Sphere free energy of vector models to order $1/N$

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We calculate the large-N expansion of the sphere free energy $F = -\log Z_{S^d}$ of the $\mathrm{O}(N)\phi^4$ and the Gross-Neveu $(\bar{\psi}\psi)^2$ conformal field theories (CFTs) to order $1/N$. Analytic regularization of these theories requires consistently shifting the UV scaling dimension of the auxiliary field: this can only be done by modifying its kinetic term. This modification combines with the counterterms to give the result that matches the ϵ expansion, resolving a puzzle raised by Tarnopolsky. These F's can be written compactly in terms of the anomalous dimensions, for both the short- and long-range versions of these CFTs. We also provide various technical results including a computation of the counterterms on the sphere and a neat derivation of the sphere free energy of a free conformal field. Finally, we observe that the long-range CFT becomes the short-range CFT at exactly the point where its $\tilde{F} = -\sin\frac{\pi d}{2} F$ is maximized as a function of the vector's scaling dimension.

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I. INTRODUCTION

We study the vector models. The $\mathrm{O}(N)$ ϕ^4 vector model is defined by

$$Z_{\mathrm{O}(N)} \equiv \int \mathrm{D}\phi e^{-S_{\mathrm{O}(N),\phi^4}},$$
$$S_{\mathrm{O}(N),\phi^4} \equiv \int_x \frac{1}{2}\phi_i C^{-1}\phi_i + \frac{\lambda_0}{8N}(\phi_i\phi_i)^2 \qquad (1.1)$$

and its complex fermionic cousin, the Gross-Neveu (GN) model, is defined by

$$Z_{\mathrm{GN}} \equiv \int \mathrm{D}\psi\mathrm{D}\bar{\psi} e^{-S_{\mathrm{GN},\psi^4}},$$
$$S_{\mathrm{GN},\psi^4} \equiv \int_x -\bar{\psi}_i C^{-1}\psi^i + \frac{\lambda_0}{2N}(\bar{\psi}_i\psi^i)^2, \qquad (1.2)$$

where i now runs from 1 to n, and $N \equiv n\mathrm{tr}\,\mathbb{I}_s$. In each action, $C(x,y)$ is the bare position-space propagator of the free field and $\int_x = \int \mathrm{d}^d x\sqrt{g}$. Both of these quantum field theories (QFTs) are known to provide access to an interacting Wilson-Fisher-like d-dimensional Euclidean conformal field theory (CFT) as fixed points of their renormalization group (RG) flow: for $2 < d < 4$, the $\mathrm{O}(N)$ vector CFT lies in the IR of (1.1) and the GN vector CFT lies in the UV of (1.2).

In this paper, we compute the universal part of these CFTs' sphere free energies [1]: we just do the functional integrals

$$F \equiv -\log Z|_{S^d}, \qquad (1.3)$$

in the limit of a large number N of fields $\{\phi_i, \psi^i\}$, working with a particular regularization and tuning the renormalized couplings to ensure that we are working at the Wilson-Fisher-like fixed point. By the universal part, we mean the part that does not depend on the regularization; henceforth this caveat will be dropped, and we will freely refer to the F's that we compute as the (sphere) free energy. The sphere free energy has been of great interest of late, as $\tilde{F} \equiv -\sin(\frac{\pi d}{2})F$ provides a measure of the number of degrees of freedom of an arbitrary CFT [2–4], generalizing the two-dimensional c to higher-dimensional CFTs [5,6]. Additionally, it was recently determined that the leading large-N correction to the scaling dimensions of ϕ and ψ are determined by extremization of $\tilde{F}$ over a family of CFTs [7].

Our F results, given in Sec. II, are to order $1/N$ and in arbitrary dimension d, and confirm and generalize the values conjectured by matching to the ϵ expansion in [1]. They prove to have a simple form in terms of the field scaling dimensions. Tarnopolsky [1] was not able to confirm their conjecture by a direct computation, due to mysterious factors of 3 and $\frac{5}{2}$. We resolve this issue by realizing that the analytic regularization, a shift of the scaling dimension of the auxiliary field by δ, must be

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implemented by adding terms to the action. These terms combine with the counterterms $\propto 1/\delta$ to cancel the mysterious factors.

Following the long tradition in physics of varying parameters that do not appear to be variable, there exist generalizations of these CFTs which have an explicitly nonlocal action and are called the long-range vector CFTs. They are parametrized by the scaling dimension of the vector $\Delta_{\phi/\psi} \equiv \frac{d-s}{2}$; the short-range CFTs are then specific points on these lines of nonlocal CFTs. It is straightforward to work in an arbitrary-s formalism, and so we present our results for both the long- and short-range CFTs. Notably, we observe (but do not prove) that the short-range CFTs coincide with the long-range CFTs that have extremal free energy, as parametrized by s.[1] This is particularly nice, since the long-range expressions are typically much nicer and more compact than the short-range ones.

A. Paper structure

We present the long- and short-range results together, as the analyses are almost identical. The structure of this paper is the following.

(1) In the remainder of this section, we first describe the short- and long-range CFTs and the crossover between them; then we consider their regularization; and then we compactly summarize our F's. Section I F then contains our discussion of these results.
(2) In Sec. II we immediately present the free energy results for the O(N), Gross-Neveu, and supersymmetric CFTs and compare them to each other.
(3) In Sec. III we describe the correct construction and regulation of the O(N) model (the same is done for the GN model in Appendix C).
(4) In Sec. IV we compute the universal part of the free energy for these models and comment on why it takes the form it does. We provide a compact derivation of $F_\Phi(\Delta)$, the free energy of an arbitrary generalized free conformal field of dimension Δ.

Various details are relegated to the other Appendixes.

(1) In Appendix A, we review our conventions for Gaussian integrals and free fields, as well as position-space integrals.
(2) We describe how to find the values of the counterterm in Appendix B.
(3) We compute our diagrams on the sphere using standard Mellin-Barnes integrals, which are discussed in Appendix D 2. For completeness, we also discuss how we can use the method of uniqueness (summarized in Appendix A 6) in Appendix D 1.

[1]We proved it in [8], which also extends the $\tilde{F}$-extremization noted above to all orders in $1/N$.

B. The large-N expansion and the ϵ expansion

Working in the large-N limit is made simpler by introducing a real auxiliary field $\sigma \sim :\phi_i\phi_i:$. The partition function can then be rewritten exactly using a new action,

$$Z_{\mathrm{O}(N)} \overset{\S 3}{=} \int \mathrm{D}^{(\mathrm{I})}\sigma \mathrm{D}\phi e^{-S_{\mathrm{O}(N)}},$$
$$S_{\mathrm{O}(N)} \equiv \int_x \frac{1}{2}\phi_i\left(C^{-1} - \frac{\sigma}{\sqrt{N}}\right)\phi_i - \frac{\sigma^2}{2\lambda_0}, \tag{1.4}$$

$$Z_{\mathrm{GN}} \overset{\S C}{=} \int \mathrm{D}^{(\mathrm{I})}\sigma \mathrm{D}\psi \mathrm{D}\bar{\psi} e^{-S_{\mathrm{GN}}},$$
$$S_{\mathrm{GN}} \equiv \int_x -\bar{\psi}_i\left(C^{-1} - \frac{\sigma}{\sqrt{N}}\right)\psi^i - \frac{\sigma^2}{2\lambda_0}, \tag{1.5}$$

where in slightly more generality we can take $C(x,y)$ to be the propagator of a free conformal field of dimension $\Delta^{\mathrm{UV}}_{\phi/\psi}$ (whether on the sphere or on flat space—see Appendix A for our conventions). Our strategy for computing F is to work at the critical point by taking $\lambda_0 \to \infty$, and integrate out the vector field ϕ_i. This leaves a nonlocal effective action for σ in which $1/N$ appears only as a coupling constant (a continuous parameter), allowing for a perturbative solution for F in $1/N$.

The standard conformally coupled free scalar and fermion have $\Delta^{\mathrm{UV}}_\phi = \frac{d-2}{2}$ and $\Delta^{\mathrm{UV}}_\psi = \frac{d-1}{2}$; these are implemented in curved space by

$$C^{-1}_{\mathrm{O}(N)} = -\nabla^2 + \frac{d-2}{4(d-1)}\mathcal{R} \quad \text{and} \quad C^{-1}_{\mathrm{GN}} = +\not{\nabla} \tag{1.6}$$

respectively (for flat space, these are just $C^{-1} = -\partial^2$ and $+\not{\partial}$, which yield the canonical position-space correlators $C(x,y)$ given in Appendix A 2). Now, in the IR for $2 < d < 4$, we can then ignore the irrelevant operator σ^2 in the actions above (schematically, we send the bare coupling $\lambda_0 \to \infty$); this then gives the interacting CFT with action

$$S = \int_x \frac{1}{2}\phi_i\left(C^{-1} - \frac{\sigma}{\sqrt{N}}\right)\phi_i + \text{counterterms} \tag{1.7}$$

and conformal scaling dimensions

$$\Delta_\phi = \Delta^{\mathrm{UV}}_\phi + O(1/N), \qquad \Delta_\sigma = \Delta^{\mathrm{UV}}_\sigma + O(1/N), \tag{1.8}$$

where $\Delta_\sigma = d - 2\Delta^{\mathrm{UV}}_\phi$, for which we would like to compute F.

To properly define any QFT we must specify a regularization. However, this auxiliary theory is unpleasant to regulate when working at the critical point: using a hard cutoff is too difficult, and dimensional regularization (DREG) alone does not work since the interaction $\sigma\phi_i\phi_i$

is marginal in any d. The computationally easiest way to access the interacting CFT is to combine DREG with an analytic shift in the scaling dimension of the field σ in the UV, a regularization which we call DREG + δ. This shift is not trivial to implement, particularly because the leading kinetic term for σ is dynamically generated by ϕ loops. It can only be implemented by modifying the action of the theory, which makes σ an interacting field even in the UV. Technically, then, this is an IR duality, since we are considering a different QFT that flows to the same Wilson-Fisher-like CFT in the IR: we illustrate this in Fig. 1. Away from the IR, these do not describe the same QFT; this is manifest because the UV that includes a σ field is explicitly nonlocal, as we shall see. Accounting for this modification to the action, including the counterterms that are required for the correlators to be finite, then leads to the additional contribution to F that matches the ϵ expansion.

The $1/N$ expansion works because in the limit that the small parameter is taken to zero, the UV and IR CFTs coincide (including the presence of a free field σ); hence the interacting IR theory can be solved by perturbing around the UV theory. This is identical to the case of the ϵ expansion, as demonstrated in Fig. 1—except for the fact that the UV theory is not the CFT of N free scalars. However, unlike in the ϵ-expansion case, finite $1/N$ does not also provide a regularization of the loop integrals. Hence, an additional regularization δ is required—this is the source of the technical complexity of the $1/N$ expansion compared to the ϵ expansion.

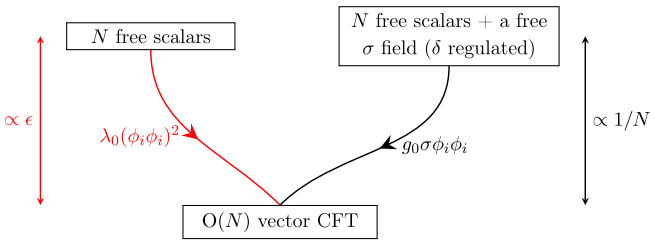


FIG. 1. A schematic of our IR duality in theory space: there are two different QFTs that flow to the O(N) vector CFT. The standard UV completion of N free scalars is on the left. The Wilson-Fisher (WF) CFT found by perturbing this theory by $(\phi_i\phi_i)^2$, (1.1), can be solved for finite N by using DREG, working perturbatively in $\epsilon = 4 - d$. In the large-N limit, DREG no longer suffices: we need a QFT (3.13) with a new regulator δ that improves the UV behavior—this is shown on the right. Now $\sigma\phi_i\phi_i$ is the perturbing relevant operator, and we can solve the IR CFT by working perturbatively in $1/N$. Computationally, we can solve for the O(N) theory by perturbing about either of these theories because in each case the length of the flow (i.e., the size of the distance between the UV and IR CFTs, as measured by their conformal data) is proportional to that small parameter ϵ or $1/N$. We ignore the subtlety that in some d's the roles of free UV and interacting IR might be reversed [9].

C. Which CFTs?

We will also compute F for the CFTs defined by (1.7) but with C a conformal propagator with generic scaling dimension $\Delta_\phi^{\rm UV}$. These are a more general family of CFTs that exist in arbitrary dimension, though they may be nonunitary and/or nonlocal, and reaching them requires careful tuning of counterterms [4,10–12]. We therefore remain entirely agnostic about whether our RG flow from the free theory is toward the UV or the IR, and simply compute F for the Wilson-Fisher-like CFT in each case, working directly at the fixed point.

For the O(N) models, we can choose C to give

(1) $\Delta_\phi^{\rm UV} = \frac{d}{2} - k$ for integer k. Equation (1.7) then describes a local CFT (albeit manifestly nonunitary for $k > 1$). We refer to these as the short-range (SR) CFTs, though they are also called the $\Box^k$ CFTs [13] or Lifshitz points [10]. For these $C^{-1} \sim (-\partial^2)^k = \Box^k$, as described in Appendix A 3.

(2) $\Delta_\phi^{\rm UV} = \frac{d-s}{2}$ for arbitrary s. Equation (1.7) then describes nonlocal Wilson-Fisher-like CFTs: these are called the long-range (LR) CFTs [14–23]. For these, we have a fractional Laplacian $C^{-1} \sim (-\partial^2)^{s/2} = \Box^{s/2}$, as described in Appendix A 3.

On the sphere, the operators $C^{-1} \sim \Box^{s/2}$ that define a free scalar of the given dimension are in reality more complicated and will be discussed later.

As was discussed in [10], the $\Box^k$ O(N) CFT is found by flowing to the IR of (1.1) with $C^{-1} \sim \Box^k$ for $2k < d < 4k$. There are likely issues with canonical quantization and Ostrogradsky instabilities in all such higher derivative theories, but [10] also commented that they are probably well defined as Euclidean theories constructed via the path integral.

Crossover from long- to short-range: An order of limits. In the SR case, the interacting CFT has $\Delta_\phi = \Delta_\phi^{\rm UV} + \hat\gamma_\phi$, with some anomalous dimension $\hat\gamma_\phi$; in the LR case, the interacting CFT has $\Delta_\phi = \Delta_\phi^{\rm UV} = \frac{d-s}{2}$ exactly: this is because quantum corrections can only generate local divergences, and hence local counterterms—therefore the nonlocal kinetic term for ϕ cannot be renormalized to any order in N^2. Somewhat surprisingly, then, these two CFTs coincide at the crossover value

$$s = s_{\rm SR} = 2k - 2\hat\gamma_\phi \tag{1.9}$$

(possibly up to the addition of N decoupled free fields [19,26], although our computations of $\tilde F$, along with those of [4], suggest that this interpretation is not quite correct). That is, the CFT data of the short-range models and the

[2]This was proven to all orders in $\epsilon \equiv s - d/2$ in [24]. It also follows from defining the LR theory as a defect in a higher-dimensional bulk [25].

long-range models are identical at the crossover points [22]. In this paper, we show that the same applies for F.

The details of how to reach these CFTs are complex. For any $s > s_{\rm SR}|_{k=1}$, the CFT defined by (1.7) will generically flow to the $k = 1$ short-range CFT, as described by [22,26]. We can understand this qualitatively in the following way. Due to the local divergences coming from the Wilsonian RG, the field theory is always trying to generate a $k = 1$ short-range kinetic term for the scalar. This would give ϕ the scaling dimension of $\frac{d-s_{\rm SR}}{2}$. However, for $s < s_{\rm SR}$ the explicit kinetic term of dimension $\frac{d-s}{2} > \frac{d-s_{\rm SR}}{2}$ wins, so ϕ gets stuck with scaling dimension $\frac{d-s}{2}$. For $s > s_{\rm SR}$, unless further counterterms are tuned to cancel those local divergences, we will just find the short-range CFT.

However, in the following we will assume that further counterterms are tuned to ensure that we can flow to a CFT with $\Delta_\phi = \frac{d-s}{2}$ even for $s > s_{\rm SR}$, as was done in [10]. This is similar to how, despite the fact that the ϕ^4 theory flows to a mean field theory for any $d > 4$, we can reach the $\mathrm{O}(N)$ CFT for $d > 4$ by a judicious choice of action [9,27].

To forestall a potential point of confusion: there are now two different ways of reaching the $\mathrm{O}(N)$ CFT. The reason for this is a difference in the order of limits: the limit $s \to 2k$ does not commute with the limit in which the regulator of this theory (δ) is removed. Thus, as we show explicitly in Appendix B 5, the short-range CFT can be reached by either

(1) (1.7), where we fix $s \to 2k$ and then take $\delta \to 0$;
(2) or (1.7), where we first take $\delta \to 0$ and then take $s \to 2k - 2\hat{\gamma}_\phi$.

The same comments apply to the analogous generalizations of the Gross-Neveu model, where $\Delta_\psi^{\rm UV} = \frac{d}{2} - k$ (for $k - \frac{1}{2}$ integer) and $\Delta_\psi^{\rm UV} = \frac{d-s}{2}$: the short- and long-range CFTs coincide at $s_{\rm SR} = 2k - 2\hat{\gamma}_\psi$.

The short-range models lie at the extremal F for the long-range models. Given that the short-range fixed points lie on the curve parametrized by s (equivalently, parametrized by Δ_ϕ), it is natural to ask whether there is anything special about the short-range fixed points from the long-range perspective. Put another way, if we only knew about the long-range CFTs, parametrized by s, how would we notice that $s_{\rm SR}$ was special? The answer is[3] that, at least to order $1/N^2$, the short-range models lie at the extrema of $F_i^{\rm LR}(s)$,

$$\left.\frac{\mathrm{d}F_i^{\rm LR}(s)}{\mathrm{d}s}\right|_{s=s_{\rm SR}} = 0, \tag{1.10}$$

for any k (a similar story holds in the ϵ expansion [8]). In particular, for the standard $k = 1$ $\mathrm{O}(N)$ model and $k = \frac{1}{2}$ GN model, $\tilde{F}_i^{\rm LR}(s)$ is maximized by $s_{\rm SR}$ in $2 < d < 4$. Since we calculate F up to order $1/N$, this means that $\Delta_\phi^{\rm SR} = \frac{d-s_{\rm SR}}{2}$ can be found to order $1/N^2$. This gives a compact encoding of the short-range vector model's conformal data, as the extremum of a particular function.

[3]There are other distinguishing features: for example, it is only for $s_{\rm SR}$ that the theory is local. Hence, it is only for $s_{\rm SR}$ that the stress-energy tensor is a local operator, appearing in the operator product expansions (OPEs) of the fundamental fields [28]—this is just as for the long-range melonic CFTs [29–31].

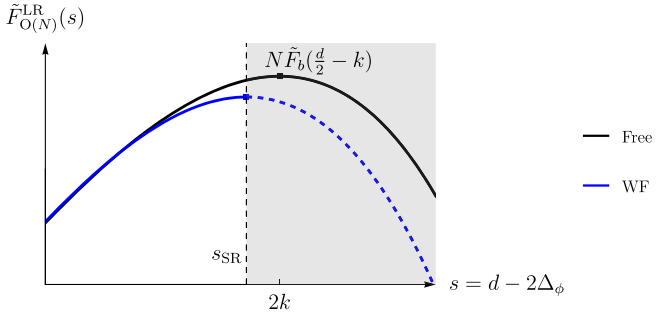


FIG. 2. A schematic of the long-range free energy $\tilde{F}^{\rm LR}(s)$ for the $\mathrm{O}(N)$ model. We have taken odd k, and assume $d > 2k$, meaning that $\tilde{F}$ is maximized to leading order in N; this is the situation for the standard ϕ^4 $\mathrm{O}(N)$ CFT ($k = 1$). The free CFT of N scalars (black line) has $\tilde{F} = N\tilde{F}_b(\frac{d-s}{2})$, which is maximal at $s = 2k$, marked with a box. The WF interacting CFT (blue line) has a maximum at $s = s_{\rm SR} = 2k - 2\hat{\gamma}_{\phi,1}/N + O(1/N^2)$, marked with a box. Both maxima are the locations of the short-range CFTs, and we have chosen a value of d such that the free local theory has a larger $\tilde{F}$. Unless counterterms are tuned, for $s > s_{\rm SR}$ we flow to the short-range model on perturbing the free scalars by ϕ^4, and so that regime of the interacting model is dashed.

For a free long-range field, it is trivial that the extrema of the long-range $\tilde{F}_{b,f}(\Delta)$ lie at the short-range scaling dimensions $\Delta = \frac{d}{2} - k$. Therefore, the situation here is *identical* to the case of a free field of dimension $\frac{d-s}{2}$: that is a one-parameter family of (generically nonlocal) CFTs for which the extrema of $\tilde{F}$ lies at the local points $s = 2k$, if $d > 2k$. Thus, the interaction just shifts that extremum by $s_{\rm SR} - 2k = -2\hat{\gamma} \sim O(1/N)$. Both lines of CFTs are illustrated in Fig. 2 for the $\mathrm{O}(N)$ model, where we have taken:

(i) odd k to show the case where $\tilde{F}$ is maximized;
(ii) $d > 2k$ to ensure that $\Delta_\phi^{\rm SR} > 0$, and so the CFT makes sense;
(iii) $4k > d$; this is the condition for the relevance of ϕ^4 as a perturbation of the free theory and defines the upper critical dimension [10].

For some values of d (but not all) in this range, we have $\tilde{F}(\Delta_\sigma^{\rm SR}) < 0$, and so the flow from the $\square^k$ free theory to the $\square^k$ WF theory satisfies the $\tilde{F}$-theorem $\tilde{F}^{\rm SR}_{\rm free} > \tilde{F}^{\rm SR}_{\rm WF}$ to leading order in N. This is the situation shown in Fig. 2: in the $k = 1$ theory, it is true for all $2 < d < 4$; in the $k = 2$ theory, it is true for $5.35 \lesssim d < 8$.

D. The shortest summary of our results

The F's are presented in Sec. II, but we wish now to emphasize their compact nature in terms of the scaling

dimensions. The interacting CFT of a bosonic ($x = b$) or fermionic ($x = f$) vector with full scaling dimension $\Delta_x \equiv \frac{d-s}{2}$ has

$$F_{\mathrm{CFT}} = NF_x(\Delta_x) + F_b(s) + \frac{F_b'(s)}{N}\left(\frac{1}{2}\Gamma_{\mathrm{tet}} + \frac{1}{3}\Gamma_{\mathrm{pr}}\right) + O(1/N^2), \tag{1.11}$$

regardless of whether it is short- to long-range. Here, $F_x(\Delta_x)$ is a standard quantity, the sphere free energy of a free boson/fermion of scaling dimension Δ_x; Γ_{tet} and Γ_{pr} are just separate s-dependent contributions to the scaling dimension of the auxiliary field σ,

$$\Delta_\sigma = d - 2\Delta_x + \frac{1}{N}(\Gamma_{\mathrm{tet}} + \Gamma_{\mathrm{pr}}) + O(1/N^2), \tag{1.12}$$

and will be defined in (2.6) and (2.15) for the O(N) and GN models respectively. s_{SR} and the corresponding $\Delta_x = \frac{d}{2} - k + \frac{\hat{\gamma}_{x,1}}{N} + \frac{\hat{\gamma}_{x,2}}{N^2} + O(1/N^3)$ in the short-range version is then picked out by the extremization condition (1.10). The full expressions for $\hat{\gamma}_{x,2}$ in particular are quite cumbersome (to say nothing of $\hat{\gamma}_{x,3}$), so this is a pleasing encapsulation.

We now discuss the correct way to regulate these theories.

E. Proper regularization

Focusing now on the O(N) model, we attack the action (1.4) by performing the quadratic ϕ integral and expanding the logarithm in powers of N

$$Z_{\mathrm{O}(N)} = Z_{\phi,\mathrm{free}}^N \int \mathrm{D}^{(\mathrm{I})}\sigma \exp(-S_\sigma) \tag{1.13a}$$

$$S_\sigma \equiv \frac{N}{2}\mathrm{tr}\,\log\left(\mathbb{I} - \frac{M}{\sqrt{N}}\right) - \frac{1}{2\lambda_0}\int_x \sigma^2 \tag{1.13b}$$

$$= -\frac{N}{2}\sum_{q=1}^{\infty}\frac{1}{N^{\frac{q}{2}}}\frac{1}{q}\mathrm{tr}\,M^q - \frac{1}{2\lambda_0}\int_x \sigma^2 \tag{1.13c}$$

where we are treating $M_{xy} = \sigma(x)C(x,y)$ as a matrix (this computation is reviewed in full in Sec. III). This is a second, explicitly nonlocal, QFT that encodes the correlators of $\sigma \sim :\phi_i\phi_i:$ in the original theory, and by construction has the same partition function $Z_{\mathrm{O}(N)}$.

