

Vacuum polarization energy of a nonzero radius cosmic string induced spacetime

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We calculate the vacuum polarization energy (VPE) of a scalar field in the background of a nonsingular cosmic string in $2 + 1$ spacetime dimensions. Our calculation expresses the VPE as a renormalized sum and integral over scattering data, which can be expressed in terms of Legendre and Bessel functions for the “ballpoint pen” model, analogous to a square well in curvature, and which can be obtained numerically for a generic string profile. We show how relationships between the local density of states, expressed in terms of the Green’s function, and the global density of states, expressed in terms of the Jost function, extend to this curved spacetime background and allow for precise implementation of perturbative renormalization conditions.

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I. INTRODUCTION

While there does not exist a fully quantized theory of gravity, it does appear possible to extend quantum field theory to a classical curved spacetime background [1], although subtleties such as Hawking radiation and its consequences for black hole information remain unresolved. Examples in which one can compute the contributions of quantum fluctuations to local densities are therefore especially valuable for gaining insight into the interplay between quantum mechanics and general relativity. In the case of the Schwarzschild metric, calculations using scattering Green’s functions such as [2–5] make it possible to demonstrate the emergence of Hawking radiation via a physically motivated choice of vacuum. However, such geometries inevitably contain singular points, making it difficult to consider global quantities that would be obtained via integration over space of local densities, and one is often restricted to calculations of local densities in regions with no matter.

In this paper we consider a simpler geometry in two space dimensions, consisting of a localized defect leading to a deficit angle at large distances. We consider a defect with a core of nonzero radius, leading to a nonsingular configuration in which we can consider regions both with and without matter. In higher dimensions, this

configuration would extend to a cosmic string geometry. We focus especially on the “ballpoint pen” model [6–8], in which a region of constant curvature is smoothly joined to a flat region with a deficit angle, forming the analog of a square well in curvature, and compute the total vacuum polarization energy (VPE) of a scalar field in this background. Local densities in this model were computed previously in [9,10] which, in principle, could be integrated over space to obtain the results found here. However, this integral converges only as a principal value at the step boundary, so it is not tractable to compute it numerically. Instead, we show how to use an analytic relationship between the integral over space of the Green’s function and the Jost function from potential scattering theory to circumvent the need to compute this integral, leading to a fully renormalized calculation in which divergences are removed via local counterterms fixed by standard renormalization conditions. Our calculation draws on scattering data derived in Refs. [11,12], but our results differ because we use only these standard subtractions to implement renormalization.

We begin by introducing the background geometry and scattering problem in Sec. II. We then show in Sec. III how to connect the local density of states, defined via the Green’s function, to the global density of states, defined via the Jost function, which is then used to compute the VPE. We carry out this calculation for the ballpoint pen model in Sec. IV and discuss our results in Sec. V. In Appendix A we explain how to extend all of these results to an arbitrary string profile using a variable phase calculation and in Appendix B we discuss aspects of analytic continuation in wave number k and angular momentum ℓ .

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II. MODEL

We consider a free scalar field ϕ of mass μ in $2+1$ spacetime dimensions, for which the action functional is

$$S = -\frac{1}{2} \int d^3x \sqrt{-g} (\nabla_\alpha \phi \nabla^\alpha \phi + \xi \mathcal{R} \phi^2 + \mu^2 \phi^2), \quad (1)$$

including a coupling to the Ricci curvature scalar $\mathcal{R}$ with strength ξ . Of particular interest is the case of conformal coupling, $\xi = \frac{1}{8}$ in two space dimensions. The equation of motion is

$$-\nabla_\alpha \nabla^\alpha \phi + \xi \mathcal{R} \phi + \mu^2 \phi = 0 \quad (2)$$

and we consider the spacetime metric [6–8]

$$ds^2 = -dt^2 + p(r)^2 dr^2 + r^2 d\theta^2 \quad (3)$$

with a deficit angle $2\theta_0$, meaning that the range of angular coordinate is $\theta = 0 \dots 2(\pi - \theta_0)$, and we define $\sigma = \frac{\pi}{\pi - \theta_0}$. To implement the deficit angle without a singularity at the origin, we introduce a profile function $p(r)$ that ranges from $\frac{1}{\sigma}$ at the origin to 1 at the string radius r_0 and beyond. The nonzero Christoffel symbols in this geometry are [6–8]

$$\Gamma_{rr}^r = \frac{p'(r)}{p(r)}, \quad \Gamma_{\theta\theta}^r = -\frac{r}{p(r)^2}, \quad \text{and} \quad \Gamma_{\theta r}^\theta = \Gamma_{r\theta}^\theta = \frac{1}{r}, \quad (4)$$

and because the geometry only has curvature in two dimensions, all the nonzero components of the Riemann and Ricci tensors

