

Anatomy of finite-volume effects on the hadronic vacuum polarization contribution to the muon $g-2$

Sakura Itatani^{1,*}, Hidenori Fukaya^{2,†} and Shoji Hashimoto^{1,3,‡}

¹*The Graduate Institute for Advanced Studies (SOKENDAI), Tsukuba, Ibaraki 305-0801, Japan*

²*Department of Physics, Graduate School of Science,
Osaka University, Toyonaka, Osaka 560-0043, Japan*

³*Institute for Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK),
Tsukuba, Ibaraki 305-0801, Japan*



(Received 14 January 2025; accepted 27 May 2025; published 13 June 2025)

The low-energy spectrum relevant to the lattice calculation of the hadronic vacuum polarization contribution to muon anomalous magnetic moment $a_\mu^{\text{HVP,LO}}$ is dominantly given by two-pion states satisfying Lüscher's finite-volume quantization condition. Finite-volume effects from those states may exhibit power-law dependence on the volume, contrary to an exponential suppression as suggested by chiral effective theory. Employing the finite-volume state decomposition of Euclidean correlators, we systematically investigate the volume dependence and identify the different volume scalings depending on the volume and Euclidean-time region. Using phenomenological inputs for $\pi\pi$ phase shift and the timelike pion form factor, we obtain an estimate for the finite-volume effects on $a_\mu^{\text{HVP,LO}}$, which is consistent with previous works. Numerical results are given for the “window” observables of a_μ .

DOI: [10.1103/2swb-krdz](https://doi.org/10.1103/2swb-krdz)

I. INTRODUCTION

The finite-volume effect is one of the most important sources of systematic error in the lattice QCD computation of the hadronic vacuum polarization (HVP) contribution to muon anomalous magnetic moment $a_\mu = (g_\mu - 2)/2$. In particular, the isovector component at the leading order, written as $a_\mu^{\text{HVP,LO}}$, gives the largest contribution and, thus, potentially involves the most significant systematic errors. For example, in the BMW Collaboration calculation in 2020 [1], the finite-volume effect at their reference lattice of size $L_{\text{ref}} \simeq 6.3$ fm is estimated as $19(3) \times 10^{-10}$, which is much larger than the error of the current experimental average, $\sim 2 \times 10^{-10}$ [2]. Given the limitation of the lattice volume one can simulate with currently available computational resources, a reliable estimate of the finite-volume effect is of great relevance to the test of the Standard Model through a_μ . In fact, an estimated finite-volume correction is added to the result to obtain the final result in [1], and its

uncertainty is estimated to be approximately the same size as the experimental error.

Some theoretical estimates of the correction due to the finite spatial and temporal extent of the lattice are available from chiral perturbation theory (ChPT) [3–6] or from more general approaches [7,8], and a correction that is suppressed at least as $\exp(-m_\pi L)$ is found in the leading order, for the pion mass m_π and spatial extent of the lattice L . There is, on the other hand, an argument that suggests a dependence of power law at least from the region of large Euclidean-time separation of the current correlator that enters in the evaluation of HVP [7,8]. This is understood as follows. In a finite spatial volume, the energy levels are quantized according to Lüscher's condition [9], and their level spacing scales as an inverse power of L . A sum of the contributions from these quantized states could naturally develop a power-law dependence like $1/L^\alpha$, especially from the lowest end of the spectrum. This effect could become most important for large Euclidean-time separations, where only a few lowest-energy levels dominate, and is relevant for the estimate of $a_\mu^{\text{HVP,LO}}$, as it receives a significant contribution from the time separation of the order of the inverse muon mass $1/m_\mu \sim 2$ fm. The transition of such power-law dependence to the exponential one as expected from other analyses and its numerical impact are still to be understood.

In this work, we systematically study the effect of finite volume on $a_\mu^{\text{HVP,LO}}$ using the framework to construct finite-volume Euclidean correlation functions from the

* Contact author: itasaku@post.kek.jp

† Contact author: hukaya@het.phys.sci.osaka-u.ac.jp

‡ Contact author: shoji.hashimoto@kek.jp

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

phenomenological input. The two-pion states in a finite volume are identified using Lüscher's formula [9], and their transition matrix element from the vacuum is given using the Lellouch-Lüscher factor [10,11] as explicitly formulated in [12]. In a finite box, the rotational symmetry is violated, and different angular momentum states can mix. In the present case of the angular momentum $J = 1$ $\pi\pi$ states, the states of $J = 3$ can mix, for example. We neglect such effects, assuming that their effect on the estimate of the finite-volume effect is not significant.

An estimate of $a_\mu^{\text{HVP,LO}}(L)$ at a finite volume L is then obtained using the time-momentum representation [13] from the Euclidean correlation functions. Phenomenological inputs are needed for the two-pion scattering phase shift $\delta_1^I(k)$ (for isospin $I = 1$ and angular momentum $J = 1$) and for the timelike pion form factor $F_\pi(s)$. In our analysis, we use the commonly adopted Gounaris-Sakurai model [14]. We also assume that inelastic state contributions, such as those from four pions and other higher-energy states, can be neglected when estimating the finite-volume effect. This framework has been applied in [1,11,15–19] to estimate the finite-volume effect; our work extends the framework to include essentially all energy $\pi\pi$ eigenstates (see Sec. IV C for details), while most of the previous works treated only a small number of lowest states so that the volume analyzed was limited or other methods had to be combined in the analysis.

Contributions from different length scales are commonly studied using the “window” quantities [20], as described in the community white paper [21]. Three length scales are considered: short-distance window (SD), intermediate window (W), and long-distance window (LD). Their region of Euclidean-time integral in the time-momentum representation is defined by $[0, 0.4]$, $[0.4, 1.0]$, and $[1.0, \infty]$ fm, respectively, allowing a transition region of width 0.15 fm implemented with a sigmoid function. (Details are discussed in Sec. V B.) So far, the simulation results from various groups have been confirmed to agree in the intermediate window (see [22], for example, for a summary of the results from [6,18,19,23–25]). The intermediate window is designed such that the discretization effect and the finite-volume effect are less important. Similar comparison of lattice data in the long-distance window is still missing (see the recent results [26,27]), and the subject of this work is the finite-volume effect that is expected to become the most significant in this window.

The dominant contribution in the long-distance region arises from the two-pion states, for which the interaction vanishes in the low-energy limit. The long-distance physics is, therefore, expected to be well described by noninteracting pion states as the zeroth-order approximation. The two-pion contribution can be written analytically in that limit (for large Euclidean-time separation), and it provides a reference for extrapolation toward the infinite-volume limit. This analysis is described in Sec. III.

For more realistic estimates of the finite-volume effect, the resonance enhancement of the $\gamma\pi^+\pi^-$ vertex due to the ρ meson in the timelike pion form factor may play an important role. Such an effect could not be fully incorporated within the framework of ChPT, and the method used in this work is needed.

We focus on the leading finite-volume effects on the isovector channel, while contributions from other flavors or from isospin breaking are ignored. Finite-volume effects for these states are expected to be much smaller, as the mass of involved states is larger.

The structure of this paper is as follows. Section II summarizes the standard formulas in the lattice calculation of $a_\mu^{\text{HVP,LO}}$. Then, in Sec. III, we outline the analytic calculation applicable for noninteracting pions. The analysis is extended to the interacting case in Sec. IV, and the results for the finite-volume effects for $a_\mu^{\text{HVP,LO}}$ including the predictions for the window quantities are presented in Sec. V. Conclusions with some discussion are given in Sec. VI.¹

II. TWO-PION CONTRIBUTION TO $a_\mu^{\text{HVP,LO}}$

In this section, we summarize the necessary formulas for the evaluation of $a_\mu^{\text{HVP,LO}}$ to define the notations.

The leading-order (the order of α^2 of fine structure constant α) hadronic vacuum polarization contribution to the muon anomalous magnetic moment, $a_\mu^{\text{HVP,LO}}$, is written as an integral of the vacuum polarization function $\Pi(Q^2)$ over spacelike momentum squared Q^2 :

$$a_\mu^{\text{HVP,LO}} = 4\alpha^2 \int_0^\infty dQ^2 K_E(Q^2) [\Pi(Q^2) - \Pi(0)] \quad (1)$$

with a known kernel function $K(Q^2)$ [28] obtained from an integral of the vertex function in Fig. 1. It is given as

$$K_E(Q^2) = \frac{1}{m_\mu^2} \hat{Q}^2 \cdot Z(\hat{Q}^2)^3 \frac{1 - \hat{Q}^2 Z(\hat{Q}^2)}{1 + \hat{Q}^2 Z(\hat{Q}^2)^2} \quad (2)$$

with

$$Z(\hat{Q}^2) = -\frac{\hat{Q}^2 - \sqrt{\hat{Q}^4 + 4\hat{Q}^2}}{2\hat{Q}^2}, \quad \hat{Q}^2 \equiv \frac{Q^2}{m_\mu^2} \quad (3)$$

for the muon mass m_μ . The spacelike momentum kernel function $K_E(Q^2)$ behaves as $1/(\hat{Q}^2)^3$ for large $\hat{Q}^2$, so the contribution from the large-energy region is highly suppressed.

¹Compared to the previous version posted on arXiv, we corrected an error in the calculation of the Lellouch-Lüscher factor and fixed precision issues in the numerical computation of the zeta function. This affected the main results and conclusions.

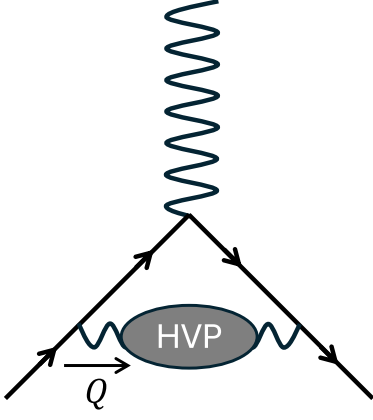


FIG. 1. Leading-order hadronic vacuum polarization contribution to muon $g - 2$.

The vacuum polarization function $\Pi(Q^2)$ is defined through

$$(Q_\mu Q_\nu - Q^2 \delta_{\mu\nu})\Pi(Q^2) = \int d^4x e^{iQx} \langle 0 | j_\mu^{\text{em}}(x) j_\nu^{\text{em}}(0) | 0 \rangle \quad (4)$$

with electromagnetic current $j_\mu^{\text{em}}(x)$. By a Fourier transform in the time direction, the representation (1) can be rewritten in the form of an integral over Euclidean time [13], which is more convenient for lattice QCD calculations:

$$a_\mu^{\text{HVP,LO}} = 4\alpha^2 m_\mu \int_0^{\tau_c} d\tau \tau^3 G(\tau) \tilde{K}_E(\tau). \quad (5)$$

An Euclidean correlator $G(\tau)$ is also obtained by a Fourier transform as

$$G(\tau) \equiv \int d\mathbf{x} \langle 0 | j_z^{\text{em}}(\tau, \mathbf{x}) j_z^{\text{em}}(0) | 0 \rangle, \quad (6)$$

where we take the electromagnetic currents $j_\mu^{\text{em}}(\tau, \mathbf{x})$ in a spatial direction $\mu = z$. It is projected to zero spatial momentum and separated in Euclidean time τ . When obtained in a finite volume, we denote the correlator by $G(\tau, L)$ and the corresponding $a_\mu^{\text{HVP,LO}}$ obtained through (5) as $a_\mu^{\text{HVP,LO}}(L)$. The kernel function $\tilde{K}_E(\tau)$ is written in terms of the corresponding Euclidean momentum-space representation $K_E(\omega^2)$ (2) as [28]

$$\tilde{K}_E(\tau) \equiv \frac{2}{m_\mu \tau^3} \int_0^\infty \frac{d\omega}{\omega} K_E(\omega^2) \left[\omega^2 \tau^2 - 4 \sin^2 \frac{\omega \tau}{2} \right]. \quad (7)$$

The upper limit τ_c of the integral (5) should be set to infinity, but in practical lattice calculations the integral is cut off at a certain value of τ_c . As shown in Fig. 2, the integrand peaks around $\tau = 1$ fm and rapidly decays

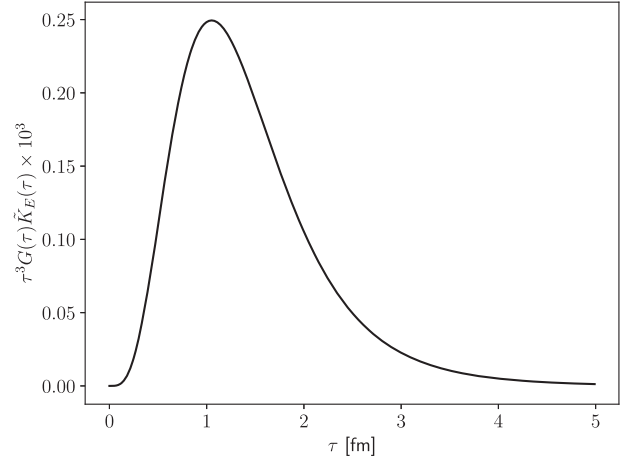


FIG. 2. Integrand of the time-momentum representation (5). The Gounaris-Sakurai model for the timelike pion form factor is used as described in the following sections.

toward larger τ . The contributions from the time intervals $[0, 1]$, $[1, 2]$, and $[2, 5]$ fm are about 34%, 39%, and 27%, respectively, and that from the region beyond 5 fm is below 1%. The LD region beyond 1 fm (with a smearing of 0.15 fm as discussed above) [21] roughly accounts for 60% of the total $a_\mu^{\text{HVP,LO}}$.

