

Nonanalytic structure of effective potential at finite temperature on compactified space

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We thoroughly investigate nonanalytic terms in the finite-temperature effective potential in one-loop approximation on a D -dimensional spacetime, $S_\tau \times R^{D-(p+1)} \times \prod_{i=1}^p S_i^1$, using a mode recombination formula. Such nonanalytic terms cannot be expressed as positive powers of field-dependent mass squared. The formula provides a clear separation of the effective potential into a part that contains the nonanalytic terms and a part that is purely analytic, and clarifies the origin of the nonanalytic terms. We obtain all the nonanalytic terms and show that only two types of nonanalytic terms arise from the modes with zero Matsubara frequency. For a real scalar field with periodic boundary conditions, if the number of noncompact spatial dimensions is odd (even), there are odd powers of M ($\log M$ terms) but no $\log M$ terms (no odd powers of M). For fermions with general boundary conditions, we find that neither of the two types appears. These results clarify the nonanalytic structure of the finite-temperature effective potential on the spacetime with compactified spatial dimensions.

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I. INTRODUCTION

Quantum field theory at finite temperature provides a fundamental framework for studying phase-transition phenomena in high-energy physics and has played a central role in advancing our understanding of the subject. In their pioneering analysis of the finite-temperature effective potential, Dolan and Jackiw [1] found that the scalar effective potential contains terms that cannot be expressed as positive powers of the field-dependent mass squared, namely, nonanalytic terms. The terms act as the source of the first-order phase transitions [2,3] and their magnitude directly controls the strength of the transition. They are, for instance, indispensable in an electroweak baryogenesis scenario [4] relying on the first-order electroweak phase transition. In exploring cosmological evolution scenarios [5], it is necessary to understand how symmetries are broken during the cooling of the Universe. In this respect, the study of nonanalytic contributions to the finite-temperature effective potential plays a crucial role. Nonanalytic terms are therefore of substantial physical significance.

Quantum field theories with compactified extra dimensions constitute one of the most promising approaches for

exploring physics beyond the Standard Model. Gauge-Higgs unification provides a representative example [6], where, once again, nonanalytic terms at finite-temperature effective potential are crucial for realizing the first-order phase transition [7]. At the same time, theories with compact extra dimensions also serve as powerful theoretical tools for field-theoretic studies, providing a well-defined framework for such investigations. Dimensional reduction offers a prime example [8,9] and moreover, these theories are known to exhibit rich and interesting phase structures [10,11].

In our previous two papers, referred to as I [12] and II [13], we determined all the power-type nonanalytic terms in the one-loop finite-temperature effective potential of a scalar field obeying periodic boundary conditions on a D -dimensional spacetime, $S_\tau^1 \times R^{D-(p+1)} \times \prod_{i=1}^p S_i^1$. In particular, we developed a new important formula called a mode recombination formula in the paper II and this formula enables a reformulation of the finite-temperature effective potential, thereby elucidating the origin of the nonanalytic terms. Although the origin of the nonanalytic terms is initially expressed using the modified Bessel functions of the second kind with multiple mode summations, we showed that, by employing the integral representations for the function together with the analytical extension of the mode summations, it can be recast as contour integrals in the complex plane. The subsequent evaluation of the relevant residues makes it possible to extract all the power-type nonanalytic terms in a straightforward manner.

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The mode recombination formula clearly shows that the emergence of nonanalytic terms is closely related to the existence of zero modes, making it a powerful tool for understanding the physical origin of such terms. As an immediate corollary, we also showed that fermion fields, whose zero modes are absent due to their statistics, exhibit no nonanalytic terms.

In this paper we derive a new remarkable expression of the finite-temperature effective potential in one-loop approximation by using extensively the mode recombination formula. The formula enables a clear separation of the effective potential into a part that contains nonanalytic terms and a part that remains purely analytic ones. We demonstrate that the effective potential possesses two physically important types of nonanalytic terms with respect to the field-dependent mass: power type and logarithmic one. These nonanalytic terms are obtained by the evaluation of the residues in a systematic and comprehensive manner. In the case of a real scalar field obeying the periodic boundary condition for the spatial $S_i^1 (i = 1, \dots, p)$ direction, we find that the nonanalytic structure of the finite-temperature effective potential turns out to be highly characteristic; the two types never appear simultaneously.

We further show that, for fermion fields with arbitrary boundary conditions for the spatial S_i^1 direction, the application of the mode recombination formula renders the finite-temperature effective potential into a form in which nonanalytic terms can be analyzed more transparently. As a consequence, in addition to the absence of the power-type nonanalytic terms as shown in the paper II, we find that the logarithmic-type nonanalytic term does not appear either regardless of the values of D and $p + 1$. The result in the case of the fermion also holds for the case of the scalar that satisfies the antiperiodic boundary condition for at least one spatial S_i^1 direction.

The organization of this paper is as follows. Section II provides the setup for analyzing nonanalytic terms in the finite-temperature effective potential and summarizes the method of the mode recombination formula developed in the paper II, followed by its application to the effective potential. In Sec. III we determine all the nonanalytic terms in the effective potential of the scalar field with the periodic boundary conditions. Section IV presents the corresponding analysis for the fermion fields. Section V contains conclusions and discussions. We derive conditions under which evaluations of integrals by residues are applicable in Appendix A and demonstrate the evaluations in Appendixes B and C.

II. SETUP AND MODE RECOMBINATION FORMULA

Let us present the setup for the discussion in this paper and the mode recombination formula which plays the

important role to study the nonanalytic terms in the finite-temperature effective potential on the spacetime with compactified spatial dimensions. Here, we refer to the contents of those in Secs. 2 and 3 of the paper II. The familiar readers can directly go to the next section.

A. Setup

We first study the effective potential at finite temperature in one-loop approximation on the D -dimensional spacetime, $S_\tau^1 \times R^{D-(p+1)} \times \prod_{i=1}^p S_i^1$ for real scalar and fermion fields. We employ the Euclidean time formulation for quantum field theory at finite temperature. Then, the Euclidean time direction is compactified on S_τ^1 whose coordinate and circumference are denoted by τ and L_0 , respectively. The L_0 is the inverse temperature, $L_0 = T^{-1}$. The spatial p directions are compactified on the p numbers of S^1 . Their coordinates are $y^i (i = 1, \dots, p)$ and L_i is the circumference of each S_i^1 . The $R^{D-(p+1)}$ is the flat Euclidean space and its coordinate is $x^k (k = 1, \dots, D - (p + 1))$.

The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial_N \phi)^2 - \frac{m_s^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \bar{\psi}(i\Gamma_N \partial_N + m_f)\psi + g\phi\bar{\psi}\psi, \quad (2.1)$$

where the N stands for $N = (\tau, k, i)$ and $\phi(\psi)$ is the scalar (fermion) field whose bulk mass is $m_s(m_f)$. The g is the Yukawa coupling.

We specify the boundary conditions for the S_τ^1 and $S_i^1 (i = 1, \dots, p)$ directions. The boundary condition for the S_τ^1 direction is defined by

$$\Phi(\tau + L_0, x^k, y^i) = e^{2\pi i \eta_0} \Phi(\tau, x^k, y^i) \quad (2.2)$$

for a given field $\Phi(\tau, x^k, y^i)$. The parameter η_0 is definitely determined by quantum statistics to be 0 (periodic) for the scalar field or to be $\frac{1}{2}$ (antiperiodic) for the fermion field. On the other hand, the boundary condition for the spatial S_i^1 direction is imposed by

$$\Phi(\tau, x^k, y^i + L_i) = e^{2\pi i \eta_i} \Phi(\tau, x^k, y^i), \quad (2.3)$$

where the parameter η_i can take 0 or $\frac{1}{2}$ for the real scalar field and can be arbitrary for the fermion field.

We employ the standard prescription to calculate the effective potential at finite temperature in one-loop approximation. By taking up the quadratic terms in the shifted Lagrangian around the constant field φ for the scalar field ϕ in Eq. (2.1), one needs to evaluate

$$V_{\text{eff}} = (-1)^f \mathcal{N} \frac{1}{2} \left(\prod_{i=0}^p \frac{1}{L_i} \sum_{n_i=-\infty}^{\infty} \right) \int \frac{d^{D-(p+1)} p_E}{(2\pi)^{D-(p+1)}} p_E \times \log \left[p_E^2 + \left(\frac{2\pi}{L_0} \right)^2 (n_0 + \eta_0)^2 + \sum_{i=1}^p \left(\frac{2\pi}{L_i} \right)^2 (n_i + \eta_i)^2 + M^2(\varphi) \right] \quad (2.4)$$

in order to obtain the effective potential on $S_\tau^1 \times R^{D-(p+1)} \times \prod_{i=1}^p S_i^1$ in one-loop approximation. Here, the $M(\varphi)$ is the field-dependent mass of the scalar (fermion) field

$$M^2(\varphi) = m_s^2 + \frac{\lambda}{2} \varphi^2 \quad (M(\varphi) = m_f + g\varphi). \quad (2.5)$$

Hereafter, we denote $M(\varphi)$ by M for simplicity. The p_E is the $D - (p + 1)$ -dimensional Euclidean momentum. The f is the fermion number that is 0 (1) for the boson (fermion) and the $\mathcal{N}$ is the on-shell degrees of freedom. The n_0 stands for the Matsubara mode arising from the S_τ^1 and the Kaluza-Klein mode $n_i (i = 1, \dots, p)$ comes from each $S_i^1 (i = 1, \dots, p)$. The parameter $\eta_i (i = 0, 1, \dots, p)$ is given in Eqs. (2.2) and (2.3).

We make use of the zeta-function regularization in order to evaluate Eq. (2.4). We can write

$$V_{\text{eff}} = (-1)^f \mathcal{N} \frac{1}{2} \left(-\frac{d}{ds} I(s) \right) \Big|_{s \rightarrow 0}, \quad (2.6)$$

where

$$I(s) \equiv \left(\prod_{i=0}^p \frac{1}{L_i} \sum_{n_i=-\infty}^{\infty} \right) \int \frac{d^{D-(p+1)} p_E}{(2\pi)^{D-(p+1)}} \left[p_E^2 + \left(\frac{2\pi}{L_0} \right)^2 (n_0 + \eta_0)^2 + \sum_{i=1}^p \left(\frac{2\pi}{L_i} \right)^2 (n_i + \eta_i)^2 + M^2 \right]^{-s}. \quad (2.7)$$

We first use the formula

$$A^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-At} \quad (2.8)$$

and perform the Gaussian p_E integration. Then, we employ the Poisson summation

$$\sum_{n_j=-\infty}^{\infty} e^{-\left(\frac{2\pi}{L_j}\right)^2 (n_j + \eta_j)^2 t} = \sum_{m_j=-\infty}^{\infty} \frac{L_j}{2\pi} \left(\frac{\pi}{t} \right)^{\frac{1}{2}} e^{-\frac{(m_j L_j)^2}{4t} + 2\pi i m_j \eta_j} \quad (2.9)$$

to obtain

$$V_{\text{eff}} = (-1)^{f+1} \mathcal{N} \frac{\pi^{\frac{D}{2}}}{2(2\pi)^D} \sum_{m_0=-\infty}^{\infty} \cdots \sum_{m_p=-\infty}^{\infty} \times \int_0^\infty dt t^{-\frac{D}{2}-1} e^{-\frac{1}{4t}[(m_0 L_0)^2 + \cdots + (m_p L_p)^2] - M^2 t + 2\pi i(m_0 \eta_0 + \cdots + m_p \eta_p)}. \quad (2.10)$$

We call $m_j (j = 0, 1, \dots, p)$ in Eq. (2.9) the winding modes, while n_0 and $n_i (i = 1, 2, \dots, p)$ are the Matsubara and the Kaluza-Klein modes, respectively.

It is convenient to separate each sum over m_j in Eq. (2.10) into the zero winding mode ($m_j = 0$) and the nonzero winding ones ($m_j \neq 0$) and to express Eq. (2.10) into the form

$$V_{\text{eff}} = \sum_{n=0}^{p+1} F^{(n)D}(M) = \sum_{n=0}^{p+1} \sum_{0 \leq i_1 < i_2 < \cdots < i_n \leq p} F_{L_{i_1}, L_{i_2}, \dots, L_{i_n}}^{(n)D}(M), \quad (2.11)$$

where

$$F_{L_{i_1}, L_{i_2}, \dots, L_{i_n}}^{(n)D}(M) = (-1)^{f+1} \mathcal{N} \frac{\pi^{\frac{D}{2}}}{2(2\pi)^D} \sum'_{m_{i_1}=-\infty}^{\infty} \dots \sum'_{m_{i_n}=-\infty}^{\infty} \times \int_0^{\infty} dt t^{-\frac{D}{2}-1} e^{-\frac{1}{4t}[(m_{i_1} L_{i_1})^2 + \dots + (m_{i_n} L_{i_n})^2] - M^2 t + 2\pi i(m_{i_1} \eta_{i_1} + \dots + m_{i_n} \eta_{i_n})}. \quad (2.12)$$

The prime of the summation in $\sum'_{m_j=-\infty}^{\infty}$ means that the zero winding mode ($m_j = 0$) is removed. By using the formula

$$\int_0^{\infty} dt t^{-\nu-1} e^{-At-\frac{B}{t}} = 2 \left(\frac{A}{B}\right)^{\frac{\nu}{2}} K_{\nu}(2\sqrt{AB}), \quad (2.13)$$

where the $K_{\nu}(z)$ is the modified Bessel function of the second kind, Eq. (2.12) for $n \geq 1$ becomes

$$F_{L_{i_1}, L_{i_2}, \dots, L_{i_n}}^{(n)D}(M) = (-1)^{f+1} \mathcal{N} \frac{2^n}{(2\pi)^{\frac{D}{2}}} \sum_{m_{i_1}=1}^{\infty} \dots \sum_{m_{i_n}=1}^{\infty} \left(\frac{M^2}{(m_{i_1} L_{i_1})^2 + \dots + (m_{i_n} L_{i_n})^2} \right)^{\frac{D}{4}} \times K_{\frac{D}{2}} \left(\sqrt{M^2 \{(m_{i_1} L_{i_1})^2 + \dots + (m_{i_n} L_{i_n})^2\}} \right) \cos(2\pi m_{i_1} \eta_{i_1}) \dots \cos(2\pi m_{i_n} \eta_{i_n}). \quad (2.14)$$

$F^{(0)D}$ in Eq. (2.11) is the contribution from all the zero winding modes $m_0 = m_1 = \dots = m_p = 0$ in Eq. (2.10) and is calculated to be

$$F^{(0)D}(M) = (-1)^{f+1} \mathcal{N} \frac{\pi^{\frac{D}{2}}}{2(2\pi)^D} \int_0^{\infty} dt t^{-\frac{D}{2}-1} e^{-M^2 t} = (-1)^{f+1} \mathcal{N} \frac{\pi^{\frac{D}{2}}}{2(2\pi)^D} \Gamma\left(-\frac{D}{2}\right) (M^2)^{\frac{D}{2}}, \quad (2.15)$$

where we have used Eq. (2.8). For $D = \text{even}$, if one makes use of the dimensional regularization and $\overline{\text{MS}}$ renormalization, for example, then, we obtain

$$F^{(0)D}(M) = (-1)^{f+1} \mathcal{N} \frac{(-1)^{\frac{D}{2}}}{2^{D+1} \pi^{\frac{D}{2}} (\frac{D}{2})!} M^D \left\{ -\log \frac{M^2}{\mu^2} + \sum_{r=1}^{\frac{D}{2}} \frac{1}{r} \right\}, \quad (2.16)$$

where the scale μ is a sequence of possible renormalization conditions. On the other hand, for $D = \text{odd}$, with the help of the formula

$$\Gamma\left(-\frac{D}{2}\right) \Big|_{D=\text{odd}} = \frac{(-1)^{\frac{D+1}{2}} 2^{\frac{D+1}{2}}}{D!!} \sqrt{\pi}, \quad (2.17)$$

we have

$$F^{(0)D}(M) = (-1)^{f+1} \mathcal{N} \frac{(-1)^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}} \pi^{\frac{D-1}{2}} D!!} M^D. \quad (2.18)$$

For $D = \text{even}$ (odd), $F^{(0)D}(M)$ has the logarithmic(power)-type nonanalytic terms, as shown in Eqs. (2.16) and (2.18).