$$\begin{aligned} R_{\theta r \theta}^r &= -R_{\theta \theta r}^r = R_{\theta\theta} = g_{\theta\theta} \frac{\mathcal{R}}{2} \quad \text{and} \\ R_{r\theta r}^\theta &= -R_{r\theta}^\theta = R_{rr} = g_{rr} \frac{\mathcal{R}}{2} \end{aligned} \quad (5)$$

can be expressed in terms of the curvature scalar $\mathcal{R} = \frac{2}{r} \frac{p'(r)}{p(r)^3}$. It obeys the Gauss-Bonnet theorem in two dimensions,

$$\begin{aligned} \int_0^{r_0} p(r) dr \int_0^{2\pi/\sigma} r d\theta \mathcal{R} \\ = \frac{4\pi}{\sigma} \int_0^{r_0} \frac{p'(r)}{p(r)^2} dr \\ = -\frac{4\pi}{\sigma} \frac{1}{p(r)} \Big|_{r=0}^{r=r_0} = 4\pi \left(1 - \frac{1}{\sigma}\right) = 4\theta_0, \end{aligned} \quad (6)$$

for any $p(r)$ with the boundary conditions given above.

Acting on any scalar χ , the covariant derivatives simply become ordinary derivatives, while for second derivatives we have nontrivial contributions from the Christoffel symbols,

$$\begin{aligned} \nabla_\theta \nabla_\theta \chi &= \partial_\theta^2 \chi - \Gamma_{\theta\theta}^r \partial_r \chi, \quad \nabla_r \nabla_r \chi = \partial_r^2 \chi - \Gamma_{rr}^r \partial_r \chi, \quad \text{and} \\ \nabla_r \nabla_\theta \chi &= \nabla_\theta \nabla_r \chi = \partial_\theta \partial_r \chi - \Gamma_{r\theta}^\theta \partial_\theta \chi, \end{aligned} \quad (7)$$

and so covariant derivatives with respect to θ can be nonzero even if χ is rotationally invariant. In particular, we have the Laplace-Beltrami operator

$$(g^{\theta\theta} \nabla_\theta \nabla_\theta + g^{rr} \nabla_r \nabla_r) \chi = \frac{1}{r^2} \left(\frac{\partial^2 \chi}{\partial \theta^2} + \mathcal{D}_r^2 \right) \chi, \quad (8)$$

where $\mathcal{D}_r = \frac{r}{p(r)} \frac{\partial}{\partial r}$ is the radial derivative. The equation of motion for ϕ then becomes

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{r^2} \mathcal{D}_r^2 - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \mu^2 + \xi \mathcal{R} \right) \phi = 0. \quad (9)$$

We define the Green's function $G(\mathbf{r}, \mathbf{r}', k)$ for wave number k , which obeys

$$\begin{aligned} \left(-\frac{1}{r^2} \mathcal{D}_r^2 - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \xi \mathcal{R} - k^2 \right) G(\mathbf{r}, \mathbf{r}', k) \\ = \frac{1}{rp(r)} \delta(r - r') \delta(\theta - \theta'), \end{aligned} \quad (10)$$

and we furthermore define the changes in radius and area as

$$\Delta r = \int_0^\infty (p(r) - 1) dr \quad (11)$$

and

$$\Delta A = \frac{2\pi}{\sigma} \int_0^\infty r(p(r) - 1) dr, \quad (12)$$

respectively.

We can then decompose the Green's function in a partial wave basis as

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}', k) &= \sum_{\ell=0}^\infty {}'G_\ell(\mathbf{r}, \mathbf{r}', k) \\ &= \frac{i\sigma}{2} \sum_{\ell=0}^\infty {}'\psi_{k,\ell}^{\text{reg}}(r_<) \psi_{k,\ell}^{\text{out}}(r_>) \cos[\sigma\ell(\theta - \theta')], \end{aligned} \quad (13)$$

where the subscripts indicate the larger and smaller of the two radii and the prime indicates that the $\ell = 0$ term in the sum is counted with a weight of one-half, which arises because we restrict to $\ell \geq 0$. This sum is written in terms of the regular and outgoing scattering wave functions, which are solutions to the radial wave equation

$$\left(-\frac{1}{r^2}\mathcal{D}_r^2 + \frac{\ell^2\sigma^2}{r^2} + \xi\mathcal{R}\right)\psi_{\ell,k}(r) = k^2\psi_{\ell,k}(r), \quad (14)$$

normalized according to the Wronskian relation

$$\frac{d}{dr}(\psi_{k,\ell}^{\text{reg}}(r))\psi_{k,\ell}^{\text{out}}(r) - \psi_{k,\ell}^{\text{reg}}(r)\frac{d}{dr}(\psi_{k,\ell}^{\text{out}}(r)) = \frac{p(r)}{r}, \quad (15)$$

where the outgoing wave function approaches the Hankel function $H_{\sigma\ell}^{(1)}(kr)$ at large distances.

