

Worldsheet from worldline

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We take a step toward a “microscopic” derivation of gauge-string duality. In particular, using mathematical techniques of Strebel differentials and discrete exterior calculus, we obtain bosonic string worldsheet action for string embedded in $d + 1$ -dimensional asymptotically anti-de Sitter space from multiloop Feynman graphs of quantum field theory of scalar matrices in d -dimensions in the limit of diverging loop number. Our work is building on the program started by ’t Hooft in 1974, this time including the holographic dimension, which we show to emerge from the continuum of Schwinger parameters of Feynman diagrams.

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Introduction. Despite being widely accepted and having passed many nontrivial tests, gauge-gravity duality [1–3] lacks a satisfactory derivation as an equivalence between quantum gauge theory and string theory. In particular, apart from certain low-dimensional examples and integrable class of quantum field theories, e.g., [4–8], we are unaware of any microscopic derivation of the string worldsheet directly from the gauge theory path integral. This derivation would entail, in a concrete sense, completing the program started by ’t Hooft’s seminal paper [9], that is, to derive the string path integral—this time including the holographic dimension—from the ribboned Feynman graphs of gauge theories.

A promising starting point for this endeavor is Gopakumar’s observation [10–12]. See [6,13–21] for an incomplete list of other examples and alternative approaches. In particular, [12] proposed to use Strebel differentials to map the skeleton graphs of n -point functions of gauge invariant operators in free conformal field theory (CFT) to the worldsheet of a putative string theory, see also [22,23]. Furthermore, in [24,25] a curious similarity between (discretized) open and closed strings and a graphical representation of one and two trees that appear in Symanzik polynomial representation of high-loop Feynman graphs (see, e.g., [25–29]) were observed and it was proposed that embedding of the string in the holographic dimension arises from a continuum limit of the Schwinger parameters of graphs with large number of loops.

We merge these observations in this Letter and explicitly construct the string worldsheet action that arises from the

continuum limit of Feynman graphs of a quantum field theory (QFT) in d dimensions. We show that the target space of the string that corresponds to the QFTs we consider in this Letter is asymptotically $d + 1$ -dimensional anti-de Sitter space (AdS_{d+1}).

Generating function. Consider expansion of a QFT generating function in Feynman diagrams. We restrict ourselves to QFTs of $N \times N$ matrices that are massless scalars with a generic polynomial type interaction. Contribution of a graph Γ is given by

$$\mathcal{Z}_\Gamma \propto \int \prod_i d^d X_i \prod_{e_{ij} \in \Gamma} G(X_i, X_j) \prod_{v_i \in \mathcal{S}_\Gamma} J(X_i) \quad (1)$$

with the proportionality constant including all factors of the couplings and symmetry factors, e_{ij} and v_i denoting the edges and the vertices, and J denoting the sources. We assume, following [9], that Feynman graphs are put in ribboned form and grouped as discretized string world-sheets with genus g . We first rewrite (1) in a form that allows us to read off the corresponding (discretized) worldsheet action.

A key role is played by the Schwinger parametrization of the propagators $G(X_i - X_j)$ with Schwinger parameters σ_{ij} . If there are m edges between a pair of vertices, then we can merge them in an effective propagator [11] written in terms of a single Schwinger parameter as

$$G(X_i, X_j)^m = \frac{i^m \Gamma(\frac{d}{2} - 1)^m}{2^{\frac{md}{2}} \Gamma(m(\frac{d}{2} - 1))} \cdot \int_0^\infty d\sigma_{ij} \sigma_{ij}^{m(\frac{d}{2}-1)-1} e^{-\frac{\sigma_{ij}(X_i-X_j)^2}{4}}. \quad (2)$$

We assume vanishing masses throughout this Letter—see [25] for generalization of this expression to massive

