

Parity-dependent Casimir forces and Hall currents for a confined Dirac field

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We study a massless Dirac field subjected to two alternative boundary conditions on two parallel thin walls, in $d + 1$ dimensions. The two configurations correspond to the system being even or odd under reflection about the midplane between the two walls, and lead to qualitatively different behaviors. The even (symmetric) configuration produces an attractive Casimir force, whereas the odd (antisymmetric) one yields repulsion, in agreement with a general theorem linking parity to the sign of the fermionic Casimir effect. We complement this result by studying two phenomena associated with the vacuum fluctuations responsible for the Casimir interaction, both of which are also sensitive to parity: the correlation between currents concentrated on the walls, and the induced bulk current under the influence of an external electric field. For the latter we show that, in $2 + 1$ dimensions, an induced transverse (Hall-like) current arises, whose spatial profile inherits the symmetry of the confining potential.

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I. INTRODUCTION

In the presence of nontrivial boundary conditions, the vacuum fluctuations of a quantum field may give rise to macroscopic manifestations, with the Casimir [1,2] and related effects being among the most distinctive ones.

Except for very special cases, like highly diluted media [3–5], the Casimir force between two bodies cannot be obtained as the result of a superposition [6–8]. This, in turn, implies that the relationship between that force and the geometry (shapes and positions) of the intervening bodies, becomes generally rather involved.

In this context, what is perhaps the most relevant property of the effect is its *sign*, namely, whether it is an attractive or repulsive force. An important theoretical result [9] relates this property to the behavior under a parity transformation of the system. Indeed, it states that a sufficient condition for the force between two dielectric or conducting objects to be attractive is the existence of a reflection symmetry with respect to a plane. In a recent work [10], we have studied a Dirac field in the presence of a background having support on two disconnected spatial regions, interpreted as lumps of matter that, in appropriate limits, impose bag model boundary conditions [11–13] on their respective frontiers. The main result was that if a background is odd under

reflection about the midplane between the two objects, they repel.

In this work, we provide a concrete study of the consequences, for some relevant physical observables, of the system being even or odd under such a reflection. The model consists of two thin walls, each one of them imposing boundary conditions on its two faces. Since this leaves open the possibility of symmetric and antisymmetric setups, it is natural to compare their physical properties. In particular, we study the correlator between current operators concentrated on the walls, and we also study the vacuum currents, inside the slab, induced by an external, constant, uniform electric field which is perpendicular to the planes of the walls.

This paper is organized as follows: in Sec. II, we define the physical system, including the action, the singular potential, and the conventions adopted. In Sec. III, we review the form of the Casimir energy for the two parity configurations and also show how the Casimir pressure for this system may be expressed in terms of the vacuum expectation value of a surface current operator, i.e., concentrated on one of the two walls. This is indeed even more clear in the case of $2 + 1$ dimensions. Based on this feature, in Sec. IV we deal with the vacuum correlation function between those surface currents at the two walls, an object which measures the effect of simultaneous deformations of the two boundaries. In Sec. V, we compute the current induced by an external electric field perpendicular to the walls, and discuss its relation to the parity anomaly in $2 + 1$ dimensions. Results for the full fermionic propagator in the presence of one and two walls, required for our concrete calculations, are presented in Appendix A. The generalization of results for

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the Casimir energy to the case of singular potentials with more general coupling constants is commented in Appendix B.

Finally, in Sec. VI we present a general discussion of the results obtained, and our conclusions.

II. THE MODEL

We begin by defining the real-time action we shall consider in this paper; it describes a Dirac field $\psi, \bar{\psi}$, including its coupling to an external gauge field A , and a “potential” term V , used to impose the boundary conditions

$$\mathcal{S}(\psi, \bar{\psi}; V, A) = \int d^n x \bar{\psi}(x) [i\not{D} - V(x^d)] \psi(x), \quad (1)$$

where $n = d + 1$ is the number of space-time dimensions, with $x = (x_{\parallel}, x^d)$ denoting the coordinates. Here, we have used $x_{\parallel}$ as a shorthand notation for $x^0, x^1, \dots, x^{d-1}$, natural space-time coordinates for the constant- x^d planes, where the boundary conditions are imposed. Indices from the beginning of the Greek alphabet shall be used for indices labeling the first d space-time indices.

To that end we shall assume the form of V to be as follows:

$$V(x^d) = g_L \delta(x^d - a_L) + g_R \delta(x^d - a_R). \quad (2)$$

The dimensionless constants $g_{L,R}$ characterize the strength of the singular interactions and $a_{L,R}$ the positions of the parallel infinite walls, all perpendicular to the x^d axis. This singular potential is used for imposing MIT bag boundary conditions on each side of each (zero-width) plane, as we explain at the end.

The covariant derivative: $D_{\mu} \equiv \partial_{\mu} + ieA_{\mu}$ introduces the minimal coupling with an electric charge e to an electromagnetic field, while, in our conventions, the γ -matrices satisfy $(\gamma^{\mu})^{\dagger} = \gamma^0 \gamma^{\mu} \gamma^0$ and $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$, with $g^{\mu\nu} = \text{diag}(+1, -1, \dots, -1)$ the inverse metric (we have followed the general usage of not writing explicitly the $2^{\lfloor \frac{d}{2} \rfloor} \times 2^{\lfloor \frac{d}{2} \rfloor}$ identity matrix on the right hand side). The fermions are, for all n , in an irreducible representation of Clifford algebra.

As a matter of convenience, we shall use a shorter notation for cases where the external gauge field is absent; namely, $\mathcal{S}(\psi, \bar{\psi}; V)$ for $\mathcal{S}(\psi, \bar{\psi}; V, 0)$.

Let us now discuss the boundary conditions: as already mentioned, the system we consider contains fermions, and there is a slab, defined by two parallel planes which impose MIT bag boundary conditions. We recall that, for a static surface Σ , those conditions are usually implemented as follows:

$$(1 + i\hat{n}(x) \cdot \vec{\gamma})\psi(x)|_{x \in \Sigma} = 0, \quad (3)$$

where $\hat{n}(x)$ is a unit normal vector to Σ . It is worth noting that, in the absence of singularities in the field and in the equations it satisfies, both the inner and outer normal vectors do the job of leading to the vanishing of the normal current. Namely,

$$n_{\mu}(x) \bar{\psi}(x) \gamma^{\mu} \psi(x)|_{x \in \Sigma} = 0, \quad \text{with} \quad n^{\mu}(x) \equiv (0, \hat{n}(x)). \quad (4)$$

In the case of the bag model, there is usually a natural choice, corresponding to the normal pointing toward the exterior of the bag. The reason for that is the following: assume a bag for a spatial region U is implemented by a “potential well” $V_U(\mathbf{x})$ which in the conventions of our model corresponds to

$$V_U(\mathbf{x}) \equiv V_0 \theta_U(\mathbf{x}), \quad (5)$$

where $\theta_U(\mathbf{x})$ vanishes when $\mathbf{x} \in U$ and equals 1 otherwise. It is then rather straightforward to see that, when $V_0 \rightarrow +\infty$ the only remaining solutions must satisfy the bag condition with a normal which points outward of U . Conversely, the inner normal appears when $V_0 \rightarrow -\infty$. Both lead to a vanishing current on the boundary, and are both compatible with the self-adjointness of the Dirac Hamiltonian in the bag, as explained in [13].

Here, however, we consider surfaces which impose boundary conditions on both of their faces. These may be generated by means of a singular potential localized on the surface. Indeed, for the $x^d = 0$ plane, we consider the Dirac equation with a potential

$$V(x^d) = g \delta(x^d). \quad (6)$$

By an immediate generalization to d dimensions of the analysis presented in [14,15], we get

$$i\gamma^d [\psi(x_{\parallel}, \epsilon) - \psi(x_{\parallel}, -\epsilon)] - \frac{g}{2} [\psi(x_{\parallel}, \epsilon) + \psi(x_{\parallel}, -\epsilon)] = 0. \quad (7)$$

Defining the projectors

$$\mathcal{P}_{\pm} \equiv \frac{1}{2} (1 \pm i\gamma^d), \quad (8)$$

one gets, in the special cases $g = \pm 2$, the conditions

$$\begin{aligned} \mathcal{P}_{\pm} \psi(x_{\parallel}, \mp \epsilon) &= 0 \quad \text{if } g = 2 \\ \mathcal{P}_{\pm} \psi(x_{\parallel}, \pm \epsilon) &= 0 \quad \text{if } g = -2, \end{aligned} \quad (9)$$

which correspond to bag boundary conditions with normal vector $\hat{n}(x) = (\vec{0}_{\parallel}, \pm 1)$ at $x^d = \pm \epsilon$ (with the signs consistently chosen for each side of the wall). This agrees with the explanation in [13] since, at least for a potential which

approximates the δ , when considering its two “boundaries” at $\pm\epsilon$, one obtains bag boundary conditions with opposite orientations of the normal vector on each side.