B. Mode recombination formula

The mode recombination formula developed in the paper II plays the important role to study the nonanalytic terms. We give a brief review of the formula here for the reader's convenience.

Let us first recast Eq. (2.11) into

$$\begin{aligned} V_{\text{eff}} &= F^{(0)D}(M) + \sum_{n=1}^{p+1} \sum_{0 \leq i_1 < i_2 < \dots < i_n \leq p} F_{L_{i_1}, L_{i_2}, \dots, L_{i_n}}^{(n)D}(M) \\ &= F^{(0)D}(M) + \left\{ F_{L_0}^{(1)D}(M) + \sum_{n=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} (F_{L_{i_1}, \dots, L_{i_n}}^{(n)D}(M) + F_{L_0, L_{i_1}, \dots, L_{i_n}}^{(n+1)D}(M)) \right\}. \end{aligned} \quad (2.19)$$

The terms in the curly brackets¹ are obtained by focusing on the scale L_0 in $L_{i_j} (i_j = 0, \dots, p)$ and separating the term associated with L_0 from the one without L_0 on the right-hand side of Eq. (2.11). We have proved in the paper II

$$F_{L_{i_1}, \dots, L_{i_n}}^{(n)D}(M) + F_{L_0, L_{i_1}, \dots, L_{i_n}}^{(n+1)D}(M) = \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}), \quad (2.20)$$

where

$$M_{(0)}^2 \equiv M^2 + \left(\frac{2\pi}{L_0} \right)^2 (n_0 + \eta_0)^2. \quad (2.21)$$

We call Eq. (2.20) the mode recombination formula and the formula holds irrespective of whether fields are fermions or scalars and also of boundary conditions for the spatial $S_i^1 (i = 1, \dots, p)$ direction. It should be noted that, as shown in the paper II, the Matsubara mode n_0 is recovered instead of the winding mode m_0 by using the Poisson summation (2.9) inversely.

By using Eq. (2.20), Eq. (2.19) becomes

$$V_{\text{eff}} = F^{(0)D}(M) + \left\{ F_{L_0}^{(1)D}(M) + \sum_{n=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) \right\}. \quad (2.22)$$

We can repeat the same procedure by focusing next on the scales $L_1, L_2, \dots, L_{p-1}$ sequentially for the term with the multiple mode summations like the third term in Eq. (2.22) and we have finally arrived at the expression (3.11) in the paper II for the effective potential V_{eff} . Moreover, we have also shown in the paper II that the first and second terms in Eq. (2.22) can be recast into

$$F^{(0)D}(M) + F_{L_0}^{(1)D}(M) = \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F^{(0)D-1}(M_{(0)}), \quad (2.23)$$

which formally follows from the formula (2.20) with $n = 0$. Then, one has

$$V_{\text{eff}} = \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F^{(0)D-1}(M_{(0)}) + \sum_{n=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}). \quad (2.24)$$

Equations (2.22) and (2.24) have already been obtained in the paper II and we rewrite them into more convenient forms in the next section, making the subsequent discussion more transparent.

Equation (2.14) contains the modified Bessel function of the second kind involving the multiple mode summations. The integral representation for the function in the complex plane is the powerful tool to study the nonanalytic terms in the effective potential as shown in the papers I and II. We also follow the same strategy in the present paper and introduce the integral representation for the modified Bessel function of the second kind in the complex plane [14,15]

$$K_\nu(x) = \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma(t) \Gamma(t-\nu) \left(\frac{x}{2} \right)^{-2t+\nu}. \quad (2.25)$$

¹ $F_{L_0}^{(1)D}(M)$ corresponds to the finite-temperature effective potential obtained by Dolan and Jackiw [1].

Here, the constant c should be understood to be a point located on the real axis which is greater than all the poles of the gamma functions in the integrand.

Applying the formula (2.25) to Eq. (2.14) together with Eq. (2.21), we obtain the integral form in the complex plane for the third term of Eq. (2.22)

$$\begin{aligned} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) &= (-1)^{f+1} \mathcal{N} \frac{2^n \pi^{\frac{D-1}{2}}}{L_0^D} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D-1}{2}\right) \sum_{n_0=-\infty}^{\infty} \left\{ (n_0 + \eta_0)^2 + \left(\frac{ML_0}{2\pi}\right)^2 \right\}^{-t+\frac{D-1}{2}} \\ &\quad \times \left(\frac{\pi}{L_0}\right)^{-2t} S^{(n)}(t; L_{i_1}, \dots, L_{i_n}), \end{aligned} \quad (2.26)$$

where we have defined

$$S^{(n)}(t; L_{i_1}, \dots, L_{i_n}) \equiv \Gamma(t) \sum_{m_{i_1}=1}^{\infty} \cdots \sum_{m_{i_n}=1}^{\infty} \{(m_{i_1} L_{i_1})^2 + \cdots + (m_{i_n} L_{i_n})^2\}^{-t} \cos(2\pi m_{i_1} \eta_{i_1}) \cdots \cos(2\pi m_{i_n} \eta_{i_n}). \quad (2.27)$$

The right-hand side of Eq. (2.23) is recast by using Eq. (2.15) with Eq. (2.21) into

$$\begin{aligned} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F^{(0)D-1}(M_{(0)}) &= \frac{1}{L_0} (-1)^{f+1} \mathcal{N} \frac{\pi^{\frac{D-1}{2}}}{2(2\pi)^{D-1}} \Gamma\left(-\frac{D-1}{2}\right) \sum_{n_0=-\infty}^{\infty} (M_{(0)}^2)^{\frac{D-1}{2}} \\ &= (-1)^{f+1} \mathcal{N} \frac{\pi^{\frac{D-1}{2}}}{2L_0^D} \Gamma\left(-\frac{D-1}{2}\right) \sum_{n_0=-\infty}^{\infty} \left\{ (n_0 + \eta_0)^2 + \left(\frac{ML_0}{2\pi}\right)^2 \right\}^{\frac{D-1}{2}}. \end{aligned} \quad (2.28)$$

The mode sum over n_0 in Eqs. (2.26) and (2.28) has the analytical extension [16]. Depending on the value of η_0 that specifies the boundary condition for the S_t^1 direction, the form of the analytical extension is different and it is preferable to treat the fermion and the scalar separately.

Equations (2.22) and (2.26) are the starting point in the present paper to investigate the nonanalytic structure of the effective potential for the case of the scalar field with the periodic boundary condition for the spatial S_i^1 direction. On the other hand, Eqs. (2.24), (2.26), and (2.28) will be used for studying the nonanalytic terms in the effective potential for the case of the fermion field with arbitrary boundary condition for the spatial S_i^1 direction.

III. NONANALYTIC TERMS FOR SCALAR FIELD WITH PERIODIC BOUNDARY CONDITION

In this section we analyze the nonanalytic structure of the effective potential V_{eff} in the case of a real scalar field ($f=0, \eta_0=0, \mathcal{N}=1$) with the periodic boundary condition ($\eta_1 = \cdots = \eta_p = 0$) for the spatial $S_i^1 (i=1, \dots, p)$ direction. To this end, we further recast Eq. (2.22) into a more convenient form.

A. New form of V_{eff}

We separately write the zero mode ($n_0 = 0$) part and the nonzero mode ($n_0 \neq 0$) one in Eq. (2.26) to yield

$$\begin{aligned} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) &= \frac{1}{L_0} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M) - \frac{2^n \pi^{\frac{D-1}{2}}}{L_0^D} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D-1}{2}\right) \\ &\quad \times \sum_{n_0=1}^{\infty} \left\{ n_0^2 + \left(\frac{ML_0}{2\pi}\right)^2 \right\}^{-t+\frac{D-1}{2}} \left(\frac{\pi}{L_0}\right)^{-2t} \tilde{S}^{(n)}(t; L_{i_1}, \dots, L_{i_n}), \end{aligned} \quad (3.1)$$

where we have defined

$$\tilde{S}^{(n)}(t; L_{i_1}, \dots, L_{i_n}) = \Gamma(t) \sum_{m_{i_1}=1}^{\infty} \cdots \sum_{m_{i_n}=1}^{\infty} \{(m_{i_1} L_{i_1})^2 + \cdots + (m_{i_n} L_{i_n})^2\}^{-t}. \quad (3.2)$$

The mode sum over n_0 in Eq. (3.1) has the analytical extension [16] and it is given by

$$\begin{aligned} \Gamma(t) \sum_{n_0=1}^{\infty} \{n_0^2 + z^2\}^{-t} &= -\frac{1}{2} \frac{\Gamma(t)}{z^{2t}} + \frac{\sqrt{\pi}}{2} \frac{\Gamma(t - \frac{1}{2})}{z^{2(t - \frac{1}{2})}} \\ &+ \frac{\pi^{2t}}{\sqrt{\pi} 2\pi i} \int_{c_1 - i\infty}^{c_1 + i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \zeta(2t_1 - 2t + 1) \Gamma(t_1) (\pi z)^{-2t_1}. \end{aligned} \quad (3.3)$$

After changing the variable $\bar{t} = t - \frac{D-1}{2}$ in Eq. (3.1) and then applying Eq. (3.3) to the second term in Eq. (3.1), we obtain

$$\begin{aligned} & -\frac{2^n}{L_0 \pi^{\frac{D-1}{2}}} \frac{1}{2\pi i} \int_{c - \frac{D-1}{2} - i\infty}^{c - \frac{D-1}{2} + i\infty} dt \left\{ -\frac{1}{2} \Gamma(t) \left(\frac{ML_0}{2\pi}\right)^{-2t} + \frac{\sqrt{\pi}}{2} \Gamma\left(t - \frac{1}{2}\right) \left(\frac{ML_0}{2\pi}\right)^{-2(t - \frac{1}{2})} \right. \\ & + \frac{\pi^{2t}}{\sqrt{\pi} 2\pi i} \int_{c_1 - i\infty}^{c_1 + i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \zeta(2t_1 - 2t + 1) \Gamma(t_1) \left(\frac{ML_0}{2}\right)^{-2t_1} \Big\} \\ & \times \left(\frac{\pi}{L_0}\right)^{-2t} \tilde{S}^{(n)}\left(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n}\right), \end{aligned} \quad (3.4)$$

where we have denoted $\bar{t}$ as t again.

We deform the t_1 integration path in Eq. (3.4) to encircle all the poles of the integrand and evaluate the t_1 integral by the residue theorem. For this to be possible, the condition

$$0 < \frac{ML_0}{2\pi} < 1 \quad (3.5)$$

must be satisfied. This follows from the requirement that the contribution from the large left semicircle attached to the original t_1 integral must vanish for the residue theorem to apply. The condition (3.5) will be derived in Appendix A.

Among the poles $t_1 = t - \frac{1}{2} - \ell$ ($\ell = 0, 1, \dots$) of $\Gamma(t_1 - t + \frac{1}{2})$, only the pole $t_1 = t - \frac{1}{2}$ is relevant because of $\zeta(-2\ell) = 0$ ($\ell = 1, 2, \dots$), which always follows from the combination $\Gamma(t_1 - t + \frac{1}{2}) \zeta(2t_1 - 2t + 1)$ and is frequently used throughout our discussions. The residue at the pole $t_1 = t - \frac{1}{2}$ with $\zeta(0) = -1/2$ cancels the second term in Eq. (3.4) and that at the pole $t_1 = t$ of $\zeta(2t_1 - 2t + 1)$ with $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ does the first term in Eq. (3.4). Thus, we are left with the contribution from the pole $t_1 = -\bar{n}$ ($\bar{n} = 0, 1, \dots$) of $\Gamma(t_1)$ and Eq. (3.1) becomes²

$$\begin{aligned} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) &= \frac{1}{L_0} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M) - \frac{2^n}{\pi^{\frac{D}{2}} L_0} \sum_{\bar{n}=0}^{\infty} \frac{(-1)^{\bar{n}}}{\bar{n}!} \left(\frac{ML_0}{2}\right)^{2\bar{n}} \frac{1}{2\pi i} \int_{c - \frac{D-1}{2} - i\infty}^{c - \frac{D-1}{2} + i\infty} dt \Gamma\left(-t - \bar{n} + \frac{1}{2}\right) \zeta(-2t - 2\bar{n} + 1) \\ & \times \left(\frac{1}{L_0}\right)^{-2t} \tilde{S}^{(n)}\left(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n}\right). \end{aligned} \quad (3.6)$$

There may appear the poles in the multiple mode summations, $\tilde{S}^{(n)}(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n})$ that contribute to the above t integral. It does not, however, change the $(M^2)^{\bar{n}}$ dependence in Eq. (3.6) because of the nonexistence of M in $\tilde{S}^{(n)}(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n})$. Hence, the second term in Eq. (3.6), which arises from the nonzero modes ($n_0 \neq 0$), is entirely analytic with respect to M and is expressed in the positive powers of M^2 . This means that any nonanalytic term, if present, arises from the first term in Eq. (3.6) that comes from the contribution of the zero mode ($n_0 = 0$). We recognize that the zero mode is crucial for the existence of the nonanalytic term. The effective potential (2.22) with Eq. (3.6) becomes