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scalars. This procedure constructs the so-called “skeleton graph” $\bar{\Gamma}$ of Γ [11] with all duplicated edges fused into a single one and each skeleton edge assigned a length given by the corresponding σ_{ij} . The partition function is then expressed as

$$\mathcal{Z}[J] = \sum_{\bar{\Gamma}} \int d[X] d[\sigma] \mathcal{P}_{\bar{\Gamma}} e^{-\frac{1}{4} \sum \sigma_{ij} (X_i - X_j)^2}, \quad (3)$$

where $[X]$ and $[\sigma]$ denote the collection of all vertex positions and graph metrics of the skeleton graph and $\mathcal{P}_{\bar{\Gamma}}$ is a polynomial in σ_{ij} that depends on the graph structure, coupling constants, and sources.

To determine it, we introduce auxiliary variables ξ_i at the vertices v_i and represent each edge connected to a vertex by $i\xi_i$ so that for each propagator between v_i and v_j we must include a factor of $-\xi_i \xi_j$. It is straightforward to show that

$$\mathcal{P}_{\bar{\Gamma}} = \frac{1}{n!} \left(i \prod_i U(-i\partial_{\xi_i}) \right) \prod_{\bar{e}_{ij} \in \bar{\Gamma}} \sum_m \frac{(-1)^m \Gamma(\frac{d}{2} - 1)^m}{2^{\frac{md}{2}} \Gamma(m(\frac{d}{2} - 1))} \cdot \sigma_{ij}^{m(\frac{d}{2}-1)-1} \frac{(\text{Tr}(\xi_i^\dagger \xi_j + \xi_j^\dagger \xi_i))^m}{2(m!)} \Big|_{\xi=0} \quad (4)$$

gives a nonvanishing contribution only when the number of edges connected to the vertex v_i is precisely equal to the number of derivatives and only when the coordination number of the vertices equal to the one on some diagram Γ whose skeleton diagram is $\bar{\Gamma}$. The symmetry factors of the graph are correctly accounted for by the factors of $1/n!$ and $1/m!$ in this expression. The latter arises from swapping of propagators between the same pair of vertices and the former from swapping of vertices. This gives the correct factor as long as we sum over all nonequivalent permutations of vertices. This sum will be implemented below as part of the integral over the moduli space, which should now be the moduli space of Riemann surfaces with n marked points counting separately all nonequivalent permutations of these points.

We recognize the exponential in (3) as a kinetic term

$$S_K = \frac{1}{4} \sum_{ij} \sigma_{ij} (X_i - X_j)^2, \quad (5)$$

and rewrite $\mathcal{P}_{\bar{\Gamma}}$ above as an interaction potential

$$S_I[\xi, \theta] = - \sum \log(iU(\theta_i)) - i\xi_i \theta_i + f(\sigma_{ij}^{\frac{d}{2}-1} \xi_i \xi_j), \quad (6)$$

where the function f is

$$f(\sigma_{ij}^{\frac{d}{2}-1} \xi_i \xi_j) = \log \left(\sum_m \frac{\Gamma(\frac{d-2}{2})^m (-1)^m}{2^{\frac{md}{2}} \Gamma(m(\frac{d-2}{2}))} \sigma_{ij}^{m(\frac{d}{2}-2)} (\xi_i \xi_j)^m \right) \quad (7)$$

and the extra factor of $\frac{1}{\sigma_{ij}}$ has been absorbed into the measure for σ . To obtain the expression (6), we followed the trick

$$(\cdots)|_{\xi=0} = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\theta e^{-i\xi\theta} (\cdots)$$

and performed integration by parts inside the path integral.

Triangulation and continuum. We view the sum of the kinetic and potential terms in (5) and (6) as discretized worldsheet action of the dual string theory and take the continuum limit to obtain the continuous worldsheet theory. This should be done with care, employing the technology of Strebel differentials and discrete exterior calculus, which we shortly review below and in more detail in Supplemental Material [30].