We note that the rather formal derivation we considered, namely, getting the matching conditions from the integration of Dirac’s equation, may be related to the more exhaustive study of different singular potentials for the Dirac equation in $1 + 1$ dimensions, considered in [16]. Indeed, the transition matrix $\mathbb{T}$ from the left to the right of square pulse potential centered on $x^d = 0$, which tends to a δ function of height λ when its width tends to 0 is

$$\mathbb{T} = e^{-\lambda}\mathcal{P}_+ + e^{\lambda}\mathcal{P}_- \quad (10)$$

which yields the same conditions we derived, under the correspondence

$$e^\lambda = \frac{1 + g/2}{1 - g/2}. \quad (11)$$

In most of this paper, we shall use the special cases corresponding to bag boundary conditions, where $|g_{L,R}| = 2$. So we will set $g_{L,R} = 2\eta_{L,R}$, with $\eta_{L,R} = \pm 1$, since the sign choice affects the physics. However, in the next section, we present a more general result, valid for any real values of the constants.

III. CASIMIR ENERGY

One of the most immediate consequences of the boundary conditions imposed is in the vacuum energy which, since it depends on the distance a between the walls, leads to Casimir forces. The vacuum energy of the confined Dirac field can be computed via the functional determinant of the operator appearing in the action (1), with the singular potential inducing nontrivial boundary conditions at the walls. The result, after renormalization, depends on the separation a between the walls and on the signs $\eta_{L,R}$.

For the case of two walls located at $x^d = a_L$ and $x^d = a_R \equiv a_L + a$, with coupling constants g_L and g_R , the Casimir energy per unit area, $\mathcal{E}(a)$, may be obtained as follows: we first define

$$e^{i\Gamma(V)} = \frac{\mathcal{Z}(V)}{\mathcal{Z}_0} \quad (12)$$

with

$$\mathcal{Z}(V) = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i\mathcal{S}(\bar{\psi}, \psi; V)} \quad \text{and} \quad \mathcal{Z}_0 \equiv \mathcal{Z}(V)|_{V=0}. \quad (13)$$

Since the result of integrating out the fields may be formally put in terms of functional determinants

$$\Gamma(V) = -i \log \left[\frac{\det(i\cancel{\partial} - V)}{\det(i\cancel{\partial})} \right]. \quad (14)$$

Since $V(x^d)$ depends only on the perpendicular coordinate, the system is translation-invariant along the d parallel space-time directions, so that the effective action factors as $\Gamma(V) = -\mathcal{E}(a)V_{\parallel}$, with $V_{\parallel}$ the (large) parallel space-time volume and $\mathcal{E}(a)$ the energy per unit parallel area. Fourier-decomposing the determinant along these directions, and dropping an a -independent constant (which accounts for the self-energy of the walls), we get

$$\begin{aligned} \mathcal{E}(a) &= i \int \frac{d^d p_{\parallel}}{(2\pi)^d} \log \left[\frac{\det(i\gamma^d \partial_d + \cancel{p}_{\parallel} - V)}{\det(i\gamma^d \partial_d + \cancel{p}_{\parallel})} \right] \\ &= i \int \frac{d^d p_{\parallel}}{(2\pi)^d} \text{Tr} \log [(i\gamma^d \partial_d + \cancel{p}_{\parallel})^{-1} \\ &\quad \times (i\gamma^d \partial_d + \cancel{p}_{\parallel} - V)]. \end{aligned} \quad (15)$$

Subtracting from $\mathcal{E}(a)$ its limit when $a \rightarrow \infty$ leads to the interaction Casimir energy. Since V has support on a finite set of points, the Fredholm determinant in (15) collapses to that of a spinor wall-index matrix already (see Appendix A).

For arbitrary couplings g_L, g_R

$$\mathcal{E}(a) = i \int \frac{d^d p_{\parallel}}{(2\pi)^d} \log \det M(p_{\parallel}), \quad (16)$$

with $M(p_{\parallel})$ given explicitly in Appendix B. Its determinant evaluates, in any number of dimensions, to

$$\begin{aligned} \det M(p_{\parallel}) &= \left[\frac{(4 + g_L^2)(4 + g_R^2)}{16} \right]^{2^{[n/2]-1}} [1 + \rho_L \rho_R e^{2iaP}]^{2^{[n/2]-1}}, \end{aligned} \quad (17)$$

with $P \equiv \sqrt{p_{\parallel}^2}$ and the “reflection coefficients”

$$\rho_i \equiv \frac{4g_i}{4 + g_i^2}, \quad i = L, R. \quad (18)$$

Subtracting an a -independent term, and performing the Wick rotation, the renormalized Casimir energy becomes

$$\mathcal{E}(a) = -2^{[n/2]-1} \int \frac{d^d p_{\parallel}}{(2\pi)^d} \log [1 + \rho_L \rho_R e^{-2a|p_{\parallel}|}]. \quad (19)$$

The reflection coefficients satisfy $|\rho_i| \leq 1$, with the bound saturated only when $|g_i| = 2$; in that limit $\rho_i \rightarrow \eta_i$ and (19) reduces to

$$\mathcal{E}(a) = -2^{[n/2]-1} \int \frac{d^d p_{\parallel}}{(2\pi)^d} \log [1 + \eta_L \eta_R e^{-2a|p_{\parallel}|}]. \quad (20)$$

Equation (20) is therefore the perfect-confinement limit of the more general expression. For $|g_i| < 2$ the magnitude of the Casimir energy is reduced, while its sign is still

determined by $\text{sgn}(\rho_L \rho_R) = \text{sgn}(g_L g_R)$. In particular, for symmetric couplings $g_L = g_R = g$, $|\mathcal{E}(a)|$ reaches its maximum precisely at $|g| = 2$; this was shown by explicit evaluation in $1 + 1$ and $3 + 1$ dimensions in [17], and is consistent with (19).

Throughout the rest of the paper we work exclusively in the bag-condition limit $|g_{L,R}| = 2$, where (20) means that the force is attractive when $\eta_L = \eta_R$ and repulsive when $\eta_L = -\eta_R$. The repulsive case is consistent with the theorem established in [10], which states that a sufficient condition for repulsion is that the potential coupling the fermionic field to the objects be odd under spatial reflection. A further reason for this restriction is that the surface-current correlators of Sec. IV and the induced bulk current of Sec. V depend on the full propagator within the slab, which factorises only at $|g_{L,R}| = 2$ (cf. Appendix A).

Moreover, in this case the Casimir energy is exactly $2^{\lfloor \frac{n}{2} \rfloor}$ times the Casimir energy of a scalar field obeying Dirichlet boundary conditions, but with opposite sign (repulsive rather than attractive).

A simple way to understand this result is that the odd version of the potential effectively imposes Dirichlet conditions on half of the $2^{\lfloor \frac{n}{2} \rfloor}$ spinor components on the inner sides of the walls, while leaving the remaining components unconstrained there. This “half” factor is compensated by the factor of 2 appearing in the power of the determinant in the fermionic functional integral relative to the bosonic counterpart.

A. The pressure in terms of a bilinear concentrated on one of the walls

Let us now explore the relationship between the phenomenon of Casimir interaction and the vacuum expectation values of fermionic bilinears concentrated on one of the planes. We know that this must indeed be possible, since one could calculate a force by evaluating the corresponding flux of the energy-momentum tensor. But we use here a more expedite procedure, namely, we derive the Casimir force per unit area, or pressure $\mathcal{P}(a)$, from (15). Indeed, applying the same kind of procedure as in [15], we derive

$$\begin{aligned} \mathcal{P}(a) &= -\partial_a \mathcal{E}(a) \\ &= -ig_R \int \frac{d^d p_{\parallel}}{(2\pi)^d} \text{tr}[\gamma^d \not{p}_{\parallel} \tilde{S}_F(p_{\parallel}; a_R, a_R)], \end{aligned} \quad (21)$$

where $\tilde{S}_F(p_{\parallel}; x^d, y^d)$ is the fermionic propagator in the presence of two walls, derived in Appendix A. This already shows that the pressure depends on a fermionic bilinear, concentrated on one of the plates. Of course, the opposite result would have been obtained had we considered the derivative with respect to a_L instead. In the particular case of $n = 2 + 1$ dimensions, since $\gamma^d \gamma^\alpha = -i\epsilon^{\alpha\beta} \gamma_\beta$, we can even refine the previous assertion to saying that the pressure depends on the fermionic current on the wall.