In the long-distance region, the Euclidean correlator is dominated by two-pion states and written as

$$G(\tau, L) = \sum_n |\langle 0 | j_z^{\text{em}} | \pi\pi, n \rangle_V|^2 e^{-E_{\pi\pi, n} \tau} \quad (8)$$

in a finite volume $V = L^3$. Here, the isovector channel is assumed and an isospin index for the two-pion state $|\pi\pi, n\rangle_V$ is suppressed. The states are discretized in a finite volume, and the energy of n th state $|\pi\pi, n\rangle_V$ is denoted as $E_{\pi\pi, n}$.

In the infinite-volume limit, where the two-pion states are simply labeled by the momenta $\mathbf{k}$ and $\mathbf{k}'$ of the pions in the asymptotic state, the matrix elements in (8) are expressed in terms of the pion form factor $F_\pi(s)$:

$$\begin{aligned} \langle 0 | j_\mu | \pi_{\mathbf{k}} \pi_{\mathbf{k}'}; \text{in} \rangle &= -\langle \pi_{\mathbf{k}} \pi_{\mathbf{k}'}; \text{out} | j_\mu | 0 \rangle \\ &= e^{i\delta_1^1(\mathbf{k}-\mathbf{k}')} (\mathbf{k} - \mathbf{k}')_\mu F_\pi(s). \end{aligned} \quad (9)$$

Here, we consider the case of the center-of-mass frame $\mathbf{k} + \mathbf{k}' = 0$. The momentum transfer is then $s = 4(m_\pi^2 + \mathbf{k}^2)$. The phase shift of the scattering is denoted as $\delta_1^1(\mathbf{k} - \mathbf{k}')$.

We describe the method to construct the finite-volume correlation function $G(\tau, L)$ from phenomenological inputs for $F_\pi(s)$ and $\delta_1^1(\mathbf{k})$ in Sec. IV.

III. TWO-PION CONTRIBUTION IN THE NONINTERACTING CASE

Before analyzing more realistic cases including interactions between pions, we consider the case of noninteracting pions, in order to gain some idea about possible asymptotic functional forms of the Euclidean correlator and $a_\mu^{\text{HVP,LO}}(L)$. It should provide a reasonable approximation at low energy, which is most relevant to the finite-volume effect, because the pion interaction vanishes in the (massless and) low-energy limit.

A. Current correlator

For noninteracting pions, the two-pion states in (8) are identified by their relative momentum $\mathbf{k}$. The pion form factor in (9) is $|F_\pi(s)| = 1$ and the phase shift is $\delta_1^1 = 0$. For a finite volume $V = L^3$, the Euclidean-time correlation function is written as (see Appendix A 3 in [15] as well as Sec. 2.2.3 in [8])

$$G(\tau, L) = \frac{1}{L^3} \sum_{\mathbf{k}} \frac{k_z^2 e^{-2\sqrt{\mathbf{k}^2 + m_\pi^2} \tau}}{\mathbf{k}^2 + m_\pi^2}, \quad (10)$$

where the sum is taken over $k_i = 2\pi l_i/L$ with integers l_i . In order for a numerical evaluation, this expression is useful when $\tau/L \gg 1$, where the summation over $\mathbf{k}$ converges quickly. When $\tau/L \sim 1$ or smaller, the following expression is more adequate for numerical evaluation.

Using the Poisson resummation formula for an arbitrary function $f(p)$,

$$\begin{aligned} \sum_l f\left(\frac{2\pi l}{L}\right) &= \int dp \sum_l \delta\left(p - \frac{2\pi l}{L}\right) f(p) \\ &= L \sum_n \int \frac{dp}{2\pi} f(p) e^{inpL}, \end{aligned} \quad (11)$$

we obtain

$$G(\tau, L) = \sum_{\mathbf{n}} \int \frac{d^3 p}{(2\pi)^3} \frac{p_z^2 e^{-2\sqrt{\mathbf{p}^2 + m_\pi^2} \tau}}{\mathbf{p}^2 + m_\pi^2} e^{in \cdot \mathbf{p} L}, \quad (12)$$

where three-component vector $\mathbf{n} = (n_x, n_y, n_z)$ consists of integers. Invariance under 90° rotations allows us to replace p_z^2 by $\mathbf{p}^2/3$, and using an integral over angular components of $\mathbf{p}$, which we denote by θ_p and φ_p ,

$$\int_0^{2\pi} d\varphi_p \int_0^\pi d\theta_p \sin \theta_p e^{i|\mathbf{n}|pL \cos \theta_p} = \frac{4\pi \sin(|\mathbf{n}|pL)}{|\mathbf{n}|pL}, \quad (13)$$

we have

$$G(\tau, L) = \frac{m_\pi^3}{6\pi^2} \sum_{\mathbf{n}} \int_0^\infty dy \frac{y^4}{y^2 + 1} \frac{\sin(m_\pi L |\mathbf{n}| y)}{m_\pi L |\mathbf{n}| y} e^{-2m_\pi \tau \sqrt{y^2 + 1}}, \quad (14)$$

where we have changed a variable as $p(\equiv |\mathbf{p}|) = ym_\pi$. Note that the $\mathbf{n} = \mathbf{0}$ part corresponds to the infinite-volume limit $L = \infty$. The nonzero $|\mathbf{n}|$ terms correspond to the effect of pion wrapping around the finite-volume lattice. In practice, the sum over $|\mathbf{n}|$ is truncated.

The effect of pion form factor, or the $\gamma\pi^+\pi^-$ vertex form factor, may be included by multiplying $|F_\pi(4(m_\pi^2 + \mathbf{k}^2))|^2$ to the sum over $\mathbf{k}$ in (10). This amounts to inserting $|F_\pi(4m_\pi^2(1 + y^2))|^2$ in (14). However, this may cause a problem when $|F_\pi(s)|^2$ is a rapidly changing function, as in the case for the ρ -meson resonance. Since the integrand in (14) contains a highly oscillating term $\sin(m_\pi L |\mathbf{n}| y)$, the integral could vary drastically for different $|\mathbf{n}|$ unless $|\mathbf{n}|$ is large enough to cancel out the oscillation. The condition corresponds to $|\mathbf{n}| \gg 2\pi/\Delta k_\rho L$, where Δk_ρ represents the momentum range in which the form factor varies rapidly. For the physical ρ -meson resonance, Δk_ρ is roughly $m_\rho \Gamma/8k_\rho \sim 40$ MeV. (Here, ρ -meson mass $m_\rho = 770$ MeV and width $\Gamma = 150$ MeV as well as the corresponding pion momentum $k_\rho = \sqrt{m_\rho^2/4 - m_\pi^2} = 360$ MeV are introduced.) For example, when $L = 6$ fm, the condition implies $|\mathbf{n}| \gg 5$. Thus, many terms have to be kept in the wraparound expansion to introduce the resonancelike pion form factor.

Now we consider the finite-volume correction $\Delta G(\tau, L) \equiv G(\tau, L) - G(\tau, \infty)$:

$$\begin{aligned} \Delta G(\tau, L) &= \frac{m_\pi^3}{12\pi^2} \sum_{\mathbf{n} \neq \mathbf{0}} \int_{-\infty}^\infty dy \frac{y^3}{y^2 + 1} \frac{e^{iym_\pi L |\mathbf{n}|} e^{-2m_\pi \tau \sqrt{y^2 + 1}}}{2im_\pi L |\mathbf{n}|} \\ &\quad + \text{c.c.} \end{aligned} \quad (15)$$

When $m_\pi \tau \gg 1$, the small y^2 region dominates the integral, and

$$\begin{aligned} &\int_{-\infty}^\infty dy \frac{y^3}{y^2 + 1} \frac{e^{iym_\pi L |\mathbf{n}|}}{2im_\pi L |\mathbf{n}|} \exp\left[-2m_\pi \tau \sqrt{y^2 + 1}\right] \\ &\sim \int_{-\infty}^\infty dy \frac{y^3}{2im_\pi L |\mathbf{n}|} \exp\left[-2m_\pi \tau (1 + y^2/2) + iym_\pi L |\mathbf{n}| y\right] \\ &= \frac{e^{-2m_\pi \tau - \frac{m_\pi^2 L^2 n^2}{4\tau}}}{2im_\pi L |\mathbf{n}|} \int_{-\infty}^\infty dy y^3 \exp\left[-m_\pi \tau \left(y - \frac{iL |\mathbf{n}|}{2\tau}\right)^2\right] \\ &= \frac{e^{-2m_\pi \tau - \frac{m_\pi^2 L^2 n^2}{4\tau}}}{2im_\pi L |\mathbf{n}|} \int_{-\infty}^\infty dz \left(z + \frac{iL |\mathbf{n}|}{2\tau}\right)^3 \exp[-m_\pi \tau z^2], \end{aligned} \quad (16)$$

where the y integral is analytically continued to the complex plane and replaced by that of $z \equiv y - \frac{iL|n|}{2\tau}$. Dropping odd functions in z , we obtain

$$\begin{aligned}
 \Delta G(\tau, L) &= \frac{2m_\pi^3}{12\pi^2} \sum_{n \neq 0} \frac{e^{-2m_\pi\tau - \frac{m_\pi L^2 n^2}{4\tau}}}{4m_\pi\tau} \int_{-\infty}^{\infty} dz \left(3z^2 - \frac{L^2|n|^2}{4\tau^2} \right) \exp[-m_\pi\tau z^2] \\
 &= \frac{m_\pi^3}{6\pi^2} \sum_{n \neq 0} \frac{e^{-2m_\pi\tau - \frac{m_\pi L^2 n^2}{4\tau}}}{4m_\pi\tau} \left(-\frac{3}{\tau} \frac{\partial}{\partial m_\pi} - \frac{L^2|n|^2}{4\tau^2} \right) \int_{-\infty}^{\infty} dz \exp[-m_\pi\tau z^2] \\
 &= \frac{m_\pi^3}{6\pi^2} \sum_{n \neq 0} \frac{e^{-2m_\pi\tau - \frac{m_\pi L^2 n^2}{4\tau}}}{4m_\pi\tau} \left(-\frac{3}{\tau} \frac{\partial}{\partial m_\pi} - \frac{L^2|n|^2}{4\tau^2} \right) \sqrt{\frac{\pi}{m_\pi\tau}} \\
 &= \sqrt{\frac{m_\pi}{\pi^3\tau^5}} \frac{e^{-2m_\pi\tau}}{48} \sum_{n \neq 0} \left(3 - \frac{m_\pi L^2|n|^2}{2\tau} \right) \exp\left(-\frac{m_\pi L^2 n^2}{4\tau}\right), \tag{17}
 \end{aligned}$$

which is Eq. (A15) in [15]. Here, using

$$\begin{aligned}
 \sum_{n \neq 0} \left(3 - \frac{m_\pi L^2|n|^2}{2\tau} \right) \exp\left(-\frac{m_\pi L^2 n^2}{4\tau}\right) \\
 = \left(3 + 2m_\pi \frac{\partial}{\partial m_\pi} \right) \sum_{n \neq 0} \exp\left(-\frac{m_\pi L^2 n^2}{4\tau}\right)
 \end{aligned}$$

and a formula of the elliptic theta function

$$\vartheta_3(0, q) = 2 \sum_{n=1}^{\infty} q^{n^2} + 1 = \sum_{n=-\infty}^{\infty} q^{n^2},$$

we obtain a compact expression:

$$\begin{aligned}
 \Delta G(\tau, L) &= \sqrt{\frac{m_\pi}{\pi^3\tau^5}} \frac{e^{-2m_\pi\tau}}{48} \left(3 + 2m_\pi \frac{\partial}{\partial m_\pi} \right) \\
 &\times \left[\left(\vartheta_3\left(0, e^{-\frac{m_\pi L^2}{4\tau}}\right) \right)^3 - 1 \right]. \tag{18}
 \end{aligned}$$

