(2 + 1)-dimensional gravity in asymptotically AdS spacetimes with spatial wormhole slices: Reduced phase space dynamics and the BTZ black hole

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We solve Einstein's equations with negative cosmological constant in $2+1$ dimensions in the Hamiltonian formulation. The spacetime has the topology of $\Sigma \times \mathbf{R}$, where $\mathbf{R}$ corresponds to the time direction and Σ is a cylinder $\mathbf{R} \times \mathbf{S}^1$, and the spacetime metric satisfies asymptotically AdS boundary conditions. We address the question of gauge invariance by fixing the maximal slicing and spatial harmonic gauge conditions and demonstrate that there are no residual small diffeomorphisms in this gauge. We explicitly solve the Hamiltonian and momentum constraints and the gauge conditions to obtain a two-dimensional reduced phase space. We also solve for the lapse and shift functions in the above choice of gauge. For simplicity, and with the Bañados-Teitelboim-Zanelli (BTZ) black hole in mind, we restrict the solution of the momentum constraints and the lapse and shift functions to be independent of $\mathbf{S}^1$. More generally, one can consider an infinite set of solutions labeled by the angular momentum around the $\mathbf{S}^1$ which are related to the Virasoro algebra of the boundary CFT and boundary gravitons, but we do not consider this generalization here. In asymptotically AdS spacetimes besides the standard Wheeler-DeWitt equations there is a Schrödinger equation corresponding to the boundary Arnowitt-Deser-Misner Hamiltonian. We express this Hamiltonian in terms of the reduced phase space variables and discuss its classical solutions and quantization. We exhibit the wave functions and a continuous positive energy spectrum. Each energy eigenvalue E corresponds to a BTZ black hole of mass $M = E/2$. This identification is based on the fact that the classical solution of the reduced phase space dynamics gives rise to a spacetime that is related to the two-sided BTZ black hole by a diffeomorphism.

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I. INTRODUCTION

It is well known that the theory of gravitation in $2+1$ dimensions described by Einstein's equations has no local degrees of freedom [1–5]. In particular, this implies that there is no Newton's law of gravitation in the weak-field limit and no gravitational force. It has been shown by Bañados, Teitelboim, and Zanelli [6] that black hole solutions exist in $(2+1)$-dimensional Einstein gravity when the cosmological constant is negative; however, the singularity in the interior of the black hole is not a curvature singularity and simply separates regions containing closed time curves. This feature of the $(2+1)$-dimensional black hole may also be attributed to the absence of a local gravitational force.

Though there are no local degrees of freedom, the topology of the $(2+1)$-dimensional spacetime could give rise to global degrees of freedom. In this paper, we restrict our attention to spacetimes of the form $\Sigma \times \mathbf{R}$, where Σ is a cylinder. For this case, we systematically obtain an explicit description of these global degrees of freedom and their dynamics in the Hamiltonian formulation of Einstein's theory using Dirac's method for constrained Hamiltonian systems [7,8].

The Hamiltonian formulation of general relativity is applicable to spacetimes of the form $\Sigma \times \mathbf{R}$; we review basic aspects in Sec. II. The phase space variables are the spatial metric g_{ij} on the two-dimensional Σ and its conjugate momentum π^{ij}. In addition, there are two auxiliary fields: the lapse N and the shift N^i, which correspond to the time-time component and time-space components of the spacetime metric. In the Hamiltonian formulation, the six Einstein's equations in $2+1$ dimensions separate nicely into three constraint equations which do not involve time derivatives of g_{ij} and π^{ij} and three

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dynamical equations which involve first-order time derivatives of g_{ij} and π^{ij}.

The three constraints—the Hamiltonian constraint $\mathcal{H}_\perp \approx 0$ and the two momentum constraints $\mathcal{H}_i \approx 0$, $i = 1, 2$—generate the diffeomorphisms of the $(2+1)$-dimensional spacetime. We solve the constraint equations by choosing gauge conditions for each of the three equations. These are the maximal slicing condition $g_{ij}\pi^{ij} = 0$ for the Hamiltonian constraint and the spatial harmonic condition $\partial_i(\sqrt{g}g^{ij}) = 0$ for the momentum constraints. In Sec. IV, we demonstrate that these fix small diffeomorphisms completely without any residual freedom in the case that the spatial slices are cylinders or wormholes of the form $\mathbf{R} \times \mathbf{S}^1$.

On a given spatial slice with the topology of the cylinder, the solutions g_{ij} and π^{ij} of the constraints and gauge conditions are determined completely in terms of two numbers m and T, which constitute the reduced phase space degrees of freedom. The number m is constrained to be nonzero and represents the modulus of constant negative curvature metrics on the cylinder with asymptotically AdS (AAdS) boundary conditions, and the quantity $p_m = -\frac{4\pi^2}{\kappa^2}\frac{T}{m^2}$ turns out to be the variable conjugate to m. Thus, the reduced phase space is two-dimensional with coordinates (m, p_m). We describe the solutions of the constraint equations and the structure of the reduced phase space in Sec. V.

The preservation of the maximal slicing and spatial harmonic gauge conditions under Hamiltonian evolution imposes differential equations on the lapse N and shift N^i auxiliary fields. These differential equations are elliptic in nature and have unique solutions given appropriate boundary conditions for N and N^i. We solve these equations explicitly in Sec. VI for a given spatial slice in terms of the reduced phase space data m and T.

So far, we have described the solutions of the constraint equations on a given spatial slice. However, to assemble the spacetime solution of Einstein equations, we have to describe the dynamics of the data g_{ij}, π^{ij}, N and N^i on the spatial slices. The dynamics of g_{ij}, π^{ij}, N and N^i are completely determined by the dynamics of the reduced phase space variables m and p_m. In turn, the dynamics in reduced phase space is governed by the reduced Hamiltonian which is obtained by plugging in the solutions of the constraints and gauge conditions into the expression of the original Hamiltonian of general relativity.

We describe the reduced Hamiltonian and the classical solutions of the reduced Hamiltonian equations of motion in Sec. VII. We also show that this reduced phase space is (classically) related by a canonical transformation to the phase space variables for spherically symmetric spacetimes that were discussed by Kuchar [9] some time ago.

We discuss the quantization of the classical reduced phase space and solve the Schrödinger equation in Sec. VIII. The Schrödinger equation is written for the reduced phase space dynamics, and the time coordinate in the Schrödinger equation corresponds to the time that labels maximal slices. The energy levels E correspond to Bañados-Teitelboim-Zanelli (BTZ) black holes of mass $M = E/2$. This identification is based on the fact (as we shall explicitly show in Sec. IX) that the classical solution of the reduced phase space gives rise to a spacetime related to the two-sided BTZ black hole by a diffeomorphism.

The geometry of the spacetime solutions obtained thus far turns out to be that of the Kruskal extension of the nonrotating BTZ black hole in $2+1$ dimensions [6,10] with a particular value of the Arnowitt-Deser-Misner (ADM) mass determined by the initial conditions for the reduced phase space trajectories. In Sec. IX, we explicitly obtain the diffeomorphism from the solution obtained in the Hamiltonian approach above to the Kruskal extension of the BTZ black hole.

The maximally sliced solution has many interesting features which we elucidate in Sec. IX, some of which are (i) the spatial slices cut across the event horizons of the BTZ black hole and (ii) the maximally sliced solution avoids the past and future singularities of the black hole completely. The second point above implies that the fate of the black hole singularity upon including quantum effects cannot be directly addressed by the quantum theory based on the reduced phase space obtained by maximal slicing [11]. In the companion paper [12], we have studied the related question of the behavior of probe quantum fields on the classical maximally sliced geometry near the boundaries of the maximal slicing coordinate patch.

It must be emphasized that the presence of the event horizons in the maximally sliced solutions becomes obvious only after the diffeomorphism to the Kruskal extension of the BTZ black hole. It is of interest to note that this fact seems analogous to an observation made sometime ago in [13] in the context of two-dimensional string theory and its description in terms of fermions: the presence of the stringy black hole [14,15] of the $c = 1$ matrix model in the fermionic description was apparent only after a nontrivial transformation of the phase space variables in the fermionic theory.

We conclude with a discussion of some interesting questions and directions for future research in Sec. X. A set of Appendixes supplements the results in the main text with details of calculations.

II. PRELIMINARIES: THE HAMILTONIAN FORMULATION

We describe the Hamiltonian formulation of general relativity in $d+1$ dimensions for spacetimes which are AAdS. The Hamiltonian formulation can be written for spacetimes which take the form $\Sigma \times \mathbf{R}$, where $\mathbf{R}$ corresponds to time and Σ is a d-dimensional spatial manifold. The spacetime metric can be written in the ADM form:

$$ds^2 = -N(x,t)^2 dt^2 + g_{ij}(x,t)(dx^i + N^i(x,t)dt)(dx^j + N^j(x,t)dt). \quad (1)$$

The basic dynamical variables are the Riemannian metric g_{ij} on a constant t spatial slice Σ, and the phase space is described by g_{ij} and its conjugate momentum π^{ij}. The Poisson brackets are

$$\{g_{ij}(x), \pi^{kl}(x')\} = \delta^{(k}_{(i}\delta^{l)}_{j)}\delta(x,x'). \quad (2)$$

The Hamiltonian is

$$H[N,N_i] = \int_\Sigma d^d x (N\mathcal{H}_\perp + N_i\mathcal{H}^i) + H_\partial, \quad (3)$$

where

(i) $\mathcal{H}_\perp$ and $\mathcal{H}^i$ are, respectively, the Hamiltonian and momentum constraints

$$\mathcal{H}_\perp = \frac{\kappa^2}{\sqrt{g}}\left(\pi^{ij}\pi_{ij} - \frac{1}{d-1}\pi^2\right) - \frac{1}{\kappa^2}\sqrt{g}(R - 2\Lambda),$$
$$\mathcal{H}^i = -2D_j\pi^{ij}. \quad (4)$$

Here, $\pi = g_{ij}\pi^{ij}$ is the trace of π^{ij}.

(ii) N and N_i are the auxiliary lapse and shift fields, respectively, with the lapse N being constrained to be nowhere zero on Σ in the classical theory.

(iii) The constant $\kappa^2 = 16\pi G_N$ is the gravitational coupling which is set to 1 unless indicated otherwise, and $\Lambda = -d(d-1)/2\ell^2$ is the cosmological constant with ℓ the AdS length scale.

(iv) H_∂ contains boundary terms that must be included to make the functional partial derivatives of H with respect to g_{ij} and π^{ij} well defined [16–18]. The structure of the boundary terms depend on the type of boundary conditions that one imposes on the fields [19]. When all fields satisfy Dirichlet boundary conditions, with the spacetime metric approaching the AdS_{d+1} metric at the asymptotic boundaries, we have

$$H_\partial = \oint_{\partial\Sigma} d^{d-1}x\left(-\frac{2}{\kappa^2}N\sqrt{\sigma}k + 2\hat{r}_i N_j \pi^{ij}\right) + H_{\rm ct}, \quad (5)$$

where σ is the determinant of the induced metric on $\partial\Sigma$, $\hat{r}^i$ is the outward pointing unit normal of the boundary $\partial\Sigma$ and $k = g^{ij}D_i\hat{r}_j$ is the trace of the extrinsic curvature of the boundary.

The term $H_{\rm ct}$ contains counterterms which are required to subtract the divergences that occur due to the noncompactness of the AAdS spacetimes [22–28]. In the $d = 2$ case that is studied in this paper, the counterterm turns out be of the form

$$d = 2: \; H_{\rm ct} = \frac{2}{\kappa^2}\oint_{\partial\Sigma} dx N\sqrt{\sigma}\frac{1}{\ell}. \quad (6)$$

The lapse and shift are Lagrange multiplier fields and their equations of motion give the Hamiltonian and momentum constraints

$$\mathcal{H}_\perp \approx 0, \qquad \mathcal{H}_i \approx 0, \quad (7)$$

where the $\approx$ symbol means we have to impose them weakly in phase space, i.e., impose them only after computing any Poisson brackets involving the expressions $\mathcal{H}_\perp$ and $\mathcal{H}_i$. These are first-class constraints since their Poisson brackets are again proportional to a combination of these constraints. The algebra of the constraints is [29,30]

$$\{\mathcal{H}_\perp(x), \mathcal{H}_\perp(x')\} = (g^{ij}\mathcal{H}_j(x) + g^{ij}\mathcal{H}_j(x'))\delta_{,i}(x,x'),$$
$$\{\mathcal{H}_i(x), \mathcal{H}_\perp(x')\} = \mathcal{H}_\perp(x)\delta_{,i}(x,x'),$$
$$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_i(x)\delta_{,j}(x,x') + \mathcal{H}_j(x)\delta_{,i}(x,x'). \quad (8)$$

The remaining Hamiltonian equations of motion are the dynamical equations [31]

$$\dot{g}_{ab} = 2Ng^{-1/2}\left(\pi_{ab} - \frac{1}{d-1}g_{ab}\pi\right) + 2D_{(a}N_{b)},$$
$$\dot{\pi}^{ab} = \sqrt{g}\left(-N\left(R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab}\right) + D^aD^bN - g^{ab}D^2N\right)$$
$$+ \frac{N}{2\sqrt{g}}g^{ab}\left(\pi^{ij}\pi_{ij} - \frac{1}{d-1}\pi^2\right)$$
$$- \frac{2N}{\sqrt{g}}\left(\pi^{ac}\pi_c{}^b - \frac{1}{d-1}\pi\pi^{ab}\right)$$
$$- D_i(\pi^{ib}N^a) - D_j(\pi^{aj}N^b) + D_p(\pi^{ab}N^p). \quad (9)$$

Plugging in the expression for π^{ij} in terms of $\dot{g}_{ij}$ into the $\dot{\pi}^{ij}$ equation, we get the three second-order dynamical Einstein equations for $\ddot{g}_{ij}$.

There are two approaches to the quantum theory that one can adopt. One is to work with the original phase space degrees of freedom $g_{ij}(x)$ and $\pi^{ij}(x)$ and quantize this phase space with the standard symplectic form $\int_\Sigma d^d x \delta g_{ij}(x) \wedge \delta\pi^{ij}(x)$. The Hamiltonian and momentum constraints are then promoted to operators and they are imposed on the appropriate wave functionals. The description of the original phase space degrees of freedom $g_{ij}(x)$ and $\pi^{ij}(x)$ relies on the ADM space + time split of the spacetime metric (1) which has constant t slices as the manifold Σ, and g_{ij} is the spatial metric on the constant t slices. There is a crucial assumption here that such an ADM split exists globally on Σ or at least in the appropriate regions of interest in the spacetime.

The quantum theory can then be described in terms of the wave functionals of the spatial metrics on the spatial slice Σ, $\Psi[g_{ij}]$. In the standard way, the conjugate momentum $\pi^{ij}(x)$ acts on these wave functionals as the functional derivative $-i\hbar\frac{\delta}{\delta g_{ij}(x)}$. The Schrödinger equation is then

$$i\hbar\frac{\partial}{\partial t}\Psi[g_{ij},t]=\hat{H}[N,N_i]\Psi[g_{ij},t], \tag{10}$$

where t is the time coordinate at the boundary and that agrees there with the time coordinate that appears in the ADM split (1). Here $\hat{H}[N,N_i]$ is the expression $H[N,N_i]$ (3) for all choices of N and N_i that satisfy appropriate boundary conditions (see Sec. III) and with the fields g_{ij},π^{ij} promoted to operators and ordered with a particular prescription. The first-class constraints $\mathcal{H}_\perp\approx 0$, $\mathcal{H}_i\approx 0$ become constraints on the wave functionals [33]:

$$\hat{\mathcal{H}}_\perp\Psi[g_{ij},t]=0,\qquad \hat{\mathcal{H}}_i\Psi[g_{ij},t]=0, \tag{11}$$

which are the Wheeler-DeWitt equations. Plugging the conditions (11) back into the Schrödinger equation (10), the volume integrals simply vanish and we are left with the boundary terms in the Hamiltonian:

$$i\hbar\frac{\partial}{\partial t}\Psi[g_{ij},t]=\hat{H}_\partial\Psi[g_{ij},t]. \tag{12}$$

In this paper we do not adopt the above approach to quantization. Instead, we adopt a different route where we impose the Hamiltonian and momentum constraints and the corresponding gauge conditions on the classical degrees of freedom g_{ij} and π^{ij} and obtain a classical reduced phase space. This reduced phase space is equipped with an appropriate symplectic form that is induced by the original $\int_\Sigma \delta g_{ij}(x)\wedge\delta\pi^{ij}(x)$ on the reduced phase space. One then attempts to quantize the reduced phase space equipped with the above symplectic form using some method of quantization that is appropriate to the context, if any are available. We adopt this *reduced phase space quantization* approach in this paper [34].