A remark on the meaning of the propagator at the wall is in order. The propagator obtained in Appendix A is, in general, discontinuous across each wall: it contains a piece carrying a $\text{sgn}(x^d - y^d)$ factor, reflecting the well-known Šeba-type jump in the Dirac wave function induced by a δ -potential [18]. The trace appearing in (21), however, projects out exactly the discontinuous components: using the decomposition of the propagator $\tilde{S}_F = A\mathbb{1} + B\hat{p}_\alpha \gamma^\alpha + C\gamma^d + D\hat{p}_\alpha \gamma^\alpha \gamma^d$ given in Appendix A, together with the Dirac algebra (with $g^{dd} = -1$), one finds in any dimension

$$\text{tr}[\gamma^d \not{p}_{\parallel} \tilde{S}_F(p_{\parallel}; x^d, y^d)] = -2^{\lfloor n/2 \rfloor} D(p_{\parallel}; x^d, y^d) \sqrt{p_{\parallel}^2}. \quad (22)$$

Of the four scalar coefficients, only the smooth coefficient D contributes to the pressure. D at coincident points on the wall is, in turn, continuous. From the explicit expressions in Appendix A,

$$\begin{aligned} D(p_{\parallel}; a_R, a_R) &= \begin{cases} (\eta/2) \tan\left(\sqrt{p_{\parallel}^2} a\right) & \text{(symmetric case)} \\ (\eta/2) \cot\left(\sqrt{p_{\parallel}^2} a\right) & \text{(antisymmetric case)} \end{cases} \end{aligned} \quad (23)$$

neither of which involves any sign function. The choice of side of the wall at which to evaluate the propagator is also unambiguous: for $|g_{L,R}| = 2$ the wall is impermeable, and the pressure exerted on the right wall is determined by the propagator on its inner face, the only one in causal contact with the slab interior. This realizes the partial cancellation expected on the grounds that, in the same sense in which a δ -potential acts on a Dirac field as a δ' -potential would on a scalar [16,18], the projected combination relevant for the pressure is well defined.

IV. CORRELATION BETWEEN CURRENTS

As we have just discussed, the Casimir pressure may be expressed in terms of the vacuum expectation value of a surface current. Let us now see that the correlation between the two fermionic currents, each one concentrated on a wall, also plays a role in the Casimir interaction. Indeed, by following a procedure which is entirely analogous to the one of [15], but with time-independent deformations, we now assume a small deformation of each boundary, in such a way that L and R correspond to $x^d = -(\frac{a}{2} + \xi_L(\vec{x}_{\parallel}))$ and $x^d = +(\frac{a}{2} + \xi_R(\vec{x}_{\parallel}))$, respectively. Here, $\vec{x}_{\parallel}$ are the first $d - 1$ spatial coordinates. Besides, we assume that the deformations have zero spatial average. Were such an average different from zero, it could be removed by a redefinition of a .

Then, from (12) and (13), we obtain for the lowest order correction to the vacuum energy involving both deformations, E_{LR}

$$E_{LR} = \int d^d \vec{x}_{\parallel} \int d^d \vec{y}_{\parallel} \xi_L(\vec{x}_{\parallel}) \gamma(\vec{x}_{\parallel} - \vec{y}_{\parallel}) \xi_R(\vec{y}_{\parallel}), \quad (24)$$

where $\gamma(\vec{x}_{\parallel}) = \int \frac{d^{d-1} \vec{k}_{\parallel}}{(2\pi)^{d-1}} e^{i\vec{k}_{\parallel} \cdot \vec{x}_{\parallel}} \int \frac{dk^0}{2\pi} \tilde{\gamma}(k^0, \vec{k}_{\parallel})$ and

$$\tilde{\gamma}(k_{\parallel}) = 4i\eta_L \eta_R \int \frac{d^d p_{\parallel}}{(2\pi)^d} \text{tr} \left[\gamma^d \tilde{S}_F \left(p_{\parallel} + k_{\parallel}; -\frac{a}{2}, \frac{a}{2} \right) \right. \\ \left. \times (\not{p}_{\parallel} + \not{k}_{\parallel}) \gamma^d \not{p}_{\parallel} \tilde{S}_F \left(p_{\parallel}; \frac{a}{2}, -\frac{a}{2} \right) \right]. \quad (25)$$

For $d = 2$, the energy reduces to

$$E_{LR} = 4i\eta_L \eta_R \int dx^1 \int dy^1 A_L^0(x^1) \Pi_{00}(x^1 - y^1) A_R^0(y^1), \quad (26)$$

where we introduced the vacuum polarization tensor

$$C^{\alpha\beta}(x_{\parallel} - y_{\parallel}) \equiv \left\langle J^{\alpha} \left(x_{\parallel}, -\frac{a}{2} \right) J^{\beta} \left(y_{\parallel}, \frac{a}{2} \right) \right\rangle = - \int_{\not{p}_{\parallel} \not{y}_{\parallel}} e^{-i(p_{\parallel} - k_{\parallel}) \cdot (x_{\parallel} - y_{\parallel})} \text{tr} \left[\gamma^{\alpha} \tilde{S}_F \left(p_{\parallel}; -\frac{a}{2}, \frac{a}{2} \right) \gamma^{\beta} \tilde{S}_F \left(k_{\parallel}; \frac{a}{2}, -\frac{a}{2} \right) \right]. \quad (28)$$

We will mostly consider the equal-time correlator, i.e., we set $x_{\parallel} - y_{\parallel} = (0, \vec{x}_{\parallel})$, which isolates the spatial dependence induced by the boundaries and avoids mixing it with retardation effects. Other time orderings can be obtained by keeping the full $x_{\parallel} - y_{\parallel}$ dependence and performing the corresponding analytic continuation.

Using the expressions for the propagator from Appendix A and performing the Dirac traces¹ the equal-time correlator between currents at the walls is

$$C^{\alpha\beta}(\vec{x}_{\parallel}) = - \frac{2^{\lfloor \frac{d-1}{2} \rfloor}}{(2\pi a^2)^d} \left(\frac{|\vec{x}_{\parallel}|}{a} \right)^{2-d} \left\{ \delta_{\eta_L \eta_R} \mathcal{U}^{\alpha\beta} \mathcal{I}_d \left[\cosh(\cdot), \frac{|\vec{x}_{\parallel}|}{a} \right] \right. \\ \left. + \bar{\delta}_{\eta_L \eta_R} \mathcal{V}^{\alpha\beta}(\vec{x}_{\parallel}) \mathcal{I}_{d+2} \left[\sinh(\cdot), \frac{|\vec{x}_{\parallel}|}{a} \right] \right\}, \quad (29)$$

where $\delta_{\eta_L \eta_R}$ is equal to one when $\eta_L = \eta_R$, zero otherwise, and $\bar{\delta}_{\eta_L \eta_R}$ the opposite. We have also defined

$$\mathcal{I}_d[F(\cdot); t] = \left(\int_0^{\infty} ds \frac{s^{d/2} J_{\frac{d-2}{2}}(st)}{F(s)} \right)^2, \quad (30)$$

and the tensorial structure is given by

¹Using that, for $n = 2 + 1$ dimensions the gamma matrices can be chosen as $\gamma^{\mu} = (\sigma_1, i\sigma_2, i\sigma_3)$, we have $\gamma^{\mu} \gamma^{\nu} = g^{\mu\nu} \mathbb{1} - i\epsilon^{\mu\nu\rho} \gamma_{\rho}$ for $\mu, \nu, \rho \in \{0, 1, 2\}$. This gives $\text{tr}(\gamma^{\alpha} \gamma^{\beta} \gamma^2) = -2i\epsilon^{\alpha\beta 2}$ and $\text{tr}(\gamma^{\alpha} \gamma^{\rho} \gamma^{\beta} \gamma^{\lambda} \gamma^2) = -2i(g^{\alpha\rho} \epsilon^{\beta\lambda 2} + g^{\beta\lambda} \epsilon^{\alpha\rho 2})$ for $\alpha, \beta, \rho, \lambda \in \{0, 1\}$.

$$\tilde{\Pi}_{\alpha\beta}(k_{\parallel}) = \int \frac{d^d p_{\parallel}}{(2\pi)^d} \text{tr} \left[\gamma_{\alpha} \tilde{S}_F \left(p_{\parallel} + k_{\parallel}; -\frac{a}{2}, \frac{a}{2} \right) \right. \\ \left. \times \gamma_{\beta} \tilde{S}_F \left(p_{\parallel}; \frac{a}{2}, -\frac{a}{2} \right) \right], \quad (27)$$

and $A^0 = \partial_1 \xi$. Note that in this static case, the geometric gauge field has only temporal component. Besides, it vanishes when the deformation is constant. A constant contribution would in turn amount to a change in a , and therefore cannot appear as a deformation. Thus, the interaction between the deformations depends on the correlation between the surface currents on the two walls: a vacuum polarization tensor, which will also play an important role in Sec. V.