Strebel differentials. The theory of Strebel differentials [31] provides an isomorphism between the space of metric ribboned graphs and the moduli space on punctured Riemann surfaces with a positive real number assigned to every puncture. To every propagator with length σ corresponds a vertical strip in the complex plane of width σ [12,22]. Then, these strips are glued into a manifold as depicted in Fig. 1 and detailed in Supplemental Material [30]. The vertices of Γ become punctures in the manifold and a positive real number a_i associated with the

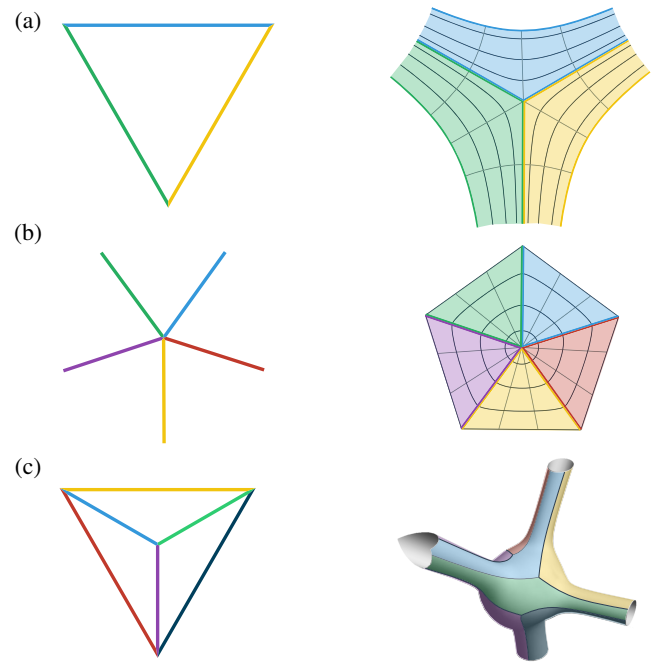


FIG. 1. Gluing of the strips at (a) a face and (b) a vertex. In (c) we show an example of the Riemann surface obtained from a simple graph of genus 0 with four vertices corresponding to a sphere with four infinite cylinders attached.

i th puncture is given by the sum of all the lengths of edges incident to the corresponding vertex $a_i = \sum_j \sigma_{ij}$. This construction naturally provides a metric for the Riemann surface given by the flat metric on each strip, corresponding to a particular gauge fixing of Weyl and diffeomorphism symmetries. This metric exhibits a double pole around the punctures (vertices of Γ)

$$g_{\text{Str}} = \frac{a_i^2}{4\pi^2} \frac{dud\bar{u}}{|u|^2}. \quad (8)$$

Discrete exterior calculus. The theory of Strebel differentials provides the worldsheet as a Riemann surface, but it does not determine the worldsheet action from (5) and (6). For this, one first needs to extend the map between double poles of the Riemann surface (8) and location of vertices in space time X_i^μ to entire functions $X^\mu(u, \bar{u})$ on the Riemann surface. We can address this issue through the machinery of discrete exterior calculus used to solve geometric problems on manifolds through numerical methods where the specified data are necessarily finite [32].

Consider a Riemann surface Σ with a triangulation, see below, in terms of points v_i , intervals e_{ij} , and triangles f_{ijk} such that the edges of f_{ijk} are e_{ij} , e_{jk} , and e_{ki} and the end points of e_{ij} are v_i and v_j . One discretizes data on the manifold by integrating over the relevant simplices of our triangulation. That is, one integrates a function over the points v_i , a 1-form over the intervals e_{ij} and a 2-form over the triangles f_{ijk} . This allows us to use Stokes's theorem without issue. For instance, take the 1-form df specified over the edges e_{ij} , then one has

$$(df)_{e_{ij}} = \int_{e_{ij}} df = \int_{\partial e_{ij}} f = (f)_j - (f)_i. \quad (9)$$

This defines for us a discrete exterior derivative which is unambiguous in the discretization lengths by virtue of Stokes's theorem.