One can make a connection between these two ways to proceed by appealing to the connection between the "quantum equations" (10) in the semiclassical limit and the Einstein equations (9), using the Hamilton-Jacobi method. For a recent exposition of this connection, see [32].

In this paper, we specialize to the case of $d=2$, i.e., $(2+1)$-dimensional asymptotic spacetimes. It is a known fact that Einstein gravity in $2+1$ dimensions is nondynamical; i.e., it does not contain local dynamical degrees of freedom. However, there are usually global degrees of freedom arising from the topology of the spacetime, including boundaries. We shall see the consequences of this at various stages of our analysis.

The Hamiltonian theory is analyzed by first solving the Hamiltonian and momentum constraints by imposing the maximal slicing, spatial harmonic gauge conditions, which we discuss in Sec. IV. This way, we obtain a classical reduced phase space in which the constraints are automatically satisfied. In the $(2+1)$-dimensional setting of the current paper, this reduced phase space turns out to be finite dimensional and described by degrees of freedom that refer to global topological aspects of the spacetime. Since the constraints that appear in the volume integrals of the Hamiltonian (3) have been solved, the Hamiltonian for the reduced phase space degrees of freedom simply becomes the boundary term H_∂ evaluated on the reduced phase space.

III. ASYMPTOTIC ADS BOUNDARY CONDITIONS

The asymptotic AdS boundary conditions that we specify make use of an areal radial coordinate r defined in a neighborhood of a given asymptotic boundary. The boundary is approached as $r\to\infty$. This definition of AAdS boundary conditions is sufficient for our purposes, though a coordinate independent version can be given [37]. The global AdS_{2+1} metric is

$$ds^2=-\left(\frac{r^2}{\ell^2}+1\right)dt^2+\left(\frac{r^2}{\ell^2}+1\right)^{-1}dr^2+r^2d\varphi^2. \tag{13}$$

The above metric has the following asymptotic ($r\to\infty$) values for the lapse, shift, spatial metric and conjugate momenta:

$$\mathring{g}_{ij}dx^idx^j=\frac{\ell^2}{r^2}dr^2+r^2d\varphi^2,$$
$$\mathring{\pi}^{ij}=0,\qquad \mathring{N}=\frac{r}{\ell},\qquad \mathring{N}_i=0, \tag{14}$$

where we have used a circle over the symbols to indicate that they are asymptotic values for the reference AdS_{2+1} metric.

It then seems plausible to require that the variables g_{ij}, π^{ij}, N and N_i of our solutions of Einstein's equations approach the global asymptotic AdS values $\mathring{g}_{ij}$, $\mathring{\pi}_{ij}$, $\mathring{N}$, and $\mathring{N}_i$, respectively, with some *a priori* unspecified falloffs in r as $r\to\infty$, i.e., as one approaches a given asymptotic boundary in the locally defined areal radial coordinate (in case the solution is in a different coordinate system, one has to first do a coordinate transformation to an areal radial coordinate near the asymptotic boundary and then match with the boundary conditions specified above).

However, we shall see that the spacetime solutions we obtain the maximal slicing spatial harmonic gauge are such that the momentum component π^{rr} does not go to zero at the asymptotic boundaries but to a time-dependent constant. Thus, we need to relax the above requirement to

allow for such boundary values of the momenta. The consequences of relaxing the boundary conditions are interesting to explore.

It turns out that the metric g_{ij} indeed approaches the AdS values in our solutions, and we continue with the requirement that g_{ij} approach the asymptotic AdS value $\mathring{g}_{ij}$. The following consequence of this will be useful later. For solutions of the Hamiltonian constraint $\mathcal{H}_\perp = 0$, the above boundary conditions (including the relaxation on the boundary values of the momenta) imply that the scalar curvature R of g_{ij} approaches 2Λ as one approaches a given asymptotic boundary:

$$R \to 2\Lambda = -\frac{2}{\ell^2} \quad \text{as } r \to \infty. \tag{15}$$

IV. GAUGE FIXING

The constraints $\mathcal{H}_\perp \approx 0$ and $\mathcal{H}_i \approx 0$ are first-class constraints [40] on phase space and generate gauge transformations. These gauge transformations are generated with the Hamiltonian (3) with some choice for the auxiliary fields N and N^i, and they correspond to the diffeomorphisms of spacetime which are trivial at the asymptotic boundaries—the *small diffeomorphisms*.

One fixes this gauge freedom by choosing gauge conditions, one for each of the constraints. The solutions of the gauge conditions and first-class constraints describe the reduced phase space which contain the truly gauge invariant degrees of freedom; an important requirement is that gauge freedom is completely fixed without residual small diffeomorphisms that preserve the gauge conditions. Note that large diffeomorphisms (those diffeomorphisms that are nontrivial at the boundaries but preserve the asymptotic AdS boundary conditions) need not be affected by the gauge fixing procedure.

A. The gauge conditions

In this work, we choose the *maximal slicing gauge* for the Hamiltonian constraint $\mathcal{H}_\perp$ and the *spatial harmonic gauge* for the momentum constraints $\mathcal{H}^i$:

$$\begin{aligned} &\text{maximal slicing: } \pi = 0, \\ &\text{spatial harmonic gauge: } \mathcal{D}^j \coloneqq \partial_i(\sqrt{g}g^{ij}) = 0. \end{aligned} \tag{16}$$

We show next that the maximal slicing gauge condition completely fixes the small gauge transformations generated by the Hamiltonian constraint $\mathcal{H}_\perp$ globally in phase space. Though we prove this result in $d = 2$, it is readily generalized to $d > 2$ and we emphasize that this result holds in $d > 2$ as well.

A similar general result does not hold for the spatial harmonic gauge conditions. In this work, we specialize to $d = 2$ and spatial slices with the topology of a cylinder $\Sigma = \mathbf{R} \times \mathbf{S}^1$. In this case, we demonstrate the spatial harmonic gauge conditions completely fix the small gauge transformations generated by the momentum constraints $\mathcal{H}_i$ globally in phase space. These are indeed the small spatial diffeomorphisms on Σ, and imposing the harmonic gauge conditions chooses a harmonic coordinate system on the spatial slice $\Sigma = \mathbf{R} \times \mathbf{S}^1$.

However, we must note that the analysis for the spatial harmonic gauge conditions typically fails for two-dimensional spatial surfaces with more complicated topology and also in $d > 2$. That is, the spatial harmonic gauge conditions cannot be imposed globally in phase space. This is tied to the fact that spatial harmonic coordinate systems typically fail beyond a certain region on the spatial slice when the coordinate differentials become linearly dependent [41]. We note that, in general, it is not known whether there exists a gauge condition that holds globally in phase space for the momentum constraints.

B. The gauge fixing procedure

To check that the gauge conditions completely fix the gauge transformations generated by the Hamiltonian (3), we must show that the Poisson brackets of the gauge conditions and the constraints are not all (weakly) zero. It is straightforward to compute the various Poisson brackets on the constrained surface $\mathcal{H}_\perp \approx 0$, $\mathcal{H}_i \approx 0$, $\pi \approx 0$, $\mathcal{D}^j \approx 0$:

$$\begin{aligned} &\{\mathcal{H}_\perp(x), \pi(x')\} \approx \kappa^{-2}\sqrt{g}(D^2 + 2\Lambda - \kappa^4 g^{-1}\pi_{ij}\pi^{ij})\delta(x, x'), \\ &\{\mathcal{H}_i(x), \pi(x')\} \approx \pi(x')D_i\delta(x, x') \approx 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &\{\mathcal{H}_\perp(x), \mathcal{D}^j(x')\} \approx 2\kappa^2\partial'_k(\pi^{jk}(x)\delta(x, x')), \\ &\{\mathcal{H}^i(x), \mathcal{D}^j(x')\} \approx -2D_k\partial'_l\Big(\sqrt{g}\Big(\frac{1}{2}(g^{ij}g^{kl} + g^{jk}g^{il} \\ &\qquad - g^{ik}g^{jl})\delta(x, x')\Big)\Big). \end{aligned} \tag{18}$$

To verify that our gauge conditions completely fix the small diffeomorphism gauge freedom, we must show that, given a spacetime diffeomorphism $(N = \xi_\perp, N_i = \xi_i)$ that

(i) is *a small diffeomorphism*, i.e., $\xi_\perp, \xi_i \to 0$ at the boundaries with sufficiently fast falloffs, and
(ii) preserves the gauge conditions (16), i.e., $\{H[\xi_\perp, \xi_i], \pi(x')\} = 0$, $\{H[\xi_\perp, \xi_i], \mathcal{D}^j(x')\} = 0$,

is the trivial diffeomorphism, i.e., $\xi_\perp = 0$, $\xi_i = 0$ everywhere on Σ. The condition $\{H[\xi_\perp, \xi_i], \pi(x')\} = 0$ gives

$$\sqrt{g}\left(D^2 + 2\Lambda - \kappa^4\frac{1}{g}\pi_{ij}\pi^{ij}\right)\xi_\perp = 0. \tag{19}$$

Multiplying by $\xi_\perp$, integrating over Σ, integrating by parts in the D^2 term, and discarding the boundary term since $\xi_\perp$ is a small diffeomorphism [42], we get

$$\int_\Sigma d^2x\sqrt{g}\left(D_i\xi_\perp D^i\xi_\perp+\left(\kappa^4\frac{1}{g}\pi^{ij}\pi_{ij}-2\Lambda\right)\xi_\perp^2\right)=0. \tag{20}$$

Every term on the left-hand side is positive definite, and, hence, (20) is satisfied only if $\xi_\perp=0$. Thus, the only solution to (19) is $\xi_\perp=0$.

Next, the condition $\{H[\xi_\perp,\xi_i],\mathcal{D}^j(x')\}=0$ gives the differential equation

$$\partial_l(2\kappa^2\xi_\perp\pi^{kl}+\sqrt{g}(2D^{(k}\xi^{l)}-g^{kl}D_i\xi^i))=0. \tag{21}$$

Since we have already established that $\xi_\perp=0$ previously, the first term in the above equation drops out.

We now specialize to the spatial slice Σ being a cylinder. In Sec. V below, we will see that any metric g_{ij} on the cylinder which is AAdS is conformally related to a flat Cartesian metric f_{ij} of the form $d\theta^2+m^2d\varphi^2$. Note that the differential operator in (22) is conformally invariant. Thus, the equation can be written for the conformally flat Cartesian metric f_{ij}:

$$\partial_l(\sqrt{f}(2\partial^{(k}\xi^{l)}-f^{kl}\partial_i\xi^i))=0. \tag{22}$$

Contracting the above equation with ξ_k, integrating over Σ, integrating by parts and discarding the boundary term since ξ_k is a small diffeomorphism, we get

$$\int_\Sigma d^2x\sqrt{f}(\partial_{(k}\xi_{l)}-f_{kl}\partial\cdot\xi)(\partial^{(k}\xi^{l)}-f^{kl}\partial\cdot\xi)=0. \tag{23}$$

Since the integrand is positive definite, the above condition implies the conformal Killing equation

$$\partial_{(k}\xi_{l)}-f_{kl}\partial\cdot\xi=0. \tag{24}$$

Since the only small diffeomorphism which satisfies the conformal Killing equation is the trivial diffeomorphism, we have shown that $\{H[\xi_\perp,\xi_i],\mathcal{D}^j(x')\}=0$ implies that $\xi_i=0$ for the case of the cylinder $\Sigma=\mathbf{S}^1\times\mathbf{R}$.

One can now solve the constraints and gauge conditions simultaneously to obtain the reduced phase space [43]. The Poisson bracket and reduced Hamiltonian on reduced phase space can be obtained from the original Lagrangian $\pi^{ij}\dot{g}_{ij}-H$ by substituting the solutions of the constraints and gauge conditions.

In the next section, we describe the reduced phase space dynamics of $(2+1)$-dimensional gravity with spatial slice $\Sigma=\mathbf{S}^1\times\mathbf{R}$, which satisfy the AAdS boundary conditions discussed in Sec. III. In Sec. IX, we show that the spacetime solutions that correspond to the solutions in reduced phase space are diffeomorphic to the fully extended nonrotating BTZ black hole [6,10].

V. THE REDUCED PHASE SPACE

We follow the Lichnerowicz-York method of decomposing the metric g_{ij} and conjugate momentum π^{ij} as

$$g_{ij}=e^{2\lambda}\tilde{g}_{ij},\qquad \pi^{ij}=e^{-2\lambda}\tilde{\pi}^{ij}, \tag{25}$$

with the conformal factor λ satisfying $\lambda\to 0$ as one approaches the boundaries. We then solve the Hamiltonian and momentum constraints subject to the maximal slicing and spatial harmonic gauge conditions.

A. The Hamiltonian constraint

The condition $\lambda\to 0$ at the boundaries requires $\tilde{g}_{ij}$ to satisfy the AAdS boundary conditions near the boundaries, in particular, that its Ricci scalar $\tilde{R}$ approach $2\Lambda<0$ near the boundaries. However, it is always possible to Weyl transform any such metric to one which satisfies $\tilde{R}=2\Lambda$ *everywhere* on Σ due to the existence and uniqueness of solutions to Yamabe's equation [44]. Thus, we take $\tilde{g}_{ij}$ to satisfy $\tilde{R}=2\Lambda$ everywhere. Recall that the maximal slicing condition is $\pi=0$. This in turn imposes $\tilde{g}_{ij}\tilde{\pi}^{ij}=\tilde{\pi}=0$.

Substituting the decomposition (25) in the Hamiltonian constraint along with the maximal slicing condition, we get the Lichnerowicz equation for λ:

$$2\tilde{D}^2\lambda=-\kappa^4\tilde{g}^{-1}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{\pi}^{ij}\tilde{\pi}^{kl}e^{-2\lambda}+2\Lambda(1-e^{2\lambda}). \tag{26}$$

It can be shown that, given a $\tilde{g}_{ij}$ and $\tilde{\pi}^{ij}$ satisfying $\tilde{R}=2\Lambda$ and $\tilde{\pi}=0$ everywhere on Σ, the above Lichnerowicz equation has a unique solution λ which satisfies $\lambda\to 0$ as one approaches the AAdS boundaries [38,45–48]. We review the proof of this statement in Appendix A.

In the case of $\Sigma=\mathbf{S}^1\times\mathbf{R}$, the space of constant negative curvature metrics which satisfy the AAdS boundary conditions at both boundaries is very simple to describe. The spatial harmonic gauge condition $\partial_i(\sqrt{g}g^{ij})=0$ is conformally invariant so that $\tilde{g}_{ij}$ must also satisfy it. It turns out that any metric $\tilde{g}_{ij}$ that satisfies $\tilde{R}=2\Lambda$ everywhere on the cylinder and the spatial harmonic gauge condition is of the form

$$\tilde{g}_{ij}dx^idx^j=\frac{\ell^2}{\cos^2\theta}(d\theta^2+m^2d\varphi^2),\quad \text{with}\quad m\neq 0. \tag{27}$$

We outline the proof in Appendix B. Here, $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$ is a longitudinal coordinate with $\pm\pi/2$ being the boundaries of the cylinder and $\varphi\in[0,2\pi]$ is an angular coordinate.

Thus, the space of metrics satisfying $\tilde{R}=2\Lambda$ on the cylinder and the spatial harmonic gauge condition is the space of the parameter $m\neq 0$ that appears in the metric (27).