We consider the ballpoint pen profile function [6–8]

$$p^{\text{pen}}(r) = \begin{cases} \left[\sigma^2 - \frac{r^2}{r_0^2}(\sigma^2 - 1)\right]^{-1/2} & r < r_0 \\ 1 & r > r_0, \end{cases} \quad (16)$$

where r_0 is the string radius, which gives constant curvature $\mathcal{R} = \frac{2(\sigma^2-1)}{r_0^2}$ inside and zero curvature outside. It has geometrical factors

$$\Delta r^{\text{pen}} = \left(\frac{\arctan \sqrt{\sigma^2 - 1}}{\sqrt{\sigma^2 - 1}} - 1\right)r_0 \quad \text{and} \quad \Delta A^{\text{pen}} = -\frac{\pi r_0^2}{\sigma} \frac{\sigma - 1}{\sigma + 1} \quad (17)$$

and its scattering wave functions are given in terms of the Legendre functions of the first and second kind $P_{\nu(k)}^{-\ell}$ and $Q_{\nu(k)}^{\ell}$ and the Bessel and Hankel functions $J_{\sigma\ell}$ and $H_{\sigma\ell}^{(1)}$ by

$$\psi_{k,\ell}^{\text{pen}}(r) = \begin{cases} \theta(r_0 - r)A_{k,\ell}^{\text{pen}}P_{\nu(k)}^{-\ell}\left(\frac{1}{\sigma p^{\text{pen}}(r)}\right) + \theta(r - r_0)[J_{\sigma\ell}(kr) + B_{k,\ell}^{\text{pen}}H_{\sigma\ell}^{(1)}(kr)], & \text{regular solution,} \\ \theta(r_0 - r)[C_{k,\ell}^{\text{pen}}P_{\nu(k)}^{-\ell}\left(\frac{1}{\sigma p^{\text{pen}}(r)}\right) + D_{k,\ell}^{\text{pen}}Q_{\nu(k)}^{\ell}\left(\frac{1}{\sigma p^{\text{pen}}(r)}\right)] + \theta(r - r_0)H_{\sigma\ell}^{(1)}(kr), & \text{outgoing solution} \end{cases} \quad (18)$$

where $\theta(x)$ is the Heaviside step function. We have introduced

$$\nu(k) = -\frac{1}{2} + \frac{1}{2}\sqrt{(1 - 8\xi) + \frac{4k^2r_0^2}{\sigma^2 - 1}} \quad (19)$$

and

$$A_{k,\ell}^{\text{pen}}D_{k,\ell}^{\text{pen}} = \frac{2i(-1)^\ell}{\pi\sigma} \quad (20)$$

$$\begin{aligned} \frac{C_{k,\ell}^{\text{pen}}}{D_{k,\ell}^{\text{pen}}} &= -\frac{(\sigma^2 - 1)Q_{\nu(k)'}^{\ell}\left(\frac{1}{\sigma}\right)H_{\sigma\ell}^{(1)}(kr_0) + \sigma kr_0 Q_{\nu(k)}^{\ell}\left(\frac{1}{\sigma}\right)H_{\sigma\ell'}^{(1)}(kr_0)}{(\sigma^2 - 1)P_{\nu(k)'}^{-\ell}\left(\frac{1}{\sigma}\right)H_{\sigma\ell}^{(1)}(kr_0) + \sigma kr_0 P_{\nu(k)}^{-\ell}\left(\frac{1}{\sigma}\right)H_{\sigma\ell'}^{(1)}(kr_0)} \\ B_{k,\ell}^{\text{pen}} &= -\frac{(\sigma^2 - 1)P_{\nu(k)'}^{-\ell}\left(\frac{1}{\sigma}\right)J_{\sigma\ell}(kr_0) + \sigma kr_0 P_{\nu(k)}^{-\ell}\left(\frac{1}{\sigma}\right)J'_{\sigma\ell}(kr_0)}{(\sigma^2 - 1)P_{\nu(k)'}^{-\ell}\left(\frac{1}{\sigma}\right)H_{\sigma\ell}^{(1)}(kr_0) + \sigma kr_0 P_{\nu(k)}^{-\ell}\left(\frac{1}{\sigma}\right)H_{\sigma\ell'}^{(1)}(kr_0)}, \end{aligned} \quad (21)$$

where a prime denotes a derivative with respect to the function's argument and we have assumed $\ell \geq 0$. While all four coefficients can be obtained by matching the wave function and its first derivative at $r = r_0$, here we show only the combinations of these coefficients that we will need; the product $A_{k,\ell}^{\text{pen}}D_{k,\ell}^{\text{pen}}$ is particularly simple due to the Wronskian relation. We can then extract the Jost function $F_{\ell}(k)$ [13,14] from the $B_{k,\ell}^{\text{pen}}$ coefficient as [11]

