

Soft vanishing of the scattering amplitudes for cylindrical gravitational waves

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The graviton S matrix has a famous soft pole. We show that the S matrix for cylindrical gravitational waves has a soft zero. The soft pole for ordinary gravitons comes from a Ward identity for supertranslation symmetry at asymptotic infinity. We show that the soft zero for cylindrical gravitational waves comes from a Ward identity for Geroch symmetry at asymptotic infinity. Because it is a zero rather than a pole, there is no memory effect. Overall, this soft zero is a manifestation of Geroch symmetry and of the extraordinary simplicity of cylindrical gravitational waves.

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I. INTRODUCTION

In recent years, there has been a lot of interest in “infrared triangles” [1]. An infrared triangle is a triangle of ideas relating asymptotic symmetries, soft theorems, and memory effects. The interest in infrared triangles comes partly from the fact that they could be a stepping stone toward understanding holography for asymptotically flat gravity. The present paper came from trying to understand if there is an infrared triangle for cylindrical gravitational waves. Cylindrical gravitational waves enjoy a somewhat mysterious infinite dimensional symmetry called Geroch symmetry [2–8]. We wanted to understand if Geroch symmetry fits into an infrared triangle.

Cylindrical gravitational waves are effectively two dimensional. Our approach will be to dimensionally reduce general relativity to get a nonlinear sigma model and then to quantize the sigma model. A more physically realistic approach would be to quantize gravity first and then dimensionally reduce.¹ However, we are going to confine our study to tree-level physics. Tree-level physics is essentially classical physics and the sigma model describes classical cylindrical gravitational waves perfectly.

Our main result will be to show that cylindrical gravitational waves have a soft zero. This stands in stark contrast to the famous soft pole of ordinary gravitons. We begin by computing the tree-level two-particle S matrix for cylindrical

gravitational waves directly using Feynman diagrams. We observe that a soft zero is visible in these amplitudes. In the second part of the paper we provide a symmetry explanation for this soft zero. This zero is similar to an Adler zero because the dimensionally reduced theory is a nonlinear sigma model. However, it is not exactly an ordinary Adler zero because one of the symmetries we consider [Eq. (4.1)] is not exactly a shift symmetry and because the theory does not have r -translation invariance [Eq. (2.2)].

We will formulate the soft zero as a Ward identity for Geroch symmetry at asymptotic infinity. This is a piece of the infrared triangle for cylindrical gravitational waves. The original examples of infrared triangles were for gauge theories and asymptotically flat gravity² [18–26]. Those works established a template for relating soft theorems to asymptotic symmetries. We will follow that template closely in this work. Since the underlying template is the same, let us just comment on the differences.

First, our symmetry is Geroch symmetry. This is a global symmetry of the sigma model. Thus, these are not just asymptotic symmetries, they are really “everywhere symmetries.” Of course, they also act asymptotically and we will phrase our discussion in those terms so that we can see the connection to infrared triangles.

Second, let us clarify that the Geroch group is infinite dimensional but we are only ever going to use three Geroch transformations in this paper. One of the Geroch transformations we consider is a shift symmetry. The other is a slightly more complicated version of a shift symmetry which reduces to a shift symmetry in the free theory. In the future, it would be interesting to work out the full set of S -matrix constraints coming from Geroch symmetry.

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¹An interesting middle path is to dimensionally reduce the classical theory to three spacetime dimensions [9,10].

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²There has also been interesting work on the relationship between soft theorems and symmetries for scalar fields [11–17].

So far we have discussed soft theorems and asymptotic symmetries. The third corner of the infrared triangle is the memory effect. However, since cylindrical gravitational waves have a zero instead of a pole, there is no memory effect. In other words, there is no dynamical way to cause a transition between Geroch-related vacua. This stands in stark contrast to the memory effect for ordinary gravitons. In the case of ordinary gravitons, hard radiation can cause transitions between supertranslation-related vacua.