For small $\zeta \equiv e^{-m_\pi L^2/4\tau}$, the elliptic theta function scales as $\vartheta_3(0, \zeta) \rightarrow 1 + 2\zeta + \dots$, so that $\Delta G(\tau, L)$ approaches zero as $\exp(-m_\pi L^2/4\tau)$. When $L \sim 4\tau$, this is consistent with the well-known estimate of the finite-volume scaling as $\exp(-m_\pi L)$. However, we find differences in the shorter and longer distances. In the region of large time separation $\tau \gtrsim L/4$, the exponential suppression becomes weaker than $\exp(-m_\pi L)$ due to a factor of $L/4\tau$, while for $\tau \lesssim L/4$ the suppression is stronger. Thus, the effect on $a_\mu^{\text{HVP,LO}}$ depends on which region of τ one probes by the window quantity, for instance.

When ζ is not so small, say, $\zeta > 0.05$, the elliptic theta function is well approximated by $\vartheta_3(0, \zeta) \sim \sqrt{\pi/(-\ln \zeta)}$ [29]. It means that the square brackets in (18) contain a

term $(4\pi\tau/m_\pi)^{3/2}(1/L)^3$ for relatively small volumes $m_\pi L^2/4\tau \lesssim 3$. It is remarkable that a power-law dependence appears in the large-volume scaling: $\sim 1/L^3$ for a fixed τ . In (18), this power-law dependence is accidentally canceled due to the term $(3 + 2m_\pi \partial/\partial m_\pi)$, but it can remain, in general. In fact, we find a dependence like $1/(m_\pi L)^3$ for the interacting case (see Figs. 15 and 16 in Sec. V).

In general, the Euclidean correlator $\Delta G(\tau, L)$ has a complex dependence on volume and time separation due to the infinite sum over n in (17). The scaling of $a_\mu^{\text{HVP,LO}}$ could, therefore, be complicated, as it is given by a weighted integral of $G(\tau, L)$ over τ . Namely, it may behave as a power law at relatively small L , say, $m_\pi L \sim 6$, turn to exponential for larger L , and then decrease even faster for asymptotically large L . The explicit calculation is shown below.

B. Contribution $a_\mu^{\text{HVP,LO}}$

Let us define the finite-volume correction for $a_\mu^{\text{HVP,LO}}$ as

$$\Delta a_\mu^{\text{HVP,LO}}(L) \equiv a_\mu^{\text{HVP,LO}}(L) - a_\mu^{\text{HVP,LO}}(\infty), \tag{19}$$

where $a_\mu^{\text{HVP,LO}}(L)$ is obtained from the finite-volume Euclidean correlator $G(\tau, L)$ using (5).

Figure 3 shows $-\Delta a_\mu^{\text{HVP,LO}}(L)$ constructed from the noninteracting two-pion correlator (18), which is obtained under assumption $m_\pi\tau \gg 1$. Since this condition is not always satisfied in the integral (5), we also show a decomposition of the integral into different regions of τ . The different lines in Fig. 3 represent contributions from a given Euclidean-time range: $[0, 1]$ (dotted), $[1, 2]$ (dot-dashed), $[2, 5]$ (dashed), and $[5, \infty]$ fm (cyan). The condition $m_\pi\tau \gg 1$ would be satisfied reasonably well for $\tau \simeq 2$ fm or larger. The finite-volume effect is negative, and we plot $-\Delta a_\mu^{\text{HVP,LO}}(L)$.

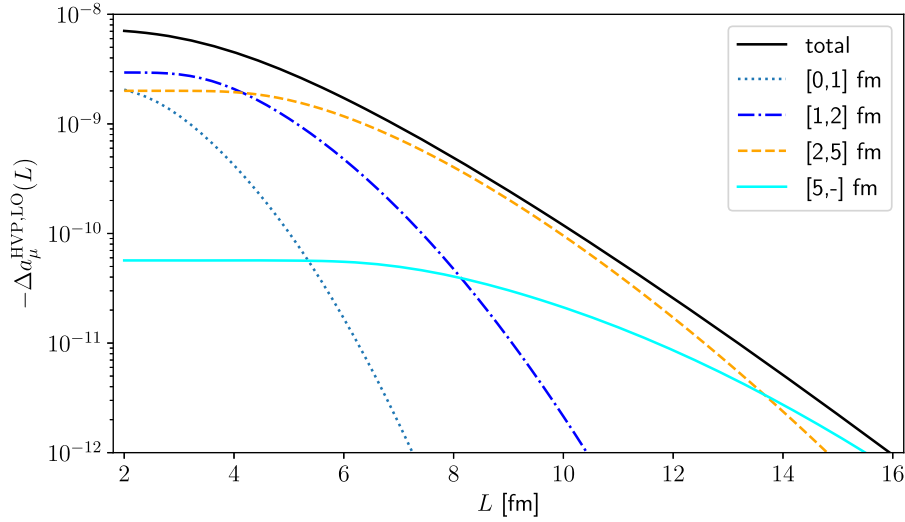


FIG. 3. $|\Delta a_\mu^{\text{HVP,LO}}(L)| = -\Delta a_\mu^{\text{HVP,LO}}(L)$ constructed from noninteracting pion correlator (18). The total (black curve) is divided into different Euclidean-time contributions: [0, 1] (dotted curve), [1, 2] (dot-dashed curve), [2, 5] (dashed curve), and [5, ∞] fm (cyan curve). Pion mass m_π is set to 140 MeV.

The most significant contribution comes from the Euclidean-time range [1, 2] or [2, 5] fm, depending on the volume size L . Namely, the range [1, 2] fm is dominant for $L \lesssim 4$ fm, while the other range [2, 5] fm becomes far more important beyond $L \sim 5$ fm. Note that $a_\mu^{\text{HVP,LO}}$ itself is dominated by the time interval $\sim 1/m_\mu$, which is typically around 1–2 fm.

We also find that $-\Delta a_\mu^{\text{HVP,LO}}(L)$ decreases faster than exponential for large L . The curve for each time range shows a curvature, most prominently for the smallest time range, [0, 1] fm, but for others as well. This is understood from the asymptotic form of (18), i.e., $\exp(-m_\pi L^2/4\tau)$ with τ in the limited range. Since the integral (5) is dominated by the time range around $1/m_\mu$, one might

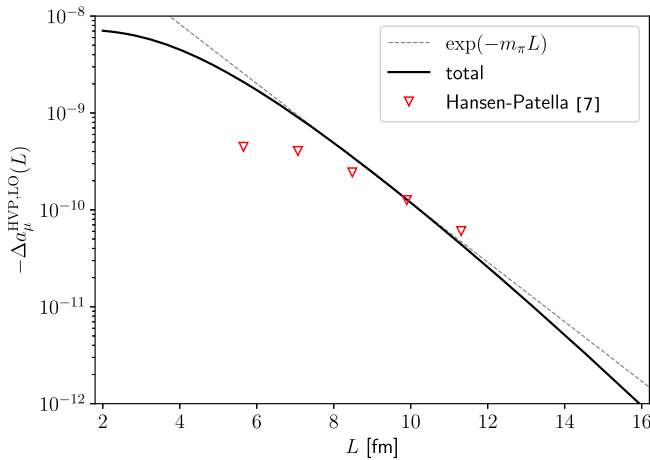


FIG. 4. $-\Delta a_\mu^{\text{HVP,LO}}(L)$ constructed from noninteracting pion correlator (18). An estimate by Hansen and Patella [7] [with an input $F_\pi(s) = 1$] is shown by down triangles. Also plotted is a line to show the slope of $\exp(-m_\pi L)$ (dashed line).

think that the finite-volume effect scales as $\exp(-m_\pi m_\mu L^2/4)$ for asymptotically large L , but the explicit calculation shows a more complicated convolution of different time separations as demonstrated in the plot.

For the smaller L region, on the other hand, the decrease of $|\Delta a_\mu^{\text{HVP,LO}}(L)|$ toward larger volumes is much flatter, suggesting nonexponential or powerlike behavior. This is again consistent with the expectation from the functional form of the correlator, as discussed above.

In Fig. 4, the total $-\Delta a_\mu^{\text{HVP,LO}}(L)$ is compared with the previous result by Hansen and Patella [7] [with $F_\pi(s) = 1$]. Our estimate is close to Hansen-Patella for $L \simeq 8$ –10 fm but decreases more rapidly for large L and increases more significantly for smaller volumes. Also shown in Fig. 4 is a line showing the slope of $\exp(-m_\pi L)$ as suggested in [7]. The magnitude of this exponential line is tuned such that it runs through the point near $L = 10$ fm. The estimate based on noninteracting pions shows a significantly different dependence on L from the simple exponential suggested in previous studies.

In chiral perturbation theory, the finite-volume effect is often estimated with a truncated sum over the integers $\mathbf{n}$ in (12) that represent how many times pions wrap around the volume in each direction. For each $|\mathbf{n}|$, the L dependence is given by $e^{-m_\pi |\mathbf{n}| L} / \sqrt{m_\pi |\mathbf{n}| L}$ [3,5], and the total is obtained by a sum over $|\mathbf{n}|$ taking into account a multiplicity for each $|\mathbf{n}|$. Figure 5 demonstrates the effect of the truncation. We again use the approximation $m_\pi \tau \gg 1$, and the two-point function is given by (18). It turns out that the truncation at $|\mathbf{n}| = 1$ leads to a significant underestimate even at $L \sim 6$ or 8 fm. The higher-order effects are also numerically important for small L 's. This plot is obtained with the approximation $m_\pi \tau \gg 1$, but similar results would be expected for the full estimate.

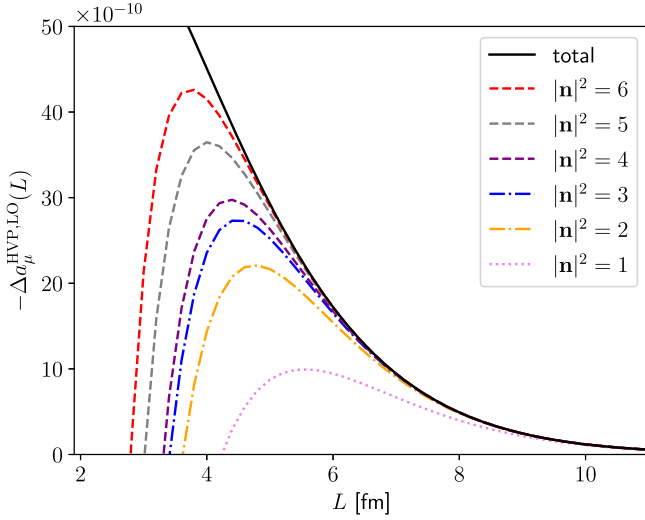


FIG. 5. $-\Delta a_\mu^{\text{HVP,LO}}(L)$ constructed from noninteracting pion correlator (18). The contributions of wraparound effects are added only up to a certain value of $|n|^2$.