Another crucial operation is the Hodge dual. We first define the dual $\star\sigma^k$ of a k simplex σ^k by the convex span of the circumcenters of all the simplices that contain σ^k and σ^k itself, see Fig. 2. The Hodge dual of a k -form α is then defined by

$$(\star\alpha)_{\star\sigma^k} = \frac{|\star\sigma^k|}{|\sigma^k|} (\alpha)_{\sigma^k}, \quad (10)$$

where $|\cdot|$ stands for the volume in the corresponding dimension.¹ The last operation we need is the wedge product [32]

¹Note that $\star\alpha$ is not discretized in the same triangulation as α , but instead in a dual version of it.

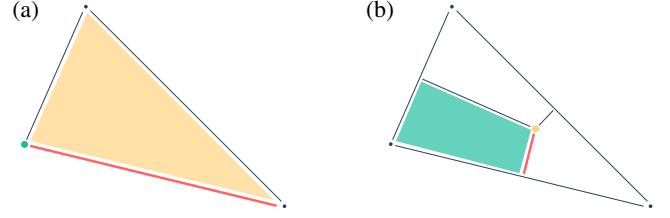


FIG. 2. Depiction of part of the triangulation (a) with its dual shown in (b). Cells painted in the same color are dual to each other.

$$(\alpha \wedge \beta)_{\sigma^k \cup \star\sigma^k} = \frac{|\sigma^k \cup \star\sigma^k|}{|\sigma^k| |\star\sigma^k|} (\alpha)_{\sigma^k} (\beta)_{\star\sigma^k}, \quad (11)$$

where $\sigma^k \cup \star\sigma^k$ is the smallest convex set containing both σ^k and $\star\sigma^k$. If we decrease the size of our triangulation, we can expect all these operations to converge [33,34] since surfaces are locally flat and thus well approximated by subsets of $\mathbb{R}^2$.

Triangulation. Finally, we need to provide a triangulation of the Riemann surface—which we obtained using Strebel differentials above—on which discrete exterior calculus can operate. This is tricky, as the punctures are located at the end points of semi-infinite cylinders, see Fig. 1. We can, however, remove these cylinders by a Weyl transformation with a factor of $\epsilon^2 |u|^2$ in the region $0 < |u| < 1$ on the Strebel metric (8). This leaves a flat punctured disk that can be completed by restoring the marked point. Disks corresponding to different poles are identified at the horizontal curves at $|u| = 1$ defining a continuous Weyl transformation on the entire manifold. The resulting manifold is a collection of sections of flat disks with opening angles $\frac{2\pi\sigma_{ij}}{a_i}$ and radii $\frac{a_i\epsilon}{2\pi}$ identified along their curved edges, see Fig. 3. The factor of ϵ enables us to adjust the volume V of the manifold, i.e., to keep it fixed even when we introduce more poles or change the a_i . One finds

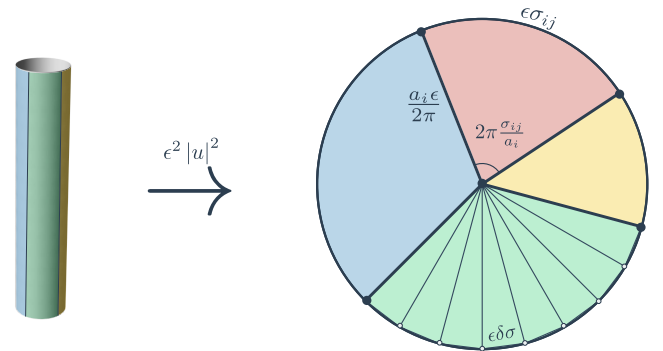


FIG. 3. After we perform a Weyl transformation, the infinite cylinders become flat disks split into sections that are identified along its curved edge. We triangulate the manifold by splitting these sections into small triangles.

$$V = \sum_i \frac{\epsilon^2 a_i^2}{4\pi} \Rightarrow \epsilon = \sqrt{\frac{4\pi V}{\sum a_i^2}} \sim \frac{1}{\sqrt{n}}, \quad (12)$$

where n is the number of vertices in the graph. What we achieved so far is not quite a triangulation because sections of disks are not simplices, but we can split each section further into small triangles that approximate the disks arbitrarily well. Note that, at strong coupling, where diagrams with $n \gg 1$ dominate the partition function, the mesh describing our manifold becomes finer and finer.