²We exchange the order of the sum over $\bar{n}$ and the t integral in Eq. (3.6). The condition (3.5) ensures the convergence of the sum, and one can show that the t integral along the imaginary axis is well defined by using the discussion in the Appendix of the paper I together with the one in Appendix A of this paper. We also apply the exchange in subsequent equations, although we do not state it explicitly each time. Since the technical details of the exchange lie outside the scope of this paper, we do not pursue them further.

$$\begin{aligned}
V_{\text{eff}} = & F^{(0)D}(M) + F_{L_0}^{(1)D}(M) + \sum_{n=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} \frac{1}{L_0} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M) \\
& - \sum_{n=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} \frac{2^n}{\pi^{\frac{D}{2}} L_0} \sum_{\bar{n}=0}^{\infty} \frac{(-1)^{\bar{n}}}{\bar{n}!} \left(\frac{ML_0}{2} \right)^{2\bar{n}} \frac{1}{2\pi i} \int_{c-\frac{D-1}{2}-i\infty}^{c-\frac{D-1}{2}+i\infty} dt \\
& \times \Gamma\left(-t - \bar{n} + \frac{1}{2}\right) \zeta\left(-2t - 2\bar{n} + 1\right) \left(\frac{1}{L_0}\right)^{-2t} \tilde{S}^{(n)}\left(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n}\right). \quad (3.7)
\end{aligned}$$

We repeat the same discussion as the Sec. II B focusing next on the scale L_1 for the third term in Eq. (3.7) and apply the analytical extension (3.3) to the mode sum over n_1 associated with the scale L_1 . Then, we evaluate the t_1 integral like Eq. (3.4) by the residue theorem to yield

$$\begin{aligned}
\sum_{n=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M) = & F_{L_1}^{(1)D-1}(M) + \sum_{n=1}^{p-1} \sum_{2 \leq i_1 < i_2 < \dots < i_n \leq p} \frac{1}{L_1} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-2}(M) \\
& - \sum_{n=1}^{p-1} \sum_{2 \leq i_1 < i_2 < \dots < i_n \leq p} \frac{2^n}{\pi^{\frac{D-1}{2}} L_1} \sum_{\bar{n}=0}^{\infty} \frac{(-1)^{\bar{n}}}{\bar{n}!} \left(\frac{ML_1}{2} \right)^{2\bar{n}} \frac{1}{2\pi i} \int_{c-\frac{D-2}{2}-i\infty}^{c-\frac{D-2}{2}+i\infty} dt \\
& \times \Gamma\left(-t - \bar{n} + \frac{1}{2}\right) \zeta(-2t - 2\bar{n} + 1) \left(\frac{1}{L_1}\right)^{-2t} \tilde{S}\left(t + \frac{D-2}{2}; L_{i_1}, \dots, L_{i_n}\right). \quad (3.8)
\end{aligned}$$

One needs the condition $0 < ML_1/(2\pi) < 1$ for the t_1 integration to be evaluated by the residue theorem, which will be discussed in Appendix A. Let us note that in Eq. (3.8) the second term is the contribution from the zero mode ($n_1 = 0$), while the third one comes from the nonzero modes ($n_1 \neq 0$). We observe again that the contribution from the nonzero modes depends on the positive powers of M^2 , so that it is analytical, and the nonanalytic terms potentially originate from the first and the second terms. This procedure can be continued in order up to L_{p-1} by applying the mode recombination formula successively p times to the term with the multiple mode summations like $F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-2}(M)$ in Eq. (3.8).

Then, we finally derive a new remarkable expression for the effective potential

$$\begin{aligned}
V_{\text{eff}} = & F^{(0)D}(M) + \sum_{k=0}^p \left(\frac{1}{\prod_{i=0}^{k-1} L_i} \right) F_{L_k}^{(1)D-k}(M) \\
& - \sum_{k=1}^p \sum_{n=1}^{p+1-k} \sum_{k \leq i_1 < i_2 < \dots < i_n \leq p} \frac{2^n}{\pi^{\frac{D-(k-1)}{2}}} \frac{1}{\prod_{j=0}^{k-1} L_j} \sum_{\bar{n}=0}^{\infty} \frac{(-1)^{\bar{n}}}{\bar{n}!} \left(\frac{ML_{k-1}}{2} \right)^{2\bar{n}} \frac{1}{2\pi i} \int_{c-\frac{D-k}{2}-i\infty}^{c-\frac{D-k}{2}+i\infty} dt \\
& \times \Gamma\left(-t - \bar{n} + \frac{1}{2}\right) \zeta(-2t - 2\bar{n} + 1) \left(\frac{1}{L_{k-1}}\right)^{-2t} \tilde{S}^{(n)}\left(t + \frac{D-k}{2}; L_{i_1}, \dots, L_{i_n}\right). \quad (3.9)
\end{aligned}$$

It should be understood that $\prod_{i=0}^{k-1} L_i = 1$ for $k = 0$ in the second term of Eq. (3.9). We observe from Eq. (3.9) that the mode recombination formula enables the clear separation between the terms that contain the nonanalytic terms and entirely analytic ones. Hence, we conclude that the nonanalytic terms in V_{eff} are given by

$$V_{\text{eff}}|_{\text{n.a.}} = F^{(0)D}(M)|_{\text{n.a.}} + \sum_{k=0}^p \left(\frac{1}{\prod_{i=0}^{k-1} L_i} \right) F_{L_k}^{(1)D-k}(M)|_{\text{n.a.}}, \quad (3.10)$$

where the abbreviation denoted by “n.a.” means *nonanalytic terms*. Let us remind the reader again that the second term in Eq. (3.10) is the contributions from the zero modes ($n_0 = n_1 = \dots = n_p = 0$).

We repeatedly evaluate the t_1 integral of the type appearing in Eq. (3.4) in order to obtain Eq. (3.9). For this to be possible, one needs the condition

$$0 < \frac{ML_i}{2\pi} < 1 \quad (i = 0, 1, \dots, p). \quad (3.11)$$

From the above argument alone, we obtain only the conditions up to $i = p - 1$, but as we will see below and in Appendix A, Eq. (3.11) also specifies the conditions under which the residue theorem for computing the nonanalytic terms in Eq. (3.10) can be applied.

B. Nonanalytic terms

The first term in Eq. (3.10) is already calculated to be given by Eqs. (2.16) and (2.18). We analyze what types of nonanalytic terms appear in the second term of Eq. (3.10). From Eqs. (2.14) and (2.25), $F_{L_k}^{(1)D-k}(M)$ can be recast as

$$\begin{aligned} F_{L_k}^{(1)D-k}(M) &= \frac{-2}{(2\pi)^{\frac{D-k}{2}}} \sum_{m_k=1}^{\infty} \left(\frac{M^2}{(m_k L_k)^2} \right)^{\frac{D-k}{4}} K_{\frac{D-k}{2}}(m_k M L_k) \\ &= \frac{-2}{(2\pi)^{\frac{D-k}{2}}} \left(\frac{M^2}{2} \right)^{\frac{D-k}{2}} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D-k}{2}\right) \Gamma(t) \zeta(2t) \left(\frac{M L_k}{2} \right)^{-2t}, \end{aligned} \quad (3.12)$$

where we have used the definition of the zeta function $\zeta(z) = \sum_{m=1}^{\infty} m^{-z}$. By deforming the contour in the complex t plane so as to enclose all the poles of the integrand, we evaluate the t integral using the residue theorem. For this to be possible, as discussed in Appendix A, one needs the condition, $0 < M L_k / (2\pi) < 1$ ($k = 0, \dots, p$), which is the same as Eq. (3.11).

In the paper II we are only interested in the power-type nonanalytic terms with respect to M , that is, the odd power of M and calculate the integral by the residue theorem by focusing only on the single pole of the integrand. We did not studied other nonanalytic terms in the paper. In the present paper we thoroughly investigate other type of nonanalytic terms, so that the higher-order pole of the integrand must be taken into account. In fact, as we will see below, the double pole generates the logarithmic-type nonanalytic term, $\log M$.

Computing the t integral by the residue theorem in Eq. (3.12) is straightforward and the details of the calculation are given in Appendix B. We present the results here. For $(D, k) = (\text{even}, \text{even}), (\text{odd}, \text{odd})$, we obtain

$$\begin{aligned} F_{L_k}^{(1)D-k}(M) &= \frac{(-1)(-1)^{\frac{D-k}{2}}}{2^{\frac{D-k}{2}} \pi^{\frac{D-k-2}{2}} (D-k-1)!!} \frac{M^{D-k-1}}{L_k} \\ &+ \frac{(-1)(-1)^{\frac{D-k}{2}}}{2^{D-k} \pi^{\frac{D-k}{2}} (\frac{D-k}{2})!} M^{D-k} \left\{ \log\left(\frac{M L_k}{4\pi}\right) - \frac{1}{2} \left(\psi(1) + \psi\left(\frac{D-k}{2} + 1\right) \right) \right\} \\ &+ \frac{(-1)(-1)^{\frac{D-k}{2}}}{2^{D-k} \pi^{\frac{D-k}{2}}} M^{D-k} \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell} \Gamma(2\ell+1) \zeta(2\ell+1)}{\Gamma(\ell + \frac{D-k}{2} + 1) \Gamma(\ell+1)} \left(\frac{M L_k}{4\pi} \right)^{2\ell} \\ &+ \frac{-1}{\pi^{\frac{D-k}{2}} L_k^{D-k}} \sum_{m=0}^{\frac{D-k}{2}-1} \frac{(-1)^m}{m!} \Gamma\left(\frac{D-k}{2} - m\right) \zeta(D-k-2m) \left(\frac{M L_k}{2} \right)^{2m}. \end{aligned} \quad (3.13)$$

The first term, the power-type nonanalytic term, arises from the residue at the single pole $t = \frac{1}{2}$ of $\zeta(2t)$ and the second term containing the logarithmic-type nonanalytic term originates from the residue at the double pole $t = 0$ of the part $\Gamma(t - \frac{D-k}{2})\Gamma(t)$ in the integrand. The other two terms are analytic. We denote the nonanalytic terms in Eq. (3.13) by

$$F_{L_k}^{(1)D-k}(M) \Big|_{\text{n.a.}}^{\text{pow}} \equiv \frac{(-1)(-1)^{\frac{D-k}{2}}}{2^{\frac{D-k}{2}} \pi^{\frac{D-k-2}{2}} (D-k-1)!!} \frac{M^{D-k-1}}{L_k}, \quad (3.14)$$

$$F_{L_k}^{(1)D-k}(M) \Big|_{\text{n.a.}}^{\log} \equiv \frac{(-1)(-1)^{\frac{D-k}{2}}}{2^{D-k} \pi^{\frac{D-k}{2}} (\frac{D-k}{2})!} M^{D-k} \log\left(\frac{M L_k}{4\pi}\right). \quad (3.15)$$

For $(D, k) = (\text{even}, \text{odd}), (\text{odd}, \text{even})$, we have

$$\begin{aligned}
 F_{L_k}^{(1)D-k}(M) &= \frac{(-1)^{\frac{D-k+1}{2}}}{2^{\frac{D-k+1}{2}} \pi^{\frac{D-k-1}{2}} (D-k)!!} M^{D-k} \\
 &+ \frac{(-1)(-1)^{\frac{D-k-1}{2}}}{2^{D-k-1} \pi^{\frac{D-k-1}{2}} (\frac{D-k-1}{2})!} \frac{M^{D-k-1}}{L_k} \left\{ -\log(ML_k) + \frac{1}{2} \left(\gamma_E + \psi \left(\frac{D-k-1}{2} + 1 \right) \right) \right\} \\
 &+ \frac{-1}{\pi^{\frac{D-k}{2}} L_k^{D-k}} \sum_{\substack{m=0 \\ (m \neq \frac{D-k-1}{2})}}^{\infty} \frac{(-1)^m}{m!} \Gamma \left(\frac{D-k}{2} - m \right) \zeta(D-k-2m) \left(\frac{ML_k}{2} \right)^{2m}.
 \end{aligned} \tag{3.16}$$

The first term is the power-type nonanalytic term that comes from the residue at the single pole $t = 0$ of $\Gamma(t)$ and the second term including the logarithmic-type nonanalytic term arises from the residue at the double pole $t = \frac{1}{2}$ of the part $\Gamma(t - \frac{D-k}{2})\zeta(2t)$ in the integrand. The last term is analytic. We represent the nonanalytic terms in Eq. (3.16) as

$$F_{L_k}^{(1)D-k}(M) \Big|_{\text{n.a.}}^{\text{pow}} \equiv \frac{(-1)^{\frac{D-k+1}{2}}}{2^{\frac{D-k+1}{2}} \pi^{\frac{D-k-1}{2}} (D-k)!!} M^{D-k}, \tag{3.17}$$

$$F_{L_k}^{(1)D-k}(M) \Big|_{\text{n.a.}}^{\text{log}} \equiv \frac{(-1)^{\frac{D-k-1}{2}}}{2^{D-k-1} \pi^{\frac{D-k-1}{2}} (\frac{D-k-1}{2})!} \frac{M^{D-k-1}}{L_k} \log(ML_k). \tag{3.18}$$

We have evaluated the t integral in Eq. (3.12) by the residue theorem and have found the possible types of the nonanalytic terms in $F_{L_k}^{(1)D-k}(M)$ to be the power type or the logarithmic one. $F_{L_k}^{(1)D-k}(M)$ contains not only the nonanalytic terms but also the analytic ones. This implies that although $F_{L_k}^{(1)D-k}(M)$ is the contribution from the zero-mode sector through the process of applying the mode recombination formula, it incorporates more contributions in addition to the nonanalytic terms. We will come back to this point again at the end of this section.