We here have an infinite number of couplings, but to any fixed order in $1/N$ we need only consider a finite number of them; since we want to calculate F to order $1/N$, we go only to $q = 4$. Since the ϕ^2 mass term is tuned to vanish by assumption, the σ tadpole must vanish: $\mathrm{tr}\,M = 0$. Thus, the σ propagator that is generated is the inverse of $-\frac{1}{2}C(x,y)^2$, treated as a matrix, which is

$$\langle\sigma(x)\sigma(y)\rangle = \frac{C_{\sigma,0}}{s(x,y)^{2\Delta_\sigma^{\mathrm{UV}}}} + O(1/N), \quad \Delta_\sigma^{\mathrm{UV}} \equiv d - 2\Delta_\phi^{\mathrm{UV}}, \tag{1.14}$$

for some normalization $C_{\sigma,0}$. This is a conformal propagator if we take $s(x,y)$ to be either the chordal distance on the sphere, defined in Appendix A 3, or $|x-y|$ in flat space. Of course, for the usual short-range model $\Delta_\phi^{\mathrm{UV}} = \frac{d-2}{2}$, so σ^2 has the naive irrelevant scaling dimension $4 > d$; hence we are justified in dropping it when approaching the IR. We drop it for all the theories that we consider, as we want to work at the conformal fixed point.

If it were easy to regularize this action with a hard cutoff, the computation of F would now be straightforward. It is not. Instead, we opt for an analytic regularization scheme DREG + δ, where we analytically continue in two parameters:

(1) DREG: a standard technique in QFT, we essentially evaluate integrals in dimension d low enough that they converge and analytically continue back to the region d of interest. As usual, this allows us to ignore power-law divergences.
(2) A shift of the UV scaling dimension of σ: $\Delta_\sigma^{\mathrm{UV}} \mapsto \Delta_\sigma^{\mathrm{UV}} - \delta$.

The reason for the $\Delta_\sigma^{\mathrm{UV}}$ shift is that DREG no longer suffices to regulate this action once we go beyond the leading order of the critical vector model; this is because DREG works by making a marginal interaction slightly relevant, which turns logarithmic divergences $\log\Lambda$ into poles $1/\epsilon$. Unfortunately, $\sigma\phi_i\phi_i$, the bare interaction, is marginal and hence has logarithmic divergences for all d.

When computing the free energy, the $\Delta_\sigma^{\mathrm{UV}}$ shift must be implemented consistently in order to find the correct value of $\tilde{F}$; this can only be done by modifying the kinetic term in the action such that the field has scaling dimension $\Delta_\sigma^{\mathrm{UV}} - \delta$. First, we drop the irrelevant σ^2 term from (1.13). We are trying to subtract off a dynamically generated kinetic term; hence we need to add an $O(N^0)$ term to the action, of schematic form

$$S_{\sigma,\delta} = \frac{1}{g_0^2}\int_{x,y}\frac{1}{2}\sigma(x)\left(\frac{c_\delta}{s(x,y)^{2(d-\Delta_\sigma^{\mathrm{UV}}+\delta)}} - \frac{c\mu^{2\delta}}{s(x,y)^{2(d-\Delta_\sigma^{\mathrm{UV}})}}\right)\times\sigma(y) \tag{1.15}$$

for some constants c, c_δ. The first term is the new kinetic term, and the second should almost entirely subtract off the old one. If this seems strange, recall that the renormalization procedure generally involves adding some new UV physics. This is usually done by changing the bare coupling: adding a counterterm. Here, we are changing the UV physics more explicitly by actually modifying the kinetic term in the UV; this improves the UV behavior of

the diagrams, turning the logarithmic divergences into $1/\delta$ poles, as required.

Before this, σ only appeared linearly in the $\lambda_0 \to \infty$ action, and the normalization of the σ field could therefore be changed freely to set the coefficient of $-\frac{Z_\phi}{\sqrt{N}}\sigma\phi_i\phi_i$ to 1. With this new quadratic term, the normalization of σ is now meaningful. To canonically normalize the field, we set $\sigma_{\text{new}} = \sigma_{\text{old}}/g_0$, making the interaction $\frac{Z_\phi g_0}{\sqrt{N}}\sigma\phi_i\phi_i$. The critical point of this regulated theory now lies in the limit $g_0 \to \infty$, which is the deep IR. We work directly at the critical point; to control this limit, we write $g_0 = \mu^\delta g Z_g$, where Z_g takes the form of a power series in $1/N$, with each term containing $1/\delta$ poles: thus as $\delta \to 0$, we implement the aforementioned $g_0 \to \infty$.

In the effective action for σ the second term of (1.15) partially cancels with the dynamically generated kinetic term for σ, leaving a residual nonlocal quadratic σ term with a coefficient $\sim Z_{g,1}/(\delta N)$. This must be treated as an interaction term; taking care to take the correct order of limits, which is $N \to \infty$ first, followed by $\delta \to 0$, we find that the diagram that it multiplies is of order δ. Hence, we find an order

$$\sim \frac{\delta}{\delta}\frac{Z_{g,1}}{N} \tag{1.16}$$

correction to the naive free energies of these models.

F. Discussion

In this paper we computed the free energy of the vector models in the large-N limit to order $1/N$, confirming the conjectures of [1], which matched the ϵ-expansion results of [3,4]. The mismatch of [1] was resolved by a careful accounting of the analytic regulator, a shift in the UV scaling dimension; this combined with the counterterms to give a contribution, essentially due to a factor $\lim_{\delta\to 0}\delta/\delta$. This demonstrates another danger of regularizing by analytically continuing in an operator dimension, on top of the known danger of breaking supersymmetry [32]. A similar approach to ours here could also be applied to, for example, the $\mathrm{O}(N)^r$ models of [33] or large-N QED_d [34] [which could then be checked by comparison to the $d = 3$ result (5.8) of [35]].

Our results also were naturally generalizable to the long-range vector models and the $\square^k$ CFTs. A notable feature of all of these F's at this order is their simplicity when expressed in terms of the scaling dimensions, Eq. (1.11). This fact follows naturally from how vacuum diagrams are obtained from two-point function diagrams by the addition of a propagator. At the next order in N there are many more diagrams, with some giving non-ζ contributions [36,37]; however, it would be interesting to confirm whether the contributions to the long-range model continue to be of the form $F \supset F'(s)(\frac{1}{m_i}\frac{\Gamma_i}{N^2})$ for $\frac{\Gamma_i}{N^2} \subset \gamma_\sigma$, with analogous results for the short-range model coming from the limit $s \to s_{\text{SR}}$; this could be done relatively straightforwardly by comparison to the ϵ expansion. We also observed that the short-range models lie at the extrema of the long-range free energy, giving a minimal encoding of the otherwise cumbersome Δ_ϕ^{SR}; it would be nice to prove this in general.

It is worth noting that, in agreement with [4], we found no sign of the additional scalar field χ noted in [22,26] at the point of crossover from the long-range to the short-range theory: we found

$$\lim_{s\to s_{\text{SR}}} F_{\text{LR}}(s) = F_{\text{SR}}, \tag{1.17}$$

where our F_{SR} also matched the ϵ expansion. The expectation from [26] was that

$$\lim_{s\to s_{\text{SR}}} F_{\text{LR}}(s) \overset{?}{=} F_{\text{SR}} + N F_b\left(\Delta_\chi = \frac{d+s}{2}\right), \tag{1.18}$$

since χ_i was speculated to be a vector of N decoupled generalized free fields with scaling dimensions equal to the shadow of ϕ_i [and so $F_b(\frac{d+s}{2}) = -F_b(\frac{d-s}{2}) = -F_b(\Delta_\phi)$]. This does not agree with our analysis and so requires further study.

Now, CFTs are usually considered on flat space. However, F must be computed on the sphere, as otherwise the free energy diverges with the volume: this CFT is the same as the flat-space one in the sense that it shares the same conformal data $\{(-\Delta_i, \rho_i), C_{ijk}\}$ as the flat-space CFT. To justify this: we expect it to be generically possible to couple a flat-space CFT to a background metric $g_{\mu\nu}$ in such a way that the conformal symmetry descends from Weyl symmetry [38,39]. If so, then because the sphere is related to flat space by a Weyl map, $g^{S^d}_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$, then the CFTs must be the same. We can also show by explicit calculation that the CFT data are the same in either space, which is what we have done for this paper. Thus, the quantity $\tilde{F}$ can be associated with the abstract CFT. However, in general, that coupling to g may be nontrivial (or even impossible): we must know the curvature couplings, which we can ignore for the flat-space CFT. At our order $1/N$, we need only consider the curvature couplings coming from the kinetic term. However, at subleading orders we must consider the curvature counterterms, as has been seen already in the ϵ expansion [3,4].

Finally, there is debate at present over whether the critical $\mathrm{O}(N)\phi^4$ vector model and the critical $\mathrm{O}(N)$ nonlinear sigma model are indeed the same. However, it was optimistically speculated that the two theories match to all orders in the $1/N$ expansion [40], in which case the computations here apply to both.

II. THE SPHERE FREE ENERGIES

We now summarize our results for all the vector models of interest. In the following, we will make heavy use of

$$A(\Delta) \equiv \frac{\Gamma(\frac{d}{2}-\Delta)}{\Gamma(\Delta)}, \qquad A_f(\Delta) \equiv \frac{\Gamma(\frac{d}{2}-\Delta+\frac{1}{2})}{\Gamma(\Delta+\frac{1}{2})}, \quad (2.1)$$

which come from the Fourier transforms of bosonic and fermionic propagators respectively (see Appendix A 5). We shall see that it is more compact to first present the long-range case, and then give the short-range version. We note that the (universal part of the) free energy density with respect to the regularized volume of hyperbolic space,[4]

$$\tilde{F} \equiv \frac{1}{2}\frac{\mathrm{vol}S^{d+1}}{\mathrm{vol}\mathbb{H}^{d+1}}[F]_{\text{universal}} = -\sin\left(\frac{\pi d}{2}\right)F|_{S^d,\text{DREG}}$$
$$= \sin\left(\frac{\pi d}{2}\right)\log Z|_{S^d,\text{DREG}} \quad (2.2)$$

is the more natural quantity to consider. $\tilde{F}$ is believed to be finite, independent of the sphere radius, with a smooth analytic continuation in d, and satisfy for a general RG flow

$$\tilde{F}_{\text{UV}} > \tilde{F}_{\text{IR}}; \quad (2.3)$$

it is therefore thought to serve as a measure of the number of degrees of freedom of a (unitary) CFT [2,3]. We shall move freely between the two, writing F in general but $\tilde{F}$ when we want to consider its sign. Happily, we write the CFT F's below in such a way that we need only map $F_x(\Delta) \mapsto \tilde{F}_x(\Delta)$ to find the $\tilde{F}$ s. Note also that the $+O(1/N^2)$ s are left implicit for all the F's below.

A. O(N) vector model

The free energy of a free scalar field of scaling dimension Δ is known to be

$$F_b(\Delta) \equiv \int_{\frac{d}{2}}^{\Delta} \mathrm{d}\Delta F_b'(\Delta),$$
$$F_b'(\Delta) \equiv -\frac{\pi}{\sin(\frac{\pi d}{2})\Gamma(d+1)}\frac{1}{A(\Delta)A(d-\Delta)}. \quad (2.4)$$

We give a compact derivation of F for an arbitrary conformal field in Sec. IV B. There are two features of note.

(i) $\tilde{F}_b'(\frac{d}{2}-k)=0$. When modified with the usual factor, $\tilde{F}_b''(\frac{d}{2}-k)(-1)^k > 0$ for all $d > 2k$: hence the standard free field is a maximum of $\tilde{F}_b(\Delta)$. This fact is of great importance for the short-range vector models.

(ii) We have chosen the standard $F_b(\frac{d}{2})=0$, agreeing with the identity $\mathrm{tr}\,\delta(x-y)=0$ that is forced upon us by the lack of a regulating scale in DREG.

1. Long-range O(N)

In the IR of the long-range O(N) vector model for generic s we find

$$\Delta_\phi = \frac{d-s}{2}, \qquad \Delta_\sigma = s + \frac{\Gamma_{\text{tet}}+\Gamma_{\text{pr}}}{N} + O(1/N^2), \quad (2.5)$$

where Δ_ϕ is exactly $\frac{d-s}{2}$ to all orders in $1/N$, and the γ_σ contributions come from the tetrahedral and prism diagrams respectively,[5]

$$\Gamma_{\text{tet}}(s) \equiv -\frac{4}{\Gamma(\frac{d}{2})}\frac{A(\frac{d-s}{2})^2}{A(d-s)},$$
$$\Gamma_{\text{pr}}(s) \equiv \frac{8}{\Gamma(\frac{d}{2})}\frac{A(d-\frac{3s}{2})A(s)A(\frac{d-s}{2})^3}{A(d-s)^2}. \quad (2.6)$$

The free energy is

$$F^{\text{LR}}_{\text{O}(N)} = NF_b\left(\frac{d-s}{2}\right) + F_b(s) + \frac{F_b'(s)}{N}\left(\frac{1}{2}\Gamma_{\text{tet}} + \frac{1}{3}\Gamma_{\text{pr}}\right), \quad (2.7)$$

which we can rewrite using (2.5)

$$F^{\text{LR}}_{\text{O}(N)} = NF_b(\Delta_\phi) + F_b(\Delta_\sigma) - \frac{F_b'(s)}{N}\left(\frac{1}{2}\Gamma_{\text{tet}} + \frac{2}{3}\Gamma_{\text{pr}}\right) + O(1/N^2) \quad (2.8)$$

to make it clear that the correction term is not just absorbed by the F_b's.

2. Short-range O(N)

There are two different ways of finding the free energy of the short-range model:

(1) Compute directly for the short-range O(N) CFT.
(2) Take the limit of the long-range results,

$$\Delta_\phi = \frac{d-s}{2} \to \Delta_\phi^{\text{SR}} = \frac{d-2}{2} + \frac{\hat{\gamma}_{\phi,1}}{N} + O(1/N^2),$$
$$\hat{\gamma}_{\phi,1} \equiv \frac{2F_b'(2)}{F_b''(\frac{d-2}{2})} = \frac{-2\Gamma(d-2)}{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-2)\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2}+1)}. \quad (2.9)$$

As we show in Sec. IV C 3, the results found in the two cases are identical; but, crucially, the counterterms required

[4]The computation of the regularized vol $\mathbb{H}^{d+1}$ is discussed in the Appendixes of [7], but amounts to computing the volume of hyperbolic space in DREG such that it is finite for generic d. The factor of $\frac{1}{2}\mathrm{vol}\,S^{d+1}$ in (2.2) is not strictly necessary, being smooth and positive for all $d \geq 0$, but is there by convention from [2].

[5]These results, as well as the later counterterms and normalizations, correct various typos in [22].

in each case differ. Either way, we find

$$F^{\rm SR}_{{\rm O}(N)} = NF_b(\Delta^{\rm SR}_\phi) + F_b(\Delta^{\rm SR}_\sigma) - \frac{F_b'(2)}{N}\left(\frac{1}{2}\Gamma_{\rm tet} + \frac{2}{3}\Gamma_{\rm pr}\right)\Big|_{s=2},$$
$$\Delta^{\rm SR}_\sigma = 2 + \frac{\Gamma_{\rm tet} + \Gamma_{\rm pr} - 2\hat\gamma_{\phi,1}}{N}\Big|_{s=2} + O(1/N^2)\Big), \tag{2.10}$$

which also follows from $s \to s_{\rm SR} \equiv 2 - 2\frac{\hat\gamma_{\phi,1}}{N} + O(1/N^2)$ in (2.8). Because $F_b'(\frac{d-2}{2}) = 0$, we only need $\hat\gamma_\phi$ to leading order in N, so this expands

$$\begin{aligned} F^{\rm SR}_{{\rm O}(N)} &= NF_b\left(\frac{d-2}{2}\right) + F_b(2) \\ &\quad + \frac{F_b'(2)}{N}\left(\frac{F_b''(\frac{d-2}{2})}{F_b'(2)}\frac{\hat\gamma^2_{\phi,1}}{2} - 2\hat\gamma_{\phi,1} + \frac{1}{2}\Gamma_{\rm tet} + \frac{1}{3}\Gamma_{\rm pr}\right) \\ &\overset{(2.9)}{=} NF_b\left(\frac{d-2}{2}\right) + F_b(2) \\ &\quad + \frac{F_b'(2)}{N}\left(\frac{1}{2}\Gamma_{\rm tet} + \frac{1}{3}\Gamma_{\rm pr} - \hat\gamma_{\phi,1}\right)\Big|_{s=2}, \end{aligned} \tag{2.11}$$

where at this order we can evaluate the Γ's for $s = 2$. This result perfectly matches the conjecture of [1],

$$F^{\rm SR}_{{\rm O}(N)} = NF_b\left(\frac{d-2}{2}\right) + F_b(2) + \left(\frac{3-d}{3(d-2)(d-1)} + \frac{1}{d}\right) \times \frac{\hat\gamma_{\phi,1}}{N}. \tag{2.12}$$

B. Gross-Neveu model

Turning now to the fermionic vector model, we recall that the free energy of a single complex component of a free Dirac fermion of dimension Δ is

$$F_f(\Delta) \equiv \int_{\frac{d}{2}}^{\Delta} {\rm d}\Delta F_f'(\Delta),$$
$$F_f'(\Delta) \equiv + \frac{\pi}{\sin(\frac{\pi d}{2})\Gamma(d+1)}\frac{2}{A_f(\Delta)A_f(d-\Delta)}, \tag{2.13}$$

where the 2 in the numerator comes from the complex nature of the field. This also satisfies $\tilde F_f'(\frac{d}{2} - k) = 0$ and $\tilde F''(\frac{d}{2} - k)(-1)^{k-\frac{1}{2}} < 0$ for all $d > 2k - 1$ for half-integer k. Taking the GN model to have a $U(n)$ symmetry, our definition of $N \equiv n{\rm tr}\,\mathbb{I}_s$ [41] means that N free Dirac fermions contribute $NF_f(\frac{d-1}{2})$ to the free energy.

1. Long-range GN model

In the IR of the complex Gross-Neveu vector model, for generic s

$$\Delta_\psi \equiv \frac{d-s}{2}, \qquad \Delta_\sigma = s + \frac{\Gamma^{\rm GN}_{\rm tet}}{N} + O(1/N^2), \tag{2.14}$$

where

$$\Gamma^{\rm GN}_{\rm tet} \equiv \frac{2}{\Gamma(d/2)}\frac{A_f(\frac{d-s}{2})^2}{A(d-s)}, \tag{2.15}$$

and the equivalent of $\Gamma^{\rm GN}_{\rm pr} = 0$ because there is no cubic vertex for σ in the σ-only QFT. Then the free energy is

$$F^{\rm LR}_{\rm GN} = NF_f\left(\frac{d-s}{2}\right) + F_b(s) + \frac{F_b'(s)}{N}\left(\frac{1}{2}\Gamma^{\rm GN}_{\rm tet}\right) \tag{2.16a}$$
$$= NF_f(\Delta_\psi) + F_b(\Delta_\sigma) - \frac{F_b'(s)}{N}\left(\frac{1}{2}\Gamma^{\rm GN}_{\rm tet}\right). \tag{2.16b}$$

2. Short-range GN model

As before, we could also compute the free energy of the short-range GN CFT directly, but the results are identical to extremizing the LR F. The extremum lies at $s \to s_{\rm SR} \equiv 1\text{–}2\frac{\hat\gamma_{\psi,1}}{N} + O(1/N^2)$, such that

$$\begin{aligned} \Delta_\psi \to \Delta^{\rm SR}_\psi &= \frac{d-1}{2} + \frac{\hat\gamma_{\psi,1}}{N} + O(1/N^2), \\ \hat\gamma_{\psi,1} &\equiv \frac{2F_b'(1)}{F_f''(\frac{d-1}{2})} = \frac{\Gamma(d-1)}{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)^2\Gamma(\frac{d}{2}+1)}, \\ \Delta^{\rm SR}_\sigma &= 1 + \frac{\Gamma^{\rm GN}_{\rm tet} - 2\hat\gamma_{\psi,1}}{N}. \end{aligned} \tag{2.17}$$

Plugging this in to (2.16b) the short-range free energy is

$$F^{\rm SR}_{\rm GN} = NF_f(\Delta^{\rm SR}_\psi) + F_b(\Delta^{\rm SR}_\sigma) - \frac{F_b'(s)}{N}\left(\frac{1}{2}\Gamma^{\rm GN}_{\rm tet}\right) \tag{2.18a}$$
$$\overset{(2.17)}{=} NF_f\left(\frac{d-1}{2}\right) + F_b(1) + \frac{F_b'(1)}{N}\left(\frac{1}{2}\Gamma^{\rm GN}_{\rm tet} - \hat\gamma_{\psi,1}\right)\Big|_{s=1}, \tag{2.18b}$$

where again we need only evaluate $\Gamma^{\rm GN}_{\rm tet}$ for $s = 1 + O(1/N)$. Again, we match [1],

$$F_{\rm GN}^{\rm SR} = NF_f\left(\frac{d-1}{2}\right) + F_b(1) + \frac{1}{d}\frac{\hat{\gamma}_{\psi,1}}{N}. \tag{2.19}$$

C. Supersymmetric model

For completeness, we now present the short-range[6] supersymmetric (SUSY) version of the O(N) model—the theory of $N+1$ chiral superfields with an O(N)-symmetric superpotential $W = \frac{\lambda_0}{2}\Sigma\Phi_i\Phi_i$ presented in [1], which has action [42]

$$S_{\rm SUSY} = \int \mathrm{d}^d x\left[\int \mathrm{d}^4\theta(\bar{\Phi}_i\Phi_i + \bar{\Sigma}\Sigma) - \frac{\lambda_0}{2}\int \mathrm{d}^2\theta\Sigma\Phi_i\Phi_i - \frac{\lambda_0}{2}\int \mathrm{d}^2\bar{\theta}\,\bar{\Sigma}\,\bar{\Phi}_i\bar{\Phi}_i\right]. \tag{2.20}$$

For the IR CFT, we know that the superpotential must have R-charge 2, so $\Delta_\Sigma + 2\Delta_\Phi = d-1$. Localization then permits an exact calculation of the free energy in terms of the unknown Δ's,

$$F_{\rm SUSY} = NF_S(\Delta_\Phi) + F_S(\Delta_\Sigma), \tag{2.21}$$

defined in terms of the free energy of a free chiral superfield

$$F_S(\Delta) = 2F_b(\Delta) + 2F_f\left(\Delta + \frac{1}{2}\right) + 2F_b(\Delta+1), \tag{2.22a}$$

$$F_S'(\Delta) = \frac{1}{\sin(\frac{\pi d}{2})}\frac{\pi}{\Gamma(d-1)}\frac{1}{A(\Delta)}\frac{1}{A(d-1-\Delta)}, \tag{2.22b}$$

containing a complex scalar, a fermion with four real components, and an auxiliary complex scalar. As before, F_f is the free energy for a single complex fermion component, and we have analytically continued in d while keeping the dimension of our fermions fixed [2,43]. $\tilde{F}$-extremization then tells us that we can determine Δ_Φ by extremizing this exact function $F_{\rm SUSY}(\Delta_\Phi, \Delta_\Sigma)$, subject to the constraint [2].

Foreshadowing the next section, we slightly generalize the kinetic term for Φ_i to be $\sim\Phi_i D^{2(k-1)}\Phi_i$ (and add in suitable counterterms). We can then expand around $\Delta_\Phi = \frac{d}{2} - k + \hat{\gamma}_\Phi$ and $\Delta_\Sigma = 2k - 1 - 2\hat{\gamma}_\Phi$ in the large-N limit. Using $F_S'(\frac{d}{2}-k) = 0$, extremization finds that

$$\hat{\gamma}_{\Phi,1} = \frac{2F_S'(2k-1)}{F_S''(\frac{d}{2}-k)},$$
$$\hat{\gamma}_{\Phi,2} = -\hat{\gamma}_{\Phi,1}^2\left(\frac{F_S'''(\frac{d}{2}-k)}{2F_S''(\frac{d}{2}-k)} + \frac{2F_S''(2k-1)}{F_S'(2k-1)}\right),$$
$$\Delta_\Sigma = 2k - 1 - 2\hat{\gamma}_\Phi, \tag{2.23}$$

which agrees with [44,45]. Hence, compactly, the expansion of (2.21) begins

$$F_{\rm SUSY} = NF_S\left(\frac{d}{2}-k\right) + F_S(2k-1) + \frac{F_S'(2k-1)}{N}(-\hat{\gamma}_{\Phi,1}) \tag{2.24a}$$

which is almost identical to (2.27), albeit with both Γ's set to zero. This continues with

$$+\frac{1}{N^2}\left(2F_S''(2k-1) + \frac{1}{6}F_S'''\left(\frac{d}{2}-k\right)\hat{\gamma}_{\Phi,1}\right)\hat{\gamma}_{\Phi,1}^2. \tag{2.24b}$$

D. $\Box^k$ CFTs

The results for the free energy of the Wilson-Fisher-like (bosonic) $\Box^k$ CFTs and (fermionic) $\phi\Box^{k-1/2}$ CFT are obtained in the same way as the other short-range CFTs. In the former, k is integer; in the latter, $k-1/2$ is an integer.