$$F_{\ell}^{\text{pen}}(k) = \frac{(kr_0)^{\ell}}{(\sigma^2 - 1)^{\ell/2}} \frac{i^{\ell(\sigma-1)}\pi}{2i\sigma^{3/2}} \left[(\sigma^2 - 1)P_{\nu(k)'}^{-\ell}\left(\frac{1}{\sigma}\right)H_{\sigma\ell}^{(1)}(kr_0) + \sigma kr_0 P_{\nu(k)}^{-\ell}\left(\frac{1}{\sigma}\right)H_{\sigma\ell'}^{(1)}(kr_0) \right] \quad (22)$$

by using

$$1 + 2B_{k,\ell}^{\text{pen}} = \frac{F_{\ell}(-k)}{F_{\ell}(k)}, \quad (23)$$

where the normalization is chosen to give the asymptotic limit

$$\lim_{k \rightarrow \infty} F_\ell(k) e^{ik\Delta r} = 1 \quad (24)$$

with $F(-k) = F(k)^*$ for k real. We define the S -matrix

$$S_\ell(k) = \frac{F(-k)}{F(k)}, \quad (25)$$

which obeys

$$\lim_{k \rightarrow \infty} e^{-2ik\Delta r} S_\ell(k) = 1 \quad \text{and} \quad \lim_{\ell \rightarrow \infty} (-1)^{\ell(\sigma-1)} S_\ell(k) = 1. \quad (26)$$

We note that it is also straightforward to extract analogous scattering data for the “flowerpot” model, in which the curvature is concentrated at a δ -function ring of radius r_0 . However, such a configuration has divergent total energy due to this sharp limit [15]. In Appendix A we explain how to obtain scattering data numerically for an arbitrary continuous profile $p(r)$.

III. POINT STRING AND DENSITY OF STATES

In order to pass from local densities, which can be computed from the Green’s function, to global quantities like the total energy, we connect the integral over space of the Green’s function to the Jost function. More specifically, $\frac{2k}{i}$ times the imaginary part of the Green’s function gives a local density of states, and the corresponding global density of states is given by $\frac{1}{\pi}$ times the k derivative of the Jost function [15]. Since the global density of states diverges with volume, we must consider differences, for example, by subtracting the corresponding results for empty space, to obtain a meaningful result. In curved space, this subtraction is complicated by the changes to the background geometry.

To form a simple comparison, we consider the “point string” configuration [16–18], which is the $r_0 \rightarrow 0$ limit of the ballpoint pen. It consists of flat spacetime with the same deficit angle for $r > 0$ and the associated curvature concentrated at $r = 0$. We can similarly construct its Green’s function from the wave functions

$$\psi_{k,\ell}^{\text{point}}(r) = \begin{cases} J_{\sigma\ell}(kr), & \text{regular solution} \\ H_{\sigma\ell}^{(1)}(kr), & \text{outgoing solution} \end{cases} \quad (27)$$

and we obtain, for $\ell \geq 0$,

$$\begin{aligned} \frac{2k}{i\pi} \int_0^\infty \frac{2\pi r}{\sigma} dr \left(G_\ell(r, r, k) p(r) - G_\ell^{\text{point}}(r, r, k) \right) \\ = \frac{i}{\pi} \frac{d}{dk} \log \left((ikr_0)^{\ell(\sigma-1)} F_\ell(k) \right) \end{aligned} \quad (28)$$

for complex k in the upper half-plane, where the nonzero imaginary part of k is necessary for convergence of the integral on the left-hand side. This formula connects the local and global density of states, generalizing to curved spacetime the relationship between the spatial integral of the Green’s function and the Jost function derived in Ref. [15]. In the limit where k approaches the real axis from above, the real part of this equation represents the density of states in channel ℓ ,

$$\Delta\rho_\ell(k) = \rho_\ell(k) - \rho_\ell^{\text{point}}(k) = \frac{1}{\pi} \frac{d}{dk} \delta_\ell(k) \quad (29)$$

where we have defined the phase shift as

$$\delta_\ell(k) = \frac{1}{2i} \log S_\ell = -\arg F_\ell(k) \quad (30)$$

for k real.

On dimensional grounds, the vacuum energy density of the point string goes like $\frac{1}{r}$ for $\mu = 0$, giving a divergent total energy. Nonetheless, we will be able to use it as an intermediate step when renormalizing the VPE of the ballpoint pen or other nonsingular configurations.

IV. VACUUM POLARIZATION ENERGY

The VPE can be constructed as a sum over the zero-point energies $\frac{1}{2} \sqrt{k^2 + \mu^2}$ for all the modes of oscillation of the scalar field. To put this sum in a tractable form for calculation in an unbounded region, we rewrite it as an integral weighted by the continuum density of states, as calculated above. (We would also need to include the explicit contribution from bound states, but there are none for the configurations we consider.) Such calculations are typically divergent, requiring that we introduce local counterterms with coefficients defined through perturbative renormalization conditions. First, we must make a free space (cosmological constant) subtraction to account for the change in area, which multiplies the change in the free density of states in two dimensions $\Delta\rho^{\text{free}}(k) = 2 \sum_{\ell=0}^\infty \Delta\rho_\ell^{\text{free}}(k) = \frac{k\Delta A}{2\pi}$ for $k \geq 0$. This subtraction represents the difference in total energy between the point string and empty space, which is divergent when integrated over k . As we will see below, when we combine this quantity with the difference between the nonzero radius string and the point string, which has a corresponding divergence as described in the previous section, we obtain a finite result.