B. The momentum constraints

It is convenient to express our equations in terms of the extrinsic curvature

$$\tilde{K}^{ij} = \kappa^2 \frac{1}{\sqrt{\tilde{g}}}(\tilde{\pi}^{ij} - \tilde{g}^{ij}\tilde{\pi}). \tag{28}$$

The maximal slicing condition $\pi = 0$ translates to $\tilde{K} = \tilde{g}_{ij}\tilde{K}^{ij} = 0$. Making use of $\tilde{K} = 0$ and the harmonic coordinate system (θ, φ) introduced above, the momentum constraints for pure gravity can be simplified to

$$\partial_\theta \tilde{K}_{\theta\varphi} + \frac{1}{m^2}\partial_\varphi \tilde{K}_{\varphi\varphi} = 0, \qquad \partial_\varphi \tilde{K}_{\theta\varphi} - \partial_\theta \tilde{K}_{\varphi\varphi} = 0. \tag{29}$$

While it is possible to explicitly write down the most general solutions to these equations, we restrict ourselves to solutions which are independent of φ since we are eventually interested in maximally slicing the BTZ solution [49]. Thus, the solution is

$$\tilde{K}_{\theta\theta} = \frac{T}{m^2}, \qquad \tilde{K}_{\varphi\varphi} = -T, \qquad \tilde{K}_{\theta\varphi} = 0, \tag{30}$$

where T is a real number independent of θ and φ. The usual AAdS boundary conditions require $\tilde{K}_{ij} = 0$ at the boundaries since the AdS metric is static. However, to get nontrivial dynamics, we need to relax the boundary conditions and allow for nonzero boundary values of $\tilde{K}_{ij}$.

The solutions to the momentum constraints are thus given by

$$\tilde{\pi}^{\theta\theta} = -\kappa^{-2}\frac{\cos^2\theta}{\ell^2}\frac{T}{m}, \qquad \tilde{\pi}^{\varphi\varphi} = \kappa^{-2}\frac{\cos^2\theta}{\ell^2}\frac{T}{m^3}, \qquad \tilde{\pi}^{\theta\varphi} = 0. \tag{31}$$

C. The Lichnerowicz equation

Substituting the metric (27) and the solutions (31) into the Lichnerowicz equation, we get

$$\tilde{D}^2\lambda = -\frac{1}{\ell^2} - \frac{T^2\cos^4\theta}{m^4\ell^4}e^{-2\lambda} + \frac{1}{\ell^2}e^{2\lambda}. \tag{32}$$

Since the coefficients of the above equation are independent of φ, it can be truncated to the space of λ which are independent of φ. Since there is a unique solution λ for the above equation which satisfies $\lambda \to 0$ at the boundaries [38,45–48], it must be the solution that is independent of φ. Defining

$$C = \frac{T}{\ell m^2}, \qquad \bar{\lambda}(\theta) = \lambda(\theta) - \log\cos\theta - \frac{1}{2}\log|C|, \tag{33}$$

we get the ordinary differential equation

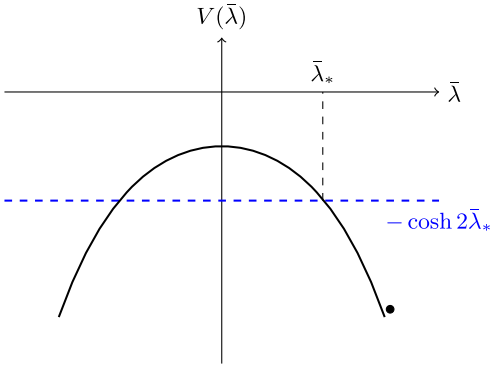


FIG. 1. The effective mechanics problem for the Lichnerowicz equation. The energy level $-\cosh 2\bar{\lambda}_*$ (blue dashed line) is such that the particle (black dot) stays on the positive side of the potential $V(\bar{\lambda}) = -\cosh 2\bar{\lambda}$.

$$\frac{1}{|C|}\bar{\lambda}'' = 2\sinh 2\bar{\lambda}, \tag{34}$$

which is the equation for a particle of mass $1/|C|$ with coordinate $\bar{\lambda}$ and potential $-\cosh 2\bar{\lambda}$, with θ being the time parameter ($' = \frac{d}{d\theta}$). Integrating the equation above once, we get the "energy conservation" equation

$$\frac{1}{2|C|}\bar{\lambda}'^2 - \cosh 2\bar{\lambda} = -\cosh 2\bar{\lambda}_*, \tag{35}$$

where $-\cosh 2\bar{\lambda}_*$ is the total "energy" of the particle whereas the two terms on the left-hand side are the kinetic and potential energy, respectively. The particle starts at "time" $\theta = -\pi/2$ at $\bar{\lambda} = \infty$, reaches the turning point $\bar{\lambda}_*$ at $\theta = 0$ and returns to $\bar{\lambda} = \infty$ at $\theta = \pi/2$. See Fig. 1 for a sketch of the trajectory of the particle. The energy conservation equation gives

$$\bar{\lambda}' = \pm\sqrt{2|C|}\sqrt{\cosh 2\bar{\lambda} - \cosh 2\bar{\lambda}_*}, \tag{36}$$

corresponding to the $\theta > 0$ and $\theta < 0$ branches of the trajectory, respectively. The above first-order equation can be solved exactly and gives the following expression for the conformal factor:

$$e^{\lambda(\theta)} = \sqrt{\frac{C}{k_*}}\frac{\cos\theta}{\mathrm{sn}(\sqrt{\frac{C}{k_*}}(\frac{\pi}{2} - \theta); |k_*|)}, \tag{37}$$

where $\mathrm{sn}(x; k_*)$ is Jacobi's sn function and is a function of C determined by the transcendental equation

$$\frac{\pi^2 C}{4} = k_* K(|k_*|)^2, \tag{38}$$

where $K(|k_*|)$ is Jacobi's complete elliptic integral of the first kind. The function $k_*(C)$ satisfies

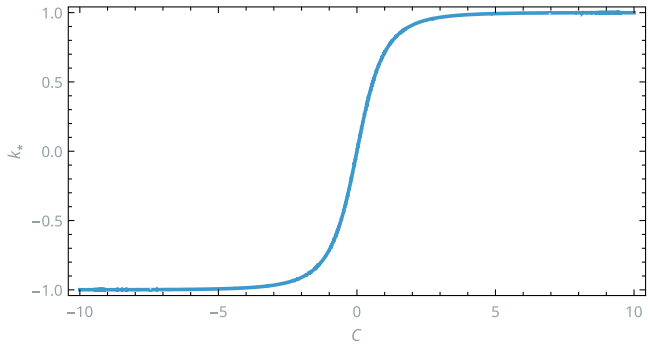


FIG. 2. The function $k_*(C)$.

$$\begin{aligned}&\mathrm{sgn}(k_*) = \mathrm{sgn}(C),\\ &k_*(C) = C + \mathcal{O}(C^2) \quad \text{for } C \approx 0,\\ &k_*(C) \sim \pm(1 - 8\mathrm{e}^{-\pi\sqrt{|C|}}) \quad \text{as } C \to \pm\infty \end{aligned} \tag{39}$$

and is numerically plotted in Fig. 2.

Substituting $\theta = 0$ in (37), using (38) and $\mathrm{sn}(K(|k_*|);|k_*|) = 1$, we see that $k_*(C)$ is related to the turning point value $\bar{\lambda}_*$ of $\bar{\lambda}$:

$$|k_*| = \mathrm{e}^{-2\bar{\lambda}_*}. \tag{40}$$

The solution (37) is a smooth, even function of θ which is well defined everywhere in the range $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and satisfies the boundary conditions $\lambda(\pm\frac{\pi}{2}) = 0$. See Fig. 3 for a plot of $\lambda(\theta)$ for the value $C = 0.5$. The solutions of the constraint equations and gauge conditions are thus parametrized by two constants m and T which form the reduced phase space. The final form of the solutions are

$$g_{\theta\theta} = \mathrm{e}^{2\lambda(\theta)}\frac{\ell^2}{\cos^2\theta}, \quad g_{\varphi\varphi} = \mathrm{e}^{2\lambda(\theta)}\frac{\ell^2 m^2}{\cos^2\theta}, \quad g_{\theta\varphi} = 0, \tag{41}$$

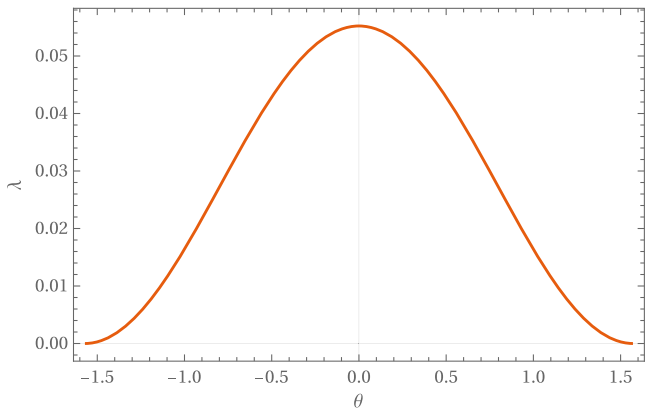


FIG. 3. $\lambda(\theta)$ for $C = 0.5$ and $k_* = 0.448$.

$$\begin{aligned}\pi^{\theta\theta} &= \kappa^{-2}\mathrm{e}^{-2\lambda(\theta)}\frac{\cos^2\theta}{\ell^2}\frac{T}{m}, \qquad \pi^{\theta\varphi} = 0,\\ \pi^{\varphi\varphi} &= -\kappa^{-2}\mathrm{e}^{-2\lambda(\theta)}\frac{\cos^2\theta}{\ell^2}\frac{T}{m^3}. \end{aligned} \tag{42}$$

D. The description of the reduced phase space

We see that, in the maximal slicing spatial harmonic gauge, the spatial metric g_{ij} is given entirely in terms of the single real parameter m. The cotangent direction p_m in π^{ij} that corresponds to changes in m can be obtained as follows:

$$p_m \equiv \int_\Sigma \mathrm{d}^2x \pi^{ij}\frac{\partial g_{ij}}{\partial m}. \tag{43}$$

Plugging in the expressions in (41) and (42), we get

$$p_m = -\frac{4\pi^2}{\kappa^2}\frac{T}{m^2}. \tag{44}$$

The reduced phase space is thus two-dimensional with the coordinates (m, p_m). We shall see in Sec. VII B that the $\pi^{ij}\dot{g}_{ij}$ term in the unreduced, original phase space action gives rise to the combination $p_m\dot{m}$ when we substitute into it the expressions (41) and (42). From this, we can read off the symplectic form on the reduced phase space to be $\mathrm{d}p_m \wedge \mathrm{d}m$.

VI. LARGE DIFFEOMORPHISMS: THE LAPSE AND SHIFT

So far, in the context of gauge fixing, we have been concerned with the Hamiltonian function $H[N, N_i]$ (3) for small diffeomorphisms $(N = \xi_\perp, N_i = \xi_i)$. We have seen that the gauge conditions (16) completely fix the small diffeomorphism freedom. However, it is possible that there are residual diffeomorphisms which preserve the gauge conditions that do not fall off to zero as one approaches the asymptotic AdS boundaries. These are the *large diffeomorphisms*. These can be obtained by again looking at the differential equations (19) and (21) but dropping the condition that the diffeomorphism (N, N_i) falls off to zero at the asymptotic boundaries. The equations are

$$\begin{aligned}\sqrt{g}(D^2 - K_{ij}K^{ij} + 2\Lambda)N = 0,\\ \partial_l(\sqrt{g}(2NK^{kl} + 2D^{(k}N^{l)} - g^{kl}D_iN^i)) = 0, \end{aligned} \tag{45}$$

where recall that K^{ij} is the extrinsic curvature defined in terms of the conjugate momentum as

$$K^{ij} = \kappa^2\frac{1}{\sqrt{g}}(\pi^{ij} - g^{ij}\pi). \tag{46}$$

We assume that the N and N^i are independent of φ; these are the simplest solutions to work with, and eventually we shall see that the spacetime with this lapse and shift corresponds to the maximally sliced, nonrotating BTZ solution. The more general equations are of interest for uncovering the boundary Virasoro asymptotic symmetries that are present in $2+1$ AAdS spacetimes [18]. We do not consider them here and address this interesting problem in future work.

Plugging in the solutions of the constraint equations (41) and (42), the differential equations become

$$\frac{\mathrm{d}^2N}{\mathrm{d}\theta^2} - (4|C|\cosh 2\bar{\lambda})N = 0,$$
$$\frac{\mathrm{d}^2N^\theta}{\mathrm{d}\theta^2} = -2\mathrm{sgn}(T)\ell^{-1}\frac{\mathrm{d}}{\mathrm{d}\theta}(\mathrm{e}^{-2\bar{\lambda}}N), \qquad \frac{\mathrm{d}^2N^\varphi}{\mathrm{d}\theta^2} = 0. \tag{47}$$

A. The solution for the lapse

Consider the expression

$$N(\theta) = \bar{\lambda}' m\left(c + b\int_{\theta_0}^{\theta}\frac{\mathrm{d}\vartheta}{\bar{\lambda}'^2(\vartheta)}\right), \tag{48}$$

with $\bar{\lambda}(\theta)$ the solution of the Lichnerowicz equation and θ_0 some reference value on the spatial slice. It is straightforward to check that this function satisfies the equation for the lapse.

Recall that

$$\bar{\lambda}' = \pm\sqrt{2|C|(\cosh 2\bar{\lambda} - \cosh 2\bar{\lambda}_*)}, \tag{49}$$

with the sign $\pm$ corresponding to the ranges $\theta > 0$ and $\theta < 0$, respectively. Thus, there are two branches for the solution, one for each part of the spatial slice. We choose the lower limit θ_0 in the integral to be $\pm\pi/2$ and the integration constants (c, b) and $(-\tilde{c}, -\tilde{b})$, respectively, for the $\theta > 0$ and $\theta < 0$ branches. Thus, the lapse is given by

$$N(\theta) = \begin{cases} |\bar{\lambda}'|m\left(c + b\int_{\frac{\pi}{2}}^{\theta}\frac{\mathrm{d}\vartheta}{\bar{\lambda}'^2(\vartheta)}\right) & \theta > 0 \\ -|\bar{\lambda}'|m\left(-\tilde{c} - \tilde{b}\int_{-\frac{\pi}{2}}^{\theta}\frac{\mathrm{d}\vartheta}{\bar{\lambda}'^2(\vartheta)}\right) & \theta < 0. \end{cases} \tag{50}$$

Demanding smoothness of the solution at $\theta = 0$ by equating the function values and derivatives at $\theta = 0$, we get the relations

$$b = \tilde{b} = b_*\frac{\tilde{c} + c}{2}, \tag{51}$$

where

$$b_* = -\frac{(\frac{C}{k_*})^{3/2}(k_*^2 - 1)^2}{2E(|k_*|) + (k_*^2 - 1)K(|k_*|)}. \tag{52}$$

Thus, the solution for the lapse has two free parameters c and $\tilde{c}$ which is appropriate for a solution of a second-order differential equation. The solution is an even function of θ when $\tilde{c} = c$.

1. The sign of the lapse

Let us look at the equation for the lapse:

$$D^2N = (K_{ij}K^{ij} - 2\Lambda)N. \tag{53}$$

Suppose c and $\tilde{c}$ are both positive. In this case, since the lapse is positive and grows in an unbounded manner at both the boundaries, it must have at least one minimum in the interior. At any minimum point, we have $D^2N > 0$. The differential equation then implies that the minimum value of the lapse must be positive since $(K_{ij}K^{ij} - 2\Lambda) \geq 0$. At any maxima of N in the interior, the same argument says that the maximum values of N must be negative. This is clearly an impossible situation and hence there is only one (global) minimum for N in the interior at which it is positive. It then follows that N is positive everywhere on Σ and never attains a zero.

2. AAdS boundary conditions for the lapse

In Sec. IX B below where we introduce an areal radial coordinate r on the spatial slice Σ, we show that the choice $c = \tilde{c} = 1$ corresponds to a lapse which approaches the asymptotic AdS value $\frac{r}{\ell}$ in the limit $r \to \infty$ (see Sec. III for the boundary conditions). With this choice we get a foliation of the asymptotic AdS spacetime in which the bulk time matches the boundary time. However, other values for c or $\tilde{c}$ are still allowed by the formalism. In fact a general choice of c and $\tilde{c}$ gives vector fields that implements a more general large diffeomorphism of the spacetime foliation. With this understanding, we proceed with parameters c and $\tilde{c}$ being arbitrary for now. We will put $c = \tilde{c} = 1$ from Sec. VII C onward.

