

Gravitational metric of a star

Poul H. Damgaard¹, Hojin Lee², Kanghoon Lee³, and Tabasum Rahnema³

¹*Niels Bohr International Academy, Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*

²*School of Physics, Korea Institute for Advanced Study, 85 Hoegi-ro, Dongdaemun-Gu, Seoul 02455, Korea*

³*Quantum Universe Center, Korea Institute for Advanced Study, 85 Hoegi-ro, Dongdaemun-Gu, Seoul 02455, Korea*



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Solving the classical equations of motion in general relativity recursively, we consider the metric of a spatially localized and stationary source of matter. Having in mind a star of general composition, we characterize it by means of its infinite set of mass and current multipoles. Specializing to de Donder gauge we set up the recursive equations that produce the metric outside the star to any desired order in perturbation theory, expanded both in Newton's constant and in the order of multipoles. Up to second post-Minkowskian order we express the result to any order in the multipole expansion in terms of generalized (tensor) bubble integrals in momentum space and a corresponding simple expansion in inverse distances. In a special corner of the space of multipoles we recover the Kerr black hole solution to the given order. By tweaking just slightly the multipoles away from the Kerr limit the metric will describe stars that are Kerr-like and yet are not black holes. A subtlety with respect to the gauge ambiguity of de Donder gauge is also pointed out.

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I. INTRODUCTION

What is the space-time metric outside a star? It is an elementary property of Newtonian gravity that a spherically symmetric mass distribution leads to a gravitational potential that at any distance r from the center is equivalent to that of a pointlike source of mass equal to the total mass inside a sphere of that radius. In general relativity, an analogous statement is Birkhoff's theorem, which implies that a spherically symmetric vacuum solution of the Einstein equations is that of a Schwarzschild black hole, i.e., as if sourced by a pointlike mass at the origin. But real stars are not perfectly spherically symmetric, and it is thus of both theoretical and experimental interest to determine the effects of deviations from spherical symmetry. Again, Newtonian gravity gives us a suitable tool with which to analyze the issue: the multipole expansion of the mass distribution. In this approach, the arbitrary mass distribution inside a given finite volume is parametrized by its infinite sequence of multipole moments, with higher multipole moments corresponding to weaker and weaker

contributions to the gravitational field at large distances away from the star. The multipole expansion in that case is simply that of an expansion in inverse powers of radius r .

In general relativity, the situation is more involved. We can still consider a distribution of matter sources inside a finite three-volume and ask for the gravitational metric outside that finite volume. Intuition might tell us that at very large distances the vacuum solution could simplify. Should it perhaps approach that of a black hole, and if so, how could this be borne out in detail? A multipole expansion is again a suitable starting point, but there are issues that should be confronted first. One concerns the possibility of gravitational radiation. Even if the mass distribution of the star is not perfectly axisymmetric but deviates just slightly from this symmetry, we can consider the metric as effectively time independent in an adiabatic domain of timescales, where we can ignore the effect of gravitational radiation. In this paper we refrain from going into the technical details on how to define such an adiabatic time interval during which it is sensible to ignore gravitational radiation. Instead, we appeal to known astrophysical sources as well as associated astrophysical timescales over which gravitational radiation remains negligible. This may be so even though there can still be clear observational signatures specific to Einstein's general theory of relativity and the deviations from Newtonian physics. As an example we can have in mind our own Sun which definitely causes

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measurable general relativistic effects without measurable gravitational radiation being one of them. These considerations lead us to focus on the case of *stationary* matter sources and corresponding time-independent space metric, the vacuum solutions of the Einstein equations outside a finite region of three-space which contain all the matter sources. The study of metrics by means of the multipole expansion in general relativity dates back to Geroch [1] and Hansen [2] and, in particular, to the comprehensive treatment of Thorne [3], who introduced the notion of asymptotically Cartesian and mass-centered coordinates. By restricting ourselves to stationary sources, we allow for uniform rotations. Such stationary matter distributions inside a finite volume we will define to be our star. In a particular limit, and as we will clarify below, we recover the multipole expansion of a Kerr black hole.

The most important difference with a Newtonian multipole problem is obviously the nonlinearity of general relativity. In Newtonian gravity, the gravitational field outside a star can be described by its infinite set of multipole moments and, hence, by a simple expansion in terms of powers of $1/r$, each term corresponding to a given multipole moment. The gravitational field outside a Newtonian star is thus in immediate one-to-one correspondence with the multipole moments, and we can conversely read off each multipole moment as the coefficient of a given $1/r^n$ falloff at infinity. It follows from solving the associated Poisson equation, which is exact. In general relativity, the gravitational interactions imply that this is so only at the linearized level. Beyond, the various moments that characterize the mass distribution and currents of the stationary sources are still in one-to-one correspondence with the gravitational metric away from such a bounded source, but this is filtered by the gravitational self-interactions. The n th-order term of the expansion with falloff $1/r^n$ will then, in the weak-field approximation, have an additional infinite expansion in powers of Newton's constant G . This makes the multipole formalism of general relativity a double expansion: in $1/r^n$ and in powers of G . It is customary to refer to this as a multipolar post-Minkowskian expansion. Its use by far surpasses the immediate goals of this paper, which is restricted to the application to stationary space-times. Away from the stationary limit the multipole expansion is a crucial analytical tool with which to describe gravitational radiation from localized sources of matter, and there is a large and impressive literature on the subject (see, e.g., [4–13] for a partial list). Finding the gravitational metric of a compact object such as a neutron star of course also has a long history, dating back to the seminal work of Tolman, Oppenheimer, and Volkoff [14,15]. The aim was initially to find the gravitational metric outside a spherical source of matter corresponding to a given equation of state. There is also a large literature on the external gravitational field of rotating stars and how to map from a given equation of state

onto a set of multipole moments of more general distributions of matter with cylindrical symmetry; see, e.g., Refs. [16–21].

We shall here revisit the multipole expansion of general relativity in the light of the recursive approach of Ref. [22]. Briefly stated, this allows us to express in a compact manner the iterative nature of the perturbative solution to Einstein's field equations, giving each new order in G in terms of integrals over lower-order expressions. The integrations arise due to the solutions of field equations being most conveniently obtained by Fourier transforming to momentum space. For the stationary space-times, we consider in this paper those integrations that are over three-dimensional space only, as in [22]. This considerably simplifies computations, and the required integrals are all of a simple kind that we label as generalized bubble integrals (tensor integrals of rational form corresponding to having massless propagators in quantum field theory language). These bubble integrals arise simply when we introduce the inverse Laplacian in momentum space, and they are loop integrals only in that sense. At intermediate stages, we need to regularize the integrals, and it is natural to choose dimensional regularization, which is ideally suited for this problem. One of the computational obstacles in the early approach to the multipole expansion was indeed the use of the more cumbersome Hadamard regularization, which in addition requires nontrivial amendments beyond the first leading orders (the use of dimensional regularization for the multipole problem in general relativity was advocated earlier in Refs. [23,24]). We note already here the use of both terminology and tools from relativistic quantum field theory. This paper is indeed one more step toward reformulating analytical solutions to the Einstein field equations by means of a recursive quantum field theoretic technique. Great strides have already been made based on the quantum field theoretical amplitude-based approach [25–37]. It considers the scattering of pointlike particles that interact only gravitationally. Extracting the classical terms of such computations is a known procedure, and the common lore in recent years has been that it is far simpler to first consider the full quantum field theoretical amplitude problem of the post-Minkowskian expansion, and subsequently extract only the classical terms, than it is to solve the classical equations of motion directly. We challenge such a position in this paper by considering a problem with no simple and immediate description in terms of the scattering of pointlike massive particles. An alternative formulation preceding the iterative approach of Ref. [22] that we also pursue here is the off-shell recursive method of Refs. [38–40]. It contains similar iterative structures originating in the perturbative formalism, but it also includes all quantum corrections and is thus closely related to the full amplitude-based formalisms. One quantum field theoretical approach that is much closer to our perturbative setup is the worldline formalism for classical

general relativity, which precisely selects the classical pieces to each order in G [41–48]. These worldline formulations can include dissipative effects due to gravitational radiation as well and are, in fact, in one-to-one correspondence [49] with the Kosower-Maybee-O’Connell prescription for amplitudes [27] which also includes dissipative backreaction from the gravitational field.

The worldline formalisms are, however, still rooted in pointlike descriptions of massive particles that interact gravitationally. The link between the multipole expansion and new effective couplings to the worldline action is part of the program of describing extended objects in general relativity through what is known as the gravitational skeleton method, whereby the worldline becomes dressed with higher-order couplings to the gravitational field [41]. Instead, we here proceed directly to the solution of the classical equations of motion, now using to advantage the recent progress in integration techniques from scattering amplitude computations. This bypasses the need for effective field theory ideas, matching conditions, and other mappings between the fundamental degrees of freedom of classical Einstein gravity and those of the effective field theory. Everything is determined by classical physics, classical equations of motion, and the coupling constant G . The only assumption needed is that a perturbative expansion around flat Minkowski space is meaningful so that the multipole expansion is applicable.

We choose the post-Minkowskian multipole expansion of stationary matter sources to illustrate the advantages of this recursive formalism. The resulting solution is algebraic in nature, with the solution being provided by tensor integrals of increasing rank. Inevitably, the actual solution becomes of rapidly growing complexity with this rank. Fourier transformation back to three-space yields a corresponding complicated tensor structures of Cartesian coordinates, and the expansion rapidly clogs up due to the numerous different tensors that are allowed and which will appear. Such is the perturbative multipole formalism since each new order is of higher rank, and the nonlinear nature of the field equations leads to a mixing of these tensor structures at higher orders in G . It is also not our aim here to provide detailed expressions to very high order in the post-Minkowskian expansion because they are not by themselves particularly illuminating. We shall nevertheless present what we believe is the first complete second-order post-Minkowskian expression for stationary space-times based on mass multipoles up to quadrupoles, and current multipoles up to dipole order, in order to illustrate the relative ease with which we obtain these results. Pushing this to higher orders can be done straightforwardly. It has been argued that such highly accurate multipole expansions for celestial mechanics in our Solar System will soon be required to match the observational accuracy of submicroarcsecond astrometry [13,50].

II. THE POST-MINKOWSKIAN MULTIPOLE EXPANSION

From Newtonian gravity (or electrostatics) we are familiar with the multipole expansion, typically expressed in terms of spherical harmonics $Y_{\ell m}(\theta, \phi)$ or associated Legendre polynomials $P_{\ell}^m(\cos \theta)$. For our purposes, the alternative formulation in terms of symmetric trace-free (STF) tensors (see Ref. [3] for a clear exposition of the properties of such STF tensors) is simpler to deal with. We shall briefly review the relation between the two formulations below. STF tensors arise very naturally when taking Cartesian coordinate derivatives of $1/r$ and are thus very naturally linked to a multipole expansion. In addition, such STF tensors satisfy a number of beautiful relations that simplify analytical expressions. As for spherical harmonics, which are scalar functions on the sphere, STF tensors appear because of their interpretation in terms of representations of the rotation group in three dimensions. In order to make the paper self-contained, we begin this section with an overview of the relation between these two formulations, and simultaneously, we introduce our choice of variables, conventions, and notation. A number of very useful relations among STF tensors are relegated to Appendix A.

It is well known that a convenient choice for gravitational perturbation theory is an expansion around flat space of the gothic (or Landau-Lifshitz) tensor density

$$\mathfrak{g}^{\mu\nu}(x^\mu) \equiv \sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \quad (1)$$

which selects harmonic coordinates x^μ by the gauge condition $\partial_\nu(\mathfrak{g}^{\mu\nu}) = 0$. Perturbation theory is set up by expanding the deviation from Minkowski space as an infinite series in G :

$$h^{\mu\nu} = \sum_{n=1}^{\infty} G^n h_{(n)}^{\mu\nu} \quad (2)$$

and solving the equations of motion order by order in G . Then each term $h_{(n)}^{\mu\nu}(t, x, y, z)$ of the series in harmonic coordinates admits a finite multipolar expansion associated with the $O(3)$ group of rotations of the spatial coordinates [which leave invariant $r \equiv ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}$ and $t \equiv x^0$], i.e.,

$$h_{(n)}^{\mu\nu}(x^\mu) = \sum_{l=0}^{\infty} h_{(n),L}^{\mu\nu}(r, t) \hat{n}^L(\theta, \phi). \quad (3)$$

Using standard notation, L denotes a multi-index $i_1 i_2 \dots i_l$, and thus $n^L := n^{i_1} n^{i_2} \dots n^{i_l}$ with $n^i \equiv x^i/r$ ($i = 1, 2, 3$). A hat on Cartesian tensors indicates that we take the STF part. Thus, $\hat{n}^L$ denotes the STF part of n^L and as an example $\hat{n}^{ij} = n^i n^j - \delta^{ij}/3$. We refer to Appendix A for general STF tensor expressions and useful relations among them.

The sum appearing on the right-hand side of Eq. (3) is equivalent to a finite expansion in spherical harmonics $Y_l^m(\theta, \phi)$ as explained in detail in Ref. [3].

