-1}(qz) & \mp(ik_2 + k_1)Y_{\nu-1}(qz) & \pm iqY_\nu(qz) \end{pmatrix}, \\ \mathcal{C}_<^{(\pm)}(z') &= \frac{\pi\sqrt{z'}}{2} \begin{pmatrix} \mp i\omega H_\nu^{(\pm)}(qz) & 0 & \pm iqH_{\nu-1}^{(\pm)}(qz) & \pm(ik_2 - k_1)H_\nu^{(\pm)}(qz) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\omega H_{\nu-1}^{(\pm)}(qz) & (ik_1 - k_2)H_{\nu-1}^{(\pm)}(qz) & qH_\nu^{(\pm)}(qz) \end{pmatrix}, \end{aligned}$$

where, with a slight abuse of notation, we employ the symbols $H_\alpha^{(\pm)}(x) = J_\alpha(x) \pm iY_\alpha(x)$ to denote the Hankel functions of the first and second kind, respectively. Using Eq. (58), one can construct $G_{\mathbb{D}_k}(z, z', \omega)$, though we avoid reporting its explicit form since it is quite heavy to read and not particularly informative. On the contrary, we focus on Eq. (60) which, removing the ε -dependence from the integrand, translates to

$$\begin{aligned} \rho_{\mathbb{D}_k}(\omega) &\doteq -\frac{1}{2\pi i} (G_{\mathbb{D}_k}(z, z', \omega) - G_{\mathbb{D}_k}^\dagger(z', z, \omega)) \\ &= \frac{\sqrt{zz'}}{2} \Theta(\omega^2 - |\mathbf{k}|^2) \begin{pmatrix} \omega J_\nu^\nu(z, z') & 0 & -qJ_{\nu-1}^\nu(z, z') & -\tilde{k}J_\nu^\nu(z, z') \\ 0 & \omega Y_{\nu-1}^{\nu-1}(z, z') & \tilde{k}^\star Y_{\nu-1}^{\nu-1}(z, z') & -qY_{\nu-1}^{\nu-1}(z, z') \\ -qJ_{\nu-1}^{\nu-1}(z, z') & -i\tilde{k}Y_{\nu-1}^{\nu-1}(z, z') & \frac{1}{\omega}(J_{\nu-1}^\star Y_{\nu-1})(z, z') & \frac{q}{\omega}\tilde{k}[J_{\nu-1}^{\nu-1}(z, z') - Y_{\nu-1}^{\nu-1}(z, z')] \\ -\tilde{k}^\star J_\nu^\nu(z, z') & -qY_{\nu-1}^\nu(z, z') & \frac{q}{\omega}\tilde{k}^\star[J_{\nu-1}^\nu(z, z') - Y_{\nu-1}^\nu(z, z')] & \frac{1}{\omega}(Y_\nu^\star J_\nu)(z, z') \end{pmatrix}, \end{aligned} \quad (63)$$

where we have employed the following compact notation

$$\begin{aligned} J_\beta^\alpha(z, z') &= J_\alpha(qz)J_\beta(qz'), & Y_\beta^\alpha &= Y_\alpha(qz)Y_\beta(qz'), \\ (J_\alpha^\star Y_\beta)(z, z') &= q^2 J_\alpha^\alpha(z, z') + |\mathbf{k}|^2 Y_\beta^\beta(z, z'), \\ \tilde{k} &= ik_1 + k_2, & \tilde{k}^\star &= -ik_1 + k_2. \end{aligned}$$

Notice that $|\tilde{k}| = |\mathbf{k}|$. One can verify by direct inspection that

$$\rho_{\mathbb{D}_k}(\omega) = u_1(z)u_1(z')^\dagger + u_2(z)u_2(z')^\dagger,$$

where $\{u_1, u_2\}$ is a set of complete eigenfunctions that fulfill the upper-half MIT-bag boundary conditions:

$$u_1(z) = \sqrt{\frac{z}{2\omega}} \begin{bmatrix} \omega J_\nu(qz) \\ 0 \\ -qJ_{\nu-1}(qz) \\ (ik_1 - k_2)J_\nu(qz) \end{bmatrix}, \quad u_2(z) = \sqrt{\frac{z}{2\omega}} \begin{bmatrix} 0 \\ \omega Y_{\nu-1}(qz) \\ (ik_1 + k_2)Y_{\nu-1}(qz) \\ -qY_\nu(qz) \end{bmatrix}.$$

As one can infer both from this expression and from Eq. (63), there is a simple pole at $\omega = 0$. A direct computation yields that the residue associated to it is

$$\rho_{\mathbb{D}_k}^{\text{dis}}(0) = |\mathbf{k}|^2 \sin(\pi\nu) \frac{2\sqrt{zz'}}{\pi} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K_{\nu-1}(|\mathbf{k}|z)K_{\nu-1}(|\mathbf{k}|z') & \frac{ik_1+k_2}{|\mathbf{k}|} K_{\nu-1}(|\mathbf{k}|z)K_\nu(|\mathbf{k}|z') \\ 0 & 0 & \frac{-ik_1+k_2}{|\mathbf{k}|} K_\nu(|\mathbf{k}|z)K_{\nu-1}(|\mathbf{k}|z') & K_\nu(|\mathbf{k}|z)K_\nu(|\mathbf{k}|z') \end{pmatrix}, \quad (64)$$

where $K_\alpha(x)$ is the modified Bessel function of the second kind. This is tantamount to considering the square integrable eigenfunction of $\mathbb{D}_k$, associated to the eigenvalue $\omega = 0$, which fulfills the upper-half MIT-bag boundary condition, namely

$$\rho_{\mathbb{D}_k}^{\text{dis}}(0) = v(z)v(z)^\dagger, \quad v(z) = |\mathbf{k}| \sqrt{\frac{2\sin(\pi\nu)z}{\pi}} \begin{bmatrix} 0 \\ 0 \\ K_{\nu-1}(|\mathbf{k}|z) \\ -\frac{ik_1-k_2}{|\mathbf{k}|} K_\nu(|\mathbf{k}|z) \end{bmatrix}.$$

