

Non-Abelian Casimir effect for plates, symmetrical tube, and box on the lattice

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We present nonperturbative results of the Casimir potential in non-Abelian SU(3) gauge theory in $(2 + 1)D$ and $(3 + 1)D$ in the confined and deconfined phase. For the first time, geometries beyond parallel plates in $(3 + 1)D$ are explored and we show that the Casimir effect for the symmetrical tube and symmetrical box is attractive. The Casimir potential for the tube differs from the massless noninteracting scalar field theory prediction, where a repulsive Casimir potential is expected. Unlike the parallel plate geometry where the plate size is fixed, in the case of the tube and box, the sizes of the faces forming the walls of the geometries changes with separation distance. We propose various methods that can be used to account for the energy contributions from creating the boundaries. We show that increasing the temperature from a confined to a deconfined phase does not alter the Casimir potential. This observation is consistent with prior work suggesting that the region inside the walls is a boundary induced deconfined phase.

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I. INTRODUCTION

A fascinating consequence of quantum fluctuations of the electromagnetic and matter fields is the existence of the Casimir effect [1,2]. First conclusively observed in an experiment by Lamoreaux [3] it refers in its simplest form to the fact that two ideally conducting plates, placed at some distance d apart in a vacuum will experience a small (on the order of $\sim 10^{-3} \text{ Nm}^{-2}$ for plates placed $\sim 1 \mu\text{m}$ apart), but finite attractive pressure toward each other.

Two seemingly separate but ultimately intimately related explanations are discussed in the literature (for reviews see e.g. [4,5]). The most common one is found in introductory texts on quantum field theory and understands the Casimir effect as a result of imposing boundary conditions on the fluctuating electromagnetic field, which in turn modify the vacuum and thus reveal zero-point energy differences. Another point of view considers fluctuating dipoles in

the plates, which lead to a van der Waals like interaction. Interestingly both pictures arise naturally from different coarse graining of a microscopic theory of matter and gauge fields. When we integrate out the matter fields, only fluctuating gauge fields remain, modified by the presence of the plates, on the other hand, if we integrate out the gauge field, fluctuating dipoles with weak residual interactions ensue.

In this study we adopt the fluctuating gauge field point of view as it straightforwardly lends itself to an investigation based on Monte Carlo simulations on a lattice. To recap, in this picture the Casimir effect can be understood as follows; the presence of the neutral conducting plates in the vacuum imposes boundary conditions on the stationary modes of the electromagnetic field in vacuum. However, between the plates, the long wavelength ($\lambda > d$) modes get frozen out, thus limiting the number of modes that can exist within the cavity between the plates as opposed to the number of modes in free space (outside the plates). The limited number of modes between the plates corresponds to a lower vacuum energy density in that region, leading to a pressure difference and subsequently the attractive force.

In Abelian gauge theories, studies of the Casimir effect have been extended to various geometries and generalized to materials with arbitrary dielectrics and rough surfaces at finite temperature [6]. However, such geometries cannot be easily tested experimentally or computed analytically,

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with recent works exploring machine learning techniques [7]. Thermal fluctuations result in radiation pressure which adds thermal corrections to the Casimir force [8–10]. The thermal wavelength is given by, $\lambda_T = \hbar c / (2\pi k_B T)$ and $\lambda_T \simeq 1.2 \mu\text{m}$ at room temperature. Hence the relevant scale where thermal fluctuations become important is at separation distances, $d \gtrsim 1 \mu\text{m}$ where the wavelength of such fluctuations can fit inside the contained geometry.

In the abovementioned generalization, Lifshitz also predicted that there is a geometrical element to the Casimir effect, where some configurations result in a repulsive force. An example of such a configuration with a repulsive potential is the double spherical cavity where an inner conducting spherical shell of radius r is surrounded by an outer spherical shell with radius, $R \rightarrow \infty$ where the outer boundary is removed to spatial infinity [11–13]. A repulsive force has also been observed for a symmetrical tube in a weakly coupled, massless noninteracting scalar field computed in Ref. [14] where finite volume effects on the equation of state are studied using Dirichlet boundary conditions. This study suggests that the resulting zero-temperature Casimir pressure is *attractive* for a parallel plate and box configurations, and *repulsive* for a tube configuration in the scalar theory. Understanding whether such a geometric dependence of the Casimir force exists in non-Abelian gauge theories motivates this present work.

Other studies in QED have also revealed interesting effects resulting from the presence of boundaries, such as the Scharnhorst effect [15,16], where radiative corrections for parallel plate geometry lead to photons between the plates (moving perpendicular to the plates) travelling faster than the vacuum speed of light, c . The Abelian Casimir effect has also been studied on the lattice in an attempt to overcome inaccuracies due to e.g., thermal corrections in perturbative analytic methods for general geometries which are usually formulated using simplifications such as fixed boundary conditions. See, for example Ref. [17] where Green's functions are used to describe the boundary effects on vacuum fluctuations in quantum field theory and Refs. [18,19] where the Chern-Simons action is employed.