The same quantity $-\Delta a_\mu^{\text{HVP,LO}}(L)$, without the truncation in $|n|$, is plotted as a function of $1/(m_\pi L)^3$ in Fig. 6. Near the infinite-volume limit, the finite-volume correction vanishes exponentially or even faster as described above, while it increases significantly for finite volumes from $1/(m_\pi L)^3 \sim 0.005$, which corresponds to $m_\pi L \simeq 6$. A linear increase can be seen for the contribution from [2, 5] fm, for example, below $1/(m_\pi L)^3 \simeq 0.01$. Then, it saturates to a constant due to the cancellation mentioned above.

The overall size, of the order of 20×10^{-10} at the nominal volume size $m_\pi L \simeq 4$, or $1/(m_\pi L)^3 \simeq 0.016$, is very significant compared to the target precision of about a few times 10^{-10} .

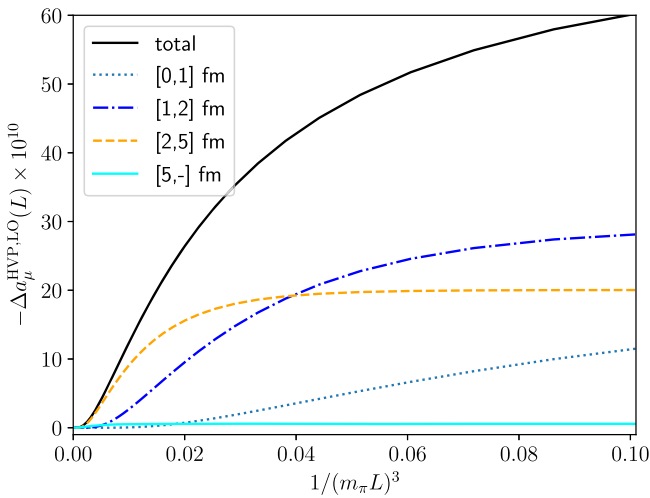


FIG. 6. $-\Delta a_\mu^{\text{HVP,LO}}(L)$ constructed from noninteracting pion correlator (18) as a function of $1/(m_\pi L)^3$. Different lines are represented as in Fig. 3.

IV. TWO-PION CONTRIBUTION IN THE INTERACTING CASE

In this section, we extend the estimate of the finite-volume effects to the case of interacting pions. Instead of performing lattice QCD simulations, which is computationally expensive, we use the phenomenologically known $\pi\pi$ phase shift $\delta_1^1(k)$ and timelike pion form factor $F_\pi(s)$ to construct the Euclidean correlator. In doing so, we neglect the inelastic effects due to four-pion states and higher, since their contribution to $a_\mu^{\text{HVP,LO}}$ is subleading (10% or less compared to the $\pi\pi$ contribution) and so is the finite-volume effect. The Euclidean correlator (8) on a finite volume is constructed from the energy spectrum $E_{\pi\pi,n}$ and the matrix element $\langle 0 | j_z^{\text{em}} | \pi\pi, n \rangle_V$. They are obtained by applying Lüscher's condition for finite-volume states.

A. Finite-volume energy spectrum

The two-pion spectrum on a finite volume satisfies Lüscher's condition [9]:

$$\delta(k) + \phi\left(\frac{kL}{2\pi}\right) = n\pi, \quad (20)$$

where $k \equiv |\mathbf{k}|$ and $\delta(k)$ is a shorthand notation of $\delta_1^1(k)$ for the present case of $I = J = 1$. The two-pion state energy to satisfy (20) is obtained as $E_{\pi\pi,n} = 2\sqrt{m_\pi^2 + k^2}$ from k for each n . The function $\phi(q)$ is defined by $\tan\phi(q) = -\pi^{3/2}q/\mathcal{Z}_{00}(1;q)$ with Lüscher's generalized zeta function $\mathcal{Z}_{00}(s;q) = (1/\sqrt{4\pi})\sum_l (l^2 - q^2)^{-s}$. On the right-hand side of (20), n is a non-negative integer.

We compute $\phi(q)$ numerically up to $q = \sqrt{30.0} \approx 5.5$, beyond which we face a numerical instability. We follow the prescription given in Appendix C in [9]. The numerical integral necessary to compute $\mathcal{Z}_{00}(s;q)$ is performed with the best possible precision available in SciPy and is cross-checked using Mathematica. The numerical instability is

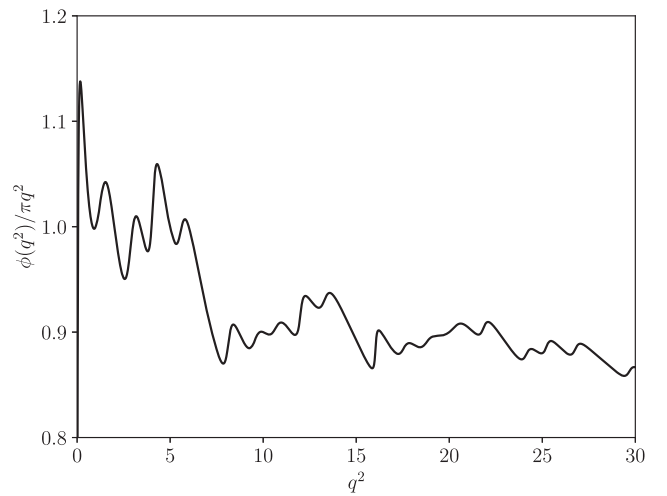


FIG. 7. $\phi(q)/\pi q^2$ computed for q^2 up to 35.0.

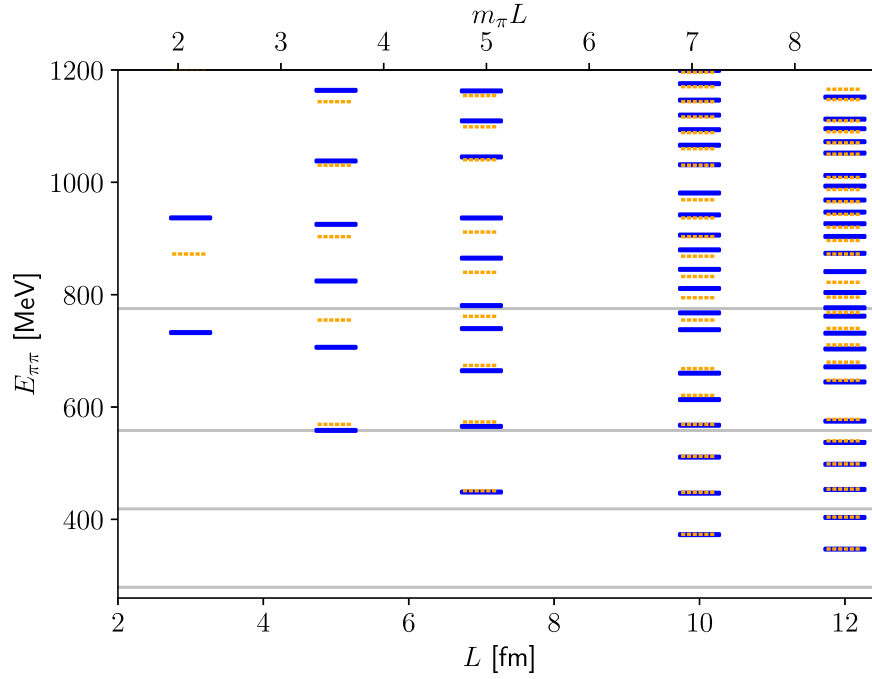


FIG. 8. Energy spectrum of the $\pi\pi$ states on finite volumes of size $L = 3, 5, 7, 10$, and 12 fm. The spectrum with (blue lines) and without (orange dotted lines) $\pi\pi$ interaction is shown for each volume. Gray lines correspond to $2m_\pi$, $3m_\pi$, $4m_\pi$, and m_ρ , from the bottom to top. The pion and rho-meson masses are set to their physical values: $m_\pi = 139.6$ MeV and $m_\rho = 775$ MeV.

observed as a strong cancellation among different terms when q is too large. The results are confirmed with the numerical table in [30] for $q^2 = 0.1\text{--}9.0$. Near $q^2 = 0$, especially below 0.1, our calculation of $\phi(q)$ agrees with Eqs. (A.4) and (A.5) in [30]. The results for $\phi(q)/\pi q^2$ are shown in Fig. 7. It is close to 1 at the first approximation, but a complicated structure is visible. In the following analysis, n is limited to $n \leq 26$, which means that the highest-energy $\pi\pi$ state for a given volume is limited. We also assume that the contribution of higher partial waves of $l \geq 3$ can be ignored.

Figure 8 shows the energy spectrum of the $\pi\pi$ states on finite volumes. The $\pi\pi$ phase shift $\delta(k)$ for those of interacting spectrum (blue lines) is taken from the Gounaris-Sakurai model (see below). The noninteracting $\pi\pi$ spectrum is also shown (orange dotted lines), which is obtained by setting $\delta(k) = 0$. The highest-energy state possible with $n \leq 26$ is slightly below 1200 MeV at $L = 12$ fm, while the covered energy range extends much higher for smaller volumes.

B. Matrix element from Gounaris-Sakurai model

The matrix element $\langle 0 | j_z^{\text{em}} | \pi\pi, n \rangle_V$ is reconstructed from the electromagnetic pion form factor $|F_\pi(s)|$ with the Lellouch-Lüscher formula [10,11] as described in [12]:

$$|F_\pi(E_{\pi\pi,n}^2)|^2 = \left(q\phi'(q) + k \frac{\partial \delta(k)}{\partial k} \right) \frac{3\pi E_{\pi\pi,n}^2}{2k^5} |\langle 0 | j_z^{\text{em}} | \pi\pi, n \rangle_V|^2. \quad (21)$$

We assume the Gounaris-Sakurai (GS) model [14] for the pion form factor. It is written as

$$F_\pi^{\text{GS}}(s) = \frac{-m_\rho^2 - \Pi_\rho(0)}{s - m_\rho^2 - \Pi_\rho(s)}, \quad (22)$$

where $\Pi_\rho(s)$ represents the ρ -meson self-energy as defined in the model. In the GS model, $\Pi_\rho(s)$ is obtained using the (twice-subtracted) dispersion relation with an input

$$\text{Im} \Pi_\rho(s) = -\frac{g_{\rho\pi\pi}^2}{6\pi} \frac{k^3}{\sqrt{s}}, \quad (23)$$

where $k = \sqrt{s/4 - m_\pi^2}$. The $\rho\pi\pi$ coupling $g_{\rho\pi\pi}$ is set using the experimentally available $\rho \rightarrow \pi\pi$ decay width as $\Gamma_{\rho\pi\pi} = (g_{\rho\pi\pi}^2/6\pi)(k_\rho^3/m_\rho^2)$. In this analysis, we took $\Gamma_{\rho\pi\pi} = 149$ MeV, which corresponds to $g_{\rho\pi\pi} = 5.976$.

The pion form factor (21) has a peak at the physical ρ -meson mass as shown in Fig. 9. The sudden drop near the peak due to the ρ - ω mixing is not captured by the GS model (black curve), which contains the $I = 1$ states only, but the overall shape of the experimental data is well reproduced. The deficit of the GS model (black curve) compared to the experimental data is mainly attributed to the inelastic states [31,32]. Since $|F_\pi(s)|$ is greatly enhanced due to the ρ resonance, the contribution to $a_\mu^{\text{HVP,LO}}$ can be larger compared to the noninteracting case $|F_\pi(s)| = 1$.