As mentioned in the Introduction, we define our star by a stationary and spatially localized source of matter. The resulting spacetime metric in the vacuum outside the star will then be time independent. Such a definition includes isolated and nonevolving compact bodies, such as non-radiating stars and also black holes. Far from the spatially localized sources, one can expand the metric in terms of multipole moments in order to describe the gravitational field generated by them. These multipole moments characterize the spatial distribution of mass and currents in a gauge-invariant (coordinate-invariant) way. At each order in G , there is an infinite set of STF tensors representing mass M and current S multipoles:

$$\{M_L, S_L\}, \quad L = i_1 \dots i_l.$$

After solving the Einstein equations with these sources, the multipoles thus determine the entire external field and they are essential for precision modeling of gravitational interactions. In the asymptotic zone of a stationary source, this multipole expansion is time independent.

The time-independent mass and current multipoles are defined by moments of the energy-momentum tensor [8,13].

(1) *Mass multipole moments.*

$$M^L = \int d^3x (T^{00}(\mathbf{x}) + T^{ii}(\mathbf{x})) \hat{x}^L, \quad (4)$$

where $\hat{x}^L \equiv x^{(i_1} x^{i_2} \dots x^{i_l)}$ denotes the STF part of the multi-index product and the angle brackets represent the STF projection. Note that we should distinguish $\hat{x}^L$ and x^L , which x^L is the conventional tensor product $x^L = x^{i_1} x^{i_2} \dots x^{i_l}$.

(2) *Current multipole moments.*

$$S^L = \int d^3x x^{(L-1} \epsilon_{jk}^i x^j T^{0k}(\mathbf{x}), \quad (5)$$

where $x^{L-1} = x^{i_1} x^{i_2} \dots x^{i_{l-1}}$.

We note that a classic paper describing the approximate metric of a rotating stationary matter distribution is that of Hartle and Thorne [16]. This predates the multipole expansion literature and shows how to obtain an approximate metric for, for instance, a neutron star by a slowly rotated Schwarzschild solution, truncated at leading terms.

III. THE RECURSIVE SOLUTION

Here we briefly review the iterative method that was used to compute the Schwarzschild metric to all orders in Ref. [22]. It boils down to systematizing the perturbative solution to the Einstein equations of motion in de Donder

gauge based on the expansion of Eq. (2). We have already provided the first step in the previous section. Restricting ourselves to stationary solutions, we here consider time-independent sources of matter that give rise to an energy-momentum tensor $T_{\mu\nu}(\mathbf{x})$ which, as indicated, is only a function of the spatial coordinates $\mathbf{x}$. We have collected a number of useful identities involving STF tensors and their relation to Fourier-transformed variables in Appendix B. Using results from there, and to leading order in G , the equations of motion then reduce to Eq. (B16) and with the multipole expanded solution of Eq. (B19).

It is convenient to introduce the Fourier transforms of $h^{\mu\nu}$ which we call currents. Thus, corresponding to the expansion of Eq. (2), we also introduce an ordered sum

$$J^{\mu\nu}(\mathbf{k}) = \sum_{n=1}^{\infty} G^n J_{(n)}^{\mu\nu}(\mathbf{k}), \quad (6)$$

where

$$J_{(n)}^{\mu\nu}(\mathbf{k}) \equiv \int d^3x h_{(n)}^{\mu\nu}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}. \quad (7)$$

For simplicity of notation we define $J_{(n)|\mathbf{k}}^{\mu\nu} \equiv J_{(n)}^{\mu\nu}(\mathbf{k})$. Using the terminology of Ref. [22] we call n the rank of the expansion. For rank 1 we have linearized gravity, and the currents are simply the Fourier transforms of Eq. (B19). Higher-order terms derive from the gravitational self-interactions that are included in the Einstein tensor. Expanding the equations of motion order by order in G and making use of the de Donder gauge condition, we collect terms in the Landau-Lifshitz pseudotensor

$$\tau_{LL}^{\mu\nu} = \sum_{n=1}^{\infty} G^{n+1} \tau_{LL(n)}^{\mu\nu}. \quad (8)$$

Matching powers of G this defines a hierarchy of equations. The expansion of the Landau-Lifshitz pseudotensor starts at one order higher than the expansion of $h^{\mu\nu}$ itself, and the equations become recursive. Each new order of $h_{(n)}^{\mu\nu}$ is given in terms of combinations of previously computed terms of up to and including those of rank $(n-1)$. The usefulness of the currents of Eq. (6) derives simply from the need to solve the Green function problem for the n th term, thus making it convenient to first go into momentum space, perform the needed convolutions, and then transform back to real space. The intermediate step of Fourier-space convolutions is the analog of loop integrations in the amplitude approach. Beyond the first few orders, the expansion becomes algebraically unwieldy. This problem can be solved by introducing the auxiliary set of metric tensor variables described in Ref. [22]. It bypasses at intermediate steps the inversion of $\mathfrak{g}^{\mu\nu}$ through a doubling of metric variables by the introduction of a new tensor density $\tilde{\mathfrak{g}}_{\mu\nu}$ which is only

identified as the inverse of $\mathbf{g}^{\mu\nu}$ (i.e., $\mathbf{g}^{\mu\nu}\tilde{\mathbf{g}}_{\nu\rho} = \delta^\mu_\rho$) at the end. Such a simplified method becomes essentially mandatory to introduce at high orders in G but it is not needed here.

For the 2PM calculations of this paper, we need only the first nontrivial terms of the Landau-Lifshitz pseudotensor

$\tau_{LL(1)}^{\mu\nu}$. The Einstein equation at the 2PM order is

$$\square h_{(2)}^{\mu\nu}(\mathbf{x}) = -\tau_{LL(1)}^{\mu\nu}, \quad (9)$$

where, if we denote $h = h_\rho{}^\rho$,

$$\begin{aligned} \tau_{LL(1)}^{\mu\nu} = & 2\partial^\rho h^{\sigma(\mu} \partial_\sigma h_{\rho}{}^{\nu)} - 2\partial_\rho (h_\sigma{}^{(\mu} \partial^\nu h^{\rho\sigma)}) + 2h_\rho{}^{(\mu} \square h^{\nu)\rho} + \frac{1}{2} \partial^\mu h^{\rho\sigma} \partial^\nu h_{\rho\sigma} - \partial_\rho (h^{\rho\sigma} \partial_\sigma h^{\mu\nu}) - \frac{1}{4} \partial^\mu h \partial^\nu h \\ & - \frac{1}{2} h^{\mu\nu} \square h + \eta^{\mu\nu} \left[\frac{1}{8} \partial^\rho h \partial_\rho h + \frac{1}{4} h \square h + \frac{1}{4} \partial^\rho h^{\sigma\tau} \partial_\rho h_{\sigma\tau} - \frac{1}{2} h^{\rho\sigma} \square h_{\rho\sigma} + \frac{1}{2} \partial^\tau h^{\rho\sigma} \partial_\rho h_{\tau\sigma} + h^{\rho\sigma} \partial_\rho \partial^\tau h_{\sigma\tau} \right], \end{aligned} \quad (10)$$

which we will use in the next section. We point out the important fact that, from this stage and further toward higher orders in G , all dynamics resides entirely in the gravitational self-interactions of general relativity. The higher-order terms are thus universal (invariant functions of the particular values of the infinite set of multipole moments that arise from the energy-momentum tensor) and reflecting precisely how general relativity glues these different multipole moments together based on the principle of general covariance.

IV. TRUNCATED MULTIPOLE EXPANSIONS

Having set up the recursive relations, we are now ready to apply them to the multipole expansion. In contradistinction to the Schwarzschild case [22] the recursion is now double: both in order of multipoles and in order of G . The recursion relations expand in powers of G , given what can be viewed as initial data in terms of 1PM multipoles. This is analogous to the Schwarzschild solution, where the initial condition is provided by a single pointlike matter source at the origin. That Schwarzschild case corresponds to a time-independent metric of nonvanishing h^{00} , all other components of h vanishing at that initial order. For the general multipole expansion, the source is rather characterized by an infinite set of physical parameters that describe the Taylor expansion of the metric at the linearized level. In practice, one will truncate at a given order in multipoles and then generate the post-Minkowskian expansion in G based on this finite number of initial values. This double expansion rapidly clogs up due to the numerous different tensor structures that result from the couplings between different multipoles, together with spatial indices. In order not to lose sight of the underlying simple recursive structures in the post-Minkowskian expansion, it is therefore illustrative to begin with an example to 2PM order, truncated to quadrupole mass multipole order and leading current multipole order. As we shall see, this already produces new results, and it will then become clear how it generalizes to arbitrary orders.

A. Rank 1

We recall that we label the power n of the coupling constant G as the rank. Rank 1 is what in quantum field theory language would be labeled tree level, and $h_{(1)}^{\mu\nu}$ is the solution to the classical equations of motion to order G . This is the starting point of the iteration and we have the simple relation Eq. (B16) of linearized gravity. The solutions (B19) of this equation give our initial conditions for the recursion relations.

Hence, from (7) we get the rank-1 currents up to the quadrupole moments

$$\begin{aligned} J_{(1)|\mathbf{k}}^{00} &= \int_{\mathbf{x}} \left(\frac{4M}{r} + \frac{6M_{ab}}{r^3} \hat{n}_{ab} \right) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= 16\pi \left(\frac{M}{|\mathbf{k}|^2} - \frac{M_{ab} \hat{k}^{ab}}{2|\mathbf{k}|^2} \right), \\ J_{(1)|\mathbf{k}}^{0i} &= \int_{\mathbf{x}} \left(-2\epsilon_{iab} n^a \frac{S^b}{r^2} \right) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= i\epsilon_{iab} \frac{8\pi S^b k^a}{|\mathbf{k}|^2}, \end{aligned} \quad (11)$$

and in addition $\mathcal{J}_{(1)|\mathbf{k}}^{ij} = 0$.

B. Rank 2

From Einstein equation at 2PM order (9), we first consider the contributions to the time components $h_{(2)}^{00}$ under the stationary condition and the harmonic gauge:

$$\begin{aligned} \nabla^2 h_{(2)}^{00} &= \frac{7}{8} \partial_i h_{(1)}^{00} \partial_i h_{(1)}^{00} + \frac{5}{4} h_{(1)}^{00} \partial^2 h_{(1)}^{00} \\ &\quad - \frac{1}{2} \partial_i h_{(1)}^{0j} \partial_i h_{(1)}^{0j} - \frac{3}{2} \partial_i h_{(1)}^{0j} \partial_j h_{(1)}^{0i} - h_{(1)}^{0i} \partial^2 h_{(1)}^{0i}, \end{aligned} \quad (12)$$

which by iteration gives us the currents. To help displaying how the recursion works, we split up $h_{(2)}^{00}$ into two parts: $h_{(2)}^{00[a]}$, which depends only on $h_{(1)}^{00}$, and $h_{(2)}^{00[b]}$, which

depends only on $h_{(1)}^{0i}$:

$$h_{(2)}^{00} = h_{(2)}^{00[a]} + h_{(2)}^{00[b]}, \quad (13)$$

where $h_{(2)}^{00[a]}$ and $h_{(2)}^{00[b]}$ satisfy

$$\begin{aligned} \nabla^2 h_{(2)}^{00[a]} &= \frac{7}{8} \partial_i h_{(1)}^{00} \partial_i h_{(1)}^{00} + \frac{5}{4} h_{(1)}^{00} \partial^2 h_{(1)}^{00}, \\ \nabla^2 h_{(2)}^{00[b]} &= -\frac{1}{2} \partial_i h_{(1)}^{0j} \partial_i h_{(1)}^{0j} - \frac{3}{2} \partial_i h_{(1)}^{0j} \partial_j h_{(1)}^{0i} \\ &\quad - h_{(1)}^{0i} \partial^2 h_{(1)}^{0i}. \end{aligned} \quad (14)$$

Let us consider first the iterative contributions from $J_{(2)|-k_1}^{00[a]}$, which is associated with $\partial^2 h_{(2)}^{00[a]}$:

$$\begin{aligned} J_{(2)|-k_1}^{00[a]} &= \frac{1}{|k_1|^2} \int_{k_2} \left[\frac{5}{4} |k_2|^2 - \frac{7}{8} k_{12} \cdot k_2 \right] \\ &\quad \times J_{(1)|-k_{12}}^{00} J_{(1)|k_2}^{00}. \end{aligned} \quad (15)$$

Using the bubble integral formula in Eq. (C4), we have

$$\begin{aligned} \frac{1}{\pi^2} J_{(2)|-k}^{00[a]} &= \frac{14M^2}{|k|} + \frac{7M_{ab}M_{ab}|k|^3}{40} + \frac{3M_{ab}^2 \hat{k}^{ab}|k|}{8} \\ &\quad - \frac{21MM_{ab} \hat{k}^{ab}}{4|k|} + \frac{21M_{ab}M_{cd} \hat{k}^{abcd}}{256|k|}. \end{aligned} \quad (16)$$

We next find the contributions from the three terms involving the $h_{(1)}^{0i}$ components, viz., those arising from the current multipoles. We label them by superscript $[b]$:

$$\begin{aligned} |k_1|^2 J_{(2)|-k_1}^{00[b]} &= \int_{k_2} \left[\left(\frac{k_{12} \cdot k_2}{2} - |k_2|^2 \right) \delta^{ij} + \frac{3k_2^i k_{12}^j}{2} \right] \\ &\quad \times J_{(1)|-k_{12}}^{0i} J_{(1)|k_2}^{0j}. \end{aligned} \quad (17)$$