We then focus on the correlation between currents at the walls ($x^d = -\frac{a}{2}$ and $y^d = \frac{a}{2}$)

$$\mathcal{U}^{\alpha\beta} = g^{\alpha\beta} - \delta_{d2} \eta_L \epsilon^{\alpha\beta 2} \quad (31)$$

$$\mathcal{V}^{\alpha\beta}(x) = g^{\alpha\beta} + 2 \frac{x^{\alpha} x^{\beta}}{-x^2} - \delta_{d2} \eta_L \frac{x_{\rho} x_{\lambda}}{-x^2} (g^{\alpha\rho} \epsilon^{\beta\lambda 2} + g^{\beta\rho} \epsilon^{\alpha\lambda 2}). \quad (32)$$

The correlation between charge densities is always negative: a positive fluctuation at one wall tends to be accompanied by a negative fluctuation at the other. This reflects the tendency of the vacuum to balance charge fluctuations across the boundaries. Regarding the tensorial structure, a parity-breaking term arises in two spatial dimensions, and the antisymmetric case introduces an angular dependence. Let us analyze separately the most relevant cases of $d = 2$ and $d = 3$.

In two spatial dimensions with $x_{\parallel} = (0, x)$ the correlation between currents is

$$C^{\alpha\beta}(x) = - \frac{1}{(2\pi a^2)^2} \left\{ \delta_{\eta_L \eta_R} \begin{pmatrix} 1 & -\eta_L \\ \eta_L & -1 \end{pmatrix} \left(\int_0^{\infty} ds \frac{s J_0\left(\frac{sx}{a}\right)}{\cosh(s)} \right)^2 \right. \\ \left. + \bar{\delta}_{\eta_L \eta_R} \begin{pmatrix} 1 & \eta_L \\ \eta_L & 1 \end{pmatrix} \left(\int_0^{\infty} ds \frac{s J_1\left(\frac{sx}{a}\right)}{\sinh(s)} \right)^2 \right\}, \quad (33)$$

$$+ \bar{\delta}_{\eta_L \eta_R} \begin{pmatrix} 1 & \eta_L \\ \eta_L & 1 \end{pmatrix} \left(\int_0^{\infty} ds \frac{s J_1\left(\frac{sx}{a}\right)}{\sinh(s)} \right)^2 \right\}, \quad (34)$$

and the behavior of the integrals can be seen in Fig. 1, where the spatial dependence of the current-current correlation is shown. One observes that an even potential leads to a positively correlated response, whereas an odd configuration

produces a negative correlation. This is consistent with the sign of the Casimir force: in analogy with Ampère's law, parallel (antiparallel) currents are associated with attractive (repulsive) interactions.

In three spatial dimensions the integrals can be explicitly computed, so the correlation between currents is, with $x_{\parallel} = (0, x, y)$ denoting the difference between the coordinates on each plane

$$\mathcal{C}^{\alpha\beta}(x, y) = -\frac{2}{(2\pi a^2)^3} \frac{a}{r} \left\{ \delta_{\eta_L \eta_R} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{\pi^3 a \tanh^2(\frac{\pi r}{2a})}{8r \cosh^2(\frac{\pi r}{2a})} \right. \\ \left. + \bar{\delta}_{\eta_L \eta_R} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{x^2 - y^2}{r^2} & \frac{2xy}{r^2} \\ 0 & \frac{2xy}{r^2} & \frac{y^2 - x^2}{r^2} \end{pmatrix} \frac{\pi a^3 (\sinh(\frac{\pi r}{a}) - \frac{\pi r}{a})^2}{2r^3 (1 + \cosh(\frac{\pi r}{a}))^2} \right\} \Big|_{r=\sqrt{x^2+y^2}}. \quad (35)$$

Note that, when the potential is odd, there is an interesting phenomenon involving the spatial dependence of the correlation function. To that end, it is convenient to choose an appropriate coordinate system. Since (x, y) and $(-y, x)$ are eigenvectors of the nondiagonal block in the correlation matrix, introducing polar coordinates on the plane: $\hat{r} = \frac{1}{r}(x, y)$ and $\hat{\phi} = \frac{1}{r}(-y, x)$ we see that

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{x^2 - y^2}{r^2} & \frac{2xy}{r^2} \\ 0 & \frac{2xy}{r^2} & \frac{y^2 - x^2}{r^2} \end{pmatrix} \quad (36)$$

becomes diagonal

$$M' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (37)$$

This means that the radial components and the tangential components of the currents are separately correlated. Namely, in the odd case, the only nonvanishing correlations are

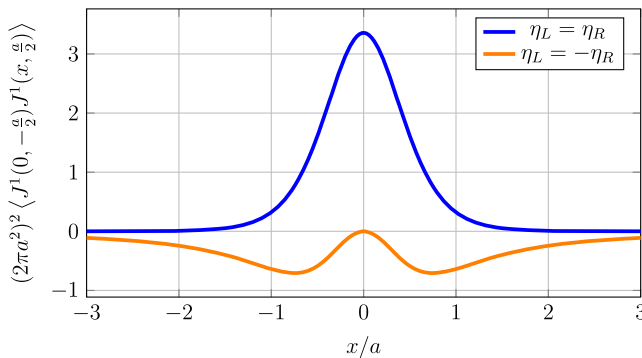


FIG. 1. Position dependence of correlation between parallel current densities at the walls in $d = 2$.

$$\mathcal{C}_{rr}^{odd}(x, y) = -\mathcal{C}_{\phi\phi}^{odd}(x, y) \\ = -\frac{\pi}{(2\pi a^2)^3} \left(\frac{a}{r}\right)^4 \left[\frac{\sinh(\frac{\pi r}{a}) - \frac{\pi r}{a}}{1 + \cosh(\frac{\pi r}{a})} \right]^2. \quad (38)$$

One way to have a pictorial representation of the previous result is that fluctuations that look like a radial flow from (or toward) a point on a plane are negatively correlated. On the other hand, fluctuations looking like circular streams have a positive correlation.

V. INDUCED BULK CURRENT

The induced current is defined as the vacuum expectation value of the fermionic current $e\bar{\psi}(x)\gamma^\mu\psi(x)$ (with $e < 0$ the charge of the electron) in the presence of an external electromagnetic field A_μ . We are interested in the lowest order contribution, i.e., the linear response, which turns out to be

$$j^\mu(x) = \langle e\bar{\psi}(x)\gamma^\mu\psi(x) \rangle_A |_{1\text{st order}} \\ = ie^2 \int d^n y \Pi^{\mu\nu}(x, y) A_\nu(y), \quad (39)$$

where $\Pi^{\mu\nu}(x, y) = \text{tr}(\gamma^\mu S_F(x, y)\gamma^\nu S_F(y, x))$ is the vacuum polarization tensor. For a uniform electric field perpendicular to the walls $\vec{E} = (\vec{0}_{\parallel}, E)$, with the gauge choice $A_\mu(x) = (-x^d E, \vec{0})$, the induced current takes the form

$$j^\mu(x^d) = -ie^2 E \int_{-\infty}^{\infty} dy^d y^d \tilde{\Pi}^{\mu 0}(0_{\parallel}; x^d, y^d), \quad (40)$$

with

$$\tilde{\Pi}^{\mu 0}(0_{\parallel}; x^d, y^d) \\ = \int \frac{d^d p_{\parallel}}{(2\pi)^d} \text{tr}[\gamma^\mu \tilde{S}_F(p_{\parallel}; x^d, y^d) \gamma^0 \tilde{S}_F(p_{\parallel}; y^d, x^d)]. \quad (41)$$

