

Preinflationary QCD axion stars after moduli domination

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The growth of adiabatic density perturbations during an era of early matter domination induces $\mathcal{O}(1)$ fluctuations in preinflationary QCD axion dark matter across a broad, string-theory-motivated parameter space. Remarkably, at Λ CDM matter-radiation equality the scale of these perturbations coincides with the quantum Jeans scale, so they collapse to solitonic “axion stars.” These axion stars have densities up to 10^4 eV⁴, and, including their surrounding halos, they contain as much as 50% of dark matter. Direct searches for a smooth axion background can be suppressed, but transient enhancements or indirect astrophysical signals at axion masses $m_a \lesssim 10^{-5}$ eV would point to a nonstandard cosmological history.

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I. INTRODUCTION

The QCD axion is a well-motivated extension of the Standard Model of particle physics that solves the strong CP problem [1–3]. Moreover, candidate axions are generic in string theory [4–7] and a QCD axion inevitably forms at least a component of dark matter.

There are two classes of axion cosmology, known as the pre- and postinflationary scenarios [8]. We focus on the preinflationary case, where the Peccei-Quinn (PQ) symmetry is broken during inflation and not subsequently restored. This is natural for axions arising from p -form fields integrated over compactification cycles, since these constructions have no 4D phase in which the PQ symmetry is unbroken [9–11]. Such realizations are attractive because the axion shift symmetry can be protected up to exponentially suppressed effects (similar protection can occur in some postinflationary theories [12,13]).

In the preinflationary scenario, the axion field is nearly homogeneous over the observable Universe at the end of inflation. Axion dark matter is produced by the misalignment mechanism, with the abundance depending on the initial misalignment angle θ_0 [14–16]. We assume that the QCD axion makes up all the dark matter. In a standard

cosmological history, this corresponds to an axion decay constant $f_a \simeq 10^{12}$ GeV for $\theta_0 \simeq 1$ [17,18].

However, observations only require a standard cosmological history below temperatures of about 5 MeV [19,20], shortly before big bang nucleosynthesis (BBN), and earlier epochs are largely unconstrained.

String theory compactifications contain scalar fields, known as moduli, that control the size and shape of the extra dimensions. Also produced by misalignment, these typically come to dominate the energy density of the early Universe, leading to an era of early matter domination (EMD) [21–24]. They decay by operators suppressed by the Planck scale $M_{\text{Pl}} \equiv G^{-1/2}$ at a rate $\Gamma \simeq m_\phi^3/M_{\text{Pl}}^2$, where m_ϕ is the modulus mass. Moduli are often light, e.g. $m_\phi \sim 10^2$ TeV, yielding a low reheating temperature

$$T_{\text{rh}} \sim 5 \text{ MeV} \left(\frac{m_\phi}{10^2 \text{ TeV}} \right)^{3/2}, \quad (1)$$

defined by $\Gamma_\phi = \sqrt{\frac{4\pi^3}{45} g_\star(T_{\text{rh}})} T_{\text{rh}}^2/M_{\text{Pl}}$, after which standard radiation domination begins.

EMD with $T_{\text{rh}} \lesssim \text{GeV}$ reduces the axion relic abundance, allowing larger f_a for a given θ_0 [25–29]. This is potentially advantageous for string theory models, which often lead to values of $f_a \gtrsim 10^{14}$ GeV that would otherwise require $\theta_0 \ll 1$ [30–32]. But θ_0 is not directly observable, so a discovery of an axion with large f_a would be, at most, a tentative hint of EMD.

In this paper, we show that EMD has another, more robust, observational signature in QCD axion dark matter, in the form of distinctive substructure. References [33–36] previously studied axion substructure from EMD, with

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Ref. [35] including wave effects during EMD. We build on these by demonstrating the central role of wave effects both during EMD and around Λ CDM matter-radiation equality (MRE), and by performing numerical simulations from early times through MRE.¹

II. AXION DARK MATTER AND EMD

We assume that EMD is driven by a single light modulus ϕ , as in, e.g., the Large Volume Scenario [46–48]. Then, during EMD the temperature T decreases with increasing scale factor R as $T \propto R^{-3/8}$ [27,49].

The axion begins oscillating at $T \simeq T_\star$, defined by $m_a(T_\star) = 3H(T_\star)$, with H the Hubble parameter and $m_a(T)$ the temperature-dependent axion mass. For our purposes it suffices to take

$$m_a(T) = \begin{cases} m_a & \text{if } T < \Lambda \\ m_a(\Lambda/T)^{\alpha/2} & \text{otherwise,} \end{cases} \quad (2)$$

with $\Lambda = 100$ MeV, $\alpha = 8$, and m_a the zero-temperature mass [50]. Provided that $T_\star \gtrsim 100$ and $T_{\text{rh}} \lesssim 100$ MeV, the observed relic abundance requires

$$f_a \simeq 10^{14} \text{ GeV} \left(\frac{10 \text{ MeV}}{T_{\text{rh}}} \right)^{4/3} \theta_0^{-4/3}. \quad (3)$$

For $T_{\text{rh}} \sim 10$ MeV values of f_a between 10^{14} and 10^{16} GeV are possible with $\theta_0 \gtrsim 0.1$. In this range,

$$T_\star \simeq 140 \text{ MeV} \left(\frac{T_{\text{rh}}}{10 \text{ MeV}} \right)^{1/4} \left(\frac{10^{14} \text{ GeV}}{f_a} \right)^{1/8}, \quad (4)$$

is self-consistently greater than 100 MeV.

III. STRUCTURE FORMATION DURING EMD

We consider the minimal scenario in which the initial modulus perturbations are the scale-invariant adiabatic fluctuations from inflation, extrapolated to EMD scales.²

During matter domination, sub-horizon modulus density perturbations grow as $\delta_\phi \propto R$, while superhorizon modes are frozen. For modes that enter the horizon during EMD, the dimensionless linear power spectrum of the modulus is therefore

¹Substructure from EMD has been studied for other dark matter candidates, and can lead to new observational signals [37–40]. However, in many theories, the strongest effects are erased after the modulus decays [37,41–45]. An axion retains information because of its nonthermal production and wave nature.

²While the axion mass is changing, perturbations in the temperature also lead to some axions being excited to finite momentum modes, prior to the times we consider. In Appendix A we show that this effect does not significantly modify our conclusions.

$$\Delta_\phi^2(k) \simeq \Delta_0^2 \left(\frac{k}{RH} \right)^4, \quad (5)$$

where k is the comoving wave number and $\Delta_0^2 = (\frac{2}{3})^2 A_s$ with $A_s \simeq 2 \times 10^{-9}$ [35,51].

Axions constitute a tiny fraction of the total energy density throughout EMD. Consequently, perturbations in their energy density do not significantly affect the gravitational potential (and their self-interactions are also negligible). Axions do however fall into the gravitational wells of the modulus perturbations. Treating the modulus as an external source, nonrelativistic subhorizon axion energy density perturbations in Newtonian gauge $\delta_a(\mathbf{k}) \equiv \delta_{a,k}$ evolve as

$$\delta''_{a,k} + \mathcal{H}\delta'_{a,k} + (c_{s,a}^2 k^2 \delta_{a,k} - 4\pi G R^2 \bar{\rho}_{\text{tot}} \delta_{\phi,k}) = 0, \quad (6)$$

where $'$ denotes a derivative with respect to conformal time, $\mathcal{H} = R'/R$, $\bar{\rho}_{\text{tot}}$ is the average total energy density, and $c_{s,a} = k/(2m_a R)$ is the axion sound speed [35,52]. Equation (6) follows from the conservation of the energy-momentum tensor, or equivalently from the nonrelativistic limit of the axion field equation of motion after a Madelung transformation (see Appendix B).

The nonzero $c_{s,a}$ reflects the axion's wave nature (the analogous effects for the modulus are negligible on the scales of interest). Balancing wave (quantum) pressure against gravity in Eq. (6) identifies an effective axion Jeans scale in the background of the modulus:

$$k_J^{(\text{phys})} = \frac{k_J}{R} \equiv \left[16\pi G \bar{\rho}_{\text{tot}} m_a^2 \left(\frac{\delta_{\phi,k}}{\delta_{a,k}} \right) \right]^{1/4}. \quad (7)$$

Axion modes at $k < k_J$ grow, while those at $k > k_J$ oscillate. Unusually, k_J depends on the ratio $\delta_{\phi,k}/\delta_{a,k}$. This is because a larger perturbation in the modulus density sources a deeper gravitational well, allowing a greater axion density to build up before its quantum pressure matches the gravitational force.