Similarly, upon substitution of the corresponding 1PM currents from (11), this gives

$$\begin{aligned} |k_1|^2 J_{(2)|-k_1}^{00[b]} &= \int_{k_2} \left[\frac{32\pi^2 k_{12} \cdot k_2}{|k_2|^2 |k_{12}|^2} (k_2 \cdot k_{12} |S|^2 - (k_{12} \cdot S)(k_2 \cdot S)) \right. \\ &\quad \left. - \frac{96\pi^2}{|k_2|^2 |k_{12}|^2} (k_2 \cdot (k_{12} \times S))^2 \right], \end{aligned} \quad (18)$$

where $k_2 \cdot (k_{12} \times S) = \epsilon_{ijk} k_2^i k_{12}^j S^k$. Evaluating the integrals we obtain

$$J_{(2)|k}^{00[b]} = \frac{5\pi^2 |S|^2 |k|}{3} - \frac{7\pi^2 S^i S^j}{4|k|} \hat{k}^{ij}. \quad (19)$$

The total 2PM contribution is thus $J_{(2)|k}^{00} = J_{(2)|k}^{00[a]} + J_{(2)|k}^{00[b]}$ with the result

$$\begin{aligned} J_{(2)|k}^{00} &= \frac{14\pi^2 M^2}{|k|} + \frac{7\pi^2 M_{ab}M_{ab}|k|^3}{40} + \frac{5\pi^2 |S|^2 |k|}{3} \\ &\quad + \frac{3\pi^2}{4} \left[\frac{M_{ab}^2 |k|^2}{2|k|} - \frac{7MM_{ab}}{|k|} - \frac{7S_a S_b}{3|k|} \right] \hat{k}^{ab} \\ &\quad + \frac{21\pi^2 M_{ab}M_{cd} \hat{k}^{abcd}}{256|k|}. \end{aligned} \quad (20)$$

We next move on to find the currents for h^{0i} to order G^2 . From the equations of motion to this order,

$$\nabla^2 h_{(2)}^{0i} = \frac{1}{2} h_{(1)}^{0i} \partial^2 h_{(1)}^{00} + \partial_j h_{(1)}^{0i} \partial_j h_{(1)}^{00} - \partial_i h_{(1)}^{0j} \partial_j h_{(1)}^{00}, \quad (21)$$

and the corresponding recursion is

$$\begin{aligned} J_{(2)|-k_1}^{0i} &= \frac{1}{|k_1|^2} \int_{k_2} \left[\left(\frac{1}{2} |k_2|^2 - k_{12} \cdot k_2 \right) \delta^{ij} + k_{12}^i k_2^j \right] \\ &\quad \times J_{(1)|-k_{12}}^{0j} J_{(1)|k_2}^{00}, \end{aligned} \quad (22)$$

which for the currents gives

$$\begin{aligned} J_{(2)|-k}^{0i} &= i\pi^2 |k| \left[\frac{2M\epsilon_{ijk}}{|k|^2} - \frac{\epsilon_{ijl} M_{lk}}{2} - \frac{M_{il} \epsilon_{ljk}}{8} \right] S^j k^k \\ &\quad + \frac{\epsilon_{iab} M_{cd}}{16|k|} S^a k^{bcd}. \end{aligned} \quad (23)$$

We finally turn to the h^{ij} components at 2PM order. While not present at order G , these terms are generated by iteration through the equations of motion:

$$\begin{aligned} \nabla^2 h_{(2)}^{ij} &= \partial_k h_{(1)}^{0i} \partial_k h_{(1)}^{0j} - 2\partial^k h_{(1)}^{0(i} \partial^j) h_{(1)}^{0k} - \frac{1}{4} \partial_i h_{(1)}^{00} \partial_j h_{(1)}^{00} \\ &\quad + \partial_i h_{(1)}^{0k} \partial_j h_{(1)}^{0k} + \frac{\delta^{ij}}{8} [\partial_k h_{(1)}^{00} \partial_k h_{(1)}^{00} - 4\partial_k h_{(1)}^{0\ell} \partial_k h_{(1)}^{0\ell} \\ &\quad + 4\partial_k h_{(1)}^{0l} \partial_l h_{(1)}^{0k} + 2h_{(1)}^{00} \nabla^2 h_{(1)}^{00} - 8h_{(1)}^{0k} \nabla^2 h_{(1)}^{0k}]. \end{aligned} \quad (24)$$

Deriving the recursion by taking the Fourier transform, we get the corresponding currents by solving the recursion

$$\begin{aligned}
-\pi^{-2}J_{(2)|k}^{ij} = & M^2 A_{1,1}^{ij} + \frac{M(\mathbf{k} \cdot \mathbf{M} \cdot \mathbf{k})}{16|\mathbf{k}|} A_{5,7}^{ij} + \frac{M}{8|\mathbf{k}|} B^{ij} - \frac{MM^{ij}|\mathbf{k}|}{8} - \frac{(\mathbf{k} \cdot \mathbf{M} \cdot \mathbf{k})^2}{512|\mathbf{k}|} A_{5,1}^{ij} + \frac{(\mathbf{k} \cdot \mathbf{M} \cdot \mathbf{k})}{128|\mathbf{k}|} B^{ij} - \frac{(\mathbf{k} \cdot \mathbf{M} \cdot \mathbf{M} \cdot \mathbf{k})}{128|\mathbf{k}|^{-2}} A_{4,13}^{ij} \\
& + \frac{|\mathbf{k}|}{192} \left[\frac{7}{2} (\mathbf{k} \cdot \mathbf{M} \cdot \mathbf{k}) M^{ij} - 5(M \cdot \mathbf{k})^i (M \cdot \mathbf{k})^j \right] + \frac{(M \cdot \mathbf{k})_k l}{192} B^{ijk} - \frac{5l^2 M_{ab} M_{ab}}{768} A_{1,1}^{ij} - \frac{l^3}{192} M_{ij}^2 \\
& + \frac{(\mathbf{S} \times \mathbf{k})_i (\mathbf{S} \times \mathbf{k})_j}{2|\mathbf{k}|} + \frac{2(\mathbf{S} \cdot \mathbf{k})^2}{16} A_{1,3}^{ij} + \frac{|\mathbf{S}|^2 |\mathbf{k}|^2}{16} A_{11,3}^{ij} + \frac{5}{8} \left[|\mathbf{k}| S^i S^j - \frac{2\mathbf{k} \cdot \mathbf{S}}{|\mathbf{k}|} k^{(i} S^{j)} \right], \quad (25)
\end{aligned}$$

where $\mathbf{k} \cdot \mathbf{M} \cdot \mathbf{k} = k^i M_{ij} k^j$, $(M \cdot \mathbf{k})_i = M_{ij} k^j$ and

$$\begin{aligned}
A_{\alpha,\beta}^{ij} &= \frac{\alpha k_1^i k_1^j}{k^3} - \frac{\beta \delta^{ij}}{k}, \\
B^{ijk} &= k^j M^{ik} + k^i M^{jk}, \quad B^{ij} = B^{ijk} k_k. \quad (26)
\end{aligned}$$

C. Stationary gravitational metric at 2PM order

The iterative solution has provided us with all stationary 2PM currents that include mass multipoles and the leading current multipole corresponding to angular momentum. The only remaining task in order to extract the metric is to Fourier transform back to space coordinates. Before listing those expressions, we here provide a few details on how to solve the required integrals. The same method used to derive, e.g., Eq. (C6).

Using this, we can now provide the corresponding expressions for the metric tensor:

$$\begin{aligned}
h_{(2)}^{00} = & \frac{7M^2}{r^2} + \frac{21MM_{ab}}{r^4} \hat{n}_{ab} - \frac{5|\mathbf{S}|^2}{3r^4} + \frac{7S^a S^b}{r^4} \hat{n}^{ab} \\
& + \frac{63M_{ab} M_{cd} \hat{n}^{abcd}}{4r^6} + \frac{9M_{ab}^2 \hat{n}^{ab}}{r^6} + \frac{21M_{ab} M_{ab}}{10r^6} \quad (27)
\end{aligned}$$

and

$$\begin{aligned}
h_{(2)}^{0i} = & \frac{2M(\mathbf{S} \times \mathbf{n})^i}{r^3} + \frac{3\tilde{S}_{ia} M_{bc} \hat{n}^{abc}}{2r^5} \\
& - \frac{19\tilde{S}_{ia} (M \cdot \mathbf{n})_a}{10r^5} + \frac{M_{ij} (\mathbf{S} \times \mathbf{n})^j}{2r^5}, \quad (28)
\end{aligned}$$

where $M_{ab}^2 = M_{ac} M_{cb}$ and $\tilde{S}_{ia} = \epsilon_{iab} S^b$.

The h^{ij} components at order G^2 are considerably more lengthy. We decompose the term as

$$h_{(2)}^{ij} = \frac{1}{r^2} H_2^{ij} + \frac{1}{r^4} H_4^{ij} + \frac{1}{r^6} H_6^{ij} \quad (29)$$

and each component is

$$\begin{aligned}
H_2^{ij} &= M^2 \left[\hat{n}^{ij} + \frac{1}{3} \delta^{ij} \right], \\
H_4^{ij} &= MM_{ab} \left[\frac{15}{2} \hat{n}^{ijab} + \frac{11}{7} \delta^{ij} \hat{n}^{ab} \right] - \frac{12}{7} MM^{a(i} \hat{n}_a^{j)} + \left[\frac{9}{2} \hat{n}^{ijab} - \frac{13}{7} \delta^{ij} \hat{n}^{ab} \right] S_a S_b - 2\hat{n}^{ab} \tilde{S}_{ia} \tilde{S}_{jb} \\
&+ \frac{32}{7} S^a \hat{n}_a^{(i} S^{j)} - \frac{S^i S^j}{15} - |\mathbf{S}|^2 \left[\frac{20}{7} \hat{n}^{ij} - \frac{2}{15} \delta^{ij} \right], \\
H_6^{ij} &= M_{ab} M_{cd} \left[\frac{75}{4} \hat{n}^{ijabcd} + \frac{9\delta^{ij}}{44} \hat{n}_{abcd} \right] - \frac{6M_{ij}^2}{35} - \frac{90}{11} M_{ab}^2 \hat{n}^{ijab} + M_{ab} M^{ab} \left[\frac{11}{70} \delta^{ij} - \frac{25}{84} \hat{n}^{ij} \right] \\
&- \frac{5}{21} M_a^{2(i} \hat{n}^{j)}_b + \frac{18}{11} M^{bc} M^{a(i} \hat{n}^{j)}_{abc} + \left[\frac{10}{21} M^{ab} M^{ij} - \frac{23}{42} M^{a(i} M^{j)b} + \frac{29}{42} \delta^{ij} M_{ab}^2 \right] \hat{n}_{ab}. \quad (30)
\end{aligned}$$

We have checked that all these expressions match with the literature [9,12].

V. REDUCTION TO THE KERR METRIC

As a further check on the expressions given above, we can consider the black hole limit where our metric should reduce to the Kerr metric in harmonic gauge [51,52]. This is a degenerate corner of the general multipole expansion in which the current multipoles and mass multipoles are not independent but scale with one parameter, the reduced angular momentum $a = J/M$. We start by listing the standard form of harmonic gauge $h^{\mu\nu}$ for the Kerr metric:

$$\begin{aligned} h^{00} &= 1 + \frac{R^2[a^2(3\bar{M}^2 + 4\bar{M}R + R^2) + (\bar{M} + R)^4]}{a^2 - \bar{M}^2 + R^2} \\ &\quad + \frac{a^2 z^2}{a^2 z^2 + R^4}, \\ h^{0i} &= -\frac{2a\bar{M}R^2(\bar{M} + R)\epsilon_{ij3}x^j}{(a^2 z^2 + R^4)(a^2 - \bar{M}^2 + R^2)}, \\ h^{ij} &= \frac{\bar{M}^2 R^2}{R^4 + a^2 z^2} \zeta^i \zeta^j, \end{aligned} \quad (31)$$

where $\bar{M} = GM$ and the defining equation for R^2 is

$$R^2 = r^2 - a^2 \left(1 - \frac{z^2}{R^2}\right), \quad (32)$$

and $r \equiv \sqrt{x^2 + y^2 + z^2}$. In addition,

$$\zeta^i = \left(\frac{Rx + ay}{R^2 + a^2}, \frac{-ax + Ry}{R^2 + a^2}, \frac{z}{R} \right). \quad (33)$$

We remind that we can read off the multipoles for Kerr by expanding the terms of order G and a in powers of $1/r$. As mentioned earlier, there are indeed no contributions to h^{ij} at order G , but there are terms in h^{ij} with both odd and even powers of a at order G^2 and higher. This will become an important point when we make contact with our general multipole expansion, as will be discussed below.