These lattice simulations have been extended to *compact* QED which exhibits properties, such as confinement in $(2+1)\text{D}$ that make it a valuable starting point before addressing non-Abelian gauge theories. We refer the reader to Refs. [20,21] for the case of parallel conducting wires in $(2+1)\text{D}$ and Ref. [22] for parallel conducting plates in $(3+1)\text{D}$ compact $U(1)$ gauge theory using ideally conducting parallel metallic boundaries. In these studies, the resulting action describes photon dynamics which characterize the perturbative effects and the formation of Abelian monopole dynamics describing the nonperturbative effects. The monopoles lead to mass-gap generation which screens the Casimir potential at large separation distances. At short separation distances, the monopole density between the boundaries is diminished into a dilute

gas of monopole-antimonopole pairs and the region between the wires/plates goes through an induced deconfining phase transition.

Non-Abelian Yang-Mills theories are more complicated than their Abelian counterparts since they are nonlinear, i.e. self-interacting. Let us briefly review the formulation and findings of Ref. [23], where the Casimir effect in non-Abelian gauge theory is studied on the lattice for perfectly conducting static parallel wires in $(2+1)\text{D}$ $SU(2)$ at zero temperature using chromoelectric boundaries.¹ The resulting attractive Casimir potential was found to be described by the empirical fitting function,

$$V_{\text{Cas}}^{\text{fit}}(R) = -(N_c^2 - 1) \frac{\zeta(3)}{16\pi} \frac{1}{R^{(\nu+2)} \sigma^{(\nu+1)}} e^{-M_{\text{Cas}} R}, \quad (1)$$

with fit parameters, ν and M_{Cas} , where σ is the string tension and N_c is the number of colors. The parameter ν describes the anomalous scaling dimension of the potential at short separation distances compared to the expected tree-level behavior,

$$V_{\text{Cas}}^{\text{tree}}(R) = -(N_c^2 - 1) \frac{\zeta(3)}{16\pi} \frac{1}{R^2}, \quad (2)$$

describing the Casimir energy of $(N_c^2 - 1)$ noninteracting copies of a monopole-free $U(1)$ gauge theory analogous to a free scalar field with Dirichlet boundary conditions [24]. The authors of [23] concluded that similarly to the finite volume geometry and the formulation in compact QED, this Casimir geometry is bound to induce a smooth deconfining phase transition between the wires. We explore the presence of this smooth deconfining phase transition for other geometries.

The zero temperature Yang-Mills theory exhibits non-perturbative dynamic mass-gap generation in both $(2+1)\text{D}$ and $(3+1)\text{D}$, similarly to compact QED. This nonperturbative mass-gap generation results in an effective screening of the Casimir potential at large separation distances quantified by the parameter M_{Cas} (the Casimir mass) in the exponential of Eq. (1). The mass-dependent exponential decay of the Casimir potential is consistent with the Casimir energy decay for a massive scalar particle obtained in a gauge-invariant Hamiltonian formulation of $(2+1)\text{D}$ non-Abelian gauge theories [25].

Interestingly, the resulting Casimir mass corresponding to the zero energy of the $(2+1)\text{D}$ gauge theory with perfect conductor parallel wires [23],

$$M_{\text{Cas}}^\infty = 1.38(3)\sqrt{\sigma} < M_{0++} = 4.718(43)\sqrt{\sigma} \quad (3)$$

¹On the lattice, this requires the coupling at the surface of the perfect conductor material to be different from the vacuum inverse coupling, which corresponds to increasing the lattice coupling at the boundaries by a Lagrange multiplier λ , such that the tangential field component vanishes, i.e. $\text{Tr}[P] \rightarrow 1$.