We are ready to calculate the nonanalytic terms in V_{eff} according to Eq. (3.10). We introduce the useful notation to study the nonanalytic term and discuss separately the cases of even and odd values of D and $p+1$. For the case of $D = \text{even}$, let us recall the notations defined in the paper II, which is convenient to discuss the power-type nonanalytic terms

$$A_{2n+1}^{\text{pow}} \equiv \frac{1}{\prod_{i=0}^{2n-1} L_i} F_{L_{2n+1}}^{(1)D-2n-1}(M) \Big|_{\text{n.a.}}^{\text{pow}} = \frac{(-1)^{\frac{D-2n}{2}}}{2^{\frac{D-2n}{2}} \pi^{\frac{D-2n-2}{2}} (D-2n-1)!!} \frac{M^{D-2n-1}}{L_0 \cdots L_{2n}}, \tag{3.19}$$

$$B_{2n}^{\text{pow}} \equiv \frac{1}{\prod_{i=0}^{2n-1} L_i} F_{L_{2n}}^{(1)D-2n}(M) \Big|_{\text{n.a.}}^{\text{pow}} = \frac{(-1)(-1)^{\frac{D-2n}{2}}}{2^{\frac{D-2n}{2}} \pi^{\frac{D-2n-2}{2}} (D-2n-1)!!} \frac{M^{D-2n-1}}{L_0 \cdots L_{2n}}, \tag{3.20}$$

where we have newly added the superscript “pow” in Eqs. (3.19) and (3.20). One immediately finds

$$A_{2n+1}^{\text{pow}} + B_{2n}^{\text{pow}} = 0 \quad \text{for } n = 0, 1, \dots \tag{3.21}$$

We further introduce the notations for the logarithmic-type nonanalytic terms defined by

$$A_{2n-1}^{\text{log}} \equiv \frac{1}{\prod_{i=0}^{2n-2} L_i} F_{L_{2n-1}}^{(1)D-(2n-1)}(M) \Big|_{\text{n.a.}}^{\text{log}} = \frac{(-1)^{\frac{D-2n}{2}}}{2^{D-2n} \pi^{\frac{D-2n}{2}} (\frac{D-2n}{2})!} \frac{M^{D-2n}}{L_0 \cdots L_{2n-1}} \log(ML_{2n-1}), \tag{3.22}$$

$$B_{2n}^{\text{log}} \equiv \frac{1}{\prod_{i=0}^{2n-1} L_i} F_{L_{2n}}^{(1)D-2n}(M) \Big|_{\text{n.a.}}^{\text{log}} = \frac{(-1)(-1)^{\frac{D-2n}{2}}}{2^{D-2n} \pi^{\frac{D-2n}{2}} (\frac{D-2n}{2})!} \frac{M^{D-2n}}{L_0 \cdots L_{2n-1}} \log \left(\frac{ML_{2n}}{4\pi} \right), \tag{3.23}$$

where we have used Eqs. (3.18) and (3.15), respectively. We find

$$A_{2n-1}^{\log} + B_{2n}^{\log} = \frac{(-1)^{\frac{D-2n}{2}}}{2^{D-2n} \pi^{\frac{D-2n}{2}} (\frac{D-2n}{2})! L_0 \cdots L_{2n-1}} M^{D-2n} \times \log\left(\frac{4\pi L_{2n-1}}{L_{2n}}\right) \quad \text{for } n = 0, 1, \dots \quad (3.24)$$

The $n = 0$ contribution in Eq. (3.24) should be read as

$$A_{2n-1}^{\log}|_{n=0} + B_{2n}^{\log}|_{n=0} = \frac{(-1)^{\frac{D}{2}}}{2^D \pi^{\frac{D}{2}} (\frac{D}{2})!} M^D \log\left(\frac{4\pi}{\mu L_0}\right), \quad (3.25)$$

where we have defined

$$A_{2n-1}^{\log}|_{n=0} \equiv F^{(0)D}(M) \Big|_{\text{n.a.}}^{\log} \equiv \frac{(-1)^{\frac{D}{2}}}{2^D \pi^{\frac{D}{2}} (\frac{D}{2})!} M^D \log \frac{M}{\mu}, \quad (3.26)$$

$$B_{2n}^{\log}|_{n=0} \equiv F_{L_0}^{(1)D}(M) \Big|_{\text{n.a.}}^{\log} \equiv \frac{(-1)(-1)^{\frac{D}{2}}}{2^D \pi^{\frac{D}{2}} (\frac{D}{2})!} M^D \log\left(\frac{ML_0}{4\pi}\right). \quad (3.27)$$

Here, we have used Eq. (2.16) with $f = 0, \mathcal{N} = 1$ for Eq. (3.26), and Eq. (3.15) with $k = 0$ for Eq. (3.27). The $\log M$ -dependence in Eqs. (3.24) and (3.25) cancels out, leaving alternatively the nonanalytic dependence on the scale ratios, L_{2n-1}/L_{2n} and $1/\mu L_0$. In particular, the cancellation of the $\log M$ term in Eq. (3.25) occurs between the zero-temperature contribution (3.26) and the finite-temperature one (3.27), which has been pointed out by Dolan and Jackiw in their paper [1]. This implies that the finite-temperature effective potential in the case of no spatial S_i^1 direction ($p = 0$) does not have the $\log M$ dependence, but it still possesses the nonanalytic structure given by the logarithmic term, $\log(1/\mu L_0)$. As we will see below, this is also the case for fermions.

Equipped with Eqs. (3.19), (3.20), (3.22) and (3.23), we can calculate the nonanalytic terms in the effective potential V_{eff} by Eq. (3.10) for $D = \text{even}$.

C. $(D, p+1) = (\text{even}, \text{odd})$

The nonanalytic terms in V_{eff} are written by

$$V_{\text{eff}}|_{\text{n.a.}} = \sum_{n=0}^{\frac{p}{2}} \left\{ A_{2n-1}^{\log} + B_{2n}^{\log} \right\} + \sum_{n=0}^{\frac{p}{2}-1} \left\{ A_{2n+1}^{\text{pow}} + B_{2n}^{\text{pow}} \right\} + B_{2n}^{\text{pow}}|_{n=\frac{p}{2}} \\ = \sum_{n=0}^{\frac{p}{2}} \frac{(-1)^{\frac{D-2n}{2}}}{2^{D-2n} \pi^{\frac{D-2n}{2}} (\frac{D-2n}{2})! L_0 \cdots L_{2n-1}} M^{D-2n} \log\left(\frac{4\pi L_{2n-1}}{L_{2n}}\right) + \frac{(-1)(-1)^{\frac{D-p}{2}}}{2^{\frac{D-p}{2}} \pi^{\frac{D-p-2}{2}} (D-p-1)!! L_0 \cdots L_p} M^{D-p-1}, \quad (3.28)$$

where we have used Eqs. (3.21), (3.24) and (3.20) with $n = \frac{p}{2}$. The $n = 0$ contribution in the sum over n of Eq. (3.28) is given by Eq. (3.25). The effective potential has the power-type nonanalytic term [12,13] arising from $B_{2n}^{\text{pow}}|_{n=\frac{p}{2}}$ in this case. The $\log M$ term cancels out and what remains is the nonanalytic term that depends on the scale ratios, L_{2n-1}/L_{2n} and $1/\mu L_0$.

D. $(D, p+1) = (\text{even}, \text{even})$

In this case the nonanalytic terms are given by

$$V_{\text{eff}}|_{\text{n.a.}} = \sum_{n=0}^{\frac{p-1}{2}} \left\{ A_{2n-1}^{\log} + B_{2n}^{\log} \right\} + A_{2n-1}^{\log}|_{n=\frac{p+1}{2}} + \sum_{n=0}^{\frac{p-1}{2}} \left\{ A_{2n+1}^{\text{pow}} + B_{2n}^{\text{pow}} \right\} \\ = \sum_{n=0}^{\frac{p-1}{2}} \frac{(-1)^{\frac{D-2n}{2}}}{2^{D-2n} \pi^{\frac{D-2n}{2}} (\frac{D-2n}{2})! L_0 \cdots L_{2n-1}} M^{D-2n} \log\left(\frac{4\pi L_{2n-1}}{L_{2n}}\right) + \frac{(-1)^{\frac{D-p-1}{2}}}{2^{D-p-1} \pi^{\frac{D-p-1}{2}} (\frac{D-p-1}{2})! L_0 \cdots L_p} M^{D-p-1} \log(ML_p), \quad (3.29)$$

where we have used Eqs. (3.21), (3.24) and (3.22) with $n = \frac{p+1}{2}$. Equation (3.25) should be understood as the $n = 0$ contribution in the n sum of Eq. (3.29). The effective potential possesses no power-type nonanalytic term, which is consistent with the results in the papers I and II. On the other hand, the $\log M$ term arising from $A_{2n-1}^{\log}|_{n=\frac{p+1}{2}}$ survives and other nonanalytic contributions, $\log(L_{2n-1}/L_{2n})$ and $\log(1/\mu L_0)$ also appear.

Let us proceed to study the case of $D = \text{odd}$. We use the notations defined in the paper II, which is useful to discuss the power-type nonanalytic terms

$$C_{2n-1}^{\text{pow}} \equiv \frac{1}{\prod_{i=0}^{2n-2} L_i} F_{L_{2n-1}}^{(1)D-(2n-1)}(M) \Big|_{\text{n.a.}}^{\text{pow}} = \frac{(-1)(-1)^{\frac{D-(2n-1)}{2}}}{2^{\frac{D-2n+1}{2}} \pi^{\frac{D-2n-1}{2}} (D-2n)!!} \frac{M^{D-2n}}{L_0 \cdots L_{2n-1}}, \quad (3.30)$$

$$D_{2n}^{\text{pow}} \equiv \frac{1}{\prod_{i=0}^{2n-1} L_i} F_{L_{2n}}^{(1)D-2n}(M) \Big|_{\text{n.a.}}^{\text{pow}} = \frac{(-1)^{\frac{D-(2n-1)}{2}}}{2^{\frac{D-2n+1}{2}} \pi^{\frac{D-2n-1}{2}} (D-2n)!!} \frac{M^{D-2n}}{L_0 \cdots L_{2n-1}}, \quad (3.31)$$

where we have newly added the superscript “pow” in Eqs. (3.30) and (3.31). We see that the relation

$$C_{2n-1}^{\text{pow}} + D_{2n}^{\text{pow}} = 0 \quad \text{for } n = 0, 1, \dots \quad (3.32)$$

holds. In the case of $n = 0$ in Eq. (3.32), it should be understood as

$$C_{2n-1}^{\text{pow}}|_{n=0} \equiv F^{(0)D}(M) \Big|_{\text{n.a.}}^{\text{pow}} \equiv \frac{(-1)(-1)^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}} \pi^{\frac{D-1}{2}} D!!} M^D, \quad (3.33)$$

$$D_{2n}^{\text{pow}}|_{n=0} \equiv F_{L_0}^{(1)D}(M) \Big|_{\text{n.a.}}^{\text{pow}} = \frac{(-1)^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}} \pi^{\frac{D-1}{2}} D!!} M^D, \quad (3.34)$$

where we have used Eq. (2.18) with $f = 0, \mathcal{N} = 1$ and Eq. (3.17) with $k = 0$, respectively, which confirms the relation (3.32) for $n = 0$.

In addition to them, we further define the notations

$$C_{2n+1}^{\log} \equiv \frac{1}{\prod_{i=0}^{2n} L_i} F_{L_{2n+1}}^{(1)D-(2n+1)}(M) \Big|_{\text{n.a.}}^{\log} = \frac{(-1)(-1)^{\frac{D-2n-1}{2}}}{2^{D-2n-1} \pi^{\frac{D-2n-1}{2}} (\frac{D-2n-1}{2})!} \frac{M^{D-2n-1}}{L_0 \cdots L_{2n}} \log\left(\frac{ML_{2n+1}}{4\pi}\right), \quad (3.35)$$

$$D_{2n}^{\log} \equiv \frac{1}{\prod_{i=0}^{2n-1} L_i} F_{L_{2n}}^{(1)D-2n}(M) \Big|_{\text{n.a.}}^{\log} = \frac{(-1)^{\frac{D-2n-1}{2}}}{2^{D-2n-1} \pi^{\frac{D-2n-1}{2}} (\frac{D-2n-1}{2})!} \frac{M^{D-2n-1}}{L_0 \cdots L_{2n}} \log(ML_{2n}), \quad (3.36)$$

where we have used Eqs. (3.15) and (3.18), respectively. Then, we find

$$C_{2n+1}^{\log} + D_{2n}^{\log} = \frac{(-1)^{\frac{D-2n-1}{2}}}{2^{D-2n-1} \pi^{\frac{D-2n-1}{2}} (\frac{D-2n-1}{2})!} \frac{M^{D-2n-1}}{L_0 \cdots L_{2n}} \log\left(\frac{4\pi L_{2n}}{L_{2n+1}}\right) \quad \text{for } n = 0, 1, \dots \quad (3.37)$$

The $\log M$ dependence drops out, leaving the nonanalytic dependence on the scale ratio L_{2n}/L_{2n+1} instead.

With the help of Eqs. (3.30), (3.31), (3.35) and (3.36), one can calculate the nonanalytic terms in the effective potential V_{eff} according to Eq. (3.10) for $D = \text{odd}$.

E. $(D, p+1) = (\text{odd}, \text{odd})$

The nonanalytic terms in the effective potential V_{eff} are given by

$$\begin{aligned} V_{\text{eff}}|_{\text{n.a.}} &= \sum_{n=0}^{\frac{p}{2}-1} \{C_{2n+1}^{\log} + D_{2n}^{\log}\} + D_{2n}^{\log}|_{n=\frac{p}{2}} + \sum_{n=0}^{\frac{p}{2}} \{C_{2n-1}^{\text{pow}} + D_{2n}^{\text{pow}}\} \\ &= \sum_{n=0}^{\frac{p}{2}-1} \frac{(-1)^{\frac{D-2n-1}{2}}}{2^{D-2n-1} \pi^{\frac{D-2n-1}{2}} (\frac{D-2n-1}{2})!} \frac{M^{D-2n-1}}{L_0 \cdots L_{2n}} \log\left(\frac{4\pi L_{2n}}{L_{2n+1}}\right) + \frac{(-1)^{\frac{D-p-1}{2}}}{2^{D-p-1} \pi^{\frac{D-p-1}{2}} (\frac{D-p-1}{2})!} \frac{M^{D-p-1}}{L_0 \cdots L_p} \log(ML_p), \end{aligned} \quad (3.38)$$

where we have used Eqs. (3.32), (3.37) and (3.36) with $n = \frac{p}{2}$. We observe that there is no power-type nonanalytic term [12,13]. The $\log M$ term arising from $D_{2n}^{\log}|_{n=\frac{p}{2}}$ survives and the $\log(L_{2n}/L_{2n+1})$ term also appears.