Indeed, an important point to mention is the following. The full Kerr solution provides one compact vacuum solution to Einstein equation outside a suitable source with spin. From that closed-form solution one cannot identify the combinations of different multipole moments that are behind the compact expression for the metric. Analyzing how the Kerr metric builds up perturbatively, order by order in G and spin a , we can obtain that map. This will allow us to trace precisely which multipole moments contribute to given orders of the Kerr metric expansion. Such an analysis has not been carried out before.

We now need to identify the Kerr mass and current multipoles. An axisymmetric source such as that of a Kerr black hole with a chosen symmetry axis s^i implies invariance under only the subgroup $SO(2)_s \subset SO(3)$ of rotations about s^i . Since rank- l STF tensors form irreducible $SO(3)$ representations, imposing invariance under this

axial $SO(2)$ selects a one-dimensional invariant subspace within it. In such a case, the only rotationally invariant STF direction is that of the symmetry axis (the spin s axis).

Therefore, at each l , the multipole tensor must be proportional to the unique axis-aligned STF tensor built solely from the unit axis vector:

$$M_L = a_l \hat{s}_L. \quad (34)$$

One can find the detailed proof of this statement from symmetry in Appendix D. This implies, if we choose the symmetry axis to be in the 3 direction, the only independent component is the all- z one $M_{33\dots 3}$, while any component carrying an x or y index is excluded by the symmetry (and the remaining diagonal pieces are fixed by the trace-free condition).

By expanding (31) at the $\mathcal{O}(G)$,

$$\begin{aligned} h_{(1)}^{00} &= \sum_{l=0}^{\infty} \frac{4(-1)^l M a^{2l}}{r^{2l+1}} P_{2l}(\hat{n} \cdot \hat{s}), \quad \hat{n} \cdot \hat{s} = \frac{z}{r}, \\ &= 4 \sum_{l=0}^{\infty} \frac{(-1)^l M a^{2l}}{r^{2l+1}} \frac{(4l-1)!!}{2l!} \hat{n}^L \hat{s}_L, \end{aligned} \quad (35)$$

where, in the last step, we employed the identity in Eq. (B21) with $L = i_1, \dots, i_{2l}$.

We now compare with the general form in Eq. (B22) and, by matching coefficients, find

$$M_L = (-1)^l M a^{2l} \hat{s}_L. \quad (36)$$

Therefore,

$$\begin{aligned} \text{odd: } M_{2l+1} &= 0, \\ \text{even: } M_{2l} &= M(-1)^l a^{2l}. \end{aligned} \quad (37)$$

The Kerr current multipole expressions can be read off similarly. In summary, the Kerr multipoles corresponding to an axisymmetric rank- l STF tensor aligned with the spin axis s can be grouped together in the complex combination [1,2]

$$\mathcal{M}_l \equiv M_l + iS_l, \quad \mathcal{M}_l = M(ia)^l, \quad (38)$$

corresponding to

$$M_L = (-1)^l a^{2l} M \hat{s}^L; \quad S^{aL} = (-1)^l M a^{2l+1} \hat{s}^{aL}. \quad (39)$$

Let s^i be the unit spin direction of Kerr, chosen along the 3 direction: $s^i \equiv \delta^{i3}$. We then have for the current dipole and the mass monopole and quadrupoles:

$$S^a = Ma\delta^{a3} = (0, 0, Ma),$$

$$M_{ab} = -Ma^2\hat{s}_{ab} = -Ma^2\left(s_a s_b - \frac{1}{3}\delta_{ab}\right). \quad (40)$$

With the symmetry axis aligned with the z axis, we thus have for the mass quadrupole

$$M_{zz} = -\frac{2}{3}Ma^2, \quad M_{xx} = M_{yy} = +\frac{1}{3}Ma^2.$$

We begin with a comparison between h^{00} of our multipole expansion and h^{00} of the Kerr metric. Since we have kept mass multipoles up to and including quadrupoles but only the leading current multipole (spin), and since we now expand in the common parameter a , we need some care. Through iterations, we can recover all terms of order a^2 [arising from combinations of mass monopole-quadrupole (MM_{ab}) or current dipole-dipole ($S_a S_b$)] and we can also recover all terms of order a^4 arising from mass quadrupole-quadrupole ($M_{ab}M_{cd}$). But there are also potentially iterative terms coming from mass monopole-hexadecapole (MM_{abcd}) and current dipole-octupole ($S_a S_{bcd}$) combinations. Those pieces we have not computed for the 2PM order. One could imagine also terms of order a^4 from

further combinations like $S_a S_b S_{cd}$, etc., but such contributions cannot arise at order G^2 since this is the first iteration in G that hence links only two multipoles.

Because we have truncated our multipole expansion at mass quadrupole and current dipole order, we do not have all the pieces that contribute to order a^4 for h^{00} at 2PM order. It turns out, however, that we have more information available than what can be read off from the “exact” form of the Kerr metric in Eq. (31). This follows from the analysis of [53], which shows how to build up the Kerr metric from the perturbative expansions in both g and a , given a (1PM) source for a Kerr black hole. In detail, this can be done by attaching “charges” $Q[n]$ to all multipoles so that $Q[0]$ multiplies the mass monopole and $Q[2]$ multiplies the mass quadrupole (and in general all $Q[n]$ for n even multiply the mass multipoles). Likewise, $Q[1]$ multiplies the current dipole, and in general $Q[n]$ multiplies the current multipoles for n odd. This way of identifying how different multipole combinations build the Kerr metric (31) from a Kerr source gives us the possibility to check our 2PM computations for all components of $h^{\mu\nu}$ up to order a^4 . We list all of the components for reference in Appendix E. For instance, for the case at hand, the expansion of the 2PM Kerr metric $h_{2\text{PM}}^{00}$ truncated at $\mathcal{O}(a^4)$ is given by [53]

$$M^{-2}h_{(2)}^{00}|_{\mathcal{O}(a^4)} = \frac{7Q[0]^2}{r^2} + \frac{7(r^2 - 3z^2)}{r^6}Q[0]Q[2] - \frac{(4r^2 - 7z^2)}{r^6}Q[1]^2 + \frac{7(r^2 - 3z^2)^2}{4r^{10}}Q[2]^2 - \frac{(4r^4 - 33r^2z^2 + 35z^4)}{r^{10}}Q[1]Q[3] + \frac{7(3r^4 - 30r^2z^2 + 35z^4)}{4r^{10}}Q[0]Q[4]. \quad (41)$$

We are now ready to compare this with the general multipole expansions of Eq. (27) by substituting values for mass and current multipoles for the Kerr black hole given in Eq. (39). We recover the Kerr metric for the specific choice

$$Q[l] = a^l. \quad (42)$$

The coefficients to the various combinations of these Q ’s in this expansion of the metric correspond one to one with the corresponding combinations of multipole moments. They show how the Kerr metric builds up perturbatively by combining those different multipoles through the recursion.

Collecting the results up to quadrupole moments $h_{(2)}^{00}|_{\text{Quad}}$, we have

$$h_{(2)}^{00}|_{\text{Quad}} = \frac{M^2}{r^2} \left[7 + \frac{a^2(3 - 14n_3^2)}{r^2} + \frac{a^4(1 - 3n_3^2)^2}{4r^4} \right], \quad (43)$$

with $n_3 = \frac{z}{r}$. We can now clearly see how our general expansion reduces to the Kerr values through the match to the first four coefficients given in Eq. (41) after substitution

of the identification (42). In greater generality, assigning these Q labels at the 1PM source level of the Kerr metric allows us to identify and extract different multipole coefficients in the metric at any higher PM order and higher multipole orders.

We now turn to a similar check on our h^{0i} expressions at 2PM order. At leading order in a only the current dipole and mass monopole, i.e., MS_a , contribute. From (28) we immediately obtain at order a :

$$h_{(2)}^{0i} = -2M\epsilon^{ij3}S_3\frac{n_j}{r^3} = -2Ma\frac{\epsilon^{ij3}x_j}{r^3}. \quad (44)$$

Moving on to order a^3 (only odd orders of a can appear due to parity) we can have combinations $M_{ab}S_c$, MS_{abc} , and MM_{abc} . Of these, we can only check the terms arising from quadrupole-dipole contributions.

Then we can write

$$h_{(2)}^{0i} = M^2a\epsilon_{iab}n^aS^b \left[-\frac{2}{r^3} - \frac{a^2}{2r^5}(3n_3^2 + 1) \right]. \quad (45)$$

Comparing with the component expressions for the Kerr metric at 2PM order listed in Appendix E, we find a perfect match to order a^3 from our general multipole expansion.

We finally turn to a comparison of our h^{ij} expressions in the Kerr limit. At 2PM order we can potentially get terms of order a (from combinations MS_a), order a^2 (from combinations $S_a S_b$ and MM_{ab}), order a^3 (from combinations

$S_a M_{bc}$ and MS_{abc}), and order a^4 (from combinations $M_{ab} M_{cd}$, MM_{abcd} , $M_{abc} S_d$, and MS_{abcd}). Of these, we are not comparing with any terms involving M_{abc} , M_{abcd} , S_{abc} or S_{abcd} . We now check which of the remaining kinds of combinations will appear in our iterative solution.

As we have chosen the spin axis, s^i to be in the 3 direction ($s^i \equiv \delta^{i3}$), we have

$$h_{(2)}^{ij} = \frac{M^2}{r^2} n^i n^j + \frac{M^2 a^2}{r^4} [(5n_3^2 - 11)n^i n^j - 2v^i v^j + 8n_3 s^{(i} n^{j)} - 3s^i s^j + 3(1 - n_3^2)\delta^{ij}] + \frac{3M^2 a^4}{4r^6} \left[(1 - 18n_3^2 + 25n_3^4)n^i n^j + 2\left(n_3^2 - \frac{1}{3}\right)s^i s^j + 8(n_3 - 2n_3^3)s^{(i} n^{j)} + 2n_3^2(1 - n_3^2)\delta^{ij} \right], \quad (46)$$

where $v^i = \epsilon^i_{a3} n^a$.

This matches with the Kerr metric at mass quadrupole-quadrupole order $\mathcal{O}(a^4)$, as listed in Appendix E.

We thus find agreement with the Kerr metric in the Kerr limit of multipoles in Eq. (39) for all terms we have computed at 2PM order. There are two ways in which to view these confirmations of results. The first is to see this as only a consistency check on two similar computations: the perturbative solution based on the Kerr source of Ref. [53] (which is based on the same iterative formalism) and our more general multipole expansion here. These two calculations should clearly agree in the Kerr limit. Another way to consider it is to think beyond the Kerr black hole and view the deformations from the Kerr metric as representing interesting compact astrophysical objects whose metrics can resemble Kerr black holes to as high a degree of accuracy as we wish. This second viewpoint highlights the interesting scenario that a compact star can seem arbitrarily similar to a Kerr black hole, and yet it will not be a black hole. Approaching the star from a far distance, all multipoles may appear to be those of the Kerr metric, except for a few multipoles that are slightly different. Then there will be no horizons, and the star will reveal its true identity only very close to the would-be horizons. The existence of such a black hole mimicker should not be ruled out by fundamental principles in general relativity. Neither should it be ruled out by astrophysical production mechanisms that would force the stellar evolution to end at a Kerr black hole if sufficiently close to it in terms of multipoles. It would not seem to involve careful tuning to realize such an astrophysical compact object. Examples of external metrics of rotating compact objects that differ by arbitrarily small amounts from those of Kerr black holes have been considered in, e.g., Refs. [54,55]. It is worth stressing that a way to distinguish spinning black holes from mimickers in gravitational wave signals would be to extract individual multipole moments from the data. Although impossible in the near future, and very difficult in the long run, this would provide a quantifiable measure of the degree of confidence

one has in assigning the identities of genuine black hole solutions to the binary systems. The idea put forth in this paper is that for stationary sources one can compute the full multipole expansion beyond linearized gravity to in principle arbitrarily high order in G and, thus, be able to identify not only potential deviations from black hole metrics in the data, but also pinpoint precisely which combinations of multipole moments will be impacted by such potential deviations. In practice, the distinction between black holes and very compact astrophysical objects may be almost immaterial from the point of view of gravitational wave physics but, of course, of major importance for astrophysics.

Computing the Kerr metric perturbatively from scattering amplitudes and worldline formalisms is a different line of attack which assigns spin to a pointlike object either at small quantized levels or for spins of classical magnitudes through effective couplings to curvature terms [56–68]. In the amplitude-based approach this is achieved through the addition of nonminimal couplings to higher-spin fields, and an approach similar in spirit is taken in the worldline formalisms. The link between these two constructions is through the higher-derivative nonminimal couplings, which again map to the set of multipoles of the present paper.