B. The solution for the shift

The shift equations are much simpler to solve. The solution for N^φ is

$$N^\varphi(\theta) = \left(\frac{\pi}{2} - \theta\right)a_\varphi + \left(\frac{\pi}{2} + \theta\right)\tilde{a}_\varphi, \tag{54}$$

for constants a_φ and $\tilde{a}_\varphi$. The solution for N^θ is

$$N^\theta = \left(\frac{\pi}{2}-\theta\right)a_\theta + \left(\frac{\pi}{2}+\theta\right)\tilde{a}_\theta - 2\ell^{-1}\mathrm{sgn}(T)\int_0^\theta \mathrm{d}\vartheta \mathrm{e}^{-2\bar{\lambda}}N, \tag{55}$$

where a_θ and $\tilde{a}_\theta$ are constants and N is the solution (50).

The simplest boundary conditions that one can impose on the shift are to set them to be zero at the boundaries $\theta = \pm\frac{\pi}{2}$. In particular, this is justified for the component N^θ since the θ coordinate takes values in the finite $[-\pi/2, \pi/2]$, and a nonzero N^θ at the boundaries will move the boundaries which is inconsistent.

Setting the shift to be zero at the boundaries determines the values of the constants a_θ, a_φ, $\tilde{a}_\theta$, and $\tilde{a}_\varphi$. For the choice $c = \tilde{c} = 1$ on the lapse (50) that implements the AAdS boundary conditions, we get

$$a_\varphi = \tilde{a}_\varphi = 0, \qquad a_\theta = -\tilde{a}_\theta = -\frac{2}{\pi\ell}\mathrm{sgn}(T)\int_0^{\pi/2}\mathrm{d}\vartheta \mathrm{e}^{-2\bar{\lambda}}N. \tag{56}$$

VII. THE HAMILTONIAN EQUATIONS OF MOTION IN REDUCED PHASE SPACE

A. The reduced Hamiltonian

The reduced phase space consists of the metric and momentum that satisfy the constraints and gauge conditions. The Hamiltonian (3) on such configurations reduces to the boundary term:

$$H_\partial = -\frac{2}{\kappa^2}\oint_{\partial\Sigma}\mathrm{d}\varphi N\sqrt{\sigma}\left(k-\frac{1}{\ell}\right) + 2\oint_{\partial\Sigma}\mathrm{d}\varphi \hat{r}_i N_j \pi^{ij}. \tag{57}$$

The second term involving the shift does not contribute when the shift satisfies the boundary conditions $N^i = 0$ on $\partial\Sigma$. On the solutions (41) and (42) with the lapse (50), the first term evaluates to

$$H(m,T) = \frac{2\pi(c+\tilde{c})}{\kappa^2}\frac{T}{\ell}\left(k_*(T/\ell m^2) + \frac{1}{k_*(T/\ell m^2)}\right). \tag{58}$$

Next, we discuss the Hamiltonian equations of motion in reduced phase space. Since the Hamiltonian depends only on the phase space variables T and m, and not on any time parameter, it will be conserved under time evolution. Define

$$M = 2\pi\frac{T}{\ell}\left(k_*(T/\ell m^2) + \frac{1}{k_*(T/\ell m^2)}\right). \tag{59}$$

This conserved quantity is the "dimensionless mass," i.e., $M = \kappa^2 M_{\mathrm{ADM}}$, where M_{ADM} is the ADM mass of the solution measured at either asymptotic boundary. The ADM mass in asymptotically AdS_3 spacetime is defined as

$$M_{\mathrm{ADM}} = -\frac{2}{\kappa^2}\oint \mathrm{d}\varphi N\sqrt{\sigma}(k-\ell^{-1}). \tag{60}$$

B. The reduced phase space action and equations of motion

The reduced phase space action is obtained from the action in the original, unreduced phase space

$$\mathcal{S} = \int \mathrm{d}\bar{t}\int_\Sigma \mathrm{d}^2x(\pi^{ij}\dot{g}_{ij} - H), \tag{61}$$

where H is the original Hamiltonian (3), and plugging in the gauge-fixed expressions (41) and (42) for the metric g_{ij}:

$$g_{\theta\theta} = \mathrm{e}^{2\lambda(\theta)}\frac{\ell^2}{\cos^2\theta}, \quad g_{\varphi\varphi} = \mathrm{e}^{2\lambda(\theta)}\frac{\ell^2 m^2}{\cos^2\theta}, \quad g_{\theta\varphi} = 0 \tag{62}$$

and momentum π^{ij}:

$$\pi^{\theta\theta} = \kappa^{-2}\mathrm{e}^{-2\lambda(\theta)}\frac{\cos^2\theta}{\ell^2}\frac{T}{m}, \qquad \pi^{\theta\varphi} = 0, \qquad \pi^{\varphi\varphi} = -\kappa^{-2}\mathrm{e}^{-2\lambda(\theta)}\frac{\cos^2\theta}{\ell^2}\frac{T}{m^3}. \tag{63}$$

Note that we have introduced the time coordinate $\bar{t}$ to exclusively label the maximal slices. Earlier in Eq. (1), t was used as a general time coordinate for the ADM split.

The $\pi^{ij}\dot{g}_{ij}$ term in the action becomes

$$\int_\Sigma \mathrm{d}^2x\pi^{ij}\dot{g}_{ij} = p_m\dot{m} = -\frac{4\pi^2}{\kappa^2}\frac{T\dot{m}}{m^2}. \tag{64}$$

In arriving at the above equation, we have explicitly used the maximal slicing gauge condition $\pi = \pi^{ij}g_{ij} = 0$. Hence, there is no contribution from this term to the Hamiltonian which is given entirely by the surface term. In another choice of gauge, the symplectic term $\pi^{ij}\dot{g}_{ij}$ could also contribute to the Hamiltonian.

Let us define a new set of phase space coordinates:

$$Q = 4\pi^2\frac{T}{m^2} \quad \text{and} \quad P = m \tag{65}$$

and also define

$$\bar{H} = \kappa^2 H. \tag{66}$$

The reduced phase space equations of motion can then be derived from the action

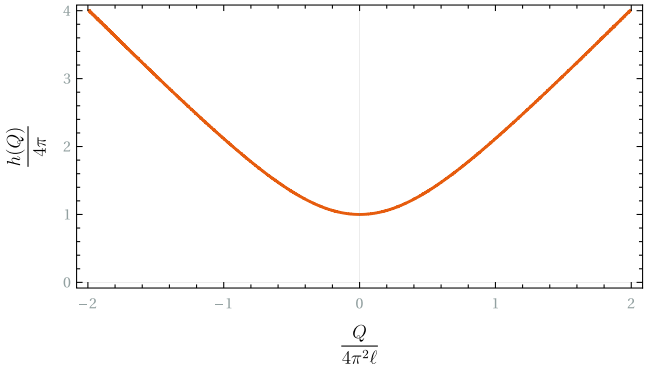


FIG. 4. A plot of $\frac{1}{4\pi}h(Q)$ vs $\frac{Q}{4\pi^2\ell}$ for $c = \tilde{c} = 1$.

$$\mathcal{S}[Q,P] = \frac{1}{\kappa^2}\int \mathrm{d}\bar{t}(P\dot{Q} - \bar{H}(Q,P)), \tag{67}$$

where the Hamiltonian when expressed in terms of Q and P is

$$\bar{H}(Q,P) = P^2 h(Q), \tag{68}$$

where

$$h(Q) = \frac{c+\tilde{c}}{2\pi\ell}Q\left(k_*\left(\frac{Q}{4\pi^2\ell}\right) + k_*\left(\frac{Q}{4\pi^2\ell}\right)^{-1}\right). \tag{69}$$

The function $h(Q)$ is plotted in Fig. 4.

The equations of motion are

$$\dot{Q} = 2Ph(Q), \qquad \dot{P} = -(c+\tilde{c})\left(\frac{M}{Q} + \frac{\pi^2 b_* P^2}{Q}\right), \tag{70}$$

where b_* is the quantity given by (52) which we reproduce below for convenience:

$$b_* = -\frac{(\frac{C}{k_*})^{3/2}(k_*^2-1)^2}{2E(|k_*|) + (k_*^2-1)K(|k_*|)}. \tag{71}$$

The equations for m and T can be obtained by using the definition $Q = 4\pi^2\frac{T}{m^2}$:

$$\begin{aligned}\dot{T} &= -\frac{c+\tilde{c}}{2}b_* m^3,\\ \dot{m} &= -(c+\tilde{c})\left(\frac{Mm^2}{4\pi^2 T} + \frac{b_* m^4}{4T}\right).\end{aligned} \tag{72}$$

In Appendix C, we show that our reduced phase space coordinates (Q,P) are related by a canonical transformation to the phase space variables discussed by Kuchar [9] for spherically symmetric spacetimes applied to the BTZ black hole [50].

C. Phase space trajectories

From this section onward we will consider AAdS boundary conditions which correspond to $c = \tilde{c} = 1$. As discussed above, the quantity M (59) remains constant under time evolution. Since the range of k_* is $0 < |k_*| < 1$, there is an upper bound for $|T|$ under time evolution which we call T_∞:

$$|T| \le T_\infty = \frac{M\ell}{4\pi}. \tag{73}$$

Further, the quantity b_* is negative as can be seen from (71). The $\dot{T}$ equation of motion then implies that T monotonically increases as $\bar{t}$ increases. Hence, as $\bar{t} \to \pm\infty$, T must approach $\pm T_\infty$. The $\dot{m}$ equation of motion then implies that m decreases monotonically to 0 as $\bar{t} \to \pm\infty$. In addition, the above implies that $C = T/\ell m^2 \to \pm\infty$ so that $k_*(C) \to \pm 1$. The conserved quantity M then has a simple expression in terms of T_∞ which one obtains by taking $\bar{t} \to \infty$:

$$\frac{M}{2\pi} = \frac{2T_\infty}{\ell}. \tag{74}$$

We shall see in Sec. IX that the maximally sliced solution is diffeomorphic to a region of the Kruskal extension of the BTZ black hole so that M corresponds to the mass of the BTZ black hole. We have thus given an interpretation of the mass of the BTZ black hole as the initial condition for one of the reduced phase space variables.

We can solve the equations of motion explicitly as follows. Note that the expression (59) for M gives a quadratic equation for k_* whose solution gives an expression in terms of M and T. This is given by

$$k_*(T) = \frac{R_-^2(T)}{\ell T} = \frac{\ell T}{R_+^2(T)} = \mathrm{sgn}(T)\frac{R_-(T)}{R_+(T)}, \tag{75}$$

where

$$R_\pm^2(T) \equiv \ell T_\infty\left(1 \pm \sqrt{1 - \frac{T^2}{T_\infty^2}}\right). \tag{76}$$

Using $C = T/\ell m^2$ and the above expression for $k_*(T)$, we get

$$\dot{T} = \frac{R_+}{\ell^3}\frac{(R_+^2 - R_-^2)^2}{2R_+^2 E(\frac{R_-}{R_+}) + (R_-^2 - R_+^2)K(\frac{R_-}{R_+})}, \tag{77}$$

where we have suppressed the dependence of $R_\pm$ on T for brevity. This is an equation for T that involves only T on the right-hand side (apart from the constant of motion M) which makes it easier to integrate. One can then plug this solution into the equation for m in (72) and integrate the $\dot{m}$

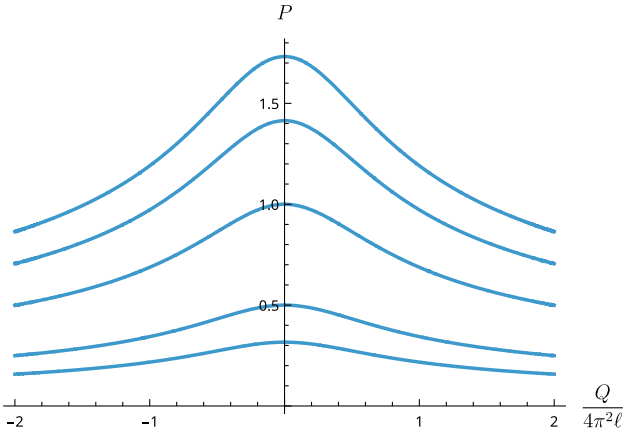


FIG. 5. Phase space trajectories corresponding to $\frac{M}{2\pi} = 3, 2, 1, \frac{1}{4}, \frac{1}{10}$ from top to bottom. Though not evident from the plots, all trajectories satisfy $P \to 0$ as $Q \to \pm\infty$. The value at $Q = 0$ is $P = m = \sqrt{M/2\pi}$.

equation to obtain m as a function of $\bar{t}$. Some sample phase space trajectories are plotted numerically in Fig. 5.

Before moving on, we note that while the total area under a phase space trajectory is divergent, one can still get a finite area by integrating from $Q = 0$ to $Q = Q_c$:

$$\int_0^{Q_c} P\mathrm{d}Q = \int_0^{Q_c} \sqrt{\frac{\bar{H}}{h(Q)}}\mathrm{d}Q = \sqrt{2M}\int_0^{Q_c} \frac{\mathrm{d}Q}{\sqrt{h(Q)}}, \tag{78}$$

where Q_c is some cutoff. The particular phase space trajectory under consideration is uniquely labeled by the mass M. We will revisit this fact later in Sec. VIII.

D. The collapse of the lapse

The large $\bar{t}$ behavior of T and m has an important consequence on the lapse $N(\bar{t}, \theta)$ of our solution. We can estimate the behavior of T and m for large $\bar{t}$. Plugging in $T \approx T_\infty$ into Eq. (77), we get

$$\dot{T} \approx 4\left(\frac{T_\infty}{\ell}\right)^{3/2}\left(1 - \frac{T}{T_\infty}\right), \tag{79}$$

which we can integrate explicitly to get

$$T = T_\infty(1 - B\mathrm{e}^{-2\sqrt{2}\eta\bar{t}}), \tag{80}$$

where we have defined η:

$$\eta = \frac{1}{\ell}\left(\frac{2T_\infty}{\ell}\right)^{1/2} = \frac{1}{\ell}\sqrt{\frac{M}{2\pi}}, \tag{81}$$

which will be identified with the surface gravity of the BTZ black hole. B in (80) is a constant of integration which we cannot estimate from the large $\bar{t}$ behavior and has to be determined from an exact integration of the $\dot{T}$ equation (77) either numerically or by an analysis similar to the one in [51] for the maximally sliced Schwarzschild black hole [52].

Further, as $\bar{t} \to +\infty$, since m becomes very small and $T \to T_\infty$, one can drop the m^4 term in the $\dot{m}$ equation of motion and get

$$\dot{m} \approx -\frac{Mm^2}{2\pi^2 T_\infty} = -\frac{2}{\pi\ell}m^2, \tag{82}$$

which gives

$$m(\bar{t}) \sim \frac{\pi\ell}{2\bar{t}} \quad \text{as } \bar{t} \to \infty. \tag{83}$$

The value of the lapse $N(\bar{t}, \theta)$ at the center of the wormhole throat $\theta = 0$ takes the following form:

$$N(\theta = 0, \bar{t}) = \frac{\ell\dot{T}(\bar{t})}{2\sqrt{T_\infty^2 - T^2}} \approx \ell\eta\sqrt{B}\mathrm{e}^{-\sqrt{2}\eta\bar{t}}, \quad \text{as } \bar{t} \to \infty. \tag{84}$$

The lapse is driven to zero as $\bar{t} \to \infty$, a phenomenon termed as the *collapse of the lapse* [52]. In the context of the mapping of the solution to the Kruskal extension of the BTZ black hole, this means that the proper time as measured by $N\mathrm{d}\bar{t}$ is driven to zero. This implies that the maximally sliced spacetime obtained in this section will be mapped to only a subregion of the Kruskal extension of the BTZ black hole, as we demonstrate in Sec. IX.