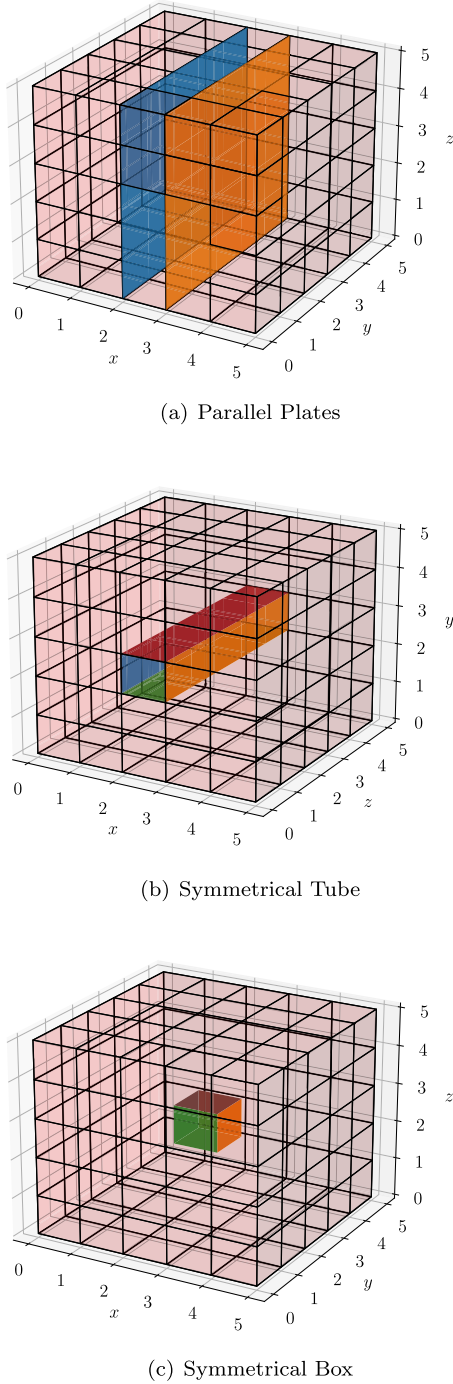


FIG. 1. Geometry of chromoelectric boundary conditions on the geometries of parallel plates, symmetrical and box in a $(3+1)$ D cubic lattice.

is less than the lowest 0^{++} glueball mass in $(2+1)$ D $SU(2)$ gauge theory [26]. In principle, the lowest glueball mass, $M_{0^{++}}$ should be the lowest mass in the system, hence this result suggests that the dominant degrees of freedom in the non-Abelian theory are a lighter gluonic state, possibly forming a “light-gluon” plasma due to the boundary-induced deconfining phase transition in the region between the wires.

This low Casimir mass has also been obtained in an alternative study employing analytical methods to show that the propagator (viewed in terms of traversing from one boundary to the other and back) of the gauge-invariant gluon field in $SU(2)$ corresponds to the propagator of a massive scalar field in $(2+1)$ D, with a mass corresponding to the Casimir mass [25]. The same study also shows that the $(2+1)$ D Casimir mass has physical implications for the $(3+1)$ D theory. It is shown that the Casimir mass in two dimensions is equal to the high temperature magnetic mass due to screening effects in the three-dimensional theory. Hence studies of the Casimir effect in $(2+1)$ D gauge theories can be used as a probe of the magnetic screening mass of a pure gauge QCD plasma.

In motivation of our study of the Casimir effect in non-Abelian gauge theories, we highlight its importance in quantum field theory. For example, in quantum chromodynamics, the Casimir effect has been pivotal in building our understanding of the phenomenologically relevant MIT bag model [27–29] of hadrons which describes hadron spectroscopy, confinement and other hadron properties.

In this work, we study the Casimir effect in non-Abelian gauge theories in $(2+1)$ D and $(3+1)$ D for the gauge group $SU(3)$ using lattice techniques. Inspired by the first-principle numerical simulations of Ref. [23], we apply chromoelectric boundary conditions on the lattice to formulate new as of yet unexplored types of geometries. We extend the results of the Casimir potential in $(3+1)$ D gauge theories for parallel conducting plates recently computed in Ref. [30] to $SU(2)$ and provide new results for the Casimir potential in $(3+1)$ D for a symmetrical tube and box, respectively. These geometries are shown in Fig. 1. The asymmetrical tube and box are studied in Ref. [31].

Moreover, we explore the Casimir energy dependence on the temperature given that Yang-Mills theories in infinite volume exhibit a phase transition where the theory deconfines into a plasma of gluons. As such, we investigate the Polyakov loop (deconfinement order parameter) as we change temperatures.

II. BOUNDARIES AND ELECTROMAGNETIC FIELDS

The energy-momentum tensor associated with the Yang-Mills Lagrangian in Minkowski space,

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a}, \quad (4)$$

is given by,

$$T^{\mu\nu} = -F^{\mu\alpha,a} F_{\alpha}^{\nu,a} + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta}^a F^{\alpha\beta,a}, \quad (5)$$

where $F_{0i} = E_i$, $F_{ij} = \epsilon_{ijk} B_k$ and $\frac{1}{4} F_{ij} F_{ij} = \frac{1}{2} B_i B_i$. We work on a Euclidean lattice using the coordinate system

notation, (x, y, z, t) , thus need to transform the field-strength tensor in Eq. (5) into this new coordinate system. The resulting energy density in Euclidean space whose local expectation value describes the energy of vacuum fluctuations of the gluon field is,