As before, the quickest way to find the results is to plug $s = 2k - 2\hat{\gamma}_{\phi,1}/N$ into (2.7) and $s = 2k - 2\hat{\gamma}_{\psi,1}/N$ into (2.16a). Adding in the supersymmetric results for comparison, the leading anomalous dimension of the vector is

$$\hat{\gamma}_{\phi,1} \equiv \frac{2F_b'(2k)}{F_b''(\frac{d}{2}-k)}, \quad \hat{\gamma}_{\psi,1} \equiv \frac{2F_b'(2k)}{F_f''(\frac{d}{2}-k)}, \quad \hat{\gamma}_{\Phi,1} \equiv \frac{2F_S'(2k-1)}{F_S''(\frac{d}{2}-k)}. \tag{2.25}$$

As was described in [7], this leading correction to the vector anomalous dimension manifestly follows from an analogous $\tilde{F}$-extremization, despite the lack of supersymmetry in the first two cases. The σ field has scaling dimensions that cannot be determined by $\tilde{F}$-extremization,

$$\Delta_\sigma^{\Box^k} = 2k + \frac{\Gamma_{\rm tet} + \Gamma_{\rm pr} - 2\hat{\gamma}_{\phi,1}}{N},$$
$$\Delta_\sigma^{\partial\Box^{k-1/2}} = 2k + \frac{\Gamma_{\rm tet}^{\rm GN} + 0 - 2\hat{\gamma}_{\psi,1}}{N},$$
$$\Delta_\Sigma^{D^{2(k-1)}} = 2k - 1 + \frac{0 + 0 - 2\hat{\gamma}_{\Phi,1}}{N}, \tag{2.26}$$

which leads to free energies

[6]The supersymmetric long-range models do not appear to be known. However, by analogy with the O(N) and GN models, we can speculate that the long-range vector superconformal field theory (SCFT) is a SCFT of superfields with scaling dimensions $\Delta_\Phi = \frac{d-s}{2}$ and $\Delta_\Sigma = s - 1 + O(1/N^2)$. It should be reachable by RG flow from a UV free theory for suitable ranges of s and d: the scaling dimension of Φ should be unchanged from the UV, whereas that of Σ should be modified from a value less than $s-1$ to $\Delta_\Sigma = s - 1$.

$$
\begin{aligned}
F^{\Box^k}_{\mathrm{O}(N)} &= NF_b\left(\frac{d}{2}-k\right)+F_b(2k)\\
&\quad+\frac{F_b'(2k)}{N}\left(\frac{1}{2}\Gamma_{\mathrm{tet}}+\frac{1}{3}\Gamma_{\mathrm{pr}}-\hat{\gamma}_{\phi,1}\right),\\
F^{\bar{\sigma}\Box^{k-1/2}}_{\mathrm{GN}} &= NF_f\left(\frac{d}{2}-k\right)+F_b(2k)\\
&\quad+\frac{F_b'(2k)}{N}\left(\frac{1}{2}\Gamma^{\mathrm{GN}}_{\mathrm{tet}}+0-\hat{\gamma}_{\psi,1}\right),\\
F^{D^{2(k-1)}}_{\mathrm{SUSY}} &= NF_S\left(\frac{d}{2}-k\right)+F_S(2k-1)\\
&\quad+\frac{F_S'(2k-1)}{N}(0+0-\hat{\gamma}_{\Phi,1}),
\end{aligned}
\tag{2.27}
$$

where again at this order we need only evaluate the Γ's to leading order in $1/N$ (i.e., for integer s). Note also that these results follow from evaluating Eqs. (2.8) and (2.16b) for the short-range scaling dimensions.

E. Curious similarities

It was observed by Gracey [45] that the bosonic, fermionic, and supersymmetric computations of anomalous dimensions are all extremely similar, with two, one, and zero of the same diagrams contributing in each case. We highlight this here for the free energy also. The only slight complication is that we are considering a real bosonic model and a complex fermionic model. We must therefore take:

(i) $N_b = N$ and $N_f = 2N$, which counts the number of real degrees of freedom of the complex fermionic field. However, since ψ is complex, we also map $F_b \mapsto \frac{1}{2}F_f$. Thus, $NF_b \mapsto NF_f$.

(ii) $\tilde{A}_b = A$ and $\tilde{A}_f \equiv -iA_f$ [accounting for the $-i$ coming from the fermion's Fourier transform in (A40)].

Then the expressions for $\hat{\gamma}_{\phi/\psi,1}$ and Γ_{tet}/N become identical in the bosonic and fermionic cases, with $\gamma^{\mathrm{LR}}_{\phi/\psi}=0$ and

$$
\begin{aligned}
\gamma^{\mathrm{SR}}_{\phi/\psi} \supset \frac{\hat{\gamma}_{\phi/\psi,1}}{N} &= \frac{2F_b'(2k)}{NF_x''(\frac{d}{2}-k)},\\
\gamma_\sigma \supset \frac{\Gamma_{\mathrm{tet}}}{N} &= -\frac{4}{N_x\Gamma(d/2)}\frac{\tilde{A}_x(\frac{d-s}{2})^2}{A(d-s)}
\end{aligned}
\tag{2.28}
$$

where, as above, $x = b$ for the $\mathrm{O}(N)$ model and $x = f$ for the GN model. Indeed, we shall later explicitly see this correspondence for the $O(1/N^2)$ correction to $\gamma^{\mathrm{SR}}_{\phi/\psi}$, in (3.25). Of course, Γ_{pr} has no analog in the GN model, as the identity $\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho]=0$ (which holds for general $d\neq 3$ [43]) means that there is no three-point interaction term for σ (by regularity in dimension this holds also in $d=3$).[7] Thanks to our compact forms for F, we see that the free energies of these models are also similar.

Turning to the supersymmetric model, (2.27) shows that we can obtain both the γ_i's and F simply by promoting the $2F_b'(\Delta_\sigma^{\mathrm{UV}})/F_i''(\Delta_\phi^{\mathrm{UV}})$ in (2.25) to $2F_S'(\Delta_\Sigma^{\mathrm{UV}})/F_S''(\Delta_\Phi^{\mathrm{UV}})$ and setting both $\Gamma_{\mathrm{pr,tet}}=0$. We could now treat, for example, the Popović model [43,46]; the computation there will be analogous, save for a now fermionic auxiliary field, and so we stop here.

F. The short-range theories are the extrema of the long-range theories

Finally, we note that the short-range theories lie at the extrema of the long-range theories. That is, if we parametrize the long-range models by s, then the short-range models satisfy

$$
\left.\frac{\mathrm{d}\tilde{F}^{\mathrm{LR}}(s)}{\mathrm{d}s}\right|_{s=s_{\mathrm{SR}}}=0,
\tag{2.29}
$$

which determines

$$
s_{\mathrm{SR}} = 2k-2\hat{\gamma}
\tag{2.30}
$$

to order $1/N^2$—i.e., including the unwieldy expressions $\hat{\gamma}_{\phi/\psi,2}$ [which are given in (3.24) and (3.25)]. This provides a principle for picking out the short-range model; $\tilde{F}^{\mathrm{LR}}(s)$ also appears to be a minimal encoding of the $\hat{\gamma}$'s to this order, in the sense that the extrema of a relatively simple function encode the complex $\hat{\gamma}$'s. Calculating the second derivative in each case, we find

$$
\left.\frac{\mathrm{d}^2\tilde{F}^{\mathrm{LR}}_{\mathrm{O}(N)}(s)}{\mathrm{d}s^2}\right|_{s=s_{\mathrm{SR}}}=\frac{N}{4}\tilde{F}_b''\left(\frac{d}{2}-k\right)+O(N^0),
\tag{2.31}
$$

$$
\left.\frac{\mathrm{d}^2\tilde{F}^{\mathrm{LR}}_{\mathrm{GN}}(s)}{\mathrm{d}s^2}\right|_{s=s_{\mathrm{SR}}}=\frac{N}{4}\tilde{F}_f''\left(\frac{d}{2}-k\right)+O(N^0).
\tag{2.32}
$$

Therefore, in the bosonic theory in $d>2k$, we find a maximum for odd k and a minimum for even k; in the fermionic theory, we find a maximum for $k=1/2,5/2,\ldots$ and a minimum for $k=3/2,7/2,\ldots$.

III. MODEL SETUP AND CONFORMAL DATA

In this section, we construct the regulated action of the $\mathrm{O}(N)$ CFT. A regularization is required for the theory to be well defined; we choose to use an analytic regularization and give the values that the necessary counterterms take at the critical point (the CFT). We also provide the two-point function normalizations, which are useful for identifying the correspondence between the long- and short-range theories.

[7] As an aside: for the short-range versions in $d=3$ the parallel is even closer, because in the $\mathrm{O}(N)$ model $\Gamma_{\mathrm{pr}}|_{s=2,d=3}=0$.

A. Conformal propagators and their inverses

Consider the unit-normalized conformal propagators of fermions and bosons in flat space,

$$G_\phi(x) \equiv \frac{1}{|x|^{2\Delta_\phi}}, \qquad G_\psi(x) \equiv \frac{\not{x}}{|x|^{2\Delta_\psi+1}}. \tag{3.1}$$

Their inverses are straightforward: they are proportional to the shadow propagators, which are the conformal propagators for operators of scaling dimension $d-\Delta$. This is clear from inverting the matrix $G(x,y) \equiv G(x-y)$, which can be done in flat space by diagonalizing (via a Fourier transform), taking the reciprocal of that diagonal momentum-space correlator, and Fourier transforming back

$$\begin{aligned} G_\phi^{-1}(x) &= \frac{1}{\mathcal{N}_b(\Delta_\phi)} \frac{1}{x^{2(d-\Delta_\phi)}}, \\ G_\psi^{-1}(x) &= \frac{1}{\mathcal{N}_f(\Delta_\psi)} \frac{\not{x}}{x^{2(d-\Delta_\psi)+1}}, \end{aligned} \tag{3.2a}$$

where for bosons and fermions

$$\mathcal{N}_\phi \equiv \mathcal{F}_{2\Delta_\phi,0}\mathcal{F}_{2(d-\Delta_\phi),0} = \pi^d A(\Delta_\phi)A(d-\Delta_\phi), \tag{3.2b}$$

$$\mathcal{N}_\psi \equiv \mathcal{F}_{2\Delta_\psi,1}\mathcal{F}_{2(d-\Delta_\psi),1} = -\pi^d A_f(\Delta_\psi)A_f(d-\Delta_\psi), \tag{3.2c}$$

for the usual $\mathcal{F}_{2\Delta,s}$ defined in (A40). Comparing to Eqs. (2.4) and (2.13), it is evident that $F'_{b/f}(\Delta) = f(d)/\mathcal{N}_{b/f}(\Delta)$—we shall prove this in general in Sec. IV B. On replacing $|x-y| \to s(x,y)$, this inversion relation (3.2a) also holds on the sphere [39].

B. Constructing the ϕ^4 action

Consider the O(N)ϕ^4 action in d Euclidean dimensions on the sphere S^d. Leaving all fields and parameters bare for the moment, we write the partition function and action as

$$S_{\phi^4} = \int_x \frac{1}{2}\phi_i C^{-1}\phi_i + \frac{\lambda_0}{8N}(\phi_i\phi_i)^2, \quad Z_{\mathrm{O}(N)} = \int \mathrm{D}\phi e^{-S_{\phi^4}}, \tag{3.3}$$

where C^{-1} is the bare kinetic term and the index $i = 1,\ldots,N$. Taking, for example, the short-range vector model with $k=1$, the kinetic term of the conformally coupled scalar is

$$C^{-1} = -\nabla^2 + \frac{d-2}{4(d-1)}\mathcal{R}. \tag{3.4}$$

Here $\int_x$ indicates $\int \mathrm{d}^d x\sqrt{g}$, and the Ricci scalar curvature of the sphere is $\mathcal{R} = d(d-1)/R^2$. We use the sphere's stereographic coordinates, so $\mathrm{d}s^2 = \frac{4R^2}{(1+x^2)^2}dx^\mu dx^\mu$. Our conventions are reviewed in Appendix A. For arbitrary s, C^{-1} is the conformal Laplacian of biweight $(\Delta_\phi^{\mathrm{UV}}, d-\Delta_\phi^{\mathrm{UV}})$ on S^d, which we discuss in Appendix A 3. This ensures that, for $\lambda_0 = 0$, (3.3) describes N generalized free fields of conformal scaling dimension $\Delta_\phi^{\mathrm{UV}}$.

There is a quadratic mass term that should be here, which is always relevant, and which must be tuned away to reach the CFT. However, we work in a renormalization scheme (DREG + δ) where we can set the bare value of these parameters to zero. Indeed, every relevant interaction should be added: hence for the short-range $\Box^k$ CFTs every term $\phi\Box^{m<k}\phi$ must also be tuned—but in DREG their coefficients just must be tuned to zero. Also, in some discrete set of dimensions, some monomials become just renormalizable; if we work in generic dimension d this can be avoided, since the quantities that we calculate are regular in dimension [47]. At this order in $1/N$ it is also not necessary to consider any of the curvature couplings, unlike in the ϵ expansion [3,4]. The same comments apply in the case of the long-range models.

C. The action with the auxiliary field

As usual, we introduce a pair of real auxiliary fields. The first is σ, which is integrated along an imaginary contour; the second is $\rho(x)$, which we set to equal $\frac{1}{2\sqrt{N}}\phi_i(x)\phi_i(x)$ using a δ function implemented by σ,

$$1 = \int \mathrm{D}\rho\delta\left(\rho - \frac{1}{2\sqrt{N}}\phi_i\phi_i\right) = \int \mathrm{D}\rho\mathrm{D}^{(\mathrm{I})}\sigma e^{-\int_x \sigma(\rho - \frac{1}{2\sqrt{N}}\phi_i\phi_i)}. \tag{3.5}$$

This particular normalization of this bare ρ is chosen in order to ensure that eventually σ has an order N^0 two-point function. As reviewed in Appendix A 1, it is conventional to implement the δ function by defining σ with an integral running along the imaginary axis, rather than explicitly adding i's to the action: hence if the functional integral were discretized, it would be $\int \mathrm{D}^{(\mathrm{I})}\sigma = \prod_x \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\sigma(x)}{2\pi i}$. Placing this inside Z,

$$\begin{aligned} Z_{\mathrm{O}(N)} = \int \mathrm{D}\phi\mathrm{D}^{(\mathrm{I})}\sigma\mathrm{D}\rho \exp -\Bigg(\int_x \frac{1}{2}\phi_i C^{-1}\phi_i - \frac{1}{2\sqrt{N}}\sigma\phi_i\phi_i \\ + \frac{1}{2}\lambda_0\rho^2 + \sigma\rho\Bigg). \end{aligned} \tag{3.6}$$

Factoring $\frac{1}{2}\lambda_0(\rho + \sigma/\lambda_0)^2$, we can exactly integrate out the quadratic ρ, and so

$$\begin{aligned} Z_{\mathrm{O}(N)} = Z_\rho \int \mathrm{D}\phi\mathrm{D}^{(\mathrm{I})}\sigma \exp \\ -\left(\int_x \frac{1}{2}\phi_i C^{-1}\phi_i - \frac{1}{2\sqrt{N}}\sigma\phi_i\phi_i - \frac{\sigma^2}{2\lambda_0}\right), \end{aligned} \tag{3.7}$$

where we can drop $\log Z_\rho \equiv -\frac{1}{2}\mathrm{tr}\,\log(\lambda_0\delta(x-y)) = 0$ in DREG. Since $\lambda_0 > 0$, the potential for σ might appear to be upside down; however, σ runs along an imaginary contour here, and so the σ potential is indeed bounded below.

Because λ_0 is a relevant coupling, in the $k = 1$ theory for $2 < d < 4$ in the IR we expect the last term to drop out. That is, to reach the conformal IR we send the fixed (bare) $\lambda_0 \to \infty$, obtaining an IR action (which of course must be regulated)

$$Z_{\mathrm{O}(N)} = \int \mathrm{D}\phi\mathrm{D}^{(\mathrm{I})}\sigma \exp -\left(\int_x \frac{1}{2}\phi_i\left(C^{-1} - \frac{1}{\sqrt{N}}\sigma\right)\phi_i\right). \tag{3.8}$$

Analogous comments apply for higher-k theories.

D. Analytically regulating the auxiliary action

Without a regulator, this partition function (3.8) is divergent. If we were willing to use hard-cutoff renormalization, we could use it to work exactly at the critical point (see Ref. [10] for comments on this approach). However, doing the required loop integrals is not feasible with a hard cutoff, and so we seek an alternative analytic regulator. This procedure is briefly described in [47], but was originally motivated in [48]; see Ref. [49] and Sec. 4.3 in [50] for later presentations.

Because σ appears in only one place here, to leading order in N its scaling dimension is $\Delta_\sigma = d - 2\Delta_\phi^{\mathrm{UV}}$. Hence, the interaction is marginal in any d, and therefore, as described in the Introduction, analytically continuing in the dimension does not work. However, what if we could analytically shift the UV scaling dimension of σ to $s - \delta$, with $\delta \to 0$, while keeping that of ϕ at $\frac{d-s}{2}$. The bare scaling dimension of $\sigma\phi_i\phi_i$ would then be $d - \delta$ to leading order, so we could renormalize as usual, canceling the divergences at each order in $1/N$. In analogy to the textbook perturbative quantum field theory, the controlling parameter of the formal expansion, which is usually the coupling constant g is here $1/N$; the divergences encoded by $1/\epsilon \sim \log\Lambda$ poles are the $1/\delta$s. The analytic continuation in dimension ensures that power-law divergences can be ignored (i.e., implicitly canceled by some Lorentz-invariant local counterterm). We use a minimal subtraction (MS) scheme where we minimally subtract the $1/\delta$ poles that appear at each order in N.

Recall from (1.14) that the propagator for σ that is generated is $K(x,y)$, where

$$K^{-1}(x,y) \equiv -\frac{1}{2}\left(\frac{C_{\phi,0}}{s(x,y)^{2\Delta_\phi^{\mathrm{UV}}}}\right)^2. \tag{3.9}$$

The shift is then done by using a different propagator for σ in the Feynman rules, identical to K except for a small shift in $\Delta_\sigma^{\mathrm{UV}}$,

$$K_\delta(x,y) = \frac{C_{\sigma,0}}{s(x,y)^{2(s-\delta)}}. \tag{3.10}$$

This can only be implemented in practice by modifying the action: we add $\frac{1}{g_0^2}\frac{1}{2}\sigma K_\delta^{-1}\sigma$ and almost entirely subtract off the old kinetic term with a $-\frac{\mu^{2\delta}}{g_0^2}\frac{1}{2}\sigma K^{-1}\sigma$. However, the cancellation cannot be made perfect. If the cancellation were perfect, it would be possible to absorb the Z_g coupling by a redefinition of σ—but we need Z_g to cancel the divergences. This yields a residual additional two-point counterterm for σ that is $\propto \frac{1}{N\delta}K^{-1}$. Using DeWitt notation for compactness, the new action is

$$\begin{aligned} S_{\mathrm{temp}} = &\int_{x,y}\frac{Z_\phi}{2}\phi_x^i\left(C_{xy}^{-1} - \frac{\sigma_x\delta_{xy}}{\sqrt{N}}\right)\phi_y^i \\ &+ \frac{1}{g_0^2}\frac{1}{2}\int_{x,y}\sigma_x(K_{\delta,xy}^{-1} - \mu^{2\delta}K_{xy}^{-1})\sigma_y. \end{aligned} \tag{3.11}$$

As described in the Introduction, in standard QFT counterterms are a modification of physics at the microscopic scale. Evidently, in the limit $g_0 \to \infty$ we recover (3.8), the interacting CFT; thus the addition of these terms, an order $K_\delta^{-1} - \mu^{2\delta}K^{-1} = O(\delta)$ modification of the original Lagrangian, is also a modification of UV physics.

Rescaling $\sigma_{\mathrm{new}} = \sigma_{\mathrm{old}}/g_0$ to canonically normalize σ, the DREG + δ-regulated action which henceforth will be used for all computations is[8]

$$\begin{aligned} S_{\mathrm{O}(N),\delta} \equiv &\int_{x,y}\frac{Z_\phi}{2}\phi_x^i\left(C_{xy}^{-1} - \frac{Z_g g\mu^\delta}{\sqrt{N}}\sigma_x\delta_{xy}\right)\phi_y^i \\ &+ \frac{1}{2}\int_{x,y}\sigma_x(K_{\delta,xy}^{-1} - \mu^{2\delta}K_{xy}^{-1})\sigma_y. \end{aligned} \tag{3.13}$$

We make the following comments:

(i) We treat C^{-1} and K_δ^{-1} as the bare kinetic terms; the other terms are a three-point and two-point interaction respectively. The latter is slightly subtle, since insertions of K^{-1} are not suppressed by factors of $1/N$, but instead cancel with ϕ loops as we describe in Appendix B 6. We avoid this subtlety: in our calculation of the counterterms, by using the

[8] More conventionally, we would write $S_{\mathrm{O}(N),\delta} = S_\phi + S_{\mathrm{int}} + S_{\sigma,\delta}$ for

$$\begin{aligned} S_\phi &= \int_{x,y}\frac{Z_\phi}{2}\phi_i(x)C^{-1}(x,y)\phi_i(y), \\ S_{\mathrm{int}} &= -\int_x \frac{Z_\phi Z_g g\mu^\delta}{2\sqrt{N}}\sigma(x)\phi_i(x)\phi_i(x), \\ S_{\sigma,\delta} &= \int_{x,y}\frac{1}{2}\sigma(x)(K_\delta^{-1}(x,y) - \mu^{2\delta}K^{-1}(x,y))\sigma(y). \end{aligned} \tag{3.12}$$

two-particle irreducible (2PI) formalism; when calculating F, by integrating out ϕ entirely and working with the σ theory.

(ii) K_δ here implements the scaling dimension shift by δ, and $K = K_{\delta=0}$.

(iii) C_{xy}^{-1} and K_{xy}^{-1} are the inverses of C_{xy} and K_{xy} treated as matrices and are given explicitly in (B8).

(iv) We have written $g_0 = Z_g g \mu^\delta$, but set the dimensionless $g = 1$ henceforth. The counterterms Z_ϕ and Z_g have a perturbative expansion in $1/N$ around 1, given in (3.22) and (3.15) for the short- and long-range cases. We choose to use the MS scheme, so each order in N contains only poles in δ.

(v) We could instead nonminimally subtract; however, we are happy for the moment to deal with conformal fields ϕ and σ that do not have unit-normalized propagators and therefore only comment on this in Appendix B 5.

(vi) $Z_\phi \neq 1$ is only required in the short-range models, as nonlocal terms are not renormalized. This is because Wilsonian renormalization generates only correction terms that are polynomial in momenta—that is, local derivative couplings in position space, and hence a nonlocal term (specifically, bilocal) is unrenormalized [10].

(vii) For the same reason, K_δ^{-1} and K^{-1} do not need counterterms in generic d.

(viii) Taking the limit $\delta \to 0$ corresponds to removing the regulator.

Our desired partition function is then

$$Z_{\mathrm{O}(N)} \equiv e^{-F_{\mathrm{O}(N)}} \equiv \lim_{\delta\to 0} \int \mathrm{D}\phi \mathrm{D}^{(\mathrm{I})}\sigma\, e^{-S_{\mathrm{O}(N),\delta}}. \tag{3.14}$$

Since we want to work exactly at the critical point, we must tune Z_ϕ and Z_g as functions of δ. If we wanted to work away from the critical point, we would need to introduce coefficients in front of both K terms [51] [see also [49] and Eq. (4.106) in [50]], making the action multiplicatively renormalizable—but we do not want to. The critical values of our counterterms must be determined by a standard RG analysis; though standard, we perform it in Appendix B 3, as inconsistent regulators have been used in the literature. We take care to use the same shift by δ in all diagrams; we also choose to use the 2PI formalism, meaning that we need to consider fewer diagrams.

E. General s conformal data

The counterterms at the infrared critical point for generic s are

$$Z_\phi = 1, \qquad Z_g = 1 + \frac{Z_{g,1}}{N\delta} + \mathrm{O}(1/N^2), \qquad Z_{g,1} \equiv -\frac{1}{2}\left(\Gamma_{\mathrm{tet}} + \frac{1}{2}\Gamma_{\mathrm{pr}}\right). \tag{3.15}$$

With these, we can safely take the limit $\delta \to 0$. Then, to the N-orders shown, the two-point functions are

$$\begin{aligned}
\langle\sigma(x)\sigma(y)\rangle &= \frac{C_\sigma}{s(x,y)^{2\Delta_\sigma}}, \qquad \Delta_\sigma = s + \gamma_\sigma, \qquad \gamma_\sigma = \frac{\Gamma_{\mathrm{tet}} + \Gamma_{\mathrm{pr}}}{N}, \qquad C_\sigma = C_{\sigma,0}\left(1 + \frac{A_{\sigma,1}}{N}\right), \\
\langle\phi_i(x)\phi_j(y)\rangle &= \frac{C_\phi \delta_{ij}}{s(x,y)^{2\Delta_\phi}}, \qquad \Delta_\phi = \frac{d-s}{2}, \qquad C_\phi = C_{\phi,0}\left(1 + \frac{A_{\phi,1}}{N} + \frac{A_{\phi,2}}{N^2}\right),
\end{aligned} \tag{3.16}$$

where in the long-range model the scaling dimension of Δ_ϕ is exact to all orders in N. The lack of anomalous dimension here follows from the nonrenormalization of the nonlocal kinetic term mentioned above and was proven rigorously by [24].