Putting together all these data and taking into account Remark 2.1, the spectral decomposition of the δ -distribution reads

$$\delta(z - z') = \rho_{\mathbb{D}_k}^{\text{dis}}(0) + \int_{\mathbb{R}} \rho_{\mathbb{D}_k}(\omega) d\omega, \quad (65)$$

where $\rho_{\mathbb{D}_k}(\omega)$ and $\rho_{\mathbb{D}_k}^{\text{dis}}(0)$ are, respectively, as per Eqs. (63) and (64). Eventually, the sought advanced-minus-retarded Green's operator reads, setting $x = (t, \mathbf{x}, z)$, as follows:

$$G(x, x') = \int_{\mathbb{R}^2} \frac{d\mathbf{k}}{2\pi} e^{i[\omega(t-t') - \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')] } \left\{ \rho_{\mathbb{D}}^{\text{dis}}(0) + \int_{|\mathbf{k}|}^{+\infty} d\omega [\rho_{\mathbb{D}}(\omega) + \rho_{\mathbb{D}}(-\omega)] \right\} \Upsilon^0. \quad (66)$$

The physical significance of this $\omega = 0$ mode is not obvious *a priori*. From the above analysis, we can only conclude that there exists a family of self-adjoint boundary conditions (see Remark 5.1 for a detailed classification), that admits a square integrable, stationary, zero-energy eigenfunction as a possible wave function. Establishing whether or not this mode yields physically inconsistent scenarios is possible only through a careful analysis at the level of physical observables associated to the quantum field theory of the Dirac field. Once more, this is possible once causality is guaranteed: see Remark 4.1.

Remark 5.2. In order to infer in which cases it is necessary to take into account the existence of a pole at $\omega = 0$ in the construction of the advanced and retarded propagators, one should first focus on Eq. (15). Setting $\omega = 0$, the square integrable solutions can be constructed out of the building blocks

$$\varphi_1(z) = \sqrt{z} \begin{bmatrix} K_\nu(|\mathbf{k}|z) \\ -\frac{ik_1-k_2}{|\mathbf{k}|} K_{\nu-1}(|\mathbf{k}|z) \\ 0 \\ 0 \end{bmatrix}, \quad \varphi_2(z) = \sqrt{z} \begin{bmatrix} 0 \\ 0 \\ K_{\nu-1}(|\mathbf{k}|z) \\ -\frac{ik_1-k_2}{|\mathbf{k}|} K_\nu(|\mathbf{k}|z) \end{bmatrix}. \quad (67)$$

Their rescaled boundary values as per Eq. (42) are

$${}_R\varphi_1(0) = \begin{bmatrix} 2^{\nu-1}|\mathbf{k}|^{-\nu}\Gamma(\nu) \\ -\frac{ik_1-k_2}{|\mathbf{k}|}2^{-\nu}|\mathbf{k}|^{\nu-1}\Gamma(1-\nu) \\ 0 \\ 0 \end{bmatrix}, \quad {}_R\varphi_2(0) = \begin{bmatrix} 0 \\ 0 \\ 2^{-\nu}|\mathbf{k}|^{\nu-1}\Gamma(1-\nu) \\ -\frac{ik_1-k_2}{|\mathbf{k}|}2^{\nu-1}|\mathbf{k}|^{-\nu}\Gamma(\nu) \end{bmatrix}. \quad (68)$$

Hence, every linear combination $\varphi_{L^2}(z) = a\varphi_1(z) + b\varphi_2(z)$, with $a, b \in \mathbb{C}$, has to fulfill

$${}_R\varphi_{L^2}^{(1)}(0) = M {}_R\varphi_{L^2}^{(2)}(0), \quad (69)$$

where $M = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \in \mathcal{M}(\mathbb{C}, 2)$ is such that

$$\left(\frac{|\mathbf{k}|}{2}\right)^{2\nu-1} \frac{\Gamma(1-\nu)}{\Gamma(\nu)} c_1 - \frac{ik_1-k_2}{|\mathbf{k}|} c_2 = \frac{ik_1+k_2}{|\mathbf{k}|} c_3 + \left(\frac{|\mathbf{k}|}{2}\right)^{1-2\nu} \frac{\Gamma(\nu)}{\Gamma(1-\nu)} c_4.$$

The example detailed above abides by Eq. (69) with $M = 0$.