$$T^{00} = \frac{1}{2}(\overrightarrow{E^2} - \overrightarrow{B^2}). \quad (6)$$

In the parallel plate geometry where the two plates are placed at x_0 and x_1 , separated by a distance R , one can explore the symmetries of the Euclidean field strength tensor by exploiting the rotational symmetries of the geometry in four-dimensional Euclidean space. The three possible rotations result in the following equivalence relations in the expectation values of the field components,

$$\text{F1: } \langle B_z^2 \rangle = \langle B_y^2 \rangle \quad \text{and} \quad \langle E_z^2 \rangle = \langle E_y^2 \rangle \quad (7)$$

$$\text{F2: } \langle B_z^2 \rangle = \langle E_x^2 \rangle \quad \text{and} \quad \langle B_x^2 \rangle = \langle E_z^2 \rangle \quad (8)$$

$$\text{F3: } \langle B_y^2 \rangle = \langle E_x^2 \rangle \quad \text{and} \quad \langle B_x^2 \rangle = \langle E_y^2 \rangle \quad (9)$$

where F1 is obtained from the rotation about the xt plane, F2 about the xz plane and F3 about the xy plane.

The electromagnetic field strength tensor is related to the lattice *plaquette* variable according to,

$$U_{\mu\nu}(n) = 1 + ia^2 F_{\mu\nu}(n) - \frac{a^4}{2} (F_{\mu\nu}(n))^2 + \dots, \quad (10)$$

where a is the lattice spacing, and the continuum definition of the field strength tensor generalized from the definition in electrodynamics has been used and the forward finite differences method is employed for the derivative terms. On the lattice, the plaquette is represented by the product of four link variables forming a square,

$$U_{\mu\nu}(n) = U_\mu(n) U_\nu(n + \hat{\mu}) \times U_{-\hat{\mu}}(n + \hat{\mu} + \hat{\nu}) U_{-\hat{\nu}}(n + \hat{\nu}), \quad (11)$$

$$= U_\mu(n) U_\nu(n + \hat{\mu}) U_\mu(n + \hat{\nu})^\dagger U_\nu(n)^\dagger. \quad (12)$$

The sum of all possible plaquettes on the lattice describes the Wilson gauge action, with each plaquette counted with a single orientation. The Wilson action is given by,

$$S[U] \equiv \frac{\beta}{N_c} \sum_{n \in \Lambda} \sum_{\mu < \nu} \text{Re Tr}[1 - U_{\mu\nu}(n)], \quad (13)$$

where the sum is performed over all lattice points containing a plaquette and the Lorentz indices. The prefactor, β is the inverse coupling,

$$\beta = \frac{2N_c}{g^2}, \quad (14)$$

and provides the physical scale on the lattice. The electric-type boundary conditions are enforced by the following condition [23];

$$\beta_P = \begin{cases} \beta, & P \notin \mathcal{S} \\ \lambda_w \beta, & P \in \mathcal{S}, \end{cases} \quad (15)$$

requiring the coupling at the surface of the perfect conductor material (given by the limit, $\lambda_w \rightarrow \infty$) to be different from the vacuum inverse coupling, where the subscript P refers to plaquette and $\mathcal{S}$ refers to the world volume of the conductor material. We define $S_{P_{ij}}^{(k)}$ as the action of the k th configuration with plaquettes P_{ij} lying on the world volume of the boundary.

In the absence of the plates, the vacuum fields components in Eq. (6) are indistinguishable and fluctuate around a common expectation value as expected for a zero temperature lattice. We show the numerical results of the field expectation values in the presence of plates at varying separation distances normalized by subtracting the vacuum field expectation values in Fig. 2. The field components equivalence relations provided in Eqs. (7)–(9) can be seen directly from all the subfigures, where the error bars are omitted since they are small. We observe that field components on the world volume of the plates, E_z , E_y and B_x increase in magnitude as the distance to the plates is decreased. We also observe that the remaining field components also react to the presence of the plates and are slightly chipped. This behavior is expected from the structure of the local action, where updating a single link variable affects the six plaquettes attached to it.

Note that there is a clear symmetry in the orientation of the equivalent field fluctuations at the position of the plates because these are artificially modified through the boundary condition. This is accompanied by an asymmetry in the fields around the plates as a consequence of the naïve forward finite differences approach in taking the derivatives in the Wilson field-strength formulation given in Eq. (10). This topic is subject to ongoing research within the lattice community, see for example Ref. [32] for an expansive discussion.