We will need the following geometric data, which we can easily deduce from Fig. 3. Denoting the poles by indices i and j and the auxiliary vertices introduced to make the disks into triangles by an index a , we obtain $|e_{ia}| = a_i \epsilon / 2\pi$ for length of the auxiliary edges and $|\star e_{ia}| = \epsilon \delta \sigma / 2$ for their duals. Their convex span has area $|e_{ia} \cup \star e_{ia}| = a_i \delta \sigma \epsilon^2 / 8\pi$. The area of the dual of a pole is $|\star v_i| = a_i^2 \epsilon^2 / 16\pi$ and for the auxiliary vertices lying between disk i and disk j we find $|\star v_a| = 3(a_i + a_j) \delta \sigma \epsilon^2 / 16\pi$.

Continuum. We are ready to take the continuum limit of the discrete action (5) and (6). Discrete exterior derivative of X^μ is given by

$$(dX^\mu)_{e_{ia}} = \frac{a_i}{a_i + a_j} (X_j - X_i), \quad (13)$$

where the vertex v_a lies between the disks i and j and we chose to linearly interpolate the known values of X^μ to the auxiliary vertices on the edges of the disk. Following the definitions above, one finds

$$(adX^\mu \wedge \star dX_\mu)_{e_{ia} \cup \star e_{ia} + e_{ja} \cup \star e_{ja}} = \frac{\pi \delta \sigma}{4} (X_j - X_i)^2. \quad (14)$$

Summing over the entire manifold yields

$$S_K = \frac{1}{\pi} \int d^2 z \sqrt{\det g} a \nabla_\alpha X^\mu \nabla_\beta X^\nu g^{\alpha\beta} \eta_{\mu\nu}, \quad (15)$$

up to corrections of order $\epsilon \sim 1/\sqrt{n}$.

For the continuum limit of the potential term (6) we need to find a continuum version of the σ 's. There are generically $6g - 6 + 3n$ independent σ 's due to the correspondence with the space $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$. A total of n of them are expressible in terms of a and $6g - 6 + 2J$, where J is the number of external sources still present in the stringy calculations as moduli of the worldsheet. This leaves $2(n - J)$ to be integrated out, which can be rearranged into two more real functions t_1 and t_2 . We define these to be given by the interpolation of the $\frac{\sigma_{ij}}{a_i}$ for the independent σ_{ij} , which are not one of the $6g - 6 + 2J$ moduli of the worldsheet. At the external sources where there cannot be any independent σ , we just set t_i to 0. There are still

some σ at every vertex that are not captured by these functions. This is either because they correspond to moduli of the worldsheet or because they can be expressed in terms of a . The first kind we will ignore for the moment since they are clearly subleading in $\frac{1}{n}$. For the second kind, we note that a typical skeleton vertex has order 3 unless some of the σ are fine tuned to 0, thus we can construct a third function given by $t_3 = 1 - t_1 - t_2$ to capture them. The other quantity in (6) for which we need a continuum approximation is $\xi_i \xi_j$. To make sense of the δ functions in f_p , let us assume that we smear them in a rotationally symmetric fashion on the region dual to v_i . Then it is straightforward to show that $(\frac{d\xi \wedge \star d\xi}{\star n f_p})_{\tilde{v}_{ia}} = \frac{\pi}{2} (\xi_i - \xi_a)^2$ by just dividing the values of the two top forms integrated on $\star \tilde{v}_{ia} \equiv e_{ia} \cup \star e_{ia}$ and similarly $(\frac{da \wedge \star d\xi}{\star n f_p})_{\tilde{v}_{ia}} = \frac{\pi}{2} \frac{a_i^2 (a_i - a_j) (\xi_i - \xi_j)}{(a_i + a_j)^2}$. Interpolating these to v_a , we directly find