When $T \gg 100$ MeV the axion mass is suppressed, so the Jeans scale is small and rapidly increasing, and the relevant modes grow negligibly. Once the mass saturates, the evolution is governed by a critical scale

$$k_{\text{crit}} \equiv R(16\pi G \bar{\rho}_{\text{tot}} m_a^2)^{1/4}, \quad (8)$$

which grows as $R^{1/4}$. Modes at $k < k_{\text{crit}}$ reach $\delta_a \simeq \delta_\phi$, so the axion density power spectrum $\Delta_a^2(k)$ tracks $\Delta_\phi^2(k)$. Meanwhile, modes at $k > k_{\text{crit}}$ are limited to $\delta_{a,k}/\delta_{\phi,k} \simeq (k_{\text{crit}}/k)^4$ by the Jeans barrier.

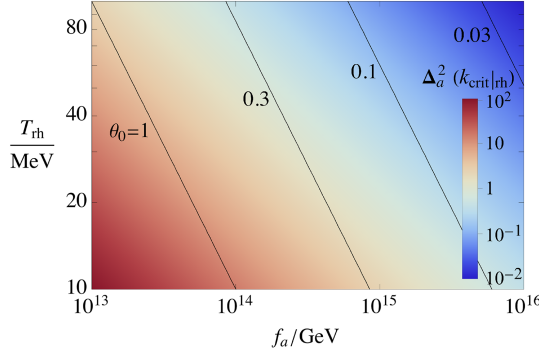


FIG. 1. The peak amplitude of the axion linear power spectrum at the end of EMD $\Delta_a^2(k_{\text{crit}}|_{\text{rh}})$. The misalignment angle θ_0 for the axion to comprise the full dark matter abundance is also shown.

Given that $\Delta_\phi^2 \propto k^4$, we obtain

$$\Delta_a^2(k) \sim \begin{cases} \Delta_0^2(k/(RH))^4 & \text{if } k < k_{\text{crit}}(t) \\ \Delta_0^2(k_{\text{crit}}/(RH))^4 (k_{\text{crit}}/k)^4 & \text{otherwise.} \end{cases} \quad (9)$$

The axion power at reheating peaks at $k_{\text{crit}}|_{\text{rh}}$, with

$$\begin{aligned} \Delta_a^2(k_{\text{crit}}|_{\text{rh}}) &\sim \Delta_\phi^2(k_{\text{crit}}|_{\text{rh}}) \\ &\sim 400 \left(\frac{10 \text{ MeV}}{T_{\text{rh}}} \right)^4 \left(\frac{10^{14} \text{ GeV}}{f_a} \right)^2. \end{aligned} \quad (10)$$

Consequently, as shown in Fig. 1, broad classes of EMD scenarios lead to large inhomogeneities in the axion field.

If ϕ perturbations on the scale k_{crit} go nonlinear, some axions become gravitationally bound to the resulting collapsed structures (simulations show that structures at smaller spatial scales have little effect on the axion power spectrum, as the axion's de Broglie wavelength would exceed the object's size). These axions seed power on smaller comoving length scales. For some (f_a, T_{rh}) modes slightly below k_{crit} also reach $\Delta_a^2(k) \sim 1$.

IV. TO MRE

As the modulus decays at the end of EMD the Universe becomes radiation dominated and the gravitational potential decreases. Axions that were not bound to modulus fluctuations coast with the redshifting peculiar velocities acquired during EMD. Meanwhile, axions that were bound to collapsed modulus structures become unbound and free-stream, while retaining their comoving momentum [34].

Numerical simulations, which we discuss below, show that if $\Delta_a^2(k_{\text{crit}}|_{\text{rh}}) \lesssim 1$ at reheating, the axion power spectrum remains approximately unchanged and retains a peak near $k_{\text{crit}}|_{\text{rh}}$. If $\Delta_a^2(k_{\text{crit}}|_{\text{rh}}) \gtrsim 1$ at reheating, the axion power spectrum relaxes to have a peak of order unity. Provided $\Delta_a^2(k_{\text{crit}}|_{\text{rh}})$ is not many orders of magnitude larger than one, the peak is within a factor of a few of $k_{\text{crit}}|_{\text{rh}}$. This

reflects the characteristic momentum scale of the axions at reheating: since the axion energy density is quadratic in the field, it inherits structure on comparable scales.

The axions then evolve freely until MRE, when their density perturbations begin to dominate. The axion quantum Jeans scale at this time is $k_J|_{\text{mre}} = R_{\text{mre}}(16\pi G\bar{\rho}_{\text{mre}}m_a^2)^{1/4}$. Crucially, this parametrically coincides with the peak of the axion power spectrum:

$$\begin{aligned} \frac{k_{\text{crit}}|_{\text{rh}}}{k_J|_{\text{mre}}} &= \frac{R_{\text{rh}}(16\pi G\bar{\rho}_{\text{tot,rh}}m_a^2)^{1/4}}{R_{\text{mre}}(16\pi G\bar{\rho}_{\text{mre}}m_a^2)^{1/4}} \\ &\simeq \frac{R_{\text{rh}}T_{\text{rh}}}{R_{\text{mre}}T_{\text{mre}}} \simeq 1, \end{aligned} \quad (11)$$

where we used $RT = \text{const.}$ during radiation domination, and neglected corrections from changes in the number of relativistic degrees of freedom and transitions between epochs [these modify Eq. (11) by a factor less than 1.5].

V. AXION STARS

The largest axion density contrast perturbations, which are typically $\mathcal{O}(1)$, therefore lie at the boundary between gravitational collapse and quantum pressure support at MRE. These form axion stars: self-gravitating solitonic configurations in which gravity is balanced by quantum pressure rather than virial motion.

Ground-state axion stars are spherically symmetric with density profile

$$\rho(r) = \rho_c \chi\left(\frac{r}{\lambda_J(\rho_c)}\right), \quad (12)$$

where ρ_c is the central density, $\lambda_J(\rho_c) = 2\pi/k_J(\rho_c)$, and χ is a universal dimensionless profile with a characteristic core [53–56]. An axion star's mass M and half-mass radius $R_{1/2}$ satisfy $MR_{1/2} \simeq 3.9/(Gm_a^2)$ and $\rho_c \simeq G^3 m_a^6 M^4 / (64\pi)$, with the axion's de Broglie wavelength of order $R_{1/2}$.

The stars are produced with excited quasinormal modes. Also, they are surrounded by halos, where both quantum pressure and velocity dispersion are important and the profile deviates from a pure soliton [57,58].

Overdensities with $\delta_a > 1$ collapse at $R/R_{\text{mre}} \sim 1/\delta_a$, producing axion stars with $\rho_c \simeq \bar{\rho}_{\text{mre}}(1 + \delta_a)\delta_a^3 \sim (1 - 10^4)\text{eV}^4$ [59]. The masses of stars from perturbations on a scale k_δ are

$$M \simeq (1 + \delta_a) \left(\frac{k_J|_{\text{mre}}}{k_\delta} \right)^3 M_{J,\text{mre}}, \quad (13)$$

with $M_{J,\text{mre}} = \lambda_J(\bar{\rho}_{\text{mre}})^3 \bar{\rho}_{\text{mre}}$

$$\simeq 2 \cdot 10^{-13} M_\odot \left(\frac{f_a}{10^{14} \text{ GeV}} \right)^{3/2}. \quad (14)$$

Since $k_\delta \simeq k_J|_{\text{mre}}$, these values lie well below the maximum stable axion star mass from the quartic interaction, $10^{-11} M_\odot f_a / (10^{14} \text{ GeV})$ [60].

After MRE, k_J grows allowing smaller-scale overdensities to collapse into lower-mass, lower-density stars. Initially perturbative modes with $k < k_J|_{\text{mre}}$ also grow and collapse. The resulting structures increasingly deviate from the solitonic solution. Some early-formed stars are bound inside larger structures, in a process analogous to hierarchical structure formation, and later merge into halos sourced by modes that were superhorizon during EMD, including, eventually, galaxies.