With $A(x) \equiv \Gamma(\frac{d}{2} - x)/\Gamma(x)$ as above, the leading contributions to γ_σ are

$$\Gamma_{\mathrm{tet}} \equiv -\frac{4}{\Gamma(\frac{d}{2})}\frac{A(\frac{d-s}{2})^2}{A(d-s)}, \qquad \Gamma_{\mathrm{pr}} \equiv \frac{8}{\Gamma(\frac{d}{2})}A(s)A\left(d - \frac{3s}{2}\right)\frac{A(\frac{d-s}{2})^3}{A(d-s)^2}. \tag{3.17}$$

The leading normalizations are

$$C_{\phi,0} = \frac{\pi^{-\frac{d}{2}}2^{-s}}{A(\frac{d-s}{2})}, \qquad C_{\sigma,0} = \frac{1}{(C_{\phi,0})^2}\frac{-2}{\mathcal{N}_b(s)} = -2\frac{4^s A(\frac{d-s}{2})^2}{A(d-s)A(s)}. \tag{3.18}$$

The subleading normalization for σ is

$$
\begin{aligned}
A_{\sigma,1} &= A_{\sigma,1,\text{tet}} + A_{\sigma,1,\text{pr}} + A_{\sigma,1,\text{pil}},\\
A_{\sigma,1,\text{tet}} &= \Gamma_{\text{tet}}\left(B\left(\frac{d-s}{2}\right) - B(d-s)\right),\\
A_{\sigma,1,\text{pr}} &= \Gamma_{\text{pr}}\left(\frac{3}{2}B\left(\frac{d-s}{2}\right) + \frac{1}{2}B\left(d-\frac{3s}{2}\right) - B(d-s) - B(s)\right),\\
A_{\sigma,1,\text{pil}} &= \frac{4F_b'(s)}{F_b'(\frac{d-s}{2})} = \frac{4\mathcal{N}_b(\frac{d-s}{2})}{\mathcal{N}_b(s)} = \frac{4A(\frac{d-s}{2})A(d-\frac{d-s}{2})}{A(s)A(d-s)},
\end{aligned}
\tag{3.19}
$$

where we have defined $B(x) \equiv -\frac{\mathrm{d}}{\mathrm{d}x}\log A(x)$. For ϕ we can go one order further,

$$
\begin{aligned}
A_{\phi,1} &= -\frac{1}{2}A_{\sigma,1,\text{pil}},\\
A_{\phi,2} &= \frac{-F_b'(s)}{F_b'(\frac{d-s}{2})}\left[(\Gamma_{\text{pr}}+\Gamma_{\text{tet}})\left(B\left(\frac{d-s}{2}\right) - 2B(d-s)\right) + \Gamma_{\text{tet}}B(s) + \Gamma_{\text{pr}}B\left(d-\frac{3s}{2}\right)\right].
\end{aligned}
\tag{3.20a}
$$

This also makes it apparent that for $\Delta_\phi = \frac{d-s}{2}$,

$$
\frac{\mathrm{d}F^{\text{LR}}_{\text{O}(N)}}{\mathrm{d}\Delta_\phi} = NF_b'(\Delta_\phi)\frac{C_\phi}{C_{\phi,0}} = NF_b'(\Delta_\phi)\left(1 + \frac{A_{\phi,1}}{N} + \frac{A_{\phi,2}}{N^2} + \cdots\right).
\tag{3.20b}
$$

Combined with the observation of F-extremization, this makes it clear that the norm of the $\phi\phi$ correlator C_ϕ vanishes in the limit $s \to s_{\text{SR}}$, assuming the long-range normalization of ϕ. This was noted in [26] [around (5.11), Sec. B] (see also Sec. II.B in [52] for earlier commentary). The same equation holds for the Gross-Neveu model with $\Delta_\psi = \frac{d-s}{2}$

$$
\frac{\mathrm{d}F^{\text{LR}}_{\text{GN}}}{\mathrm{d}\Delta_\psi} = NF_f'(\Delta_\psi)\frac{C_\psi}{C_{\psi,0}} = NF_f'(\Delta_\psi)\left(1 + \frac{A_{\psi,1}}{N} + \frac{A_{\psi,2}}{N^2} + \cdots\right).
\tag{3.20c}
$$

These identities will be explained in [8].

F. The short-range models

It is clear that if we attempt to take the bare scaling dimension $s \to 2k$, all of $A_{\sigma,1,\text{pil}}$, $A_{\phi,1}$, and $A_{\phi,2}$ diverge because

$$
\lim_{s\to 2k}\frac{1}{F_b'(\frac{d-s}{2})} \sim \Gamma(-k)\Gamma(k) = \infty
\tag{3.21}
$$

for all integer k. This comes from the fact that the limits $\delta \to 0$ and $s \to 2k$ do not commute; we cannot take $\delta \to 0$ for arbitrary s and then take $s \to 2k$. Thus, to directly reach the short-range models, $s = 2k$ must be fixed first, and only then may we take the regulator to zero.

To keep the short-range model's two-point functions finite as $\delta \to 0$, we are forced to modify Z_ϕ,

$$
\begin{aligned}
Z_\phi &= 1 - \frac{\hat{\gamma}_{\phi,1}}{N\delta} + O(1/N^2)\\
Z_g &= 1 - \frac{1}{2N\delta}\left(\Gamma_{\text{tet}} + \frac{1}{2}\Gamma_{\text{pr}} - 2\hat{\gamma}_{\phi,1}\right)_{s=2k} + O(1/N^2).
\end{aligned}
\tag{3.22}
$$

Note that the product $Z_\phi Z_g = 1 - \frac{1}{2N\delta}(\Gamma_{\text{tet}} + \frac{1}{2}\Gamma_{\text{pr}})_{s=2k}$ is identical to the general s case; that is, we only need to modify the coefficient of the kinetic term of ϕ in the action.

Then the IR scaling dimensions are

$$
\begin{aligned}
\Delta_\phi &= \frac{d}{2} - k + \frac{\hat{\gamma}_{\phi,1}}{N} + \frac{\hat{\gamma}_{\phi,2}}{N^2} + O(1/N^3),\\
\Delta_\sigma &= 2k + \frac{1}{N}(\Gamma_{\text{tet}} + \Gamma_{\text{pr}} - 2\hat{\gamma}_{\phi,1})_{s=2k} + O(1/N^2),
\end{aligned}
\tag{3.23}
$$

where, as usual, $\hat{\gamma}_{\phi,1} = \frac{2F_b'(2k)}{F_b''(\frac{d}{2}-k)}$, and the $O(1/N^2)$ term is

$$
\begin{aligned}
\hat{\gamma}_{\phi,2} = \frac{\hat{\gamma}_{\phi,1}}{2}&\left[(\Gamma_{\text{tet}}+\Gamma_{\text{pr}})\left(B\left(\frac{d}{2}-k\right) - 2B(d-2k)\right)\right.\\
&\left. + \Gamma_{\text{tet}}B(2k) + \Gamma_{\text{pr}}B(d-3k)\right]\\
&- \hat{\gamma}_{\phi,1}^2 \underbrace{\left(\frac{F_b'''(\frac{d}{2}-k)}{2F_b''(\frac{d}{2}-k)} + \frac{2F_b''(2k)}{F_b'(2k)}\right)}_{(\psi^{(0)}(\frac{d}{2}-k)-\psi^{(0)}(\frac{d}{2}+k)-\frac{1}{k})+2(B(2k)-B(d-2k))}.
\end{aligned}
\tag{3.24}
$$

The remaining normalization data are given in Appendix B 4. Clearly the first term here is

$\lim_{\delta\to 0}(\delta A_{\phi,2}|_{s=2k-\delta})$, and the second term is identical to that which follows from $\tilde{F}$-extremization [compare the supersymmetric result (2.23)].[9] As discussed in the Introduction, Δ_σ here indeed coincides at this order with the scaling dimensions of the long-range model (3.16) evaluated for $s_{\rm SR}=2-2\hat{\gamma}_\phi$. Further, this value of $\hat{\gamma}_\phi$, including the $O(1/N^2)$ correction, does indeed extremize $\tilde{F}^{\rm LR}_{{\rm O}(N)}$.

For the Gross-Neveu model, following the recipe described in Sec. II E, we map $\Gamma_{\rm pr}\to 0$, $\Gamma_{\rm tet}\to\Gamma^{\rm GN}_{\rm tet}$, and $F_b(\frac{d}{2}-k)\mapsto\frac{1}{2}F_f(\frac{d}{2}-k)$. Then, we map $B(\frac{d}{2}-k)\mapsto B_f(\frac{d}{2}-k)$, where $B_f(x)\equiv-\frac{\rm d}{{\rm d}x}\log A_f(x)$, and all other B's stay the same. This yields the correct result [54]

$$\hat{\gamma}_{\psi,2}=\frac{\hat{\gamma}_{\psi,1}}{2}\Gamma^{\rm GN}_{\rm tet}\left(B(2k)+B_f\left(\frac{d}{2}-k\right)-2B(d-2k)\right)-\hat{\gamma}^2_{\psi,1}\left(\frac{F'''_f(\frac{d}{2}-k)}{2F''_f(\frac{d}{2}-k)}+\frac{2F''_b(2k)}{F'_b(2k)}\right). \tag{3.25}$$

IV. FREE ENERGY CALCULATION

We have correctly regulated this partition function; let us now integrate it. The computation for the Gross-Neveu model exactly parallels this computation, so we only provide some comments in Appendix D 6.

A. Integrating out ϕ

Performing the Gaussian integral (A1) of ϕ_i in (3.13), we find a

$$\frac{1}{2}{\rm Tr}_{x,i}\log\left(\delta_{ij}C^{-1}(x,y)-\frac{g_0}{\sqrt{N}}\delta_{ij}\sigma(x)\delta(x-y)\right)=\frac{N}{2}{\rm tr}\,\log\left(C^{-1}-\frac{g_0}{\sqrt{N}}\sigma\mathbb{I}\right) \tag{4.1}$$

appearing in the effective action—the ${\rm Tr}_{x,i}$ here is over space and the O(N) index, whereas all other tr's are only over space. If we now define

$$Z^N_{\phi,\rm free}\equiv\exp\left(-\frac{N}{2}{\rm tr}\,\log C^{-1}\right)=\exp\left(-NF_b\left(\frac{d-s}{2}\right)\right), \tag{4.2}$$

which we discuss in the next section, the tr $\log C^{-1}$ can be factored out

$$\frac{Z_{{\rm O}(N),\delta}}{Z^N_{\phi,\rm free}}=\int{\rm D}^{(\rm I)}\sigma\exp-\left(\frac{N}{2}{\rm tr}\,\log\left[\mathbb{I}-\frac{g_0}{\sqrt{N}}M\right]+\frac{1}{2}\int_{x,y}\sigma_x(K^{-1}_{\delta,xy}-\mu^{2\delta}K^{-1}_{xy})\sigma_y\right)$$
$$=\int{\rm D}^{(\rm I)}\sigma\exp-\left(\sum_{q=1}^{\infty}-\frac{g_0^q}{2q}\frac{{\rm tr}\,M^q}{N^{q/2-1}}+\frac{1}{2}\int_{x,y}\sigma_x(K^{-1}_{\delta,xy}-\mu^{2\delta}K^{-1}_{xy})\sigma_y\right). \tag{4.3a}$$

We again use the matrix $M(x,y)=\sigma(x)C(x,y)$ to simplify notation. Note that the value of the counterterm Z_ϕ does not impact the universal part of the free energy, as it disappeared when ϕ was integrated out.

The $q=1$ term $\int_x\sigma(x)C(x,x)$ ruins conformality if it is nonzero [47]. However, by assumption, we are working at the critical point, and therefore must tune the relevant interaction $+\frac{1}{2}m_0^2\phi_i\phi_i$. Adding such an interaction to the action (3.13) produces a σ tadpole term that we can use to cancel the $q=1$ term. Happily, in DREG this is unnecessary as $C(x,x)=0$ [39], and so the σ tadpole automatically vanishes: ${\rm tr}\,M=0$, and so we can just set $m_0=0$.

Recalling that $K^{-1}(x,y)=-\frac{1}{2}C(x,y)^2$, we see that $\exp-(\frac{g_0^2}{2}\int\sigma K^{-1}\sigma)$ is the first nonzero term coming from the sum. This only partially cancels the additional K^{-1} from the regulator, since the critical $g_0^2=\mu^{2\delta}(1+\frac{2Z_{g,1}}{N\delta})$. Thus,

$$=\int{\rm D}^{(\rm I)}\sigma\exp-\left(\frac{1}{2}\int_{x,y}\sigma_x(K^{-1}_{\delta,xy}-(\mu^{2\delta}-g_0^2)K^{-1}_{xy})\sigma_y-\frac{g_0^3}{6\sqrt{N}}{\rm tr}\,M^3-\frac{g_0^4}{8N}{\rm tr}\,M^4+\cdots\right). \tag{4.3b}$$

Dropping the subleading terms in N, we find a second quadratic integral, and so

$$F_{{\rm O}(N)}=NF_b\left(\frac{d-s}{2}\right)+F_b(s-\delta)+O(1/N), \tag{4.4}$$

where we have used $\int{\rm D}^{(\rm I)}\sigma e^{-\frac{1}{2}\int\sigma K_\delta^{-1}\sigma}=\frac{1}{2}{\rm tr}\log(-K_\delta^{-1})=F_b(s-\delta)$ which, as discussed in Appendix A 1 a, holds when evaluated in DREG. We now calculate $\frac{1}{2}{\rm tr}\log C^{-1}$ in full generality.

[9] Reference [53] described this term as the contribution of the Hartree-Fock diagrams.

B. The free energy of a free conformal field of dimension Δ

We now give a straightforward computation of (the universal part of) the sphere free energy of a free bosonic/fermionic conformal field Φ of dimension Δ_Φ in an arbitrary representation. This is a standard result from the AdS/CFT literature for any Lorentz representation of Φ [7,39,55–57]; we provide here a very simple alternative derivation, which demonstrates the relationship of F to the inversion $\mathcal{N}$'s defined in (3.2a). We rely on the trivial result that for any matrix $M(\Delta)$

$$\frac{\mathrm{d}}{\mathrm{d}\Delta}\mathrm{tr}\,\log M(\Delta) = \mathrm{tr}\,M^{-1}(\Delta)\frac{\mathrm{d}}{\mathrm{d}\Delta}M(\Delta) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\mathrm{tr}\,M^{-1}(\Delta)M(\alpha)\Big|_{\alpha=\Delta} \tag{4.5}$$

which we can use if working in the range of d where everything converges.

Without loss of generality, we can consider a real conformal generalized free field Φ of arbitrary dimension Δ_Φ and representation ρ, which has a propagator G_Φ (suppressing all indices and ignoring the presence of ghosts, which may be required for some representations). Then it has a kinetic term G_Φ^{-1}, which is written down properly, including contact terms, for scalars in Appendix A 3. Performing the functional integral,

$$F_\Phi \equiv -\log\int \mathrm{D}^{(\mathrm{I})}\Phi e^{(-1)^{\mathrm{F}_\Phi}\frac{1}{2}\int_{x,y}\Phi_x G^{-1}_{\Phi,xy}\Phi_y} = \frac{1}{2}\mathrm{str}\,\log G_\Phi^{-1} = (-1)^{\mathrm{F}_\Phi}\frac{1}{2}\mathrm{tr}\,\log G_\Phi^{-1}, \tag{4.6}$$

where $(-1)^{\mathrm{F}_\Phi}$ just gives a minus sign for fermionic reps. The trace here is over space, Lorentz indices, and symmetry group indices. The normalization of this propagator does not contribute to the universal part, since $\mathrm{tr}\,\delta(x-y)=0$ in DREG—so we can take G_Φ to be unit normalized.

First, let us elaborate on the propagator inverse relation (3.2a). At separated points, it is generally true that the inverse of a conformal propagator is proportional to the "shadow propagator," which is the propagator of the field $\tilde{\Phi}$ with scaling dimension $\tilde{\Delta}_\Phi \equiv d-\Delta_\Phi$, transforming in the "reflected representation" ρ^{ref}. This is because the inverse propagator and the shadow field's propagator transform identically under conformal symmetry. Since both are unique, we must have

$$G_\Phi^{-1}(x,y) = \frac{1}{\mathcal{N}_\Phi}G_{\tilde{\Phi}}(x,y), \tag{4.7}$$

for some $\mathcal{N}_\Phi(\Delta_\Phi,\rho)$, up to contact terms that vanish for $x\neq y$ [given in, e.g., (A29)]. Because the G's are unit normalized, $\mathcal{N}_\Phi$ must be a purely representation-theoretical quantity,[10] identical for flat space and the sphere, which can be calculated explicitly by taking the inverse in momentum space [we gave the values for bosons and fermions in (3.2b) and (3.2c)].

1. Computing the trace

Differentiating and using (4.5) we find

$$F'_\Phi = \frac{\mathrm{d}F_\Phi}{\mathrm{d}\Delta_\Phi} = -(-1)^{\mathrm{F}_\Phi}\frac{1}{2}\mathrm{tr}\,G_\Phi^{-1}\frac{\mathrm{d}G_\Phi}{\mathrm{d}\Delta_\Phi} = -(-1)^{\mathrm{F}_\Phi}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\alpha}(\mathrm{tr}\,G_\Phi^{-1}G_{\Phi,\Delta_\Phi=\alpha})\Big|_{\alpha=\Delta_\Phi}, \tag{4.9}$$

where we clearly have repeatedly swapped the order of derivatives and integrals, assuming everything to converge. As mentioned above, we therefore must implicitly work in the range of d where this trace converges as the UV cutoff is taken to infinity, and then analytically continue back up: this is just DREG. Explicitly writing out the position-space part of the trace,

$$F'_\Phi = -\frac{1}{2}\frac{1}{\mathcal{N}_\Phi}\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{x,y}\mathrm{tr}_\rho\Big[G_\Phi^{-1}(x,y)\underbrace{(-1)^{\mathrm{F}_\Phi}G_{\Phi,\alpha}(y,x)}_{G_{\Phi,\alpha}(x,y)}\Big]\Big|_{\alpha=\Delta_\Phi} \tag{4.10}$$

$$= -\frac{1}{2}\frac{\mathrm{tr}_\rho(\mathbb{I}_\rho)}{\mathcal{N}_\Phi}\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{x,y}\frac{1}{s(x,y)^{2(d-\Delta_\Phi+\alpha)}}\Big|_{\alpha=\Delta_\Phi}, \tag{4.11}$$

where we have used the known behavior of conformal propagators under $x\leftrightarrow y$, and we can drop the contributions of contact terms due to $G(x,x)=0$ and $\mathrm{tr}\,\delta(x-y)=0$. By definition of the inverse, we do not need to consider the details of the suppressed Lorentz/symmetry indices, being guaranteed to end up with only the identity $\mathbb{I}_\rho$ in the representation ρ. The trace of $\mathbb{I}_\rho$ is simply the (real, by assumption) dimension of ρ, so we are left with the standard sphere integral

$$I_2(\Delta) \overset{(\mathrm{D.10})}{\equiv} \int_{x,y}\frac{1}{s(x,y)^{2\Delta}} = \frac{\pi^d}{(2R)^{2(\Delta-d)}}\frac{\Gamma(\frac{d}{2})}{\Gamma(d)}\frac{\Gamma(\frac{d}{2}-\Delta)}{\Gamma(d-\Delta)}, \tag{4.12}$$

[10] For a general representation, $\mathcal{N}_\Phi(\Delta_\Phi,\rho)$ can also be found using [58]

$$\frac{\dim_{\mathbb{R}}(\rho)}{\mathcal{N}_\Phi} = (-1)^{\mathrm{F}_\Phi}\Omega_d\mu(\Phi), \tag{4.8}$$

where $\Omega_d = 2^d\mathrm{volSO}(d)$ is a constant that drops out of all computations, and $\mu(\Phi)$ is the Plancherel measure of $\mathrm{SO}(d+1,1)$ for Φ's conformal representation $(-\Delta_\Phi,\rho)$.

i.e.,

$$F'_\Phi = -\frac{1}{2}\frac{\dim_{\mathbb{R}}(\rho)}{\mathcal{N}_\Phi}\frac{\mathrm{d}}{\mathrm{d}\alpha} I_2(d-\Delta_\Phi+\alpha)\Big|_{\alpha=\Delta_\Phi} = -\frac{1}{2}\frac{\dim_{\mathbb{R}}(\rho)}{\mathcal{N}_\Phi} I'_2(d). \tag{4.13}$$

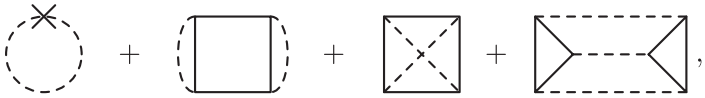
We now assume that $F_\Phi(\Delta_\Phi = \frac{d}{2}) = 0$, which follows immediately from the identity in DREG $\mathrm{tr}\log G_\Phi^{-1} = -\mathrm{tr}\log G_\Phi$. This yields

$$F_\Phi(\Delta_\Phi,\rho) = \frac{-1}{\sin(\frac{\pi d}{2})}\frac{\pi^{d+1}}{\Gamma(d+1)}\int_{\frac{d}{2}}^{\Delta_\Phi}\mathrm{d}\Delta'\frac{\dim_{\mathbb{R}}(\rho)}{\mathcal{N}_\Phi(\Delta',\rho)} \tag{4.14}$$

for any conformal generalized free field.

C. The free energy corrections

The corrections here are, as usual, minus the sum of the connected vacuum diagrams. The N-scaling that we have chosen for σ above makes it easy to see what diagrams contribute. Ignoring self-loops, which vanish, they are

$-\delta F_{\mathrm{O}(N),1/N} \equiv$  , (4.15)

$I_{\mathrm{ct}} : \frac{1}{2}$ $I_{\mathrm{pil}} : \frac{1}{4}$ $I_{\mathrm{tet}} : \frac{1}{8}$ $I_{\mathrm{pr}} : \frac{1}{12}$

which we term the counterterm diagram, the pillow diagram, the tetrahedron diagram, and the prism diagram respectively. The cross indicates an insertion of $-\mu^{2\delta}\frac{2Z_{g,1}}{N\delta}K^{-1}$, thus making the diagram order $1/N$. We would have also considered the diagram , with symmetry factor $1/8$, but it is automatically zero even for finite δ (and even with additional infinitesimal shifts of the propagators), something which can be checked using its Mellin-Barnes representation.

1. The noncounterterm contributions

The diagrams, evaluated with general conformal propagators, are

$$I_{\mathrm{pil}} = \frac{1}{N}\frac{1}{4}\int_{x_{1,2,3,4}}\frac{C_\sigma^2}{(s_{12}s_{34})^{2\Delta_\sigma}}\frac{C_\phi^4}{(s_{12}s_{23}s_{34}s_{41})^{2\Delta_\phi}}, \tag{4.16a}$$

$$I_{\mathrm{tet}} = \frac{1}{N}\frac{1}{8}\int_{x_{1,2,3,4}}\frac{C_\sigma^2}{(s_{13}s_{24})^{2\Delta_\sigma}}\frac{C_\phi^4}{(s_{12}s_{23}s_{34}s_{41})^{2\Delta_\phi}}, \tag{4.16b}$$

$$I_{\mathrm{pr}} = \frac{1}{N}\frac{1}{12}\int_{x_{1\cdots6}}\frac{C_\phi^3}{(s_{12}s_{23}s_{31})^{2\Delta_\phi}}\frac{C_\sigma^3}{(s_{14}s_{25}s_{36})^{2\Delta_\sigma}}\frac{C_\phi^3}{(s_{45}s_{56}s_{64})^{2\Delta_\phi}}, \tag{4.16c}$$

where $s_{ij} = s(x_i,x_j)$ is the sphere chordal distance; we leave the coefficients and scaling dimensions C_i and Δ_i general for the moment. These diagrams are computed with MBRESOLVE, following exactly the procedure described in [3], which we comment on in Appendix D 2. In particular, we take advantage of the sphere's symmetries to fix one of the integrated points. This simplifies the computation, and also, somewhat surprisingly, means that no further regulators (analogous to the η shifts in Appendix B 3) are required.

The shift of $\Delta_\sigma^{\mathrm{UV}}$ to $s-\delta$ ensures that $\Delta_\sigma + 2\Delta_\phi = d - \delta \neq d$ and also means that the Mellin-Barnes representations of the diagrams are finite. Defining a small parameter $\varkappa$,

$$\varkappa \equiv d - \Delta_\sigma - 2\Delta_\phi \ll 1, \tag{4.17}$$

that then allows a compact presentation of the results for arbitrary Δ_ϕ. Performing the Mellin-Barnes integrals, and expanding for small $\varkappa$ to linear order, we find expressions of the form $\prod_i \Gamma(\alpha_i)^{p_i} \times (1 + \sum_i c_i p_i x \psi^{(0)}(\alpha_i))$ for some p_i's and c_i's. Since in general

$$\prod_i \Gamma(\alpha_i)^{p_i}\left(1 + \sum_i c_i p_i \varkappa \psi^{(0)}(\alpha_i) + O(\varkappa^2)\right) = \prod_i \Gamma(\alpha_i + c_i\varkappa)^{p_i} + O(\varkappa^2), \tag{4.18}$$

we can write the evaluations of (4.16) compactly (albeit nonuniquely),

$$\frac{I_{\mathrm{tet}}}{C_\sigma^2 C_\phi^4} = \frac{3}{8N}\frac{2\pi^{2d}}{\Gamma(d)} e^{2\gamma\varkappa}(2R)^{4\varkappa}\Gamma\left(2\varkappa - \frac{d}{2}\right) A\left(\Delta_\sigma + \frac{\varkappa}{3}\right) A\left(\Delta_\phi + \frac{\varkappa}{3}\right)^2 + O(\varkappa^2); \tag{4.19a}$$

$$\frac{I_{\mathrm{pr}}}{C_\sigma^3 C_\phi^6} = \frac{\frac{5}{2}}{12N}\frac{2\pi^{3d}}{\Gamma(d)} e^{3\gamma\varkappa}(2R)^{6\varkappa}\Gamma\left(3\varkappa - \frac{d}{2}\right) A\left(\Delta_\sigma + \frac{\varkappa}{5}\right)^3 A\left(\Delta_\phi + \frac{3\varkappa}{5}\right)^3 A\left(-\frac{d}{2} + 3\Delta_\phi + \frac{3\varkappa}{5}\right) + O(\varkappa^2). \tag{4.19b}$$

We need to evaluate these diagrams at $\varkappa = \delta$ and for $C_i = C_{i,0}$; however, these expressions are valid for any small shift of these scaling dimensions.