Case 2: Proper MIT-bag condition. We consider now the MIT-bag conditions as per Remark 3.1, namely $\Upsilon^1 {}_R\Psi(0) = {}_R\Psi(0)$ where we are considering once more the rescaled spinors. Splitting $\Psi = \Psi^{(1)} \oplus \Psi^{(2)}$ allows us to rewrite this condition as

$$\begin{aligned} \Upsilon^1 {}_R\Psi(0) = {}_R\Psi(0), \quad &\Leftrightarrow \begin{cases} \hat{\gamma}^0 {}_R\Psi^{(1)}(0) = -{}_R\Psi^{(1)}(0), \\ \hat{\gamma}^0 {}_R\Psi^{(2)}(0) = {}_R\Psi^{(2)}(0), \end{cases} \\ &\Leftrightarrow (\mathbf{1} + \hat{\gamma}^0)_R \Psi^{(1)}(0) - i(\mathbf{1} - \hat{\gamma}^0)_R \Psi^{(2)}(0) = 0, \end{aligned}$$

which, comparing with Table II, corresponds to $U = -\hat{\gamma}^0$. The procedure is identical to that of the previous case, and therefore, we limit ourselves to highlight that comparing this boundary condition with Eq. (69) entails that the pole at $\omega = 0$ does not occur. In addition, the eigenfunctions that fulfill this boundary condition are $\{w_1, w_2\}$, the first two columns of $\mathcal{F}_\omega(z)$ as per Eq. (61). In this case, the advanced-minus-retarded Green's operator reads, setting $x = (t, \mathbf{x}, z)$, as follows:

$$G(x, x') = \int_{\mathbb{R}^2} \frac{d\mathbf{k}}{2\pi} e^{i[\omega(t-t') - \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')] } \int_{\mathbb{R}} d\omega [\rho_{\mathbb{D}_k}(\omega) + \rho_{\mathbb{D}_k}(-\omega)] \Upsilon^0, \quad (70)$$

where

$$\rho_{\mathbb{D}_k}(\omega) = \Theta(\omega^2 - |\mathbf{k}|^2) (w_1(z)w_1(z')^\dagger + w_2(z)w_2(z')^\dagger).$$

Following the same rationale as Ref. [10], the absence of a point spectrum for $\mathbb{D}_k$ codified in the pole at $\omega = 0$ entails that we can use Eq. (70) to construct the two-point correlation function of the ground state of a Dirac fields. This identifies unambiguously a state for the underlying algebra of fields. In order to avoid too many technical, algebraic details in the main body of this paper, we review succinctly its construction in Appendix B, commenting in particular on the property of being F-local; see Ref. [13].

More precisely, we look for two bidistributions $\omega_2^\pm$ such that their integral kernel is a solution of

$$D_x \omega_2^+(x, y) = 0, \quad D_y \omega_2^-(x, y) = 0, \quad (71)$$

where D is the Dirac operator and where we also impose the proper MIT-bag boundary conditions. In addition, we require the canonical anticommutation relations to hold true, namely

$$\omega_2^+(x, y) + \omega_2^-(y, x) = iG(x, y), \quad (72)$$

where $G(x, y)$ is as per Eq. (66). A concrete realization of these conditions is

$$\begin{aligned} \omega_2^+(x, x') &= \int_{\mathbb{R}^2} \frac{d\mathbf{k}}{2\pi} e^{i[\omega(t-t') - \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')] } \\ &\quad \times \int_{|\mathbf{k}|}^{+\infty} d\omega [w_1(z)\bar{w}_1(z') + w_2(z)\bar{w}_2(z')], \\ \omega_2^-(x, x') &= G(x', x) - \omega_2^+(x', x), \end{aligned} \quad (73)$$

Remark 5.3. We highlight that the two-point function for a Dirac field abiding by the proper MIT-bag boundary condition has been also discussed in Ref. [22], and a direct analysis of this reference compared to ours shows that the two constructions yield the same outcome.

We conclude this section by stressing that Eq. (73) identifies a ground state for the Dirac field subject to the proper MIT-bag boundary conditions, as one can infer

either by observing that only positive frequencies occur in the mode decomposition or by employing the definition given in Ref. [7]. At last, we remark that Eq. (73) fits with the construction discussed in Ref. [32]. Therein it is proven that when considering the Dirac equation on any manifold with bounded geometry, a two-point function for the Dirac field for which ω_2^+ is built out of only positive-energy solutions, while ω_2^- is built out of only negative-energy solutions, determines always a quasifree, Hadamard ground state.

VI. CONCLUSIONS

In this paper, we have investigated massive Dirac fields on AdS_n , $n \geq 2$, both in the global and in the Poincaré patch. By generalizing a procedure first outlined in Ref. [15] in the two-dimensional scenario, we have classified the admissible boundary conditions which can be assigned at the conformal boundary. Among them, we have individuated those going under the name of MIT-bag in the literature as well as a much larger class which we have tagged as generalized MIT-bag boundary conditions. All these options entail the well posedness of the mixed initial-boundary value problem as well as the existence of two natural candidates for playing the roles of advanced and retarded propagators for the Dirac operator. In turn, these allow us to codify the canonical anticommutation relations and to construct ω_2 , the two-point correlation function of an underlying ground state with prescribed boundary conditions, provided that no bound state modes are present, a feature possible when considering some among the generalized MIT-bag boundary conditions. We have applied this paradigm to two concrete examples in PAdS_4 , but the procedure carries no *a priori* limitation and it could be applied in all dimensions. If one considers any globally hyperbolic open region of the underlying background, by using Ref. [7], we can conclude that ω_2 is of local Hadamard form. Hence, this is a natural candidate to construct a local and covariant algebra of Wick polynomials, the building block to deal with interactions at a perturbative level. In addition, in sharp contrast with the scalar case, it appears that when bound state modes are absent, no IR singularity appears, hence avoiding the potential pathological features first highlighted in Ref. [33].