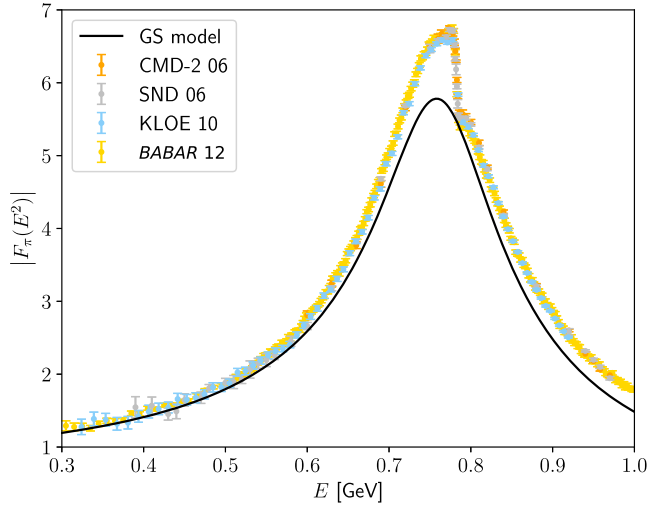


FIG. 9. Timelike pion form factor from the experimental data of $e^+e^- \rightarrow \pi^+\pi^-$ (CMD-2 06 [33,34], SND 06 [35], KLOE [36], and BABAR [37]) and from the GS model (solid line).

The phase shift $\delta(k)$ in the GS model is written as

$$\frac{k^3}{\sqrt{s}} \cot \delta(k) = k^2 h(\sqrt{s}) - k_\rho^2 h(m_\rho) + b(k^2 - k_\rho^2), \quad (24)$$

where $k_\rho = \sqrt{m_\rho^2/4 - m_\pi^2}$. The function $h(\sqrt{s})$ and a constant b are written, respectively, as

$$h(\sqrt{s}) = \frac{2}{\pi} \frac{k}{\sqrt{s}} \ln \left(\frac{\sqrt{s} + 2k}{2m_\pi} \right), \quad (25)$$

$$b = -h(m_\rho) - \frac{24\pi}{g_{\rho\pi\pi}^2} - \frac{2k_\rho^2}{m_\rho} \frac{dh}{d\sqrt{s}} \Big|_{\sqrt{s}=m_\rho}, \quad (26)$$

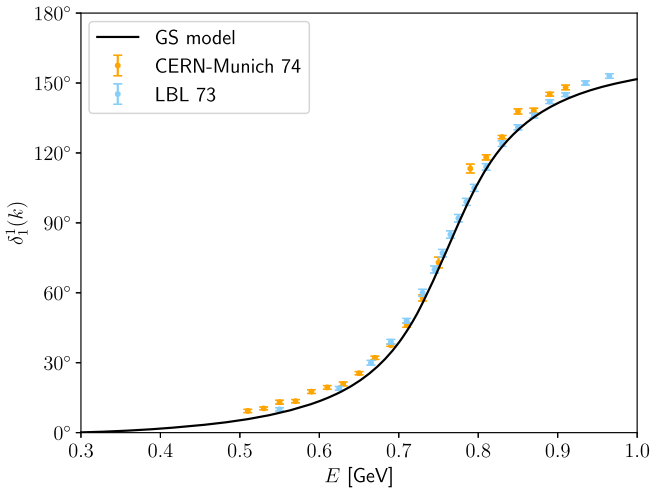


FIG. 10. Phase shift from the experimental data (CERN-Munich 74 [38] and LBL 73 [39]), compared with the GS model (solid line).

and the ρ -meson mass is set with the condition

$$\text{Re}\Pi_\rho(s)|_{s=m_\rho^2} = 0, \quad \frac{d}{ds} \text{Re}\Pi_\rho(s)|_{s=m_\rho^2} = 0. \quad (27)$$

Some details of the Gounaris-Sakurai model are found in Appendix A in [16].

Figure 10 shows the experimental data for $\delta(k) = \delta_1^0(k)$ and the corresponding GS model. The phase shift passes through 90° around the mass of the ρ meson. The GS model describes the experimental data quite closely.

C. Sum over the low-lying $\pi\pi$ states

We reconstruct the finite-volume Euclidean correlator $G(\tau, L)$ using (8) with finite-volume energy $E_{\pi\pi, n}$ and matrix element $\langle 0 | j_z^{\text{em}} | \pi\pi, n \rangle_V$ as described above. Since the major contribution to $a_\mu^{\text{HVP, LO}}(L)$ arises from energy levels that are not too high compared to m_μ due to the suppression of the kernel function $K_E(Q^2)$ in large Q^2 , we expect that the sum over n saturates rapidly beyond $E_{\pi\pi} \sim 1$ GeV. The finite-volume effect $\Delta a_\mu^{\text{HVP, LO}}(L)$ should be even more dominated by low-energy states.

In Fig. 11, we plot $a_\mu^{\text{HVP, LO}}(L)$ constructed from a finite number of low-lying $\pi\pi$ states. That is, the sum (8) is truncated at an upper limit $n = 3, 5, 10, 20$, and 26 when constructing the finite-volume correlator $G(\tau, L)$. The τ integral is divided into $[0, 1]$ (top panel), $[1, 2]$ (middle), and $[2, 10]$ fm (bottom) as in the analysis of the non-interacting case. The estimate of $a_\mu^{\text{HVP, LO}}(L)$ is well saturated by a smaller number of states, even 3 or 5 , for smaller volumes, $L \sim 3\text{--}4$ fm, as expected from the corresponding energy spectrum (Fig. 8); i.e., the energy of the fifth-lowest state is already above 1 GeV for $L = 3$ and 5 fm. Going to larger volumes, the saturation becomes slower, but we find that the difference between $n = 20$ and 26 is very small: The symbols (open and filled circles) are nearly overlapping in the plot. Numerically, the difference between $n = 20$ and 26 is 3×10^{-10} , 1×10^{-10} , and 0.1×10^{-10} for the time ranges $[0, 1]$, $[1, 2]$, and $[2, 10]$ fm, respectively, for the largest volume $L = 12$ fm.

The dependence on the upper limit of the energy sum, $E_{\pi\pi, \text{cut}}$, can be understood from the dispersion integral representation of $a_\mu^{\text{HVP, LO}}$, which is written as $\sim \int_0^\infty ds/s K(s) R(s)$ with the kernel $K(s)$ and the R ratio $R(s)$ (see, e.g., [21]). For large $s = E^2$, the kernel scales as $K(s)/s \sim 1/s^2$ and the two-pion contribution to $R(s)$ decreases as $1/s^2$, so we expect the cutoff dependence of $1/E_{\pi\pi, \text{cut}}^6$. It should also be nearly independent of the volume, as it is a UV effect. Figure 12 shows $a_\mu^{\text{HVP, LO}}(L)$ as a function of $1/E_{\pi\pi, \text{cut}}^6$. The blue curve represents the dependence in the infinite-volume limit, while the filled symbols show the truncated sum for $L = 10, 11$, and 12 fm. We can confirm that the shape is nearly independent of the volume.

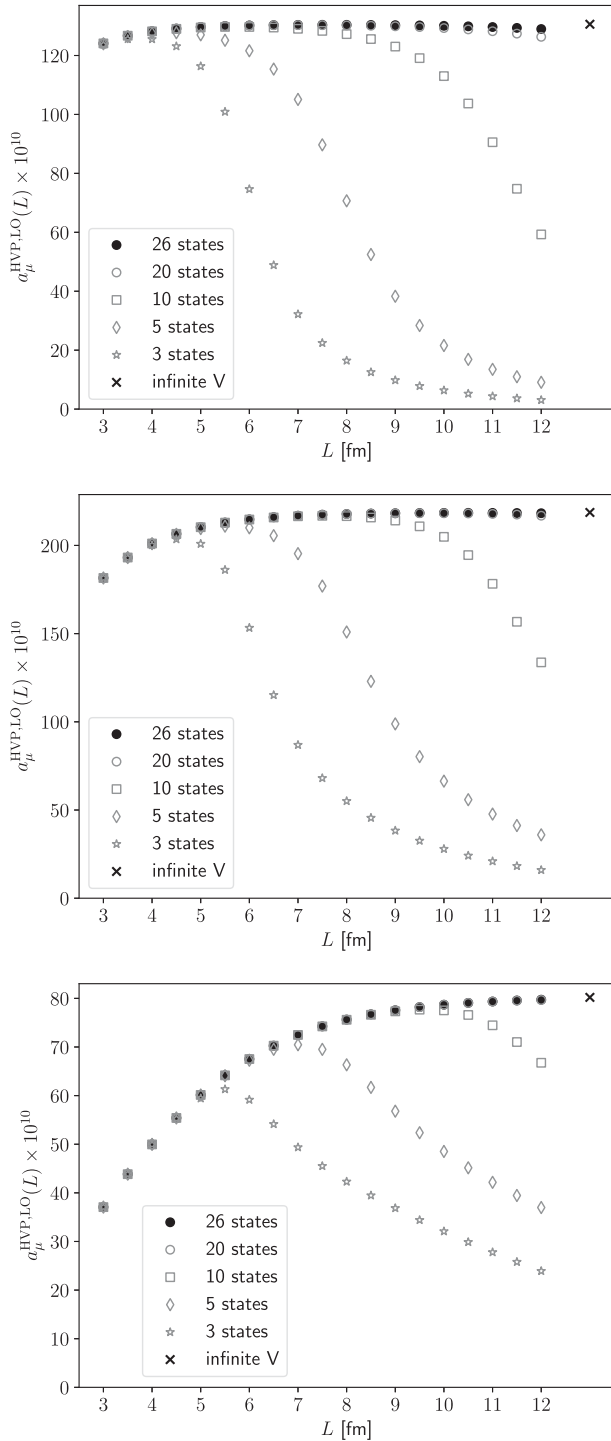


FIG. 11. Saturation of the sum over $\pi\pi$ states for $a_\mu^{\text{HVP,LO}}(L)$. Lowest n states are summed to construct the correlator $G(\tau, L)$: $n = 3, 5, 10, 20$, and 26 . Results for partial time integrals in the range $[0,1]$ fm (top panel), $[1,2]$ fm (middle), $[2,10]$ fm (bottom) are shown as a function of L . Crosses represent the infinite-volume limit including all possible states.

We, therefore, extrapolate the results for $a_\mu^{\text{HVP,LO}}(L)$ toward $n \rightarrow \infty$ assuming the $E_{\pi\pi,\text{cut}}$ dependence obtained in the infinite-volume limit. In practice, we fit the 3–5 data

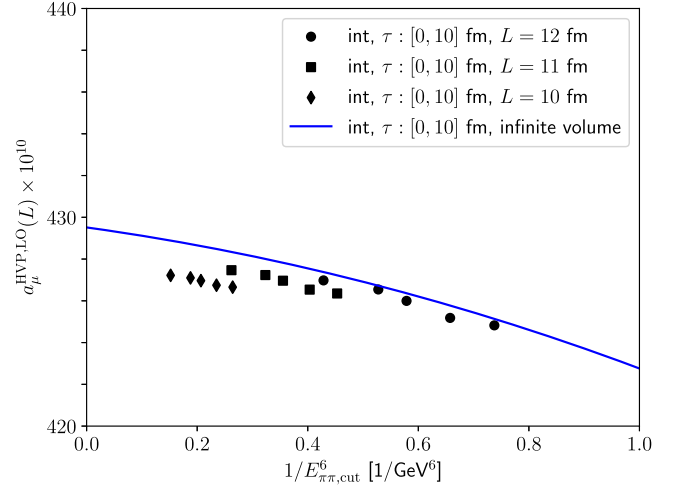


FIG. 12. $a_\mu^{\text{HVP,LO}}(L)$ reconstructed with a truncated sum over states in (8). Results for $L = 12$ fm (circles), 11 fm (squares) and 10 fm (diamonds) with $\tau_c = 10$ fm are plotted as a function of $1/E_{\pi\pi,\text{cut}}^6$, where $E_{\pi\pi,\text{cut}}$ is the highest energy $E_{\pi\pi,n}$ included. Data points correspond to $n = 22$ – 26 (from right to left). The infinite-volume limit is shown by a blue curve.

points of the highest $E_{\pi\pi,\text{cut}}$ with this scaling form (taking $E_{\pi\pi,\text{cut}} = E_{\pi\pi,n}$) plus a constant shift representing the finite-volume effect. The deviation among the fits with 3–5 points is 0.07×10^{-10} for the worst case of $L = 12$ fm and much smaller for 11 and 10 fm. We take an average of the three fit results as an estimate for $a_\mu^{\text{HVP,LO}}(L)$. The lowest cutoff energy involved in the extrapolation is about 1.1 GeV for $L = 12$ fm. We use the maximum deviation from the average as an estimate of systematic error. This estimate of error is added for both directions, positive and negative, to be conservative.