If we are only interested in what happens between the walls, the limits of (40) can be replaced by $(-\frac{a}{2}, \frac{a}{2})$. Taking into

account the symmetry properties of the coefficients found in Appendix A, after performing the Dirac traces² we get

$$\tilde{\Gamma}^{\mu 0}(0_{\parallel}; x^d, y^d) = \delta^{\mu 0} \text{tr} \mathbb{1} \int \frac{d^d p_{\parallel}}{(2\pi)^d} \left[A^2 - C^2 + \frac{p_0^2 + \vec{p}_{\parallel}^2}{p_0^2 - \vec{p}_{\parallel}^2} (B^2 - D^2) \right] \quad (42)$$

$$+ 4i\delta^{\mu 1} \delta^{d2} \int \frac{d^d p_{\parallel}}{(2\pi)^d} \left[\frac{p_0^2 + p_1^2}{p_0^2 - p_1^2} BD - AC \right], \quad (43)$$

This expression shows that there is only induced spatial current for the case of $2 + 1$ dimensions, and it is transversal to the electric field. After the Wick rotation $p^0 \rightarrow ip_E^0$, the first term in (43)—the one involving the product BD —vanishes by angular integration in the Euclidean parallel-momentum plane, as we now show. The coefficients B and D depend on $p_{\parallel}$ only through $\sqrt{p_{\parallel}^2}$ (cf. Appendix A), and hence become functions of the Euclidean radial variable $p_E \equiv \sqrt{(p_E^0)^2 + p_1^2}$ alone, with no dependence on the angular variable φ defined by $(p_E^0, p_1) = p_E(\sin \varphi, \cos \varphi)$. The kinematic factor multiplying them, in turn, transforms as

$$\frac{p_0^2 + p_1^2}{p_0^2 - p_1^2} \xrightarrow{\text{Wick}} -\frac{p_1^2 - (p_E^0)^2}{p_E^2} = -\cos(2\varphi), \quad (44)$$

and $\int_0^{2\pi} \cos(2\varphi) d\varphi = 0$, so its integration over the angular variable annihilates the BD contribution. The AC term carries no analogous angular factor (after Wick rotation, A and C are also functions of p_E only) and is thus unaffected. The induced spatial current therefore reduces to

$$j^1(x) \equiv \eta_R \frac{e^2 E}{4\pi} \tilde{j}_{\eta_L \eta_R} \left(\frac{x}{a} \right) = \eta_R \frac{e^2 E}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} dt t \int_0^{\infty} ds s f \left(s; \frac{x}{a}, t; \eta_L \eta_R \right), \quad (45)$$

where now we denote x to the coordinate transversal to the walls, and with

$$f(s; z, t; +1) = 2 \frac{\cosh[s(z+t)]}{\cosh(s)} \left[\text{sgn}(z-t) e^{-s|z-t|} + e^{-s} \frac{\sinh[s(z-t)]}{\cosh(s)} \right], \quad (46)$$

$$f(s; z, t; -1) = 2 \frac{\sinh[s(z+t)]}{\sinh(s)} \left[\text{sgn}(z-t) e^{-s|z-t|} - e^{-s} \frac{\sinh[s(z-t)]}{\sinh(s)} \right]. \quad (47)$$

Evaluating the integrals yields

$$\begin{aligned} \tilde{j}_{+1}(x) = & -\frac{1}{2} - \frac{1}{6} \log 2 + 6 \log A_{\text{GK}} + \frac{1}{2} \log \left[\frac{\Gamma(\frac{3-2x}{4}) \Gamma(\frac{3+2x}{4})}{\Gamma(\frac{1-2x}{4}) \Gamma(\frac{1+2x}{4})} \right] \\ & + -\frac{x}{4} \left[\psi_0 \left(\frac{1-2x}{4} \right) - \psi_0 \left(\frac{1+2x}{4} \right) - \psi_0 \left(\frac{3-2x}{4} \right) + \psi_0 \left(\frac{3+2x}{4} \right) \right] \\ & + -\frac{1-4x^2}{64} \left[\psi_1 \left(\frac{1-2x}{4} \right) + \psi_1 \left(\frac{1+2x}{4} \right) - \psi_1 \left(\frac{3-2x}{4} \right) - \psi_1 \left(\frac{3+2x}{4} \right) \right], \end{aligned} \quad (48)$$

$$\tilde{j}_{-1}(x) = \frac{1}{2} \log \left[\frac{\Gamma(\frac{1+2x}{2})}{\Gamma(\frac{1-2x}{2})} \right] - \frac{x}{2} \left[\psi_0 \left(\frac{1-2x}{2} \right) + \psi_0 \left(\frac{1+2x}{2} \right) \right] + -\frac{1-4x^2}{16} \left[\psi_1 \left(\frac{1-2x}{2} \right) - \psi_1 \left(\frac{1+2x}{2} \right) \right], \quad (49)$$

where $A_{\text{GK}} \approx 1.2824\dots$ is the Glaisher-Kinkelin constant, which arises from the regularization of the infinite product of Γ functions, and ψ_0, ψ_1 are the polygamma functions of order 0 and 1 respectively. We can see in Fig. 2 that an even confining potential produces an even current density, while an odd confining potential leads to an odd distribution. This means that an external electric field with an even confining potential produces a net induced current in the direction perpendicular to the electric field.

Regarding the charge density, one can verify that in both configurations the distribution is an odd function of x^d for

every dimension, so the total induced charge between the walls vanishes, as expected.

A. Effective Hall conductivity and the parity anomaly

The results of the previous subsection admit a natural interpretation in the language of the parity anomaly in $2 + 1$ dimensions [19,20]. In this dimensionality, a single massless Dirac fermion cannot be regularized in a way that preserves both gauge invariance and parity; the parity anomaly manifests through a half-integer contribution to the Hall conductivity. For the contribution to the parity anomaly of the borders in a four dimensional theory, see [21], where sign-dependent boundary conditions (analog to our choice for $\eta_{L,R}$) also appear.

²This expression is valid for $d = 1, 2, 3$.

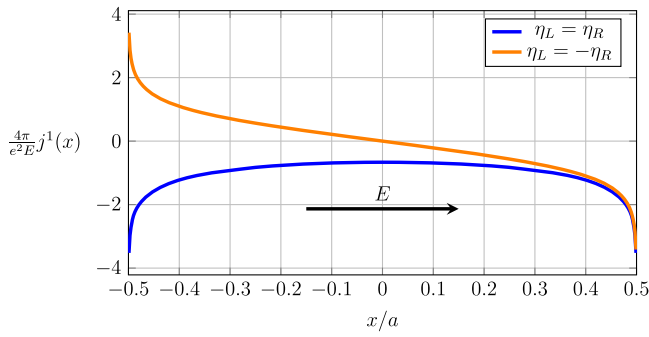


FIG. 2. Spatial current density in the presence of an external electric field perpendicular to the walls.

In our system, the boundary conditions imposed by the singular potential project out half of the spinor components at each wall. For the symmetric configuration ($\eta_L = \eta_R$), the same projector $\mathcal{P}_\pm = (1 \pm i\gamma^d)/2$ is imposed at both boundaries, effectively breaking parity within the slab. This is the mechanism behind the nonvanishing transverse current: the fermionic vacuum in the slab develops a parity-odd response to the external electric field.

An effective Hall conductivity can be defined by integrating the current density across the slab

$$J^1 \equiv \frac{1}{a} \int_{-a/2}^{a/2} dx^d j^1(x^d) = \eta_R \frac{e^2 E}{4\pi} \int_{-1/2}^{1/2} dx \tilde{j}_{\eta_L \eta_R}(x) \equiv \sigma_H E. \quad (50)$$

For the antisymmetric case, the odd symmetry of $\tilde{j}_{-1}(z)$ under $z \rightarrow -z$ guarantees that $\sigma_H = 0$, consistent with the fact that the odd potential preserves parity.