VI. SIMULATIONS

To test the analysis above and determine numerical coefficients, we perform simulations from EMD through MRE. The key results are presented here, with further details in Appendix C.

We solve the evolution of the axion and modulus fields from $T = 100 \text{ MeV}$ on a periodic lattice in a cosmological background. The box size and resolution are such that all modes are subhorizon and nonrelativistic, reducing the dynamics to the Schrödinger-Poisson system. The modulus is initialised as a Gaussian random field with power spectrum of Eq. (5), while the axion is initially homogeneous. Modulus decay is modeled by uniformly reducing its energy density and transferring this to a homogeneous radiation component. The lattice UV scale is taken well above $k_{\text{crit}}|_{\text{rh}}$. To control UV artifacts, we use a reduced modulus mass, such that its Jeans scale lies above $k_{\text{crit}}|_{\text{rh}}$ throughout. We have verified that systematic uncertainties from this modification, as well as the finite box size, timestep, and lattice spacing, are negligible.

During EMD, $\Delta_a^2(k) \simeq \Delta_\phi^2(k)$ for $k \lesssim k_{\text{crit}}$, while power at $k \gtrsim k_{\text{crit}}$ is initially suppressed but grows if modes around k_{crit} become nonlinear. After reheating, Δ_a^2 develops the expected peak near $k_{\text{crit}}|_{\text{rh}} \simeq k_J|_{\text{mre}}$, both in the perturbative and nonlinear cases. For $T_{\text{rh}} = 20 \text{ MeV}$, the peak is at $k \simeq 1.5 k_J|_{\text{mre}}$ for $f_a = 10^{15} \text{ GeV}$. It shifts to smaller $k/k_J|_{\text{mre}}$ for larger f_a ($0.5 k_J|_{\text{mre}}$ for $f_a = 10^{16} \text{ GeV}$) and to larger k for smaller f_a ($3 k_J|_{\text{mre}}$ for $f_a = 3 \times 10^{14} \text{ GeV}$), corresponding to more linear and more nonlinear evolution respectively.

During radiation domination the axion power spectrum is nearly constant. Shortly before MRE, bound structures form, indicated by the decoupling of the densest regions from Hubble expansion and the growth of small-scale power.

In Fig. 2 we show the spherically averaged density profiles of the 20 densest objects at $R = R_{\text{mre}}$ for $f_a = 10^{15} \text{ GeV}$ with $T_{\text{rh}} = 20 \text{ MeV}$ in a typical simulation. Each profile is rescaled by its central density ρ_c and $\lambda_J(\rho_c)$, so that solitonic configurations take a universal form. For $r \lesssim \lambda_J(\rho_c)$, the profiles closely follow this,

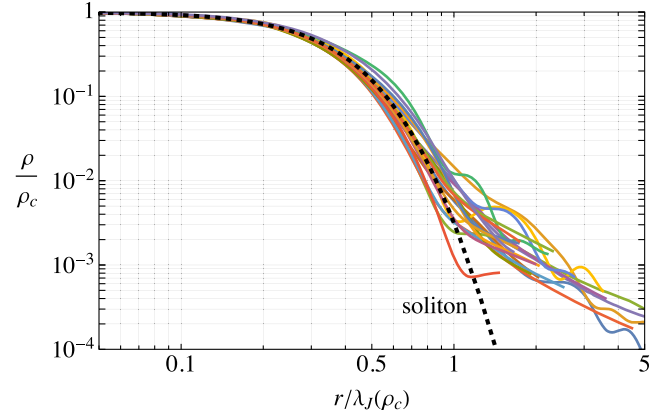


FIG. 2. Spherically averaged density profiles of the densest objects at MRE in an example simulation (colored), compared to the solitonic axion star prediction (dashed black).

indicating substantial axion star cores. At larger radii they deviate, resembling halos but with the axion de Broglie wavelength comparable to their size. In the central regions, the radial component of the quantum pressure balances the gravitational potential gradient.

Figure 3 shows the fraction of axions in stars, and in stars plus surrounding halos, for different f_a , with $T_{\text{rh}} = 20 \text{ MeV}$. Stars are defined as regions whose profiles match the soliton form within a factor of 1.25 (relaxing the agreement criterion to 1.5 increases the fraction of axions in stars by roughly a factor of 1.3). Halos are taken to extend to $5\bar{\rho}_a$, which is approximate since their edges are not sharply defined. Simulations end once the densest solitons are no longer resolved.

The fraction of axions in stars exceeds 15% for $f_a = 10^{15} \text{ GeV}$, with most forming around MRE. The densest stars appear first, followed by less-dense, lower-mass ones. The early stars have masses a few times $M_{J,\text{mre}}$

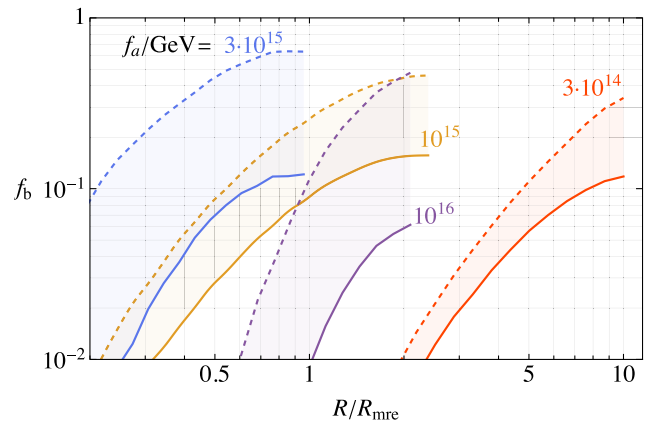


FIG. 3. The fraction f_b of axions bound in axion stars (solid) and in stars and their halos (dashed) as a function of the scale factor for different f_a , with $T_{\text{rh}} = 20 \text{ MeV}$.

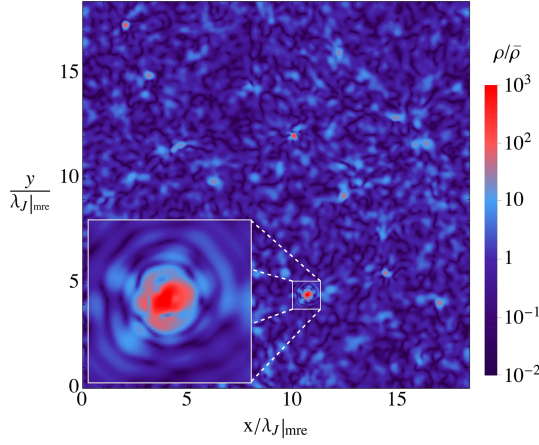


FIG. 4. The axion energy density through a slice of a simulation, relative to the mean density $\bar{\rho}$, at $R/R_{\text{mre}} = 2$ for $f_a = 10^{15}$ GeV and $T_{\text{rh}} = 20$ MeV.

and densities up to $10^5 \bar{\rho}_{\text{mre}}$, while later stars are around $M_{J,\text{mre}}$. The total bound fraction reaches 50%.

The qualitative behavior is similar for 3×10^{14} GeV $\lesssim f_a \lesssim 10^{16}$ GeV with $T_{\text{rh}} = 20$ MeV, corresponding to $\Delta_a^2(k_{\text{crit}|_{\text{rh}}})$ from less than one to nonlinear at reheating. For f_a near the lower end of this range, the peak is shifted to slightly larger $k/k_{J|mre}$, delaying formation of stars until a little after MRE. For f_a towards the upper end, the spectrum is slightly more infrared so the bound objects are further from isolated solitons (with soliton fraction of about 8% for $f_a = 10^{16}$ GeV). In all cases, the total bound fraction exceeds 30%. Smaller f_a cannot be probed due to too much power reaching the lattice scale. More generally, the properties of the bound structures are controlled by $\Delta_a^2(k_{\text{crit}|_{\text{rh}}})$, with different (f_a, T_{rh}) giving similar results when this is fixed.

A slice of the axion density soon after MRE is plotted in Fig. 4, showing a typical star and halo.

VII. DISCUSSION

A key open question is the subsequent evolution of axion stars and their host halos. These may grow through accretion and mergers, potentially approaching the stability limit, or be disrupted as they are incorporated into larger structures [61–64]. Analytic estimates for postinflationary axions suggest the solitons that form around MRE are relatively robust [65], but dedicated simulations are required. Improved numerical methods, such as adaptive meshes, will be important for resolving these dynamics, as well as probing regimes with $\Delta_a^2(k_{\text{crit}|_{\text{rh}}}) \gg 1$.