The infinite-volume limit to compare (crosses in Fig. 11) is obtained using the Euclidean correlator directly reconstructed from the pion form factor:

$$G(\tau) = \int_0^\infty dE E^2 \rho(E^2) e^{-E\tau}, \quad (28)$$

where

$$\rho(E^2) = \frac{1}{48\pi^2} \left(1 - \frac{4m_\pi^2}{E^2}\right)^{3/2} |F_\pi^{\text{GS}}(E)|^2 \quad (29)$$

is the spectral function representing the $\pi\pi$ states.

D. Results for $a_\mu^{\text{HVP,LO}}$

The results of the decomposition (8) for $a_\mu^{\text{HVP,LO}}$ are shown in Fig. 13. The results are shown for different values of the upper limit τ_c in the time-momentum integral (5): $\tau_c = 1, 2$, and 10 fm. For the largest volumes, beyond $m_\pi L \gtrsim 6$, we find that the finite-volume estimate is already

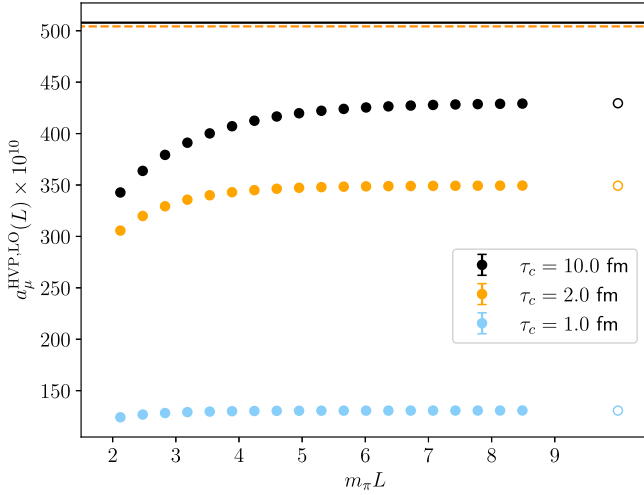


FIG. 13. $\pi\pi$ contribution to $a_\mu^{\text{HVP,LO}}$ for different volumes. The results for three different time integral upper limit are shown, *i.e.* $\tau_c = 1, 2$, and 10 fm. The infinite-volume results are shown on the right by open symbols for each τ_c . Two horizontal lines represent the estimates from dispersion analysis (solid line: DHMZ 19 [40], dashed line: KNT 19 [41]).

close (within roughly 5×10^{-10}) to the infinite-volume limit (open circles).

In the same plot, the estimates from the dispersion relation using the experimental data of $e^+e^- \rightarrow \pi^+\pi^-$ are shown by horizontal lines; two groups, DHMZ 19 [40] and KNT 19 [41], are consistent with each other. The results of our analysis are significantly lower even with the largest $\tau_c = 10$ fm, for which the integral has almost reached the value of $\tau_c \rightarrow \infty$. In fact, the difference between $\tau_c = 5$ and 10 fm is invisible at the scale of this plot. The deficit compared to the dispersion method may be attributed to the GS model for the pion form factor. As shown in Fig. 9, the GS model significantly underestimates the experimental data near the ρ resonance. However, this inconsistency would not largely affect our analysis of finite-volume effects, since the agreement of the pion form factor is better in the lower-energy region, which is most responsible for finite-volume effects.

Figure 13 clearly shows that the finite-volume effect looks significant for $m_\pi L \lesssim 6$, which corresponds to $L \lesssim 8$ fm, especially when the large time separation, $\gtrsim 1$ fm, is included in the integral (5). The size of this finite-volume effect is larger than a few times 10^{-10} , the target precision of lattice calculation of $a_\mu^{\text{HVP,LO}}$. In the following section, we analyze the finite-volume effect $\Delta a_\mu^{\text{HVP,LO}}$ in more detail.

V. FINITE-VOLUME EFFECTS FOR $a_\mu^{\text{HVP,LO}}$

In this section, we present our estimate of the finite-volume effects for $a_\mu^{\text{HVP,LO}}$. The results for the total and sharply cut time windows are shown first, and those for the smeared windows follow.

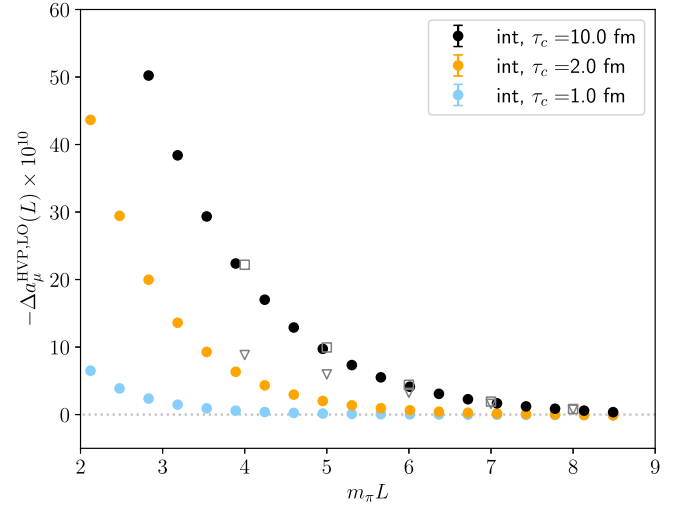


FIG. 14. Finite-volume effect for the muon anomalous magnetic moment $-\Delta a_\mu^{\text{HVP,LO}}(L)$. Estimates are obtained with phenomenological $\pi\pi$ phase shift and time-like pion form factor described by the Gounaris-Sakurai model. Results are plotted for three values of upper limit τ_c in the Euclidean-time integral (5): $\tau_c = 1, 2$, and 10 fm. The time integral is nearly saturated for $\tau_c = 10$ fm. Estimates by Hansen and Patella [7,8] are also plotted (triangles: leading order, $e^{-m_\pi L}$ [7], squares: including up to $e^{-3m_\pi L}$ [8]).

A. Total and sharply cut time windows

Figure 14 shows the finite-volume effect $-\Delta a_\mu^{\text{HVP,LO}}(L)$ as a function of $m_\pi L$. The infinite-volume limit constructed with (28) is subtracted from the finite-volume results.

The finite-volume effect $-\Delta a_\mu^{\text{HVP,LO}}(L)$ increases rapidly for $m_\pi L \simeq 6$ or smaller, especially when a region of large Euclidean-time separation is included ($\tau_c = 2$ and 10 fm). The size of $-\Delta a_\mu^{\text{HVP,LO}}(L)$ is as large as 22×10^{-10} for $m_\pi L = 4$, which is often considered a safe choice to suppress the systematic error due to the finite-volume size. Since the target precision for $a_\mu^{\text{HVP,LO}}$ is very high, *i.e.*, at the level of 5×10^{-10} or better, the size of the systematic effect as we find here has to be carefully estimated and subtracted.

The integral in the time-momentum representation (5) is saturated to an excellent precision at $\tau_c = 10$ fm (black points in Fig. 14), and these data points can be considered an estimate of total $-\Delta a_\mu^{\text{HVP,LO}}(L)$.

The estimates of Hansen and Patella obtained using effective field theory [7,8] are also plotted in Fig. 14. They include the estimate of a wraparound effect of the pion in one direction giving $e^{-m_\pi L}$ effects [7] (triangles), as well as those involving multiple wraps up to the term of $e^{-3m_\pi L}$ [8] (squares). (Contributions of the order of $e^{-\sqrt{2+\sqrt{3}}m_\pi L}$ are neglected, though. It is, therefore, effectively summed up to $e^{-\sqrt{3}m_\pi L}$.) These points can be compared with the full $-\Delta a_\mu^{\text{HVP,LO}}(L)$, *i.e.*, our estimate at $\tau_c = 10$ fm. Our results

reproduce those of Hansen-Patella including $e^{-\sqrt{3}m_\pi L}$ [8], implying that the both approaches are compatible for $m_\pi L \simeq 5$ –8. It also suggests that the wraparound effect beyond those included is not significant for these volumes.

In the BMW Collaboration calculation [11], the finite-volume effect is estimated in two steps. Their reference lattice has a size $(L_{\text{ref}}, T_{\text{ref}}) \simeq (6.3, 9.4)$ fm, which corresponds to $(m_\pi L_{\text{ref}}, m_\pi T_{\text{ref}}) \simeq (4.4, 6.7)$. From there, they estimate the difference from a bigger volume of $L_{\text{big}} = T_{\text{big}} \simeq 10.8$ fm ($m_\pi L_{\text{big}} = m_\pi T_{\text{big}} \simeq 7.6$) and then use chiral effective theory [8] to estimate the difference from the infinite volume. The first step, L_{ref} to L_{big} , is estimated as $18(2) \times 10^{-10}$ mainly using actual simulation data on a coarse lattice, while the second is smaller than 1×10^{-10} . Our corresponding estimates are 13.7×10^{-10} and 1.0×10^{-10} , respectively. We find a reasonable agreement for both. In the more recent publication [22], the estimate for $L_{\text{ref}} \rightarrow L_{\text{big}}$ from actual simulations is updated to $8.9(9) \times 10^{-10}$. Their own estimate for $L_{\text{ref}} \rightarrow \infty$ is also updated using a method similar to our work but with the pion form factor driven from Omnes formula as $9.1(2) \times 10^{-10}$, which is smaller than our analysis 14.8×10^{-10} . The difference needs to be understood in detail.

In the recent calculation by the RBC/UKQCD Collaboration [26], the largest volume lattice at the physical point has $m_\pi L = 5.2$, for which we expect a correction of about 8.0×10^{-10} .

Another recent calculation by the Fermilab Lattice, HPQCD, and MILC Collaborations [27] employ lattices up to $L = 5.8$ fm, which corresponds to $m_\pi L = 4.0$. Our estimate of the finite-volume correction is about

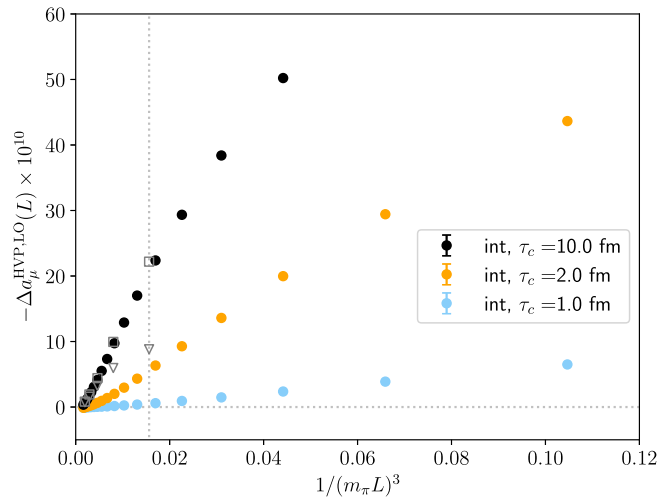


FIG. 15. Finite-volume effect for the muon anomalous magnetic moment $-\Delta a_\mu^{\text{HVP,LO}}(L)$ plotted as a function of $1/(m_\pi L)^3$. The vertical dotted line shows $m_\pi L = 4$. Other details are the same as in Fig. 14.

20×10^{-10} , but it may not be simply applied to their calculation using the staggered fermion, for which a dedicated formulation is required [6,19].