A. Gauge redundancy

We finally point out an important caveat with respect to the matching of our general multipole expansion with that of the Kerr metric in the Kerr limit. While we find agreement to the 2PM order with the specific combinations of multipole moments described above, there are also terms in the expansion of Eq. (31) that we cannot reproduce, and we should explain why. As an illustration, consider terms proportional to a at the 2PM contributions to h^{ij} . If we expand Eq. (31) to that order, we find the term

$$h_{(2)}^{12} = \frac{M^2 a}{r^5} (y^2 - x^2), \quad (47)$$

which can only arise from a combination of mass monopole and current dipole, MS_a . However, as can be seen from (24), a term of that kind will not be generated by our recursion. This apparent inconsistency of our recursive method is resolved once it is realized that the exact de Donder gauge Kerr metric of Eq. (31) is only one particular form of the Kerr metric in de Donder gauge, an issue explained very clearly in Ref. [69] for the case of the Schwarzschild black hole. Because the harmonicity condition on coordinates is preserved under a shift of those coordinates by any harmonic function, we have the freedom to add such a 2PM residual gauge vector ξ^i :

$$x^i \rightarrow x^i + \xi^i, \quad (48)$$

where $\square \xi^i = 0$. The gothic h^{ij} 's do not transform as tensors under such an infinitesimal transformation but rather

$$h^{ij} \rightarrow h^{ij} - \partial^i \xi^j - \partial^j \xi^i + \eta^{ij} \partial_k \xi^k. \quad (49)$$

If we choose

$$\xi^i = -M^2 a \epsilon^{ij3} \frac{x^j}{3r^3}, \quad i, j \in \{1, 2\}, \quad (50)$$

which indeed obeys $\square \xi^i = 0$, then h^{12} transforms as

$$h_{(2)}^{12} \rightarrow h_{(2)}^{12} - \frac{M^2 a}{r^5} (y^2 - x^2), \quad (51)$$

and it thus cancels the order- a term of Eq. (47). This phenomenon was noticed in Ref. [53] and it shows that the standard form of the Kerr metric in harmonic gauge Eq. (31), which nicely presents a compact nonperturbative version of the harmonic-gauge Kerr metric, achieves the compact closed-form expression by including terms that can be removed by a coordinate change while still retaining the harmonic gauge.

In fact, this gauge redundancy persists to infinite order in a . Although we do not need it here, we note that the corresponding 2PM gauge vector to all orders in a is most conveniently expressed in terms of the variable R , and it reads

$$\xi^i = \frac{M^2}{2a^2} \left[\frac{Ra}{R^2 + a^2} + \tan^{-1} \left(\frac{R}{a} \right) - \frac{\pi}{2} \right] \epsilon^{ij3} x^j, \quad (52)$$

with the relation between R and r as given in Eq. (32). The existence of this gauge vector implies that at 2PM order all terms with odd powers of a are gauge artifacts within the de Donder gauge. Indeed, our recursive solution does not generate such terms.

VI. GENERALIZATION TO HIGHER MULTIPOLES

We finally show how the above iteration generalizes to any number of multipoles. Since the $\mathcal{O}(G^2)$ equations of motion are unchanged, the recursion relations are unaltered, but the currents that enter these relations will include more terms. Correspondingly, new convolution integrals will arise. They are fortunately all of what can be called generalized bubble integrals, the nature of which follows from the fact that we are solving the massless equations of motion for the metric fluctuations. The 2PM solution for the metric involving an arbitrary number of higher multipoles can be given entirely in terms of these generalized bubble integrals. Here we outline the solution for h^{00} to order G^2 and including all orders in the multipoles.

We begin by writing the 1PM current multipole expansion for h^{00} from Eq. (B22):

$$J_{(1)|\mathbf{k}}^{00} = 4 \sum_{l=0}^{\infty} \frac{(-1)^{2l} (2l-1)!!}{l!} M_L \int_x \frac{\hat{n}_L}{r^{l+1}} e^{-ik \cdot x}. \quad (53)$$

A useful consequence of M_L being an STF tensor is that its contraction with any other tensor T_L satisfies $M_L T_L = M_L \hat{T}_L$, where $\hat{T}_L$ denotes the STF projection of T_L . This means that we can switch freely between hatted $\hat{\mathbf{k}}$ and unhatted $\mathbf{k}$ variables in the above expression for the 1PM current:

$$J_{(1)|\pm\mathbf{k}}^{00} = 16\pi \sum_{l=0}^{\infty} \frac{(\pm i)^l}{l!} M_L \frac{\hat{k}^L}{|\mathbf{k}|^2}. \quad (54)$$

Momentum space expressions for the current multipole components follow analogously. With the help of the Fourier transform formula Eq. (C6), the 1PM current corresponding $h_{(1)}^{01}$ can be derived from Eq. (B22):

$$J_{(1)|\pm\mathbf{k}}^{0i} = 16\pi \sum_{l=0}^{\infty} \frac{(\pm i)^l l}{(l+1)!} \epsilon_{iab} \frac{S_{bL-1} \hat{k}^{aL-1}}{|\mathbf{k}|^2}, \quad (55)$$

together with $J_{(1)|\pm\mathbf{k}}^{ij} = 0$.

A. Iterations for rank-2 currents

The recursive equations of motion for h^{00} to order G^2 have already been given in Eq. (12). As in Sec. IV, we divide up the corresponding 2PM currents according to an additional index a and b that tracks contributions from the mass and current multipoles separately. Then from Eq. (15) we get the 2PM current from iterated mass multipole pieces:

$$J_{(2)|-k_1}^{00[a]} = \frac{(16\pi)^2}{|k_1|^2} \sum_{l,l'=0}^{\infty} \frac{(-1)^l i^{l+l'} M_L M_{L'}}{l!l'!} \times \int_{k_2} \left[-\frac{7}{8} k_{12} \cdot k_2 + \frac{5}{4} |k_2|^2 \right] \frac{k_{12}^L k_2^{L'}}{|k_{12}|^2 |k_2|^2}, \quad (56)$$

where L and L' are multi-indices and we write $k_2^e k_2^{L'} = k_2^{eL'}$. Considering multinomial expansion for the term k_{12}^L ,

$$(k_1 + k_2)^L = \sum_{q=0}^L \binom{L}{q} k_1^{L-Q} k_2^Q, \quad (57)$$

where $|Q| = q$. Working implicitly in dimensional regularization which removes all scale-free integrals, we get

$$J_{(2)|-k_1}^{00[a]} = \frac{(16\pi)^2}{|k_1|^2} \sum_{l,l'=0}^{\infty} \sum_{q=0}^l \frac{7(-1)^{l+1} i^{l+l'}}{8l'!q!(l-q)!} I_{k_1,q}^{[a]}, \quad (58)$$

where

$$I_{k_1,q}^{[a]} \equiv k_1^{eP} M_{PQ} M_{L'} \mathcal{I}_{k_1}^{eL'Q}, \quad L = PQ, \quad (59)$$

and

$$\mathcal{I}_{k_1}^{eL'Q} = \int_{k_2} \frac{k_2^{eL'Q}}{|k_{12}|^2 |k_2|^2} = \sum_{r=0}^{\lfloor \frac{q}{2} \rfloor} C_{n,r} T_{r|k_1}^{eL'Q} |k_1|^{2r-1}. \quad (60)$$

Here $C_{n,r}$ and $T_{r|k_1}^{eL'Q}$ are defined in (C4). Substituting these into $I_{k_1,q}^{[a]}$, we have

$$I_{k_1,q}^{[a]} = k_1^{eP} M_{PQ} M_{L'} \sum_{r=0}^{\lfloor \frac{q}{2} \rfloor} C_{n,r} T_{r|k_1}^{eL'Q} |k_1|^{2r-1}. \quad (61)$$

The remaining task is to evaluate

$$k^{eP} M_{PQ} M_{L'} T_{r|k}^{eL'Q}. \quad (62)$$

At this stage the STF property is crucial. Since the mass multipoles $M_{L'}$ and M_L are STF individually, any Kronecker delta in $T_{r|k_1}^{eL'Q}$ that contracts two indices within L' , or two indices within $L = PQ$, gives zero. Hence the only non-vanishing contractions are of two types: first, Kronecker deltas connecting one index in L' with one index in Q ; second, one delta involving the distinguished index e , contracting e with one index in L' or in Q .

To be specific, let s denote the number of Kronecker deltas in $T_{r|k}^{eL'Q}$ that are contracted with the mass multipoles. Type I represents the case where all r Kronecker deltas are contracted with M , such that $r = s$. In type II, s of the Kronecker deltas are contracted with the multipole, while

the single remaining delta is contracted with M_{PQ} or $M_{L'}$, and $r = s + 1$. Denoting the set of these contracted multi-indices by S , where $|S| = s$, the respective types can be written as

$$\begin{aligned} \text{type I: } & \delta^{i_1 j_1} \dots \delta^{i_s j_s} M_{i_1 \dots i_s A} M_{j_1 \dots j_s B} = M_{SA} M_{SB}, \\ \text{type IIa: } & \delta^{e e'} \delta^{i_1 j_1} \dots \delta^{i_s j_s} M_{e' i_1 \dots i_s A} M_{j_1 \dots j_s B} = M_{eSA} M_{SB}, \\ \text{type IIb: } & \delta^{e e'} \delta^{i_1 j_1} \dots \delta^{i_s j_s} M_{i_1 \dots i_s A} M_{e' j_1 \dots j_s B} = M_{SA} M_{eSB}, \end{aligned} \quad (63)$$

where we decomposed the multi-indices according to

$$\begin{aligned} \text{type I: } & L' = SA, \quad Q = SR, \\ & L = PQ = SB, \quad B = PR, \\ \text{type IIa: } & L' = eSA, \quad Q = SR, \\ & L = PQ = SB, \quad B = PR, \\ \text{type IIb: } & L' = SA, \quad Q = eSR, \\ & L = PQ = eSB, \quad B = PR, \end{aligned} \quad (64)$$

Contracting with all the remaining k vectors, every surviving term reduces to the common tensor structure $M_{SA} k^A M_{SB} k^B$ by symmetry.

We now proceed to evaluate the contractions explicitly, first considering type I. In this case all s deltas in $T_{r|k}^N$ must connect L' with Q . After contraction with the external factors $k^e k^P$, one obtains

$$\begin{aligned} k^{eP} T_{s|k}^{eL'Q} M_{L'} M_{PQ} &= N_1 |k|^2 M_{SA} k^A M_{SB} k^B, \\ N_1 &= \binom{l'}{s} \binom{q}{s} s! = \frac{l'!q!}{s!(l'-s)!(q-s)!}. \end{aligned} \quad (65)$$

We next consider type II. Besides the s contractions between L' and Q , there must be one additional Kronecker delta involving the index e . For type IIa case, the number of such terms is

$$N_2 = \frac{l'!q!}{s!(l'-s-1)!(q-s)!} = (l'-s)N_1. \quad (66)$$

For type IIb case, the number of those terms is

$$N_{2'} = \frac{l'!q!}{s!(l'-s)!(q-s-1)!} = (q-s)N_1. \quad (67)$$

Combining these two results, we have

$$\begin{aligned}
& k^a k^P T_{s+1|k}^{aL'Q} M_{L'} M_{PQ} \\
&= [N_2 + N_2'] M_{SA} k^A M_{SB} k^B \\
&= (l' + q - 2s) N_1 M_{SA} k^A M_{SB} k^B.
\end{aligned} \tag{68}$$

Finally, combining (65) and (68), we find

$$\begin{aligned}
I_{k,q}^{[a]} &= (-1)^n \sum_{s=0}^{\min(l',q)} N_1 [C_{n,s} + (l' + q - 2s) C_{n,s+1}] \\
&\quad \times |k|^{2s+1} M_{SA} k^A M_{SB} k^B.
\end{aligned} \tag{69}$$

Evaluating $C_{n,s} + (l' + q - 2s) C_{n,s+1}$, we obtain

$$C_{n,s} + (l' + q - 2s) C_{n,s+1} = \frac{(-1)^s \Gamma(l' + q - s + \frac{1}{2})}{2^{s+4} \sqrt{\pi} \Gamma(l' + q + 1)}. \tag{70}$$

It follows that $I_{k,q}^{[a]}$ is given by

$$\begin{aligned}
I_{k,q}^{[a]} &= \sum_{s=0}^{\min(l',q)} C_{n,s}^{[a]} |k|^{2s+1} M_{SA} k^A M_{SB} k^B, \\
C_{n,s}^{[a]} &= \frac{(-1)^{n+s} l'! q!}{2^{s+4} \sqrt{\pi} s! (l' - s)! (q - s)!} \frac{\Gamma(n - s - \frac{1}{2})}{\Gamma(n)},
\end{aligned} \tag{71}$$

where $n = l' + q + 1$. Substituting the results into (58), we obtain

$$\begin{aligned}
J_{(2)|-k_1}^{00[a]} &= \frac{(16\pi)^2}{|k_1|^2} \sum_{l,l'=0}^{\infty} \frac{7(-1)^{l+1} l^{l'}}{8l! l'!} \sum_{q=0}^l \binom{l}{q} \\
&\quad \times \sum_{s=0}^{\min(l',q)} C_{n,s}^{[a]} |k_1|^{2s+1} M_{SA} k_1^A M_{SB} k_1^B.
\end{aligned} \tag{72}$$

Equation (58) provides us with the full 2PM contribution from all mass multipoles in terms of a generalized (tensor) bubble integral. We now turn to those coming from current multipoles. We read off the corresponding 2PM pieces for $h_{(2)}^{00}$ from Eq. (12), giving us the corresponding momentum space integral

$$\begin{aligned}
J_{(2)|-k_1}^{00[b]} &= \frac{1}{|k_1|^2} \int_{k_2} \left[\frac{k_1 \cdot k_2 - |k_2|^2}{2} J_{(1)|-k_{12}}^{0i} J_{(1)|k_2}^{0i} \right. \\
&\quad \left. + \frac{3}{2} k_2^i k_{12}^j J_{(1)|-k_{12}}^{0i} J_{(1)|k_2}^{0j} \right].
\end{aligned} \tag{73}$$