VIII. QUANTIZATION OF THE REDUCED PHASE SPACE

Recall the reduced phase space action (67) written in (m, T) variables:

$$\mathcal{S}[m, T] = \int \mathrm{d}\bar{t}\left(-\frac{4\pi^2}{\kappa^2}\frac{T}{m^2}\dot{m} - H(m, T)\right). \tag{85}$$

The first term in the action above can be interpreted as arising from the following symplectic form on a phase space with coordinates (m, T):

$$\frac{4\pi^2}{\kappa^2}\frac{\mathrm{d}T \wedge \mathrm{d}m}{m^2}, \tag{86}$$

which is the symplectic form on the hyperbolic upper half-plane. It is interesting that the Hamiltonian formulation of gravity naturally provides the reduced phase space to be the hyperbolic upper half-plane with its natural symplectic form. The quantization of the reduced phase space may be carried out via the method of coadjoint orbits since the upper half-plane is indeed a coadjoint orbit of $\mathrm{SL}(2, \mathbf{R})$.

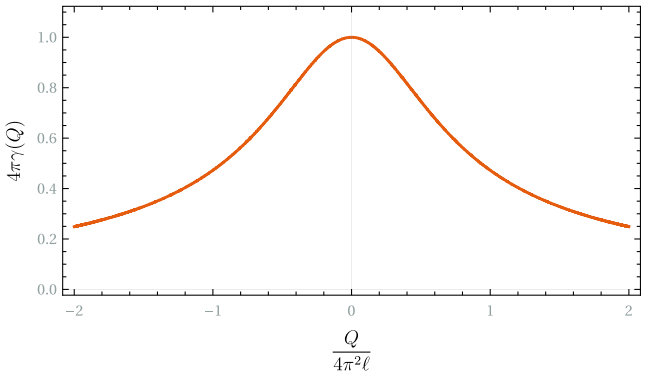


FIG. 6. A plot of $4\pi\gamma(Q)$ vs $\frac{Q}{4\pi^2\ell}$.

It would also be interesting to identify the origin of this $\mathrm{SL}(2,\mathbf{R})$.

Presently this approach seems difficult to carry out explicitly and we opt for directly setting up the Schrödinger equation and solving it in terms of the coordinates

$$Q = 4\pi^2 \frac{T}{m^2} \quad \text{and} \quad P = m \tag{87}$$

introduced before. Henceforth we interpret Q and P as "position and momentum," respectively. Recall the reduced phase space action

$$\mathcal{S}[Q,P] = \frac{1}{\kappa^2}\int \mathrm{d}\bar{t}(P\dot{Q} - \bar{H}(P,Q)), \tag{88}$$

where the classical Hamiltonian $\bar{H}(P,Q)$ takes the form of the kinetic energy of a free particle on a one-dimensional space with inverse metric $h(Q)$:

$$\bar{H}(P,Q) = h(Q)P^2, \tag{89}$$

with

$$h(Q) = \frac{Q}{\pi\ell}\left(k_*\left(\frac{Q}{4\pi^2\ell}\right) + \frac{1}{k_*(\frac{Q}{4\pi^2\ell})}\right). \tag{90}$$

We have put $c = \tilde{c} = 1$ in the above and will keep this choice henceforth.

The quantization of a particle in curved space is well known. A consistent choice of the quantum Hamiltonian is the Laplacian on the one-dimensional manifold with metric $\gamma(Q) \equiv h^{-1}(Q)$:

$$\hat{H} = -\hbar_{\mathrm{eff}}^2 h^{1/2}\partial_Q(h^{1/2}\partial_Q) = -\hbar_{\mathrm{eff}}^2 \frac{1}{\sqrt{\gamma}}\partial_Q\left(\frac{1}{\sqrt{\gamma}}\partial_Q\right). \tag{91}$$

The metric $\gamma(Q)$ is plotted in Fig. 6.

Note that $\hbar_{\mathrm{eff}} \equiv \hbar\kappa^2$ is the usual Planck area. The Schrödinger equation on the position space wave functions $\psi(Q,\bar{t})$ is then

$$\mathrm{i}\hbar_{\mathrm{eff}}\frac{\partial\psi(Q,\bar{t})}{\partial\bar{t}} = -\hbar_{\mathrm{eff}}^2\frac{1}{\sqrt{\gamma}}\partial_Q\left(\frac{1}{\sqrt{\gamma}}\partial_Q\psi(Q,\bar{t})\right). \tag{92}$$

Since the Hamiltonian operator is time independent, we can solve the time-independent Schrödinger equation by separating variables $\psi(Q,\bar{t}) = \mathrm{e}^{-\mathrm{i}E\bar{t}/\hbar_{\mathrm{eff}}}\psi_E(Q)$:

$$-\hbar_{\mathrm{eff}}^2\frac{1}{\sqrt{\gamma}}\partial_Q\left(\frac{1}{\sqrt{\gamma}}\partial_Q\psi_E(Q)\right) = E\psi_E(Q). \tag{93}$$

Let us define a new coordinate $\bar{Q}$ via

$$\bar{Q}(Q) = \int_0^Q \mathrm{d}q\sqrt{\gamma(q)}. \tag{94}$$

Since $\sqrt{\gamma(Q)} \sim Q^{-1/2}$ as $Q \to \infty$, the coordinate $\bar{Q}$ also has the same range $(-\infty,\infty)$ as Q. In terms of $\bar{Q}$, we get the simple equation

$$-\hbar_{\mathrm{eff}}^2\frac{\partial^2}{\partial\bar{Q}^2}\psi_E(\bar{Q}) = E\psi_E(\bar{Q}), \tag{95}$$

whose spectrum is positive, i.e., $E \geq 0$, with the eigenfunctions

$$\psi_p(\bar{Q}) = \mathrm{e}^{-\mathrm{i}p\bar{Q}/\hbar_{\mathrm{eff}}}, \quad \text{with} \quad p\in\mathbf{R} \quad \text{and} \quad E = p^2. \tag{96}$$

These wave functions are δ-function normalized:

$$(\psi_p,\psi_{p'}) = \int_{-\infty}^{\infty}\mathrm{d}\bar{Q}\psi_p^*(\bar{Q})\psi_{p'}(\bar{Q}) = \delta(p-p'). \tag{97}$$

The quantum mechanics of the reduced phase space in the $\bar{Q}$ variable is that of a free particle. The transition amplitude to go from $\bar{Q}_1$ to $\bar{Q}_2$ is well known:

$$K(\bar{Q}_2,\bar{t}_2;\bar{Q}_1,\bar{t}_1) = \frac{\exp(\frac{\mathrm{i}(\bar{Q}_2-\bar{Q}_1)^2}{4\hbar_{\mathrm{eff}}(\bar{t}_2-\bar{t}_1)})}{\sqrt{4\pi\mathrm{i}\hbar_{\mathrm{eff}}(\bar{t}_2-\bar{t}_1)}}. \tag{98}$$

The kernel in the "momentum" basis is the Fourier transform (here $\bar{Q} = \bar{Q}_2 - \bar{Q}_1$)

$$\begin{aligned} K(p,\bar{t}_2-\bar{t}_1) &= \int_{-\infty}^{\infty}\mathrm{d}\bar{Q}K(\bar{Q}_2,\bar{t}_2;\bar{Q}_1,\bar{t}_1)\mathrm{e}^{\mathrm{i}p\bar{Q}/\hbar_{\mathrm{eff}}} \\ &= \exp\left(-\mathrm{i}p^2(\bar{t}_2-\bar{t}_1)/\hbar_{\mathrm{eff}}\right) \\ &= \exp\left(-\mathrm{i}2M(\bar{t}_2-\bar{t}_1)/\hbar_{\mathrm{eff}}\right). \end{aligned} \tag{99}$$

Since the Hamiltonian we have diagonalized is the boundary ADM Hamiltonian times κ^2, its eigenfunctions represent quantum states labeled by a real number p with energy $E = p^2$. As we will discuss later in Sec. IX, the

classical solutions in the reduced phase space correspond to the two-sided BTZ black hole solution. Hence we can identify the energy eigenvalue with twice the mass of the BTZ black hole, i.e., $p^2 = 2M$.

In the Q representation the wave functions are

$$\psi_p(Q) = \psi_p(\bar{Q}(Q)), \quad \text{with} \quad \bar{Q}(Q) = \int_0^Q \mathrm{d}q\sqrt{\gamma(q)}, \tag{100}$$

and are normalized as

$$\int_{-\infty}^{\infty} \mathrm{d}Q\sqrt{\gamma(Q)}\psi_p^*(Q)\psi_{p'}(Q) = \delta(p-p'). \tag{101}$$

The transition amplitude in terms of Q can be obtained from (98):

$$K(Q_2,\bar{t}_2;Q_1,\bar{t}_1) = \frac{\exp\left(\frac{\mathrm{i}\left(\int_0^{Q_2}\mathrm{d}q\sqrt{\gamma(q)}-\int_0^{Q_1}\mathrm{d}q\sqrt{\gamma(q)}\right)^2}{4\hbar_{\mathrm{eff}}(\bar{t}_2-\bar{t}_1)}\right)}{\sqrt{4\pi\mathrm{i}\hbar_{\mathrm{eff}}(\bar{t}_2-\bar{t}_1)}}. \tag{102}$$

The general wave function would be a linear combination of the above eigenstates:

$$\psi(Q) = \int_{-\infty}^{\infty} \mathrm{d}p C_p \psi_p(Q). \tag{103}$$

Upon taking the Fourier transform,

$$\tilde{\psi}(P) = \int_{-\infty}^{\infty} \mathrm{d}Q \mathrm{e}^{-\mathrm{i}QP/\hbar_{\mathrm{eff}}}\psi(Q), \tag{104}$$

we obtain wave functions $\tilde{\psi}(P)$ which are functions of P.

Recall that the coordinate $P = m$ was classically restricted to be nonzero; however, the quantum theory is completely well defined for $P \in \mathbf{R}$.

A. An exact classical-quantum correspondence

The phase of the energy eigenfunction with energy $E = 2M$ is proportional to

$$p\bar{Q} = p\int_0^Q \frac{\mathrm{d}q}{\sqrt{h(q)}} = \sqrt{2M}\int_0^Q \frac{\mathrm{d}q}{\sqrt{h(q)}}. \tag{105}$$

Curiously this is the same as the "regulated" area under a phase space trajectory that we calculated in Sec. VII C:

$$\int_0^{Q_c} P\mathrm{d}Q = \sqrt{2M}\int_0^{Q_c} \frac{\mathrm{d}Q}{\sqrt{h(Q)}}, \tag{106}$$

with the identification $Q = Q_c$.

Thus we see that each energy eigenstate labeled by $E = 2M$ corresponds to a classical geometry which is diffeomorphic to the two-sided BTZ black hole of mass M (as we show in the next section).

IX. THE DIFFEOMORPHISM TO THE EXTENDED BTZ BLACK HOLE

We have obtained a spacetime solution of Einstein's equations in the Hamiltonian formulation in the maximal slicing, spatial harmonic gauge. The spacetime is the time evolution of a wormhole geometry with two asymptotic AdS ends, with the spacetime metric in the ADM form by

$$\begin{aligned}\mathrm{d}s^2 = &-N(\bar{t},\theta)^2\mathrm{d}\bar{t}^2 + \frac{\ell^2\mathrm{e}^{2\lambda(\bar{t},\theta)}}{\cos^2\theta}(\mathrm{d}\theta^2 + N^\theta(\bar{t},\theta)\mathrm{d}\bar{t})^2 \\ &+ \frac{\ell^2\mathrm{e}^{2\lambda(\bar{t},\theta)}}{\cos^2\theta}m(\bar{t})^2\mathrm{d}\varphi^2.\end{aligned} \tag{107}$$

This spacetime solution has ADM mass $\frac{M}{\kappa^2}$ (59) which is the value of the ADM Hamiltonian on the above solution. It is straightforward to show that the above solution is in fact diffeomorphic to a region R_0 of the full Kruskal extension of the nonrotating BTZ black hole solution in $2+1$ dimensions. First, we give a short review of the fully extended BTZ black hole solution. In this section, we take $\kappa^2 = 1$ for convenience.

A. A review of the fully extended BTZ black hole

The BTZ black hole metric is

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\varphi^2, \quad f(r) = \frac{r^2}{\ell^2} - \frac{M}{2\pi}. \tag{108}$$

The spacetime has an asymptotic AdS boundary as $r \to \infty$. The metric has a coordinate singularity at the horizon $r = R_h = \ell\sqrt{M/2\pi}$ at which the time translation Killing vector becomes null. The above metric is valid for the range

$$\text{region I}: R_h < r < \infty, \quad -\infty < t < \infty, \quad 0 \le \varphi < 2\pi. \tag{109}$$

To go past the null horizon, one first defines the tortoise coordinate r_*:

$$r_* = \int_\infty^r \frac{\mathrm{d}\rho}{f(\rho)} = \frac{1}{2\eta}\log\frac{r-R_h}{r+R_h}, \tag{110}$$

where $\eta = R_h/\ell^2$ is the surface gravity of the black hole. The Kruskal-Szekeres coordinates are defined as

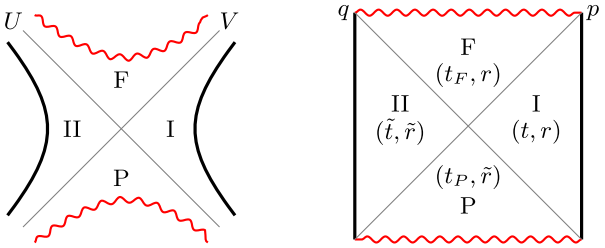


FIG. 7. The four regions of the maximally extended BTZ black hole shown in the Kruskal and Penrose diagrams. The local BTZ-type coordinates in each region are shown in the Penrose diagram, with $r > R_h$ in region I, $r < R_h$ in region F, $\tilde{r} > R_h$ in region II and $\tilde{r} < R_h$ in region P.

$$U = -\eta^{-1}\ \exp(-\eta(t - r_*)) = -\eta^{-1}\sqrt{\frac{r - R_h}{r + R_h}}e^{-\eta t},$$
$$V = \eta^{-1}\ \exp(\eta(t + r_*)) = \eta^{-1}\sqrt{\frac{r - R_h}{r + R_h}}e^{\eta t}. \tag{111}$$

In region I, we have $U < 0$ and $V > 0$. The past horizon $t \to -\infty$, $r \to R_h$ is described by the surface $V = 0$ whereas the future horizon $t \to \infty$, $r \to R_h$ is described by the surface $U = 0$. The BTZ black hole metric in terms of the Kruskal-Szekeres coordinates is

$$ds^2 = -4\frac{R_h^2}{\ell^2}\frac{dU dV}{(1+\eta^2 UV)^2} + R_h^2\left(\frac{1-\eta^2 UV}{1+\eta^2 UV}\right)^2 d\varphi^2. \tag{112}$$

As is clear, the metric has no singularities at the horizons $UV = 0$ and is valid for the range $|\eta^2 UV| < 1$, with $U, V \in \mathbf{R}$. This is the fully extended BTZ black hole. There are four regions corresponding to the different signs of U and V which we designate regions I, F, II and P. These are displayed in Fig. 7.

In each of the regions, we can find BTZ-like coordinates in which the metric takes the form (108). These coordinates are also displayed in Fig. 7. It is also useful to define the Penrose coordinates (p, q):

$$p = \arctan(\eta V), \qquad q = \arctan(\eta U). \tag{113}$$

The Penrose diagram of the fully extended BTZ black hole is displayed in the second panel in Fig. 7. The entire extended black hole spacetime fits inside the finite region in (p, q) space where the p and q coordinates are in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The singularities are at $p + q = \pm\frac{\pi}{2}$ and the boundaries at $p - q = \pm\frac{\pi}{2}$. The horizons are at $p = 0$ and $q = 0$.

B. The diffeomorphism

The strategy to obtain the diffeomorphism is as follows. We first express our maximally sliced solution (107) in terms of an areal radial coordinate since it facilitates an easy comparison with the static BTZ metric in (108).

Note that doing this takes us out of the spatial harmonic gauge with coordinates (θ, φ) whereas the maximal slicing gauge condition is still valid.