In sum, the fact that cylindrical gravitational waves have a soft zero instead of a soft pole is a manifestation of Geroch symmetry and of the extraordinary simplicity of cylindrical gravitational waves. This simplicity would seem to make cylindrical gravitational waves an attractive target for holography.³

II. SIGMA MODEL

The two-dimensional spacetime metric is

$$ds^2 = -dt^2 + dr^2. \quad (2.1)$$

Space is a half-line ($r > 0$). Future and past infinity, $\Sigma^\pm$, are the lines at $t = \pm\infty$ with coordinate r . The Lagrangian of the sigma model is [4–6,8,28–30]

$$L = -\frac{r}{2} \partial_a X_1 \partial^a X_1 - \frac{r}{2} e^{2X_1} \partial_a X_2 \partial^a X_2. \quad (2.2)$$

For notational simplicity we have set the coupling $g = 1$. Here $X_1 = X_1(t, r)$ and $X_2 = X_2(t, r)$ are scalar fields. They encode the two physical polarizations of the graviton. Equation (2.2) has a global symmetry

$$\delta X_1 = -X_2, \quad \delta X_2 = \frac{1}{2}(1 - e^{-2X_1} + X_2^2). \quad (2.3)$$

In the free theory, this is just an $\text{SO}(2)$ rotation of the vector X_μ . Equation (2.3) is a simple example of a Geroch transformation. The full set of Geroch transformations is infinite dimensional. Most of the Geroch symmetries are hidden symmetries in the sense that they involve nonlocal transformations of the fields.

The sigma model reduces to a free theory on $\Sigma^\pm$. In the free theory, the Noether charge for Eq. (2.3) is

$$\mathcal{Q}^\pm = \int_{\Sigma^\pm} dr r [X_2 \partial_t X_1 - X_1 \partial_t X_2]. \quad (2.4)$$

The mode expansion of the free field is (suppressing the $\pm$)

$$X(t, r) = \frac{1}{\sqrt{2}} \int_0^\infty dk [a(k) e^{-ikt} + a(k)^\dagger e^{ikt}] J_0(kr). \quad (2.5)$$

This is an expansion in energy eigenmodes.⁴ The creation and annihilation operators, $a^\dagger$ and a , obey

$$[a_\mu(k), a_\nu(k')^\dagger] = \delta_{\mu\nu} \delta(k - k'). \quad (2.6)$$

Inserting Eq. (5.5) into (2.4) and doing the r integral gives

$$\mathcal{Q} = i \int dk [a_1(k)^\dagger a_2(k) - a_2(k)^\dagger a_1(k)]. \quad (2.7)$$

Thanks to this symmetry, the two-particle tree-level $\mathcal{S}$ matrix takes the form⁵

$$\mu\rho \mathcal{S}_{\nu\gamma} = \begin{array}{c} \begin{array}{ccc} k_1 & \mu & \nu & k_3 \\ & \diagdown & \diagup & \\ & \text{---} \text{---} \text{---} & & \\ & \diagup & \diagdown & \\ k_2 & \rho & \gamma & k_4 \end{array} \\ \end{array} = \delta_{\mu\rho} \delta_{\nu\gamma} \sigma_1 + \delta_{\mu\nu} \delta_{\rho\gamma} \sigma_2 + \delta_{\mu\gamma} \delta_{\nu\rho} \sigma_3. \quad (2.8)$$

So there are just three functions to consider: σ_1 , σ_2 , and σ_3 . And in fact the σ_i 's are all related by crossing relations, so there is really only one independent function. Indeed, $\sigma_2(k_1, \dots, k_4)$ and $\sigma_3(k_1, \dots, k_4)$ are simply related by interchanging k_3 and k_4 . And $\sigma_1(k_1, \dots, k_4)$ and $\sigma_2(k_1, \dots, k_4)$ are simply related by interchanging k_2 and k_3 and putting in an overall minus sign.

III. FEYNMAN DIAGRAMS

The goal of this section is to compute σ_1 using Feynman diagrams. Expanding the Lagrangian (2.2) gives an infinite

series of interaction terms:

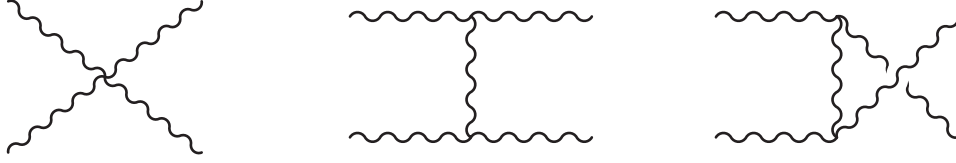
$$-r X_1 \partial_a X_2 \partial^a X_2 - r X_1^2 \partial_a X_2 \partial^a X_2 + \dots \quad (3.1)$$

The leading terms are a cubic interaction and a quartic interaction. These are the only interactions that contribute to the two-particle tree-level $\mathcal{S}$ matrix.