F. $(D, p+1) = (\text{odd}, \text{even})$

In this case the nonanalytic terms in V_{eff} are given by

$$\begin{aligned}
 V_{\text{eff}}|_{\text{n.a.}} &= \sum_{n=0}^{\frac{p-1}{2}} \left\{ C_{2n+1}^{\log} + D_{2n}^{\log} \right\} + \sum_{n=0}^{\frac{p-1}{2}} \left\{ C_{2n-1}^{\text{pow}} + D_{2n}^{\text{pow}} \right\} + C_{2n-1}^{\text{pow}}|_{n=\frac{p+1}{2}} \\
 &= \sum_{n=0}^{\frac{p-1}{2}} \frac{(-1)^{\frac{D-2n-1}{2}}}{2^{D-2n-1} \pi^{\frac{D-2n-1}{2}} (\frac{D-2n-1}{2})!} \frac{M^{D-2n-1}}{L_0 \cdots L_{2n}} \log \left(\frac{4\pi L_{2n}}{L_{2n+1}} \right) \\
 &\quad + \frac{(-1)(-1)^{\frac{D-p}{2}}}{2^{\frac{D-p}{2}} \pi^{\frac{D-p-2}{2}} (D-p-1)!!} \frac{M^{D-p-1}}{L_0 \cdots L_p}, \tag{3.39}
 \end{aligned}$$

where we have used Eqs. (3.32), (3.37) and (3.30) with $n = \frac{p+1}{2}$. V_{eff} has the power-type nonanalytic term arising from $C_{2n-1}^{\text{pow}}|_{n=\frac{p+1}{2}}$ [12,13]. The $\log M$ dependence cancels out and the nonanalytic dependence on the scale ratio L_{2n}/L_{2n+1} emerges instead. Let us note that the overall sign $(-1)^{(D-p-1)/2}$ in the $\log M$ term in Eqs. (3.29) and (3.38) takes $(D-p-1)/2 = 1(-1)$ for $D-p-1 = 4m(4m+2)$ with $m = 1, 2, \dots$

We obtained all the nonanalytic terms that cannot be expressed in the positive powers of M^2 in the effective potential V_{eff} for the case of the scalar field satisfying the periodic boundary condition for the spatial $S_i^1 (i = 1, \dots, p)$ direction. The effective potential has two types of nonanalytic terms: power type and logarithmic one. We demonstrated that the appearance of these terms is highly characteristic, that is, the power-type nonanalytic term exists in the case of $D-(p+1) = \text{odd}$, while the logarithmic-type one does in the case of $D-(p+1) = \text{even}$. The two types never appear simultaneously. We summarize the results in Table I.

It may be appropriate to comment on the nonanalytic terms, which have not been mentioned in the paper II. The zero modes ($n_0 = n_1 = \dots = n_p = 0$) are crucial for the existence of the nonanalytic terms and in fact the second term $F_{L_k}^{(1)D-k}(M)$ in Eq. (3.10) containing the nonanalytic terms arises from the zero-mode sector through the process of applying the mode recombination formula. Since the

infrared behavior of the theory may be dominated by the zero mode, such an infrared nonanalyticity is expected to be obtained directly by the contribution from the zero mode. We show that this is indeed the case below. By setting $n_0 = n_1 = \dots = n_p = 0, \eta_0 = \eta_1 = \dots = \eta_p = 0$ and $f = 0, \mathcal{N} = 1$ in Eq. (2.4), we have

$$V_{\text{eff}}^{\text{zero mode}} = \frac{1}{2} \frac{1}{L_0 \cdots L_p} \int \frac{d^{D-(p+1)} p_E}{(2\pi)^{D-(p+1)}} \log(p_E^2 + M^2). \tag{3.40}$$

We follow the prescription to calculate the effective potential in Sec. II and immediately obtain

$$V_{\text{eff}}^{\text{zero mode}} = -\frac{\pi^{\frac{D-(p+1)}{2}}}{2(2\pi)^{D-(p+1)}} \Gamma\left(-\frac{D-(p+1)}{2}\right) \frac{M^{D-p-1}}{L_0 \cdots L_p}. \tag{3.41}$$

For $D-(p+1) = \text{odd}$, by using Eq. (2.17), we find

$$V_{\text{eff}}^{\text{zero mode}}|_{D-(p+1)=\text{odd}} = \frac{(-1)(-1)^{\frac{D-p}{2}}}{2^{\frac{D-p}{2}} \pi^{\frac{D-p-2}{2}} (D-p-1)!!} \frac{M^{D-p-1}}{L_0 \cdots L_p}. \tag{3.42}$$

This is the same with the second terms in Eqs. (3.28) and (3.39). On the other hand, for $D-(p+1) = \text{even}$, the gamma function in Eq. (3.41) has the pole and we employ, for example, the dimensional regularization and the $\overline{\text{MS}}$ renormalization to yield

$$\begin{aligned}
 V_{\text{eff}}^{\text{zero mode}}|_{D-(p+1)=\text{even}} &= \frac{(-1)^{\frac{D-p-1}{2}}}{2^{D-p-1} \pi^{\frac{D-p-1}{2}} (\frac{D-p-1}{2})!} \\
 &\quad \times \frac{M^{D-p-1}}{L_0 \cdots L_p} \log \frac{M}{\mu}, \tag{3.43}
 \end{aligned}$$

TABLE I. Nonanalytic structure of the effective potential on the spacetime, $S_i^1 \times R^{D-(p+1)} \times \prod_{i=1}^p S_i^1$ for the scalar field with the periodic boundary condition for the $S_i^1 (i = 1, \dots, p)$ direction. The “non” in the table means that there is no corresponding type of nonanalytic terms. The numbers in the parentheses stand for the equation numbers in the text.

$(D, p+1)$	(even, odd)	(even, even)	(odd, odd)	(odd, even)
Power type	(3.28)	non	non	(3.39)
$\log M$ type	non	(3.29)	(3.38)	non

where the scale μ is a set of possible renormalization conditions. This is equivalent to the second terms in Eqs. (3.29) and (3.38) if we identify the scale μ as L_p^{-1} .

The above calculation shows that the nonanalytic terms are simply obtained by the contribution from only the zero mode in Eq. (2.4). Even though $F_{L_k}^{(1)D-k}(M)$ that potentially contains the nonanalytic terms also comes from the zero-mode sector through the process of applying the mode recombination formula, due the existence of the winding mode m_k in $F_{L_k}^{(1)D-k}(M)$, it includes additional quantum effects to result the third and fourth terms in Eq. (3.13) [or the third term in Eq. (3.16)] in addition to the nonanalytic terms. Therefore, our approach for studying the nonanalytic

terms leads to a larger contribution than the approach that takes into account only the zero mode.

IV. NONANALYTIC TERMS FOR FERMION FIELD WITH ARBITRARY BOUNDARY CONDITION

A. New form of V_{eff}^f

Let us study the nonanalytic structure of the effective potential V_{eff}^f in the case of a fermion ($f = 1, \eta_0 = \frac{1}{2}$) with arbitrary boundary condition for the spatial $S_i^1 (i = 1, \dots, p)$ direction. It is convenient to use the expression (2.24) and we consider the second term whose integral representation in the complex plane is given by Eq. (2.26)

$$\begin{aligned} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)(D-1, f)}(M_{(0)}) &= \mathcal{N} \frac{2^n \pi^{\frac{D-1}{2}}}{L_0^D} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D-1}{2}\right) \sum_{n_0=-\infty}^{\infty} \left\{ \left(n_0 + \frac{1}{2}\right)^2 + \left(\frac{ML_0}{2\pi}\right)^2 \right\}^{-t+\frac{D-1}{2}} \\ &\quad \times \left(\frac{\pi}{L_0}\right)^{-2t} S^{(n)}(t; L_{i_1}, \dots, L_{i_n}). \end{aligned} \quad (4.1)$$

Hereafter, we assign the superscript “ f ” in the relevant quantity like $F_{L_{i_1}, \dots, L_{i_n}}^{(n)(D-1, f)}(M_{(0)})$ in order to make it clear that we are considering the case of fermions.

The mode sum over n_0 has the analytical extension [13,16]

$$\begin{aligned} \Gamma(t) \sum_{n_0=-\infty}^{\infty} \left\{ \left(n_0 + \frac{1}{2}\right)^2 + z^2 \right\}^{-t} &= \sqrt{\pi} \frac{\Gamma(t - \frac{1}{2})}{z^{2(t-\frac{1}{2})}} - \frac{4\pi^{2t}}{\sqrt{\pi}} \frac{1}{4\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \zeta(2t_1 - 2t + 1) \Gamma(t_1) (\pi z)^{-2t_1} \\ &\quad + \frac{4(2\pi)^{2t}}{\sqrt{\pi}} \frac{1}{4\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \zeta(2t_1 - 2t + 1) \Gamma(t_1) (2\pi z)^{-2t_1} \\ &= \sqrt{\pi} \frac{\Gamma(t - \frac{1}{2})}{z^{2(t-\frac{1}{2})}} - \frac{4\pi^{2t}}{\sqrt{\pi}} \frac{1}{4\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \eta(2t_1 - 2t + 1) \Gamma(t_1) (\pi z)^{-2t_1}, \end{aligned} \quad (4.2)$$

where we have defined the so-called eta function

$$\eta(t) \equiv (1 - 2^{1-t}) \zeta(t). \quad (4.3)$$

The characteristic point in Eq. (4.2), compared with Eq. (3.3), is that the eta function appears instead of the zeta function and there is no counterpart corresponding to the first term arising from the zero mode ($n_0 = 0$). These come from the fact that in the case of fermions the zero mode is removed due to the antiperiodic boundary condition for the Euclidean time direction followed from quantum statistics. After changing the variable $\bar{t} = t - \frac{D-1}{2}$ in Eq. (4.1) and inserting Eq. (4.2) into it, we obtain

$$\begin{aligned} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)(D-1, f)}(M_{(0)}) &= \frac{1}{L_0} \mathcal{N} \frac{2^n}{\pi^{\frac{D-1}{2}}} \frac{1}{4\pi i} \int_{c-\frac{D-1}{2}+i\infty}^{c-\frac{D-1}{2}-i\infty} dt \left\{ \sqrt{\pi} \Gamma\left(t - \frac{1}{2}\right) \left(\frac{ML_0}{2\pi}\right)^{-2(t-\frac{1}{2})} \right. \\ &\quad \left. - \frac{4\pi^{2t}}{\sqrt{\pi}} \frac{1}{4\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \eta(2t_1 - 2t + 1) \Gamma(t_1) \left(\frac{ML_0}{2\pi}\right)^{-2t_1} \right\} \\ &\quad \times \left(\frac{\pi}{L_0}\right)^{-2t} S^{(n)}\left(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n}\right), \end{aligned} \quad (4.4)$$

where we have denoted $\bar{t}$ as t again.

We deform the t_1 -integration path in Eq. (4.4) so that it encloses all the poles of the integrand and then evaluate the t_1 integral by the residue theorem. For this to be possible, as discussed in Appendix A, the condition

$$0 < \frac{ML_0}{\pi} < 1 \quad (4.5)$$

must be satisfied. Among the poles $t_1 = t - \frac{1}{2} - \ell$ ($\ell = 0, 1, \dots$) of $\Gamma(t_1 - t + \frac{1}{2})$, only the pole $t_1 = t - \frac{1}{2}$ is relevant because of $\eta(-2\ell) = 0$ ($\ell = 1, 2, \dots$), which always follows from the combination $\Gamma(t_1 - t + \frac{1}{2})\eta(2t_1 - 2t + 1)$ and is frequently used throughout our discussions. The residue at the pole $t_1 = t - \frac{1}{2}$ with $\eta(0) = 1/2$ cancels the first term in Eq. (4.4). The eta function does not possess any pole unlike the zeta function. We end up with the contribution from the pole $t_1 = -\bar{n}$ ($\bar{n} = 0, 1, \dots$) of $\Gamma(t_1)$, so that Eq. (4.4) becomes

$$\begin{aligned} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)(D-1, f)}(M_{(0)}) &= -\mathcal{N} \frac{2^n}{\pi^{\frac{D}{2}} L_0} \frac{1}{2\pi i} \int_{c-\frac{D-1}{2}-i\infty}^{c-\frac{D-1}{2}+i\infty} dt \sum_{\bar{n}=0}^{\infty} \frac{(-1)^{\bar{n}}}{\bar{n}!} \left(\frac{ML_0}{2}\right)^{2\bar{n}} \Gamma\left(-t - \bar{n} + \frac{1}{2}\right) \eta(-2t - 2\bar{n} + 1) \\ &\times \left(\frac{1}{L_0}\right)^{-2t} S^{(n)}\left(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n}\right). \end{aligned} \quad (4.6)$$

There may be the poles in the multiple mode summations $S^{(n)}(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n})$ that contribute to the above t integral. It does not, however, affect the $(M^2)^{\bar{n}}$ dependence due to the nonexistence of M in the summations. We observe that all the terms in Eq. (4.6) are analytic and can be expressed in the positive powers of M^2 . Hence, there exist no nonanalytic terms in Eq. (4.6).

Let us next study the first term in Eq. (2.24), whose explicit form is given by Eq. (2.28) with $f = 1$ and $\eta_0 = \frac{1}{2}$

$$\frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F^{(0)(D-1, f)}(M_{(0)}) = \mathcal{N} \frac{\pi^{\frac{D-1}{2}}}{2L_0^D} \Gamma\left(-\frac{D-1}{2}\right) \sum_{n_0=-\infty}^{\infty} \left\{ \left(n_0 + \frac{1}{2}\right)^2 + \left(\frac{ML_0}{2\pi}\right)^2 \right\}^{\frac{D-1}{2}}. \quad (4.7)$$

The analytical extension for the mode sum over n_0 is given by setting $t = -\frac{D-1}{2}$, $z = \frac{ML_0}{2\pi}$ in Eq. (4.2) as

$$\begin{aligned} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F^{(0)(D-1, f)}(M_{(0)}) &= \mathcal{N} \frac{\pi^{\frac{D}{2}}}{2(2\pi)^D} \Gamma\left(-\frac{D}{2}\right) M^D \\ &- \mathcal{N} \frac{1}{L_0^D} \frac{1}{\pi^{\frac{D}{2}} 2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 + \frac{D}{2}\right) \eta(2t_1 + D) \Gamma(t_1) \left(\frac{ML_0}{2}\right)^{-2t_1}. \end{aligned} \quad (4.8)$$

The first term in Eq. (4.8) is precisely the $F^{(0)(D, f)}(M)$ from Eq. (2.15) with $f = 1$. On the other hand, the second term in Eq. (4.8) is equivalent to $F_{L_0}^{(1)(D, f)}(M)$. To see this, we set $f = 1$, $n = 1$, $\eta_0 = \frac{1}{2}$ in Eq. (2.14), identifying L_{i_1} as L_0 and use Eq. (2.25) together with the definition of the eta function $\eta(z) = \sum_{m=1}^{\infty} (-1)^{m-1} m^{-z}$. Then, we have

$$F_{L_0}^{(1)(D, f)}(M) = -\mathcal{N} \frac{1}{(2\pi)^{\frac{D}{2}}} \left(\frac{M^2}{2}\right)^{\frac{D}{2}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D}{2}\right) \Gamma(t) \eta(2t) \left(\frac{ML_0}{2}\right)^{-2t}. \quad (4.9)$$

And if we change the variable $t_1 = t - \frac{D}{2}$ in Eq. (4.9), we arrive at the second term in Eq. (4.8).