We reckon that this work represents just the first step in the analysis of the interplay between boundary conditions and Dirac fields. There are several open questions which should be tackled in future works and, among these, we highlight the following:

- (i) From a structural viewpoint, the advanced and retarded propagators $\mathcal{G}^\pm$, constructed in Sec. IV, abide by all the defining properties of a fundamental solution, but it is not manifest that given $f \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-1}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})$, we obtain

$$\text{supp}(\mathcal{G}^\pm(f)) \subseteq J^\mp(\text{supp}(f)).$$

In globally hyperbolic spacetime, this feature is codified directly in the definition of the advanced and retarded propagators, playing a key role in proving their uniqueness. When one is working with a mode decomposition, this property is not manifest, but it can be proven using energy estimates. It has been highlighted that even in the scalar case, this approach does not work, and only recently an alternative procedure has been devised in Ref. [18]. Yet, since in this reference, it is particularly important that the dynamics are codified by a second-order partial differential equation, a direct application to Dirac fields is not immediate. Hence, it would be desirable to devise a workaround which allows us to prove the support property also for the propagators of a Dirac field,

- (ii) Still from the structural viewpoint, one of the pillars of the algebraic approach to quantum field theory goes under the name of the Radzikowski theorem [34,35]. This guarantees the equivalence between the local and global Hadamard form of a two-point correlation function. The first establishes the existence of a codified form of the integral kernel $\omega_2(x, x')$ provided that x and x' lie in the same geodesic neighborhood. This guarantees that the singular part of ω_2 is covariant; namely it is completely determined in terms of the underlying geometry and of the equation of motion. At the same time, the global Hadamard form codifies the singular structure of ω_2 globally, using the language of microlocal analysis. The interplay and equivalence of these two conditions is foundational for establishing a perturbative, local, and covariant quantization scheme for interacting field theories; see, e.g., Ref. [36]. Yet all these results rest on the assumption of global hyperbolicity and, in presence of timelike boundaries such as in AdS_n , more sophisticated techniques need to be employed. In the scalar case, the global structure of the two-point function has been studied in Refs. [30,31] and a generalization of the Radzikowski theorem has been proven in Ref. [18]. Yet all these works strongly rely on the underlying equation being of second order, hence requiring a substantial modification to be applied to Dirac fields. Furthermore, in Ref. [37], Hadamard states with MIT boundary conditions have been studied, and a comparison with our results is certainly desirable.
- (iii) From a physical perspective, we would like to draw the attention on two open questions. In view of finding a connection with the AdS/CFT correspondence, one should explore the possibility of considering a larger class of boundary conditions, in which

the bulk field restricted to the boundary acts a source of a dynamics thereon. In the scalar scenario, dynamical boundary conditions of Wentzell type have been already investigated; see, e.g., Refs. [38–40]. Yet, it is unclear whether a similar option exists also when considering field theories whose dynamics are codified by Dirac type operators, and a detailed analysis should be undertaken. A second remarkable and recent physical application explores the interplay between quantum field theory and relativistic quantum information by coupling a field to an Unruh-De Witt detector, which is a two-level system moving along a worldline of the underlying background. The possibility of having different boundary conditions has far-reaching consequences which have been investigated when considering scalar fields, even in connection to specific phenomena such as the anti-Unruh and anti-Hawking effects; see, e.g., Refs. [41–46]. Since Dirac fields admit boundary conditions which are structurally very different from the scalar counterparts, one can expect qualitative and quantitative differences which we deem worth to be investigated.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: THE DIRAC EQUATION ON CURVED BACKGROUNDS

The goal of this appendix is to recall succinctly the geometric construction of spinors on $(\mathcal{M}, g)$, an n -dimensional Lorentzian, orientable, time-orientable, smooth manifold endowed with a metric g , where $\mathcal{M}$ has possibly a nonempty boundary; see also Ref. [47]. By $F\mathcal{M}$, we denote the frame bundle associated to $\mathcal{M}$, while $\varepsilon \in \Gamma(F\mathcal{M})$ is a smooth section whose components $\{\varepsilon_a(p)\}_{a=0,\dots,n-1}$ identify an ordered, orthonormal, orthochronous basis for $T_p\mathcal{M}$. In addition, if we fix $\{e_\mu\}_{\mu=0,\dots,n-1}$, a local coordinate system for $\mathcal{M}$, we say that ε is a vielbein if

$$g^{\mu\nu} = -\eta^{ab}(\varepsilon_a)^\mu(\varepsilon_b)^\nu, \quad \forall a, b, \mu, \nu = 0, \dots, n-1, \quad (\text{A1})$$

where $g_{\mu\nu}$ and η_{ab} denote the components, respectively, of the metric tensor of $\mathcal{M}$ and of Minkowski spacetime with