The same results for $-\Delta a_\mu^{\text{HVP,LO}}(L)$ are plotted as a function of $1/(m_\pi L)^3$ in Fig. 15. Away from the large-volume regime $1/(m_\pi L)^3 \gtrsim 0.01$, the finite-volume correction seems well described by a linear function of $1/(m_\pi L)^3$ for $\tau_c = 1.0$ and 2.0 fm. Including larger τ , i.e., $\tau_c = 10.0$ fm, the dependence is not that simple, but we can still see a component behaving as $1/(m_\pi L)^3$. This reminds us of the possible $1/L^3$ dependence of $\Delta a_\mu^{\text{HVP,LO}}(L)$ (discussed in Sec. III) for the noninteracting model. Closer to the infinite-volume limit $1/(m_\pi L)^3 \sim 0.01$, they deviate from the linear dependence and approach zero more rapidly.

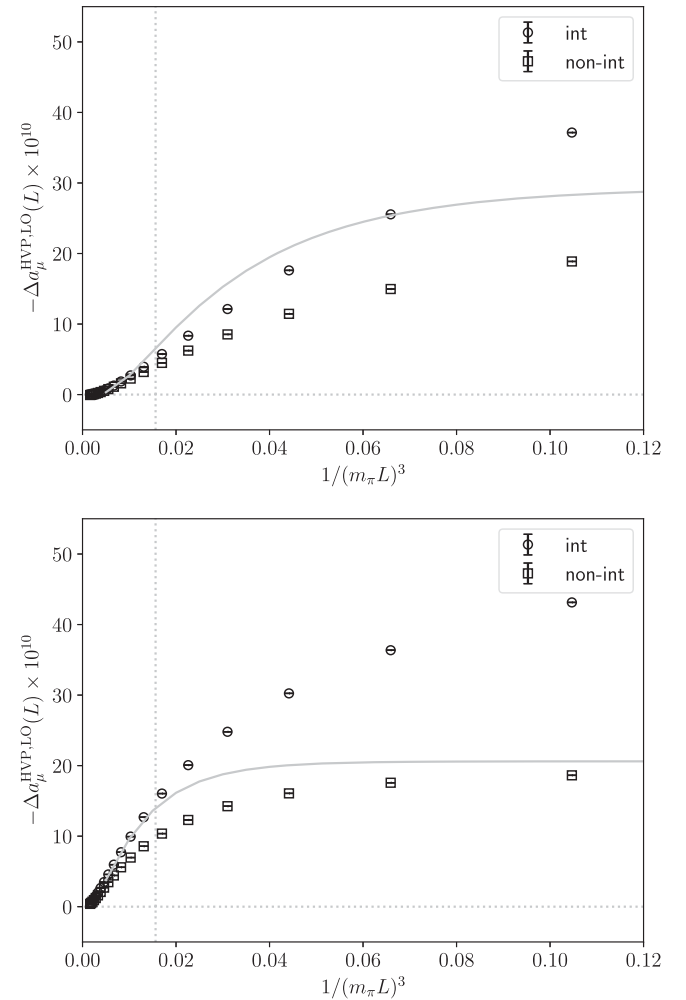


FIG. 16. Finite-volume effect for $a_\mu^{\text{HVP,LO}}$ evaluated in the time range $[1,2]$ fm (top panel) and in $[2,10]$ fm (bottom panel). Estimates with and without $\pi\pi$ interactions are shown (circles and squares, respectively). The error bar is from the extrapolation in the energy cutoff $E_{\pi\pi,\text{cut}}$ (see Sec. IV C). The gray curve is obtained with the approximate formula for non-interacting $\Delta G(\tau, L)$, Eq. (18), that is valid for $m_\pi \tau \gg 1$.

To see this explicitly, we show a comparison between the finite-volume effect with and without the $\pi\pi$ interactions in Fig. 16. The plots represent the contributions in the Euclidean-time range $[1, 2]$ (top panel) and $[2, 10]$ fm (bottom panel). The noninteracting $\pi\pi$ contribution (squares) is evaluated numerically with inputs $\delta(k) = 0$ and $F_\pi(s) = 1$, and no additional approximation such as $m_\pi\tau \gg 1$ is involved. The approximate formula (18) for $m_\pi\tau \gg 1$ (as shown in Fig. 6) is also plotted by a gray curve. As mentioned associated with (18), the noninteracting evaluation flattens for large $1/(m_\pi L)^3$ due to an accidental cancellation, while the interacting case shows a linear increase and the result turns out to be significantly larger compared to the noninteracting case.

Our result for $-\Delta a_\mu^{\text{HVP,LO}}$ depends on the details of the $\pi\pi$ interaction, i.e., the phase shift and timelike pion form factor. In order to see the size of its dependence, we repeat the analysis with slightly modified parameters in the Gounaris-Sakurai model. That is, we shift the ρ -meson mass parameter m_ρ by $-20/+20$ MeV, and check how much it affects the resulting $-\Delta a_\mu^{\text{HVP,LO}}$. The results are $+2.9/-2.6$, $+0.6/-0.6$, $+0.14/-0.11$, and $+0.05/-0.06$ for $L = 3, 5, 7$, and 10 fm, respectively in units of 10^{-10} . Compared to the finite-volume effect itself, the shift gives a minor effect.

B. Smeared time windows

For convenience, we also provide the results for the smeared time windows proposed by [20] and commonly adopted by many collaborations. Using the smeared Heaviside function

$$\Theta(\tau, \tau'; \Delta) \equiv \frac{1}{2} \left[1 + \tanh \frac{\tau - \tau'}{\Delta} \right], \quad (30)$$

the leading-order HVP contribution $a_\mu^{\text{HVP,LO}}$ can be decomposed into SD, W, and LD pieces as $a_\mu^{\text{HVP,LO}} = a_\mu^{\text{SD}} + a_\mu^{\text{W}} + a_\mu^{\text{LD}}$. Their time-momentum representations of (5) are given, respectively, as

$$\begin{aligned} a_\mu^{\text{SD}} &= 4\alpha^2 m_\mu \int_0^{\tau_c} d\tau [1 - \Theta(\tau, \tau_0; \Delta)] \tau^3 G(\tau) \tilde{K}_E(\tau), \\ a_\mu^{\text{W}} &= 4\alpha^2 m_\mu \int_0^{\tau_c} d\tau [\Theta(\tau, \tau_0; \Delta) - \Theta(\tau, \tau_1; \Delta)] \tau^3 G(\tau) \tilde{K}_E(\tau), \\ a_\mu^{\text{LD}} &= 4\alpha^2 m_\mu \int_0^{\tau_c} d\tau \Theta(\tau, \tau_1; \Delta) \tau^3 G(\tau) \tilde{K}_E(\tau). \end{aligned} \quad (31)$$

The separation points are set as $\tau_0 = 0.4$ fm and $\tau_1 = 1.0$ fm, and the width Δ of the smearing is set to 0.15 fm. We take $\tau_c = 10$ fm, which nearly saturates the integral up to ∞ .

Similarly to Figs. 14 and 15, we plot the estimates of the finite-volume effect for each time window in Fig. 17 (versus $m_\pi L$) and in Fig. 18 [versus $1/(m_\pi L)^3$]. Different windows are distinguished by different colors.

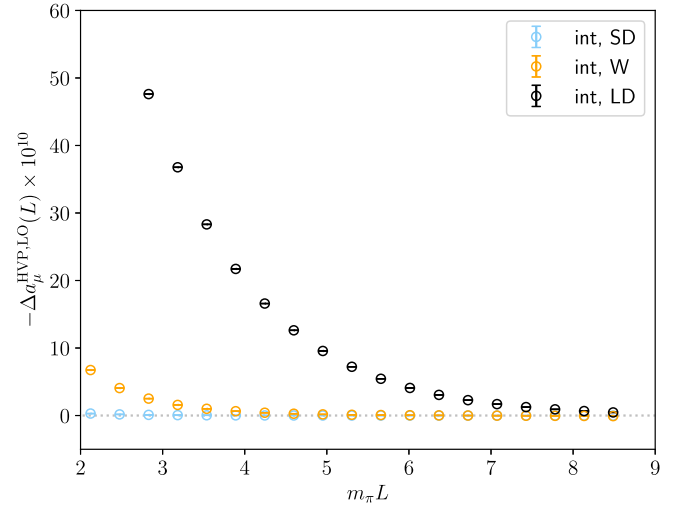


FIG. 17. Finite-volume effect for $a_\mu^{\text{HVP,LO}}$ evaluated for different Euclidean-time windows, plotted as a function of $m_\pi L$. The error bar is from the extrapolation in the energy cutoff $E_{\pi\pi,\text{cut}}$ (see Sec. IV C).

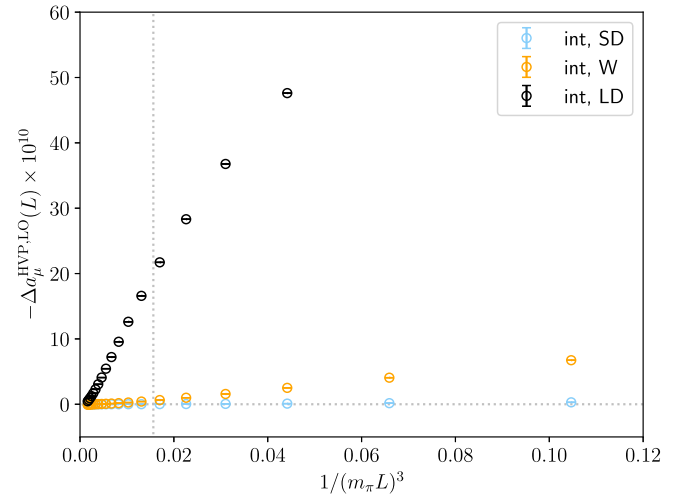


FIG. 18. Finite-volume effect for $a_\mu^{\text{HVP,LO}}$ evaluated for different Euclidean-time windows, plotted as a function of $1/(m_\pi L)^3$. The error bar is from the extrapolation in the energy cutoff $E_{\pi\pi,\text{cut}}$ (see Sec. IV C).

The finite-volume effect is most significant for the LD window, and its dependence on $m_\pi L$ is very similar to those of sharp-cut windows.

The numerical values are listed in Table I. The errors are from the extrapolation in the energy cutoff $E_{\pi\pi,\text{cut}}$ (see Sec. IV C). Other sources of errors, such as those from the input parameters as well as from the ignored inelastic states, are not shown.

VI. CONCLUSION

In this paper, the finite-volume effect for the contribution of leading-order hadronic vacuum polarization to the muon

TABLE I. $-\Delta a_\mu^{\text{HVP,LO}} \times 10^{10}$ for three Euclidean-time windows, SD, W, and LD, as well as a sum over them corresponding to the total contribution. The errors are from the extrapolation in the energy cutoff $E_{\pi\pi,\text{cut}}$ (see Sec. IV C). If the error is not written, it is smaller than the last digit shown. The last line is $a_\mu^{\text{HVP,LO}} \times 10^{10}$ in the infinite volume.

L [fm]	SD	W	LD	Total
3.0	0.293	6.737	79.756	86.787
3.5	0.166	4.066	61.558	65.790
4.0	0.097	2.508	47.611	50.216
4.5	0.058	1.571	36.769	38.397
5.0	0.035	0.997	28.311	29.343
5.5	0.021	0.639	21.715	22.375
6.0	0.013	0.413	16.588	17.013
6.5	0.008	0.267	12.621	12.896(1)
7.0	0.004	0.170	9.562	9.736(1)
7.5	0.001	0.109(1)	7.221	7.331(2)
8.0	0.000(1)	0.071(2)	5.443	5.513(3)
8.5	-0.002(1)	0.042(3)	4.085	4.125(5)
9.0	-0.003(1)	0.020(4)	3.055	3.072(7)
9.5	-0.005(2)	0.001(6)	2.276(1)	2.272(10)
10.0	-0.007(2)	-0.014(9)	1.690(2)	1.669(14)
10.5	-0.009(3)	-0.028(12)	1.247(4)	1.210(19)
11.0	-0.011(3)	-0.043(15)	0.916(6)	0.863(25)
11.5	-0.013(4)	-0.062(20)	0.659(8)	0.584(33)
12.0	-0.017(5)	-0.086(24)	0.458(11)	0.355(41)
∞	12.31	120.3	296.9	429.5

$g - 2$, $a_\mu^{\text{HVP,LO}}$, is studied in detail using phenomenological inputs. Unlike previous works [3,7,8], we explicitly construct the Euclidean correlator in a box assuming that only two-pion states contribute without any truncation of the pion's wrapping-around effect. The phenomenological inputs are the $\pi\pi$ phase shift and the timelike pion form factor, both of which can be consistently parametrized using the Gounaris-Sakurai model.