We can solve the pillow exactly in terms of hypergeometric functions, and a complete expression is presented in (D19). The expression for the pillow is $O(\delta)$ for generic s, but $O(\delta^0)$ for $s = 2k$. This reflects the fact that the pillow only contributes to the anomalous dimensions in the short-range models.

$$\frac{I_{\text{pil}}}{C_\sigma^2 C_\phi^4} = \begin{cases} \frac{1}{4N}\frac{2\pi^{2d}\Gamma(\frac{d}{2})\Gamma(-\frac{d}{2})}{\Gamma(d)}\frac{\Gamma(\frac{s}{2})\Gamma(-\frac{s}{2})}{\Gamma(\frac{d-s}{2})\Gamma(\frac{d+s}{2})}\varkappa + O(\varkappa^2) & s \neq 2k \\ \frac{12}{4N}\frac{\pi^{2d}}{\Gamma(d)}\frac{(-1)^k}{4k}\frac{\Gamma(-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2}-k)\Gamma(\frac{d}{2}+k)} + O(\varkappa) & s = 2k. \end{cases} \tag{4.20}$$

We now fix to

$$\Delta_\phi \to \frac{d-s}{2}, \qquad \Delta_\sigma \to s - \delta, \tag{4.21}$$

and therefore set

$$\varkappa = \delta, C_\sigma C_\phi^2 \to C_{\sigma,0}(C_{\phi,0})^2 \overset{(3.18)}{=} -2/\mathcal{N}_b(s), \tag{4.22}$$

which makes

$$I_{\text{tet}} = -3 \times \frac{1}{2N} F_b'(s)\Gamma_{\text{tet}} + O(\delta),$$
$$I_{\text{pr}} = -\frac{5}{2} \times \frac{1}{3N} F_b'(s)\Gamma_{\text{pr}} + O(\delta), \tag{4.23a}$$

and

$$I_{\text{pil}} = \begin{cases} O(\delta) & s \neq 2k \\ 3 \times \frac{1}{N} F_b'(2k)\hat{\gamma}_{\phi,1} + O(\delta) & s = 2k. \end{cases} \tag{4.23b}$$

2. The counterterm contribution

The counterterm diagram is

$$I_{\text{ct}} = \frac{1}{2}\left(-\mu^{2\delta}\frac{2Z_{g,1}}{N\delta}\right)\text{tr}\, K^{-1}K_\delta \tag{4.24}$$

which can be computed using $I_2(\Delta)$,

$$\text{tr}\, K^{-1}K_\delta = \frac{1}{\mathcal{N}_s}\int_{x,y}\frac{1}{s(x,y)^{2((d-s)+s-\delta)}} = \frac{1}{\mathcal{N}_s}I_2(d-\delta). \tag{4.25}$$

Taking the limit $\delta \to 0$, we find a finite correction from the counterterm for any s,[11]

$$F_{\text{O}(N)} \supset -I_{\text{ct}} = \frac{Z_{g,1}}{N}\frac{1}{\mathcal{N}_s}\frac{2}{-\sin(\frac{\pi d}{2})}\frac{\pi^{d+1}}{\Gamma(d+1)} + O(\delta) = \frac{2Z_{g,1}}{N}F_b'(s) + O(\delta). \tag{4.27}$$

3. Assembling F

Recalling that Z_g differs in each of the short- and long-range cases, we see that the extra factors of 3 and $5/2$ that appeared in the integral evaluations cancel with the contributions from $Z_{g,1}$ in the $1/N$ correction to F. Writing the counterterm contributions on a separate row, we find

$$F_{\text{O}(N),1/N} = \begin{matrix} -I_{\text{tet}} - I_{\text{pr}} - I_{\text{pil}} \\ -I_{\text{ct}} \end{matrix}$$
$$\frac{F_{\text{O}(N),1/N}}{F_b'(s)/N} = \begin{matrix} +3 \times \frac{1}{2}\Gamma_{\text{tet}} & +\frac{5}{2} \times \frac{1}{3}\Gamma_{\text{pr}} \\ -2 \times \frac{1}{2}\Gamma_{\text{tet}} & -\frac{3}{2} \times \frac{1}{3}\Gamma_{\text{pr}} \end{matrix} + \begin{cases} 0 & s \neq 2k \\ +3 \times \frac{1}{2}(-2\hat{\gamma}_{\phi,1}) & s = 2k, \\ -2 \times \frac{1}{2}(-2\hat{\gamma}_{\phi,1}) & \end{cases} \tag{4.28}$$

where we have evaluated $Z_{g,1}$ using (3.22) for SR and (3.15) for LR. Thus, we obtain

$$F_{\text{O}(N)}^{\text{SR},\square^k} = NF_b\left(\frac{d}{2}-k\right) + F_b(2k) + \frac{F_b'(2k)}{N}\left(\frac{1}{2}(-2\hat{\gamma}_{\phi,1}) + \frac{1}{2}\Gamma_{\text{tet}} + \frac{1}{3}\Gamma_{\text{pr}}\right)_{s=2k}, \tag{4.29a}$$

$$F_{\text{O}(N)}^{\text{LR}} = NF_b\left(\frac{d-s}{2}\right) + F_b(s) + \frac{F_b'(s)}{N}\left(\frac{1}{2}\Gamma_{\text{tet}} + \frac{1}{3}\Gamma_{\text{pr}}\right). \tag{4.29b}$$

[11]This same correction also follows from doing the Gaussian σ integral and expanding

$$F_{\text{O}(N)} \supset \frac{1}{2}\text{tr}\,\log\left(K_\delta^{-1} + \frac{2Z_{g,1}\mu^{2\delta}}{N\delta}K^{-1}\right) = \frac{1}{2}\text{tr}\,\log K_\delta^{-1} + \frac{Z_{g,1}\mu^{2\delta}}{N\delta}\text{tr}\, K^{-1}K_\delta + O(1/N^2). \tag{4.26}$$

Using $F'_b(\frac{d}{2}-k)=0$ and $\hat{\gamma}_\phi = 2F'_b(2k)/F''_b(\frac{d}{2}-k)$, the identity

$$F^{\mathrm{LR}}_{\mathrm{O}(N)}(s=2k-2\hat{\gamma}_\phi) = F^{\mathrm{SR},\square^k}_{\mathrm{O}(N)} \tag{4.30}$$

becomes obvious, as does the compact form (1.11). We conclude with two comments about why F takes this form.

4. Why do anomalous dimensions appear in F?

The schematic form for the free energy corrections $\sim\frac{F'_b(s)}{N}\Gamma_i$ can be understood by studying the two-point function equations. Manifestly, any $\langle\sigma\sigma\rangle$ diagram can be converted into a vacuum diagram by closing the loop. If $\Sigma_{xy}=\langle\sigma(x)\sigma(y)\rangle$ is the σ propagator, then any vacuum diagram satisfies

$$I_i \propto \int_{xy}\frac{\delta I_i}{\delta\Sigma_{xy}}\Sigma_{xy}; \tag{4.31}$$

but this functional derivative is just an amputated self-energy diagram, as described in Appendix B. The self-energy diagrams that we find are, as usual for a self-energy diagram, proportional to the inverse propagator, with a pole that gives the anomalous dimension contribution of that diagram,

$$\frac{\delta I_i}{\delta\Sigma_{xy}} \propto \Sigma^{-1}_{xy}s(x,y)^{2m\delta}\times\left(\frac{\Gamma_i}{N}\frac{1}{\delta}+O(\delta^0)\right) \tag{4.32}$$

for some integer m—this is clear from inspection of Eqs. (B20c) and (B20d). Performing the integral in (4.31) then yields the finite result

$$I_i \propto \frac{1}{N}\frac{\Gamma_i}{\mathcal{N}_b(s)}\lim_{\delta\to0}\frac{1}{\delta}I_2(d-m\delta)\sim\frac{F'_b(s)}{N}\Gamma_i. \tag{4.33}$$

This heuristic argument explains how the vacuum diagrams pick out the poles of the two-point function diagrams. However, it does not explain the particular numerical factors of $\frac{1}{2}$ and $\frac{1}{3}$ that we find in the final F's (4.29)—those require a full calculation that also accounts for the counterterms.

D. F for the UV of our alternative field theory

Recall that the way that the $4-\epsilon$ expansion works is that the QFT (1.1) describes both the free theory of N fields, with $g=0$, and the interacting Wilson-Fisher CFT with $g\sim\epsilon$ in $d=4-\epsilon$. Because these two theories collide for $\epsilon=0$, taking ϵ to be small, we solve for the WF CFT by perturbing around the solvable free CFT. The WF free energy is [3]

$$F_{\mathrm{O}(N),d=4-\epsilon}=F_{\mathrm{free}}+O(\epsilon^2),\qquad F_{\mathrm{free}}=NF_b\left(\frac{d}{2}-1\right). \tag{4.34}$$

The corrections are proportional to a positive power of the small parameter that controls the deviation of the CFT from some solvable theory (in this case, a free field theory).

The same occurs in the large-N case. Evidently, the DREG + δ-regulated field theory's free energy can be calculated by sending $g\to0$ and $\mu\to0$ in (3.13), giving

$$F_{\mathrm{O}(N),g=0}=NF_b\left(\frac{d-s}{2}\right)+F_b(s-\delta). \tag{4.35}$$

Recall that we want to improve the UV convergence of this theory, and therefore we choose $\delta>0$; hence sending $\mu\to0$ has led to the K_δ^{-1} dominating over $-K^{-1}$. Strictly speaking, this involves working away from the critical point, and so really we should be considering coupling constants in front of the σ kinetic terms and finding their β functions, but this would have the same effect. The interacting free energies that we found are clearly perturbations around (4.35) proportional to the small parameter $1/N$, just as in the ϵ-expansion case.

Equation (4.35) tells us that in the UV this regulated theory has an extra scalar of dimension s, compared to the ϕ^4 theory. This makes sense, as in the large-N limit we access the O(N) CFT from an alternative field theory that is a distance $1/N$ away, as was shown in Fig. 1. This is what we mean when we say that the analytic regulator makes σ dynamical in the UV. However, this extra scalar does not have the $\Delta=\frac{d+s}{2}$ required to be the extra scalar χ proposed in [26] to resolve problems with the crossover from short- to long-range.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: CONVENTIONS

For convenience, we summarize here our conventions.

1. Gaussian integrals

The Gaussian integrals are

$$\begin{aligned}\int \mathrm{D}\phi e^{-\frac{1}{2}\phi_i[C^{-1}]_{ij}\phi_j+J_i\phi_i} &= \prod_i\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi_i}{\sqrt{2\pi}}e^{-\frac{1}{2}\phi^TC^{-1}\phi+J^T\phi}\\ &= (\det C^{-1})^{-\frac{1}{2}}e^{+\frac{1}{2}J^TCJ}\\ &= \exp\left(-\frac{1}{2}\log\det C^{-1}+\frac{1}{2}J^TCJ\right)\end{aligned} \tag{A1a}$$

and its real Grassmann cousin

$$\int \mathrm{D}\psi e^{+\frac{1}{2}\psi_i[C^{-1}]_{ij}\psi_j+\eta_i\psi_i} = \int \mathrm{d}\psi_n \cdots \mathrm{d}\psi_1 e^{+\frac{1}{2}\psi^T C^{-1}\psi_j+\eta_i\psi_i} = (\det\, C^{-1})^{\frac{1}{2}} e^{+\frac{1}{2}\eta^T C\eta}$$
$$= \exp\left(+\frac{1}{2}\log\ \det\, C^{-1} + \frac{1}{2}\eta^T C\eta\right), \tag{A1b}$$

in the conventions of [41]. The complex version is

$$\int \mathrm{D}\psi\mathrm{D}\bar{\psi} e^{\bar{\psi}_i[C^{-1}]^i_j\psi^j+\bar{\eta}_i\psi^i+\bar{\psi}_i\eta^i} = \int \prod_i \mathrm{d}\psi^i \mathrm{d}\bar{\psi}_i e^{\bar{\psi}C^{-1}\psi+\bar{\eta}\psi+\bar{\psi}\eta} = (\det\, C^{-1}) e^{-\bar{\eta}C\eta}$$
$$= \exp\left(+\log\ \det\, C^{-1} - \bar{\eta}C\eta\right). \tag{A1c}$$

We stress the different signs in the action, $S_b = \frac{1}{2}\phi C^{-1}\phi$ and $S_f = -\bar{\psi}C^{-1}\psi$; the latter sign follows purely from our chosen conventions for Grassmann integrals.

If we consider these integrals as written in DeWitt notation, these are exactly infinite-dimensional functional integrals. That is, if we upgrade $\phi_i \mapsto \phi(x)$, $[C^{-1}]_{ij} \mapsto C^{-1}(x,y)$, $\mathbb{I} = \delta_{ij} \mapsto \delta(x-y)$ and $\sum_i \mapsto \int_x$, these expressions describe the functional Gaussian integrals—of course, we have been careful to define the bosonic measure $\mathrm{D}\phi$ with the factors of $(\sqrt{2\pi})^\infty$. We evaluated the universal part of these determinants for conformal sphere propagators C in Sec. IV B.

a. Imaginary Gaussian integrals

Recall the definition of the Dirac δ function,

$$\delta(H) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}\rho}{2\pi} e^{i\rho H} \overset{\sigma\equiv -i\rho}{=} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\sigma}{2\pi i} e^{-\sigma H}. \tag{A2}$$

Upgrading to the functional version, it is conventional to define a field $\sigma(x)$ with an imaginary contour [59], such that

$$\delta(H(x)) = \int \mathrm{D}^{(\mathrm{I})}\sigma e^{-\int_x \sigma(x)H(x)}$$
$$\equiv \left(\prod_x \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}\sigma(x)}{2\pi i}\right) e^{-\int_x \sigma(x)H(x)}. \tag{A3}$$

Alternatively, we could keep σ on a real contour at the cost of i's in the action; by convention, we do not do so.

The σ action takes the form $\frac{1}{2}\sigma K^{-1}\sigma$, for $-K^{-1}$ positive definite. Due to the imaginary contour and the normalization of $\mathrm{D}^{(\mathrm{I})}\sigma$,

$$\int \mathrm{D}^{(\mathrm{I})}\sigma e^{-\frac{1}{2}\int_{x,y}\sigma_x K^{-1}_{xy}\sigma_y + \int_x J_x\sigma_x}$$
$$\overset{\rho=i\sigma}{=} \left(\prod_x \int_{-\infty}^{\infty} \frac{\mathrm{d}\rho_x}{\sqrt{2\pi}}\right) e^{-\int_{x,y}\frac{1}{2}\rho_x(-K^{-1}_{xy})\rho_y - i\int_x J_x\rho_x - \int_x \frac{1}{2}\log 2\pi}$$
$$\overset{(\mathrm{A.1})}{=} e^{-\frac{1}{2}\mathrm{tr}\,\log(-K^{-1}) - \frac{1}{2}\log(2\pi)\mathrm{tr}\,\mathbb{I}} e^{-\frac{1}{2}J^T(-K)J}. \tag{A4}$$

$\sum_x 1 = \mathrm{tr}\,\mathbb{I} = \int_{x,y}\delta(x-y)$ manifestly gives a UV divergent contribution to the free energy on the sphere. However, this divergence vanishes in DREG; equivalently, it can be canceled by a local counterterm that modifies the constant part of the action, $S \mapsto S - \int_x \frac{1}{2}\log(2\pi)$. Therefore, the universal part of this integral is simply its computation in DREG, which we write schematically as

$$\int \mathrm{D}^{(\mathrm{I})}\sigma e^{-\frac{1}{2}\int \sigma K^{-1}\sigma + \int J\sigma} = e^{-\frac{1}{2}\mathrm{tr}\,\log(-K^{-1}) + \frac{1}{2}J^T K J}. \tag{A5}$$

In the main text, we calculated this trace for $-K^{-1} \mapsto -K^{-1}_\delta = \frac{1}{2}(C_{\phi,0})^2/s(x,y)^{2(d-s+\delta)}$.

2. Kinetic terms and position space

In position space, on an arbitrary geometry, we write

$$\int_x = \int \mathrm{d}^d x\sqrt{g(x)}. \tag{A6}$$

When working in flat space, we can transform to momentum space, where we define

$$\int_p = \int \frac{\mathrm{d}^d p}{(2\pi)^d}. \tag{A7}$$

For the bosonic theory, we write

$$S_b = +\frac{1}{2}\int_x \phi C^{-1}\phi, \tag{A8}$$

where any fields without argument are assumed to be at x. As usual, the propagator is the matrix inverse of the kinetic term C^{-1}. So, for the flat space $C^{-1} = -\partial^2$,

$$\langle\phi(p)\phi(q)\rangle = C(p,q) = C(p)\delta(p+q) = +\frac{1}{p^2}\delta(p+q) \tag{A9}$$

$$\langle\phi(x)\phi(y)\rangle = C(x,y) = C(x-y), \qquad C(x) \equiv \frac{C_{\phi,0}}{|x|^{2\Delta_\phi^{\mathrm{UV}}}},$$
$$\Delta_\phi^{\mathrm{UV}} \equiv \frac{d-2}{2}. \tag{A10}$$

In the fermionic models, we keep the dimension of spinor space $\mathrm{tr}\,\mathbb{I}_s$ (i.e., the dimension of the γ matrices) general, folding it into N as described in the main text. Our Euclidean γ matrix conventions and Gaussian integral conventions are those of [47], such that the free fermion action is

$$S_f = -\int_x \bar\psi C^{-1}\psi, \tag{A11}$$

with $\bar\psi \equiv -\psi^\dagger\gamma_0$, and of course $\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$. Once again, for $C^{-1} = \not\partial$,

$$\langle\psi(p)\bar\psi(q)\rangle = C(p,q) = C(p)\delta(p+q) = +\frac{1}{i\not p}\delta(p+q) \tag{A12}$$

$$\langle\psi(x)\bar\psi(y)\rangle = C(x,y) = C(x-y), \qquad C(x) \equiv \frac{C_{\psi,0}\not x}{|x|^{2\Delta_\psi^{\mathrm{UV}}+1}},$$
$$\Delta_\psi^{\mathrm{UV}} \equiv \frac{d-1}{2}, \tag{A13}$$

where position- and momentum-space propagators are distinguished implicitly by argument.

We heavily overload our notation, such that we can freely move between the local derivative implementations $C^{-1} = -\partial^2$ and $\not\partial$, the real space matrix $C(x,y) = C(x-y)$, and the momentum space $C(p,q) = C(p)\delta(p+q)$. Thus, the most general kinetic term is

$$\frac{1}{2}\int_x \phi C^{-1}\phi = \frac{1}{2}\int_{x,y}\phi(x)C^{-1}(x,y)\phi(y) \tag{A14}$$

which in flat space would be $\frac{1}{2}\int_x \phi(x)(-\partial^2)^{\frac{d}{2}-\Delta_\phi^{\mathrm{free}}}\phi(x) = \frac{1}{2}\int_p \tilde\phi(p)(p^2)^{\frac{d}{2}-\Delta_\phi^{\mathrm{free}}}\tilde\phi(-p)$.

For a fermion in flat space, we have $C^{-1} = \not\partial$ for the canonical free fermion, with

$$C^{-1} = \not\partial(-\partial^2)^{\frac{d-1}{2}-\Delta_\psi^{\mathrm{free}}} = \frac{\not\partial}{\sqrt{-\partial^2}}(-\partial^2)^{\frac{d}{2}-\Delta_\psi^{\mathrm{free}}} \tag{A15}$$

for general $\Delta_\psi^{\mathrm{free}}$.

3. Sphere kinetic terms for scalars

We work in stereographic coordinates on the Euclidean sphere S^d. These coordinates are defined by

$$\mathrm{d}s^2 = \Omega(x)^2\mathrm{d}x^\mu\mathrm{d}x^\mu, \qquad \Omega(x) \equiv \frac{2R}{1+x^2},$$
$$\mathcal{R}_{\mu\nu} = \frac{d-1}{R^2}g_{\mu\nu}, \qquad \mathcal{R} = \frac{d(d-1)}{R^2}. \tag{A16}$$

The volume of the unit sphere then follows from

$$R^d\mathrm{vol}S^d \equiv \int_x 1 = \int \mathrm{d}^dx\sqrt{g} = R^d\frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}. \tag{A17}$$

We define the sphere chordal distance to be

$$s(x,y) \equiv \sqrt{\Omega(x)\Omega(y)}|x-y|. \tag{A18}$$

If we want a free propagator of the form

$$C(x,y) \equiv \frac{C_\phi}{s(x,y)^{2\Delta}} \tag{A19}$$

on the sphere, we can schematically write

$$S = \frac{1}{2}\int_{x,y}\phi(x)C^{-1}(x,y)\phi(y),$$
$$C^{-1}(x,y) \approx \frac{1}{C_\phi}\frac{1}{\mathcal{N}_b(\Delta)}\frac{1}{s(x,y)^{2(d-\Delta)}}. \tag{A20}$$

However, this is not quite right, as this integral is (a) divergent, requiring contact terms to make it converge and (b) clearly zero for $\Delta = \frac{d}{2} - k$. The correct way to write the inverse C^{-1} that satisfies

$$\int_y C(x,y)C^{-1}(y,z) = \frac{1}{\sqrt{g(x)}}\delta(x-z) \tag{A21}$$

is with the conformal Laplacian of biweight $(\Delta, d-\Delta)$ on S^d, which we will now explain, following [39]. Defining the standard $\zeta = \frac{d}{2} - \Delta = \frac{s}{2}$, we can obtain (A19) from

$$S = \frac{1}{2}\int_x \phi(x)C^{-1}\phi(x), \qquad C^{-1} = \frac{\mathcal{D}_\zeta}{C_\phi\mathcal{F}_{2\Delta,0}},$$
$$\mathcal{D}_\zeta = \frac{1}{R^{2\zeta}}\frac{\Gamma(R\mathcal{D}_{\frac{1}{2}}+\frac{1}{2}+\zeta)}{\Gamma(R\mathcal{D}_{\frac{1}{2}}+\frac{1}{2}-\zeta)}, \tag{A22}$$

where $\mathcal{D}_{\frac{1}{2}}$ is defined with the Laplace-Beltrami operator $\nabla^2_{S^d}$,

$$\mathcal{D}_{\frac{1}{2}} \equiv \sqrt{-\nabla^2_{S^d} + \left(\frac{d-1}{2R}\right)^2}. \tag{A23}$$

On the sphere, $-\nabla^2_{S^d}$ has eigenvalues ω_n with degeneracy g_n for $n \geq 0$,

$$\begin{aligned} \omega_{n\geq 0} &= \frac{1}{R^2} n(n+d-1), \\ g_n &\equiv \frac{(n+d-2)!(2n+d-1)}{n!(d-1)!} \\ &= \left(1+\frac{2n}{d-1}\right)\binom{d+n-2}{n}. \end{aligned} \tag{A24}$$

Thus, $\mathcal{D}_{\frac{1}{2}}$ and $\mathcal{D}_\zeta$ have eigenvalues with the same degeneracy

$$\omega_n^{(\frac{1}{2})} \equiv \frac{1}{R}\left(\frac{d-1}{2}+n\right), \qquad \omega_n^{(\zeta)} \equiv \frac{1}{R^{2\zeta}} \frac{\Gamma(n+\frac{d}{2}+\zeta)}{\Gamma(n+\frac{d}{2}-\zeta)}. \tag{A25}$$

To calculate the universal part of the free energy, we compute

$$F_b\left(\Delta \equiv \frac{d}{2}-\zeta\right) \equiv \frac{1}{2}\mathrm{tr}\,\log \mathcal{D}_\zeta = \frac{1}{2}\sum_{n=0}^{\infty} g_n \log \omega_n^{(\zeta)}. \tag{A26}$$

This matches (2.4) as computed using the methods of Sec. IV B after differentiating with respect to ζ; the resulting sum contains terms of the form 0^{-d} and so is finite for $d < 0$—it can then be analytically continued up to the d of interest.

We make three observations:

(i) $\mathcal{D}_k$ is manifestly a finite polynomial of order k in $(\mathcal{D}_{\frac{1}{2}})^2$ for integer k, beginning

$$\mathcal{D}_{k\in\mathbb{Z}} = (\mathcal{D}^2_{\frac{1}{2}})^k + \frac{k}{12}\frac{(1-4k^2)}{R^2}(\mathcal{D}^2_{\frac{1}{2}})^{k-1} + \cdots, \tag{A27}$$

which makes explicit the intuition that for the $\Box^k$ models $C^{-1} \sim (-\Box)^k + \cdots$. It also matches (3.4) for $\zeta = k = 1$ (the constant term serves to remove the zero mode).

(ii) $\mathcal{D}_\zeta \to (\mathcal{D}_{\frac{1}{2}})^{2\zeta}$ in the flat-space limit of $R \to \infty$ (equivalently, in the large-momentum limit), making explicit the intuition that the flat-space $C^{-1} \sim (-\Box)^{s/2}$.