For the symmetric case, integrating the analytic expression for $\tilde{j}_1(z)$ yields

$$\sigma_H = -\eta_R \frac{e^2}{4\pi}, \quad (51)$$

which is the natural parity anomaly scale, and it results from the bulk modes inhabiting the slab. The propagator coefficients in the symmetric configuration (Appendix A) have poles at $\cos(\sqrt{p_\parallel^2} a) = 0$, i.e., at $p_\parallel^2 = m_n^2$ with

$$m_n = \left(n + \frac{1}{2}\right) \frac{\pi}{a}, \quad n = 0, 1, 2, \dots, \quad (52)$$

which are precisely the effective $(2+1)$ -dimensional masses of the discrete tower of eigenmodes of the perpendicular Dirac operator under the symmetric bag boundary conditions. Each such mode, viewed in the parallel directions as an effective $(2+1)$ -dimensional massive Dirac fermion, contributes its own parity-anomaly response to the Hall conductivity; with in (51) resulting from the sum of these contributions, with the regularization implemented implicitly

by the propagator-based derivation through the Wick rotation and the standard $i\epsilon$ prescription.

In the antisymmetric configuration the analogous poles lie at $\sin(\sqrt{p_\parallel^2} a) = 0$, giving the spectrum $m_n = n\pi/a$ ($n = 1, 2, \dots$); the parity-conjugate boundary conditions of this case organize the modes in $\pm m_n$ pairs, whose individual parity-anomaly contributions cancel pairwise, consistent with $\sigma_H = 0$.

Finally, we note that the divergence of the current density near the walls is of logarithmic type, similar to the behavior of the vacuum expectation value of the stress-energy tensor near boundaries with Dirichlet-type conditions.

VI. DISCUSSION AND CONCLUSIONS

We have studied the vacuum physics of a massless Dirac field confined between two parallel walls in $d+1$ dimensions, realized through singular potentials with coupling constants $g_{L,R} = \pm 2$ that implement MIT bag boundary conditions. The two physically inequivalent configurations, corresponding to a confining potential that is even ($\eta_L = \eta_R$) or odd ($\eta_L = -\eta_R$) under reflection about the midplane, control the qualitative behavior of all observables.

For the Casimir energy, the even configuration yields an attractive force proportional to $2^{\lfloor n/2 \rfloor}$ times the scalar Dirichlet result, while the odd configuration produces a repulsive force of the same magnitude.

The vacuum current-current correlator between the walls, $\mathcal{C}^{\alpha\beta}$, displays a rich tensorial structure. In arbitrary dimension, a parity-even piece proportional to $\eta^{\alpha\beta}$ (or the more involved tensor $\mathcal{V}^{\alpha\beta}$) mediates the standard vacuum-dressed Coulomb and Ampère interactions between charges and currents on the walls. In $2+1$ dimensions, a parity-odd term proportional to $\epsilon^{\alpha\beta 2}$ appears, reflecting the parity anomaly of a single Dirac fermion. We showed that this term generates an effective Chern-Simons-like coupling between probe gauge fields on the two walls, present only in the symmetric configuration.

The most striking observable consequence emerges in the induced current: an external electric field perpendicular to the walls drives a transverse (Hall) current that flows parallel to the walls and perpendicular to the applied field, exclusively in $2+1$ dimensions. For the symmetric configuration, the current density is an even function of position, yielding a net integrated Hall current characterized by an effective conductivity $\sigma_H = -\eta_R \frac{e^2}{4\pi}$. For the antisymmetric configuration, the current density is odd, giving vanishing total current in agreement with parity conservation.

Several directions merit further investigation. The inclusion of a fermion mass m would introduce a gap and is expected to drive the Hall conductivity toward different quantization values as $ma \rightarrow \infty$, interpolating between different topological regimes when m is finite. The extension to finite temperature would allow contact with the thermal Casimir effect and the behavior of the Hall response in a

finite-temperature Dirac system. Finally, the interplay between the vacuum-mediated inter-wall interaction and the dynamical electromagnetic field, including the backreaction on the photon propagator within the slab, could reveal nontrivial modifications to the effective low-energy theory of the gauge field in the presence of fermionic boundary conditions.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: FERMION PROPAGATOR

In this Appendix we present the explicit form of the Dirac propagator in the presence of one and two walls.

For a matrix differential operator

$$\mathcal{O}(x, y) = \mathcal{O}_0(x - y) - \mathcal{V}(x)\delta(x - y) \quad (\text{A1})$$

$$= \int_z [\delta(x - z)\mathbb{1} - \mathcal{V}(x)\mathcal{O}_0^{-1}(x - z)]\mathcal{O}_0(z - y) \quad (\text{A2})$$

$$\equiv \int_z K(x, z)\mathcal{O}_0(z - y), \quad (\text{A3})$$

the inverse of $\mathcal{O}(x, y)$ is given by

$$\mathcal{O}^{-1}(x, y) = \int_z \mathcal{O}_0^{-1}(x - z)K^{-1}(z, y). \quad (\text{A4})$$

For the inverse of K we have

$$\begin{aligned} K^{-1} &= \frac{1}{1 - \mathcal{V}\mathcal{O}_0^{-1}} = \sum_{n=0}^{\infty} (\mathcal{V}\mathcal{O}_0^{-1})^n \\ &= 1 + \sum_{n=1}^{\infty} (\mathcal{V}\mathcal{O}_0^{-1})^n \\ &= 1 + \mathcal{V}\mathcal{O}_0^{-1} \sum_{n=0}^{\infty} (\mathcal{V}\mathcal{O}_0^{-1})^n \\ &= 1 + \mathcal{V}\mathcal{O}_0^{-1}K^{-1} = 1 + \mathcal{V}\mathcal{O}^{-1} \end{aligned} \quad (\text{A5})$$

so for the inverse of $\mathcal{O}$ we get the implicit equation

$$\mathcal{O}^{-1}(x, y) = \mathcal{O}_0^{-1}(x - y) + \int_z \mathcal{O}_0^{-1}(x - z)\mathcal{V}(z)\mathcal{O}^{-1}(z, y). \quad (\text{A7})$$

The complete propagator is defined as the inverse of the kinetic operator (up to a factor i)

$$S_F(x, y) \equiv i[\gamma^\mu \partial_\mu - V]^{-1}(x, y), \quad (\text{A8})$$

and, as shown in the previous step, it satisfies the implicit equation

$$S_F(x, y) = S_F^{(0)}(x - y) - i \int d^n z S_F^{(0)}(x - z)V(z)S_F(z, y), \quad (\text{A9})$$

where

$$\begin{aligned} S_F^{(0)}(x - y) &\equiv i[\gamma^\mu \partial_\mu]^{-1}(x - y) \\ &= i \int \frac{d^n p}{(2\pi)^n} \frac{p_\mu \gamma^\mu}{p^2 + i\epsilon} e^{-ip \cdot (x - y)} \end{aligned} \quad (\text{A10})$$

is the free propagator for a massless Dirac field. For the configuration considered in this paper, because of the translational symmetry in the directions parallel to the walls, the full propagator can be expressed as

$$S_F(x, y) = \int \frac{d^d p_\parallel}{(2\pi)^d} e^{-ip_\parallel \cdot (x_\parallel - y_\parallel)} \tilde{S}_F(p_\parallel; x^d, y^d). \quad (\text{A11})$$

Inserting the potential (2) in (A9), the (partial) Fourier transform of the full propagator turns out to be

$$\begin{aligned} \tilde{S}_F(p_\parallel; x^d, y^d) &= \tilde{S}_F^{(0)}(p_\parallel; x^d - y^d) - i \sum_{i,j=1}^N g_i \tilde{S}_F^{(0)}(p_\parallel; x^d - a_i) \\ &\quad \times T_{ij} \tilde{S}_F^{(0)}(p_\parallel; a_j - y^d), \end{aligned} \quad (\text{A12})$$

where the (partial) Fourier transform of the free propagator is³

$$\tilde{S}_F^{(0)}(p_\parallel; z) = \frac{e^{i|z|\sqrt{p_\parallel^2}}}{2} \left(\frac{p_\alpha}{\sqrt{p_\parallel^2}} \gamma^\alpha - \text{sgn}(z) \gamma^d \right), \quad \text{with} \quad (\text{A13})$$

$$\alpha = 0, \dots, d - 1$$

and T_{ij} are matrices in the spinor space satisfying

$$\sum_{k=1}^N T_{ik} [\delta_{kj} \mathbb{1} + i g_j \tilde{S}_F^{(0)}(p_\parallel; a_k - a_j)] = \delta_{ij} \mathbb{1}. \quad (\text{A14})$$

³ $p_\parallel^2$ is implicitly defined with the Feynman prescription, namely $p_\parallel^2 \equiv p_\parallel^2 + i\epsilon$ with $\epsilon \rightarrow 0^+$.