$$\xi_i \xi_j = \left(\xi^2 + \frac{4\xi}{\pi} \frac{\nabla \xi \cdot \nabla a}{n f_p} - \frac{2}{\pi} \frac{(\nabla \xi)^2}{n f_p} \right)_{v_a} + \mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right). \quad (16)$$

Using the definition of f_p to turn (6) into an integral, we can express its continuum limit as

$$S_I = -n \int \left(\log(iU(a^{\frac{d-2}{4}} \theta)) - i\xi \theta + \sum_i f(t_i^{\frac{d-2}{2}} \xi^2) \right) \star f_p - \int C_1 \left(\frac{4}{\pi} \nabla \log \xi a^{-\frac{d-2}{4}} \cdot \nabla \log a - \frac{2}{\pi} (\nabla \log \xi a^{-\frac{d-2}{4}})^2 \right) \star 1, \quad (17)$$

where we have rescaled $\xi \rightarrow \xi a^{-\frac{d-2}{4}}$ and $\theta \rightarrow \theta a^{\frac{d-2}{4}}$ and defined $C_1 = \sum_i \xi^2 t_i^{\frac{d-2}{2}} f'(\xi^2 t_i^{\frac{d-2}{2}})$.

Putting everything together, we find the following expression for the generating function:

$$\mathcal{Z}[J] = \sum_n \frac{1}{n!} \int \mathcal{D}X^\mu \mathcal{D}\xi \mathcal{D}\theta \mathcal{D}a \mathcal{D}[g] \left| \frac{\delta \sigma}{\delta(a, g)} \right| e^{-S_K - S_I}, \quad (18)$$

where S_K and S_I are given by (15) and (17). The path integrals are understood to be discretized on the triangulation described above and $\mathcal{D}[g]$ refers to an integral over the moduli space of the Riemann surfaces with marked points.

Leaving the Jacobian $|\delta \sigma / \delta(a, g)|$ aside for a moment, we now consider the large n limit of (18). Clearly the path integrals over ξ , θ , t_1 , and t_2 will be dominated by the classical saddle of (17). These can be further be solved in powers of a around $a \rightarrow \infty$, which is also a natural limit to take, as it corresponds to the UV limit of the QFT, which we expect to be a CFT—see the Discussion section below. Restricting to the case of real fields with a single trace potential given by $U = \text{Tr}(\sum_{k!} \Phi^k + J^T \Phi)$, we find that

ξ , θ , t_1 , and t_2 are constant to leading order in a .² More general situations can be considered just as easily with similar results. Using this solution in (17) we obtain

$$S_I = C_0 + \frac{k(d+6)(d-2)}{16\pi} \int d^2z \sqrt{\det g} \frac{\nabla_a a \nabla^a a}{a^2} \quad (19)$$

for some constant C_0 and suppressing a term proportional to $\sum_i \log a_i$.

Coming back to the measure in terms of σ , it is $\prod \frac{d\sigma_{ij}}{\sigma_{ij}}$. Translated into our choice of moduli and a_i , this gives a measure for the stringy moduli given by $\frac{dt_i}{t_i}$ and for a_i we have $\frac{da_i}{a_i}$. There is another factor for the moduli of the internal vertices that is irrelevant due to the saddle point approximation performed on t_1 and t_2 . This will essentially determine C_0 . The measure for the stringy moduli turns out to coincide with the stringy measure expected from the Faddeev-Popov determinant computed by the ghost sector of the string, as can be directly checked. As for the metric for a , in a string theory we would expect a measure induced by the distance

$$|\delta a|^2 = \int d^2z \sqrt{\det g} \frac{\delta a^2}{a^2} \propto \sum_i \delta a_i^2. \quad (20)$$