(iii) For all Δ, a calculation of $C(x,x)$ in DREG gives (see Appendix D in [39])

$$C(x,x) = 0. \tag{A28}$$

For $0 < s < 2$, this conformal Laplacian can also be written as an explicitly nonlocal operator using a subtracted hypersingular integral (the r limit is essentially a principal value—with R factors corrected from [39]),

$$\mathcal{D}_{\frac{d}{2}-\Delta}\phi(x) = \lim_{r\to 0}\frac{1}{\mathcal{F}_{2(d-\Delta),0}}\int_{y,s(x,y)>r}\frac{\phi(y)-\phi(x)}{s(x,y)^{2(d-\Delta)}} + \frac{1}{R^{d-2\Delta}}\frac{\Gamma(d-\Delta)}{\Gamma(\Delta)}\phi(x). \tag{A29}$$

This is a rigorous implementation of the schematic (A20) [since $\mathcal{F}_{2\Delta,0}\mathcal{F}_{2(d-\Delta),0} = \mathcal{N}_b(\Delta)$].

For generic $0 < s \neq 2k$, we must add further derivative terms to ensure that the Green's function is exactly the conformal propagator. In flat space $g = \delta$, we find (2.13) of [60] (note that their $D^s = \mathcal{D}_{s/2}/(2\pi)^s$ in our notation),

$$\begin{aligned} \mathcal{D}_{\frac{d}{2}-\Delta}\phi(x)|_{g=\delta} &= \lim_{r\to 0}\frac{1}{\mathcal{F}_{2(d-\Delta),0}}\int_{y,|x-y|>r}\frac{1}{|x-y|^{2(d-\Delta)}}\left(\phi(y) - \sum_{n=0}^{\lfloor s/2\rfloor}|x-y|^{2n} b_n \Box^n \phi(x)\right) \\ b_n &= \frac{\Gamma(\frac{d}{2})}{2^{2n}\Gamma(n+\frac{d}{2})\Gamma(n+1)}, \end{aligned} \tag{A30}$$

which indeed satisfies

$$\int_k \phi(k)|k|^{2s}\phi(-k) = \int_x \phi(x)\mathcal{D}_{s/2}\phi(x)|_{g=\delta} \tag{A31}$$

(recalling our Fourier transform conventions). Thus, simple $|k|^s$ terms in momentum space correspond to carefully tuned nonlocal position-space actions that contain both terms $\sim\phi(x)\phi(y)$ and $\sim\phi(x)\phi(x)$.

These integral representations are not strictly valid for $s = 2k$, because, using (A40), $1/\mathcal{F}_{2(d-\Delta),0} = 0$ for $\Delta = \frac{d}{2}-k$; however, they can still be used to define the local operator $(-\partial^2)^k$ as a "resonant" limit (see Appendix A of [61]).

4. Sphere kinetic terms for fermions

For fermions, the fractional Dirac operator was defined in Proposition 2.5 in [62] (generalizing the integer-power work of [63]). It leads to a fermion sphere propagator

which is a Weyl transformation of (A13) from flat space to the sphere. In embedding coordinates $\eta^{a=1,\ldots,d+1}$, $\sum_{ab}\delta_{ab}\eta^a\eta^b = R^2$, with $(d+1)$-dimensional γ matrices $\{\alpha_a,\alpha_b\} = 2\delta_{ab}$, this is simply [34]

$$C(x,y) = C_\psi \frac{\alpha_a(\eta_1-\eta_2)^a}{|\eta_1-\eta_2|^{2\Delta_\psi+1}}. \tag{A32}$$

To get this from an action of the form $S = -\int_x \bar\psi C^{-1}\psi$, we define $\lambda \equiv \frac{d}{2}-\Delta_\psi = \frac{s}{2}$; then, taking the sphere Dirac operator to be $D = \not\nabla_{S^d}$, we write[12]

$$C^{-1} = \frac{-i\mathcal{D}_\lambda}{C_\psi \mathcal{F}_{2\Delta_\psi,1}}, \qquad \mathcal{D}_\lambda = \frac{1}{R^{2\lambda}}\frac{\Gamma(R|D|+\frac12+\lambda)}{\Gamma(R|D|+\frac12-\lambda)}\frac{D}{|D|}. \tag{A33}$$

We do not venture a subtracted hypersingular integral representation here, though it should exist.

Since D has two sets of eigenspinors $\hat\phi_{k,\pm}$ with eigenvalues and degeneracies[13] [Eqs. (3.44) and (3.58) in [64]]

$$\omega_{n\geq 0,\pm} = \pm\left(\frac{d}{2}+n\right)/R,$$
$$g_{n,\pm} = T\frac{(d+n-1)!}{n!(d-1)!} = T\binom{n+d-1}{n}. \tag{A34}$$

The same $\hat\phi_{k,\pm}$ have eigenvalue $|\omega_n^{(\pm)}|$ for $|D|$ (in the notation of Appendix A.2 in [62], $D = \nu\cdot D_{h_0}$), so

$$\omega^{(\lambda)}_{n,\pm} = \pm\frac{1}{R^{2\lambda}}\frac{\Gamma(\frac{d+1}{2}+n+\lambda)}{\Gamma(\frac{d+1}{2}+n-\lambda)}. \tag{A35}$$

We define F_f as the universal part of the free energy of a single complex component of a Dirac fermion of dimension Δ,

$$F_f\left(\Delta = \frac{d}{2}-\lambda\right) \equiv -\frac{\text{tr }\log\mathcal{D}_\lambda}{T} = -\sum_{\sigma=\pm}\sum_{n\geq 0}\frac{g_{n,\sigma}}{T}\log\omega^{(\lambda)}_{n,\sigma}$$
$$= -2\sum_{n\geq 0}\frac{g_{n,+}}{T}\log\omega^{(\lambda)}_{n,+}. \tag{A36}$$

The last equality follows from $\sum_{n\geq 0} g_{n,\pm} \propto (1-1)^{-d} = 0$. Again, after differentiating and analytically continuing from $d<0$, this matches (2.13) as computed using the methods of Sec. IV B.

We note finally that, for $\lambda = k+\frac12$, $k\in\mathbb{N}_0$,

$$\mathcal{D}_{k+\frac12}|_{R=1} = (|D|+k)\cdots(|D|-1)D(|D|+1)\cdots(|D|+k)$$
$$= (D+k)\cdots(D-1)D(D+1)\cdots(D+k), \tag{A37}$$

which holds because $\mathcal{D}_{\frac{2k+1}{2}}/D$ depends only on $|D|^2 = D^2$ [which itself is clear from $|\omega_{n\geq 0,\pm}|^2 = (\omega_{n\geq 0,\pm})^2$]; thus we match Eq. (5.13) in [63].

5. Propagators and associated functions

In flat space, we define generic boson/fermion propagators (now unit normalized) to be

$$G_\phi(x_{12}) \equiv \frac{1}{|x_{12}|^{2\Delta_\phi}} = \overset{\Delta_\phi}{\underset{x_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\bullet_{x_2}}, \quad G_\psi(x_{12}) \equiv \frac{\not{x}_{12}}{|x_{12}|^{2\Delta_\psi+1}} = \overset{\Delta_\psi}{\underset{x_1}{\bullet}\!\!\longleftarrow\!\!\bullet_{x_2}}, \tag{A38}$$

and they are implicitly differentiated by their propagators. We can Fourier transform these:

$$G_\phi(p) = \int_x e^{-ip\cdot x}G_\phi(x) = \mathcal{F}_{2\Delta_\phi,0}\frac{1}{p^{d-2\Delta_\phi}}, \qquad G_\psi(p) = \int_x e^{-ip\cdot x}G_\psi(x) = \mathcal{F}_{2\Delta_\psi,1}\frac{\not p}{p^{d-2\Delta_\psi+1}}, \tag{A39}$$

where we define the general Fourier transform of a traceless propagator [58]

$$\mathcal{F}_{\lambda,s} \equiv i^{-s}2^{d-\lambda}\pi^{d/2}\frac{\Gamma(\frac{d+s-\lambda}{2})}{\Gamma(\frac{\lambda+s}{2})},$$
$$\text{where } \mathcal{F}_{\lambda,s}(p^{\mu_1}\cdots p^{\mu_s}-\text{traces})p^{-(d-\lambda)-s} = \int_x e^{-ip\cdot x}(x^{\mu_1}\cdots x^{\mu_s}-\text{traces})x^{-\lambda-s}. \tag{A40}$$

[12]The i here follows from $C(p,q) = \langle\psi(p)\bar\psi(q)\rangle = \mathcal{F}_{2\Delta,1}C_\psi \not p/p^{d-2\Delta_\psi+1}\delta(p+q)$ after applying (A39) to the flat-space version of (A32); hence clearly $S = -\frac12\int\bar\psi(q)C^{-1}\psi(p) = -\frac12\int\bar\psi(q)(\mathcal{F}_{2\Delta,1}C_\psi)^{-1}\not p p^{d-1-2\Delta_\psi}\delta(q+p)\psi(p)$; then $p\to -i\not\nabla$ as usual.

[13]Recall that we have fixed $\text{tr}\,\mathbb{I}_s = T$, ignoring the dependence on d of the dimension of the γ matrices, which is usually $\text{tr}\,\mathbb{I}_s = 2^{\lfloor\frac{d}{2}\rfloor}$.

Then the inverse transforms follow

$$\int_p e^{ip\cdot x}\frac{1}{p^{d-2\Delta_\phi}}=\frac{\mathcal{F}_{d-2\Delta_\phi,0}}{(2\pi)^d}\frac{1}{x^{2\Delta_\phi}},\qquad \int_p e^{ip\cdot x}\frac{\not{p}}{p^{d-2\Delta_\psi+1}}=-\frac{\mathcal{F}_{d-2\Delta_\psi,1}}{(2\pi)^d}\frac{\not{x}}{x^{2\Delta_\psi+1}}. \tag{A41}$$

Since $\int_p e^{-ip\cdot x}=\delta(x)$, consistency demands that indeed $\mathcal{F}_{2\Delta,s}\mathcal{F}_{d-2\Delta,s}=i^{-2s}(2\pi)^d$; the i^{-2s} then cancels with the necessity of taking $-p$ in (A40) when inverse transforming. Hence,

$$\frac{1}{\mathcal{F}_{\lambda,s}}(x^{\mu_1}\cdots x^{\mu_s}-\text{traces})x^{-\lambda-s}=\int_p e^{ip\cdot x}(p^{\mu_1}\cdots p^{\mu_s}-\text{traces})p^{-(d-\lambda)-s}. \tag{A42}$$

The main functions that we find in this paper are particular combinations of Γ functions that arise from Fourier transforms (A39). We drop the i's, 2's, and π that almost always cancel, and so define $\mathcal{F}_{2\Delta,s}/(-i^s 2^{d-2\Delta}\pi^{d/2})$ for $s=0,1$:

$$A(\Delta)\equiv\frac{\Gamma(\frac{d}{2}-\Delta)}{\Gamma(\Delta)},\qquad A_f(\Delta)\equiv\frac{\Gamma(\frac{d}{2}-\Delta+\frac{1}{2})}{\Gamma(\Delta+\frac{1}{2})}, \tag{A43}$$

which both satisfy $A_x(\Delta)A_x(\frac{d}{2}-\Delta)=1$. From Taylor expanding these functions, we frequently also see

$$\begin{aligned}B(\Delta)&\equiv-\frac{\mathrm{d}}{\mathrm{d}\Delta}\log A(\Delta)=\psi^{(0)}(\Delta)+\psi^{(0)}\left(\frac{d}{2}-\Delta\right),\\ B_f(\Delta)&\equiv-\frac{\mathrm{d}}{\mathrm{d}\Delta}\log A_f(\Delta)=\psi^{(0)}\left(\Delta+\frac{1}{2}\right)\\&\quad+\psi^{(0)}\left(\frac{d}{2}-\Delta+\frac{1}{2}\right),\end{aligned} \tag{A44}$$

which both satisfy

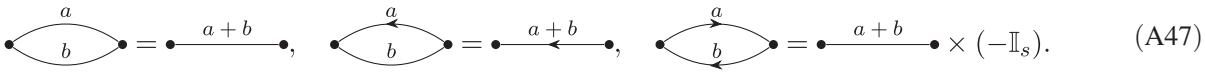
$$\begin{aligned}B_x(\Delta)&=B_x\left(\frac{d}{2}-\Delta\right),\\ A_x(\Delta+\epsilon)&=A_x(\Delta)[1-\epsilon B_x(\Delta)+O(\epsilon^2)].\end{aligned} \tag{A45}$$

For the uniqueness relations, we then define

$$\begin{aligned}U(a,b,c)&=\pi^{d/2}A(a)A(b)A(c),\\ U_{\text{bff}}(a,b,c)&=\pi^{d/2}A(a)A_f(b)A_f(c).\end{aligned} \tag{A46}$$

6. Position-space loop integrals and the method of uniqueness

To make this paper self-contained, we summarize here the method of uniqueness [65], which is used to calculate diagrams in flat space. Let us begin with the loop integrals



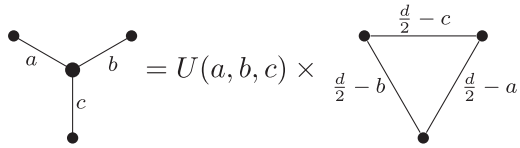
(A47)

Next, consider the propagator merging relations, which can be derived by Fourier transforming both propagators and then reverse Fourier transforming the combination. With $x_{ij}\equiv x_i-x_j$,

$$\begin{aligned}\int_{x_m}\frac{1}{|x_{1m}|^{2a}}\frac{1}{|x_{m2}|^{2b}}&=\overset{x_1}{\bullet}\overset{a}{\text{---}}\overset{x_m}{\bullet}\overset{b}{\text{---}}\overset{x_2}{\bullet}=\overset{x_1}{\bullet}\overset{a+b-\frac{d}{2}}{\text{---}}\overset{x_2}{\bullet}\times U(a,b,d-a-b)\\ \int_{x_m}\frac{\not{x}_{1m}}{|x_{1m}|^{2a+1}}\frac{1}{|x_{m2}|^{2b}}&=\overset{x_1}{\bullet}\overset{a}{\text{---}}\overset{x_m}{\bullet}\overset{b}{\text{---}}\overset{x_2}{\bullet}=\overset{x_1}{\bullet}\overset{a+b-\frac{d}{2}}{\text{---}}\overset{x_2}{\bullet}\times U_{\text{bff}}(b,a,d-a-b)\\ \int_{x_m}\frac{1}{|x_{1m}|^{2a}}\frac{\not{x}_{m2}}{|x_{m2}|^{2b+1}}&=\overset{x_1}{\bullet}\overset{a}{\text{---}}\overset{x_m}{\bullet}\overset{b}{\text{---}}\overset{x_2}{\bullet}=\overset{x_1}{\bullet}\overset{a+b-\frac{d}{2}}{\text{---}}\overset{x_2}{\bullet}\times U_{\text{bff}}(a,b,d-a-b)\\ \int_{x_m}\frac{\not{x}_{1m}}{|x_{1m}|^{2a+1}}\frac{\not{x}_{m2}}{|x_{m2}|^{2b+1}}&=\overset{x_1}{\bullet}\overset{a}{\text{---}}\overset{x_m}{\bullet}\overset{b}{\text{---}}\overset{x_2}{\bullet}=\overset{x_1}{\bullet}\overset{a+b-\frac{d}{2}}{\text{---}}\overset{x_2}{\bullet}\times(-1)U_{\text{bff}}(d-a-b,a,b).\end{aligned} \tag{A48}$$

In this final relation we have used the standard convention that γ matrix expressions are written by starting at the beginning of the arrow (x_2) and writing down each component from right to left. Then the uniqueness relation (derived in Appendix A of [66]), which is valid for $a+b+c=d$, is

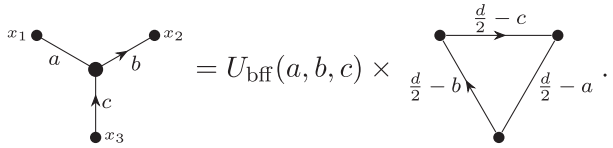
$$\int_{x_m} \frac{1}{|x_{1m}|^{2a}|x_{2m}|^{2b}|x_{3m}|^{2c}} = U(a,b,c)\frac{1}{|x_{12}|^{d-2c}|x_{13}|^{d-2b}|x_{23}|^{d-2a}}$$

$$= U(a,b,c) \times$$ 

(A49)

For a Yukawa vertex, again requiring $a+b+c=d$,

$$\int_{x_m} \frac{1}{|x_{1m}|^{2a}} \frac{\not{x}_{2m}}{|x_{2m}|^{2b+1}} \frac{\not{x}_{m3}}{|x_{m3}|^{2c+1}} = U_{\text{bff}}(a,b,c)\frac{\not{x}_{21}}{|x_{21}|^{d-2c+1}} \frac{\not{x}_{13}}{|x_{13}|^{d-2b+1}} \frac{1}{|x_{23}|^{d-2a}}$$

$$= U_{\text{bff}}(a,b,c) \times$$ 

(A50)

The order of the γ matrices remains unchanged, as we simply replace the m's by 1's: $\not{x}_{2m}\not{x}_{m3} \to \not{x}_{21}\not{x}_{13}$.

We note that these equations also hold under a map to the sphere: namely, every appearance of the metric is replaced by the sphere metric, and every appearance of $|x-y|$ is replaced by $s(x,y)$.

APPENDIX B: COUNTERTERM CALCULATIONS: 2PI FORMALISM/SKELETON DIAGRAMS

We use the action (3.13) contained in the main text and provide here a standard computation of the counterterms Z_g and Z_ϕ given in (3.15) and (3.22) via the 2PI formalism. A review of this formalism is available in [7], with a longer one in [67]. Essentially, it allows us to automatically resum self-energy diagrams and work only with the full propagators; but at its core it is just a Legendre transform with respect to a two-point source. This is very well suited to CFTs, where we know exactly the functional form of the propagators.

We now give a simplified sketch of the derivation of the 2PI effective action. For simplicity, we can consider a partition function with only a two-point source $k(x,y)$. That is, we assume that the relevant couplings are tuned such that the field vacuum expectation values (VEVs) are zero, so we do not need a one-point source. Thus, defining

$$\mathbf{W}[k] \equiv \log \int \mathcal{D}\varphi \exp\left(-S[\varphi] + \int_{x,y} \frac{1}{2}\varphi(x)k(x,y)\varphi(y)\right), \tag{B1}$$

the propagator in the presence of the source k is just the functional derivative

$$\mathbf{G}(x,y) = \langle \varphi(x)\varphi(y)\rangle_k = 2\frac{\delta \mathbf{W}}{\delta k(x,y)}. \tag{B2}$$

Then the Legendre transform of $\mathbf{W}[k]$ is the two-particle effective action $\Gamma[\mathbf{G}]$, defined by

$$\Gamma[\mathbf{G}] + \mathbf{W}[k] = \frac{1}{2}\int_{x,y} \mathbf{G}(x,y)k(x,y), \tag{B3}$$

such that

$$\frac{\delta\Gamma[\mathbf{G}]}{\delta \mathbf{G}(x,y)} = \frac{1}{2}k(x,y). \tag{B4}$$

Hence, the true propagator (without sources) is simply whatever solves the two-point Schwinger-Dyson equation

$$\frac{\delta\Gamma[\mathbf{G}]}{\delta \mathbf{G}(x,y)} = 0. \tag{B5}$$

The formalism is useful because $\Gamma[\mathbf{G}]$ is extremely easy to construct, as we will show shortly: we just sum up all the diagrams that are two-particle-reducible (i.e., the diagrams that do not contain a self-energy insertion on any of their lines). The two-particle effective action can be functionally differentiated to give the full two-point Schwinger-Dyson equations. This process is trivially generalized to multiple fields by adding indices to φ and k; then the skeleton diagram technique of [53,68,69], reviewed in [66], is exactly identical to just self-consistently solving these Schwinger-Dyson equations.

1. Implementing the 2PI formalism

We can ignore here the fact that σ has an imaginary contour, as the extra minus sign that would arise in

$\log(-\Sigma^{-1})$ that was noted in Appendix A 1 a just gives a constant and can be dropped. Tuning the relevant couplings to set the field VEVs to zero ($\langle\phi_i\rangle = 0$, $\langle\sigma\rangle = 0$), we find

$$\Gamma_{\text{2PI}}[G,\Sigma] = \frac{1}{2}\text{tr}\,\log G^{-1} + \frac{1}{2}\text{tr}\,G_0^{-1}G + \frac{1}{2}\text{tr}\,\log\Sigma^{-1} + \frac{1}{2}\text{tr}\,\Sigma_0^{-1}\Sigma + \Gamma_2[G,\Sigma], \tag{B6}$$

where we have also assumed that the O(N) is unbroken, meaning that $\langle\phi_i\sigma\rangle = 0$. Γ_2 is minus the sum of the two-particle irreducible graphs, and G_0 and Σ_0 are the quadratic terms in the original action,[14]

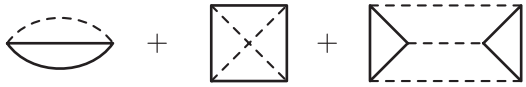
$$G_0^{-1} = \frac{Z_\phi}{C_{\phi,0}}\frac{1}{\mathcal{N}_{\frac{d-s}{2}}}\frac{1}{s(x,y)^{d+s}}, \tag{B7}$$

$$\Sigma_0^{-1} = K_\delta^{-1} - \mu^{2\delta}K^{-1} = \frac{1}{C_{\sigma,0}}\left(\frac{s(x,y)^{-2(d-s+\delta)}}{\mathcal{N}_b(s-\delta)} - \frac{\mu^{2\delta}s(x,y)^{-2(d-s)}}{\mathcal{N}_b(s)}\right), \tag{B8}$$

with the bare normalizations given by (3.18). As an aside, in the original theory (3.7) regulated with a hard cutoff, we would be using $\Sigma_0^{-1}(x,y) = -\frac{1}{\lambda_0}\delta(x-y)$. Variation of this effective action yields the equations of motion

$$G^{-1} - G_0^{-1} - 2\frac{\delta\Gamma_2}{\delta G} = 0, \qquad \Sigma^{-1} - \Sigma_0^{-1} - 2\frac{\delta\Gamma_2}{\delta\Sigma} = 0. \tag{B9}$$

At order N^0, only the melon contributes to Γ_2; at order $1/N$, we have only the tetrahedron and the prism. The pillow diagram is two-particle reducible, being a melon diagram with a self-energy insertion, and so we do not need to consider it. Thus, using the vertex rule $-Z_\phi g_0/\sqrt{N}$, we find

$-\Gamma_2 =$  $+ O(1/N^2),$

$I_{\text{mel}} : \frac{1}{4}$ $I_{\text{tet}} : \frac{1}{8}$ $I_{\text{pr}} : \frac{1}{12}$

$$I_{\text{melon}} = \frac{1}{4}\frac{Z_\phi^2 g_0^2}{N}\int_{x,y} G_{ij}(x,y)G_{ji}(y,x)\Sigma(x,y)$$

$$I_{\text{tet}} = \frac{1}{8}\frac{Z_\phi^4 g_0^4}{N^2}\int_{x_{1,2,3,4}} G_{ij}(x_1,x_2)G_{jk}(x_2,x_3)G_{kl}(x_3,x_4)G_{lm}(x_4,x_1)\Sigma(x_1,x_3)\Sigma(x_2,x_4)$$

$$I_{\text{pr}} = \frac{1}{12}\frac{Z_\phi^6 g_0^6}{N^3}\int_{x_{1,2,3,4,5,6}} (G_{ij}(x_1,x_2)G_{jk}(x_2,x_3)G_{kl}(x_3,x_1)) \times \Sigma(x_1,x_4)\Sigma(x_2,x_5)\Sigma(x_3,x_6)(G_{i'j'}(x_4,x_5)G_{j'k'}(x_5,x_6)G_{k'l'}(x_6,x_4)).$$

We note that on shell (i.e., with the solutions for G and Σ plugged in), we have that $\Gamma_{\text{2PI}}|_\star = -\log Z_{\text{O}(N)} = F_{\text{O}(N)}$ by construction. Thus, an alternative way of calculating the free energy is just to directly evaluate this at the critical point (this was done in [7]). Although slightly simpler in practice, it is conceptually clearer to stick to the standard Feynman-diagrammatic approach to the evaluation of the free energy, rather than Legendre transforming it as we have done here.

2. The two-point Schwinger-Dyson equations

The two-point Schwinger-Dyson equations follow by plugging Γ_2 in to (B9). Assuming no O(N)-symmetry breaking, so $G_{ij}(x,y) = G(x,y)\delta_{ij}$, we ansatz the conformal solution

$$\frac{\langle\phi_i(x)\phi_j(y)\rangle}{\delta_{ij}} = G(x,y) = \frac{C_\phi}{s(x,y)^{2\Delta_\phi}}, \qquad \langle\sigma(x)\sigma(y)\rangle = \Sigma(x,y) = \frac{C_\sigma}{s(x,y)^{2\Delta_\sigma}}. \tag{B10}$$

A derivative with respect to Σ of Γ_{2PI} clearly does not modify the N-scaling of the terms; a derivative with respect to G_{ij} does, but it breaks open a single ϕ loop in each term, and therefore uniformly reduces the N-scaling of each term by 1; hence

[14]We have ignored here the contact terms in (A29) which are required only to make the inversions C^{-1} and K^{-1} rigorous and could be easily restored. In any case, they can be thrown away here, as we are considering the two-point functions for separated points only.

no other diagrams will start contributing here. Henceforth, we drop all references to the flavor indices, as the N-scaling of each term is now explicit. Writing in diagrammatic form, (B9) becomes

$$G^{-1} - G_0^{-1} + \frac{Z_\phi^2 g_0^2}{N} + \frac{Z_\phi^4 g_0^4}{N^2} + \frac{Z_\phi^6 g_0^6}{N^2} = 0, \tag{B11a}$$

$$\Sigma^{-1} - \Sigma_0^{-1} + \frac{Z_\phi^2 g_0^2}{2} + \frac{Z_\phi^4 g_0^4}{2N} + \frac{Z_\phi^6 g_0^6}{2N} = 0, \tag{B11b}$$

which we evaluate using

$$\Delta_\sigma = s - \delta + \gamma_\sigma, \qquad \Delta_\phi = \frac{d-s}{2} + \gamma_\phi, \tag{B12}$$

and then take $N \to \infty$ followed by $\delta \to 0$.