Our results have implications for direct detection searches for axions with $m_a \lesssim 10^{-5}$ eV, where EMD is best motivated, e.g., [66,67]. If an $\mathcal{O}(1)$ fraction of axions are bound in substructure, and this survives disruption in the Solar neighbourhood, the smooth background is

reduced, weakening experimental signals. The densest axion stars, which have masses around $M_{J,\text{mre}}$ defined in Eq. (14), have radii

$$R_{1/2} \simeq 1.6 \cdot 10^{11} \text{ m} \left(\frac{f_a}{10^{14} \text{ GeV}} \right)^{1/2} \frac{M_{J,\text{mre}}}{M}. \quad (15)$$

Collisions between the Earth and these stars are extremely rare, with the typical interval between events

$$t_{\text{coll}} \simeq 10^4 \text{ yr} \left(\frac{f_a}{10^{14} \text{ GeV}} \right)^{1/2}, \quad (16)$$

assuming 20% of dark matter is in objects of mass $0.1 M_{J,\text{mre}}$. More frequent transient enhancements may arise from larger-scale modes ($k < k_{\text{crit}|_{\text{rh}}}$) or streams from disrupted objects [68], motivating broadband searches [66]. New indirect signals may also arise [69–80], e.g., photon emission during axion star–neutron star encounters [81,82] or from caustic microlensing [83]. Although we focused on the QCD axion, similar conclusions apply to axionlike particles with EMD. It would also be interesting to explore the impact of nongravitational couplings between the axion and modulus, as may arise in specific string compactifications.

Axion substructure also arises in the post-inflationary scenario [59,84–87]. In this case, the axion power spectrum, relative to $k_{J|mre}$, is similar to the case we have considered, leading to a comparable population of axion stars [58]. However, the axion masses in our scenario are much smaller than in the post-inflationary case in the absence of EMD ($m_a \gtrsim 10^{-4} - 10^{-3}$ eV) [88–91]. In a standard cosmological history preinflationary axions produce substructure only for $\theta_0 \simeq \pi$, corresponding to large m_a [52]. A discovery of QCD axion stars at $m_a \lesssim 10^{-5}$ eV would therefore provide strong evidence for EMD. Conversely, a discovery of a QCD axion without substructure would constrain the early evolution of the Universe beyond current bounds from BBN.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: RESONANCES DURING AXION MASS TURN-ON

Spatial inhomogeneities in the plasma temperature arising from adiabatic perturbations can resonantly excite high-momentum axion modes during the period when the axion mass is changing, $T \gtrsim 100$ MeV [92–95]. In a standard cosmological history this effect is too weak to generate significant axion substructure unless the axion is initially very close to the top of its potential with misalignment angle $\theta_0 \gtrsim \pi - 10^{-3}$. Here we extend the analysis of Ref. [93] to EMD and show that the resonant effect remains negligible.

The homogeneous mode of the rescaled axion field $\theta \equiv a/f_a$ satisfies the equation of motion

$$\theta'' + 2\mathcal{H}\theta' + R^2 m_a^2(T) \sin \theta = 0. \quad (\text{A1})$$

Unlike the main text, we parametrize the temperature dependence of the axion mass with the smooth function

$$m_a^2(T) = \frac{m_a^2}{1 + (T/\Lambda)^\alpha}, \quad (\text{A2})$$

where Λ and α are defined as in Eq. (2). We work in conformal time η , with units such that $\eta = 1$, $R = R_c$ and $H = H_c$ when $T = \Lambda$. We also assume pure modulus domination for $T > T_{\text{rh}}$, such that $R \sim \eta^2$ and $T \sim R^{-3/8}$ throughout this time. The linear Fourier space perturbation mode of the axion field $\delta\theta_k$ evolves according to

$$\delta\theta_k'' + \frac{4}{\eta}\delta\theta_k' + [k^2 + R^2 m_a^2(T) \cos \theta] \delta\theta_k = j_k, \quad (\text{A3})$$

where the source term j_k has the form

$$j_k = R^2 m_a^2(T) \sin \theta \left(2\Phi_k + \frac{\alpha}{1 + \eta^{3\alpha/4}} \frac{\delta T_k}{T} \right). \quad (\text{A4})$$

The first term corresponds to the growth due to axions falling onto the perturbations of the gravitational potential Φ_k , while the second term is associated with the effect of temperature inhomogeneities.

Temperature inhomogeneities are related to density perturbations in the radiation bath by

$$\frac{\delta T_k}{T} = \frac{1}{4} \frac{\delta \rho_{r,k}}{\bar{\rho}_r} \equiv \frac{1}{4} \delta_{r,k}. \quad (\text{A5})$$

The evolution of $\delta_{r,k} \equiv \delta_r(\mathbf{k})$ sourced by a decaying modulus has been studied in [37,41]. Subhorizon density perturbations in radiation initially grow together with the modulus; however, unlike the modulus, radiation inhomogeneities quickly saturate and oscillate with constant amplitude due to pressure. Assuming adiabatic initial conditions $\delta_{r,k}^{(0)} = -\Phi_k$, the maximum value of $\delta_{r,k}$ can be obtained numerically (see e.g. Ref. [41]):

$$\frac{\delta T_k}{T} \simeq \begin{cases} \Phi_k/4 & \text{if } k/aH < 1 \\ 9\Phi_k/4 & \text{otherwise.} \end{cases} \quad (\text{A6})$$

Φ_k is constant in time on all scales during matter domination.³ As in the main text, we take the typical amplitude of primordial gravitational potential fluctuations to be $\Phi = 3 \times 10^{-5}$, as set by quantum fluctuations during inflation, so the linear analysis remains valid until near the end of EMD.

We solve Eqs. (A1) and (A3) numerically and study the evolution of the axion energy density contrast $\delta_{a,k}$ with and without temperature inhomogeneities. As expected from Eq. (A4), at early times the effect of temperature perturbations on the growth of subhorizon modes dominates over direct gravitational interactions, but it quickly becomes negligible as the axion mass saturates.

In Fig. 5, we plot the axion energy density perturbations $\delta_{a,k}$ through the axion mass turn-on and up to reheating ($3\Lambda > T > T_{\text{rh}}$, where we assume $T_{\text{rh}} = 20$ MeV), normalised by Φ_k for modes with $k = k_{\text{crit}}|_{\text{rh}}$ and a range of f_a . We set the initial misalignment angle $\theta_0 = 0.2$ in all cases; however, $\delta_{a,k}$ is approximately independent of this choice provided $\theta_0 \lesssim 1$, so that $\cos \theta \simeq 1$ and $\sin \theta \simeq \theta$ in Eq. (A3). We show the results obtained including both terms in Eq. (A4) and only the direct effect of Φ_k . The temperature effects enhance $\delta_{a,k}$ by roughly an order of magnitude when $T = \Lambda$, while at later times their effect becomes negligible as the gravitational collapse catches up. By reheating, the effects from the δT_k contribution are completely washed out.

Figure 5 also shows the growth of $\delta_{a,k}$ assuming the axion field is homogeneous at $\eta = 1$ and without temperature fluctuations, which is analogous to the evolution implemented in our simulations (see Appendix C). For the range of decay constants probed in the simulations ($f_a \gtrsim 3 \times 10^{14}$ GeV), the density contrast quickly grows to match the values obtained by the full axion evolution across mass turn-on.

³More precisely, the saturation value can differ from Eq. (A6) by some $\mathcal{O}(1)$ factor, set by the exact value of the ratio $\Gamma_\phi/H_{\text{rh}}$. Including this factor does not affect our conclusions.