⁴There is another family of energy eigenmodes, $e^{\pm ikt} Y_0(kr)$, but they are singular at $r = 0$ so we do not include them.

⁵This follows from looking at the action of (2.7) on the two-particle in and out states, and enforcing the $\mathcal{S}$ -matrix conservation law $\langle \text{out} | (\mathcal{Q}^+ \mathcal{S} - \mathcal{S} \mathcal{Q}^-) | \text{in} \rangle = 0$.

³Some early discussion of this direction appears in [27].

FIG. 1. Tree-level contributions to $\sigma_1 = {}_{11}\mathcal{S}_{22}$, the amplitude for $X_1X_1 \rightarrow X_2X_2$ scattering.

σ_1 is the amplitude for $X_1X_1 \rightarrow X_2X_2$ scattering. It receives contributions from three diagrams (Fig. 1). The first diagram comes from the quartic interaction in (3.1). The second two diagrams only involve the cubic interaction. Note that there is no way to build an s -channel diagram for $X_1X_1 \rightarrow X_2X_2$ at tree level because the cubic interaction in (3.1) only involves a single X_1 . For a similar reason, it follows that the exchanged particle in the second two diagrams of Fig. 1 is always an X_2 .

The second and third diagrams in Fig. 1 are almost the same (just interchange k_3 and k_4), so we really only need to consider the first two diagrams. The contribution of the first diagram is

$$-4i \int d^2x r e^{-i(k_1+k_2)t} J_0(k_1r) J_0(k_2r) \partial_a [e^{ik_3t} J_0(k_3r)] \times \partial^a [e^{ik_4t} J_0(k_4r)]. \quad (3.2)$$

Expanding the derivatives and doing the t integral gives

$$-8\pi i k_3 k_4 \delta_{k_3+k_4-k_1-k_2} \int drr J_0(k_1r) J_0(k_2r) [J_0(k_3r) J_0(k_4r) + J_1(k_3r) J_1(k_4r)]. \quad (3.3)$$

To simplify this further, we use the identity

$$J_0(ar) J_0(br) = \frac{1}{2\pi} \int_{|a-b|}^{a+b} dc \frac{c J_0(cr)}{\Delta(a, b, c)}. \quad (3.4)$$

$\Delta(a, b, c)$ is the area of the triangle⁶ with side lengths a , b , and c . Using Eq. (3.4) we obtain

$$-4ik_3k_4\delta_{k_3+k_4-k_1-k_2} \int_{|k_1-k_2|}^{k_1+k_2} \frac{dkk}{\Delta(k_1, k_2, k)} \cdot \int drr J_0(kr) [J_0(k_3r) J_0(k_4r) + J_1(k_3r) J_1(k_4r)]. \quad (3.5)$$

The r integral on the second line is zero unless it is possible to form a triangle with side lengths k , k_3 , and k_4 , in which case we have

$$\int drr J_0(ar) J_0(br) J_0(cr) = \frac{1}{2\pi \Delta(a, b, c)}, \quad (3.6)$$

⁶A convenient formula is $\Delta = \frac{1}{4} \sqrt{(a+b)^2 - c^2} \sqrt{c^2 - (a-b)^2}$.

$$\int drr J_0(ar) J_1(br) J_1(cr) = \frac{b^2 + c^2 - a^2}{4\pi bc \Delta(a, b, c)}. \quad (3.7)$$

We thus obtain

$$-\frac{16i}{\pi} \delta_{k_3+k_4-k_1-k_2} \times \int_{\max(|k_1-k_2|, |k_3-k_4|)}^{k_1+k_2} \frac{kdk}{\sqrt{k^2 - (k_1 - k_2)^2} \sqrt{k^2 - (k_3 - k_4)^2}}. \quad (3.8)$$

Finally, we perform the k integral to obtain

$$-\frac{16i}{\pi} \text{Re atanh} \sqrt{\frac{k_1 k_2}{k_3 k_4}} \delta_{k_3+k_4-k_1-k_2}. \quad (3.9)$$

This is the contribution of the first diagram in Fig. 1 to σ_1 .