The only other potential divergence in two space dimensions is the tadpole counterterm, the coefficient of which is given by the integral over space of $\frac{\mathcal{R}}{48\pi}$ [9,10]. Its contribution is given by replacing the phase shift defined in Eq. (30) by its first Born approximation [15]. However, when summed over ℓ , this quantity is independent of k ,

since it is dimensionless and must be proportional to the dimensionless quantity $\int_0^\infty r p(r) \mathcal{R}(r) dr$, with no other dependence on the string background, and it is independent of μ . As a result, a unique feature of two dimensions is that

the tadpole term gives no contribution to the energy after taking the k derivative.

Putting these results together, we can compute the full VPE on the real k axis,

$$\begin{aligned} \mathcal{E} &= \int_0^\infty \frac{\sqrt{k^2 + \mu^2}}{2\pi} \left\{ 2 \sum_{\ell=0}^{\infty} \left[\frac{1}{2i} \frac{d}{dk} \log \left((-1)^{\ell(\sigma-1)} S_\ell(k) \right) \right] - \frac{k}{2} \Delta A \right\} dk \\ &= \text{Re} \int_0^\infty \frac{\sqrt{k^2 + \mu^2}}{2\pi} \left\{ 2 \sum_{\ell=0}^{\infty} \left[i \frac{d}{dk} \log \left((ikr_0)^{\ell(\sigma-1)} F_\ell(k) \right) - \frac{k \Delta A}{2\pi} \frac{1}{i(\ell + \frac{1}{2})} \right] - \frac{k}{2} \Delta A \right\} dk, \end{aligned} \quad (31)$$

where the subtraction of the second term in brackets in the second line does not affect the real part but renders the integral convergent. In the latter form, the integrand is analytic in the upper half-plane, allowing us to draw on the density of states relation in Eq. (28) to ensure that it correctly describes the VPE.

V. RESULTS AND CONCLUSIONS

By substituting the Jost function and geometrical factors for the ballpoint pen and choosing mass $\mu = 0$, using Eq. (31) we obtain the results for the total energy shown in Fig. 1. Since r_0 is the only scale in the problem for a massless field, the energy must be proportional to $\frac{1}{r_0}$, so that $r_0 \mathcal{E}$ is a function of σ and ξ only. Results for both minimal coupling ($\xi = 0$) and conformal coupling ($\xi = \frac{1}{8}$) are shown as functions of σ . We see that in both cases the total energy is everywhere negative and roughly proportional to $-\sigma(\sigma^2 - 1)$. As expected from Ref. [9], the energy is smaller in magnitude for conformal coupling than minimal coupling, but not zero in either case.

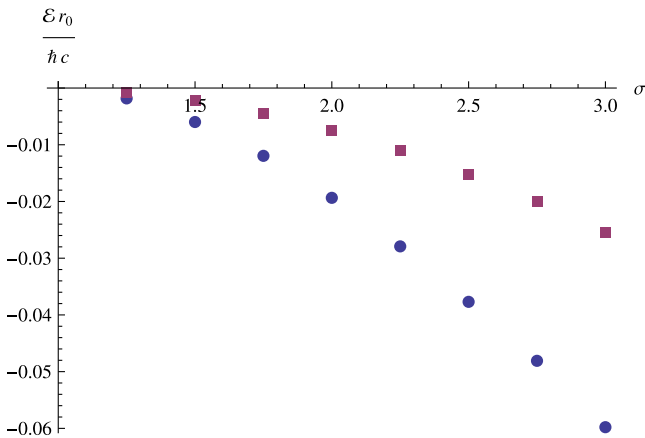


FIG. 1. Total VPE $\mathcal{E}$ in units of $\frac{1}{r_0}$ of the ballpoint pen string for a scalar field with mass $\mu = 0$, as a function of σ , computed using Eq. (31). The sum over ℓ and integral over k have been truncated and then extrapolated to infinity. Circles are for minimal coupling ($\xi = 0$) while squares are for conformal coupling ($\xi = \frac{1}{8}$).

We find results of a similar order of magnitude compared to those in Refs. [11,12] for the limit where σ is close to 1, but our results differ quantitatively because we subtract only local counterterms rather than the more complex quantities extracted from uniform asymptotic expansions and ζ -function regularization in those calculations. In particular, our results appear to vanish as $\sigma^2 - 1$ rather than $(\sigma^2 - 1)^2$ as σ approaches 1.