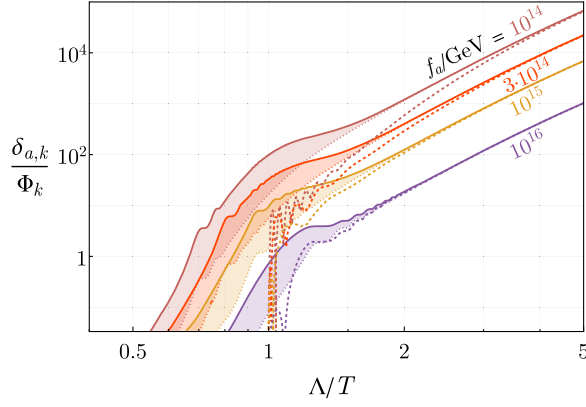


FIG. 5. Growth of axion density perturbations during axion mass turn-on and the subsequent evolution until reheating, computed in linear perturbation theory and normalized to the gravitational potential perturbation Φ_k sourcing them. Results are shown for the mode with wave number $k = k_{\text{crit}|_{\text{rh}}}$ for different f_a , with $T_{\text{rh}} = 20$ MeV in all cases. Solid curves include the effects of temperature inhomogeneities, while dotted curves are obtained by setting $\delta T_k = 0$. Dashed curves show the evolution with the axion field taken to be homogeneous at $T = \Lambda$ and neglecting temperature inhomogeneities.

We also verified numerically that temperature inhomogeneities do not have a significant effect at $k < k_{\text{crit}|_{\text{rh}}}$ or $k > k_{\text{crit}|_{\text{rh}}}$ and hence do not affect the form of the power spectrum derived in the main text, which remains peaked around k_{crit} . In particular, we find that $\delta_{a,k} \sim k^{-2}$ for modes $k \gg k_{\text{crit}|_{\text{rh}}}$, consistent with $\Delta_a^2(k) \sim k^{-4}$.

We therefore conclude that the resonant excitation can be neglected within the parameter space of interest and does not affect the conclusions of the main text.

APPENDIX B: FLUID DESCRIPTION

To study the effect of quantum pressure on the evolution of axion perturbations, we treat the axion field as a perfect fluid with equation of state $w_a = \bar{P}_a/\bar{\rho}_a \approx 0$ and sound speed $c_{s,a}^2 \equiv \delta P_a/\delta \rho_a = k^2/(4m_a^2 R^2)$. This is valid in the non-relativistic limit $k/R \ll m_a$ [96], assuming that the axion has started to oscillate and that its mass has saturated ($T \lesssim 100$ MeV). Conservation of the stress-energy tensor then gives the continuity and Euler equations in Newtonian gauge [97]

$$\delta' = -(1+w)(\nabla \cdot \mathbf{v} - 3\Phi') - 3\mathcal{H}(c_s^2 - w)\delta, \quad (\text{B1})$$

$$\mathbf{v}' = -\mathcal{H}(1-3w)\mathbf{v} - \frac{c_s^2}{1+w}\nabla\delta - \nabla\Phi, \quad (\text{B2})$$

where primes denote derivatives with respect to conformal time, η . Using that $\Phi' \ll \nabla \cdot \mathbf{v}$ in the subhorizon limit this gives the axion fluid equation of motion

$$\delta''_{a,k} + \mathcal{H}\delta'_{a,k} + c_{s,a}^2 k^2 \delta_{a,k} = -k^2 \Phi_k \quad (\text{B3})$$

for the axion fluid. Equation (B3) can equivalently be derived from the axion field equation (A3) (in the case where the mass is temperature independent) by first taking the nonrelativistic limit, as described in Appendix C, to give the Schrödinger equation (C2). Then, carrying out a Madelung transformation by writing the nonrelativistic axion field $\psi_a = \sqrt{\rho_a R^3} e^{i\theta_a}$, one obtains a perfect fluid system with density ρ_a and velocity $v_a = (Rm_a)^{-1} \nabla \theta_a$ (see e.g. [57]). Working at linear order in perturbations from homogeneity and eliminating v_a finally gives Eq. (B3).

During EMD, the gravitational perturbations are dominantly sourced by the modulus and are determined by Poisson's equation

$$\nabla^2 \Phi = 4\pi G R^2 \bar{\rho}_{\text{tot}} \delta\phi, \quad (\text{B4})$$

so that the equation for the axion perturbations becomes

$$\delta''_{a,k} + \mathcal{H}\delta'_{a,k} + c_{s,a}^2 \delta_{a,k} \left(k^2 - \frac{4\pi G R^2 \bar{\rho}_{\text{tot}}}{c_{s,a}^2} \frac{\delta\phi_k}{\delta_{a,k}} \right) = 0. \quad (\text{B5})$$

The sign of the final term determines whether the perturbations grow or oscillate, making it useful to introduce the Jeans scale

$$k_J^{(\text{phys})} = \frac{k_J}{R} \equiv \left[16\pi G \bar{\rho}_{\text{tot}} m_a^2 \left(\frac{\delta\phi_k}{\delta_{a,k}} \right) \right]^{1/4}, \quad (\text{B6})$$

which corresponds to the length scale below which quantum pressure prevents perturbations from growing. As discussed in the main text, k_J differs from the usual expression by a factor of $(\delta\phi_k/\delta_{a,k})^4$, which does not appear in the Jeans scale of the species that dominates the energy density of the Universe, since in that case the gravitational perturbations and the quantum pressure are both sourced by the same overdensity.

The ratio between the axion and modulus density contrast, obtained by solving Eq. (B3) numerically with an initially homogeneous axion field at $T_{\text{rh}}/T = 1/5$, is shown in Fig. 6 for a range of wave numbers. For $k < k_{\text{crit}}$, $\delta_{a,k}$ catches up with $\delta\phi_k$ (for superhorizon modes the growth of these adiabatic perturbations is guaranteed by Weinberg's theorem [98]). However, for $k > k_{\text{crit}}$ the growth is suppressed by the Jeans barrier to

$$\frac{\delta_{a,k}}{\delta\phi_k} = \left(\frac{k_{\text{crit}}(T)}{k} \right)^4 = \left(\frac{k_{\text{crit}|_{\text{rh}}}}{k} \right)^4 \left(\frac{T_{\text{rh}}}{T} \right)^{8/3}, \quad (\text{B7})$$

the predictions of which are displayed in dashed black. The initial decaying oscillations are due to a mode which can be found by solving the homogeneous part of Eq. (B3).

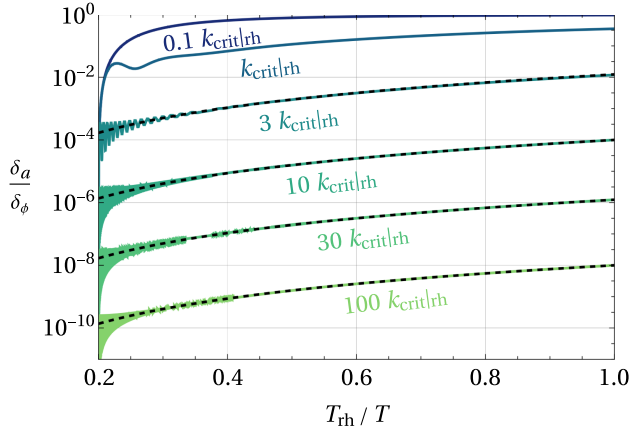


FIG. 6. The ratio between the axion and modulus density contrast during EMD, found by numerically solving Eq. (B3) for a range of wave numbers relative to the critical wave number at reheating. The black dashed lines show the Jeans barrier prediction that $\delta_a / \delta_\phi = [k_{\text{crit}}(T)/k]^4$. In all cases $T_{\text{rh}} = 20$ MeV and $f_a = 10^{14}$ GeV.

APPENDIX C: FURTHER SIMULATION DETAILS

In numerical simulations, we evolve the nonrelativistic axion field ψ_a , related to the relativistic axion field a by

$$a = \frac{1}{\sqrt{2m_a^2 R^3}} (\psi_a e^{-im_a t} + \text{c.c.}), \quad (\text{C1})$$

and analogously for the nonrelativistic modulus field ψ_ϕ , with m_ϕ in place of m_a . In the limit $\partial_t \psi_{\phi,a} \ll m_{\phi,a} \psi_{\phi,a}$ and $\partial_t^2 \psi_{\phi,a} \ll m_{\phi,a}^2 \psi_{\phi,a}$, the equations of motion for subhorizon modes reduce to

$$\begin{aligned} \left(i\partial_t + \frac{\nabla^2}{2m_a R^2} - m_a \Phi \right) \psi_a &= 0, \\ \left(i\partial_t + \frac{\nabla^2}{2m_\phi R^2} - m_\phi \Phi \right) \psi_\phi &= 0, \\ \nabla^2 \Phi &= \frac{4\pi G}{R} (|\psi_a|^2 - \langle |\psi_a|^2 \rangle \\ &\quad + |\psi_\phi|^2 - \langle |\psi_\phi|^2 \rangle), \end{aligned} \quad (\text{C2})$$

where t is the cosmological time and spatial derivatives are with respect to comoving distances. From Eq. (C1), the energy densities of the axion and modulus fields are given by $\rho_{a,\phi} = |\psi_{a,\phi}|^2 / R^3$.