Now turn to the second diagram in Fig. 1. This diagram is slightly more complicated because it has an internal line. To put in the internal line, we need the Feynman propagator:

$$\langle X_\mu(x) X_\nu(x') \rangle_0 = \delta_{\mu\nu} \frac{i}{2\pi} \int d\omega \int dk \frac{k}{\omega^2 - k^2 + i\epsilon} e^{-i\omega(t-t')} J_0(kr) J_0(kr'). \quad (3.10)$$

The vertices in this diagram come from the cubic interaction (3.1):

$$-r X_1 \partial_a X_2 \partial^a X_2. \quad (3.11)$$

Putting everything together, the contribution of this diagram is

$$-\frac{2i}{\pi} \int d\omega \int dk \frac{k}{\omega^2 - k^2 + i\epsilon} \cdot \int d^2x r e^{-ik_1t} J_0(k_1r) \partial_a [e^{ik_3t} J_0(k_3r)] \partial^a [e^{-i\omega t} J_0(kr)] \cdot \int d^2x' r' e^{-ik_2t'} J_0(k_2r') \partial_{a'} [e^{ik_4t'} J_0(k_4r')] \partial^{a'} [e^{i\omega t'} J_0(kr')]. \quad (3.12)$$

Expanding the derivatives and doing the t and ω integrals, we obtain

$$\begin{aligned}
& 8\pi i k_3 k_4 \delta_{k_3+k_4-k_1-k_2} \int dk \frac{k}{k^2 - (k_3 - k_1)^2} \\
& \cdot \int d^2 x r J_0(k_1 r) [-(k_3 - k_1) J_0(k_3 r) J_0(k r) \\
& + k J_1(k_3 r) J_1(k r)] \\
& \cdot \int d^2 x' r' J_0(k_2 r') [(k_2 - k_4) J_0(k_4 r') J_0(k r') \\
& + k J_1(k_4 r') J_1(k r')]. \tag{3.13}
\end{aligned}$$

Next, we do the r integrals using Eqs. (3.6) and (3.7) to get

$$\begin{aligned}
& \frac{8i}{\pi} \delta_{k_3+k_4-k_1-k_2} \\
& \times \int_{|k_1-k_3|}^{\min(k_1+k_3, k_2+k_4)} \frac{k dk}{\sqrt{(k_1+k_3)^2 - k^2} \sqrt{(k_2+k_4)^2 - k^2}}. \tag{3.14}
\end{aligned}$$

This integral deserves a comment. It might seem strange that we are integrating over the “momentum,” k , of the internal line in a tree diagram. But our Lagrangian (2.2) does not have r -translation invariance, so k is not conserved at vertices. Mathematically, this manifests in the fact that the triple Bessel integrals (3.6) and (3.7) are not δ functions.⁷ Performing the k integral gives

$$\frac{8i}{\pi} \text{Re atanh} \sqrt{\frac{k_1 k_3}{k_2 k_4}} \delta_{k_3+k_4-k_1-k_2}. \tag{3.15}$$

This is the contribution of the second diagram in Fig. 1 to the amplitude. The third diagram is almost the same, we just exchange k_3 and k_4 . Summing all three diagrams gives

$$\sigma_1 = -\frac{8i}{\pi} \text{Re atanh} \sqrt{\frac{k_1 k_2}{k_3 k_4}} \delta_{k_3+k_4-k_1-k_2}. \tag{3.16}$$

Now using the crossing relations discussed in the previous section, we obtain

$$\sigma_2 = \frac{8i}{\pi} \text{Re atanh} \sqrt{\frac{k_1 k_3}{k_2 k_4}} \delta_{k_3+k_4-k_1-k_2}, \tag{3.17}$$

$$\sigma_3 = \frac{8i}{\pi} \text{Re atanh} \sqrt{\frac{k_1 k_4}{k_2 k_3}} \delta_{k_3+k_4-k_1-k_2}. \tag{3.18}$$