1. One wall

For one wall placed at $x^d = a$ with arbitrary g , the propagator takes the following form:

- (i) When both points are evaluated at the same side of the wall, satisfying $\text{sgn}(x^d - a) = \text{sgn}(y^d - a)$, the propagator is

$$\begin{aligned} \tilde{S}_F(p_{\parallel}; x^d, y^d) &= \tilde{S}_F^{(0)}(p_{\parallel}; x^d - y^d) - i \text{sgn}(x^d - a) \\ &\times \frac{g}{1 + (\frac{g}{2})^2} \tilde{S}_F^{(0)}(p_{\parallel}; x^d + y^d - 2a) \gamma^d. \end{aligned} \quad (\text{A15})$$

- (ii) When each point is evaluated at a different side of the wall, so that $\text{sgn}(x^d - a) = -\text{sgn}(y^d - a)$, the propagator is

$$\tilde{S}_F(p_{\parallel}; x^d, y^d) = \frac{1 - (\frac{g}{2})^2}{1 + (\frac{g}{2})^2} \tilde{S}_F^{(0)}(p_{\parallel}; x^d - y^d). \quad (\text{A16})$$

For $g = \pm 2$, the field cannot propagate from one side of the wall to the other, so this is another way of seeing that the singular potential with that particular choice of the parameter imposes perfect bag boundary conditions.

2. Two walls with $|g| = 2$

For two walls located at $x^d = a_L$ and $x^d = a_R > a_L$, with coupling constants $g_L = 2\eta_L$ and $g_R = 2\eta_R$ respectively, for $\eta_{L,R} = \pm 1$, we need to find the inverse of a matrix whose components are given by the bracket of (A14), which is

$$\begin{pmatrix} 1 + 2i\eta_L \tilde{S}_F^{(0)}(p_{\parallel}; 0) & 2i\eta_R \tilde{S}_F^{(0)}(p_{\parallel}; a_L - a_R) \\ 2i\eta_L \tilde{S}_F^{(0)}(p_{\parallel}; a_R - a_L) & 1 + 2i\eta_R \tilde{S}_F^{(0)}(p_{\parallel}; 0) \end{pmatrix} \quad (\text{A17})$$

Using the expression for the inverse of a block matrix, after some algebra the full propagator can be expressed as

$$\tilde{S}_F(p_{\parallel}; x^d, y^d) = A \mathbb{1} + B \hat{p}_\alpha \gamma^\alpha + C \gamma^d + D \hat{p}_\alpha \gamma^\alpha \gamma^d, \quad (\text{A18})$$

where $\hat{p}_\alpha \equiv \frac{p_\alpha}{\sqrt{p_{\parallel}^2}}$, and A , B , C , and D are scalar functions that make the propagator satisfy the bag boundary conditions at both sides of each wall

$$\lim_{\epsilon \rightarrow 0^+} (\mathbb{1} \pm i\gamma^d) \tilde{S}_F(p_{\parallel}; a_i \mp \eta_i \epsilon, y^d) = 0, \quad i = L, R. \quad (\text{A19})$$

For the sake of symmetry let us take $a_L = -\frac{a}{2}$ and $a_R = \frac{a}{2}$. Since we are interested in what happens between the walls, we will only give the expressions of the coefficients for $x^d, y^d \in [-\frac{a}{2}, \frac{a}{2}]$

- (i) Symmetric case $\eta_L = \eta_R \equiv \eta$

$$A(p_{\parallel}; x^d, y^d) = -\frac{i\eta \cos \sqrt{p_{\parallel}^2}(x^d + y^d)}{2 \cos \sqrt{p_{\parallel}^2}a} \quad (\text{A20})$$

$$\begin{aligned} B(p_{\parallel}; x^d, y^d) &= \frac{e^{i\sqrt{p_{\parallel}^2}|x^d - y^d|}}{2} - \frac{e^{i\sqrt{p_{\parallel}^2}a}}{2} \\ &\times \frac{\cos \sqrt{p_{\parallel}^2}(x^d - y^d)}{\cos \sqrt{p_{\parallel}^2}a} \end{aligned} \quad (\text{A21})$$

$$\begin{aligned} C(p_{\parallel}; x^d, y^d) &= -\text{sgn}(x^d - y^d) \frac{e^{i\sqrt{p_{\parallel}^2}|x^d - y^d|}}{2} \\ &+ i \frac{e^{i\sqrt{p_{\parallel}^2}a} \sin \sqrt{p_{\parallel}^2}(x^d - y^d)}{2 \cos \sqrt{p_{\parallel}^2}a} \end{aligned} \quad (\text{A22})$$

$$D(p_{\parallel}; x^d, y^d) = \frac{\eta \sin \sqrt{p_{\parallel}^2}(x^d + y^d)}{2 \cos \sqrt{p_{\parallel}^2}a} \quad (\text{A23})$$

- (ii) Antisymmetric case $\eta_L = -\eta_R \equiv \eta$

$$A(p_{\parallel}; x^d, y^d) = \frac{i\eta \sin \sqrt{p_{\parallel}^2}(x^d + y^d)}{2 \sin \sqrt{p_{\parallel}^2}a} \quad (\text{A24})$$

$$\begin{aligned} B(p_{\parallel}; x^d, y^d) &= \frac{e^{i\sqrt{p_{\parallel}^2}|x^d - y^d|}}{2} + i \frac{e^{i\sqrt{p_{\parallel}^2}a}}{2} \\ &\times \frac{\cos \sqrt{p_{\parallel}^2}(x^d - y^d)}{\sin \sqrt{p_{\parallel}^2}a} \end{aligned} \quad (\text{A25})$$

$$\begin{aligned} C(p_{\parallel}; x^d, y^d) &= -\text{sgn}(x^d - y^d) \frac{e^{i\sqrt{p_{\parallel}^2}|x^d - y^d|}}{2} \\ &+ \frac{e^{i\sqrt{p_{\parallel}^2}a} \sin \sqrt{p_{\parallel}^2}(x^d - y^d)}{2 \sin \sqrt{p_{\parallel}^2}a} \end{aligned} \quad (\text{A26})$$

$$D(p_{\parallel}; x^d, y^d) = \frac{\eta \cos \sqrt{p_{\parallel}^2}(x^d + y^d)}{2 \sin \sqrt{p_{\parallel}^2}a} \quad (\text{A27})$$

It can be seen that for both cases A , B , and D are even under the exchange $x^d \leftrightarrow y^d$, while C is odd.

Since a wall with $|g| = 2$ completely decouples the two sides, when both points lie beyond all the walls the propagator reduces to that of a single wall

$$\tilde{S}_F(p_{\parallel}; x^d, y^d) = \begin{cases} \tilde{S}_F^{(0)}(p_{\parallel}; x^d - y^d) + i\eta_L \tilde{S}_F^{(0)}(p_{\parallel}; x^d + y^d - 2a_L)\gamma^d & \text{if } x^d, y^d < a_L \\ \tilde{S}_F^{(0)}(p_{\parallel}; x^d - y^d) - i\eta_R \tilde{S}_F^{(0)}(p_{\parallel}; x^d + y^d - 2a_R)\gamma^d & \text{if } x^d, y^d > a_R \end{cases}. \quad (\text{A28})$$

Using the same reasoning, the propagator in the presence of $N > 2$ walls can be constructed from the $N = 2$ case.

APPENDIX B: CASIMIR ENERGY FOR ARBITRARY COUPLINGS

In this Appendix we provide the detailed derivation of the closed-form expression for the Casimir energy, valid for arbitrary values of the coupling constants g_L and g_R , and show how the bag-condition limit $|g_{L,R}| = 2$ used in the body of the paper is recovered.