The background cosmological expansion is obtained by solving the Friedmann equations

$$\frac{\dot{R}}{R} = H = \sqrt{\frac{8\pi G}{3} \rho_{\text{tot}}}, \quad (\text{C3})$$

with total energy density $\rho_{\text{tot}} = \rho_{\text{rad}} + \rho_a + \rho_\phi$, where ρ_{rad} is a smooth radiation component (we neglect Standard Model matter). The energy densities evolve according to

$$\begin{aligned} \frac{d\langle \rho_\phi \rangle}{dt} &= -\Gamma_\phi \langle \rho_\phi \rangle - 3H \langle \rho_\phi \rangle, \\ \frac{d\langle \rho_r \rangle}{dt} &= +\Gamma_\phi \langle \rho_\phi \rangle - 4H \langle \rho_r \rangle, \\ \frac{d\langle \rho_a \rangle}{dt} &= -3H \langle \rho_a \rangle, \end{aligned} \quad (\text{C4})$$

where Γ_ϕ is the modulus decay rate to radiation. Given Eq. (C1), $\langle |\psi_a|^2 \rangle$ remains constant in time.

To account for modulus decay, which is not included in Eq. (C2), we uniformly rescale ψ_ϕ at each timestep such that Eq. (C4) is satisfied. To account for the radiation rapidly dispersing out of previous overdensities, we take this to be spatially homogeneous (the same approach was used in N -body simulations of particle dark matter during EMD in Ref. [45]). During EMD, $\langle |\psi_\phi|^2 \rangle \gg \langle |\psi_a|^2 \rangle$, and at the end of EMD $\langle |\psi_\phi|^2 \rangle$ falls towards zero (in practice, we stop evolving the modulus field once it contains a negligible energy density).

We solve Eq. (C2) on a lattice of fixed comoving size and lattice spacing with periodic boundary conditions. We use cubic lattices typically containing approximately 1000^3 points; given the need for multiple simulation runs, this is the largest feasible size with our available computational resources. We use a standard second-order pseudospectral algorithm, in which the part of the evolution involving the Laplacian is carried out in momentum space and the potential part in real space [99,100]. This approach conserves $\langle |\psi_{a,\phi}|^2 \rangle$ up to machine precision and does not lead to unphysical growing modes or other instabilities. All resolved modes satisfy $k/R \ll m_a$ throughout, ensuring the validity of the nonrelativistic approximation.

The power spectrum of modulus density fluctuations $P_{\delta,\phi}$ is defined by

$$\langle \delta_{\phi,k} \delta_{\phi,k'} \rangle = (2\pi)^3 \delta^{(3)}(k - k') P_{\delta,\phi}(k), \quad (\text{C5})$$

where $\delta_{\phi,k}$ is the Fourier transform of $\delta_\phi(x) = \rho_\phi(x) / \bar{\rho}_\phi - 1$. The dimensionless ϕ power spectrum Δ_ϕ^2 is given by

$$P_{\delta,\phi} = \frac{2\pi^2}{k^3} \Delta_\phi^2(k). \quad (\text{C6})$$

The axion density power spectra are defined analogously.

We begin simulations during EMD when the temperature of the Universe is $T = 100$ MeV. At this time, the modulus power spectrum is

$$\Delta_\phi^2(k) = 0.0023 \left(\frac{k}{Rm_a} \right)^4 \left(\frac{T_{\text{rh}}}{10 \text{ MeV}} \right)^8 \left(\frac{10^{14} \text{ GeV}}{f_a} \right)^4, \quad (\text{C7})$$

corresponding to the growth of subhorizon modes from an initially scale-invariant spectrum during EMD

$$\Delta_\delta^2|_{\text{entry}} = 3.36 \times 10^{-10}. \quad (\text{C8})$$

The initial axion field is taken to be spatially homogeneous. We have checked that, since axion perturbations are driven by the growing ϕ perturbations, the difference between starting from a homogeneous axion field and from one with small initial fluctuations is negligible. In particular, as shown in Appendix A, the axion power spectrum at the end of EMD is insensitive to this choice for $f_a \geq 3 \times 10^{14} \text{ GeV}$ with $T_{\text{rh}} = 20 \text{ MeV}$, as used in our simulations. For smaller f_a , more realistic initial conditions may be required for accurate late-time results.

We make one modification to the physical system: we take the modulus mass (typically tens or hundreds of TeV) to be artificially small while keeping Γ_ϕ fixed. This ensures that quantum pressure in the modulus field suppresses fluctuations at the lattice scale, which would lead to numerical artifacts. Additionally, we give m_ϕ an artificial time dependence proportional to $R^{-1/2}$ such that $k_{J,\phi}$ is time-independent during EMD. We also cut off the initial modulus power spectrum at this wave number. We choose m_ϕ such that $k_{J,\phi}$ is at least a factor of about 5 above the axion mode where there is the most power at the end of EMD, $k_{\text{crit}}|_{\text{rh}}$. With this condition satisfied, our modification affects only the UV behavior of the modulus field and does not impact the axion dynamics (specifically, the properties of the axion field at the end of EMD are unaffected).

Axion self-interactions add a term on the left-hand side of the axion equation of motion in Eq. (C2)

$$-\frac{\lambda}{(2Rm_a)^3} |\psi_a|^2 \psi_a, \quad (\text{C9})$$

with $\lambda \simeq m_a^2/f_a^2$. This is negligible compared to that from the gravitational potential due to the much larger modulus energy density at early times, and it remains unimportant around MRE, consistent with the axion stars lying well below the stability bound.

1. Systematic uncertainties

There are various sources of systematic errors in our numerical simulations. We have tested these and confirmed that the choices used for our main simulations do not lead to substantial uncertainties.

Finite timestep: In simulations, we use the dimensionless time variable $\tilde{t} = m_a \int^t dt'/R(t')^2$, where $R = 1$ at $T = 100 \text{ MeV}$. Since the timestep $\Delta\tilde{t}$ required for

numerical accuracy varies with the gravitational potential, we adapt it throughout simulations as follows. We define

$$\Delta\tilde{t}_k = 2 \frac{m_a^2}{k_{\text{max}}^2}, \quad \Delta\tilde{t}_\Phi = \frac{2}{R^2 \Phi_{\text{max}}}, \quad (\text{C10})$$

where k_{max} is the highest comoving wavenumber mode allowed by the lattice spacing. Schematically, $\Delta\tilde{t}_k$ and $\Delta\tilde{t}_\Phi$ represent the timestep over which the phase accumulated from the Laplacian and gravitational potential is $\mathcal{O}(1)$ respectively in the Schrödinger-Poisson system written in terms of $\tilde{t}$. During EMD and radiation domination we use

$$\Delta\tilde{t} = \min\{\epsilon \sqrt{\Delta\tilde{t}_k \Delta\tilde{t}_\Phi}, 10^3 \epsilon \Delta\tilde{t}_k\}, \quad (\text{C11})$$

where the second possibility ensures the timestep remains finite even during radiation domination when the gravitational potential is negligible. Near MRE, specifically after $\rho_a/\rho_{\text{rad}} = 10^{-3}$, we use

$$\Delta\tilde{t} = \min\left\{\frac{1}{3} \epsilon \sqrt{\Delta\tilde{t}_k \Delta\tilde{t}_\Phi}, 10^2 \epsilon \Delta\tilde{t}_k\right\}. \quad (\text{C12})$$

Here ϵ is a numerical parameter. We find that $\epsilon = 0.2$ provides sufficient accuracy. Rerunning simulations with identical initial conditions with $\epsilon = 0.1$ typically changes the maximum axion energy density at the end of simulations by only a few percent.⁴

Finite box size: We take the box size sufficiently large that the smallest wavenumbers k_{min} satisfy $\Delta_a^2(k_{\text{min}}) \lesssim 0.1$ and $\Delta_\phi^2(k_{\text{min}}) \lesssim 0.1$ throughout the simulations, so box-scale fluctuations remain perturbative. We have checked that increasing the box size by a factor of two from the values used in our main runs does not alter the form of Δ_a^2 at the end of EMD or after free-streaming. Similarly, provided $\Delta_a^2(k_{\text{min}})$ remains less than one around MRE, the fraction of axions bound is unaffected by the finite box size. This is consistent with results for vector dark matter produced by inflation [57] and postinflationary axions [58], which have similar spectra and dynamics around MRE.