A couple consistency checks are in order. It is clear from the Lagrangian (2.2) that there are no possible tree

diagrams for ${}_{11}\mathcal{S}_{11} = \sigma_1 + \sigma_2 + \sigma_3$, so the sum on the right-hand side should vanish. And indeed it is not hard to sum Eqs. (3.16)–(3.18) and check that $\sigma_1 + \sigma_2 + \sigma_3 = 0$. Furthermore, if we look at Eq. (2.8) we can observe $\sigma_1 + \sigma_2 + \sigma_3 = {}_{22}\mathcal{S}_{22}$. The amplitude on the right-hand side does receive contributions from three diagrams but we have computed these diagrams (we omit the details for brevity) and indeed they sum to zero.

The amplitudes we have just computed have the curious feature that they vanish in the soft limit, $k_i \rightarrow 0$. In the remainder of this work, we will show how the Geroch group provides a symmetry explanation for this soft zero.

IV. GEROCH SYMMETRY

The Lagrangian (2.2) has a pair of global symmetries:

$$\delta X_1 = -1, \quad \delta X_2 = X_2, \tag{4.1}$$

and

$$\delta X_2 = -1. \tag{4.2}$$

Equation (4.2) is a shift symmetry. Equation (4.1) is a shift symmetry in the free theory and something slightly more complicated in the interacting theory. These symmetries are examples of Geroch transformations.

The sigma model reduces to a free theory on Σ^+ . In the free theory, the Noether charges for Eqs. (4.1) and (4.2) are⁸

$$Q^+ = \int_{\Sigma^+} dr r \partial_t X. \tag{4.3}$$

There is no “hard charge” on the rhs. This means there is no memory effect for these charges. This is quite different from the story for the supertranslation symmetry of ordinary gravitons [19]. Supertranslation charges receive contributions from hard and soft gravitons.

Define a dual field, $\hat{X}$, by

$$r \partial_t X = \partial_r \hat{X}, \tag{4.4}$$

$$r \partial_r X = \partial_t \hat{X}. \tag{4.5}$$

The consistency conditions for $\hat{X}$ are just the free equations of motion, $\partial_a(r \partial^a X) = 0$. Equations (4.4) and (4.5) only define $\hat{X}$ up to a constant. We fix the constant by requiring $\hat{X}|_{\Sigma_0^+} = 0$. The Noether charge (4.3) becomes

$$Q^+ = \hat{X}|_{\Sigma_+^+}. \tag{4.6}$$

⁷Our Lagrangian does have time-translation invariance and this is why we do get an overall energy-conserving δ function, $\delta_{k_3+k_4-k_1-k_2}$.

⁸These symmetries have also been studied by Niedermaier [27], who also concluded that they are spontaneously broken in the quantum theory.

This is the asymptotic Geroch charge at the “celestial point.” In these formulas, Σ_0^+ is the point on Σ^+ at $r = 0$ and Σ_+^+ is the point on Σ^+ at $r = \infty$.

Define $\pi_\mu = r\partial_t X_\mu$. The nonvanishing commutators on Σ^+ are

$$[\pi_\mu(r), X_\nu(r')] = -i\delta_{\mu\nu}\delta(r - r'). \quad (4.7)$$

Thus

$$[Q_\mu^+, X_\nu] = -i\delta_{\mu\nu}. \quad (4.8)$$

So Q^+ generates shifts of X , as it should.

Now consider the analogous structure at past infinity. Past infinity, Σ^- , is the line at $t = -\infty$ with coordinate r . Let Σ_0^- denote the point on Σ^- at $r = 0$ and let Σ_+^- denote the point on Σ^- at $r = \infty$. As before, we require $\hat{X}|_{\Sigma_0^-} = 0$. The Geroch charges are

$$Q^- = \int_{\Sigma^-} dr r \partial_t X = \hat{X}|_{\Sigma_+^-}. \quad (4.9)$$

For notational simplicity, we use the same notation for X and $\hat{X}$ at $\Sigma^\pm$.

The shifts generated by $Q^\pm$ come from the global symmetries (4.1) and (4.2). So it is natural to require that shifts at Σ_+^+ are matched by shifts at Σ_+^- . This is our version of the “antipodal matching” condition [18]. Only here the celestial sphere is a celestial point ($\Sigma_+^\pm$), so the antipodal part of the matching condition is trivial.