The integrand in the second line of Eq. (31) is an analytic function of k in the upper half-plane, a property we used to make the connection in Eq. (28) between the local density of states as given by the Green's function and the global density of states as given by the Jost function. As a result, we might hope to use contour integration to rewrite the VPE as an integral on the imaginary axis $k = i\kappa$ arising from the cut in the $\sqrt{k^2 + \mu^2}$ term, as is standard in flat-space calculations [15,19,20]. However, we find that this integral does not converge. Perhaps related, to first extend the integral to the entire real axis before closing the contour, one typically takes advantage of the derivative of the phase shift being even, but here there are also odd terms arising from the point string and the free density of states, suggesting that k should be replaced by $\sqrt{k^2}$ in these terms.

If it is possible to extend the calculation to the imaginary axis, an additional benefit would be to make it possible to use Regge techniques to replace the sum over ℓ by an integral over imaginary $\ell = i\lambda$, which can avoid problems associated with the mismatch between sums over ℓ and $\sigma\ell$ in renormalization [9,10]. As explained in Appendix B, this replacement can be carried out successfully in our calculation, but, as with other examples of this technique, only after analytic continuation in k .

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: VARIABLE PHASE METHOD

If we instead consider an arbitrary smooth profile function $p(r)$ that interpolates between $\frac{1}{\sigma}$ at the origin and 1 at infinity, we can develop general expressions for the Green's function, Jost function, and S -matrix using the variable phase method.

We define the physical distance r_* , which is the solution to the ordinary differential equation $\frac{dr_*}{dr} = p(r)$ with $r_* = r = 0$ at the origin. We can then rewrite the wave equation in terms of the reduced wave function $\phi(r_*) = \sqrt{r}\psi(r)$ as [9,11]

$$\left[-\frac{d^2}{dr_*^2} + \frac{\sigma^2 \ell^2 - \frac{1}{4p(r)^2}}{r^2} + \left(\xi - \frac{1}{4} \right) \mathcal{R} \right] \phi_{\ell,k}(r_*) = k^2 \phi_{\ell,k}(r_*), \quad (\text{A1})$$

where r is now a function of r_* .

We define the potential including the centrifugal angular momentum term as

$$V_\ell(r_*) = \frac{\sigma^2 \ell^2 - \frac{1}{4p(r)^2}}{r^2} + \left(\xi - \frac{1}{4} \right) \mathcal{R}, \quad (\text{A2})$$

and note that near the origin, $p(r) \approx \frac{1}{\sigma}$ and $r \approx \sigma r_*$, and so the centrifugal term corresponds to angular momentum ℓ , while at large distances $p(r) \rightarrow 1$ and $r \rightarrow r_* + \Delta r$ so the centrifugal term has angular momentum $\sigma \ell$.

We define the regular and outgoing variable phase solutions [15] to Eq. (14) in terms of a Hankel function with a Riccati-type rescaling $\hat{H}_\ell^{(1)}(z) = \sqrt{z} H_\ell^{(1)}(z)$ by writing

$$\begin{aligned} \phi_{\ell,k}^{\text{reg}}(r_*) &= \frac{h_{\ell,k}(r_*)}{\hat{H}_\ell^{(1)}(kr_*)}, \\ \phi_{\ell,k}^{\text{out}}(r_*) &= g_{\ell,k}(r_*) \hat{H}_{\sigma\ell}^{(1)}(kr_*), \end{aligned} \quad (\text{A3})$$

which obey the differential equations

$$\begin{aligned} -h_{\ell,k}''(r_*) + 2k \frac{d}{dr_*} \left(\frac{\hat{H}_\ell^{(1)'}(kr_*)}{\hat{H}_\ell^{(1)}(kr_*)} h_{\ell,k}(r_*) \right) + \left(V_\ell(r_*) - \frac{\ell^2 - \frac{1}{4}}{r_*^2} \right) h_{\ell,k}(r_*) &= 0, \\ -g_{\ell,k}''(r_*) - 2k \frac{\hat{H}_{\sigma\ell}^{(1)'}(kr_*)}{\hat{H}_{\sigma\ell}^{(1)}(kr_*)} g_{\ell,k}(r_*) + \left(V_\ell(r_*) - \frac{\sigma^2 \ell^2 - \frac{1}{4}}{r_*^2} \right) g_{\ell,k}(r_*) &= 0, \end{aligned} \quad (\text{A4})$$

with boundary conditions $\lim_{r_* \rightarrow 0} h_{\ell,k}(r_*) = \lim_{r_* \rightarrow \infty} g_{\ell,k}(r_*) = 0$, $\lim_{r_* \rightarrow 0} h_{\ell,k}'(r_*) = 1$, and $\lim_{r_* \rightarrow \infty} g_{\ell,k}(r_*) = i^{(\sigma-1)\ell}$, where a prime denotes derivative with respect to the argument and the factor of $i^{(\sigma-1)\ell}$ accounts for the difference in phase between $\hat{H}_{\sigma\ell}^{(1)}(kr_*)$ and $\hat{H}_\ell^{(1)}(kr_*)$. In practice, we must start the integration for $h_{\ell,k}(r_*)$ with these boundary conditions imposed at some small value just larger than zero.