As initial conditions, we have the 1PM currents

$$J_{(1)|\pm k}^{0i} = 16\pi \sum_{l=1}^{\infty} l(\pm i)^l \frac{[\tilde{\mathbf{S}} \times \mathbf{k}]_i^{L-1}}{(l+1)! |\mathbf{k}|^2}, \tag{74}$$

where

$$[\tilde{\mathbf{S}} \times \mathbf{k}]_i^{L-1} \equiv \epsilon_{iab} S_{bL-1} k^a k^{L-1}. \tag{75}$$

Using the derivative acting on the exponentials and removing the scaleless integral,

$$J_{(2)|-k_1}^{00[b]} = \frac{(16\pi)^2}{|k_1|^2} \sum_{l,l'=1}^{\infty} \frac{(-1)^l i^{l+l'} l l'}{(l+1)! (l'+1)!} I_{k_1}^{[b]}, \tag{76}$$

where

$$I_{k_1}^{[b]} = \int_{k_2} \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2 \delta^{ij} + 3k_2^i k_{12}^j) [\tilde{\mathbf{S}} \times \mathbf{k}_{12}]_i^{L-1} [\tilde{\mathbf{S}} \times \mathbf{k}_2]_j^{L'-1}}{2|\mathbf{k}_{12}|^2 |\mathbf{k}_2|^2}. \tag{77}$$

Using the following identity:

$$\begin{aligned}
k_2^i [\tilde{\mathbf{S}} \times \mathbf{k}_{12}]_i^{L-1} &= -k_1^i [\tilde{\mathbf{S}} \times \mathbf{k}_{12}]_i^{L-1}, \\
k_{12}^j [\tilde{\mathbf{S}} \times \mathbf{k}_2]_j^{L'-1} &= k_1^j [\tilde{\mathbf{S}} \times \mathbf{k}_2]_j^{L'-1},
\end{aligned} \tag{78}$$

we can reduce the integral into

$$I_{k_1}^{[b]} = \int_{k_2} \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2 \delta^{ij} - 3k_1^i k_1^j) [\tilde{\mathbf{S}} \times \mathbf{k}_{12}]_i^{L-1} [\tilde{\mathbf{S}} \times \mathbf{k}_2]_j^{L'-1}}{2|\mathbf{k}_2|^2 |\mathbf{k}_{12}|^2}. \tag{79}$$

Using the binomial expansion (57), we have

$$k_{12}^{a(L-1)} = \sum_{p=0}^{L-1} \binom{L-1}{p} (k_1^a k_2^Q + k_1^P k_2^{aQ}), \tag{80}$$

where $L-1 = PQ$, $|P| = p$ and $|Q| = l-1-p$. Now, $I_k^{[b]}$ can be decomposed into two parts:

$$I_k^{[b]} = \frac{1}{2} \sum_{p=0}^{l-1} \binom{l-1}{p} [f_{(1)}^{aN} \mathcal{I}_k^{aN} + f_{(2)}^{abN} \mathcal{I}_k^{abN}], \tag{81}$$

where $\mathcal{I}_k^N$ is defined in (C4),

$$\begin{aligned}
f_{(1)}^{aN} &= (-1)^{n+1} (k^{(a)} k_b Y_{ii}^{b|N}) - 3k^i k^j Y_{ij}^{(aN)}, \\
f_{(2)}^{abN} &= (-1)^{n+2} k^{(a} Y_{ii}^{bN)},
\end{aligned} \tag{82}$$

and Y_{ij}^{aN} is a multiplication of two spin multipoles:

$$Y_{ij}^{aN} \equiv Y_{ij}^{aQc(L'-1)} = k^P \epsilon_{iab} \epsilon_{jcd} S_{bPQ} S_{d(L'-1)}. \tag{83}$$

Note that the parentheses on the right-hand side in (82) imply the total symmetrization of a and all the multi-indices of N for the first line and total symmetrization of a , b and all the multi-indices of N for the second line. Thus we now want evaluate $f_{(1)}^{aN}$ and $f_{(2)}^{abN}$ by eliminating T_r^N . Let us consider an arbitrary symmetric tensor X_N and its

contraction with the $T_{r|k}^N$:

$$X_N T_{r|k}^N = \frac{n!}{(2r)!(n-2r)!} (\text{Tr}^r X)_Q k^Q, \quad (84)$$

where $N = i_1 i_2 \dots i_{2r} Q$ and

$$(\text{Tr}^r X)_Q \equiv X_{a_1 a_1 \dots a_r a_r Q} \quad (85)$$

is an r -fold reduced trace in the form shown. The full contraction between X_N and $\mathcal{I}_k^N$ is thus given by

$$X_N \mathcal{I}_k^N = \sum_{r=0}^{\lfloor n/2 \rfloor} C_{n,r}^{[b]} |\mathbf{k}|^{2r-1} (\text{Tr}^r X)_Q k^Q, \quad (86)$$

where

$$C_{n,r}^{[b]} = (-1)^r \frac{\Gamma(n-r+\frac{1}{2})}{2^{r+3} \sqrt{\pi} (2r)!(n-2r)!}. \quad (87)$$

Then $I_k^{[b]}$ becomes

$$I_k^{[b]} = \frac{1}{2} \sum_{p=0}^{l-1} \binom{l-1}{p} \left[\sum_{r=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{C_{n+1,r}^{[b]}}{|\mathbf{k}|^{1-2r}} (\text{Tr}^r f_{(1)})_Q k^Q + \sum_{r'=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{C_{n+2,r'}^{[b]}}{|\mathbf{k}|^{1-2r'}} (\text{Tr}^{r'} f_{(2)})_{Q'} k^{Q'} \right]. \quad (88)$$

Hence, by combining terms, we have total $J_{(2)|-\mathbf{k}}^{00}$ as the sum of the two contributions:

$$\begin{aligned} J_{(2)|-\mathbf{k}}^{00} &= J_{(2)|-\mathbf{k}}^{00[a]} + J_{(2)|-\mathbf{k}}^{00[b]} \\ &= \frac{(16\pi)^2}{|\mathbf{k}|^2} \sum_{l,l'=0}^{\infty} \frac{7(-1)^{l+l'} l! l'!}{8l! l'!} \sum_{q=0}^l \binom{l}{q} \sum_{s=0}^{\min(l',q)} C_{n,s}^{[a]} |\mathbf{k}|^{2s+1} M_{SA} k^A M_{SB} k^B \\ &\quad + \frac{(16\pi)^2}{2|\mathbf{k}|^2} \sum_{l,l'=1}^{\infty} \frac{(-1)^l l! l'!}{(l+1)!(l'+1)!} \sum_{p=0}^{l-1} \binom{l-1}{p} \left[\sum_{r=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{C_{n+1,r}^{[b]}}{|\mathbf{k}|^{1-2r}} (\text{Tr}^r f_{(1)})_A k^A + \sum_{r'=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{C_{n+2,r'}^{[b]}}{|\mathbf{k}|^{1-2r'}} (\text{Tr}^{r'} f_{(2)})_B k^B \right]. \quad (89) \end{aligned}$$

This rather formidable expression gives the current $J_{(2)|-\mathbf{k}}^{00}$ at second post-Minkowskian order and including all mass and current multipoles. We have checked this general formula by truncating the expansion at to $l = l' = 2$ for $J_{(2)|-\mathbf{k}}^{[a]}$ and $l = l' = 1$ for $J_{(2)|-\mathbf{k}}^{[b]}$. This reproduces correctly the result of Eq. (20).

Having obtained the full current $J_{(2)|-\mathbf{k}}^{00}$ at 2PM order, the metric components $h_{(2)}^{00}$ can be straightforwardly obtained by extracting the powers of $|\mathbf{k}|$ and k components from the above results, collecting them, and applying the Fourier transform formula in (C6).

VII. CONCLUSION

Extending the recursive method of Ref. [22], we have set up the iterative equations that allow us to compute the gravitational metric of spatially localized and stationary sources to high order in G . For a pointlike source, the method allows for a computation to all orders in the coupling G , for which the Schwarzschild metric is recovered for the given gauge in the full domain of convergence of the perturbatively expanded solution. For a black hole with spin, which is sourced by a more complicated energy-momentum tensor, the iterative solution becomes more cumbersome but it can again be arranged to essentially yield the metric to any desired orders in G and spin a .

Applying the same approach to a more general spatially localized matter source, as done in this paper, shows the versatility of such an iterative scheme that is based entirely on the classical equations of motion without reference to the effective field theory of pointlike massive sources.

The general multipole expansion can be expressed in terms of an infinite series of mass multipoles and current multipoles. In the Landau-Lifshitz formalism and with the de Donder gauge condition that we impose on $h^{\mu\nu}$, the leading-order multipole expansion yields order- G contributions to h^{00} and h^{0i} but none for h^{ij} . The latter only builds up at order G^2 (and recursively at higher orders) through combinations of the lower-order terms from h^{00} and h^{0i} . In a highly degenerate corner of the space of multipoles, we recover the Kerr metric in harmonic coordinates to the given order. Interestingly, the gauge ambiguity that remains even in the chosen harmonic gauge [69] shows up already in the current-current contributions to h^{ij} at order G^2 where our iterative scheme reproduces the exact Kerr metric only up to terms that can be removed by a coordinate shift consistent with the chosen gauge.

After displaying the general iterative equations from which we can push the general multipole computation to as high orders we wish, we have truncated explicit 2PM results at mass quadrupole order and leading current multipole order simply because higher-order terms become analytically quite

unwieldy on account of the numerous different tensor structures that arise in the chosen coordinates. Although the subject of our paper obviously is far from new, we have tried to convey that our computational methods are. By Fourier transforming from metrics to currents the iterative structure that is built into the Einstein equation in perturbation theory become much simplified, and all involved integrals become analytically doable. This is where we can capitalize on all the recent developments in post-Minkowskian perturbation theory based on amplitude or worldline methods. In fact, for the multipole expansion considered here the required integrals in momentum space (loops in the language of quantum field theory) remain exceedingly simple in comparison with the integrals encountered in the two-body problem. The origin of this simplicity is easily identified: all pertinent integrals correspond to what in quantum field theory would be considered tensor integrals of massless propagators, i.e., generalized tensor bubble integrals. An interesting question that goes beyond computation of the gravitational metric of the compact object is how tidal deformations (and hence also Love numbers) can be linked to the multipole expansion, thus providing physical and measurable consequences of the expansion. Looking further ahead would be the extension of the iterative setup to the nonstationary case with, in particular, applications to the bound two-body problem in general relativity.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: USEFUL STF TENSOR IDENTITIES

In this appendix, we collect some useful relations among STF tensors. For a general rank- p tensor $A^{i_1 \dots i_p}$, the STF projection can be written as [3,6,8]

$$\hat{A}^{i_1 \dots i_p} = \sum_{k=0}^{\lfloor \frac{p}{2} \rfloor} \alpha_{p,k} \delta^{(i_1 i_2} \dots \delta^{i_{2k-1} i_{2k}} S^{i_{2k+1} \dots i_p)}_{j_1 j_1 \dots j_k j_k}, \quad (\text{A1})$$

where $S^{i_1 \dots i_p} = A^{(i_1 \dots i_p)}$ is the fully symmetrized A tensor. The k th term in the sum has k pairs of indices contracted with Kronecker deltas as indicated, and the coefficients α_k are given by

$$\alpha_{p,k} = \frac{(-1)^k p! (2p - 2k - 1)!!}{(2p - 1)!! (2k)!! (p - 2k)!}.$$

k-fold trace. For a fully symmetric rank- p tensor $S^{i_1 \dots i_p}$, one can define a k -fold trace (Tr^k) by contracting k independent pairs of indices:

$$(\text{Tr}^k S)^{i_{2k+1} \dots i_p} \equiv \delta_{i_1 i_2} \dots \delta_{i_{2k-1} i_{2k}} S^{i_1 \dots i_{2k} i_{2k+1} \dots i_p}. \quad (\text{A2})$$

Each trace contracts two indices, so a k -fold trace removes $2k$ indices and reduces the rank from p to $p - 2k$, with $0 \leq k \leq \lfloor \frac{p}{2} \rfloor$.

1. Explicit STF projections up to rank 4

a. Rank 1 ($p=1$)

$$\hat{A}^i = S^i = A^i. \quad (\text{A3})$$

b. Rank 2 ($p=2$)

$$\hat{A}^{ij} = S^{ij} - \frac{1}{3} \delta^{ij} S^{aa}. \quad (\text{A4})$$

c. Rank 3 ($p=3$)

$$\hat{A}^{ijk} = S^{ijk} - \frac{1}{5} (\delta^{ij} S^{aak} + \delta^{ik} S^{aaj} + \delta^{jk} S^{aai}). \quad (\text{A5})$$

d. Rank 4 ($p=4$)

$$\begin{aligned} \hat{A}^{ijkl} = & S^{ijkl} - \frac{1}{7} (\delta^{ij} S^{aakl} + \delta^{ik} S^{aajl} + \delta^{il} S^{aajk} \\ & + \delta^{jk} S^{aail} + \delta^{jl} S^{aaij} + \delta^{kl} S^{aaij}) \\ & + \frac{1}{35} (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) S^{aabb} \end{aligned} \quad (\text{A6})$$

with repeated indices summed over.