V. SOFT THEOREM

In this section we consider the consequences of Geroch symmetry for the semiclassical $\mathcal{S}$ matrix. Let us denote an in (out) state comprised of n (m) particles with energies k_i^{in} (k_i^{out}) on the celestial point by $|\text{in}\rangle \equiv |k_1^{\text{in}}, \dots, k_n^{\text{in}}\rangle$ ($\langle\text{out}| \equiv \langle k_1^{\text{out}}, \dots, k_m^{\text{out}}|$). The $\mathcal{S}$ matrix elements are then denoted by $\langle\text{out}|\mathcal{S}|\text{in}\rangle$. The quantum version of the classical invariance of scattering under Geroch symmetry is

$$\langle\text{out}|(Q^+\mathcal{S} - \mathcal{S}Q^-)|\text{in}\rangle = 0. \quad (5.1)$$

Defining

$$Q = Q^+ - Q^-, \quad (5.2)$$

and the time-ordered product

$$:\mathcal{QS}: = Q^+\mathcal{S} - \mathcal{S}Q^-, \quad (5.3)$$

Eq. (5.1) becomes

$$\langle\text{out}|:\mathcal{QS}:|\text{in}\rangle = 0. \quad (5.4)$$

At late times, $t \rightarrow \infty$, the field becomes free and can be approximated by the mode expansion

$$X(t, r) = \frac{1}{\sqrt{2}} \int_0^\infty dk [a_+(k)e^{-ikt} + a_+(k)^\dagger e^{ikt}] J_0(kr). \quad (5.5)$$

The creation and annihilation operators on Σ^+ , $a_+^\dagger$ and a_+ , obey

$$[a_{\mu+}(k), a_{\nu+}(k')^\dagger] = \delta_{\mu\nu}\delta(k - k'). \quad (5.6)$$

The dual field is

$$\hat{X}(t, r) = -\frac{i}{\sqrt{2}} \int_0^\infty dk [a_+(k)e^{-ikt} - a_+(k)^\dagger e^{ikt}] r J_1(kr). \quad (5.7)$$

Equation (5.5) is an expansion in energy eigenmodes. Energy eigenmodes do not have a well-defined limit at $r = \infty$. To evaluate the Geroch charges at $\Sigma_+^\pm$, it is better to use wave packets. A natural way to get wave packets is to Laplace transform the energy eigenmodes. Define

$$A_\pm(s) = \int_0^\infty dk e^{-sk} a_\pm(k), \quad (5.8)$$

$$A_\pm(\bar{s})^\dagger = \int_0^\infty dk e^{-s\bar{k}} a_\pm(k)^\dagger. \quad (5.9)$$

The inverse Laplace transforms are

$$a_\pm(k) = \int_C \frac{ds}{2\pi i} e^{sk} A_\pm(s), \quad (5.10)$$

$$a_\pm(k)^\dagger = \int_C \frac{ds}{2\pi i} e^{sk} A_\pm(\bar{s})^\dagger. \quad (5.11)$$

Insert Eqs. (5.10), (5.11) into (5.7) to get

$$\begin{aligned} \hat{X}(t, r) = & -\frac{i}{\sqrt{2}} \int_C \frac{ds}{2\pi i} \int_0^\infty dk e^{sk} [A_+(s)e^{-ikt} \\ & - A_+(\bar{s})^\dagger e^{ikt}] r J_1(kr). \end{aligned} \quad (5.12)$$

Do the k integral to get

$$\hat{X}(t, r) = -\frac{i}{\sqrt{2}} \int_C \frac{ds}{2\pi i} [A_+(s)\varphi_{-s}(t, r) - A_+(\bar{s})^\dagger \varphi_{-\bar{s}}(t, r)^*]. \quad (5.13)$$

We have defined

$$\varphi_s(t, r) = \int_0^\infty dk e^{-(s+it)k} r J_1(kr), \quad (5.14)$$

$$\varphi_{\bar{s}}(t, r)^* = \int_0^\infty dk e^{-(s-it)k} r J_1(kr). \quad (5.15)$$

Equation (5.13) is the wave packet expansion of $\hat{X}$ at Σ^+ . We recall that the notation Σ^+ means we are looking at large t . The fact that we are working at large t is reflected in the fact that we are working with free fields.