Then, we finally obtain from Eq. (2.24) with Eqs. (4.6) and (4.8) that

$$\begin{aligned} V_{\text{eff}}^f &= \mathcal{N} \frac{\pi^{\frac{D}{2}}}{2(2\pi)^D} \Gamma\left(-\frac{D}{2}\right) M^D - \mathcal{N} \frac{1}{L_0^D} \frac{1}{\pi^{\frac{D}{2}} 2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 + \frac{D}{2}\right) \eta(2t_1 + D) \Gamma(t_1) \left(\frac{ML_0}{2}\right)^{-2t_1} \\ &- \sum_{n=1}^p \sum_{1 \leq i_1 < \dots < i_n \leq p} \mathcal{N} \frac{2^n}{\pi^{\frac{D}{2}} L_0} \sum_{\bar{n}=0}^{\infty} \frac{(-1)^{\bar{n}}}{\bar{n}!} \left(\frac{ML_0}{2}\right)^{2\bar{n}} \frac{1}{2\pi i} \int_{c-\frac{D-1}{2}-i\infty}^{c-\frac{D-1}{2}+i\infty} dt \\ &\times \Gamma\left(-t - \bar{n} + \frac{1}{2}\right) \eta(-2t - 2\bar{n} + 1) \left(\frac{1}{L_0}\right)^{-2t} S^{(n)}\left(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n}\right). \end{aligned} \quad (4.10)$$

We obtain an important expression for the effective potential in the case of the fermion with arbitrary boundary condition for the spatial S_i^1 ($i = 1, \dots, p$) direction. As stated below Eq. (4.6), the third term in Eq. (4.10) is strictly analytic with respect to M . We observe that the mode recombination formula clearly separates the effective potential V_{eff}^f into a part that contains the nonanalytic terms and a part that is entirely analytic. Hence, we conclude that the nonanalytic terms, if present, can arise from the first and second terms in Eq. (4.10) and we write the nonanalytic terms in V_{eff}^f by

$$V_{\text{eff}}^f|_{\text{n.a.}} = F^{(0)(D,f)}(M)|_{\text{n.a.}} + F_{L_0}^{(1)(D,f)}(M)|_{\text{n.a.}}. \quad (4.11)$$

B. Nonanalytic terms

We have already calculated the first term in Eq. (4.10) in Sec. II and have found the power-type or the logarithmic-type nonanalytic terms. For $D = \text{even}$, it is given by Eq. (2.16) with $f = 1$

$$F^{(0)(D,f)}(M)|_{\text{n.a.}}^{\log} = -\mathcal{N} \frac{(-1)^{\frac{D}{2}}}{2^D \pi^{\frac{D}{2}} (\frac{D}{2})!} M^D \log \frac{M}{\mu}, \quad (4.12)$$

while for $D = \text{odd}$, it is done by Eq. (2.18) with $f = 1$

$$F^{(0)(D,f)}(M)|_{\text{n.a.}}^{\text{pow}} = \mathcal{N} \frac{(-1)^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}} \pi^{\frac{D-1}{2}} D!!} M^D. \quad (4.13)$$

Let us obtain the nonanalytic terms in the second term of Eq. (4.10), which is $F_{L_0}^{(1)(D,f)}(M)$. We deform the contour in the complex t_1 plane to encircle all the poles of the integrand and evaluate the integral by the residue theorem. One needs the condition (4.5) again for this to be possible, which will be derived in Appendix A. It is appropriate to compute the integral separately for even and odd D . The details of the calculation are given in Appendix C. We present the result here. For $D = \text{even}$ with Eqs. (C2), (C3) and (C5), we have

$$\begin{aligned} F_{L_0}^{(1)(D,f)}(M) = & \mathcal{N} \frac{(-1)^{\frac{D}{2}}}{2^D \pi^{\frac{D}{2}} (\frac{D}{2})!} M^D \left\{ \log \left(\frac{ML_0}{\pi} \right) - \frac{1}{2} \left(\psi(1) + \psi \left(\frac{D}{2} + 1 \right) \right) \right\} \\ & - \mathcal{N} \frac{(-1)^{\frac{D}{2}}}{2^D \pi^{\frac{D}{2}}} M^D \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell} \Gamma(2\ell+1) (1-2^{1+2\ell})}{\Gamma(\ell+\frac{D}{2}+1) \Gamma(\ell+1)} \zeta(2\ell+1) \left(\frac{ML_0}{4\pi} \right)^{2\ell} \\ & - \mathcal{N} \frac{1}{\pi^{\frac{D}{2}} L_0^D} \sum_{m=0}^{\frac{D-1}{2}} \frac{(-1)^m}{m!} \Gamma \left(\frac{D}{2} - m \right) \eta(D-2m) \left(\frac{ML_0}{2} \right)^{2m}. \end{aligned} \quad (4.14)$$

The first term in Eq. (4.14) containing the logarithmic-type nonanalytic term arises from the residue at the double pole $t_1 = -\frac{D}{2}$ of the part $\Gamma(t_1 + \frac{D}{2})\Gamma(t_1)$ in the integrand. There is no power-type nonanalytic term in Eq. (4.14), which reflects the fact that the eta function $\eta(2t_1 + D)$ does not possess the pole unlike the zeta function.

From Eqs. (4.12) and (4.14), there is no power-type nonanalytic term in the effective potential, which is consistent with the result in the paper II. We have only the logarithmic-type nonanalytic term in $F_{L_0}^{(1)(D,f)}(M)$

$$F_{L_0}^{(1)(D,f)}(M)|_{\text{n.a.}}^{\log} = \mathcal{N} \frac{(-1)^{\frac{D}{2}}}{2^D \pi^{\frac{D}{2}} (\frac{D}{2})!} M^D \log \left(\frac{ML_0}{\pi} \right), \quad (4.15)$$

so that Eq. (4.11) becomes

$$\begin{aligned}
V_{\text{eff}}^f|_{\text{n.a.}} &= F^{(0)(D,f)}(M)\Big|_{\text{n.a.}}^{\log} + F_{L_0}^{(1)(D,f)}(M)\Big|_{\text{n.a.}}^{\log} \\
&= \mathcal{N} \frac{(-1)^{\frac{D}{2}}}{2^D \pi^{\frac{D}{2}} (\frac{D}{2})!} M^D \log\left(\frac{\mu L_0}{\pi}\right). \quad (4.16)
\end{aligned}$$

The $\log M$ dependence finally cancels out, leaving alternatively the nonanalytic dependence on the scale $1/\mu L_0$.³

Let us proceed to study the second term in Eq. (4.10) for $D = \text{odd}$. As discussed in Appendix C, the integrand has no double pole in this case, so that there arise no logarithmic-type nonanalytic terms. We obtain [see Eq. (C6)]

$$\begin{aligned}
F_{L_0}^{(1)(D,f)}(M) &= -\mathcal{N} \frac{(-1)^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}} \pi^{\frac{D-1}{2}} D!!} M^D \\
&\quad - \mathcal{N} \frac{1}{\pi^{\frac{D}{2}} L_0^D} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!} \Gamma\left(\frac{D}{2} - \ell\right) \\
&\quad \times \eta(D - 2\ell) \left(\frac{ML_0}{2}\right)^{2\ell}. \quad (4.17)
\end{aligned}$$

The first term is the contribution from the single pole $t_1 = -\frac{D}{2}$ of $\Gamma(t_1 + \frac{D}{2})$ and the second one is the one from the pole $t_1 = -\ell$ ($\ell = 0, 1, \dots$) of $\Gamma(t_1)$. As expected, there is no logarithmic-type nonanalytic term in Eq. (4.17). We see that the first term, the power-type nonanalytic term, cancels out Eq. (4.13) to reproduce the result in the paper II. Therefore, the effective potential does not possess the nonanalytic terms

$$V_{\text{eff}}^f|_{\text{n.a.}} = 0 \quad \text{for } D = \text{odd}. \quad (4.18)$$

We conclude that in the case of the fermion with arbitrary boundary condition for the spatial S_i^1 ($i = 1, \dots, p$) direction, the effective potential does not have any nonanalytic term with respect to M . We summarize the results in Table II. This is quite different from the case of the scalar with the periodic boundary condition for the spatial S_i^1 direction.

If we consider the scalar field with some of the boundary condition for a spatial S_i^1 direction being antiperiodic, the effective potential in this case essentially has the same as that in the case of the fermion by regarding the spatial direction that has the antiperiodic boundary condition as the Euclidean time direction. Hence, the results in the case of the fermion immediately imply the conclusion that there are no nonanalytic terms if the scalar field obeys at least one antiperiodic boundary condition for the spatial S_i^1 direction.

³This is the same situation as Eq. (3.25), in which the $\log M$ dependence drops out.

TABLE II. Nonanalytic structure of the effective potential V_{eff}^f on the spacetime, $S_\tau^1 \times R^{D-(p+1)} \times \prod_{i=1}^p S_i^1$ for the fermion field satisfying the arbitrary boundary condition for the spatial S_i^1 ($i = 1, \dots, p$) direction. The “non” in the table means that the effective potential contains no nonanalytic term of the corresponding type.

	$D = \text{even}$	$D = \text{odd}$
Power type	non	non
$\log M$ type	non	non

V. CONCLUSIONS AND DISCUSSIONS

We have investigated the nonanalytic terms in the one-loop finite-temperature effective potentials of the scalar and fermion fields on a D -dimensional spacetime, $S_\tau^1 \times R^{D-(p+1)} \times \prod_{i=1}^p S_i^1$. These are the terms that cannot be expressed as the positive powers of the field-dependent mass squared. In our analysis we have made the extensive use of the mode recombination formula developed in the paper II, which can be applicable to fermions, scalars and arbitrary boundary conditions for the spatial direction S_i^1 ($i = 1, \dots, p$). We recast the effective potential into the remarkable expressions, Eqs. (3.9) and (4.10) that have not been obtained previously. The rewritten expression clearly separates the analytic terms from all other ones, thereby making it transparent where the nonanalytic terms of interest arise.

The relevant quantities such as $F_{L_k}^{(1)D-k}(M)$, $F_{L_0}^{(1)(D,f)}(M)$ that can potentially give rise to the nonanalytic terms in the effective potential are initially expressed in terms of the modified Bessel functions of the second kind accompanied by the mode summations. By employing the integral representations of the function together with the analytical extension of the mode summations, we rewrite them as the contour integrals in the complex plane. We evaluate the integral by the residue theorem, picking up all infinitely many poles contained in the integrand. This procedure enables us to obtain the complete set of terms that appear in the effective potential. As a result, we find that the possible nonanalytic terms with respect to M in the effective potential are of power type or logarithmic one.

The logarithmic-type nonanalytic terms newly analyzed in the present paper are found to arise from the double pole in the integrand, in contrast to the power-type nonanalytic terms that originate from the single pole as shown in the papers I and II. In the case of the scalar field obeying the periodic boundary condition for the spatial S_i^1 ($i = 1, \dots, p$) direction, we have determined all the nonanalytic terms in the effective potential by classifying the cases according to whether the spacetime dimension, D and the total number of S_i^1 , $p + 1$ are even or odd. We have clarified the significant nonanalytic structure as summarized in Table I: whenever the power type is present(absent), the

logarithmic type is absent(present). We have also studied the case of the fermion with arbitrary boundary condition for the spatial S_i^1 direction. The absence of the zero mode due to the antiperiodic boundary condition following from the quantum statistics is crucial for the nonanalytic structure and we have found that neither of the two types appears in the effective potential regardless of the values of D and $p + 1$. The results in the case of the fermion have immediately led to the conclusion that there exists no nonanalytic term in the case of the scalar satisfying the antiperiodic boundary condition for at least one spatial S_i^1 direction.⁴

We have also shown that the nonanalytic terms (3.42) and (3.43) can be obtained directly from the contribution of the zero mode alone. In our formulation based on the mode recombination formula, $F_{L_k}^{(1)D-k}(M)$ that potentially contains the nonanalytic terms also arises from the zero-mode sector in the formula. However, thanks to the winding mode m_k in $F_{L_k}^{(1)D-k}(M)$, it yields the larger contributions, the third and fourth terms in Eq. (3.13) [or the third term in Eq. (3.16)] in addition to the nonanalytic terms. Therefore, our formulation provides the richer results that cannot be obtained by considering only the zero mode. Moreover, the origin and the mechanism responsible for the emergence of the nonanalytic terms are very clear in the formulation and it serves as a powerful framework for analyzing the terms.

We have studied the nonanalytic terms that cannot be expressed by the positive powers of the field-dependent mass squared. In order to fully understand the nonanalytic structure of the effective potential, one needs to evaluate the complex t integral in the third term of Eq. (3.9) [or Eq. (4.10)]. Even though the third term is entirely analytic in M , we expect that the term contains nonanalytic contributions with power and/or logarithmic dependence on the scale $L_i (i = 0, \dots, p)$ and/or their ratio. In fact, these types of the contributions already appear, for example, in the first term of Eq. (3.28). It is of considerable interest to clarify the nonanalytic structure associated with the scale or the ratio, namely in connection with the phase-transition phenomena of the theory. A detailed investigation for fixed values of p is currently in progress and will be reported in the near future.

We have obtained the nonanalytic terms in the effective potential under the condition (3.5) [or (4.5)]. One could have new types of nonanalytic terms, which are different from the ones obtained in this paper, outside of the region

⁴We also discussed the nonanalytic terms in the case of a higher dimensional gauge field on $S_i^1 \times R^{D-(p+1)} \times \prod_{i=1}^p S_i^1$ in the paper II. The results concerning with the nonanalytic terms obtained in this paper also hold for the gauge field. The component gauge field for the spatial S_i^1 direction can acquire the vacuum expectation value to yield the fractional values of η_i in the boundary condition (2.3). It may be important to study the analytical extension in that case.

of Eqs. (3.5), (3.11) and (4.5). In this case, unlike the method adopted in this paper, it is necessary to evaluate (2.14) directly by some means. The method is not known at present, but the problem is highly challenging and interesting.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DERIVATION OF EQS. (3.5) AND (4.5)

Let us derive the conditions (3.5) and (4.5) in the text. We deform the contour in the t_1 plane of Eq. (3.4) [or Eq. (4.4)] to encircle all the poles of the integrand and evaluate the integral by the residue theorem. For this to be possible, the contribution from the large left semicircle must vanish, from which one obtains the conditions.