We evaluate these diagrams in two different ways. In Appendix D 1, we work in flat space [$s(x,y) = |x-y|$], where we can use uniqueness to directly calculate the subleading σ diagrams by hand. In Appendix D 2 we work on the sphere, where the diagrams can be evaluated using Mellin-Barnes integral representations and MBRESOLVE. Naturally, the same results are obtained in each case, though the method of uniqueness is less powerful (it no longer works for the calculation of $\hat{\gamma}_{\phi,2}$). We can then use these results to solve the two-point functions.

3. Solving the two-point functions for the long-range models, $s \neq 2k$

a. Leading order in N for $s \neq 2k$

We begin with the first corrections to the propagators. We take $g_0 = \mu^\delta Z_g g$ and freely set $X = \mu s(x,y)$ and $\mu = 1$ to reduce clutter. Stripping off the δ_{ij}'s, we find

$$G^{-1} - G_0^{-1} = -2\frac{\delta I_{\rm mel}}{\delta G} = -\frac{Z_\phi^2 g_0^2}{N} = -\frac{1}{N}\frac{Z_\phi^2 g_0^2 C_\sigma C_\phi}{X^{2(s-\delta+\gamma_\sigma+\frac{d-s}{2}+\gamma_\phi)}} + O(1/N^2) \tag{B13}$$

$$\Sigma^{-1} - \Sigma_0^{-1} = -2\frac{\delta I_{\rm mel}}{\delta \Sigma} = -\frac{Z_\phi^2 g_0^2}{2} = -\frac{1}{2}\frac{Z_\phi^2 g_0^2 C_\phi^2}{X^{2(d-s+2\gamma_\phi)}} + O(1/N). \tag{B14}$$

It is convenient to define $z = C_\sigma C_\phi^2 = z_0 + \frac{z_1}{N} + O(1/N^2)$ and self-consistent at this order to take $Z_\phi Z_g = 1 + O(1/N)$ (recall that $g = 1$). Substituting for G and Σ, we find

$$\frac{1}{\mathcal{N}_b(\frac{d-s}{2}+\gamma_\phi)} - \frac{C_\phi Z_\phi}{C_{\phi,0}}\frac{X^{-2\gamma_\phi}}{\mathcal{N}_b(\frac{d-s}{2})} + \frac{z_0}{N}X^{-2(\gamma_\sigma+2\gamma_\phi-\delta)} + O(1/N^2) = 0, \tag{B15}$$

$$\frac{1}{\mathcal{N}_b(s-\delta+\gamma_\sigma)} - \frac{C_\sigma}{C_{\sigma,0}}\left(\frac{X^{-2\gamma_\sigma}}{\mathcal{N}_b(s-\delta)} - \frac{X^{2(\delta-\gamma_\sigma)}}{\mathcal{N}_b(s)}\right) + \frac{z_0}{2}X^{-2(\gamma_\sigma+2\gamma_\phi-\delta)} + O(1/N) = 0. \tag{B16}$$

Note that $1/\mathcal{N}(\frac{d}{2}-k) = 0$, which complicates the analysis; we therefore delay discussion of $s = 2k$ to Appendix B 4 and choose to focus first on the generic and simpler case $s \neq 2k$. We then expand around $\gamma_{\sigma/\phi} \to \gamma_{\sigma/\phi,1}/N + \cdots$ and $C_{\sigma/\phi} \to C_{\sigma/\phi,0}(1 + A_{\sigma/\phi,1}/N + \cdots)$, and therefore also $z \to z_0 + z_1/N + \cdots$. First taking $N \to \infty$ and then taking $\delta \to 0$ yields

$$0 = \frac{1}{\mathcal{N}_b(\frac{d-s}{2})}\left[1 - \frac{C_{\phi,0}Z_\phi}{C_{\phi,0}}\left(1 + \frac{A_{\phi,1} - 2\gamma_\phi \log(\mu X)}{N}\right) + \frac{z_0 \mathcal{N}_b(\frac{d-s}{2}) - \gamma_{\phi,1}\frac{\mathcal{N}'_b(\frac{d-s}{2})}{\mathcal{N}_b(\frac{d-s}{2})}}{N} + O(1/N^2)\right]$$

$$0 = \frac{2}{\mathcal{N}_b(s)} + z_0 + O(1/N). \tag{B17}$$

Solving this order by order in N, we can take $Z_\phi = 1$ finite and

$$z_0 = \frac{-2}{\mathcal{N}_b(s)} \tag{B18}$$

so indeed Δ_ϕ is unmodified,

$$\gamma_{\phi,1} = 0, \qquad A_{\phi,1} = z_0 \mathcal{N}_b\left(\frac{d-s}{2}\right). \tag{B19}$$

b. Subleading order in N for $s \neq 2k$

Let us now consider the next order. Both on the sphere, $X = \mu s(x,y)$ and in flat space, $X = \mu|x-y|$, the subleading diagrams in (B11) can be calculated to order $1/N$,

$$2\frac{\delta I_{\text{tet}}}{\delta G} = \frac{Z_\phi^4 g_0^4}{N^2}\frac{C_\sigma^2 C_\phi^3}{X^{d+s-4\delta}}\left(\frac{K_1}{\delta} + \Sigma'_1 + O(\delta)\right), \tag{B20a}$$

$$2\frac{\delta I_{\text{pr}}}{\delta G} = \frac{Z_\phi^6 g_0^6}{N^2}\frac{C_\sigma^3 C_\phi^5}{X^{d+s-6\delta}}\left(\frac{K_2}{\delta} + \Sigma'_2 + O(\delta)\right), \tag{B20b}$$

$$2\frac{\delta I_{\text{tet}}}{\delta \Sigma} = \frac{Z_\phi^4 g_0^4}{2N}\frac{C_\sigma C_\phi^4}{X^{2(d-s)-2\delta}}\left(\frac{K_1}{\delta} + \Pi'_1 + O(\delta)\right), \tag{B20c}$$

$$2\frac{\delta I_{\text{pr}}}{\delta \Sigma} = \frac{Z_\phi^6 g_0^6}{2N}\frac{C_\sigma^2 C_\phi^6}{X^{2(d-s)-4\delta}}\left(\frac{K_2}{\delta} + \Pi'_2 + O(\delta)\right). \tag{B20d}$$

The exponents of X here should contain $\gamma_{\phi,\sigma}$'s, but they have been dropped as they contribute only at the next order in N. We use the conventions of [53],

$$K_1 = \frac{2\pi^d}{\Gamma(\frac{d}{2})}A(s)A\left(\frac{d-s}{2}\right)^2, \qquad K_2 = \frac{\pi^{2d}}{\Gamma(\frac{d}{2})}A(s)^3 A\left(\frac{d}{2} + \frac{d-s}{2} - s\right)A\left(\frac{d-s}{2}\right)^3, \tag{B21}$$

$$\Sigma'_1 = \frac{K_1}{2}\left(B(s) - B\left(\frac{d-s}{2}\right)\right), \qquad \Sigma'_2 = 2K_2\left(B(s) - B\left(\frac{d-s}{2}\right)\right),$$

$$\Pi'_1 = K_1\left(B(s) - B\left(\frac{d-s}{2}\right)\right), \qquad \Pi'_2 = K_2\left(4B(s) - 3B\left(\frac{d-s}{2}\right) - B\left(\frac{d}{2} + \frac{d-s}{2} - s\right)\right), \tag{B22}$$

and

$$p(\Delta) \equiv \frac{1}{\mathcal{N}_b(\Delta)} = \frac{\Gamma(d+1)}{\pi^{d+1}}\tilde{F}'_b(\Delta) = -\sin\left(\frac{\pi d}{2}\right)\frac{\Gamma(d+1)}{\pi^{d+1}}F'_b(\Delta). \tag{B23}$$

Now, recall that Z_g is divergent at order $1/N$, while Z_ϕ we can take to be finite and equal to 1,

$$Z_\phi = 1, \qquad \text{for generic } s \neq 2k. \tag{B24}$$

Thus, in minimal subtraction, we have

$$Z_\phi^2 g_0^2 = \mu^{2\delta}\left(1 + \frac{1}{N}\frac{2Z_{g,1}}{\delta} + O(1/N^2)\right). \tag{B25}$$

Putting this together for the σ equation,

$$0 = p(s - \delta + \gamma_\sigma) - \frac{C_\sigma}{C_{\sigma,0}}\left(p(s-\delta)X^{-2\gamma_\sigma} - p(s)X^{2(\delta-\gamma_\sigma)}\right) + \frac{z}{2}\left(1 + \frac{2Z_{g,1}}{N\delta}\right)X^{2(\delta-\gamma_\sigma-2\gamma_\phi)}$$
$$+ \frac{z^2}{2N}X^{4\delta}\left(\frac{K_1}{\delta} + \Pi'_1\right) + \frac{z^3}{2N}X^{6\delta}\left(\frac{K_2}{\delta} + \Pi'_2\right) + O(1/N^2). \tag{B26}$$

We now expand z and γ_σ in N and demand that each order in $1/N$ is zero to $O(\delta)$. In minimal subtraction we can therefore ignore the Σ_0^{-1} contribution as it is $O(\delta)$, and so

$$z = z_0 + \frac{z_1}{N} + O(1/N^2), \qquad \gamma_\sigma = \frac{\gamma_{\sigma,1}}{N} + O(1/N^2), \tag{B27}$$

we find at orders 1 and $1/(N\delta)$

$$p(s) + \frac{z_0}{2} = 0, \qquad \frac{z_0}{2N\delta}(2Z_{g,1} + z_0 K_1 + z_0^2 K_2) = 0. \tag{B28}$$

Looking at the coefficients of $\log X$,

$$\frac{z_0}{2N}(-2\gamma_\sigma - 4\gamma_\phi + z_0(2K_1) + z_0^2(4K_2))\log X = 0. \tag{B29}$$

Since this equation must hold for all values of X, we conclude that

$$\begin{aligned} z_0 &= -2p(s) = -2/\mathcal{N}_b(s), \\ \Delta_\sigma + 2\Delta_\phi - d &= \gamma_\sigma + 2\gamma_\phi = -\varkappa_1 = z_0 K_1 + 2z_0^2 K_2 = \Gamma_{\text{tet}} + \Gamma_{\text{pr}}, \\ 2Z_{g,1} &= -z_0 K_1 - z_0^2 K_2 = -\Gamma_{\text{tet}} - \frac{1}{2}\Gamma_{\text{pr}}. \end{aligned} \tag{B30}$$

Continuing on, we find

$$z_1 = -z_0^2\Pi_1' - z_0^3\Pi_2' + 2\varkappa_1 p'(s). \tag{B31}$$

The ϕ equation is automatically finite at this order thanks to the Z_g counterterm, so

$$\gamma_{\phi,2} = 0, \tag{B32}$$

$$\begin{aligned} A_{\phi,2} = \frac{1}{p(\frac{d-s}{2})}&(-2(\Gamma_{\text{tet}} + \Gamma_{\text{pr}})p'(s) + z_0^2(\Sigma_1' - \Pi_1') \\ &+ z_0^3(\Sigma_2' - \Pi_2')), \end{aligned} \tag{B33}$$

confirming the lack of anomalous dimension in the long-range model.

The renormalization condition here has been input by requiring that z be finite at each order in $1/N$, and it is easy to confirm by taking $C_\sigma = z/C_\phi^2$ that the above indeed matches the data in (3.5).

4. Solving the two-point functions for the short-range models, $s = 2k$

If computing directly at the short-range points (i.e., with $C^{-1} = \Box^k$), the difference is that $1/\mathcal{N}_b(\frac{d}{2} - k) = 0$. This only affects the ϕ equation of motion. The σ equation is the same, and hence to the given orders in N, some of the above results still hold, but with $s = 2k$ and, crucially, $\gamma_\phi \neq 0, Z_\phi \neq 1$:

$$\begin{aligned} z_0 &= -2/\mathcal{N}_b(s)|_{s=2k}, \quad z_1 = (\text{B.31})|_{s=2k}, \\ \gamma_\sigma + 2\gamma_\phi &= \left.\frac{\Gamma_{\text{tet}} + \Gamma_{\text{pr}}}{N}\right|_{s=2k}, \quad Z_\phi Z_g = 1 - \left.\frac{\Gamma_{\text{tet}} + \frac{1}{2}\Gamma_{\text{pr}}}{2N}\right|_{s=2k}. \end{aligned} \tag{B34}$$

The only differences are that now we also need to renormalize the ϕ propagator, since for integer k it is local and therefore can receive corrections. Our use of DREG here implies that, as usual, we have implicitly put a Lorentz-invariant counterterm for each of the possible relevant short-range propagators $\phi\Box^{0\le m<k}$, to cancel all power-law divergences (and all the irrelevant ones too, in fact—but they do not matter).

Considering the ϕ equation of motion, $1/\mathcal{N}_b(\frac{d}{2} - k) = 0$ means that Z_ϕ drops out of the equation. This is because the free conformal propagator has no inverse, because it is in an exceptional irreducible representation of the conformal group [58,70]. Instead, we avoid inverting the conformal propagators and consider the geometric series version of the ϕ Schwinger-Dyson equation (B9),

$$G = G_0 + G_0 \star \Pi \star G, \qquad \Pi_{xy} = -2\frac{\delta\Gamma_2}{\delta G_{xy}}, \tag{B35}$$

where the convolution just means matrix multiplication with respect to the indices x, y. We could have used this equation for σ as well, but it would have been messier. Using the propagator merging relations (A48), we find

$$G = \frac{1}{Z_\phi}\frac{C_{\phi,0}}{X^{d-2k}}\left(1 + \frac{Z_\phi^2 z_0}{N} X^{2\delta} U\left(\frac{d}{2}-k, \frac{d}{2}-\delta, \delta+k\right) \times U\left(\frac{d}{2}-k, \frac{d}{2}-\delta+k, \delta\right) + O(1/N^2)\right). \tag{B36}$$

To minimally subtract the divergence we choose

$$Z_\phi = 1 - \frac{\hat{\gamma}_{\phi,1}}{N\delta} + O(1/N^2), \tag{B37}$$

where indeed the new anomalous dimension for ϕ is equal to $\hat{\gamma}_{\phi,1}/N + O(1/N^2)$, with

$$\hat{\gamma}_{\phi,1} = \frac{2F_b'(2k)}{F_b''(\frac{d}{2}-k)}. \tag{B38}$$

Through (B34), it is of course necessary to also modify Z_g. Putting everything together, we find the results of Sec. III F, including $\hat{\gamma}_{\phi,2}$ which agrees with the $k=1$ vector model results in [11,53].[15] The remaining short-range normalization data can then be obtained from

$$\hat{A}_{\phi,1} = \hat{\gamma}_{\phi,1}\left(\frac{F_b'''(\frac{d}{2}-k)}{2F_b''(\frac{d}{2}-k)}\right) = \hat{\gamma}_{\phi,1}\left(\psi^{(0)}\left(\frac{d}{2}-k\right) - \psi^{(0)}\left(\frac{d}{2}+k\right) - \frac{1}{k}\right) \tag{B39}$$

and (B34).

5. A different scheme: Nonminimal subtraction

We could instead unit normalize the propagators[16]; if we do so, then the long-range model counterterms at finite δ would be identical to those of the short-range model. This would be necessary if we wanted the canonically normalized OPE coefficients, as in [11,12].

In that case, we would need to define the following while still at finite δ:

$$Z_\phi' = Z_\phi^{\overline{\text{MS}}} C_\phi, \qquad Z_g' = Z_g^{\overline{\text{MS}}} C_\sigma^{\frac{1}{2}}, \tag{B40}$$

with $C_{\phi/\sigma}$ as defined by (3.16). However, the short- and long-range theories would be distinguished by the difference that the limit

$$\lim_{\delta\to 0}\lim_{s\to 2k} \tag{B41}$$

is not the same as

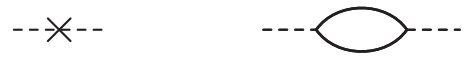
$$\lim_{s\to 2k}\lim_{\delta\to 0}, \tag{B42}$$

which is clear from consideration of $A_{\phi/\psi,1,\text{pil}}$. This shows that the difference in counterterms for the SR (3.22) and LR (3.15) cases is only due to our choice of minimal subtraction. Defining the unit-normalized arbitrary-s theory at finite δ to be $T_\delta(s)$, the SR/LR crossover is then expressed by

$$\lim_{s\to 2k-2\hat{\gamma}_\phi}\lim_{\delta\to 0} T_\delta(s) = \lim_{\delta\to 0}\lim_{s\to 2k} T_\delta(s). \tag{B43}$$

6. Non-2PI formalism

If we did not work in the 2PI formalism, the subtlety mentioned below (3.13) comes into play. If we approach (3.13) as a QFT, using $1/N$ as the expansion parameter, and start writing down all possible diagrams, the two possible $O(N^0)$ self-energy insertions on any σ line are



$\mu^{2\delta}K^{-1}(x,y)$ $\qquad$ $\frac{1}{2}\times\frac{NZ_\phi^2 g_0^2}{N}C(x,y)^2$ (B44)

Since each contribute at order N^0 (we have included the symmetry factor $\frac{1}{2}$ on the ϕ loop), we naively would need to resum them to all orders.

However, take any σ line in any diagram D, i.e.,

$$D = \int_{w,z} f(w) K_\delta(w,z) g(z) \tag{B45}$$

and insert each of these separately on that line. The sum of the two diagrams that arise is

$$\int_{w,x,y,z} f(w)K_\delta(w,x) \times \Big(\underbrace{\mu^{2\delta}K^{-1}(x,y) + \frac{1}{2}\times\frac{NZ_\phi^2 g_0^2}{N}C(x,y)^2}_{=\mu^{2\delta}(1-Z_g^2)K^{-1}(x,y)}\Big) K_\delta(y,z) \tag{B46}$$

which is order $1/N$. So, we only need to consider a finite number of insertions of $\mu^{2\delta}(1-Z_g^2)K^{-1}(x,y) \propto 1/N$—as many as are required at the given order in N [48]. This is all quite fiddly; this is why the 2PI formalism is the best approach here, as any diagram including such an insertion as this is manifestly 2PI and so need not be considered.

[15]For explicit matching with the results of Vasil'ev *et al.*: $\mu = d/2$, $\alpha = \Delta_\phi = \mu - 1 + \eta/2$, $\beta = \Delta_\sigma = 2 - \eta - \varkappa$, $\gamma_\phi = \eta/2$, and $\gamma_\sigma = -\eta - \varkappa$; thus $\Delta_\sigma + 2\Delta_\phi - d = -\varkappa$. Compare also [50], which contains a summary of O(N) model data.

[16]Unit normalization is not always the natural choice, as it can lead to spurious divergences of OPE coefficients [71].

APPENDIX C: REGULATED ACTION CONSTRUCTION FOR GROSS-NEVEU

We now perform the analysis of Sec. III for the GN model. As before, we introduce a pair of real auxiliary fields:

(1) σ, which has an imaginary contour of integration;
(2) $\rho(x)$, which we set to equal $\frac{1}{\sqrt{N}}\bar{\psi}_i(x)\psi_i(x)$.

Then we insert

$$1 = \int \mathrm{D}\rho\delta\left(\rho - \frac{1}{\sqrt{N}}\bar{\psi}_i\psi_i\right) = \int \mathrm{D}\rho\mathrm{D}^{(\mathrm{I})}\sigma e^{+\int_x \sigma(\rho - \frac{1}{\sqrt{N}}\bar{\psi}_i\psi_i)} \tag{C1}$$

into the partition function (1.2). For convenience, we have switched the sign of σ in the δ function implementation compared to the same identity in the O(N) case; this is to ensure that we end up with $\log(C^{-1} - \sigma\mathbb{I}/\sqrt{N})$ in the σ effective action, which eliminates extra minus signs in the Feynman rules in both the O(N) and GN models. We find

$$Z_{\mathrm{GN}} = \int \mathrm{D}\psi\mathrm{D}\bar{\psi}\mathrm{D}^{(\mathrm{I})}\sigma\mathrm{D}\rho \exp\left(\int_x \bar{\psi}_i\left(C^{-1} - \frac{1}{\sqrt{N}}\sigma\right)\psi_i - \frac{1}{2}\lambda_0\rho^2 + \sigma\rho\right) \tag{C2}$$

$$= \int \mathrm{D}^{(\mathrm{I})}\sigma \exp\left(+\mathrm{tr}\,\log\left(C^{-1} - \frac{1}{\sqrt{N}}\sigma\mathbb{I}\right) + \int_x \frac{1}{2\lambda_0}\sigma^2\right), \tag{C3}$$

where we again used $Z_\rho = 1$. This action now matches (1.5). In the IR, we drop the UV kinetic term for σ.[17] As before, if we used a hard cutoff, we could choose to work with this action at the critical point ($\lambda_0 \to \infty$). This is too hard, so instead we introduce the analytic regulator δ, and define

$$Z_{\mathrm{GN},\delta} = \int \mathrm{D}\psi\mathrm{D}\bar{\psi}\mathrm{D}^{(\mathrm{I})}\sigma e^{-S_{\mathrm{GN},\delta}}, \tag{C4}$$

with action

$$S_{\mathrm{GN},\delta} \equiv \int_{x,y} -Z_\psi\bar{\psi}_{i,x}\left(C^{-1}_{xy} - \frac{Z_g g\mu^\delta}{\sqrt{N}}\sigma_x\delta_{xy}\right)\psi^i_y + \frac{1}{2}\int_{x,y}\sigma_x(K^{-1}_{\delta,xy} - \mu^{2\delta}K^{-1}_{xy})\sigma_y. \tag{C5}$$

The regulator terms are identical other than a new value for $C_{\sigma,0}$,

$$K_\delta = \frac{C_{\sigma,0}}{s(x,y)^{2(s-\delta)}}, \qquad K = K_{\delta=0}. \tag{C6}$$

[17]Here this is simply a δ function. However, for the general Gross-Neveu-Yukawa theory that is marginal in $d = 2k - \epsilon$ ($k \in \mathbb{Z}$), there are other terms that must be dropped, because $S_{\mathrm{GNY}} = \int_x -\bar{\psi}_i[C^{-1} - g_0\sigma]\psi^i + \frac{1}{2}\sigma(-\partial^2)^{k-1}\sigma + \text{counterterms}$ [1]; in all cases, we find in the IR an action identical to $\lim_{\lambda_0\to\infty}$ (C3).

The minimal subtraction counterterms are

$$Z_\psi = \begin{cases} 1 & s \notin 2\mathbb{Z}+1 \\ 1 - \frac{1}{N\delta}\hat{\gamma}_{\psi,1} + O(1/N^2) & s = 2k \in 2\mathbb{Z}+1 \end{cases},$$

$$Z_\psi Z_g = 1 - \frac{1}{2N\delta}\Gamma^{\mathrm{GN}}_{\mathrm{tet}} + O(1/N^2), \tag{C7}$$

with the usual $\hat{\gamma}_{\psi,1} = 2F'_b(2k)/F''_f(\frac{d}{2} - k)$ in the short-range case. The calculation of these counterterms is identical to that of Appendix B 3, save for the differing Lorentz representation, and is therefore not reproduced here.