2. Scalar products and STF tensors

The two alternative descriptions of multipoles in terms of either spherical harmonics or STF tensors are a change of basis, and we can go back and forth between them. This is

all clearly explained in Ref. [3] and we shall just summarize a few main points here. Let

$$\mathcal{H}_l \equiv \{A^{i_1 \dots i_l} \text{symmetric and trace-free in } \mathbb{R}^3\}$$

be a set of rank- ℓ STF tensors. This space carries an irreducible representation of $SO(3)$ of dimension

$$\dim \mathcal{H}_l = 2l + 1,$$

equivalent to the usual spin- ℓ representation spanned by the spherical harmonics $\{Y_{lm}\}_{m=-l}^l$.

According to the Legendre polynomial addition theorem, $P_l(\cos \theta)$ can be written in terms of spherical harmonics

$$P_l(\cos \theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\theta_1, \phi_1) Y_{lm}^*(\theta_2, \phi_2), \quad (\text{A7})$$

where $\cos \theta = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2)$. These spherical harmonics can be written in terms of the STF tensor basis as follows [3,70]:

$$Y_{lm}(\theta_1, \phi_1) = Y_{lm}^L \hat{n}^L(\theta_1, \phi_1), \quad (\text{A8})$$

where the coefficients Y_{lm}^L are independent of the coordinates of the sphere and L is again a multi-index so that $Y_{lm}^L = Y_{lm}^{i_1 \dots i_L}$. The Cartesian tensors $\hat{n}^L$ are irreducible representations of the tensors and very useful for the analysis. An important property of these tensors is that they generate harmonic functions. Indeed, the solid harmonics $r^L \hat{n}^L$ satisfy the flat-space Laplace equation

$$\nabla^2(r^L \hat{n}^L) = 0, \quad (\text{A9})$$

which makes them the natural cartesian analog of the spherical harmonics. The scalar harmonics are orthogonal and satisfy the identity

$$Y_L^{lm*} Y_L^{lm'} = \frac{(2\ell+1)!!}{4\pi l!} \delta_{mm'}. \quad (\text{A10})$$

Using this the above relation in Eq. (A8) can be written as

$$\begin{aligned} & \sum_{m=-l}^l Y_{lm}(\theta_1, \phi_1) Y_{lm}^*(\theta_2, \phi_2) \\ &= \hat{n}^L(\theta_1, \phi_1) \sum_{m=-l}^l Y_{lm}^L Y_{lm}^*(\theta_2, \phi_2) \\ &= \frac{(2l+1)!!}{4\pi l!} \hat{n}^L(\theta_1, \phi_1) \hat{n}'^L(\theta_2, \phi_2). \end{aligned} \quad (\text{A11})$$

Hence, (A7) can be written as

$$P_l(\cos \theta) = \frac{(2l-1)!!}{l!} \hat{n}^L(\theta_1, \phi_1) \hat{n}'^L(\theta_2, \phi_2). \quad (\text{A12})$$

These identities can be used to replace angular functions in multipole expansions with STF tensor structures, and they are essential when converting expressions into the STF basis.

APPENDIX B: FOURIER TRANSFORMS AND STF TENSORS

The STF tensor formalism beautifully translates into momentum space through Fourier transforms. This is essential for our computations below, and we shall therefore spell it out in some detail. To start, consider the first relevant Fourier transform, corresponding to the monopole term $1/r$,

$$I \equiv \int d^3x \frac{e^{-ik \cdot x}}{r} = \frac{4\pi}{|k|^2}. \quad (\text{B1})$$

All Fourier transforms are defined consistently as

$$\tilde{f}(\mathbf{k}) = \int_{\mathbf{x}} e^{-ik \cdot \mathbf{x}} f(\mathbf{x}), \quad f(\mathbf{x}) = \int_{\mathbf{k}} e^{ik \cdot \mathbf{x}} \tilde{f}(\mathbf{k}), \quad (\text{B2})$$

where

$$\int_{\mathbf{x}} \equiv \int d^3x, \quad \int_{\mathbf{k}} \equiv \int \frac{d^3k}{(2\pi)^3}, \quad (\text{B3})$$

so that standard Fourier pairs such as

$$\int d^3x \frac{e^{-ik \cdot x}}{r} = \frac{4\pi}{|k|^2}, \quad \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot x}}{|k|^2} = \frac{1}{4\pi r} \quad (\text{B4})$$

are consistent. The integral extends over all of three-space, including the origin at $r = 0$. Intuitively, when we seek solutions to the Einstein equations far away from the matter sources with the center of mass at the origin of coordinates, we would not expect to be concerned with the point $r = 0$. Yet, by first Fourier transforming (and subsequently taking the inverse Fourier transform back to real space), we do encounter this spurious point. We therefore need to handle it with care and also provide expressions for $\partial_L(1/r)$, where $\partial_L = \partial_{i_1} \dots \partial_{i_L}$ for $L = i_1 \dots i_L$, that are not restricted to $r > 0$. As a starting point, consider

$$\partial_a \partial_b \frac{1}{r} = -\frac{4\pi}{3} \delta^{ab} \delta(\mathbf{x}) + \frac{1}{r^3} \left(\frac{3x^a x^b}{r^2} - \delta^{ab} \right), \quad (\text{B5})$$

where the first term is required for consistency with the point-source Poisson equation

$$\partial^2 \frac{1}{r} = -4\pi \delta(\mathbf{x}). \quad (\text{B6})$$

Hence, in terms of the Fourier integral

$$\partial_{ab} \frac{1}{r} = \partial_{ab} \int_k e^{ik \cdot x} \frac{4\pi}{|k|^2} = -\frac{4\pi}{3} \delta^{ab} \delta(x) + \frac{3}{r^3} \hat{n}^{ab}, \quad (\text{B7})$$

which implies

$$\int_k e^{ik \cdot x} 4\pi \left(\frac{k^a k^b}{|k|^2} - \frac{1}{3} \delta^{ab} \right) = -\frac{3}{r^3} \hat{n}^{ab} \quad (\text{B8})$$

and hence

$$\int_k e^{ik \cdot x} \frac{4\pi \hat{k}^{ab}}{|k|^2} = -\frac{3}{r^3} \hat{n}^{ab}, \quad (\text{B9})$$

where

$$\begin{aligned} \hat{n}^{ab} &\equiv \frac{x^{(a} x^{b)}}{r^2} = \frac{1}{r^2} \left(x^a x^b - \frac{1}{3} r^2 \delta^{ab} \right), \\ \hat{k}^{ab} &\equiv k^a k^b - \frac{1}{3} |k|^2 \delta^{ab}. \end{aligned} \quad (\text{B10})$$

This can be generalized to Fourier transforms with a multi-index L derivative

$$\int_k \frac{\partial_L (e^{ik \cdot x})}{|k|^2} = \partial_L \left(\frac{1}{4\pi r} \right), \quad (\text{B11})$$

which in turn can be used to compute Fourier transforms of tensor integrals. We shall return to this below. We note that we can absorb the local δ -function terms by redefining the derivative tensors on the left-hand side by subtracting the contribution at $r = 0$:

$$\hat{\partial}_{ab} \frac{1}{r} = \frac{1}{r^3} \left(\frac{3x^a x^b}{r^2} - \delta^{ab} \right) = \frac{3}{r^3} \hat{n}^{ab}, \quad (\text{B12})$$

where $\hat{\partial}_{ab}$ can be interpreted as a definition of the second-rank STF derivative operators

$$\hat{\partial}_{ab} \equiv \partial_a \partial_b - \frac{1}{3} \delta^{ab} \partial^2. \quad (\text{B13})$$

In particular, this implies

$$\int_x e^{-ik \cdot x} \frac{1}{r^3} \hat{n}^{ab} = -\frac{4\pi}{3|k|^2} \hat{k}^{ab}, \quad (\text{B14})$$

so that the STF property is being preserved under Fourier transformations, as it should.

In general, the Fourier transform $\mathcal{F}$ in d dimensions commutes with $SO(d)$ rotations, i.e., $\mathcal{R}[\mathcal{F}[f]] = \mathcal{F}[\mathcal{R}[f]]$ for any $\mathcal{R} \in SO(d)$, and therefore preserves the irreducible representations of $SO(d)$. Since STF tensors form irreducible representations of $SO(d)$, the Fourier transform of any

STF tensor must itself remain an STF tensor. This fact greatly constrains the structure of the result: the d -dimensional Fourier transform of $\hat{x}^L/|x|^n$, where x is a d -dimensional position vector, must take the form $\hat{k}^L \cdot f(|k|)$ for some scalar function f . The explicit form of f can be determined by employing the Schwinger parametrization together with the d -dimensional Gaussian integral formula as

$$\int d^d x e^{-ik \cdot x} \frac{\hat{x}^L}{r^m} = \frac{\pi^{\frac{d}{2}} \Gamma(\frac{d-m+2l}{2})}{i^l 2^{m-d-l} \Gamma(\frac{m}{2})} \frac{\hat{k}^L}{|k|^{d-m+2l}}. \quad (\text{B15})$$

The corresponding d -dimensional Fourier transform of an arbitrary tensor field without the STF projection is given in Appendix C 2.

For stationary sources, the Einstein equations of motion at the linearized level reduce to

$$\nabla^2 h_{(1)}^{\mu\nu}(x) = -16\pi T^{\mu\nu}(x), \quad (\text{B16})$$

which in Cartesian coordinates is solved by

$$h_{(1)}^{\mu\nu}(x) = 4 \int d^3 x' \frac{T^{\mu\nu}(x')}{|x - x'|}. \quad (\text{B17})$$

Here, we assume that the source is spatially confined to a bounded region $|x'| < r_0$ for some given r_0 so $T^{\mu\nu}(x') = 0$ for $|x'| > r_0$. Exterior to the source the solution thus admits a standard multipole expansion for $r = |x| > |x'|$ based on Taylor expanding:

$$\frac{1}{|x - x'|} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} x'^L \partial_L \left(\frac{1}{r} \right). \quad (\text{B18})$$

The expansion simplifies considerably since by assumption all multipole moments are time independent. To leading order, ignoring gravitational self-interactions, and using the definition of multipoles given in Sec. II, the solution takes the form of (using a notation in which multi-index $aL - 1 = ai_2 \dots i_l$)

$$\begin{aligned} h_{(1)}^{00}(x) &= 4 \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \frac{M_L}{r}, \\ h_{(1)}^{0i}(x) &= 4 \sum_{l=1}^{\infty} \frac{l(-1)^l}{(l+1)!} \epsilon_{iab} \partial_{aL-1} \frac{S_{bL-1}}{r}, \\ h_{(1)}^{ij}(x) &= 0, \end{aligned} \quad (\text{B19})$$

in what is known as the canonical basis. Note that the contraction between x'^L and $\partial_L(1/r)$ in Eq. (B18) satisfies

$$x'^L \partial_L \frac{1}{r} = \hat{x}'^L \partial_L \frac{1}{r} \quad (\text{B20})$$

due to the identity

$$\partial_L \frac{1}{r} = (-1)^L \frac{(2L-1)!!}{r^{L+1}} \hat{n}_L; \quad L = i_1, \dots, i_L. \quad (\text{B21})$$

From this relation, one can systematically construct the STF tensor products $\hat{x}^L$ in the definitions of the mass and current multipole moments, (4) and (5) from (B17) and (B18). These terms are all of first order in G , as also indicated by the subscript. There is no mass dipole term M_a in the expansion due to the conventional choice of frame with origin at the center of mass.

For all orders in terms of mass multipole, we can write the expansion

$$\begin{aligned} h_{(1)}^{00} &= 4 \sum_{l=0}^{\infty} \frac{(2l-1)!!}{l!} \frac{\hat{n}^L M_L}{r^{l+1}}, \\ h_{(1)}^{0i} &= 4 \sum_{l=1}^{\infty} \frac{l(2l-1)!!}{(l+1)!} \epsilon_{iab} \frac{\hat{n}^{a(L-1)} S_{b(L-1)}}{r^{l+1}}. \end{aligned} \quad (\text{B22})$$

Truncating up to the mass quadrupole term $l = 0, 2$, we have

$$h_{(1)}^{00}|_{l=0,2} = \frac{4M}{r} + \frac{6M_{ab}}{r^3} \hat{n}_{ab}. \quad (\text{B23})$$

Including only the leading spin term, the dipole ($l = 1$), we have

$$h_{(1)}^{0i}|_{l=1} = -2\epsilon_{iab} n^a \frac{S^b}{r^2}, \quad (\text{B24})$$

together with $h_{(1)}^{ij} = 0$. If we are interested in a truncation at this order in the 1PM multipole expansion, the terms of Eqs. (B23) and (B24) will serve as initial conditions for our recursion relations that will generate the higher-order corrections in G involving just these three multipoles.