The relevant part in Eq. (3.4) for the discussion is given by

$$\int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \zeta(2t_1 - 2t + 1) \Gamma(t_1) \left(\frac{ML_0}{2}\right)^{-2t_1}. \quad (\text{A1})$$

In order to examine the contribution from the large left semicircle for the contour integral (A1), we study the asymptotic behavior of $\log |I|$ in the large radius limit, where

$$I \equiv \Gamma\left(t_1 - t + \frac{1}{2}\right) \zeta(2t_1 - 2t + 1) \Gamma(t_1) \left(\frac{ML_0}{2}\right)^{-2t_1}. \quad (\text{A2})$$

To this end, we need to understand the asymptotic behavior of each function in the integrand.

Stirling's formula is given by [14]

$$\Gamma(\omega) \simeq \sqrt{2\pi} e^{-\omega} \omega^{\omega-\frac{1}{2}} (1 + \dots) \quad \text{for } |\omega| \rightarrow \infty$$

$$\text{with } |\arg \omega| < \pi, \quad (\text{A3})$$

from which it follows that

$$\begin{aligned} \log |\Gamma(\omega)| &\simeq \operatorname{Re} \left\{ \left(\omega - \frac{1}{2} \right) \log \omega - \omega + \frac{1}{2} \log(2\pi) + \dots \right\} \\ &= \left(\omega_R - \frac{1}{2} \right) \log |\omega| - (\arg \omega) \omega_I - \omega_R + \dots \end{aligned} \quad (\text{A4})$$

Here we have denoted the real (imaginary) part of ω by ω_R (ω_I).

The large left semicircle is parametrized by

$$t_1 = c + R e^{i\varphi} \quad \text{with} \quad \frac{\pi}{2} \leq \varphi \leq \frac{3}{2}\pi, \quad R \rightarrow \infty, \quad (\text{A5})$$

where $R > 0, c \in \mathbb{R}$. The asymptotic behavior⁵ of $\Gamma(t_1)$ in the limit $R \rightarrow \infty$ is obtained by using Stirling's formula (A3) with ω being identified with t_1 . Taking into account $\arg t_1 \simeq \arg \varphi \pmod{2\pi}$ in the limit $R \rightarrow \infty$, the range of validity of the formula (A3) is now given by $|\arg \varphi| < \pi$, so that the range $\frac{\pi}{2} \leq \varphi \leq \frac{3}{2}\pi$ (especially $\varphi = \pi$) lies outside of it.

In order to obtain the asymptotic behavior of the gamma function that is valid in the entire range $\frac{\pi}{2} \leq \varphi \leq \frac{3}{2}\pi$, let us consider $\Gamma(\alpha - \beta t_1)$, where $\alpha \in \mathbb{C}, \beta > 0$. By identifying ω in Eq. (A3) with $\alpha - \beta t_1$, Eq. (A4) becomes

$$\begin{aligned} \log |\Gamma(\alpha - \beta t_1)| &\simeq -\beta R \cos \varphi \log(\beta R) + \beta R (\theta \sin \varphi + \cos \varphi) \\ &\quad + \left(\alpha_R - \beta c - \frac{1}{2} \right) \log(\beta R) \end{aligned} \quad (\text{A6})$$

in the limit $R \rightarrow \infty$, where we have denoted the real part of α by α_R and $\theta = \arg(\alpha - \beta t_1)$. Since Eq. (A6) is valid for $|\theta| < \pi$ from Eq. (A3) and $\theta \simeq \arg(-e^{i\varphi})$ in the limit $R \rightarrow \infty$, Eq. (A6) is valid for $\frac{\pi}{2} \leq \varphi \leq \frac{3}{2}\pi$ with $\theta \simeq \varphi - \pi$. Thus, we can use the asymptotic formula (A6) for the entire range of the left semicircle given in Eq. (A5).

The above discussion indicates that it is appropriate to recast the $\Gamma(\alpha + \beta t_1)$ type in the integrand of Eq. (A2) into the $\Gamma(1 - \alpha - \beta t_1)$ type by using the formula

$$\Gamma(\omega)\Gamma(1-\omega) = \frac{\pi}{\sin(\pi\omega)} \quad (\text{A7})$$

in order to study the asymptotic behavior of $\log |I|$. One also needs the asymptotic behavior of the zeta function along the large left semicircle. It may be convenient to use the formula

$$\zeta(1-\omega) = 2^{1-\omega} \pi^{-\omega} \zeta(\omega) \Gamma(\omega) \cos\left(\frac{\pi}{2}\omega\right). \quad (\text{A8})$$

⁵One should keep the limit $R \rightarrow \infty$ in mind when studying the asymptotic behavior in Appendix A.

By replacing ω by $1 - \omega$ with $\omega = \alpha + \beta t_1$ in Eq. (A8), we have

$$\begin{aligned} \zeta(\alpha + \beta t_1) &= \frac{(2\pi)^{\alpha + \beta t_1}}{\pi} \zeta(1 - \alpha - \beta t_1) \Gamma(1 - \alpha - \beta t_1) \\ &\quad \times \sin\left[\frac{\pi}{2}(\alpha + \beta t_1)\right], \end{aligned} \quad (\text{A9})$$

which will be used to recast the zeta function in Eq. (A2). It is easy to see

$$\log \left| \sin \left[\frac{\pi}{2}(\alpha + \beta t_1) \right] \right| \simeq \frac{\pi}{2} \beta R |\sin \varphi| \quad (\text{A10})$$

and the asymptotic behavior of $\log |\Gamma(1 - \alpha - \beta t_1)|$ is obtained⁶ by replacing α with $1 - \alpha$ in Eq. (A6) in the limit $R \rightarrow \infty$.

Now we are ready to discuss the asymptotic behavior of $\log |I|$. Let us recast the gamma and zeta functions in Eq. (A2) by using the formulas (A7) and (A9). Then, we obtain

$$I = \pi(2\pi)^{2A} \frac{\zeta(1 - 2A - 2t_1) \Gamma(1 - 2A - 2t_1)}{\sin(\pi t_1) \Gamma(1 - A - t_1) \Gamma(1 - t_1)} \left(\frac{ML_0}{4\pi} \right)^{-2t_1}, \quad (\text{A11})$$

where we have denoted $A = \frac{1}{2} - t$. Aside from the irrelevant term, it follows that

$$\begin{aligned} \log |I| &\simeq \log |\zeta(1 - 2A - 2t_1)| + \log |\Gamma(1 - 2A - 2t_1)| \\ &\quad - \log |\sin(\pi t_1)| - \log |\Gamma(1 - A - t_1)| \\ &\quad - \log |\Gamma(1 - t_1)| + \log \left| \left(\frac{ML_0}{4\pi} \right)^{-2t_1} \right|. \end{aligned} \quad (\text{A12})$$

We note that the argument of $\zeta(1 - 2A - 2t_1)$ enters the region of absolute convergence as the R tends to infinity, so that it may be considered as a quantity of order one and thus, the first term in Eq. (A12) does not contribute to the asymptotic behavior.

Knowing that

$$\log \left| \left(\frac{ML_0}{4\pi} \right)^{-2t_1} \right| \simeq -2R \cos \varphi \log \left(\frac{ML_0}{4\pi} \right) \quad (\text{A13})$$

and using Eq. (A6), we finally obtain

$$\log |I| \simeq -\pi R |\sin \varphi| - 2R \cos \varphi \log \left(\frac{ML_0}{2\pi} \right). \quad (\text{A14})$$

⁶Let us note that $\arg(1 - \alpha - \beta t_1) \simeq \theta$ in the limit $R \rightarrow \infty$, where θ appears in Eq. (A6).

In order for the contribution to $|I|$ from the large left semicircle ($\frac{\pi}{2} \leq \varphi \leq \frac{3}{2}\pi, R \rightarrow \infty$) to be suppressed, one needs the condition

$$0 < \frac{ML_0}{2\pi} < 1, \quad (\text{A15})$$

which is Eq. (3.5) in the text.

Let us make a few comments in order. The first one is that the leading behavior $R \cos \varphi \log R$ and the subleading one $R(\theta \sin \varphi + \cos \varphi)$ in Eq. (A6) cancel out between the three gamma functions in Eq. (A12). The dominant contribution is governed by the term linear in R , as shown in Eq. (A14). The second one is that the complex integral in Eq. (3.12) is essentially the same type as Eq. (A1), so that a similar discussion leads to the condition (3.11). The last one is that if we take $\varphi = \frac{\pi}{2}(\frac{3}{2}\pi)$ in Eq. (A14), we have $\log |I| \simeq -\pi R$. This implies that the contour integral (A1)

(and (B1) defined below) along the imaginary axis is well-defined, as it should be. The three comments are also applicable to the case of the fermion, which is given below.

Let us next derive the condition (4.5) in the case of the fermion. The relevant part of the t_1 integral in Eq. (4.4) is given by

$$\int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \eta(2t_1 - 2t + 1) \Gamma(t_1) \left(\frac{ML_0}{2}\right)^{-2t_1}. \quad (\text{A16})$$

The only difference, compared with Eq. (A1), is that the zeta function in Eq. (A1) is replaced by the eta function. We denote the integrand in Eq. (A16) by I_f .

Using the formula (A7) and (4.3) together with (A9), the integrand I_f is recast as

$$I_f = \pi(2\pi)^{2A} \frac{(1 - 2^{1-2A-2t_1}) \zeta(1 - 2A - 2t_1) \Gamma(1 - 2A - 2t_1)}{\sin(\pi t_1) \Gamma(1 - A - t_1) \Gamma(1 - t_1)} \left(\frac{ML_0}{4\pi}\right)^{-2t_1}. \quad (\text{A17})$$

Up to the irrelevant terms, we have

$$\begin{aligned} \log |I_f| \simeq & \log |(1 - 2^{1-2A-2t_1})| + \log |\zeta(1 - 2A - 2t_1)| + \log |\Gamma(1 - 2A - 2t_1)| \\ & - \log |\sin(\pi t_1)| - \log |\Gamma(1 - A - t_1)| - \log |\Gamma(1 - t_1)| + \log \left| \left(\frac{ML_0}{4\pi}\right)^{-2t_1} \right|. \end{aligned} \quad (\text{A18})$$

The first term in Eq. (A18) behaves like

$$\log |(1 - 2^{1-2A-2t_1})| \simeq -2R \cos \varphi \log 2 \quad (\text{A19})$$

in the limit $R \rightarrow \infty$ and the second term can be ignored as explained before. We immediately obtain

$$\log |I_f| \simeq -\pi R |\sin \varphi| - 2R \cos \varphi \log \left(\frac{ML_0}{\pi}\right), \quad (\text{A20})$$

where we have used Eqs. (A12), (A14) and (A19). The difference of the factor 2 in the logarithm of the second term in Eq. (A20), compared with Eq. (A14), comes from the difference between the zeta function and the eta function [see Eqs. (4.3) and (A19)]. In order for the contribution to $|I_f|$ from the large left semicircle ($\frac{\pi}{2} \leq \varphi \leq \frac{3}{2}\pi, R \rightarrow \infty$) to be suppressed, one obtains the condition

$$0 < \frac{ML_0}{\pi} < 1, \quad (\text{A21})$$

which is (4.5) in the text.

It may be interesting to note that the antiperiodic boundary condition for the Euclidean time direction for the fermion can be viewed as the periodic one if we consider $2L_0$. This suggests that the condition for the scalar case (A15) is reduced to the fermion one (A21) by replacing L_0 by $2L_0$ in Eq. (A15), which is actually the case.

It is worth noting the following point. Our results (3.13), (3.16), (4.14) and (4.17) contain the infinite series. One can show that the conditions (A15) and (A21), which have been imposed to obtain the results and are derived in this appendix, exactly coincide with the radius of convergence for the series. This fact strongly supports the validity of the results using the framework to study the nonanalytic terms in this paper.

APPENDIX B: RESIDUE EVALUATION FOR $F_{L_k}^{(1)D-k}(M)$

Let us derive the results (3.13) and (3.16) in this appendix. We consider the complex integral (3.12)

$$F_{L_k}^{(1)D-k}(M) = \frac{-2}{(2\pi)^{\frac{D-k}{2}}} \left(\frac{M^2}{2}\right)^{\frac{D-k}{2}} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D-k}{2}\right) \Gamma(t) \zeta(2t) \left(\frac{ML_k}{2}\right)^{-2t}. \quad (\text{B1})$$

We deform the contour in the t plane so that it encloses all the poles of the integrand and evaluate the t integral by the residue theorem under the condition (3.11). Note that the integrand (B1) can have at most a double pole because the part $\Gamma(t)\zeta(2t)$ obviously never possesses the double pole. Thus, the double pole exists in the part $\Gamma(t - \frac{D-k}{2})\Gamma(t)$ or $\Gamma(t - \frac{D-k}{2})\zeta(2t)$ in the integrand, depending on whether $\frac{D-k}{2}$ is even or odd. It is convenient to evaluate the integral by separating the cases in which D and k are even or odd.

We first present the necessary formulas to evaluate the integral using residues. Let us consider the function $F(t) = A(t)B(t)C(t)$ and suppose that $t = \alpha$ is a double pole of the part $A(t)B(t)$ in $F(t)$ and $C(t)$ is regular at the pole. Writing

$$A(t) = \frac{A_{\text{res}}}{t - \alpha} + A_f + O(t - \alpha), \quad (\text{B2})$$

$$B(t) = \frac{B_{\text{res}}}{t - \alpha} + B_f + O(t - \alpha), \quad (\text{B3})$$

the residue of $F(t)$ at the pole $t = \alpha$ is calculated to be

$$\begin{aligned} \text{Res}F(t) \Big|_{t=\alpha} &= \frac{d}{dt} (F(t)(t - \alpha)^2) \Big|_{t=\alpha} \\ &= A_{\text{res}}B_{\text{res}}C'(\alpha) + A_{\text{res}}B_fC(\alpha) + A_fB_{\text{res}}C(\alpha). \end{aligned} \quad (\text{B4})$$

The $C'(\alpha)$ in the first term of Eq. (B4) is the origin of the logarithmic-type nonanalytic terms, as we will see below. If the pole $t = \alpha$ is the trivial zero of $C(t)$, that is, $C(\alpha) = 0$, then, we have

$$F(t) = \frac{A_{\text{res}}B_{\text{res}}}{t - \alpha} C'(\alpha) + O(t - \alpha) \quad \text{for } C(\alpha) = 0, \quad (\text{B5})$$

so that the pole $t = \alpha$ becomes a single pole of the integrand to yield

$$\text{Res}F(t) \Big|_{t=\alpha} = A_{\text{res}}B_{\text{res}}C'(\alpha) \quad \text{for } C(\alpha) = 0. \quad (\text{B6})$$

Equations (B4) and (B6) are frequently used in evaluating the integral by residues.