respect to the bases $\{e_\mu\}$ and $\{\varepsilon_a\}$, while $(\varepsilon_a)^\mu$ are such that $\varepsilon_a = \varepsilon_a^\mu e_\mu$. On top of these, we also require $\mathcal{M}$ to admit a spin structure $(S\mathcal{M}, \pi)$ where $S\mathcal{M}$ is the spin bundle; that is, a principal bundle whose structure group is $\text{Spin}_0(1, n-1)$, the double cover of $SO_0(1, n-1)$, the component connected to the identity of the Lorentz group, while $\pi: S\mathcal{M} \rightarrow F\mathcal{M}$ is a morphism of principal bundles covering the identity. Existence of such structure is tantamount to requiring that the second Stiefel-Whitney class $w_2(M) \in H_2(M; \mathbb{Z}_2)$ vanishes, while the first one $w_1(M) \in H_1(M; \mathbb{Z}_2)$ classifies the number of nonequivalent choices. In this work, we always consider manifolds for which both $w_1(M)$ and $w_2(M)$ vanish. In addition to these data, we consider on top of Minkowski spacetime $(\mathbb{R}^n, \eta)$ the standard algebra of γ -matrices γ^a , $a = 0, \dots, n-1$ such that $\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1}_{\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}}$, where $\mathbf{1}$ denotes the identity matrix acting on the spinor representation space $\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$. These can be used to define a counterpart on $(\mathcal{M}, g)$ by means of the defining identity

$$\gamma^\mu := \gamma^a(\varepsilon_a)^\mu, \quad a = 0, \dots, n-1, \quad (\text{A2})$$

which entails in turn

$$\{\gamma^\mu, \gamma^\nu\}(x) := \gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = 2g^{\mu\nu}(x)\mathbf{1}_{\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}}. \quad (\text{A3})$$

Starting from these premises, we consider on top of $\mathcal{M}$ the trivial bundle $\mathcal{D}(\mathcal{M}) \equiv \mathcal{M} \times \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$, and we denote its smooth sections as Dirac fields/spinors $\psi: \mathcal{M} \rightarrow \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$. On top of $\Gamma(\mathcal{D}\mathcal{M}) \equiv C^\infty(\mathcal{M}; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})$, we define the spin connection

$$\nabla: \Gamma(T\mathcal{M}) \times \Gamma(\mathcal{D}\mathcal{M}) \rightarrow \Gamma(\mathcal{D}\mathcal{M}),$$

such that, for all $(X, \psi) \in \Gamma(T\mathcal{M}) \times \Gamma(\mathcal{D}\mathcal{M})$, we obtain

$$\begin{aligned} \nabla_X \psi &= X^\mu \partial_\mu \psi + \frac{1}{8} X^\mu [(\varepsilon_a)_\nu \partial_\mu (\varepsilon_b)^\nu + \Gamma_{\mu\nu}^\rho (\varepsilon_a)_\rho (\varepsilon_b)^\nu] [\gamma^a, \gamma^b] \psi \\ &= X^\mu \partial_\mu \psi + \frac{1}{8} X^\mu \{[\gamma_\nu, \partial_\mu \gamma^\nu] + \Gamma_{\mu\nu}^\rho [\gamma_\rho, \gamma^\nu]\} \psi, \end{aligned} \quad (\text{A4})$$

where $\Gamma_{\mu\nu}^\rho$ denote the Christoffel symbols induced by g , while $[\cdot, \cdot]$ denotes the matrix commutator. Consequently, we can define the (massless) Dirac operator

$$D: \Gamma(\mathcal{D}\mathcal{M}) \rightarrow \Gamma(\mathcal{D}\mathcal{M}), \quad D\psi = \gamma^\mu \nabla_\mu \psi, \quad (\text{A5})$$

which abides by the Lichnerowicz-Weitzenböck formula, namely

$$DD^*\psi = D^*D\psi = \square_g \psi + \frac{1}{4} R\psi, \quad (\text{A6})$$

where D^* is the formal adjoint of D , and where $\square_g$ and R are, respectively, the D'Alembert wave operator and the scalar curvature built out of g . To conclude the section, we introduce the dual Dirac bundle $\mathcal{D}^*\mathcal{M} \equiv \mathcal{M} \times (\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})^*$, whose smooth sections are called Dirac cospinors $\tilde{\psi}: \mathcal{M} \rightarrow (\mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})^*$. We can relate spinor and cospinors via a *conjugation*,

$$\bar{\cdot}: \Gamma(D\mathcal{M}) \rightarrow \Gamma(D^*\mathcal{M}), \quad \bar{\psi} = \psi^\dagger \gamma^0, \quad (\text{A7})$$

where $\psi^\dagger$ is the adjoint of ψ , while the index of γ^0 is intended as $a = 0$ and not $\mu = 0$; see Eq. (A2). Notice that the Dirac equation $\tilde{P}\psi = (D - m)\psi = 0$ implies the anti-Dirac equation

$$\bar{\tilde{P}}\bar{\psi} \doteq (\bar{D} + m)\bar{\psi} = 0, \quad (\text{A8})$$

$$\bar{D}\bar{\psi} = -\overline{(D\psi)} = \partial_\mu \bar{\psi} \gamma^\mu - \frac{1}{8} \bar{\psi} \gamma^\mu \{[\gamma_\nu, \partial_\mu \gamma^\nu] + \Gamma_{\mu\nu}^\rho [\gamma_\rho, \gamma^\nu]\}. \quad (\text{A9})$$

To conclude this Appendix, we highlight that, in this work, we make the following choices of γ -matrices which are elements of $\mathcal{M}(2^{\lfloor \frac{n}{2} \rfloor}; \mathbb{C})$:

- (i) For $n = 2$, $\gamma^0 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- (ii) For $n = 3$, in addition to γ^0 and γ^1 as above, we set $\gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.
- (iii) For $n > 2$ even, denoting by $\hat{\gamma}^a$ the γ -matrices in the $(n-1)$ -dimensional case

$$\gamma^0 = i \begin{pmatrix} 1_{2^{\lfloor \frac{n}{2} \rfloor - 1}} & 0 \\ 0 & -1_{2^{\lfloor \frac{n}{2} \rfloor - 1}} \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \hat{\gamma}^0 \\ -\hat{\gamma}^0 & 0 \end{pmatrix},$$

$$\gamma^a = \begin{pmatrix} 0 & i\hat{\gamma}^{a-1} \\ -i\hat{\gamma}^{a-1} & 0 \end{pmatrix},$$

where $a = 1, \dots, n-1$, while $1_{\frac{n}{2}}$ is the $\frac{n}{2} \times \frac{n}{2}$ identity.

- (iv) For $n > 3$ odd,

$$\gamma^a = \hat{\gamma}^a, \quad a = 0, \dots, n-2,$$

$$\gamma^{n-1} = \begin{pmatrix} 0 & 1_{2^{\lfloor \frac{n}{2} \rfloor - 1}} \\ 1_{2^{\lfloor \frac{n}{2} \rfloor - 1}} & 0 \end{pmatrix}.$$

APPENDIX B: THE ALGEBRA OF OBSERVABLES FOR THE DIRAC FIELD ON AdS_n

The goal of this Appendix is to outline the construction of the algebra of observables associated to a Dirac field on the Poincaré patch of AdS_n , $n \geq 2$. While on globally hyperbolic manifolds with empty boundaries such construction is well established (see, e.g., Ref. [3] Chap. 3),

in presence of a nonempty timelike boundary, the scenario is rather different.

More concretely, following the rationale of Secs. III and IV, we do not work directly on PAdS_n , rather on the conformally related half-Minkowski spacetime $(\mathbb{H}^n, \eta)$ outlined in Sec. II. In order to keep contact with the geometric language used in Ref. [3], we introduce on $\mathring{\mathbb{H}}^n = \mathbb{H}^n \setminus \partial\mathbb{H}^n$ the Dirac bundle $D\mathring{\mathbb{H}}^n$. This is a trivial vector bundle $D\mathring{\mathbb{H}}^n \simeq \mathring{\mathbb{H}}^n \times \mathbb{C}^N$, which comes accompanied with its dual counterpart $D^*\mathring{\mathbb{H}}^n \simeq \mathring{\mathbb{H}}^n \times (\mathbb{C}^N)^*$. In addition, we assign on $\partial\mathring{\mathbb{H}}^n$ a boundary condition for the underlying Dirac equation, which can be codified by $U \in U(\frac{N}{2})$, an arbitrary unitary matrix with $N = 2^{\lfloor \frac{n}{2} \rfloor}$; see Eq. (44). Starting from these data, we identify a set of building blocks to construct the algebra of observables:

- (1) The space of test spinors (respectively, cospinors) as $\Gamma_0(D\mathring{\mathbb{H}}^n) \simeq C_0^\infty(\mathring{\mathbb{H}}^n; \mathbb{C}^N)$ (respectively, $\Gamma_0(D^*\mathring{\mathbb{H}}^n) \simeq C_0^\infty(\mathring{\mathbb{H}}^n; (\mathbb{C}^N)^*)$),
- (2) The advanced and retarded propagators associated to the operator P as per Eq. (12), endowed with the boundary condition U , namely $\mathcal{G}_U^\pm: \Gamma_0(D\mathring{\mathbb{H}}^n) \rightarrow \Gamma(D\mathring{\mathbb{H}}^n)$; see Sec. IV. In addition, we assume that the causal propagator $\mathcal{G}_U \doteq \mathcal{G}_U^- - \mathcal{G}_U^+$ is causally supported, i.e.,

$$\text{supp}(\mathcal{G}_U(\psi)) \subseteq J(\text{supp}(\psi)) \quad \forall \psi \in \Gamma_0(D\mathring{\mathbb{H}}^n),$$

where J denotes the union between the causal future and past in the underlying manifold.

Observe that, as a consequence of Schwartz kernel theorem, we can associate $\mathcal{G}_U$ to a bidistribution $\mathcal{G}_U \in D'(D^*\mathring{\mathbb{H}}^n \boxtimes D\mathring{\mathbb{H}}^n)$ such that working at the level of integral kernels,

$$(P \otimes \mathbb{I})G_U(x, x') = (\mathbb{I} \otimes \bar{P})G_U(x, x') = 0,$$

where $\bar{P} \doteq \bar{\partial}_\mu \Upsilon^\mu + \frac{m}{z}$ denotes the counterpart on $\mathbb{H}^n$ of $\tilde{P}$. Here the upper left-pointing arrow entails that the derivative is acting on the left. Repeating almost slavishly the same analysis carried out for P , we can conclude that $\bar{P}$ comes accompanied by advanced and retarded propagators $(\mathcal{G}_U^\pm)^*: \Gamma_0(D^*\mathring{\mathbb{H}}^n) \rightarrow \Gamma(D^*\mathring{\mathbb{H}}^n)$ abiding by the boundary conditions codified by U . As for $\mathcal{G}_U$, we assume that the causal propagator $\mathcal{G}_U^* \doteq (\mathcal{G}_U^-)^* - (\mathcal{G}_U^+)^*$ is causally supported. Using once more the Schwartz kernel theorem, we can associate to $\mathcal{G}_U^*$ a bidistribution $\mathcal{G}_U^* \in D'(D\mathring{\mathbb{H}}^n \boxtimes D^*\mathring{\mathbb{H}}^n)$ such that working at the level of integral kernels,

$$(\bar{P} \otimes \mathbb{I})G_U^*(x, x') = (\mathbb{I} \otimes P)G_U^*(x, x') = 0.$$