In the case of the symmetrical tube, exploiting rotational symmetries of the geometry results in the F3 field equivalence relation given in Eq. (9). We show a three-dimensional visualization of the field configurations for a symmetrical tube with side lengths $R_{x_{1a}} = R_{y_{1a}} = 10$ in Fig. 3. The elongated $\hat{z}$ direction has been integrated out since the fields should be the same along this direction. In Fig. 3(a) we observe that all field components fluctuate around zero in vacuum, far from where the tube is positioned. As we approach the tube (from both the outside

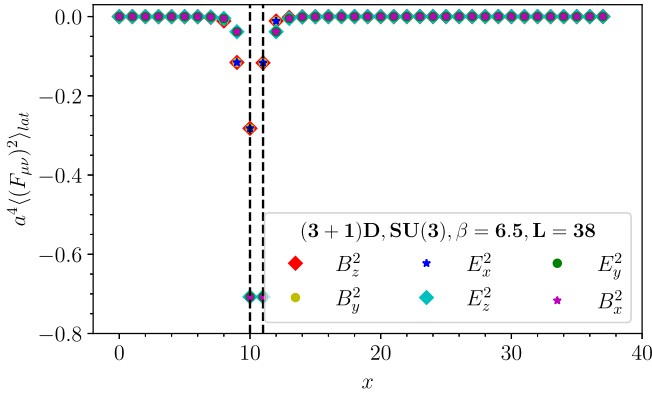
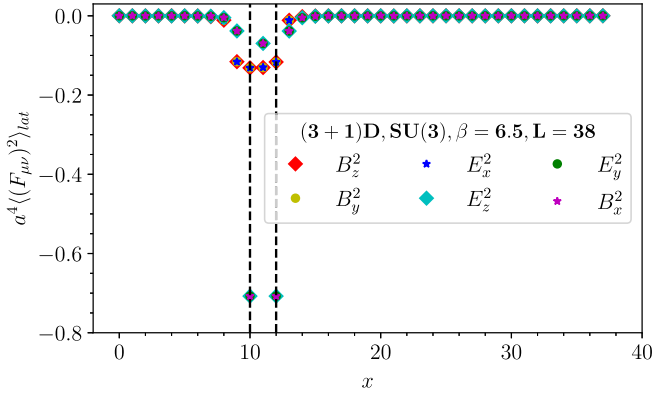
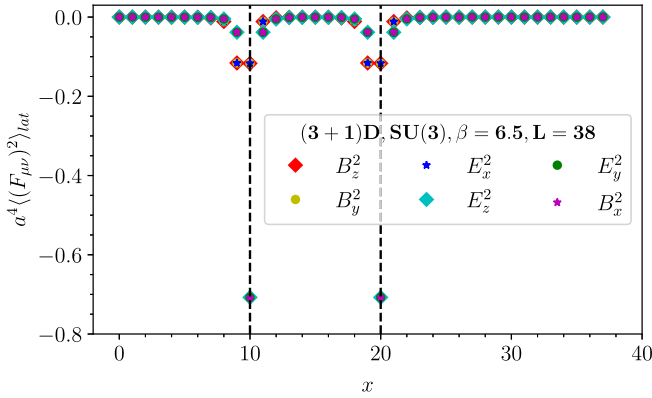
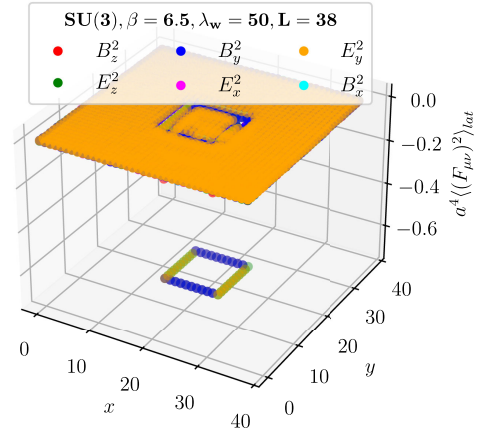
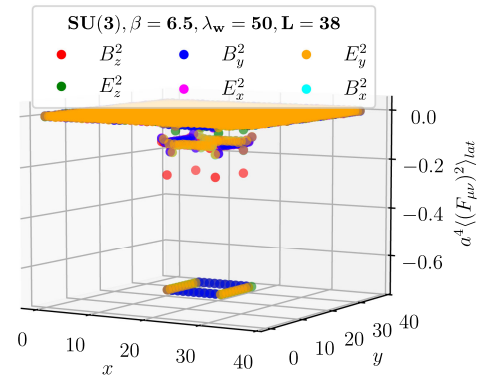
(a) Distance $R_{1a} = 1$ (b) Distance $R_{1a} = 2$ (c) Distance $R_{1a} = 10$

FIG. 2. Expectation values of the squared field strength tensor components in (3 + 1)D SU(3) along the x axis orthogonal to the plates (represented by dashed lines) placed a distance R apart.

or the inside), the surrounding plaquettes start experiencing the gauge field contributions from the boundaries of the tube and the fields change. This is seen in Fig. 3(b) where the magnitude of change increases the closer one is to the face of the tube. We also observe a relatively large change in the magnitude of the fields at the four corners of the tube,



(a) Top-view



(b) Side-view

FIG. 3. Three-dimensional expectation values of the squared field strength tensor components in (3 + 1)D SU(3) for a symmetrical tube with side lengths $R_{1a} = 10$ and faces at $x = y = 14a$ and $x = y = 24a$.

followed by maximal change on the faces of the tube. Numerical evidence of the F3 field equivalence relations is provided in Ref. [31].