Lattice spacing: Increasing the resolution by a factor of two does not affect the axion spectrum at the end of EMD, provided not too much modulus power accumulates at the lattice scale. Around MRE, finite lattice-spacing effects are not significant while the densest solitons, which are the smallest, still contain at least a few lattice points within their cores. We stop simulations when this is no longer satisfied.

Modulus mass: Since the initial $\Delta_\phi^2 \propto k^4$, we cannot place the modulus Jeans cutoff too far above $k_{\text{crit}}|_{\text{rh}}$ without generating too much UV power, which would lead to

⁴At occasional time steps the difference is larger, but these errors do not accumulate.

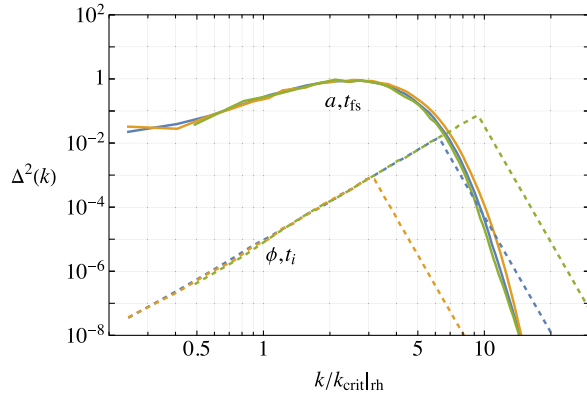


FIG. 7. The axion density power spectrum during radiation domination after free-streaming a, t_{fs} (solid lines) for $f_a = 10^{15}$ GeV and $T_{rh} = 20$ MeV, with different modulus Jeans scales and cutoffs. The corresponding initial modulus power spectra are also shown ϕ, t_i (dashed).

numerical artifacts. Typically, we take the cutoff at about $5k_{crit|_{rh}}$. We have checked that the axion power spectrum at the end of EMD is unaffected when this is varied from 2.5 to $7.5k_{crit|_{rh}}$, with the initial modulus power spectrum also cut off at the Jeans scale. As a representative example, in Fig. 7, we plot the axion density power spectrum deep into radiation domination, for different modulus Jeans scales spanning this range for $f_a = 10^{15}$ GeV and $T_{rh} = 20$ MeV.⁵ The late-time axion power spectrum takes approximately the same form in all cases.

2. Further results

Here we present further results from simulations, for axions with different values of f_a .

In Fig. 8 we plot the density power spectra of the modulus and axion during EMD, and of the axion after subsequent free-streaming. By free-streaming, we mean the regime in which the gravitational potential is negligible and the axions evolve only under their kinetic term, so they escape the previous gravitational wells.

Results are shown for $f_a = 10^{15}$ GeV and $f_a = 10^{16}$ GeV, both with $T_{rh} = 20$ MeV for which $k_{crit|_{rh}} \simeq 0.55k_{J|_{mre}}$. For $f_a = 10^{16}$ GeV, Δ_a^2 remains less than one and increases slightly during free-streaming owing to the peculiar velocities acquired during EMD. The resulting peak lies almost exactly at $k_{crit|_{rh}}$. For $f_a = 10^{15}$ GeV, Δ_a^2 is mildly nonlinear before the end of EMD, leading to enhanced power at somewhat smaller scales, $k \gtrsim k_{crit|_{rh}}$, at reheating. After free-streaming, the power spectrum relaxes to a peak with $\Delta_a^2 \sim 1$ at a scale of about $3k_{crit|_{rh}}$. The peak

is not erased by free-streaming because the axion momentum distribution is peaked around $k_{J|_{rh}}$.

In Fig. 9 we show the axion density power spectrum after free-streaming, well before MRE, for $f_a = 3 \times 10^{14}$ GeV, 10^{15} GeV, 3×10^{15} GeV, and 10^{16} GeV, all with $T_{rh} = 20$ MeV. As f_a is decreased from 10^{16} GeV, the peak approaches the nonlinear regime, as expected from Fig. 1. For smaller f_a , the modulus power spectrum at $k_{crit|_{rh}}$ becomes increasingly nonlinear. This leads to axion power at progressively smaller scales ($k \gtrsim k_{crit|_{rh}}$), due to axions becoming bound to modulus structures before the modulus decays, shifting the peak of Δ_a^2 after free-streaming to scales a factor of a few above $k_{crit|_{rh}}$.

In Fig. 10, we plot the maximum axion energy density in a simulation volume for the same parameter points as in Fig. 9, for representative runs. This is not a strictly physical observable, since it depends on the simulation box size and the particular initial conditions, but it provides a useful indication of the dynamics. Results are shown during EMD, normalised to the time-dependent mean axion density, and around MRE, normalized to the mean axion density at MRE.

During EMD, smaller f_a corresponds to a larger maximum axion energy density, reflecting the larger modulus power spectrum at $k_{crit|_{rh}}$. For $f_a = 10^{16}$ GeV and $f_a = 3 \times 10^{15}$ GeV, the axion power spectrum remains roughly less than one during EMD. After reheating at $R/R_{mre} \simeq 10^{-7}$, the maximum overdensity remains approximately constant during radiation domination in these cases. Conversely, for $f_a = 10^{15}$ GeV and $f_a = 3 \times 10^{14}$ GeV, the maximum overdensity decreases after EMD as axions free-stream out of gravitationally collapsed modulus structures.

Shortly before MRE, the largest overdensities in the axion field are about an order of magnitude denser than the background for $f_a = 3 \times 10^{15}$ GeV, 10^{15} GeV, and 3×10^{14} GeV. These overdensities collapse earliest for $f_a = 3 \times 10^{15}$ GeV, and at successively later times for smaller f_a , as indicated by the growth and subsequent saturation of $\rho_{max}/\bar{\rho}_{mre}$. This is because the peak of Δ_a^2 lies at larger $k/k_{J|_{mre}}$ for smaller f_a , so quantum pressure delays collapse until later times, when k_J has grown to match the peak scale (since $k_J \propto R^{1/4}$). Meanwhile, for $f_a = 10^{16}$ GeV, collapse occurs slightly later than for $f_a = 3 \times 10^{15}$ GeV, because the initial axion fluctuations are smaller.

These results are consistent with Fig. 3 in the main text, which shows that axion stars form later for smaller f_a . For $f_a = 10^{16}$ GeV, the fraction of axions in stars is also slightly smaller (about 5%), while the total bound fraction is similar to the other cases. This occurs because the peak of Δ_a^2 lies at slightly smaller $k/k_{J|_{mre}}$ and has a smaller amplitude, so overdensities collapse slightly later, by which time k_J has grown and exceeds the peak scale.

⁵For the largest cutoff we take a slightly smaller box size to allow a smaller lattice spacing and avoid too much power at the lattice scale.

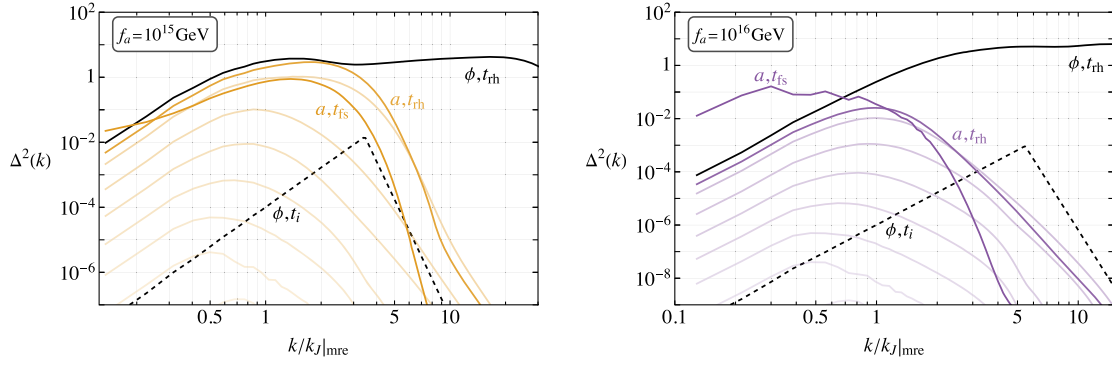


FIG. 8. The axion a and modulus ϕ density power spectra during EMD, for $f_a = 10^{15}$ GeV (left) and $f_a = 10^{16}$ GeV (right), both with $T_{\text{rh}} = 20$ MeV. The modulus power spectrum (black) is shown at the initial simulation time t_i and at the end of EMD t_{rh} . The axion power spectrum (coloured lines) is shown during EMD (partly transparent lines), at reheating, and after free-streaming t_{fs} during the subsequent radiation domination. The scales are shown in terms of the axion quantum Jeans scale at Λ CDM matter-radiation equality $k_J|_{\text{mre}}$.