Continuing in our identification of the building blocks of the algebra of observables, we introduce the following structures:

- (3) Given an isomorphism $A: \Gamma_0(D\mathbb{H}^n) \rightarrow \Gamma_0(D^*\mathbb{H}^n)$ such that $A(P\psi) = -\bar{P}(A\psi)$ for any $\psi \in \Gamma_0(D\mathbb{H}^n)$, we identify the pairings on $\Gamma_0(D\mathbb{H}^n)$ and $\Gamma_0(D^*\mathbb{H}^n)$, as follows:

$$\begin{aligned}\langle \psi, \phi \rangle_{\Gamma_0(D\mathbb{H}^n)} &= \int_{\mathbb{H}^n} (A\psi)(\phi) d\text{vol}_\eta, \quad \forall \psi, \quad \phi \in \Gamma_0(D\mathbb{H}^n), \\ \langle \zeta, \xi \rangle_{\Gamma_0(D^*\mathbb{H}^n)} &= \int_{\mathbb{H}^n} \zeta(A^{-1}\xi) d\text{vol}_\eta, \quad \forall \zeta, \quad \xi \in \Gamma_0(D^*\mathbb{H}^n),\end{aligned}$$

where $d\text{vol}_\eta$ is the volume form of $\mathbb{H}^n$ induced by the Minkowski metric η .

Following the same analysis outlined in Ref. [3], we can use all the ingredients above to endow $\Gamma_0(D\mathbb{H}^n)$ and $\Gamma_0(D^*\mathbb{H}^n)$ with the following Hermitian forms:

$$\begin{aligned}h_s^{(U)}(\psi, \phi) &= i\langle \psi, \mathcal{G}_U(\phi) \rangle_{\Gamma_0(D\mathbb{H}^n)}, \quad \forall \psi, \quad \phi \in \Gamma_0(D\mathbb{H}^n), \\ h_c^{(U)}(\zeta, \xi) &= -i\langle \zeta, \mathcal{G}_U^*(\xi) \rangle_{\Gamma_0(D^*\mathbb{H}^n)}, \quad \forall \zeta, \quad \xi \in \Gamma_0(D^*\mathbb{H}^n).\end{aligned}$$

As a consequence and following the same rationale of Ref. [3], we identify the algebra of fields for the model under scrutiny as the unital $*$ -algebra

$$\mathcal{A}(\mathbb{H}^n) = \frac{T(\mathcal{K}(\mathbb{H}^n))}{\mathcal{I}_{\mathcal{K}}(\mathbb{H}^n)},$$

where $T(\mathcal{K}(\mathbb{H}^n))$ is the tensor algebra built out of $\mathcal{K}(\mathbb{H}^n) = \Gamma_0(D\mathbb{H}^n) \oplus \Gamma_0(D^*\mathbb{H}^n)$, while $\mathcal{I}_{\mathcal{K}}(\mathbb{H}^n)$ is the two-sided $*$ -ideal generated by the following relations, given the embedding $a: \mathcal{K}(\mathbb{H}^n) \hookrightarrow T(\mathcal{K}(\mathbb{H}^n))$:

- (1) Equations of motion: $a(P\psi, 0) = a(0, \bar{P}\zeta) = 0$ for all $\psi \in \Gamma_0(D\mathbb{H}^n)$, $\zeta \in \Gamma_0(D^*\mathbb{H}^n)$, and
- (2) Canonical anticommutation relations:
 - (i) $a(\psi, 0)a(0, \zeta) + a(0, \zeta)a(\psi, 0) = h_c^{(U)}(A\psi, \zeta)$ for all $\psi \in \Gamma_0(D\mathbb{H}^n)$, $\zeta \in \Gamma_0(D^*\mathbb{H}^n)$.
 - (ii) $a(\psi, 0)a(\phi, 0) + a(0, \phi)a(\psi, 0) = 0$ for all $\psi, \phi \in \Gamma_0(D\mathbb{H}^n)$.
 - (iii) $a(0, \xi)a(0, \zeta) + a(0, \zeta)a(0, \xi) = 0$ for all $\zeta, \xi \in \Gamma_0(D^*\mathbb{H}^n)$.

The elements of the form $a(\cdot, 0)$ and $a(0, \cdot)$ are called generators of $\mathcal{A}(\mathbb{H}^n)$. Moreover, the $*$ -involution is the extension to $\mathcal{A}(\mathbb{H}^n)$ of the relation $a^*(\psi, 0) = a(0, A\psi) \forall \psi \in \Gamma_0(D\mathbb{H}^n)$. On top of $\mathcal{A}(\mathbb{H}^n)$, we can assign a quasifree state, that is a positive, normalized and linear functional $\omega: \mathcal{A}(\mathbb{H}^n) \rightarrow \mathbb{C}$ such that, for any $n \in \mathbb{N}$, given the generators $a_1, \dots, a_n \in \mathcal{A}(\mathbb{H}^n)$,