Lastly, we consider the geometry of a symmetrical box where all rotational symmetries are broken because we have fixed the spatial 3D axis, leaving only the temporal direction free. In Fig. 4(a), we show the vacuum field expectation values at a distance $R_{1a} = 4$ away from the box. The surrounding gauge fields retain their vacuum expectation values and fluctuate around zero because they are vacuum normalized. Far enough outside the box, the surrounding plaquettes do not experience any plaquette contributions from the boundaries of the box. The various electromagnetic field components are indistinguishable.

As one approaches one of the faces of the box from the outside, we start seeing a change in some of the field components at varying degrees as shown in Fig. 4(b) a distance $R_{1a} = 1$ away from the lower face of the box.

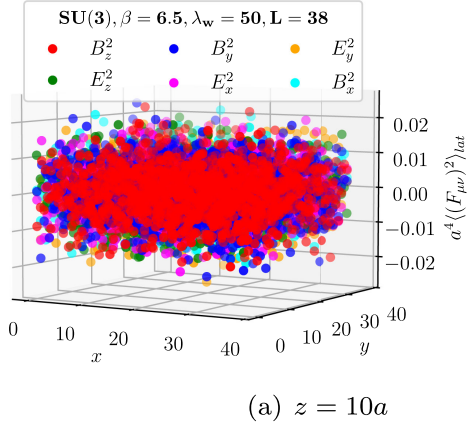
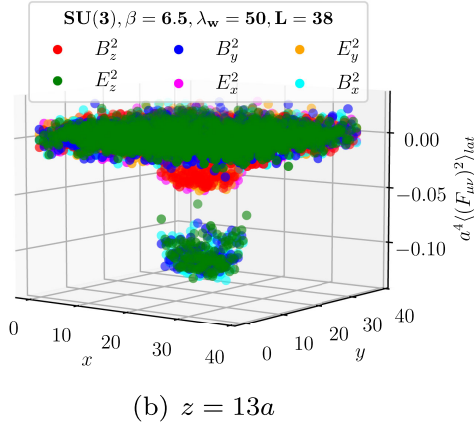
(a) $z = 10a$ (b) $z = 13a$

FIG. 4. Three dimensional expectation values of the squared field strength tensor components in $(3+1)$ D SU(3) outside a symmetrical box with side lengths $R_{1a} = 10$ and faces at $x = y = z = 14a$ and $x = y = z = 24a$.

The intensity of change in magnitude of the individual electromagnetic field components is dependent on the position around the box (i.e., different components change at varying intensities around the different faces depending on the contributing plaquettes). In this region, the vacuum plaquettes are now attached to link variables that are in contact with the boundaries of the box.

In Fig. (5), we show a cross section of the electromagnetic field components on the bottom face of the box at $z = 14a$. We observe a hollow region in the field strength indicating that all field components have the largest magnitude on the face. Similarly to the tube geometry, the field components with largest magnitude depend on the face under consideration. In this case, we are looking at a face in the xy plane, therefore the E_x , E_y and B_z experience the largest magnitude on this face. The field components on cross sections inside the box are similar to those of the tube, including the equivalence of the fields described by the F3 symmetry. We refer the reader to Ref. [31] for a comprehensive discussion on these field contributions.

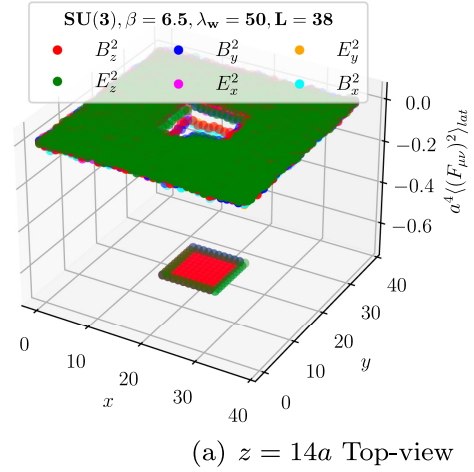
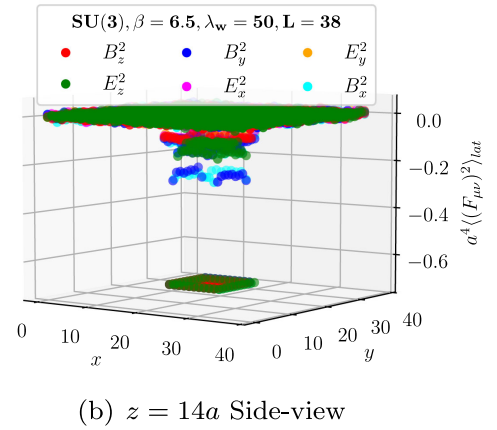
(a) $z = 14a$ Top-view(b) $z = 14a$ Side-view

FIG. 5. Three-dimensional expectation values of the squared field strength tensor components in $(3+1)$ D SU(3) on the face (at $z = 14a$) of a symmetrical box with side lengths $R_{1a} = 10$ and faces at $x = y = z = 14a$ and $x = y = z = 24a$.