The difference between this measure and the one actually arising from the field theory is expressible as a Weyl transformation of the metric. Thus, we will get some Liouville term for $\log a$. One can similarly show that the measure for X^μ suffers from the same exact problem, giving more contributions to the Liouville term for $\log a$. Overall, we find an extra contribution to the action given by

$$S_L = \int d^2z \sqrt{\det g} K_1 \frac{(\nabla a)^2}{a^2} + K_2 R \log a \quad (21)$$

with $K_1 = -\frac{16(1+9d)+8(d-2)k+(d-2)^2k^2}{1536\pi}$ and $K_2 = -\frac{2k-(k-12)d-4}{192\pi}$, and the full stringy action is given by $S = S_K + S_I + S_L$. This constitutes our final result for the worldsheet action in the large n and $a \rightarrow \infty$ limits. We find an AdS_{d+1} σ model $ds^2 = du^2 + e^{-2u} \eta_{\mu\nu} dX^\mu dX^\nu$ with $a(z) = \exp(-2u(z))$ near the boundary. Deviations from this limit relate geometry toward the interior to the renormalization group flow of the QFT. We leave study of such deviations to future work.

Discussion. We employed scalar matrix theories to illustrate a mechanism for emergence of string worldsheet from multi-loop Feynman graphs and demonstrated two basic features:

²This holds only for $d \geq 3$; otherwise, the terms that dominate the contribution from the potential are no longer the highest-order interactions.

(i) the holographic dimension arises from a continuum limit of Schwinger parameters; (ii) open-closed duality is realized by representing Strebel differentials in two equivalent ways, see Fig. 2. A generic scalar matrix, of course, is not expected to possess a geometric dual and should be supplemented with additional sectors to fulfill the hidden assumptions behind our demonstration. In particular, we ignored issues of renormalization (which can systematically be carried out in the Schwinger formalism, see [35]) and assumed that the field mass continues to vanish in the full quantum theory. One can demonstrate [25,35] that renormalization does not alter our main conclusions as long as the QFT is sufficiently close to a fixed point. Such a fixed point exists precisely when the large loop limit is well defined, see, for example, [36]. Indeed, Zamolodchikov considered the so-called “fishnet” theories and showed that large vertex limit exists at a specific coupling where the theory becomes a CFT for $(d, k) = (3, 6)$, $(4, 4)$, and $(6, 3)$, precisely the combinations argued to possess a continuum limit in [25]. In [37], the $(4, 4)$ Fishnet theory has been shown to lead to string propagation in AdS_5 using integrability techniques. Our derivation provides an alternative without resorting to integrability and it will be interesting to employ our approach to explore the fishnet and fishchain theories in more detail [38]. It would also be interesting to see how our results connect to previously studied special cases, e.g., arithmetic Riemann surfaces [39].

What happens beyond the CFT limit is of utmost importance. This can be obtained by going beyond the large a limit in the generic expressions (15) and (17). On the other hand, it is straightforward, yet cumbersome, to include fermions and gauge fields in our derivation that is essentially based on the Schwinger parametrization of the generating function, see, for example, [26]. Further generalizing our method to include renormalization properly, one could, in principle, be able to derive the worldsheet action dual to QCD.

Finally, we note that our computation is valid for arbitrary genus. In particular, (18) holds even though the Strebel metric varies, yet remains unique. It is tempting to ask whether we can extend our analysis to nonperturbative quantum gravity. In principle, our starting point, Eqs. (5) and (6), remains valid for arbitrarily small N . See [25] for a discussion on this issue.

There are various other exciting questions to explore in the future, for instance, string propagation on black holes through extending our analysis to thermal field theory and dynamical gravity through, perhaps, generalizing to ensemble of QFTs.

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