The areal radial coordinate is defined by setting $g_{\varphi\varphi}(\theta, \bar{t})$ in (107) equal to r^2 and solving for r in terms of θ and $\bar{t}$:

$$r(\theta, \bar{t}) = e^{\lambda(\theta,\bar{t})}\frac{\ell m(\bar{t})}{\cos\theta} = e^{\bar{\lambda}(\theta,\bar{t})}\sqrt{\ell|T(\bar{t})|}, \tag{114}$$

where we have used the definition $\bar{\lambda} = \lambda - \log\cos\theta - \frac{1}{2}\log\frac{|T|}{\ell m^2}$. Recall that λ is the unique solution to the Lichnerowicz equation (32), and it is an even function of θ. This implies that the above relation is a one-to-one coordinate transformation either in the range $\theta > 0$ or in the range $\theta < 0$. The range of r that corresponds to $0 < \theta < \frac{\pi}{2}$ is

$$R_+(\bar{t}) = \sqrt{\frac{\ell|T|}{|k_*|}} < r < \infty. \tag{115}$$

Thus, we need two areal radial coordinates r and $\tilde{r}$ with identical ranges to describe the entire spatial slice:

$$r = e^{\lambda(\theta)}\frac{\ell m}{\cos\theta} = e^{\bar{\lambda}(\theta)}\sqrt{\ell|T|} \quad \text{for } \theta > 0,$$
$$\tilde{r} = e^{\lambda(\theta)}\frac{\ell m}{\cos\theta} = e^{\bar{\lambda}(\theta)}\sqrt{\ell|T|} \quad \text{for } \theta < 0. \tag{116}$$

The maximally sliced spacetime metric in the coordinates $(\bar{t}, r, \varphi)$ for the region $0 < \theta < \frac{\pi}{2}$ is given by

$$ds^2 = -N^2(r, \bar{t})d\bar{t}^2 + A(r, \bar{t}) \times \left(dr + \frac{N(r, \bar{t})T(\bar{t})}{r}d\bar{t}\right)^2 + r^2 d\varphi^2, \tag{117}$$

$$A(r, \bar{t}) = \left(\frac{r^2}{\ell^2} - \frac{M}{2\pi} + \frac{T(\bar{t})^2}{r^2}\right)^{-1}, \tag{118}$$

where the lapse $N(r, \bar{t})$ is given by

$$N(r, \bar{t}) = \frac{1}{\sqrt{A(r, \bar{t})}}\left(1 - \dot{T}(\bar{t})\int_\infty^r \frac{d\rho}{\rho\left(\frac{\rho^2}{\ell^2} - \frac{M}{2\pi} + \frac{T(\bar{t})^2}{\rho^2}\right)^{3/2}}\right), \tag{119}$$

which is valid for the range (115). The same expressions in terms of $\tilde{r}$ hold for the region $-\frac{\pi}{2} < \theta < 0$ of the spatial slices.

Next, we postulate a diffeomorphism from region I of the BTZ black hole to the maximally sliced solution (117) which maps t to a function of $\bar{t}$ and r:

$$(t, r, \varphi) = (t = t(\bar{t}, r), r, \varphi). \tag{120}$$

To obtain the function $t(\bar{t}, r)$, we plug it into the BTZ metric (108) in region I and compare with the maximal slicing metric (117). In region I with $\bar{t} > 0$, we get

$$\frac{\partial t}{\partial \bar{t}} = N(r,\bar{t})A(r,\bar{t})^{1/2}, \qquad \frac{\partial t}{\partial r} = -\frac{T(\bar{t})A(r,\bar{t})^{1/2}}{rf(r)}. \tag{121}$$

The solution to the above equations is obtained in Appendix D:

$$t(r,\bar{t}) = \bar{t} + T(\bar{t})\int_r^\infty \frac{\mathrm{d}\rho}{\rho}\frac{A(\rho,\bar{t})^{1/2}}{f(\rho)}, \qquad r > R_h. \tag{122}$$

Note that the integrand has a pole at $\rho = R_h$ coming from the $f(\rho)$ in the denominator and hence $t \to \infty$ as $r \to R_h$.

Thus, the diffeomorphism between the maximal slicing solution and the static BTZ coordinate system in region I is singular and reflects the singular nature of the static BTZ coordinate system.

The diffeomorphism to Kruskal-Szekeres coordinates in region I can be obtained by using the definitions (111):

$$\begin{aligned} V &= \eta^{-1}\exp\left(\eta\bar{t} + \eta\int_r^\infty \mathrm{d}\rho \frac{T(\bar{t})A(\rho,\bar{t})^{1/2} - \rho}{\rho f(\rho)}\right), \\ &= \eta^{-1}\exp\left(\eta\bar{t} - \eta\int_r^\infty \mathrm{d}\rho \frac{\rho A(\rho,\bar{t})}{T(\bar{t})A(\rho,\bar{t})^{1/2} + \rho}\right), \\ U &= -\eta^{-1}\exp\left(-\eta\bar{t} - \eta\int_r^\infty \mathrm{d}\rho \frac{T(\bar{t})A(\rho,\bar{t})^{1/2} + \rho}{\rho f(\rho)}\right). \end{aligned} \tag{123}$$

The above formulas and can be extended to region F, $r < R_h$, by using the Cauchy principal value prescription. In this process, V remains unchanged since the integrand does not have a pole at $\rho = R_h$ whereas U picks up a minus sign. Thus, in region F, we have

$$\begin{aligned} V &= \eta^{-1}\exp\left(\eta\bar{t} - \eta\int_r^\infty \mathrm{d}\rho \frac{\rho A(\rho,\bar{t})}{T(\bar{t})A(\rho,\bar{t})^{1/2} + \rho}\right), \\ U &= \eta^{-1}\exp\left(-\eta\bar{t} - \eta\int_r^\infty \mathrm{d}\rho \frac{T(\bar{t})A(\rho,\bar{t})^{1/2} + \rho}{\rho f(\rho)}\right). \end{aligned} \tag{124}$$

Similar formulas can be written down for the diffeomorphism of the fully extended BTZ black hole in region II and region P, to the maximally sliced solution (117).

Finally, by replacing the areal radial coordinates r and $\tilde{r}$ by their expressions in terms of $\theta, \bar{t}$ in $\theta > 0$ and $\theta < 0$, respectively, we get a diffeomorphism from the spacetime solution (107) to the fully extended solution 7 in Kruskal-Szekeres coordinates. It is straightforward to show that this diffeomorphism is completely smooth everywhere in its domain of definition. However, due to the collapse of the lapse phenomenon described at the end of Sec. VII C, the maximally sliced solution (107) is only mapped to a subregion R_0 of the fully extended BTZ black hole solution. This region R_0 is shown in Fig. 8 and some representative maximal slices are drawn in Fig. 9.

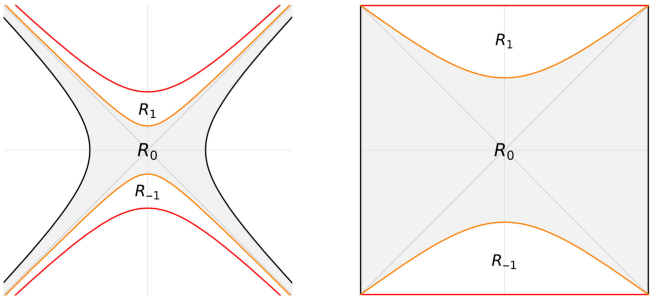


FIG. 8. The region R_0 (shaded gray) covered by the maximal slicing solution is demarcated by the orange lines in the BTZ Kruskal diagram (left) and Penrose diagram (right) of the fully extended BTZ black hole. The red curves are the singularities at $r = 0$ while the black curves are the asymptotic boundaries at infinity. The diagonal lines are the future and past horizons.

It is worth noting that the past and future horizons are both covered by the maximal slicing solution, but their presence is not apparent at all in the spacetime metric (107) in the $(\bar{t}, \theta, \varphi)$ coordinates. The presence and location of the horizons only become obvious after the diffeomorphism to the Kruskal-Szekeres coordinates of the fully extended BTZ black hole.

C. Compatibility with the asymptotic AdS boundary conditions

The part $\theta > 0$ of the spatial slice Σ contains the right boundary $\theta = +\pi/2$ and the areal radial coordinate r is a good coordinate in this patch. In the $r \to \infty$ limit, we approach the right boundary and we check whether the

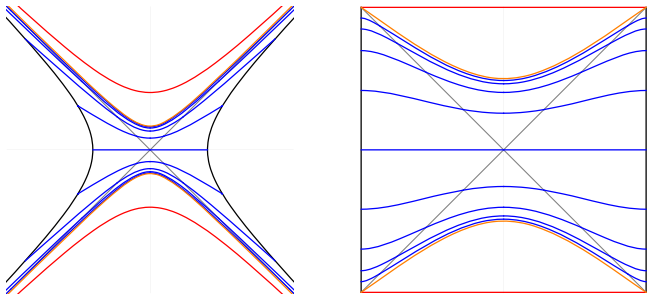


FIG. 9. Plots of constant $\bar{t} = 0, \pm 0.5, \pm 1, \pm 1.5, \pm 2$ slices, i.e., maximal slices (solid blue lines) in the BTZ Kruskal diagram (left) and Penrose diagram (right) of the fully extended BTZ black hole. The red curves are the singularities at $r = 0$ while the black curves are the asymptotic boundaries at infinity. As $\bar{t} \to \infty$, the final slice approaches $R_\infty = \ell\sqrt{M/4\pi}$ (orange curve) from below. Note that the figures are symmetric around the reflection slice $\bar{t} = 0$.

metric, momentum, lapse and shift written above approach the asymptotic AdS values in Eq. (14) in Sec. III. In a neighborhood of $r \to \infty$, we have the following asymptotic expressions for the metric:

$$g_{rr} = \frac{\ell^2}{r^2}\left(1 + \frac{R_h^2}{r^2} + \mathcal{O}(r^{-4})\right), \quad g_{\varphi\varphi} = r^2, \quad g_{r\varphi} = 0, \tag{125}$$

for the conjugate momentum:

$$\pi^{rr} = \frac{T(\bar{t})}{\ell}\left(1 - \frac{R_h^2}{2r^2} + \mathcal{O}(r^{-4})\right), \qquad \pi^{r\varphi} = 0,$$
$$\pi^{\varphi\varphi} = -\frac{\ell T(\bar{t})}{r^4}\left(1 + \frac{R_h^2}{2r^2} + \mathcal{O}(r^{-4})\right), \tag{126}$$

and for the lapse N and shift N^i:

$$N(r,\bar{t}) = \frac{r}{\ell}\left(1 - \frac{R_h^2}{2r^2} + \frac{\ell^3\dot{T}(\bar{t})}{3r^3} + \mathcal{O}(r^{-4})\right),$$
$$N^r(r,\bar{t}) = \frac{T(\bar{t})}{\ell}\left(1 - \frac{R_h^2}{2r^2} + \frac{\ell^3\dot{T}(\bar{t})}{3r^3} + \mathcal{O}(r^{-4})\right),$$
$$N^\varphi = 0. \tag{127}$$

As we can see, the conjugate momentum π^{rr} and the shift N^r do not go to zero as $r \to \infty$ but rather to constants which depend on time, in contrast to the AdS values in (14). Further, the other components of momentum and shift fall off with one power of r smaller than the falloff conditions obtained in the classic paper by Brown and Henneaux [18]. This is an intriguing aspect of the maximal slicing gauge, and it seems to require a generalization of the notion of asymptotic AdS boundary conditions. It is interesting to explore this further.

X. DISCUSSION

In this paper, we have solved Einstein's equations in the Hamiltonian formulation in the maximal slicing, spatially harmonic gauge for spacetimes of the form $\Sigma \times \mathbf{R}$ with spatial slices Σ being cylinders. Before solving the dynamical equations, we have obtained a reduced phase space for the theory which is finite dimensional. This reflects the absence of local degrees of freedom in Einstein's theory of gravitation in $2 + 1$ dimensions. The phase space variables (m, p_m) describe global degrees of freedom with the nonzero number m giving the size of the throat of the cylinder in the constant curvature metric $\tilde{g}_{ij}$.

Recall that the lapse and shift for the solutions presented in this paper were solved with the ansatz that they are independent of the angular coordinate φ. The most general solution can be written as a Fourier series in φ:

$$N(\theta,\varphi) = A_0(\theta) + \sum_{n=1}^{\infty} A_n(\theta)\cos n\varphi + B_n(\theta)\sin n\varphi,$$
$$N^i(\theta,\varphi) = C_0^i(\theta) + \sum_{n=1}^{\infty} C_n^i(\theta)\cos n\varphi + D_n^i(\theta)\sin n\varphi, \tag{128}$$

with boundary conditions used in this paper for each of the A_n, B_n and so on. The above represent the most general lapse and shift with nontrivial boundary conditions and hence the most general large diffeomorphisms that preserve the gauge conditions. These are expected to give the generators of the boundary Virasoro algebras (two for each boundary) discovered by Brown and Henneaux [18] and the canonical Poisson brackets of the corresponding Hamiltonians $H[N, N^i]$ should reproduce the Virasoro algebra with the correct central charge $c = 3\ell/2G_N$.

It is also interesting to explore the origin of the (m, p_m) reduced phase space in the Chern-Simons formulation of $(2 + 1)$-dimensional Einstein gravity with negative cosmological constant [53]. This is a Chern-Simons theory with gauge group $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$, and the m and T reduced phase space variables are most likely related to the holonomies of the Chern-Simons gauge fields around the noncontractible cycle of the spatial cylinder.

We would like to make an observation regarding the spectrum of the Hamiltonian we obtain in the quantum theory. Recall that the quantum theory is simply that of a free particle in one dimension with a particular metric on the one-dimensional line. The spectrum of the Hamiltonian is continuous, and we identified the energy eigenvalues with the mass of a corresponding BTZ black hole spacetime. In the context of AdS/CFT, the BTZ black hole in $2 + 1$ dimensions is a state in the two-dimensional unitary SCFT corresponding to the D1–D5 system, and its mass is an eigenvalue of the operator $M = 2(L_0 + \bar{L}_0)$ which has a *discrete* spectrum [54]. For the same reason the Cardy entropy formula $S = 2\pi(\sqrt{Q_1 Q_5 L_0} + \sqrt{Q_1 Q_5 \bar{L}_0})$ evaluates to a discrete answer. However, in the current calculation within effective field theory, we have a continuous spectrum in the quantization of $2 + 1$ Einstein gravity with AAdS boundary conditions. It would be interesting to find ways to remedy this discrepancy in the spectrum of the quantum BTZ black hole in effective field theory and recover the discreteness of the D1–D5 spectrum.

Finally, though we have considered pure gravity in this paper, the methods used can be extended to gravity coupled to matter fields in $2 + 1$ dimensions. In particular, the maximal slicing and spatial harmonic gauges are still a good set of gauge conditions provided that scalar fields (if present) have a nonpositive potential energy density. The reduced phase will be infinite dimensional in this case due to the matter degrees of freedom, though there will still be the finite-dimensional sector from the metric fields.

In a companion paper [12], we consider the simpler problem of a free massive scalar field probe on the classical maximally sliced black hole background obtained in this paper. Since the background solution is completely gauge fixed, there are no residual diffeomorphisms and the scalar field can be treated as a gauge invariant object. We quantize the probe scalar field in this maximal sliced fully extended BTZ black hole background and obtain a unitary Hamiltonian for the scalar field excitations. The scalar field correlation functions computed in this maximally sliced background do not exhibit thermal behavior, in contrast to the well-known case of scalar fields in the exterior region of the BTZ black hole.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: THE LICHNEROWICZ EQUATION

Here we review the proof of the existence and uniqueness of the solution to the Lichnerowicz solution in $d = 2$ using the monotone iteration method [38,45–48]. We only consider pure gravity here, but the proof can be extended to the case with scalar fields provided the potential satisfies $V(\phi) \leq 0$. We write the equations in terms of the extrinsic curvature

$$\tilde{K}^{ij} = \tilde{g}^{-1/2}(\tilde{\pi}^{ij} - \tilde{g}^{ij}\pi). \tag{A1}$$

The Lichnerowicz equation is

$$2\tilde{D}^2\lambda = \tilde{R} - \tilde{g}_{ik}\tilde{g}_{jl}\tilde{K}^{ij}\tilde{K}^{kl}\mathrm{e}^{-2\lambda} - 2\Lambda\mathrm{e}^{2\lambda}. \tag{A2}$$

Here, $\tilde{g}_{ij}$ is a metric with Ricci scalar $\tilde{R} = 2\Lambda$. Let us define

$$H(\lambda, x) \equiv \frac{1}{2}(\tilde{R} - \tilde{g}_{ik}\tilde{g}_{jl}\tilde{K}^{ij}\tilde{K}^{kl}\mathrm{e}^{-2\lambda} - 2\Lambda\mathrm{e}^{2\lambda}). \tag{A3}$$

Let $\lambda_\pm$ be a supersolution and subsolution, respectively; i.e., they satisfy

$$\tilde{D}^2\lambda_- \geq H(\lambda_-, x), \qquad \tilde{D}^2\lambda_+ \leq H(\lambda_+, x). \tag{A4}$$

What are appropriate sub- and supersolutions for the current problem? When λ is large and negative, the $\mathrm{e}^{2\lambda}$ in (A3) is negligible and the sum of the first two terms is large and negative. Thus, $\lambda_- = C_-$ for a large and negative constant C_- is an appropriate subsolution. Similarly, when λ is large and positive, the $\mathrm{e}^{2\lambda}$ term dominates if $V(\phi) \leq 0$; hence, an appropriate supersolution is $\lambda_+ = C_+$ for a large and positive constant C_+. Since the boundary condition on λ is that $\lambda \to 0$ on the boundaries, we need to take $\lambda = C_\pm$ till some cutoff near the boundaries and then smoothly, monotonically interpolate to zero as one approaches either boundary.