1. Conformal data

We find the conformal data given in Eq. (2.2) for the long-range model, which agree with [12], and the short-range data agree with [36]. The normalizations are

$$C_{\sigma/\psi} = C_{\sigma/\psi,0}\left(1 + \frac{A_{\sigma/\psi,1}}{N} + \frac{A_{\sigma/\psi,2}}{N^2}\right), \tag{C8}$$

for

$$C_{\sigma,0} = \frac{1}{(C_{\psi,0})^2}\frac{-1}{\mathcal{N}_b(s)} = -\frac{4^s A_f(\frac{d-s}{2})^2}{A(d-s)A(s)},$$
$$A_{\sigma,1} = A_{\sigma,1,\mathrm{tet}} + A_{\sigma,1,\mathrm{pil}}$$
$$A_{\sigma,1,\mathrm{tet}} = \Gamma^{\mathrm{GN}}_{\mathrm{tet}}\left(B_f\left(\frac{d-s}{2}\right) - B(d-s)\right),$$
$$A_{\sigma,1,\mathrm{pil}} = \frac{4F'_b(s)}{F'_f(\frac{d-s}{2})}, \tag{C9}$$

and

$$C_{\psi,0} = \frac{-i}{\mathcal{F}_{2(\frac{d-s}{2}),1}} = \frac{\pi^{-\frac{d}{2}}2^{-s}}{A_f(\frac{d-s}{2})}, \tag{C10}$$

$$A_{\psi,1} = -\frac{1}{2}A_{\sigma,1,\mathrm{pil}},$$
$$A_{\psi,2} = -\frac{F'_b(s)}{F'_f(\frac{d-s}{2})}\Gamma^{\mathrm{GN}}_{\mathrm{tet}}\left(B(s) + B_f\left(\frac{d-s}{2}\right) - 2B(d-s)\right). \tag{C11}$$

The short-range results follow through identically, including the short-range $O(1/N^2)$ correction (3.25) to $\hat{\gamma}_\psi$. Following the steps in Sec. IV, we find the first correction

$$\frac{F_{\mathrm{GN},1/N}}{F'_b(s)/N} = \begin{matrix} +3\times\frac{1}{2}\Gamma^{\mathrm{GN}}_{\mathrm{tet}} \\ -2\times\frac{1}{2}\Gamma^{\mathrm{GN}}_{\mathrm{tet}} \end{matrix} + \begin{cases} +3\times\frac{1}{2}(-2\hat{\gamma}_{\psi,1}) & \\ -2\times\frac{1}{2}(-2\hat{\gamma}_{\psi,1}) & s = 2k, \\ 0 & s \neq 2k, \end{cases} \tag{C12}$$

and hence the free energy agrees with [1]

$$F_{\rm GN} = -\lim_{\delta\to 0} \log Z_{{\rm GN},\delta} = (2.16a). \tag{C13}$$

We provide some notes on these vacuum diagram computations in Appendix D 6. As expected, we also find that

$$F^{\rm LR}_{\rm GN}(s = 2k - 2\hat{\gamma}_\psi) = F^{{\rm SR},\phi\Box^{k-1/2}}_{\rm GN} \tag{C14}$$

extremizes $F^{\rm LR}_{\rm GN}(s)$.

We conclude with a note that the natural expression for the $\mathcal{N} = 1$ (real) supersymmetric model is then for a real scalar superfield Σ,

$$Z_{\rm SUSY} = \int {\rm D}^{(1)}\Sigma \exp\left(-\frac{1}{2}{\rm Str}\,\log\left(C^{-1} - \frac{1}{\sqrt{N}}\Sigma\mathbb{I}\right) + \int_S \frac{1}{2\lambda_0}\Sigma^2\right) \tag{C15}$$

for the suitable $C^{-1} \sim D^2$ in superspace S; of course, this action would then need to be modified by a regulator.

APPENDIX D: EVALUATING DIAGRAMS

In this paper we have had to calculate both two-point function diagrams and vacuum diagrams—though of course, as we commented in Sec. IV C 4, these calculations are manifestly related. The former we calculated both on the sphere and in flat space, while the latter is finite (and so calculable) only on the sphere. While on the sphere we can explicitly calculate both kinds of diagrams via Mellin-Barnes integrals, for flat space the easiest approach to finding the leading anomalous dimensions is to use the uniqueness relations given above—which can be done manually. To make this paper self-contained, in this appendix we explain the calculation of flat-space diagrams via the method of uniqueness and then discuss diagram evaluation by means of Mellin-Barnes integrals.

1. Using uniqueness to calculate flat-space diagrams

The method of uniqueness only works for the σ equation (B11b); our goal will be to derive in flat space the results, Eqs. (B20c) and (B20d). Now, we cannot apply uniqueness directly to these diagrams, as none of their vertices satisfy the sum requirement $a + b + c = d$; however, we can consider diagrams which differ from the diagram we want only up to terms that are $O(\delta^2)$, but that are unique.

In flat space, let us take both of our original diagrams, throw away the anomalous dimensions, and define new diagrams (D4) and (D5), where each of the exponents $\{\frac{d-s}{2}, s\}$ have been shifted by small amounts: by some parameter η for $2\frac{\delta I_{\rm tet}}{\delta\Sigma}$ and η and η' for $2\frac{\delta I_{\rm pr}}{\delta\Sigma}$. This is carefully done such that (1) the diagrams become manifestly invariant under, separately,

$$\eta \to -\eta, \quad \text{or} \quad \eta' \to -\eta'. \tag{D1}$$

This means that the diagrams evaluated as a function of these shifts differ only at $O(\eta^2)$ or $O(\eta'^2)$ (with no cross term) from the originals,

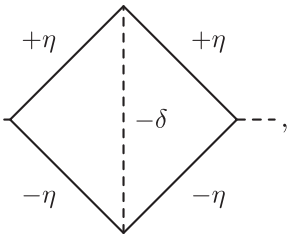
$$2\frac{\delta I_{\rm tet}}{\delta\Sigma}(\delta,\eta) = 2\frac{\delta I_{\rm tet}}{\delta\Sigma}(\delta,0) \times (1 + O(\eta^2)), \tag{D2}$$

$$2\frac{\delta I_{\rm pr}}{\delta\Sigma}(\delta,\eta,\eta') = 2\frac{\delta I_{\rm pr}}{\delta\Sigma}(\delta,0,0) \times (1 + O(\eta^2) + O(\eta'^2)). \tag{D3}$$

We also ensure that the shifts are (2) such that for some particular values $\eta, \eta' \propto \delta$ we can apply uniqueness. Since the original diagrams can only have at worst a $1/\delta$ pole, they are identical to the shifted diagrams at this order.

The required shifts must be determined by inspection and are illustrated here. The solid lines are propagators of scaling dimension $\frac{d-s}{2}$, and the dashed lines are propagators of scaling dimension s; the numbers on the lines then indicate the shifts away from that value. We define the shifted diagram

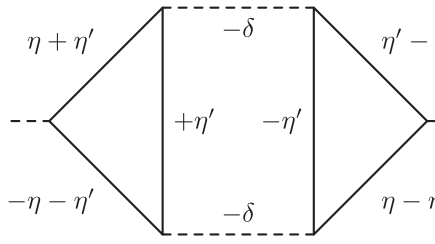
$$2\frac{\delta I_{\rm tet}}{\delta\Sigma}(\delta,\eta) \equiv \frac{g_0^4}{2N}$$



, (D4)

which is manifestly invariant under $\eta \to -\eta$. The prism diagram requires two shifts

$$2\frac{\delta I_{\rm pr}}{\delta\Sigma}(\delta,\eta,\eta') = \frac{g_0^6}{2N}$$



, (D5)

and so is a little more complicated. Because Lorentz invariance means that these diagrams are invariant under $x \leftrightarrow y$, the identities

$$2\frac{\delta I_{\rm pr}}{\delta\Sigma}(\delta,\eta,0) \equiv 2\frac{\delta I_{\rm pr}}{\delta\Sigma}(\delta,-\eta,0), \qquad 2\frac{\delta I_{\rm pr}}{\delta\Sigma}(\delta,\eta,\eta') = 2\frac{\delta I_{\rm pr}}{\delta\Sigma}(\delta,\eta,-\eta') \tag{D6}$$

are true by inspection.[18] Hence, the Taylor expansions in the η's must take the form (D2). We then notice that for $\eta = -\delta/2$ and $\eta = \eta' = -\delta$ we have unique triangles. Hence, we can calculate the desired

$$2\frac{\delta I_{\rm tet}}{\delta\Sigma}(\delta,-\delta/2) = \frac{g_0^4}{2N}\frac{C_\sigma C_\phi^4}{X^{2(d-s)-2\delta}} U(2\delta,\mu-\delta,\mu-\delta)U\left(s-\delta,\frac{d-s+\delta}{2},\frac{d-s+\delta}{2}\right), \tag{D7}$$

$$2\frac{\delta I_{\rm pr}}{\delta\Sigma}(\delta,-\delta,-\delta) = \frac{g_0^6}{2N}\frac{C_\sigma^2 C_\phi^6}{X^{2(d-s)-4\delta}} U\left(\frac{d}{2}-2\delta,\frac{d}{2}-2\delta,4\delta\right)U\left(s-\delta,\frac{d-s}{2},\frac{d-s}{2}+\delta\right) \tag{D8}$$

$$\times U\left(s-2\delta,d-\frac{3s}{2}+\delta,\frac{s}{2}+\delta\right)U\left(s-\delta,\frac{d-s}{2}-\delta,\frac{d-s}{2}+2\delta\right), \tag{D9}$$

which is easily done using the STR package [65]. But from (D2), we see that these are also the values of our desired diagrams, up to an $O(\delta)$ correction term, because $2\frac{\delta I_i}{\delta\Sigma}(\delta,0)\times\eta^2 \propto \delta$. After expanding the U's for small δ, we find exactly Eqs. (B20c) and (B20d).

2. Evaluation of diagrams on spheres via Mellin-Barnes integrals

We use stereographic coordinates on the Euclidean sphere to evaluate the various sphere integrals. For completeness, we give here two exactly solvable sphere integrals [6],

$$I_2(\Delta) \equiv \int_{x,y}\frac{1}{s(x,y)^{2\Delta}} = \frac{\pi^d}{(2R)^{2(\Delta-d)}}\frac{\Gamma(\frac{d}{2})}{\Gamma(d)}\frac{\Gamma(\frac{d}{2}-\Delta)}{\Gamma(d-\Delta)}, \tag{D10}$$

$$I_3(\Delta) \equiv \int_{x,y,z}\frac{1}{(s(x,y)s(y,z)s(z,x))^{\Delta}} = \frac{8\pi^{\frac{3(d+1)}{2}}}{\Gamma(d)}\frac{\Gamma(d-\frac{3\Delta}{2})}{\Gamma(\frac{d+1-\Delta}{2})^3}R^{3(d-\Delta)}. \tag{D11}$$

We often use the fact that

$$I_2'(d) \equiv \left.\frac{\mathrm{d}I_2}{\mathrm{d}\Delta}\right|_{\Delta=d} = \frac{2}{\sin(\frac{\pi d}{2})}\frac{\pi^{d+1}}{\Gamma(d+1)}. \tag{D12}$$

3. Mellin-Barnes notes

We use the Mellin-Barnes procedures precisely as described in [3], using MBRESOLVE [72] from the MBTOOLS collection [73,74]. In each case, we always fix one of the points, $w \to \infty$, which reduces the number of integrals that we are required to do. As mentioned above, this had the additional advantage of immediately giving a valid Mellin-Barnes representation; that is, we could avoid needing to perform the extra analytic shifts that were required in Appendix D 1. The reason for this is not clear.

4. The tetrahedron and prism diagrams

To leading order in the shifts $\Delta_\phi = \frac{d-s}{2}+\gamma_\phi$, $\Delta_\sigma = s+\gamma_\sigma$, we find

$$I_{\rm tet} = \frac{1}{8N}6C_\sigma^2 C_\phi^4 \pi^{2d}\frac{\Gamma(-\frac{d}{2})}{\Gamma(d)}\frac{\Gamma(\frac{d}{2}-s)}{\Gamma(s)}\left(\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{d-s}{2})}\right)^2(1+I_{\rm tet,corr}/N), \tag{D13}$$

if each γ is $\sim 1/N$. The next order in N is easily calculated as well,

$$\frac{I^{(1)}_{\rm tet,corr}}{N} = \left(-2(\gamma_\sigma+2\gamma_\phi)\left(\psi^{(0)}\left(-\frac{d}{2}\right)+\gamma+2\log(2R)\right) + \frac{2}{3}(\gamma_\phi-\gamma_\sigma)\left(\psi^{(0)}\left(\frac{d}{2}-s\right)-\psi^{(0)}\left(\frac{d-s}{2}\right)-\psi^{(0)}\left(\frac{s}{2}\right)+\psi^{(0)}(s)\right)\right). \tag{D14}$$

[18] Specifically, as diagrams $\mathrm{pr}_{-\eta,0} = \mathrm{pr}_{\eta,0} \overset{\star}{=} \mathrm{pr}_{\eta,0}$ and $\mathrm{pr}_{\eta,-\eta'} = \mathrm{pr}_{\eta,\eta'} \overset{\star}{=} \mathrm{pr}_{\eta,\eta'}$, where the equalities $\overset{\star}{=}$ follow from the symmetry under the exchange of x and y.

Though other ways of combining these terms exist, this can be written most compactly using (4.18). With the usual $\varkappa = d - 2\Delta_\phi - \Delta_\sigma = -2\gamma_\phi - \gamma_\sigma$,

$$\begin{aligned}\frac{I_{\text{tet}}}{C_\sigma^2 C_\phi^4} &= \frac{3}{8N}\frac{2\pi^{2d}}{\Gamma(d)} e^{2\gamma\varkappa}(2R)^{4\varkappa}\Gamma\left(2\varkappa - \frac{d}{2}\right)\frac{\Gamma(-\frac{d}{2}+\frac{2\varkappa}{3}+2\Delta_\phi)}{\Gamma(d-2\Delta_\phi-\frac{2\varkappa}{3})}\frac{\Gamma(\frac{d}{2}-\Delta_\phi-\frac{\varkappa}{3})^2}{\Gamma(\Delta_\phi+\frac{\varkappa}{3})^2} + O(1/N^2)\\ &= \frac{3}{8N}\frac{2\pi^{2d}}{\Gamma(d)} e^{2\gamma\varkappa}(2R)^{4\varkappa}\Gamma\left(2\varkappa - \frac{d}{2}\right)\frac{\Gamma(\frac{d}{2}-\Delta_\sigma-\frac{\varkappa}{3})}{\Gamma(\Delta_\sigma+\frac{\varkappa}{3})}\frac{\Gamma(\frac{d}{2}-\Delta_\phi-\frac{\varkappa}{3})^2}{\Gamma(\Delta_\phi+\frac{\varkappa}{3})^2} + O(1/N^2)\\ &= \frac{3}{8N}\frac{2\pi^{2d}}{\Gamma(d)} e^{2\gamma\varkappa}(2R)^{4\varkappa}\Gamma\left(2\varkappa - \frac{d}{2}\right)A\left(\Delta_\sigma+\frac{\varkappa}{3}\right)A\left(\Delta_\phi+\frac{\varkappa}{3}\right)^2 + O(1/N^2).\end{aligned} \tag{D15}$$

For our next diagram, we find

$$\frac{I_{\text{pr}}}{C_\sigma^3 C_\phi^6} = \frac{5}{12N}\pi^{3d}\frac{\Gamma(-\frac{d}{2})}{\Gamma(d)}\frac{\Gamma(\frac{3s}{2}-\frac{d}{2})}{\Gamma(d-\frac{3s}{2})}\left(\frac{\Gamma(\frac{d}{2}-s)}{\Gamma(s)}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{d-s}{2})}\right)^3(1 + I^{(1)}_{\text{pr,corr}}/N). \tag{D16}$$

The subleading correction to this is compactly encoded by

$$\frac{I_{\text{pr}}}{C_\sigma^3 C_\phi^6} = \frac{1}{12N}\frac{5}{2}\frac{2\pi^{3d}}{\Gamma(d)} e^{3\gamma\varkappa}(2R)^{6\varkappa}\Gamma\left(3\varkappa - \frac{d}{2}\right)A\left(\Delta_\sigma+\frac{\varkappa}{5}\right)^3 A\left(\Delta_\phi+\frac{3\varkappa}{5}\right)^3 A\left(-\frac{d}{2}+3\Delta_\phi+\frac{3\varkappa}{5}\right) + O(1/N^2). \tag{D17}$$

Of course, we can move the subleading terms around, and we note that up to γ_i^2 corrections that we did not calculate,

$$A\left(\frac{d}{2}-\Delta_\sigma+\Delta_\phi-\frac{2\varkappa}{5}\right) = A\left(d-\frac{3}{2}\Delta_\sigma-\frac{9\varkappa}{10}\right) = A\left(-\frac{d}{2}+3\Delta_\phi+\frac{3\varkappa}{5}\right). \tag{D18}$$

5. The pillow diagram for arbitrary scaling dimensions

Let us now evaluate in full generality the pillow integral for arbitrary Δ_i and C_i. We use first MBRESOLVE, then MBSUMS [75], and then the *Mathematica* SUM routine. Let us parametrize, for compactness, $d = 2\mu$, $\Delta_\sigma + 2\Delta_\phi + \varkappa = d$, and $\Delta_\phi = \Delta$. Then $I_{\text{pil}}(\mu, \varkappa, \Delta)$,

$$\begin{aligned}\frac{I_{\text{pil}}}{C_\sigma^2 C_\phi^4} = \frac{1}{4N}\frac{\pi^{4\mu}(2R)^{4\varkappa}}{\Gamma(2\mu)}\frac{\Gamma(\mu-\Delta)}{\Gamma(\Delta)}\frac{\Gamma(\mu-\varkappa)}{\Gamma(\varkappa)}\frac{\Gamma(\Delta+\varkappa-\mu)}{\Gamma(2\mu-(\Delta+\varkappa))}\Gamma(\varkappa-\mu+1)[\Gamma(\mu-\Delta)\Gamma(2\varkappa-\mu)\Gamma(\Delta+\varkappa-\mu)\\ \times {}_4\tilde{F}_3(\varkappa, \Delta+\varkappa-\mu, 2\varkappa-\mu, \mu-\Delta; \varkappa-\mu+1, \Delta+2\varkappa-\mu, \mu+\varkappa-\Delta; 1)\\ -\Gamma(\Delta)\Gamma(\mu)\Gamma(2\mu-\Delta-\varkappa){}_4\tilde{F}_3(\Delta, \varkappa, \mu, 2\mu-\Delta-\varkappa; \Delta+\varkappa, \mu-\varkappa+1, 2\mu-\Delta; 1)].\end{aligned} \tag{D19}$$

The prefactor here can be simplified using $A(x) \equiv \Gamma(\frac{d}{2}-x)/\Gamma(x)$ to

$$= \frac{1}{4N}\frac{\pi^{4\mu}(2R)^{4\varkappa}}{\Gamma(2\mu)}\frac{A(\Delta)A(\varkappa)}{A(\Delta+\varkappa-\mu)}\Gamma(\varkappa-\mu+1)\times[\cdots]. \tag{D20}$$

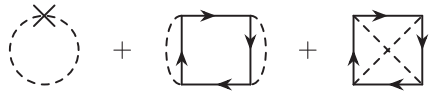
For generic s, I_{pil} vanishes $\propto \varkappa$, as given in (4.20), or alternatively by

$$I_{\text{pil}} = \frac{\pi^d}{2N}\frac{\Gamma(\frac{d}{2})\Gamma(-\frac{d}{2})}{\Gamma(d)}\mathcal{N}_b\left(\frac{d-s}{2}\right)\varkappa + O(\varkappa^2). \tag{D21}$$

However, since $\varkappa = \delta$, the existence of the counterterm means that this diagram may contribute at order $1/N^2$. In the case of s an even integer, we need to be a little more careful. *Mathematica* can only take this limit numerically, but for small $\varkappa = \delta$ we obtain from (D19) the finite result shown in (4.20).

6. Gross-Neveu for variable s

We can also calculate the arbitrary-s contributions to the free energy for the GN CFT. They are diagrammatically identical to those of (4.15) save for the absence of the prism diagram and differing symmetry factors (which here include the minus sign from the fermion loop),

$-\delta F_{\text{GN},1/N} \equiv$  .

$I^{\text{GN}}_{\text{ct}} : \frac{1}{2}$ $I^{\text{GN}}_{\text{pil}} : (-1)\frac{1}{2}$ $I^{\text{GN}}_{\text{tet}} : (-1)\frac{1}{4}$ (D22)

The counterterm diagram computation is identical to that of the O(N) model. To confirm these symmetry factors, we just expand

$$\log\int_{\sigma,\psi} e^{-\frac{1}{\sqrt{N}}\int\sigma\bar\psi_i\psi_i} \supset \frac{N}{4!(\sqrt{N})^4}\left\langle\left(\int\sigma\bar\psi\psi\right)^4\right\rangle_0^c = \frac{1}{4!N}\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle_0^c \times \underbrace{\langle(\bar\psi\psi)_1(\bar\psi\psi)_2(\bar\psi\psi)_3(\bar\psi\psi)_4\rangle_0^c}_{\equiv(-1)6L(x_i)}. \tag{D23}$$

To evaluate the fermion loop $L(x_i)$ on the sphere,[19] we use the embedding space propagator $C_{12}=C(\eta_1(x_1),\eta_2(x_2))$ defined in (A32); evaluating the traces,[20] using $\eta_i\cdot\eta_j=-\frac{1}{2}[(\eta_i-\eta_j)^2-\eta_i^2-\eta_j^2$ and $\eta_i^2=R^2$, and then mapping $|\eta_1(x_1)-\eta_2(x_2)|=s(x_1,x_2)$, we find

$$L(x_i)\equiv\frac{1}{T}\mathrm{Tr}_{S_{d+1}}[C_{12}C_{23}C_{34}C_{41}] = \frac{1}{2}C_\psi^4\frac{s_{12}^2s_{34}^2-s_{13}^2s_{24}^2+s_{14}^2s_{23}^2}{(s_{12}s_{23}s_{34}s_{41})^{2\Delta_\psi}}. \tag{D24}$$

Thus, our diagrams are

$$I_{\rm ct}^{\rm GN}=\frac{1}{2}\left(-\mu^{2\delta}\frac{2Z_{g,1}}{N\delta}\right)\mathrm{tr}\,K^{-1}K_\delta, \tag{D25a}$$

$$I_{\rm pil}^{\rm GN}=\frac{1}{N}\frac{-1}{2}\int_{x_{1,2,3,4}}\frac{C_\sigma^2}{(s_{12}s_{34})^{2\Delta_\sigma}}L(x_i), \tag{D25b}$$

$$I_{\rm tet}^{\rm GN}=\frac{1}{N}\frac{-1}{4}\int_{x_{1,2,3,4}}\frac{C_\sigma^2}{(s_{13}s_{24})^{2\Delta_\sigma}}L(x_i). \tag{D25c}$$

They can be calculated in the same way, though we note that ensuring the existence of the Mellin-Barnes representation of the pillow diagram for $s=2k, k-\frac{1}{2}\in\mathbb{Z}$ requires a modification of the powers in the numerator of L from 2 to $2+\lambda$. Taking

$$\varkappa=d-\Delta_\sigma-2\Delta_\psi, \tag{D26}$$

we find once again that the pillow-type diagram vanishes $\propto\varkappa$, and hence in our case $\propto\delta$. Specifically, for $s\neq 2k$, we find

$$F_{1/N,\rm pil}^{\rm GN}=\frac{C_\sigma^2C_\psi^4}{N}(2R)^{4\varkappa}\pi^{2d}\frac{\Gamma(\frac{d}{2})\Gamma(-\frac{d}{2})}{\Gamma(d)}\frac{(d+s^2-1)}{s^2-1}\frac{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})}{\Gamma(\frac{d-s+1}{2})\Gamma(\frac{d+s+1}{2})}\varkappa+O(\varkappa^2) = -\frac{C_\sigma^2C_\psi^4}{N}(2R)^{4\varkappa}\frac{\Gamma(\frac{d}{2})\Gamma(-\frac{d}{2})}{\Gamma(d)}\frac{(d+s^2-1)}{s^2-1}\pi^d\mathcal{N}_\psi\left(\frac{d-s}{2}\right)\varkappa+O(\varkappa^2), \tag{D27}$$

meaning that, just like the bosonic case, we have

$$I_{\rm pil}^{\rm GN}=\begin{cases}O(\delta) & s\neq 2k,\\ 3\times\frac{1}{N}F_b'(2k)\hat\gamma_{\psi,1}+O(\delta) & s=2k.\end{cases} \tag{D28}$$

Once again, due to $Z_{g,1}\supset 1/\delta$, the pillow may contribute at order $1/N^2$ in the arbitrary-s case.

The contribution from $I_{\rm tet}^{\rm GN}$ is, for any value of s,

$$F_{1/N,\rm tet}^{\rm GN}=-I_{\rm tet}^{\rm GN}=\frac{3}{2N}C_\sigma^2C_\psi^4\pi^{2d}\frac{\Gamma(-\frac{d}{2})}{\Gamma(d)}\frac{\Gamma(\frac{d}{2}-s)}{\Gamma(s)}\frac{\Gamma(\frac{s+1}{2})^2}{\Gamma(\frac{d-s+1}{2})^2}, \tag{D29}$$

which we write compactly with its $1/N$ correction as

$$F_{1/N,\rm tet}^{\rm GN}=\frac{3}{2N}C_\sigma^2C_\psi^4\frac{\pi^{2d}}{\Gamma(d)}e^{2\gamma\varkappa}(2R)^{4\varkappa}\Gamma\left(2\varkappa-\frac{d}{2}\right)A\left(\Delta_\sigma+\frac{\varkappa}{3}\right)A_f\left(\Delta_\psi+\frac{\varkappa}{3}\right)^2+O(1/N^2) \tag{D30}$$

with the usual $A_f(\Delta)=\Gamma(\frac{d}{2}-\Delta+\frac{1}{2})/\Gamma(\Delta+\frac{1}{2})$.

[19]This happens to be the same as the flat-space fermion loop, mapped from $|x-y|\to s(x,y)$; but in theories with more complex spin structure this is the correct method.

[20]The traces are over $d_{\rm embed}=d+1$-dimensional γ matrices α_i, but we keep $\mathrm{Tr}_{S_{d+1}}\alpha_i=T$; however, for example, $\alpha_i\alpha_j\alpha^i=(2-(d+1))\alpha_j$.

Finally, we note that, under an exchange of $\psi \to \phi$, with the extra factor of i noted in Sec. II E—i.e.,

$$iA_f \to A_b, \qquad C_\psi \to C_\phi, \qquad N_b = N \mapsto N_f = 2N, \tag{D31}$$

we find $F^{\mathrm{GN}}_{1/N,\mathrm{tet}} \to F^{\mathrm{O}(N)}_{1/N,\mathrm{tet}}$.

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