We can then compute the Wronskian of these two solutions,

$$\mathcal{W}_\ell(k) = \frac{\hat{H}_{\sigma\ell}^{(1)}(kr_*)}{\hat{H}_\ell^{(1)}(kr_*)} \left[\left(h_{\ell,k}'(r_*) - k \frac{\hat{H}_\ell^{(1)'}(kr_*)}{\hat{H}_\ell^{(1)}(kr_*)} h_{\ell,k}(r_*) \right) g_{\ell,k}(r_*) - h_{\ell,k}(r_*) \left(g_{\ell,k}'(r_*) + k \frac{\hat{H}_{\sigma\ell}^{(1)'}(kr_*)}{\hat{H}_{\sigma\ell}^{(1)}(kr_*)} g_{\ell,k}(r_*) \right) \right], \quad (\text{A5})$$

which is independent of r_* , so it can be evaluated at an intermediate fitting point of our choice. For fixed ℓ , the Wronskian goes to 1 as $k \rightarrow \infty$ and obeys $\mathcal{W}_\ell(-k) = \mathcal{W}_\ell(k)^*$ for k real. The Jost function is then given by $F_\ell(k) = e^{-ik\Delta r} \mathcal{W}_\ell(k)$, and from these results we obtain the Green's function as

$$G(\mathbf{r}, \mathbf{r}', k) = \frac{\sigma}{\pi} \frac{1}{\sqrt{rr'}} \sum_{\ell=0}^{\infty} \frac{\hat{H}_{\sigma\ell}^{(1)}(kr_{*>})}{\hat{H}_\ell^{(1)}(kr_{*<})} \frac{h_{\ell,k}(r_{*<}) g_{\ell,k}(r_{*>})}{\mathcal{W}_\ell(k)} \cos[\sigma \ell (\theta - \theta')]. \quad (\text{A6})$$

These expressions can then be used to calculate the total energy as in the main text, as well as to compute local densities as described in Refs. [9,10]. This approach also extends directly to complex k in the upper half-plane [15].

APPENDIX B: COMPLEX ANGULAR MOMENTUM

The deficit angle background complicates standard techniques for computing quantum corrections because it leads to a quantized angular momentum $\sigma\ell$ for integer ℓ , which must be compared to angular momentum ℓ for the free background. As a result, one often does not obtain term-by-term cancellation in the resulting sum over ℓ . In some cases, such as conical geometries [21,22] and the point string [9], this problem can be addressed by replacing the discrete ℓ sum with a continuous Kontorovich-Lebedev integral over imaginary ℓ , where the wave number k is also analytically continued to imaginary values [23,24].

Our analytic scattering data results for the ballpoint pen extend directly to complex ℓ , as does the density of states relation in Eq. (28). For quantities involving sums of analytic functions over ℓ , in particular Eq. (13) or the second line of Eq. (31), breaking the cosine into complex exponentials and then multiplying by $\frac{\pi \exp(\pm i\pi\ell)}{\sin \pi\ell}$, which has poles of unit residue at integer ℓ , allows us to equate the original sum to a contour integral in complex ℓ over a semicircle in the right half-plane. In the limit of large radius, the contribution of the semicircular arc goes to zero, leaving only the contribution on the imaginary axis, which by symmetry can then be simplified to an integral over the positive imaginary axis only. The Green's function in Eq. (13) thus becomes [9]

$$G(\mathbf{r}, \mathbf{r}', i\kappa) = \frac{1}{2\pi} \int_0^\infty \frac{id\lambda}{\sinh \frac{\lambda\pi}{\sigma}} \cosh \left[\lambda \left(\frac{\pi}{\sigma} - |\theta - \theta'| \right) \right] \left[\psi_{ik, i\frac{\lambda}{\sigma}}^{\text{reg}}(r_<) \psi_{ik, i\frac{\lambda}{\sigma}}^{\text{out}}(r_>) - \psi_{ik, -i\frac{\lambda}{\sigma}}^{\text{reg}}(r_<) \psi_{ik, -i\frac{\lambda}{\sigma}}^{\text{out}}(r_>) \right], \quad (\text{B1})$$

with $i\lambda = \ell\sigma$ and $k = i\kappa$. We note that the difference of regular functions becomes proportional to the outgoing function, so that the jump condition arises via the angular dependence while the radial dependence is continuous. Similarly, we find for the analytic continuation of the integrand in Eq. (31),