Next, define $H_c(\lambda, x) = H(\lambda, x) - c\lambda$. For a large and positive constant c, $H_c(\lambda, x)$ is a monotonically decreasing function in the interval $[C_-, C_+]$. Define a sequence of functions $\lambda_0, \lambda_1, \ldots$, with $\lambda_0 = \lambda_+$, and

$$(\tilde{D}^2 - c)\lambda_n = H_c(\lambda_{n-1}, x) \quad \text{for } n \geq 1. \tag{A5}$$

The above equations have unique solutions since $\tilde{D}^2 - c$ is negative definite. Suppose for some n, $\lambda_n \leq \lambda_+$. This certainly true for $n = 0$. Then, we have

$$\begin{aligned}(\tilde{D}^2 - c)(\lambda_{n+1} - \lambda_+) &= H_c(\lambda_n, x) - \tilde{D}^2\lambda_+ + c\lambda_+ \\ &\geq H_c(\lambda_n, x) - H(\lambda_+, x) + c\lambda_+ \\ &= H_c(\lambda_n, x) - H_c(\lambda_+, x) \geq 0. \end{aligned} \tag{A6}$$

Suppose the maximum of $\lambda_{n+1} - \lambda_+$ is positive and is attained at some point x_0. Then, at x_0, $\tilde{D}^2(\lambda_{n+1} - \lambda_+)$ is negative, and $-c(\lambda_{n+1} - \lambda_+)$ is negative as well. Thus, $(\tilde{D}^2 - c)(\lambda_{n+1} - \lambda_+)$ is negative as well, and it contradicts the above inequality. Thus, the maximum of $\lambda_{n+1} - \lambda_+$ is negative and hence $\lambda_{n+1} \leq \lambda_+$. Thus, by induction, for all n, we have $\lambda_n \leq \lambda_+$.

Similarly, suppose for some n, we have $\lambda_- \leq \lambda_n$. Then,

$$\begin{aligned}(\tilde{D}^2 - c)(\lambda_- - \lambda_{n+1}) &= \tilde{D}^2\lambda_- - c\lambda_- - H_c(\lambda_n, x) \\ &\geq H(\lambda_-, x) - c\lambda_- - H_c(\lambda_n, x) \\ &= H_c(\lambda_-, x) - H_c(\lambda_n, x) \geq 0. \end{aligned} \tag{A7}$$

If $\lambda_- - \lambda_{n+1}$ has a maximum which is positive, and it is attained at some point x_0', then $\tilde{D}^2(\lambda_- - \lambda_{n+1}) < 0$ at that point, and so is $-c(\lambda_- - \lambda_{n+1})$. This contradicts the above inequality and hence $\lambda_- - \lambda_{n+1} \leq 0$. Thus, we have shown by induction in n that

$$\lambda_- \leq \lambda_n \leq \lambda_+ \quad \text{for all } n \geq 0. \tag{A8}$$

Next, suppose, for some n, $\lambda_n \leq \lambda_{n-1}$. This certainly true for $n = 1$. Then,

$$(\tilde{D}^2 - c)(\lambda_{n+1} - \lambda_n) = H_c(\lambda_n, x) - H_c(\lambda_{n-1}, x) \geq 0. \tag{A9}$$

If the maximum of $\lambda_{n+1} - \lambda_n$, attained at a point x_0'', is positive, then $(\tilde{D}^2 - c)(\lambda_{n+1} - \lambda_n) \leq 0$ at that point which

is in contradiction with the above inequality. Thus, $\lambda_{n+1} \leq \lambda_n$. By induction, we have proved that

$$\lambda_{n+1} \leq \lambda_n \quad \text{for all } n \geq 0. \tag{A10}$$

Since the sequence $\{\lambda_n\}_{n\geq 0}$ is bounded and monotonically decreasing, it has a limit

$$\lim_{n\to\infty} \lambda_n \equiv \lambda,$$

which satisfies the limit of the iterated equation

$$(\tilde{D}^2 - c)\lambda = H_c(\lambda, x), \tag{A11}$$

which implies the Lichnerowicz equation $\tilde{D}^2\lambda = H(\lambda, x)$. Thus, a solution λ exists.

To show the uniqueness of the solution, let us first Weyl transform back to $g_{ij} = \mathrm{e}^{2\lambda}\tilde{g}_{ij}$ and $K^{ij} = \mathrm{e}^{-4\lambda}\tilde{K}^{ij}$, using the solution λ which we know exists. Suppose there is another solution $\lambda' \neq 0$. Then, there is another configuration $g'_{ij} = \mathrm{e}^{2\lambda'}\tilde{g}_{ij}$, $K'^{ij} = \mathrm{e}^{-4\lambda'}\tilde{K}^{ij}$, $\phi' = \phi$, which satisfies the Hamiltonian constraint. The two solutions are connected by the Weyl transformation $\mathrm{e}^{2\lambda''} = \mathrm{e}^{2(\lambda-\lambda')}$ such that $g'_{ij} = \mathrm{e}^{2\lambda''}g_{ij}$ and $K'^{ij} = \mathrm{e}^{-4\lambda''}K^{ij}$. The Hamiltonian constraint for g'_{ij},K'^{ij} then gives the Lichnerowicz equation for λ'':

$$2D^2\lambda'' = R - g_{ik}g_{jl}K^{ij}K^{kl}\mathrm{e}^{-2\lambda''} - 2\Lambda\mathrm{e}^{2\lambda''}. \tag{A12}$$

Here D_i is the covariant derivative compatible with g_{ij}. Since g_{ij} and K^{ij} satisfy the Hamiltonian constraint, we substitute it into the above equation to get

$$0 = 2D^2\lambda'' + g_{ik}g_{jl}K^{ij}K^{kl}(\mathrm{e}^{-2\lambda''} - 1) + 2\Lambda(\mathrm{e}^{2\lambda''} - 1). \tag{A13}$$

Note that $\Lambda < 0$. If the maximum of λ'' is strictly positive, then at the maximum point, we have $D^2\lambda'' \leq 0$, as well as $\mathrm{e}^{-2\lambda''} > 1$, so that the right-hand side is strictly negative which is a contradiction. Thus, $\lambda'' \leq 0$. Similarly, if the minimum of λ'' is strictly negative, then the right-hand side is > 0 which is again a contradiction. Thus, $\lambda'' \geq 0$. The only possibility is then $\lambda'' = 0$ so that $\lambda = \lambda'$ and there is no other distinct configuration g'_{ij}, K'^{ij} that satisfies the Hamiltonian constraint starting from $\tilde{g}_{ij}$ and $\tilde{K}^{ij}$.

APPENDIX B: THE MODULI SPACE OF CONSTANT NEGATIVE CURVATURE METRICS ON THE CYLINDER

Any two-dimensional surface with constant negative curvature can be obtained as the quotient of the hyperbolic plane by a particular Fuchsian group, i.e., a discrete subgroup of $\mathrm{PSL}(2, \mathbf{R})$ (this follows from an application of the Killing-Hopf theorem on the classification of manifolds with constant curvature to two-dimensional surfaces with constant negative curvature; see Sec. II in [55] and the recent work [56] for a nice review). In particular, for the cylinder $\mathbf{S}^1 \times \mathbf{R}$, any metric with constant negative curvature $-2/\ell^2$ is obtained by considering the quotient of the hyperbolic upper half-plane by a cyclic Fuchsian group generated by the hyperbolic transformation with length $2\pi|m|$ for some nonzero real number m. By conjugation by an appropriate $\mathrm{PSL}(2, \mathbf{R})$ element, this hyperbolic transformation can be taken to be the Möbius transformation

$$T = \begin{pmatrix} \mathrm{e}^{\pi|m|} & 0 \\ 0 & \mathrm{e}^{-\pi|m|} \end{pmatrix}, \tag{B1}$$

and the corresponding Fuchsian group is the cyclic group $\langle T \rangle$ generated by T. The fundamental domain for this action in the hyperbolic upper half-plane is the strip $1 \leq |z| < \mathrm{e}^{2\pi|m|}$. Under the identification by the cyclic group, (i) the two semicircles are mapped to each other, (ii) the arcs $-\mathrm{e}^{2\pi|m|} < x \leq -1$ and $1 \leq x < \mathrm{e}^{2\pi|m|}$ on the real line become the boundary circles, and (iii) the segment $1 \leq y < \mathrm{e}^{2\pi|m|}$ on the positive y axis becomes the closed geodesic of length $2\pi\ell|m|$ as measured by the hyperbolic metric $\ell^2 y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$.

The induced metric on the cylinder is obtained by first performing the coordinate transformation $x = \rho\sin\theta$, $y = \rho\cos\theta$, with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ so that we get $\ell^2 y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2) = \cos^{-2}\theta(\mathrm{d}\theta^2 + \rho^{-2}\mathrm{d}\rho^2)$. Next, we write $\rho = \mathrm{e}^{|m|\varphi}$ so that the metric becomes

$$\mathrm{d}\tilde{s}^2 = \frac{\ell^2}{\cos^2\theta}(\mathrm{d}\theta^2 + m^2\mathrm{d}\varphi^2). \tag{B2}$$

The identification of $\rho = 1$ and $\rho = \mathrm{e}^{2\pi|m|}$ implies that φ is an angular coordinate with period 2π.

Thus, the metrics with constant negative curvature on the cylinder are completely parametrized by the nonzero number m.

We can also obtain this result by directly solving the spatial harmonic gauge conditions and the constant curvature equation $\tilde{R} = 2\Lambda$. It is a well-known result that metric $\tilde{g}_{ij}$ can be written locally (possibly after a diffeomorphism) as $\tilde{g}_{ij} = \mathrm{e}^{2\phi}f_{ij}$ with f_{ij} a constant diagonal matrix, in which case $\tilde{g}_{ij}$ satisfies the spatial harmonic gauge condition. We make the ansatz that the above decomposition holds on the entire cylinder Σ and that the coordinate system in which it holds is the (θ, φ) coordinate system described above. The metric $\tilde{g}_{ij}$ is then of the form

$$\tilde{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{e}^{2\phi}f_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{e}^{2\phi}(a\mathrm{d}\theta^2 + b\mathrm{d}\varphi^2), \tag{B3}$$

where a and b are positive constants. Plugging the above into the constant curvature equation $\tilde{R} = 2\Lambda$ and using

the fact that the metric f_{ij} has zero curvature, we get

$$2\Lambda = -2e^{-2\phi}\left(\frac{1}{a}\partial_\theta^2 + \frac{1}{b}\partial_\varphi^2\right)\phi. \tag{B4}$$

It is easy to find a solution that is independent of the angular coordinate φ. In this case, the equation becomes, using $\Lambda = -1/\ell^2$,

$$\frac{d^2\phi(\theta)}{d\theta^2} = \frac{a}{\ell^2}e^{2\phi(\theta)}. \tag{B5}$$

It is easy to see that the solution is

$$e^{2\phi(\theta)} = \frac{\ell^2}{a\cos^2\theta}. \tag{B6}$$

It is also easy to see via an application of the maximum principle for elliptic differential equations that there is no other solution to Eq. (B4). Suppose there is another solution ϕ'. Then, there is another metric $g'_{ij} = e^{2\phi'}f_{ij}$ which satisfies $R' = 2\Lambda$ (R' is the scalar curvature of g'_{ij}). Let $\phi'' = \phi - \phi'$. Then, $g' = e^{2\phi''}\tilde{g}$, which gives

$$R' = e^{-2\phi''}(\tilde{R} - 2\tilde{D}^2\phi''), \tag{B7}$$

where $\tilde{D}_i$ is the covariant derivative compatible with $\tilde{g}_{ij}$. Plugging in $R' = \tilde{R} = 2\Lambda$, we get

$$\tilde{D}^2\phi'' + \Lambda(e^{2\phi''} - 1) = 0. \tag{B8}$$

Suppose that the maximum of ϕ'' is strictly positive. At the maximum point, we have $\tilde{D}^2\phi'' \le 0$ and $\Lambda(e^{2\phi''} - 1) < 0$ so that the lhs is strictly negative which means the equation is not satisfied at such a maximum point. Thus, the maximum value has to zero or negative and hence $\phi'' \le 0$ everywhere. Applying a similar argument to the minimum value of ϕ'', we get $\phi'' \ge 0$ everywhere. The only consistent solution is then $\phi'' = 0$ everywhere, i.e., $\phi = \phi'$ everywhere, which shows that the solution in (B6) is unique.

Thus, we get

$$\tilde{g}_{ij}dx^i dx^j = \frac{\ell^2}{\cos^2\theta}\left(d\theta^2 + \frac{b}{a}d\varphi^2\right). \tag{B9}$$

The constant b/a can be identified with the constant m^2 of the previous derivation.

A coordinate system in which the noncompactness of $\mathbf{R}\times\mathbf{S}^1$ is more manifest is obtained by setting $\tan\theta = \sinh u$ with $u \in (-\infty, \infty)$ so that the metric becomes

$$d\tilde{s}^2 = \ell^2(du^2 + m^2\cosh^2 u d\varphi^2). \tag{B10}$$

A more suggestive coordinate system, which covers only half the cylinder $u > 0$ or $u \le 0$, is the areal radial coordinate $r = \ell m\cosh u$ with $r \in (\ell m, \infty)$. In terms of the areal radial coordinate, the metric becomes

$$d\tilde{s}^2 = \frac{dr^2}{\frac{r^2}{\ell^2} - m^2} + r^2 d\varphi^2, \tag{B11}$$

which we recognize to be the exterior metric on a constant time slice of the BTZ black hole [6,10]. Starting from here, one can obtain wormholelike coordinates which cover the entire cylinder [57–59] by writing $w^2 = r^2 - \ell^2 m^2 = \ell^2 m^2\sinh^2 u$:

$$d\tilde{s}^2 = \ell^2\left(\frac{dw^2}{w^2 + \ell^2 m^2} + (w^2 + \ell^2 m^2)d\varphi^2\right). \tag{B12}$$

APPENDIX C: COMPARISON WITH KUCHAR'S PHASE SPACE FOR SPHERICALLY SYMMETRIC SPACETIME CONFIGURATIONS

In [9], Kuchar solved the Hamiltonian and momentum constraints for the configuration space of $(3+1)$-dimensional spherically symmetric asymptotically flat geometries with two asymptotically flat boundaries. Spherical symmetry implies that the momentum constraints corresponding to the two sphere coordinates are automatically satisfied. He then solved the Hamiltonian constraint and the one remaining momentum constraint to obtain a finite-dimensional reduced phase space.

The same procedure can be repeated for spherically symmetric AdS black holes in $2+1$ dimensions with asymptotic AdS boundaries with minor modifications. The result of following Kuchar's procedure of solving the constraints is again a finite- (two-) dimensional reduced phase space parametrized by one coordinate $\boldsymbol{M}$ and the conjugate momentum $\boldsymbol{P}$. The Hamiltonian turns out to be $(c + \tilde{c})\boldsymbol{M}$ and it arises from the ADM boundary terms as obtained previously in Sec. VII.