III. RESULTS

In this Section, we present results of the non-Abelian Casimir effect in SU(3) for the geometry of parallel plates as well as a symmetrical tube and box, respectively. The geometries of an asymmetrical tube and asymmetrical box are presented in Ref. [31]. Our calculations are performed on a periodic lattice volume with chromoelectric boundary conditions on the walls of our geometries. All geometries are computed for the same physical lattice volume $L\sqrt{\sigma} \sim 4$, hence the β choices. In SU(2), we use the parameters $2.427 \leq \beta \leq 2.7$ corresponding to a physical lattice spacing $0.24013(41) \leq a\sqrt{\sigma} \leq 0.09737(23)$ and in SU(3) $5.99 \leq \beta \leq 6.5$ corresponding to physical lattice spacing $0.21982(77) \leq a\sqrt{\sigma} \leq 0.10383(24)$ [33]. In the geometries of the tube and box where the plate size expands with increasing separation distance, we discuss techniques that can be used to correctly account for the energy from creating the boundaries. We show that the resulting Casimir potential is attractive for all the geometries considered.

We use the naïve Wilson action with the Metropolis algorithm to generate the field configurations. We take a minimum of 5×10^3 thermalization steps and compute the integrated autocorrelation time, which is then used to discard correlated measurements in the Markov chain and ensures that we only use statistically independent measurements. Our errors are computed using the Jackknife method. See Appendix for a discussion on the thermalization properties. A continuum extrapolation is postponed to future study.

A. The Casimir potential

1. Parallel plates

The energy density for the plates can assume three equivalent forms due to the field equivalence relations obtained from rotational symmetry discussed in the preceding chapter,

$$\varepsilon(x)_{\text{plates}} = \frac{1}{2} \sum_i [\langle B_i^2 \rangle - \langle E_i^2 \rangle], \quad i = x, y, z \quad (16)$$

$$= \frac{1}{2} [\langle B_y^2 \rangle - \langle E_y^2 \rangle] \quad (17)$$

$$= \frac{1}{2} [\langle B_z^2 \rangle - \langle E_z^2 \rangle], \quad (18)$$

where we have used the F2 symmetry in Eq. (8) to obtain Eq. (17) and F3 symmetry in Eq. (9) to obtain Eq. (18).

The lattice expression for the energy density reads,

$$\varepsilon(x)_{\text{plates}}^{\text{lat}} = \langle S_{P_{ij}} \rangle = \frac{1}{N_\tau} \sum_{N_y} \sum_{N_z} \sum_{N_\tau} S_{P_{ij}}, \quad (19)$$

where P_{ij} represents the plaquettes which lie on the world volume of the plates. We have checked that the definitions based on Eqs. (16)–(18) are equivalent in our simulations. See Ref. [31] for the potential computed using the different symmetry relations. We use $S_{P_{ij}}$ for shorthand, representing the average of $S_{P_{ij}}^{(k)}$ over all k configurations.

Note that the energy density has no direct dependence on the individual plaquette components, P_{ij} contributing to the total energy density due to the summation. We represent the energy density, $\langle S_{P_{ij}} \rangle$ with the indices to signal that only contributions from selected plaquettes are considered and for consistency with the notation in the available literature. We obtain the three-dimensional normalized Casimir energy by integrating over the normal direction and applying the normalization condition to remove the energy contribution from the plates,

$$V_{\text{Cas}}^{\text{lat}}(R) = \left[\int_{dx} \varepsilon(x)_{\text{plates}} \right]_{R-R_0} = \left[\sum_{N_x} \varepsilon(x)_{\text{plates}}^{\text{lat}} \right]_{R-R_0} \quad (20)$$

$$= \sum_{N_x} [\langle S_{P_{ij}} \rangle_R - \langle S_{P_{ij}} \rangle_{R_0}] = \langle\langle S_{P_{ij}} \rangle\rangle, \quad (21)$$

given in lattice units. On the lattice, the Casimir potential is equivalent to the normalized expectation value of the total energy of the system,

$$\langle\langle S_{P_{ij}} \rangle\rangle = \left[\frac{1}{N_\tau} \sum_{N_x} \sum_{N_y} \sum_{N_z} \sum_{N_\tau} S_{P_{ij}} \right]_{R-R_0}, \quad (22)$$

and we highlight this definition for reasons that will become clear in the subsequent Subsections.