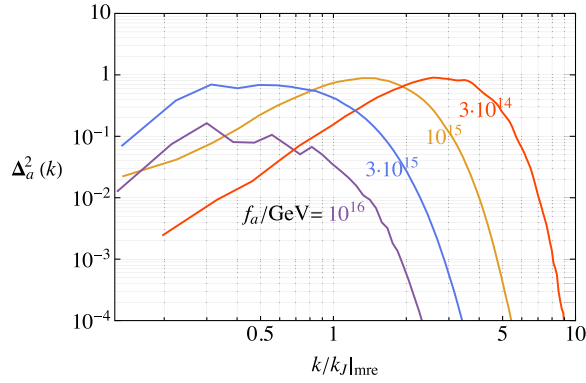


FIG. 9. The axion density power spectrum during radiation domination, for different values of f_a , with $T_{\text{rh}} = 20$ MeV.

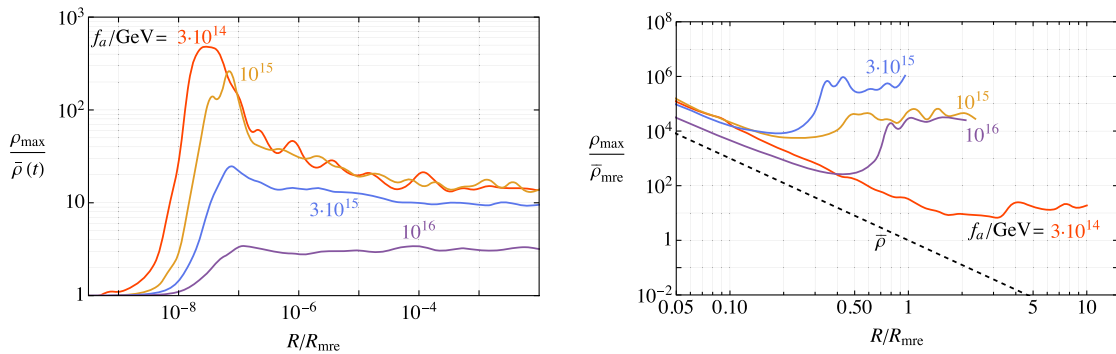


FIG. 10. The maximum axion energy density ρ_{max} in a representative simulation for different f_a with $T_{\text{rh}} = 20$ MeV, as a function of the scale factor normalized to its value at MRE R_{mre} . The left panel shows results around EMD, normalized to the evolving mean energy density $\bar{\rho}(t)$. The peak corresponds to axions gravitationally bound to large modulus perturbations. As the modulus decays, the axions free-stream out of the overdensities and relax to a Gaussian random field, corresponding to the plateau in $\rho_{\text{max}}/\bar{\rho}(t)$. The right panel shows results at late times, normalised to the mean density at MRE, $\bar{\rho}_{\text{mre}}$. The growth and subsequent plateau of $\rho_{\text{max}}/\bar{\rho}_{\text{mre}}$ corresponds to the formation of axion stars.

In Fig. 11 (left), we plot the fraction of axions in stars above given central density thresholds for $f_a = 10^{15}$ GeV and T_{rh} as a function of scale factor (with stars defined as in the main text). The densest stars, with $\rho_c \gtrsim 10^2 \text{ eV}^4$, form first, followed by progressively lighter, less dense stars at later times [corresponding to smaller δ_a or larger $k_\delta/k_J|_{\text{mre}}$ in Eq. (13)]. Even lighter stars are expected to form beyond the range of simulations.

The distribution of axion star masses at different times for $f_a = 10^{15}$ GeV and $T_{\text{rh}} = 20$ MeV is shown in Fig. 11 (right). Specifically, we plot the differential fraction of axions bound in stars of mass M . As expected, the majority of bound axions are contained in stars with masses of order $M_{J,\text{mre}}$, as defined in Eq. (14), corresponding to the mass within a Jeans length at MRE. Most of the mass is in stars with central density $\rho_c \sim 10^2 \bar{\rho}_{\text{mre}}$, while the heaviest stars have densities exceeding $10^4 \bar{\rho}_{\text{mre}}$.

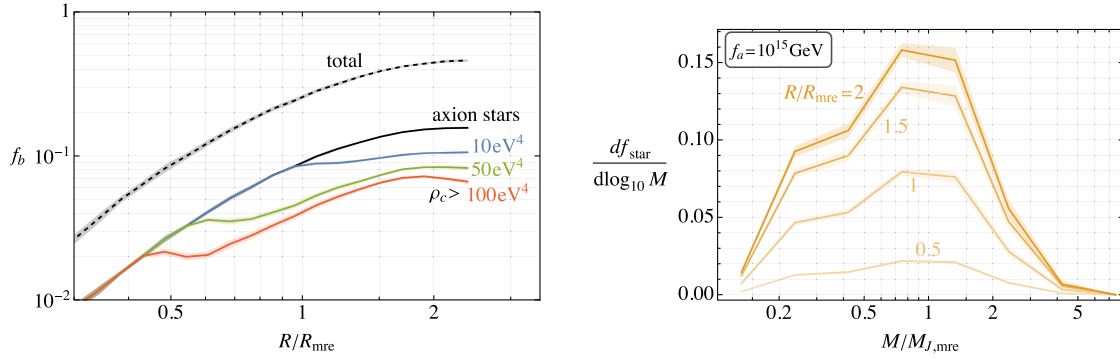


FIG. 11. *Left:* The fraction f_b of axions bound in axion stars (black), stars and the surrounding halos (black dashed), and stars with central densities exceeding the given thresholds (colored) for $f_a = 10^{15} \text{ GeV}$ and $T_{\text{rh}} = 20 \text{ MeV}$ (with statistical uncertainties). The fraction identified soon after formation is affected by the stars being produced in excited states, but this becomes less significant at late times. *Right:* The differential fraction of axions bound in stars, f_{star} , as a function of axion star mass M , evaluated at different times (R/R_{mre}), for the same f_a and T_{rh} as the left panel. The masses are normalized to $M_{J,\text{mre}}$ defined in Eq. (14).

For $f_a = 3 \times 10^{14} \text{ GeV}$, the analogous distribution of axion star masses peaks at slightly smaller values, $M/M_{J,\text{mre}} \simeq 0.5$ (note that the value of $M_{J,\text{mre}}$ depends on f_a), while for $f_a = 3 \times 10^{15} \text{ GeV}$ and 10^{16} GeV the peak remains around $M_{J,\text{mre}}$.

We have checked that, when expressed in units of $k_J|_{\text{mre}}$, the axion power spectrum after free-streaming for $T_{\text{rh}} = 10 \text{ MeV}$ and $f_a = 1.2 \times 10^{15} \text{ GeV}$, $4 \times 10^{15} \text{ GeV}$, and $4 \times 10^{16} \text{ GeV}$ match those for $T_{\text{rh}} = 20 \text{ MeV}$ and

$f_a = 3 \times 10^{14} \text{ GeV}$, 10^{15} GeV , and 10^{16} GeV respectively. Specifically, the differences in the peak position and amplitudes are at most a factor of about 2. These values are chosen so that $\Delta_a^2(k_{\text{crit}}|_{\text{rh}})$, given by Eq. (10), is matched between pairs of (f_a, T_{rh}) , indicating that this is the key quantity controlling the overall evolution of the system and the resulting substructure. A detailed exploration of the dynamics across the full range of phenomenologically interesting (f_a, T_{rh}) will be carried out in future work.

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