APPENDIX C: INTEGRALS

In this appendix, we provide explicit evaluations of the generalized tensor bubble and Fourier integrals required for the main text.

1. Tensor bubble integrals

We define an arbitrary d -dimensional tensor bubble integral by

$$\mathcal{B}_{n_1, n_2}^{(d), i_1 \dots i_n}(\ell) = \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{k^{i_1} k^{i_2} \dots k^{i_n}}{|\mathbf{k}|^{n_1} |\ell - \mathbf{k}|^{n_2}}. \quad (\text{C1})$$

This integral can be evaluated explicitly to give

$$\mathcal{B}_{n_1, n_2}^{(d), i_1 \dots i_n}(\ell) = \sum_{r=0}^{\lfloor n/2 \rfloor} C_{n_1, n_2}^{d, n, r} \frac{T_{r|\ell}^{i_1 \dots i_n}}{|\ell|^{n_{12}-d-2r}}, \quad (\text{C2})$$

where $n_{12} = n_1 + n_2$ and

$$\begin{aligned} C_{n_1, n_2}^{d, n, r} &= \frac{\Gamma(\frac{n_{12}-d}{2} - r) \Gamma(n - r + \frac{d-n_1}{2}) \Gamma(r + \frac{d-n_2}{2})}{(4\pi)^{\frac{d}{2}} 2^r \Gamma(\frac{n_1}{2}) \Gamma(\frac{n_2}{2}) \Gamma(n + d - \frac{n_{12}}{2})}, \\ T_{r|\ell}^{i_1 \dots i_n} &= \frac{n! \delta^{(i_1 i_2} \dots \delta^{i_{2r-1} i_{2r}} \ell^{i_{2r+1}} \dots \ell^{i_n})}{(2r)!(n-2r)!}. \end{aligned} \quad (\text{C3})$$

For the special case of $d = 3$ and $n_1 = n_2 = 2$, we define $\mathcal{I}_{\ell}^{i_1 i_2 \dots i_n} = \mathcal{B}_{2,2}^{(3), i_1 \dots i_n}(\ell)$. Its explicit form is

$$\begin{aligned} \mathcal{I}_{\ell}^{i_1 i_2 \dots i_n} &= \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{k^{i_1} k^{i_2} \dots k^{i_n}}{|\mathbf{k}|^2 |\ell - \mathbf{k}|^2} \\ &= \sum_{r=0}^{\lfloor n/2 \rfloor} C_{n,r} T_{r|\ell}^{i_1 i_2 \dots i_n} |\ell|^{2r-1}, \end{aligned} \quad (\text{C4})$$

where

$$C_{n,r} = \frac{(-1)^r \Gamma(n - r + \frac{1}{2})}{2^{r+3} \sqrt{\pi} n!} \quad (\text{C5})$$

and $T_{r|\ell}^{i_1 i_2 \dots i_n}$ is as defined in Eq. (C3).

2. Fourier transforms of tensor integrands

We now consider the Fourier transform in d dimensions:

$$\int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{k^{i_1} \dots k^{i_m}}{|\mathbf{k}|^n} e^{i\mathbf{k} \cdot \mathbf{x}} = \sum_{q=0}^{\lfloor m/2 \rfloor} i^{m-2q} \frac{\Gamma(\frac{d-n+2m-2q}{2})}{2^{n-m+q} \pi^{\frac{d}{2}} \Gamma(\frac{n}{2})} \frac{T_r^{i_1 \dots i_m}(\mathbf{x})}{r^{d-n+2m-2q}}, \quad (\text{C6})$$

where

$$T_r^{i_1 \dots i_m}(\mathbf{x}) = \frac{m! \delta^{(i_1 i_2} \dots \delta^{i_{2r-1} i_{2r}} \mathbf{x}^{i_{2r+1}} \dots \mathbf{x}^{i_m})}{(2r)!(m-2r)!}. \quad (\text{C7})$$

Interestingly, this coincides with the definition $T_{r|\ell}^N$ in Eq. (C3) if we replace $\mathbf{x}$ by ℓ , and $N = i_1 \dots i_m$. The tensor structure is thus identical to that of the bubble integrals.

APPENDIX D: AXISYMMETRIC STATIONARY SOURCE

Consider the general definition of mass multipoles given in Eq. (4) for the case of a source with symmetry axis chosen as the z axis so that the multipoles M^L are rank- l STF tensors invariant under the $\text{SO}(2)$ subgroup of z -axis rotations. In the far zone, the gravitational field is a scalar function on the sphere (angular dependence). Recall the

linear gravitational field generated by M_L in Eq. (B19):

$$h_{(1)}^{00}(\mathbf{x}) = \sum_{l=0}^{\infty} \frac{4}{r^{l+1}} \frac{(2l-1)!!}{l!} M_L \hat{n}^L. \quad (\text{D1})$$

Let us define $\Phi_l(\hat{n})$ as

$$h_{(1)}^{00}(\mathbf{x}) = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \Phi_l(\hat{n}). \quad (\text{D2})$$

$\Phi_l(\hat{n})$ can be written in terms of the spherical harmonics. Imposing axisymmetry $\Phi_l(\theta, \phi) = \Phi_l(\theta)$ and using the orthonormality of the spherical harmonics, we can show that

$$\Phi_l(\theta, \phi) = 4a_l \sqrt{\frac{4\pi}{2l+1}} Y_{l0}(\theta, \phi) = 4a_l P_l(\cos \theta), \quad (\text{D3})$$

where a_l is a number determined by M_L . A standard identity shown in Eq. (A12) relates Legendre polynomials to STF products as

$$P_l(\cos \theta) = \frac{(2l-1)!!}{l!} \hat{n}^L \hat{s}_L, \quad \cos \theta = \hat{n} \cdot \hat{s}. \quad (\text{D4})$$

Thus,

$$M_L \hat{n}^L = a_l \hat{n}^L \hat{s}_L. \quad (\text{D5})$$

As M_L are rank- l STF tensors and recalling that STF tensors are nothing but alternative descriptions of the spherical harmonics, we can show that any angular dependence in M_L should be solely possessed by $\hat{s}_L$:

$$M_L = a_l \hat{s}_L. \quad (\text{D6})$$

For an axisymmetric stationary source, each STF rank- l multipole M_L is fully characterized by a single number a_l . Therefore, for each l the axisymmetric subspace is one-dimensional.

To illustrate, for the choice of spin axis $\hat{s}^i = \delta^{i3}$ the $l = 4$ STF tensor can explicitly be written in terms of M_{3333} using the STF identity given in Eq. (A6). If one imposes axisymmetry: $M_{ijkl} = a_4 \hat{s}_{ijkl}$, then using $s^i = (0, 0, 1)$, we compute the independent components.

1. All-3 component

$$\hat{s}_{3333} = 1 - \frac{6}{7} + \frac{3}{35} = \frac{8}{35}, \quad (\text{D7})$$

$$M_{3333} = a_4 \frac{8}{35} \Rightarrow a_4 = \frac{35}{8} M_{3333}. \quad (\text{D8})$$

2. Mixed components

$$\begin{aligned} \hat{s}_{1133} &= \hat{s}_{2233} = 0 - \frac{1}{7} + \frac{1}{35} = -\frac{4}{35}, \\ M_{1133} &= M_{2233} = a_4 \left(-\frac{4}{35} \right) = -\frac{1}{2} M_{3333}. \end{aligned} \quad (\text{D9})$$

3. Pure transverse components

$$\begin{aligned} \hat{s}_{1111} &= \hat{s}_{2222} = \frac{3}{35}, \\ M_{1111} &= M_{2222} = a_4 \frac{3}{35} = \frac{3}{8} M_{3333}. \end{aligned} \quad (\text{D10})$$

4. Mixed transverse component

$$\hat{s}_{1122} = \frac{1}{35}, \quad M_{1122} = a_4 \frac{1}{35} = \frac{1}{8} M_{3333}. \quad (\text{D11})$$

5. Mixed odd component

M_L being a Cartesian tensor invariant under rotations about the z axis implies

$$M_{i_1 \dots i_l} = 0, \quad (\text{D12})$$

whenever the number of indices $i_k \in \{x, y\}$ is odd. That means

$$M_{1233} = -M_{1233} \Rightarrow M_{1233} = 0. \quad (\text{D13})$$

All remaining components follow from symmetry, index permutations, and tracelessness, $M_{iikl} = 0$, for all k, l (e.g., $M_{1313} = M_{1133}$, etc.). Thus, for $l = 4$, the single component M_{3333} completely determines the axisymmetric STF tensor.

APPENDIX E: 2PM KERR METRIC FROM MULTIPOLES

In this section, we list the nontrivial components of the Kerr metric at 2PM order with expansion truncated at $\mathcal{O}(a^4)$. Our notation follows that of the main text where we introduced labels $Q[n]$ to indicate the origin of the different terms of the Kerr metric based on the multipole expansion.

We list the 2PM-accurate components of the Kerr metric up to order a^4 as expressed in terms of the multipole labels $Q[n]$. For a concise expression, we introduce $x^a = \{x, y\}$ and $\tilde{x}^a = \{y, x\}$, where $a, b = 1, 2$.

(i) *Diagonal components* $h_{(2)}^{ii}|_{\mathcal{O}(a^4)}$.

$$\begin{aligned}
 M^{-2}h_{(2)}^{aa}|_{\mathcal{O}(a^4)} &= \frac{x_a^2}{r^4}Q[0]^2 + \frac{x_a^2(2r^2 - 15z^2) + r^2\tilde{x}_a^2}{2r^8}Q[0]Q[2] - \frac{x_a^2(2r^2 - 9z^2) - r^2\tilde{x}_a^2}{2r^8}Q[1]^2 \\
 &\quad + \left[\frac{3}{4r^6} - \frac{\tilde{x}_a^2 + 59z^2}{r^8} + \frac{7(4z^4 + 3\tilde{x}_a^2z^2)}{2r^{10}} + \frac{105x_a^2z^4}{4r^{12}} \right] Q[0]Q[4] \\
 &\quad - \left[\frac{3}{2r^6} - \frac{4\tilde{x}_a^2 + 47z^2}{2r^8} + \frac{2(11z^4 + 12\tilde{x}_a^2z^2)}{r^{10}} + \frac{35x_a^2z^4}{r^{12}} \right] Q[1]Q[3] \\
 &\quad + \left[\frac{3x_a^2}{r^8} + \frac{3z^2(\tilde{x}_a^2 - 8x_a^2)}{2r^{10}} + \frac{75x_a^2z^4}{4r^{12}} \right] Q[2]^2, \\
 M^{-2}h_{(2)}^{33}|_{\mathcal{O}(a^4)} &= \frac{z^2}{r^4}Q[0]^2 - \frac{(r^4 - 12r^2z^2 + 15z^4)}{2r^8}Q[0]Q[2] - \frac{(r^4 - 8r^2z^2 + 9z^4)}{2r^8}Q[1]^2 \\
 &\quad - \frac{1}{4r^{12}}(2r^6 - 45r^4z^2 + 140r^2z^4 - 105z^6)Q[0]Q[4] \\
 &\quad + \frac{1}{r^{12}}(r^6 - 18r^4z^2 + 50r^2z^4 - 35z^6)Q[1]Q[3] \\
 &\quad - \frac{1}{4r^{12}}(2r^6 - 39r^4z^4 + 108r^2z^4 - 75z^6)Q[2]^2.
 \end{aligned}$$

(ii) *Off-diagonal components* $(h_{(2)}^{ij})^{(i \neq j)}|_{\mathcal{O}(a^4)}$.

$$\begin{aligned}
 M^{-2}h_{(2)}^{12}|_{\mathcal{O}(a^4)} &= \frac{xy}{r^{12}}[4r^8Q[0]^2 + 2r^4(r^2 - 15z^2)Q[0]Q[2] - 6r^4(r^2 - 3z^2)Q[1]^2 + 4(r^4 - 42r^2z^2 + 105z^4)Q[0]Q[4] \\
 &\quad - 4(2r^4 - 24r^2z^2 + 35z^4)Q[1]Q[3] + 3(r^4 - 18r^2z^2 + 25z^4)Q[2]^2], \\
 M^{-2}h_{(2)}^{a3}|_{\mathcal{O}(a^4)} &= [4r^8Q[0]^2 + 2(7r^2 - 15z^2)r^4Q[0]Q[2] - 2(5r^2 - 9z^2)r^4Q[1]^2 + (17r^4 - 98r^2z^2 + 105z^4)Q[0]Q[4] \\
 &\quad - 4(7r^4 - 36r^2z^2 + 35z^4)Q[1]Q[3] + 3(5r^4 - 26r^2z^2 + 25z^4)Q[2]^2] \frac{x_az}{4r^{12}}.
 \end{aligned}$$

(iii) *Current components* $(h_{(2)}^{0i})|_{\mathcal{O}(a^3)}$.

$$M^{-2}h_{(2)}^{0a}|_{\mathcal{O}(a^3)} = [3(r^2 - 5z^2)Q[0]Q[3] + (r^2 + 3z^2)Q[1]Q[2]] \frac{\epsilon_{ab}x^b}{2r^8}.$$

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