$$\sum_{\ell=0}^{\infty} \left[\frac{d}{d\kappa} \log \left((\kappa r_0)^{\ell(\sigma-1)} F_\ell(i\kappa) \right) - \frac{\kappa \Delta A}{2\pi(\ell + \frac{1}{2})} \right] = \text{Im} \int_0^\infty \frac{\coth \frac{\pi\lambda}{\sigma} d\lambda}{\sigma} \left[\frac{\kappa \Delta A}{2\pi(i\frac{\lambda}{\sigma} + \frac{1}{2})} - \frac{d}{d\kappa} \log \left((\kappa r_0)^{\frac{i\lambda(\sigma-1)}{\sigma}} F_{\frac{i\lambda}{\sigma}}(i\kappa) \right) \right], \quad (\text{B2})$$

where again $i\lambda = \ell\sigma$ and $k = i\kappa$. We have checked that these integrals agree with the original sums, but because they are applicable only for imaginary $k = i\kappa$, further modification to close the contour integral would be needed to apply them to the full calculation of the energy.

We also note that when ℓ is not a real integer, for $r < r_0$ in the ballpoint pen model it is computationally preferable to take as independent solutions $P_{\nu(k)}^{-\ell}$ and $P_{\nu(k)}^{\ell}$ rather than $P_{\nu(k)}^{-\ell}$ and $Q_{\nu(k)}^{\ell}$. With this replacement made throughout, the expressions for the scattering data hold as above, except that the product of the $A_{k,\ell}$ and $D_{k,\ell}$ coefficients becomes $\frac{\pi}{2\sigma \sin \pi\ell}$.

We can also extend the variable phase method of Appendix A to complex ℓ . The calculation given there is valid when ℓ is an integer (though it also gives the correct result for ℓ any positive real number). To carry out the analytic continuation in ℓ , we must use a modified approach that is instead valid only when ℓ is not an integer. In this case, the calculation of the regular wave depends much more sensitively on the boundary conditions at the origin. As a result, we must replace the simpler boundary conditions on $h_{\ell,k}(r_*)$ given in Appendix A by those obtained from the expansion of the free regular solution

$$h_{\ell,k}(r_*) \approx r_* \left[1 + \left(\frac{\kappa r_*}{2i} \right)^{2\ell} \frac{\Gamma(-\ell)}{\Gamma(\ell)} \right] \quad (\text{B3})$$

and we must furthermore introduce overall factors of $\frac{1}{2\ell}$ in the expressions for the Wronskian in Eq. (A5) and the Green's function in Eq. (A6), which then cancel in the latter equation. [After introducing these same factors of $\frac{1}{2\ell}$, we can similarly use the boundary condition $h_{\ell,k}(r_*) \approx r_*$ for the case of nonzero integer ℓ discussed in Appendix A, along with the limit described in Eq. (B5) below for the case of $\ell = 0$.] We thus obtain the Green's function as

$$G(\mathbf{r}, \mathbf{r}', i\kappa) = \frac{\sigma}{2\pi^2 \sqrt{rr'}} \int_0^\infty \frac{d\lambda}{i\lambda \sinh \frac{\lambda\pi}{\sigma}} \cosh \left[\lambda \left(\frac{\pi}{\sigma} - |\theta - \theta'| \right) \right] \times \left[\frac{h_{\frac{i\lambda}{\sigma}, i\kappa}(r_<) g_{\frac{i\lambda}{\sigma}, i\kappa}(r_>) \hat{H}_{i\lambda}^{(1)}(i\kappa r_>)}{\hat{H}_{\frac{i\lambda}{\sigma}}^{(1)}(i\kappa r_<) \mathcal{W}_{\frac{i\lambda}{\sigma}}(i\kappa)} - \frac{h_{-\frac{i\lambda}{\sigma}, i\kappa}(r_<) g_{-\frac{i\lambda}{\sigma}, i\kappa}(r_>) \hat{H}_{-i\lambda}^{(1)}(i\kappa r_>)}{\hat{H}_{-\frac{i\lambda}{\sigma}}^{(1)}(i\kappa r_<) \mathcal{W}_{-\frac{i\lambda}{\sigma}}(i\kappa)} \right], \quad (\text{B4})$$

in the same way as in Eq. (B1), where the Green's function is now defined as the right-hand side of Eq. (A6) divided by 2ℓ and the Wronskian is now defined as the right-hand side of Eq. (A5) divided by 2ℓ , giving results that obey Eqs. (28) and (B2). While it appears that these rescalings could be avoided by appropriate redefinitions, they become necessary when $\text{Re}\ell = 0$. Such modifications are analogous to the procedure in

Ref. [15] for the $\ell = 0$ case, which uses the limit

$$\lim_{\ell \rightarrow 0} \frac{r_*}{2\ell} \left[1 + \left(\frac{kr_*}{2i} \right)^{2\ell} \frac{\Gamma(-\ell)}{\Gamma(\ell)} \right] = -r_* \ln \left(\frac{kr_*}{2i} \right) + \mathcal{O}(r_*) \quad (\text{B5})$$

to obtain the behavior of the regular solution $h_{\ell=0,k}(r_*)$ at the origin.

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