Starting from (15) and using translation invariance along the d parallel directions, the functional determinant reduces to that of an operator acting on the x^d coordinate alone. Since $V(x^d) = g_L \delta(x^d - a_L) + g_R \delta(x^d - a_R)$ has support on a finite set of points, the operator $\mathcal{O}_0^{-1}\mathcal{V}$ is of finite rank, and the Fredholm determinant collapses to that of the spinor $\otimes$ wall-index matrix already appearing in (A14), generalized to arbitrary couplings

$$M(p_{\parallel}) = \begin{pmatrix} \mathbb{1} + ig_L \tilde{S}_F^{(0)}(p_{\parallel}; 0) & ig_R \tilde{S}_F^{(0)}(p_{\parallel}; -a) \\ ig_L \tilde{S}_F^{(0)}(p_{\parallel}; +a) & \mathbb{1} + ig_R \tilde{S}_F^{(0)}(p_{\parallel}; 0) \end{pmatrix}, \quad (\text{B1})$$

each block being a $2^{\lfloor n/2 \rfloor} \times 2^{\lfloor n/2 \rfloor}$ matrix in spinor space. Modulo the a -independent self-energies, the energy density is then

$$\mathcal{E}(a) = i \int \frac{d^d p_{\parallel}}{(2\pi)^d} \log \det M(p_{\parallel}). \quad (\text{B2})$$

To make the spinor structure manifest, we use the $\text{sgn}(0) = 0$ prescription in the free propagator, so that $\tilde{S}_F^{(0)}(p_{\parallel}; 0) = \frac{1}{2} \hat{p}_{\parallel}$, and $\tilde{S}_F^{(0)}(p_{\parallel}; \pm a) = \frac{e^{iaP}}{2} (\hat{p}_{\parallel} \mp \gamma^d)$, with the shorthand

$$\hat{p}_{\parallel} \equiv \frac{P_{\alpha} \gamma^{\alpha}}{\sqrt{P_{\parallel}^2}}, \quad P \equiv \sqrt{P_{\parallel}^2}. \quad (\text{B3})$$

The relations

$$\hat{p}_{\parallel}^2 = \mathbb{1}, \quad (\gamma^d)^2 = -\mathbb{1}, \quad \{\hat{p}_{\parallel}, \gamma^d\} = 0 \quad (\text{B4})$$

will be used repeatedly. With this notation, (B1) becomes

$$M = \begin{pmatrix} \mathbb{1} + \frac{ig_L}{2} \hat{p}_{\parallel} & \frac{ig_R}{2} e^{iaP} (\hat{p}_{\parallel} + \gamma^d) \\ \frac{ig_L}{2} e^{iaP} (\hat{p}_{\parallel} - \gamma^d) & \mathbb{1} + \frac{ig_R}{2} \hat{p}_{\parallel} \end{pmatrix}. \quad (\text{B5})$$

To evaluate $\det M$ we use the Schur complement,

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \cdot \det(D - CA^{-1}B), \quad (\text{B6})$$

with $A \equiv \mathbb{1} + (ig_L/2) \hat{p}_{\parallel}$. From $\hat{p}_{\parallel}^2 = \mathbb{1}$, the eigenvalues of $\hat{p}_{\parallel}$ are ± 1 , each with multiplicity $2^{\lfloor n/2 \rfloor - 1}$, hence

$$\det A = \left(1 + \frac{g_L^2}{4}\right)^{2^{\lfloor n/2 \rfloor - 1}} = \left[\frac{4 + g_L^2}{4}\right]^{2^{\lfloor n/2 \rfloor - 1}},$$

$$A^{-1} = \frac{\mathbb{1} - (ig_L/2) \hat{p}_{\parallel}}{1 + g_L^2/4}. \quad (\text{B7})$$

The Schur complement involves the product $(\hat{p}_{\parallel} - \gamma^d)[\mathbb{1} - (ig_L/2) \hat{p}_{\parallel}](\hat{p}_{\parallel} + \gamma^d)$. Using (B4) and the identity $\gamma^d \hat{p}_{\parallel} \gamma^d = -\hat{p}_{\parallel} (\gamma^d)^2 = \hat{p}_{\parallel}$, the g_L -dependent contribution cancels exactly

$$(\hat{p}_{\parallel} - \gamma^d)(\hat{p}_{\parallel} + \gamma^d) = 2(\mathbb{1} + \hat{p}_{\parallel} \gamma^d) \equiv 4\Pi_+, \quad (\text{B8})$$

$$(\hat{p}_{\parallel} - \gamma^d) \hat{p}_{\parallel} (\hat{p}_{\parallel} + \gamma^d) = \hat{p}_{\parallel} - \gamma^d \hat{p}_{\parallel} \gamma^d = \hat{p}_{\parallel} - \hat{p}_{\parallel} = 0, \quad (\text{B9})$$

where

$$\Pi_{\pm} \equiv \frac{1}{2} (\mathbb{1} \pm \hat{p}_{\parallel} \gamma^d) \quad (\text{B10})$$

are orthogonal projectors associated with the involution $\hat{p}_{\parallel} \gamma^d$ (which squares to $-\hat{p}_{\parallel}^2 (\gamma^d)^2 = \mathbb{1}$); each has rank $2^{\lfloor n/2 \rfloor - 1}$. Therefore

$$\begin{aligned} CA^{-1}B &= \frac{(ig_L/2)(ig_R/2)e^{2iaP}}{1 + g_L^2/4} \cdot 4\Pi_+ \\ &= -\frac{g_L g_R e^{2iaP}}{1 + g_L^2/4} \Pi_+, \end{aligned} \quad (\text{B11})$$

so that the Schur complement reads

$$D - CA^{-1}B = \mathbb{1} + \frac{ig_R}{2} \hat{p}_{\parallel} + \frac{g_L g_R e^{2iaP}}{1 + g_L^2/4} \Pi_+. \quad (\text{B12})$$

The operator on the right-hand side of (B12) belongs to the two-dimensional Clifford algebra generated by $\hat{p}_\parallel$ and γ^d . This algebra has a single irreducible representation of dimension 2; the full spinor space decomposes as $2^{[n/2]-1}$ copies of it. In any such irreducible block, the choice

$$\hat{p}_\parallel \rightarrow \sigma_3, \quad \gamma^d \rightarrow i\sigma_2, \quad \hat{p}_\parallel \gamma^d \rightarrow \sigma_1 \quad (\text{B13})$$

realises (B4) and yields, for the operator in (B12), the matrix

$$\begin{pmatrix} 1 + \frac{g_L g_R e^{2iaP}}{1+g_L^2/4} \cdot \frac{1}{2}(1+1) & \frac{ig_R}{2} \\ \frac{ig_R}{2} & 1 + \frac{g_L g_R e^{2iaP}}{1+g_L^2/4} \cdot \frac{1}{2}(1-1) \end{pmatrix} \quad (\text{B14})$$

in the eigenbasis of σ_1 . Its determinant evaluates to

$$\begin{aligned} \det(D - CA^{-1}B) \\ = \left[\frac{(1+g_L^2/4)(1+g_R^2/4) + g_L g_R e^{2iaP}}{1+g_L^2/4} \right]^{2^{[n/2]-1}}. \end{aligned} \quad (\text{B15})$$

Combining (B7) and (B15) we arrive at the closed form

$$\begin{aligned} \det M(p_\parallel) \\ = \left[\frac{(4+g_L^2)(4+g_R^2)}{16} \right]^{2^{[n/2]-1}} [1 + \rho_L \rho_R e^{2iaP}]^{2^{[n/2]-1}}, \end{aligned} \quad (\text{B16})$$

with the “reflection coefficients”

$$\rho_i \equiv \frac{4g_i}{4+g_i^2}, \quad i = L, R. \quad (\text{B17})$$

The first factor in (B16) is a -independent and accounts for the self-energies of the walls; it is removed by the standard subtraction of the $a \rightarrow \infty$ value of $\mathcal{E}(a)$. Inserting (B16) in (B2) and performing the Wick rotation ($p^0 \rightarrow ip_E^0$, with the Feynman prescription $P = \sqrt{p_\parallel^2 + i\epsilon} \rightarrow i|p_\parallel|$ and $id^d p_\parallel \rightarrow -d^d p_\parallel^E$), one obtains the renormalized energy for arbitrary couplings,

$$\mathcal{E}(a) = -2^{[n/2]-1} \int \frac{d^d p_\parallel}{(2\pi)^d} \log[1 + \rho_L \rho_R e^{-2a|p_\parallel|}], \quad (\text{B18})$$

which is the formula quoted in the body of the paper. The coefficients ρ_i satisfy $|\rho_i| \leq 1$, with the upper bound saturated only when $|g_i| = 2$, where $\rho_i \rightarrow \eta_i = \pm 1$ and (B18) reduces to the bag-condition expression used throughout the paper. This realizes the expectation that the bag-condition value $|g| = 2$ is the regime of strongest Casimir interaction, in agreement with the explicit results in 1 + 1 and 3 + 1 dimensions of [17].

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