The $\pi\pi$ states in this analysis satisfy Lüscher's quantization condition, and their energy spectrum automatically reflects the wrapping-around effect to all orders, in contrast to previous studies that include only the leading terms. The Euclidean correlator $G(\tau, L)$ at a finite volume L thus constructed shows a different volume dependence for different Euclidean-time separations τ . Asymptotically, the infinite-volume limit is approached as $\exp(-m_\pi L^2/4\tau)$, while an approximate power scaling $1/L^3$ is expected in the intermediate volume region, in general.

The hadronic vacuum polarization contribution to $a_\mu^{\text{HVP,LO}}$ is obtained by a weighted integral over Euclidean time dominated in the region around $\tau \sim 1$ fm. The overall L dependence is then not simply described by a single functional form. It turns out that the finite-volume effect $\Delta a_\mu^{\text{HVP,LO}}$ shows an approximately linear dependence on $1/(m_\pi L)^3$ in the region $1/(m_\pi L)^3 > 0.01$ and rapidly decreases in the larger volume region. Their size is

consistent with the estimate reported by Hansen and Patella [8].

Our analysis fills the gap between the exponential finite-volume scaling based on, e.g., chiral effective theory and the power-law dependence expected from Lüscher's quantization condition. In the previous lattice analyses, the finite-volume effects are estimated using these different techniques depending on the Euclidean-time region of integral, but such a complication is avoided in this work, and the results are applicable for various volumes listed in Table I. Similar problems may arise for other quantities, for which the lowest-lying part of the spectrum is given by two-body states, although it became most critical for $a_\mu^{\text{HVP,LO}}$ because of its high required precision.

Similar analysis can be done for heavier pions employed in many lattice calculations, provided that the phase shift and pion form factor are available. In other words, these quantities will provide a means of estimating finite-volume effects in the lattice calculations of $a_\mu^{\text{HVP,LO}}$. When the lattice data are available on multiple volumes, this analysis would provide a highly nontrivial cross-check, as it is based on rather solid ground at least for the finite-volume effects.

Finally, we note that our analysis is not complete, as it could not take account of the inelastic states, such as four pions, and the realistic estimate can be obtained only with the full-scale simulations such as those of [1,22,26]. However, the most important contribution from the $\pi\pi$ states would be well captured by our analysis.

ACKNOWLEDGMENTS

We thank Kohtaroh Miura for numerous discussions. We are grateful to Haoyu Wang for offering a detailed numerical check for some of our results. S. H. is partly supported by MEXT as “Program for Promoting Researches on the Supercomputer Fugaku” (PMXP1020230411) and by JSPS KAKENHI, Grant No. 22H00138.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

-
- [1] S. Borsanyi *et al.*, Leading hadronic contribution to the muon magnetic moment from lattice QCD, *Nature (London)* **593**, 51 (2021).
 - [2] D. P. Aguillard *et al.* (Muon $g - 2$ Collaboration), Measurement of the positive muon anomalous magnetic moment to 0.20 ppm, *Phys. Rev. Lett.* **131**, 161802 (2023).
 - [3] C. Aubin, T. Blum, P. Chau, M. Golterman, S. Peris, and C. Tu, Finite-volume effects in the muon anomalous magnetic moment on the lattice, *Phys. Rev. D* **93**, 054508 (2016).
 - [4] J. Bijnens and J. Releford, Vector two-point functions in finite volume using partially quenched chiral perturbation theory at two loops, *J. High Energy Phys.* **12** (2017) 114.
 - [5] C. Aubin, T. Blum, M. Golterman, and S. Peris, Application of effective field theory to finite-volume effects in a_μ^{HVP} , *Phys. Rev. D* **102**, 094511 (2020).
 - [6] C. Aubin, T. Blum, M. Golterman, and S. Peris, Muon anomalous magnetic moment with staggered fermions: Is the lattice spacing small enough?, *Phys. Rev. D* **106**, 054503 (2022).
 - [7] M. T. Hansen and A. Patella, Finite-volume effects in $(g - 2)_\mu^{\text{HVP,LO}}$, *Phys. Rev. Lett.* **123**, 172001 (2019).
 - [8] M. T. Hansen and A. Patella, Finite-volume and thermal effects in the leading-HVP contribution to muonic $(g - 2)$, *J. High Energy Phys.* **10** (2020) 029.
 - [9] M. Lüscher, Two particle states on a torus and their relation to the scattering matrix, *Nucl. Phys.* **B354**, 531 (1991).
 - [10] L. Lellouch and M. Lüscher, Weak transition matrix elements from finite volume correlation functions, *Commun. Math. Phys.* **219**, 31 (2001).
 - [11] C. J. D. Lin, G. Martinelli, C. T. Sachrajda, and M. Testa, $K \rightarrow \pi\pi$ decays in a finite volume, *Nucl. Phys.* **B619**, 467 (2001).
 - [12] H. B. Meyer, Lattice QCD and the timelike pion form factor, *Phys. Rev. Lett.* **107**, 072002 (2011).
 - [13] D. Bernecker and H. B. Meyer, Vector correlators in lattice QCD: Methods and applications, *Eur. Phys. J. A* **47**, 148 (2011).
 - [14] G. J. Gounaris and J. J. Sakurai, Finite width corrections to the vector meson dominance prediction for $\rho \rightarrow e^+e^-$, *Phys. Rev. Lett.* **21**, 244 (1968).
 - [15] A. Francis, B. Jaeger, H. B. Meyer, and H. Wittig, A new representation of the Adler function for lattice QCD, *Phys. Rev. D* **88**, 054502 (2013).
 - [16] X. Feng, S. Aoki, S. Hashimoto, and T. Kaneko, Timelike pion form factor in lattice QCD, *Phys. Rev. D* **91**, 054504 (2015).
 - [17] D. Giusti, F. Sanfilippo, and S. Simula, Light-quark contribution to the leading hadronic vacuum polarization term of the muon $g - 2$ from twisted-mass fermions, *Phys. Rev. D* **98**, 114504 (2018).
 - [18] Cè, Marco *et al.*, Window observable for the hadronic vacuum polarization contribution to the muon $g-2$ from lattice QCD, *Phys. Rev. D* **106**, 114502 (2022).
 - [19] A. Bazavov *et al.* (Fermilab Lattice, HPQCD, and MILC Collaborations), Light-quark connected intermediate-window contributions to the muon $g-2$ hadronic vacuum polarization from lattice QCD, *Phys. Rev. D* **107**, 114514 (2023).
 - [20] T. Blum, P. A. Boyle, V. Gülpers, T. Izubuchi, L. Jin, C. Jung, A. Jüttner, C. Lehner, A. Portelli, and J. T. Tsang (RBC and UKQCD Collaborations), Calculation of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment, *Phys. Rev. Lett.* **121**, 022003 (2018).
 - [21] T. Aoyama *et al.*, The anomalous magnetic moment of the muon in the standard model, *Phys. Rep.* **887**, 1 (2020).
 - [22] A. Boccaletti *et al.*, High precision calculation of the hadronic vacuum polarisation contribution to the muon anomaly, [arXiv:2407.10913](https://arxiv.org/abs/2407.10913).
 - [23] C. Lehner and A. S. Meyer, Consistency of hadronic vacuum polarization between lattice QCD and the R-ratio, *Phys. Rev. D* **101**, 074515 (2020).
 - [24] G. Wang, T. Draper, K.-F. Liu, and Y.-B. Yang (χ QCD Collaboration), Muon $g-2$ with overlap valence fermions, *Phys. Rev. D* **107**, 034513 (2023).
 - [25] T. Blum *et al.* (RBC and UKQCD Collaborations), Update of Euclidean windows of the hadronic vacuum polarization, *Phys. Rev. D* **108**, 054507 (2023).
 - [26] T. Blum *et al.*, The long-distance window of the hadronic vacuum polarization for the muon $g-2$, *Phys. Rev. Lett.* **134**, 201901 (2025).
 - [27] A. Bazavov *et al.*, Hadronic vacuum polarization for the muon $g - 2$ from lattice QCD: Long-distance and full light-quark connected contribution, [arXiv:2412.18491](https://arxiv.org/abs/2412.18491).
 - [28] T. Blum, Lattice calculation of the lowest order hadronic contribution to the muon anomalous magnetic moment, *Phys. Rev. Lett.* **91**, 052001 (2003).

- [29] X. Wang, Asymptotics of the q-theta function, *Commun. Math. Anal.* **7**, 50 (2009), <https://userweb.ucs.louisiana.edu/~xxw6637/papers/CMA2009.pdf>.
- [30] M. Luscher, Signatures of unstable particles in finite volume, *Nucl. Phys.* **B364**, 237 (1991).
- [31] G. Colangelo, M. Hoferichter, and P. Stoffer, Two-pion contribution to hadronic vacuum polarization, *J. High Energy Phys.* **02** (2019) 006.
- [32] G. Colangelo, M. Hoferichter, B. Kubis, and P. Stoffer, Isospin-breaking effects in the two-pion contribution to hadronic vacuum polarization, *J. High Energy Phys.* **10** (2022) 032.
- [33] V. M. Aul'chenko *et al.*, Measurement of the $e^+e^- \rightarrow \pi^+\pi^-$ cross section with the CMD-2 detector in the 370–520-MeV c.m. energy range, *JETP Lett.* **84**, 413 (2006).
- [34] R. R. Akhmetshin *et al.* (CMD-2 Collaboration), High-statistics measurement of the pion form factor in the ρ -meson energy range with the CMD-2 detector, *Phys. Lett. B* **648**, 28 (2007).
- [35] M. N. Achasov *et al.*, Update of the $e^+e^- \rightarrow \pi^+\pi^-$ cross-section measured by SND detector in the energy region $400 < \sqrt{s} < 1000 \times \text{MeV}$, *J. Exp. Theor. Phys.* **103**, 380 (2006).
- [36] F. Ambrosino *et al.* (KLOE Collaboration), Measurement of $\sigma(e^+e^- \rightarrow \pi^+\pi^-)$ from threshold to 0.85 GeV^2 using initial state radiation with the KLOE detector, *Phys. Lett. B* **700**, 102 (2011).
- [37] J. P. Lees *et al.* (BABAR Collaboration), Precise measurement of the $e^+e^- \rightarrow \pi^+\pi^-(\gamma)$ cross section with the initial-state radiation method at BABAR, *Phys. Rev. D* **86**, 032013 (2012).
- [38] P. Estabrooks and A. D. Martin, $\pi\pi$ phase shift analysis below the K anti-K threshold, *Nucl. Phys.* **B79**, 301 (1974).
- [39] S. D. Protopopescu, M. Alston-Garnjost, A. Barbaro-Galtieri, S. M. Flatte, J. H. Friedman, T. A. Lasinski, G. R. Lynch, M. S. Rabin, and F. T. Solmitz, $\pi\pi$ partial wave analysis from reactions $\pi^+p \rightarrow \pi^+\pi^-\Delta^{++}$ and $\pi^+p \rightarrow K^+K^-\Delta^{++}$ at $7.1 \times \text{GeV}/c$, *Phys. Rev. D* **7**, 1279 (1973).
- [40] M. Davier, A. Hoecker, B. Malaescu, and Z. Zhang, A new evaluation of the hadronic vacuum polarisation contributions to the muon anomalous magnetic moment and to $\alpha(m_Z^2)$, *Eur. Phys. J. C* **80**, 241 (2020); **80**, 410(E) (2020).
- [41] A. Keshavarzi, D. Nomura, and T. Teubner, $g - 2$ of charged leptons, $\alpha(m_Z^2)$, and the hyperfine splitting of muonium, *Phys. Rev. D* **101**, 014029 (2020).