- (i) $\omega(a_1 \cdots a_n) = \omega_n(a_1, \dots, a_n) = 0$ if n is odd,
- (ii) $\omega(a_1 \cdots a_n) = \omega_n(a_1, \dots, a_n) = \sum_{\pi \in P_2(n)} \epsilon(\pi) \omega_2(a_{\pi(1)}, a_{\pi(2)}) \cdots \omega_2(a_{\pi(n-1)}, a_{\pi(n)})$ if n is even,

where $P_2(n)$ is the collection of all subdivisions of the set $\{1, \dots, n\}$ in pairs $\{(\pi(1), \pi(2)), \dots, (\pi(n-1), \pi(n))\}$ where $\pi(k-1) \leq \pi(k)$ for $k = 1, \dots, n$, while $\epsilon(\pi)$ is the sign of the associated permutation. In addition, we consider only $U(1)$ -invariant quasifree states, namely those for which $\omega_2[a_1(\cdot, 0)a_2(\cdot, 0)] = \omega_2[a_1(0, \cdot)a_2(0, \cdot)] = 0$. These identities entail that a quasifree state ω is identified

by the assignment of two continuous linear functionals,

$$\begin{aligned}\omega_2^+ &: \Gamma_0(D\mathbb{H}^n) \times \Gamma_0(D^*\mathbb{H}^n) \rightarrow \mathbb{C}, \\ \omega_2^- &: \Gamma_0(D^*\mathbb{H}^n) \times \Gamma_0(D\mathbb{H}^n) \rightarrow \mathbb{C},\end{aligned}$$

where once more we have exploited the canonical embedding of $\Gamma_0(D\mathbb{H}^n)$ and $\Gamma_0(D^*\mathbb{H}^n)$ into $\mathcal{A}(\mathbb{H}^n)$. The choice of $\omega_2^\pm$ is not free but it is subject to constraints. More precisely, they determine a $U(1)$ -invariant quasifree state if and only if setting

$$\begin{aligned}\Lambda^+(\psi, \varphi) &\doteq \omega_2^+(\psi, A\varphi), \quad \Lambda^-(\psi, \varphi) \doteq \omega_2^-(A\varphi, \psi), \\ \forall \psi, \quad \varphi &\in \Gamma_0(D\dot{\mathbb{H}}^n),\end{aligned}$$

then

- (1) On-shell condition. For all $\psi, \varphi \in \Gamma_0(D\dot{\mathbb{H}}^n)$, we obtain

$$\Lambda^\pm(P\psi, \varphi) = 0, \quad \Lambda^\pm(\psi, P\varphi) = 0.$$

- (2) Canonical anticommutation relations. For all $\psi, \varphi \in \Gamma_0(D\dot{\mathbb{H}}^n)$, we obtain

$$\begin{aligned}\Lambda^+(\psi, \varphi) + \Lambda^-(\psi, \varphi) &= h_s^{(U)}(\psi, \varphi) \\ &= i\langle \psi, \mathcal{G}_U \varphi \rangle_{\Gamma_0(D\dot{\mathbb{H}}^n)}.\end{aligned}$$

- (3) Positivity. For all $\psi \in \Gamma_0(D\dot{\mathbb{H}}^n)$, we obtain

$$\Lambda^+(\psi, \psi) \geq 0, \quad \Lambda^-(\psi, \psi) \geq 0.$$

- (4) Hermiticity. For all $\psi, \varphi \in \Gamma_0(D\dot{\mathbb{H}}^n)$, we obtain

$$\Lambda^\pm(\psi, \varphi) = \overline{\Lambda^\pm(\varphi, \psi)}.$$

We observe that among the plethora of quasifree states, not all are physically admissible, and it is therefore customary to require that both $\omega_2^\pm$ abide by the Hadamard condition, a constraint on their underlying singular structure. As mentioned in Sec. V, the two-point correlation functions identified in this work do fall in this class, but we avoid giving more details in this Appendix.

Remark B.1. At last, we discuss the interplay between $\mathcal{A}(\mathbb{H}^n)$ and F-locality; see Ref. [13]. In particular, if we consider any globally hyperbolic region $\mathcal{O} \subset \mathbb{H}^n$, it turns out that $\mathcal{O} \cap \partial\mathbb{H}^n = \emptyset$. Therefore, we can consider $\mathcal{A}(\mathbb{H}^n)|_{\mathcal{O}}$,

that it is the $*$ -subalgebra of $\mathcal{A}(\mathbb{H}^n)$ whose generators are of the form $a(\psi \oplus 0)$ and $a(0 \oplus \zeta)$ with $\psi \in \Gamma_0(DM)$ and $\zeta \in \Gamma_0(D^*M)$, such that $\text{supp}(\psi), \text{supp}(\zeta) \subset \mathcal{O}$. One can infer by direct inspection that $\mathcal{A}(\mathbb{H}^n)|_{\mathcal{O}}$ is $*$ -isomorphic to $\mathcal{A}(\mathcal{O})$, the $*$ -algebra of fields built intrinsically on $\mathcal{O}$. As a matter of fact, both P and $\bar{P}$ are left unchanged by the restriction of the underlying manifold and, since $\mathcal{O}$ does not intersect the boundary, it turns out that $\mathcal{G}_U^\pm$ must coincide with the unique advanced and retarded fundamental solution of the Dirac operator on $\mathcal{O}$; see, e.g., Ref. [3]. This suffices to draw the sought conclusion.

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