1. $(D, k) = (\text{even}, \text{even}), (\text{odd}, \text{odd})$

The first term in Eq. (3.13) arises from the single pole $t = \frac{1}{2}$ of $\zeta(2t)$ in Eq. (3.12) and is the power-type nonanalytic term that we have already obtained in Eq. (4.10) of the paper II.

The part $\Gamma(t - \frac{D-k}{2})\Gamma(t)$ possesses a double pole at $t = 0$ in this case. By using Eq. (B4) and denoting $\ell_k \equiv \frac{D-k}{2}$, the residue at the pole of Eq. (3.12) is calculated to be

$$\begin{aligned} &\frac{-2}{(2\pi)^{\frac{D-k}{2}}} \left(\frac{M^2}{2}\right)^{\frac{D-k}{2}} \frac{1}{2} \left\{ \frac{(-1)^{\ell_k}}{\ell_k!} \left(\zeta(2t) \left(\frac{ML_k}{2}\right)^{-2t} \right)' \Big|_{t=0} + \frac{(-1)^{\ell_k}}{\ell_k!} \psi(1) \zeta(0) + \frac{(-1)^{\ell_k}}{\ell_k!} \psi(\ell_k + 1) \zeta(0) \right\} \\ &= \frac{-1}{2^{D-k} \pi^{\frac{D-k}{2}}} \frac{(-1)^{\frac{D-k}{2}}}{(\frac{D-k}{2})!} M^{D-k} \left\{ \log\left(\frac{ML_k}{4\pi}\right) - \frac{1}{2} \left(\psi(1) + \psi\left(\frac{D-k}{2} + 1\right) \right) \right\}, \end{aligned} \quad (\text{B7})$$

where we have used the expansion

$$\Gamma(t) = \frac{(-1)^n}{n!} \frac{1}{t + n} + \frac{(-1)^n}{n!} \psi(n + 1) + O(t + n). \quad (\text{B8})$$

Here, the digamma function $\psi(m)$ is given by

$$\psi(m) = \sum_{r=1}^{m-1} \frac{1}{r} - \gamma_E \quad \text{and} \quad \psi(1) = -\gamma_E. \quad (\text{B9})$$

We have also used $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2} \log(2\pi)$ in the calculation and γ_E is Euler's constant. Equation (B7) is the second term in Eq. (3.13). The logarithmic-type nonanalytic term comes from $C'(\alpha)$ of Eq. (B4) in evaluating the residue at the double pole.

Let us next compute the contribution from the single pole $t = -\ell$ ($\ell = 1, 2, \dots$) of $\Gamma(t)$ and $\Gamma(t - \frac{D-k}{2})$, which corresponds to the case (B6) because of $\zeta(-2\ell) = 0$ ($\ell = 1, 2, \dots$). The straightforward calculation shows

$$\begin{aligned} & \frac{-2}{(2\pi)^{\frac{D-k}{2}}} \left(\frac{M^2}{2}\right)^{\frac{D-k}{2}} \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+\ell_k}}{(\ell+\ell_k)!} \frac{(-1)^\ell}{\ell!} 2\zeta'(-2\ell) \left(\frac{ML_k}{2}\right)^{2\ell} \\ &= \frac{(-1)(-1)^{\frac{D-k}{2}}}{2^{D-k} \pi^{\frac{D-k}{2}}} M^{D-k} \sum_{\ell=1}^{\infty} \frac{(-1)^\ell \Gamma(2\ell+1) \zeta(2\ell+1)}{\Gamma(\ell + \frac{D-k}{2} + 1) \Gamma(\ell+1)} \left(\frac{ML_k}{4\pi}\right)^{2\ell}, \end{aligned} \quad (\text{B10})$$

where we have used

$$\zeta'(-2\ell) = \frac{(-1)^\ell (2\ell)!}{2(2\pi)^{2\ell}} \zeta(2\ell+1) = \frac{(-1)^\ell \Gamma(2\ell+1)}{2(2\pi)^{2\ell}} \zeta(2\ell+1). \quad (\text{B11})$$

Eq. (B10) is the third term in Eq. (3.13).

We finally calculate the contribution from the single pole $t = \ell_k - m = \frac{D-k}{2} - m$ ($m = 0, 1, \dots, \frac{D-k}{2} - 1$) of $\Gamma(t - \frac{D-k}{2})$. We find

$$\begin{aligned} & \frac{-2}{(2\pi)^{\frac{D-k}{2}}} \left(\frac{M^2}{2}\right)^{\frac{D-k}{2}} \frac{1}{2} \sum_{m=0}^{\ell_k-1} \frac{(-1)^m}{m!} \Gamma(\ell_k - m) \zeta(2\ell_k - 2m) \left(\frac{ML_k}{2}\right)^{-2(\ell_k-m)} \\ &= \frac{-1}{\pi^{\frac{D-k}{2}} L_k^{D-k}} \sum_{m=0}^{\frac{D-k}{2}-1} \frac{(-1)^m}{m!} \Gamma\left(\frac{D-k}{2} - m\right) \zeta(D-k-2m) \left(\frac{ML_k}{2}\right)^{2m}. \end{aligned} \quad (\text{B12})$$

We have obtained the fourth term and have completed the computation of the four terms in Eq. (3.13).

2. $(D, k) = (\text{even}, \text{odd}), (\text{odd}, \text{even})$

The first term in Eq. (3.16) arises from the single pole $t = 0$ of $\Gamma(t)$ and is the power-type nonanalytic term that we have already obtained in the paper II.

The part $\Gamma(t - \frac{D-k}{2}) \zeta(2t)$ has the double pole $t = \frac{1}{2}$ in this case. We evaluate the residue at the pole and the result is given, denoting $m_k = \frac{D-k-1}{2}$, by

$$\begin{aligned} & \frac{-2}{(2\pi)^{\frac{D-k}{2}}} \left(\frac{M^2}{2}\right)^{\frac{D-k}{2}} \frac{1}{2} \left\{ \frac{(-1)^{m_k}}{m_k!} \frac{1}{2} \left(\Gamma(t) \left(\frac{ML_k}{2}\right)^{-2t} \right)' \Big|_{t=\frac{1}{2}} + \frac{(-1)^{m_k}}{m_k!} \gamma_E \Gamma\left(\frac{1}{2}\right) \left(\frac{ML_k}{2}\right)^{-1} \right. \\ & \quad \left. + \frac{(-1)^{m_k}}{m_k!} \psi(m_k+1) \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \left(\frac{ML_k}{2}\right)^{-1} \right\} \\ &= \frac{-1}{2^{D-k-1} \pi^{\frac{D-k-1}{2}} \left(\frac{D-k-1}{2}\right)!} \frac{M^{D-k-1}}{L_k} \left\{ -\log(ML_k) + \frac{1}{2} \left(\gamma_E + \psi\left(\frac{D-k-1}{2} + 1\right) \right) \right\}. \end{aligned} \quad (\text{B13})$$

This is the second term in Eq. (3.16). In the calculation we have used $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma'(\frac{1}{2}) = \Gamma(\frac{1}{2}) \psi(\frac{1}{2}) = \sqrt{\pi}(-\log 4 - \gamma_E)$. The Euler's constant, γ_E in Eq. (B13) comes from $\Gamma'(\frac{1}{2})$ and the finite term of the expansion

$$\zeta(2t) = \frac{1}{2} \frac{1}{t - \frac{1}{2}} + \gamma_E + O\left(t - \frac{1}{2}\right). \quad (\text{B14})$$

The logarithmic-type nonanalytic term arises from the $C'(\alpha)$ of Eq. (B4) in evaluating the residue at the double pole.

Let us next study the contribution from the single pole $t = \frac{D-k}{2} - m$ ($m = 0, 1, \dots$), excluding $m = \frac{D-k-1}{2}$, of $\Gamma(t - \frac{D-k}{2})$ and it is given by

$$\begin{aligned}
& \frac{-2}{(2\pi)^{\frac{D-k}{2}}} \left(\frac{M^2}{2}\right)^{\frac{D-k}{2}} \frac{1}{2} \sum_{\substack{m=0 \\ (m \neq \frac{D-k-1}{2})}}^{\infty} \frac{(-1)^m}{m!} \Gamma\left(\frac{D-k}{2} - m\right) \zeta(D-k-2m) \left(\frac{ML_k}{2}\right)^{-2(\frac{D-k}{2}-m)} \\
&= \frac{-1}{\pi^{\frac{D-k}{2}} L_k^{D-k}} \sum_{\substack{m=0 \\ (m \neq \frac{D-k-1}{2})}}^{\infty} \frac{(-1)^m}{m!} \Gamma\left(\frac{D-k}{2} - m\right) \zeta(D-k-2m) \left(\frac{ML_k}{2}\right)^{2m}.
\end{aligned} \tag{B15}$$

This is the third term and we have obtained the three terms in Eq. (3.16).

APPENDIX C: RESIDUE EVALUATION FOR $F_{L_0}^{(1)(D,f)}(M)$

Let us obtain the nonanalytic terms in the effective potential for the fermion in Sec. IV. To this end, we need to compute the integral in the second term of Eq. (4.10)

$$F_{L_0}^{(1)(D,f)}(M) = -\mathcal{N} \frac{1}{L_0^D \pi^{\frac{D}{2}} 2\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t + \frac{D}{2}\right) \eta(2t+D) \Gamma(t) \left(\frac{ML_0}{2}\right)^{-2t}, \tag{C1}$$

where we have rewritten t_1 and c_1 as t and c , respectively. We deform the contour in the t plane so as to encircle all the poles of the integrand and evaluate the t integral by the residue theorem under the condition (A16). Let us note that the eta function has no pole unlike the zeta function in Eq. (B1). We consider the integral separately, depending on whether D is even or odd.

1. 1.D= even

The double pole exists in the part $\Gamma(t + \frac{D}{2})\Gamma(t)$ in this case and taking into account $\eta(-2m) = 0 (m = 1, 2, \dots)$, it is given by $t = -\frac{D}{2}$. From (B4), we find

$$\begin{aligned}
& -\mathcal{N} \frac{1}{\pi^{\frac{D}{2}} L_0^D} \left\{ \frac{(-1)^{\frac{D}{2}}}{(\frac{D}{2})!} \left(\eta(2t+D) \left(\frac{ML_0}{2}\right)^{-2t} \right)' \Big|_{t=-\frac{D}{2}} + \frac{(-1)^{\frac{D}{2}}}{(\frac{D}{2})!} \psi\left(\frac{D}{2} + 1\right) \eta(0) \left(\frac{ML_0}{2}\right)^D + \psi(1) \frac{(-1)^{\frac{D}{2}}}{(\frac{D}{2})!} \eta(0) \left(\frac{ML_0}{2}\right)^D \right\} \\
&= \mathcal{N} \frac{(-1)^{\frac{D}{2}}}{2^D \pi^{\frac{D}{2}} (\frac{D}{2})!} M^D \left\{ \log\left(\frac{ML_0}{\pi}\right) - \frac{1}{2} \left(\psi(1) + \psi\left(\frac{D}{2} + 1\right) \right) \right\},
\end{aligned} \tag{C2}$$

where we have used $\eta(0) = \frac{1}{2}$, $\eta'(0) = \frac{1}{2} \log(\frac{\pi}{2})$. This is the first term in Eq. (4.14). We see that the logarithmic-type nonanalytic term comes from $C'(\alpha)$ of Eq. (B4) in evaluating the residue at the double pole.

Let us next study the contribution from the single pole $t = -\frac{D}{2} - \ell (\ell = 1, 2, \dots)$ of $\Gamma(\frac{D}{2} + t)$ and $\Gamma(t)$, which is the case for Eq. (B6) because of $\eta(-2\ell) = 0 (\ell = 1, 2, \dots)$. We have

$$\begin{aligned}
& -\mathcal{N} \frac{1}{\pi^{\frac{D}{2}} L_0^D} \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{\ell!} \frac{(-1)^{\ell+\frac{D}{2}}}{(\ell + \frac{D}{2})!} 2\eta'(-2\ell) \left(\frac{ML_0}{2}\right)^{-2(\frac{D}{2}-\ell)} \\
&= -\mathcal{N} \frac{(-1)^{\frac{D}{2}}}{2^D \pi^{\frac{D}{2}}} M^D \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell} \Gamma(2\ell+1) (1-2^{1+2\ell})}{\Gamma(\ell + \frac{D}{2} + 1) \Gamma(\ell+1)} \zeta(2\ell+1) \left(\frac{ML_0}{4\pi}\right)^{2\ell},
\end{aligned} \tag{C3}$$

where we have used

$$\eta'(-2\ell) = (1-2^{1+2\ell})\zeta'(-2\ell) = (1-2^{1+2\ell}) \frac{(-1)^{\ell} \Gamma(2\ell+1)}{2(2\pi)^{2\ell}} \zeta(2\ell+1) \tag{C4}$$

and Eq. (B11). This is the second term in Eq. (4.14).

Let us finally calculate the contribution from the single pole $t = -m$ ($m = 0, 1, \dots, \frac{D}{2} - 1$) of $\Gamma(t)$. We obtain

$$-\mathcal{N} \frac{1}{\pi^{\frac{D}{2}} L_0^D} \sum_{m=0}^{\frac{D}{2}-1} \frac{(-1)^m}{m!} \Gamma\left(\frac{D}{2} - m\right) \eta(D - 2m) \left(\frac{ML_0}{2}\right)^{2m}, \quad (\text{C5})$$

which is the third term and we have finished computing the three terms in Eq. (4.14).

2. $2.D = \text{odd}$

The integrand possesses no double pole in this case, so there exists no logarithmic-type nonanalytic term. The contributions from the single pole $t = -\frac{D}{2}$ of $\Gamma(t + \frac{D}{2})$

and the one $t = -\ell$ ($\ell = 0, 1, \dots$) of $\Gamma(t)$ yield⁷

$$-\mathcal{N} \frac{(-1)^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}} \pi^{\frac{D-1}{2}} D!!} M^D - \mathcal{N} \frac{1}{\pi^{\frac{D}{2}} L_0^D} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \times \Gamma\left(\frac{D}{2} - \ell\right) \eta(D - 2\ell) \left(\frac{ML_0}{2}\right)^{2\ell}. \quad (\text{C6})$$

The first term is explicitly given by Eq. (2.18) with $f = 1$ and an extra minus sign. We have calculated the two terms in Eq. (4.17).

⁷One might think that the single poles $t = -\frac{D}{2} - m$ ($m = 1, 2, \dots$) of $\Gamma(t + \frac{D}{2})$ contribute to Eq. (C1). This is not the case because the combination $\Gamma(t + \frac{D}{2})\eta(2t + D)$ has no pole there due to the property $\eta(-2m) = 0$ ($m = 1, 2, \dots$).

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