Now we want to compute $\varphi_s|_{\Sigma^+}$. First, we take the Laplace transform (5.14) to find

$$\varphi_s(t, r) = \frac{r^2}{\sqrt{r^2 + (s+it)^2} [s+it + \sqrt{r^2 + (s+it)^2}]}. \quad (5.16)$$

Then we take the $r \rightarrow \infty$ limit and find

$$\varphi_s|_{\Sigma^+} = \lim_{r \rightarrow \infty} \varphi_s = 1. \quad (5.17)$$

Taking the limit this way means we are looking at the charges at spacelike infinity rather than null infinity. An interesting question for the future is to understand if there is an analogous set of charges at null infinity.

Thus (4.6)

$$Q^+ = \hat{X}|_{\Sigma^+} = -\frac{i}{\sqrt{2}} \int_C \frac{ds}{2\pi i} [A_+(s) - A_+(\bar{s})^\dagger]. \quad (5.18)$$

These are the Geroch charges at Σ^+ in s space.

To go back to k space, insert Eqs. (5.8) and (5.9) into (5.18):

$$Q^+ = -\frac{i}{\sqrt{2}} \int_0^\infty dk \int_C \frac{ds}{2\pi i} e^{-sk} [a_+(k) - a_+(k)^\dagger]. \quad (5.19)$$

Do the s integral to get

$$Q^+ = -\frac{i}{\sqrt{2}} \int_0^\infty dk \delta(k) [a_+(k) - a_+(k)^\dagger]. \quad (5.20)$$

Do the k integral to get

$$Q^+ = -\frac{i}{\sqrt{2}} [a_+(0) - a_+(0)^\dagger]. \quad (5.21)$$

A parallel construction at Σ_+^- gives

$$Q^- = -\frac{i}{\sqrt{2}} [a_-(0) - a_-(0)^\dagger]. \quad (5.22)$$

Equations (5.21) and (5.22) are the Geroch charges at $\Sigma_+^\pm$. Recall

$$Q = Q^+ - Q^-. \quad (5.23)$$

The Ward identity (5.4) becomes

$$\langle \text{out} | : (Q^+ - Q^-) \mathcal{S} : | \text{in} \rangle = 0. \quad (5.24)$$

Using (5.21) and (5.22), this equation becomes

$$\langle \text{out} | a_+(0) \mathcal{S} | \text{in} \rangle = \langle \text{out} | \mathcal{S} a_-(0)^\dagger | \text{in} \rangle = 0. \quad (5.25)$$

We have used⁹

$$\langle \text{out} | \mathcal{S} a_-(0)^\dagger | \text{in} \rangle = -\langle \text{out} | a_+(0) \mathcal{S} | \text{in} \rangle. \quad (5.26)$$

Equation (5.25) is the soft zero for cylindrical gravitational waves. We thus obtain the desired symmetry explanation for the soft zero observed in Eqs. (3.16)–(3.18). And, conversely, Eqs. (3.16)–(3.18) lend concrete support to the soft theorem (5.25).

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⁹We do not have a general proof of this identity but it is true in the examples we have checked. In particular, it is true for the amplitudes computed in Eqs. (3.16)–(3.18). This crossing equation is a standard part of infrared triangle-style derivations of soft theorems, not unique to cylindrical gravitational waves.

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- [1] A. Strominger, Lectures on the infrared structure of gravity and gauge theory, [arXiv:1703.05448](#).
 - [2] R. P. Geroch, A method for generating solutions of Einstein's equations, *J. Math. Phys. (N.Y.)* **12**, 918 (1971).
 - [3] R. P. Geroch, A method for generating new solutions of Einstein's equation. II, *J. Math. Phys. (N.Y.)* **13**, 394 (1972).

- [4] P. Breitenlohner and D. Maison, On the Geroch group, *Ann. Inst. Henri Poincaré Phys. Theor.* **46**, 215 (1987).
- [5] H. Nicolai, Two-dimensional gravities and supergravities as integrable system, *Lect. Notes Phys.* **396**, 231 (1991).
- [6] J. H. Schwarz, Classical symmetries of some two-dimensional models, *Nucl. Phys.* **B447**, 137 (1995).