The variable $\boldsymbol{M}$ has the interpretation that it becomes the ADM mass of the spacetime solution measured at either of the asymptotic boundaries. The variable $\boldsymbol{P}$ is related to the difference in the asymptotic AdS time coordinates associated to either AAdS boundary. These interpretations are borne out of the detailed expressions for $\boldsymbol{M}$ and $\boldsymbol{P}$ in Kuchar's analysis; we have only presented the final results here.

The action in Kuchar's reduced phase space is

$$\mathcal{S}[\boldsymbol{P},\boldsymbol{M}] = \int d\bar{t}(\boldsymbol{P}\dot{\boldsymbol{M}} - (c + \tilde{c})\boldsymbol{M}). \tag{C1}$$

The equations of motion are trivial to solve and we get

$$\boldsymbol{M}(\bar{t}) = \frac{M}{\kappa^2} = \text{const}, \qquad \boldsymbol{P}(\bar{t}) = -(c+\tilde{c})\bar{t}. \quad \text{(C2)}$$

The obvious question to ask is if there is a classical canonical transformation from the coordinates $(\boldsymbol{P},\boldsymbol{M})$ to the coordinates (Q,P) that we have obtained in the present work by complete gauge fixing of the Hamiltonian and momentum constraints. Note that we have also restricted ourselves to the spherically symmetric sector when we chose the solutions of constraints and gauge fixing conditions to be independent of the angular variable φ.

It turns out that a canonical transformation indeed exists, and, in fact, we already have written the formula for $\boldsymbol{M}$ in terms of m, p_m in (59), which is

$$\boldsymbol{M}(Q,P) = \frac{P^2 Q}{2\pi\ell}\left(k_*\left(\frac{Q}{4\pi^2\ell}\right) + k_*\left(\frac{Q}{4\pi^2\ell}\right)^{-1}\right). \quad \text{(C3)}$$

With some more straightforward manipulations of Kuchar's formulas, the expression for $\boldsymbol{P}$ in terms of (Q,P) can also be obtained:

$$\boldsymbol{P} = \frac{\ell\,\mathrm{sgn}(Q)}{P}\fint_{-\pi/2}^{\pi/2}\frac{\mathrm{d}\theta}{\mathrm{e}^{2\bar{\lambda}} - (|k_*| + |k_*|^{-1})}, \quad \text{(C4)}$$

where recall that $\bar{\lambda}$ is the solution the Lichnerowicz equation. Note that there is a simple pole at $\bar{\lambda} = \frac{1}{2}\log(k_* + k_*^{-1})$ and the integral has to be evaluated by the principal value prescription.

Thus, there exists a simpler set of coordinates on the gauge fixed reduced phase space in which the phase space trajectories are straight lines parallel to the $\boldsymbol{P}$ axis.

We also give a geometrical argument why $(Q,P) \to (\boldsymbol{M},\boldsymbol{P})$ should be a canonical transformation. The area of an arbitrary region of the phase space is preserved under canonical transformations. We now consider the area below a phase space trajectory, labeled by the mass M, in both sets of coordinates and show that they are the same for all phase space trajectories.

The area under a phase space trajectory in terms of (Q,P) coordinates is given by (see Fig. 5 for reference)

$$\frac{1}{\kappa^2}\int \mathrm{d}QP(Q) = \frac{1}{\kappa^2}\int \mathrm{d}\bar{t}|\dot{Q}|P(\bar{t}) = (c+\tilde{c})\frac{M}{\kappa^2}\int \mathrm{d}\bar{t}. \quad \text{(C5)}$$

In the second step we switched to time $\bar{t}$ as the integration variable and in the third step we have used the Hamilton equation $\dot{Q} = (c+\tilde{c})M/P$. Similarly, the area under constant $\boldsymbol{M}$ straight lines is

$$\int \mathrm{d}\boldsymbol{P}\boldsymbol{M} = \int \mathrm{d}\bar{t}|\dot{\boldsymbol{P}}|\boldsymbol{M} = (c+\tilde{c})\int \mathrm{d}\bar{t}\boldsymbol{M}, \quad \text{(C6)}$$

using $\dot{\boldsymbol{P}} = -(c+\tilde{c})$ in the last step. Since $\boldsymbol{M} = M/\kappa^2$ on shell, clearly the two areas are equal:

$$\lim_{\mathcal{T}\to\infty}\frac{\int_{-\mathcal{T}}^{\mathcal{T}}\mathrm{d}p_m m(p_m)}{\int_{-\mathcal{T}}^{\mathcal{T}}\mathrm{d}\boldsymbol{P}\boldsymbol{M}} = \lim_{\mathcal{T}\to\infty}\frac{(c+\tilde{c})\frac{M}{\kappa^2}\int_{-\mathcal{T}}^{\mathcal{T}}\mathrm{d}\bar{t}}{(c+\tilde{c})\boldsymbol{M}\int_{-\mathcal{T}}^{\mathcal{T}}\mathrm{d}\bar{t}} = 1 \quad \text{(C7)}$$

for any value of M, confirming that $(Q,P) \to (\boldsymbol{M},\boldsymbol{P})$ is indeed a canonical transformation.

APPENDIX D: DIFFEOMORPHISM TO THE KRUSKAL EXTENSION OF THE BTZ BLACK HOLE

In this appendix, we show that the maximally sliced spacetime (107) obtained by solving the Hamiltonian equations of motion in the maximal slicing, spatial harmonic gauge is diffeomorphic to a region R_0 of the Kruskal extension of the BTZ black hole. We reproduce the solution below for convenience:

$$\mathrm{d}s^2 = -N(\bar{t},\theta)^2\mathrm{d}\bar{t}^2 + \frac{\ell^2\mathrm{e}^{2\lambda(\bar{t},\theta)}}{\cos^2\theta}(\mathrm{d}\theta^2 + N^\theta(\bar{t},\theta)\mathrm{d}\bar{t})^2 + \frac{\ell^2\mathrm{e}^{2\lambda(\bar{t},\theta)}}{\cos^2\theta}m(\bar{t})^2\mathrm{d}\varphi^2. \quad \text{(D1)}$$

We first introduce an areal radial coordinate in the solution (107). Using the areal radial coordinate, we describe the diffeomorphism to the fully extended BTZ black hole.

1. The maximal slicing solution in areal radial coordinates

The areal radial coordinate is defined by $r^2 = g_{\varphi\varphi}(\bar{t},\theta)$ with $g_{\varphi\varphi}$ obtained from the maximally sliced solution (D1). We get

$$r(\theta,\bar{t}) = \mathrm{e}^{\lambda(\theta,\bar{t})}\frac{\ell m(\bar{t})}{\cos\theta} = \mathrm{e}^{\bar{\lambda}(\theta,\bar{t})}\sqrt{\ell|T(\bar{t})|}, \quad \text{(D2)}$$

where we have used the definition $\bar{\lambda} = \lambda - \log\cos\theta - \frac{1}{2}\log\frac{|T|}{\ell m^2}$. Recall that λ is the unique solution to the Lichnerowicz equation (32), and it is an even function of θ. This implies that the above relation is a one-to-one coordinate transformation either in the range $\theta > 0$ or in the range $\theta < 0$. The range of r that corresponds to $0 < \theta < \frac{\pi}{2}$ is

$$R_+(\bar{t}) = \sqrt{\frac{\ell|T|}{|k_*|}} < r < \infty. \quad \text{(D3)}$$

Thus, we need two areal radial coordinates r and $\tilde{r}$ with identical ranges to describe the entire spatial slice:

$$r = \mathrm{e}^{\lambda(\theta)}\frac{\ell m}{\cos\theta} = \mathrm{e}^{\bar{\lambda}(\theta)}\sqrt{\ell|T|} \quad \text{for } \theta > 0,$$
$$\tilde{r} = \mathrm{e}^{\lambda(\theta)}\frac{\ell m}{\cos\theta} = \mathrm{e}^{\bar{\lambda}(\theta)}\sqrt{\ell|T|} \quad \text{for } \theta < 0. \quad \text{(D4)}$$

In terms of the areal radial coordinate, the metric on Σ takes the form

$$g_{ij}\mathrm{d}x^i\mathrm{d}x^j = A(r,\bar{t})\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2, \tag{D5}$$

with

$$A(r,\bar{t}) = g_{rr} = \left(\frac{r^2}{\ell^2} - \frac{M}{2\pi} + \frac{T(\bar{t})^2}{r^2}\right)^{-1}, \tag{D6}$$

the extrinsic curvature K_{ij} is

$$K_{rr} = \frac{T(\bar{t})A(r,\bar{t})}{r^2}, \quad K_{r\varphi} = 0, \quad K_{\varphi\varphi} = -T(\bar{t}), \tag{D7}$$

and the conjugate momentum π^{ij} is

$$\pi^{rr} = \frac{T(\bar{t})}{rA(r,\bar{t})^{1/2}}, \qquad \pi^{r\varphi} = 0,$$
$$\pi^{\varphi\varphi} = -\frac{T(\bar{t})A(r,\bar{t})^{1/2}}{r^3}. \tag{D8}$$

There are similar expressions in terms of $\tilde{r}$ for the $\theta < 0$ region of Σ.

Note that the above expressions are valid only for the range (115). The function $A(r,\bar{t})$ (D6) has poles at the roots of the denominator $r = R_{\pm}(\bar{t})$ which have the expressions

$$R_{\pm}(\bar{t}) = \sqrt{\ell|T||k_*|^{\mp 1}} = \ell\sqrt{\frac{M}{4\pi}}\left(1 \pm \sqrt{1 - \frac{16\pi^2T^2}{M^2\ell^2}}\right)^{1/2}. \tag{D9}$$

The parameter $M/2\pi$ is the ADM mass of the solution. The roots satisfy $R_+(\bar{t}) \geq R_-(\bar{t})$ with equality satisfied at $\bar{t} \to \infty$. Hence, the second root $R_-(\bar{t})$ lies outside the range (115).

The lapse is given by

$$N(r,\bar{t}) = \sqrt{\frac{r^2}{\ell^2} - \frac{M}{2\pi} + \frac{T(\bar{t})^2}{r^2}} \times \left(1 + \dot{T}(\bar{t})\int_\infty^r \frac{\mathrm{d}\rho}{\rho\left(\frac{\rho^2}{\ell^2} - \frac{M}{2\pi} + \frac{T(\bar{t})^2}{\rho^2}\right)^{3/2}}\right), \tag{D10}$$

which is valid for the range (115) and the same expression in terms of $\tilde{r}$ for $\theta < 0$.

The nontrivial component of the shift is N^θ which can be transformed to areal radial coordinates according to the following rule:

$$\frac{\partial r}{\partial\theta}N^\theta = \frac{\partial r}{\partial\bar{t}} + N^r. \tag{D11}$$

The term $\partial r/\partial\bar{t}$ on the right-hand side is present to account for the change in the relation of the areal radial coordinate r to θ under time evolution. The shift in areal radial coordinates comes out to be

$$N^r = \frac{NT}{r}, \qquad N^\varphi = 0. \tag{D12}$$

The spacetime metric corresponding to a given trajectory in reduced phase space is given by

$${}^{(2+1)}\mathrm{d}s^2 = -N^2(r,\bar{t})\mathrm{d}\bar{t}^2 + A(r,\bar{t})\left(\mathrm{d}r + \frac{N(r,\bar{t})T(\bar{t})}{r}\mathrm{d}\bar{t}\right)^2 + r^2\mathrm{d}\varphi^2. \tag{D13}$$

The collapse of the lapse that was described in (84) of the previous section implies that the above solution approaches a limit surface $r_{\min} = \lim_{\bar{t}\to\infty} R_+(\bar{t}) = \ell\sqrt{M/4\pi}$ as $\bar{t} \to \infty$. In particular, the local spatial volume governed by $\sqrt{g} = r\sqrt{A}$ never goes to zero since r never reaches 0. This behavior is typical of maximal slices since they tend to maximize the local volume, whereas going to smaller and smaller values tends to minimize the local volume. In higher dimensions, where $r = 0$ corresponds to a genuine curvature singularity of the volume-crushing type (e.g., the AdS-Schwarzschild solution in $3+1$ dimensions), the maximally sliced solution with wormhole spatial slices avoids the singularity altogether.

2. The diffeomorphism

The final metric (D13) is written in terms of an areal radial coordinate. This facilitates an easy map to the BTZ black hole metric in region I which is also in terms of an areal radial coordinate:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\varphi^2, \quad f(r) = \frac{r^2}{\ell^2} - \frac{M}{2\pi}. \tag{D14}$$

The diffeomorphism from region I of the BTZ black hole to the maximally sliced solution is then

$$(r,\varphi,t) = (r,\varphi,t = t(\bar{t},r)). \tag{D15}$$

Plugging $t = t(\bar{t},r)$ into the BTZ metric (108) and comparing with the maximal slicing metric (D13), we get the equations

$$\left(\frac{\partial t}{\partial\bar{t}}\right)^2 = N(r,\bar{t})^2A(r,\bar{t}), \quad \left(\frac{\partial t}{\partial r}\right)^2 = \frac{T(\bar{t})^2A(r,\bar{t})}{r^2f(r)^2}. \tag{D16}$$

In taking the square roots, we have a choice of a sign. The signs in region I have to be chosen such that increasing t corresponds to increasing $\bar{t}$ (with r fixed) and decreasing r (with $\bar{t}$ fixed). This gives, in region I

with $\bar{t} > 0$,

$$\frac{\partial t}{\partial \bar{t}} = N(r,\bar{t})A(r,\bar{t})^{1/2}, \qquad \frac{\partial t}{\partial r} = -\frac{T(\bar{t})A(r,\bar{t})^{1/2}}{rf(r)}. \quad \text{(D17)}$$

As $r \to \infty$, we have $N\sqrt{A} \to 1$. This allows the boundary condition

$$\lim_{r\to\infty} t(r,\bar{t}) = \bar{t} + t_0 \quad \text{(D18)}$$

for some arbitrary constant t_0. We typically choose $t_0 = 0$ so that the origin of time in t and $\bar{t}$ coincide. The diffeomorphism in region I is given by integrating the second equation at constant $\bar{t}$:

$$t(r,\bar{t}) = \bar{t} + T(\bar{t}) \int_r^\infty \frac{\mathrm{d}\rho}{\rho} \frac{A(\rho,\bar{t})^{1/2}}{f(\rho)}, \qquad r > R_h. \quad \text{(D19)}$$

It is easy to see that differentiating the above solution at constant $\bar{t}$ reproduces the expression for $N(r,\bar{t})A(r,\bar{t})^{1/2}$.

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$$\{\mathcal{H}_\perp(x), \mathcal{H}_\perp(x')\} = (g^{ij}\mathcal{H}_j(x) + g^{ij}\mathcal{H}_j(x'))\delta_{,i}(x,x'),$$
$$\{\mathcal{H}_i(x), \mathcal{H}_\perp(x')\} = \mathcal{H}_\perp(x)\delta_{,i}(x,x'),$$
$$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_i(x)\delta_{,j}(x,x') + \mathcal{H}_j(x)\delta_{,i}(x,x'),$$

where the derivatives on the δ functions are all with respect to the first argument.
[41] We thank E. Witten for pointing this out.

[42] For spatial metrics g_{ij} which approach the AdS_d spatial metric $\frac{dr^2}{r^2} + r^2 d\Omega_{d-1}^2$, the falloff condition on $\xi_\perp$ that ensures that the boundary contribution is zero is $\xi_\perp \sim r^{-(d-1)/2-\epsilon}$.
[43] The above analysis implies that the matrix of Poisson brackets $\mathcal{M}_{AB} = \{\mathcal{C}_A, \mathcal{C}_B\}$ is invertible, where $\mathcal{C}_A$ collectively denotes all constraints and gauge conditions. To analyze the Hamiltonian dynamics, one can alternatively work with the original phase but with the Poisson bracket replaced by the Dirac bracket constructed out of the above matrix $\mathcal{M}_{AB}$ in the standard way.
[44] Given any metric g'_{ij} whose Ricci scalar R' approaches 2Λ near the boundary, we can always find a unique ϕ such that $\tilde{g}_{ij} = e^{2\phi} g'_{ij}$ has constant Ricci scalar curvature 2Λ everywhere. This ϕ is the unique solution of the Yamabe equation $D'^2\phi = \frac{1}{2}R' - \Lambda e^{2\phi}$. See [38,45–48] for more details.
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