- [7] J. H. Schwarz, Classical duality symmetries in two-dimensions, in *STRINGS 95: Future Perspectives in String Theory* (1995), pp. 439–451, [arXiv:hep-th/9505170](#).
- [8] H. Lu, M. J. Perry, and C. N. Pope, Infinite-dimensional symmetries of two-dimensional coset models, *Nucl. Phys. B* **806**, 656 (2009).
- [9] A. Ashtekar, J. Bicak, and B. G. Schmidt, Asymptotic structure of symmetry reduced general relativity, *Phys. Rev. D* **55**, 669 (1997).
- [10] J. Bosma, M. Geiller, S. Majumdar, and B. Oblak, Radiative asymptotic symmetries of 3D Einstein-Maxwell theory, *SciPost Phys.* **16**, 092 (2024).
- [11] I. Low, Double soft theorems and shift symmetry in nonlinear sigma models, *Phys. Rev. D* **93**, 045032 (2016).
- [12] I. Low and Z. Yin, Ward identity and scattering amplitudes for nonlinear sigma models, *Phys. Rev. Lett.* **120**, 061601 (2018).
- [13] M. Campiglia, L. Coito, and S. Mizera, Can scalars have asymptotic symmetries?, *Phys. Rev. D* **97**, 046002 (2018).
- [14] Y. Hamada and S. Sugishita, Soft pion theorem, asymptotic symmetry and new memory effect, *J. High Energy Phys.* **11** (2017) 203.
- [15] D. Francia and C. Heissenberg, Two-form asymptotic symmetries and scalar soft theorems, *Phys. Rev. D* **98**, 105003 (2018).
- [16] L. Rodina, Scattering amplitudes from soft theorems and infrared behavior, *Phys. Rev. Lett.* **122**, 071601 (2019).
- [17] J. Distler, R. Flauger, and B. Horn, Double-soft graviton amplitudes and the extended BMS charge algebra, *J. High Energy Phys.* **08** (2019) 021.
- [18] A. Strominger, On BMS invariance of gravitational scattering, *J. High Energy Phys.* **07** (2014) 152.
- [19] A. Strominger and A. Zhiboedov, Gravitational memory, BMS supertranslations and soft theorems, *J. High Energy Phys.* **01** (2016) 086.
- [20] T. He, P. Mitra, A. P. Porfyriadis, and A. Strominger, New symmetries of massless QED, *J. High Energy Phys.* **10** (2014) 112.
- [21] T. He, V. Lysov, P. Mitra, and A. Strominger, BMS supertranslations and Weinberg’s soft graviton theorem, *J. High Energy Phys.* **05** (2015) 151.
- [22] D. Kapec, M. Pate, and A. Strominger, New symmetries of QED, *Adv. Theor. Math. Phys.* **21**, 1769 (2017).
- [23] T. He, P. Mitra, and A. Strominger, 2D Kac-Moody symmetry of 4D Yang-Mills theory, *J. High Energy Phys.* **10** (2016) 137.
- [24] M. Campiglia and A. Laddha, Asymptotic symmetries of QED and Weinberg’s soft photon theorem, *J. High Energy Phys.* **07** (2015) 115.
- [25] S. Pasterski, A. Strominger, and A. Zhiboedov, New gravitational memories, *J. High Energy Phys.* **12** (2016) 053.
- [26] S. Pasterski, Asymptotic symmetries and electromagnetic memory, *J. High Energy Phys.* **09** (2017) 154.
- [27] M. Niedermaier, Quantized Einstein-Rosen waves, AdS_2 , and spontaneous symmetry breaking, *Phys. Lett. B* **498**, 83 (2001).
- [28] J. H. Schwarz, Classical symmetries of some two-dimensional models coupled to gravity, *Nucl. Phys. B* **454**, 427 (1995).
- [29] D. Grumiller, D. Hofmann, and W. Kummer, Two dilaton theories in two-dimensions, *Ann. Phys. (N.Y.)* **290**, 69 (2001).
- [30] H. Lu, M. J. Perry, and C. N. Pope, Infinite-dimensional symmetries of two-dimensional coset models coupled to gravity, *Nucl. Phys. B* **806**, 656 (2009).