Due to periodic boundary conditions, the plates extend infinitely in the $\hat{y}$ and $\hat{z}$ directions. Since we consider the energy associated with a single lattice volume, we take as plate area, $A = N_z \times N_y$ which can be combined with Eq. (19) to find an expression for the energy per unit area of the plates. The physical potential scales as $V_{\text{Cas}}^{\text{phys}} = V_{\text{Cas}}^{\text{lat}}/a$, giving a well-defined expression for the total energy through an extension of the two-dimensional case proposed in Ref. [23] with the tadpole improved coupling β_I ,

$$V_{\text{Cas}}(R_{\text{phys}})/\sigma = \frac{1}{a\sqrt{\sigma}} \left(\frac{\beta_I}{\beta} \right)^4 \langle\langle S_{P_{ij}} \rangle\rangle, \quad (23)$$

where σ is the string tension and the Casimir energy per unit area of the plates scales as $V_{\text{Cas}}^{\text{phys}} = V_{\text{Cas}}^{\text{lat}}/a^3$, and the resulting potential per unit area is,

$$V_{\text{Cas}}(R_{\text{phys}})/\sqrt{\sigma^3} = \frac{1}{a^3\sqrt{\sigma^3}} \left(\frac{\beta_I}{\beta} \right)^4 \langle\langle S_{P_{ij}} \rangle\rangle, \quad (24)$$

and the string tension is used to express the potential in dimensionless units.

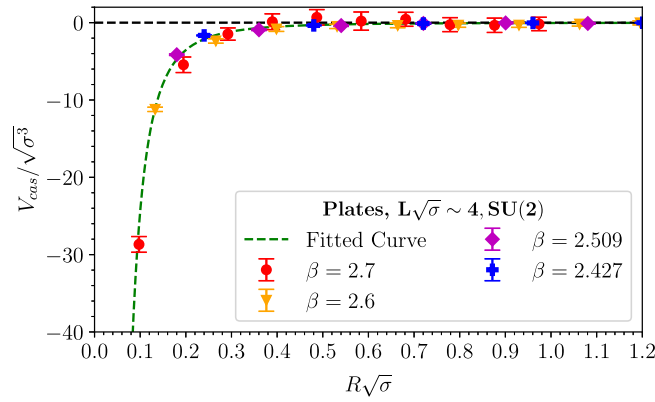
The normalization is enforced by subtracting the lattice action expectation value with the parallel plates placed at separation distance, $R_0 \rightarrow \infty$ apart, which on our finite periodic lattice we choose to be at half the lattice spatial extent, $R_0 = L/2$. For datasets where significant statistical noise is present at $R_0 = L/2$ we used the mean of the action expectation value between $0.7 < R\sqrt{\sigma} < 1.2$ for normalization. Essentially, we compute the total energy of the system with plates placed a finite distance, R apart, and from this, we subtract the total energy of the system with the plates placed as far apart as possible. At large enough distances, the Casimir interaction will diminish leaving only the vacuum energy contributions and the energy of creating the boundaries. We express this as,

$$V_{\text{Cas}}^{\text{lat}}(R) = E_R^{\text{Tot}} - E_{R_0}^{\text{Tot}} \quad (25)$$

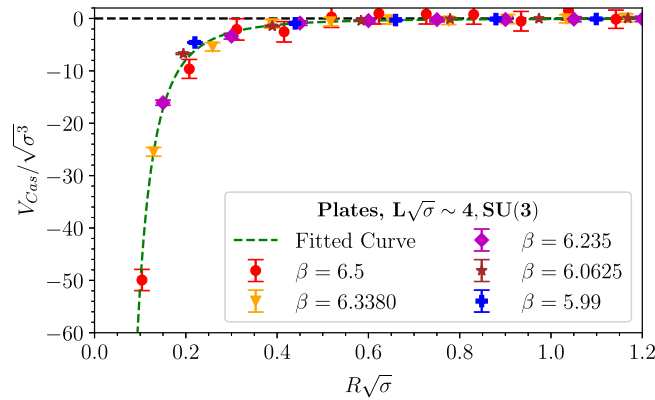
$$= [E_{\text{Cas}}(R) + E_{\text{Plates}} + E_{\text{Vac}}]_R - [E_{\text{Plates}} + E_{\text{Vac}}]_{R_0}, \quad (26)$$

where the zero-point energy is independent of R , but so is the energy contribution from the plates because the size of the plates is fixed. We will return to the discussion of the energy contributions from imposing boundary conditions in the subsequent Sections where it contributes a non-constant value. The normalization condition takes care of two energy contributions to the system, the first is the cancellation of the vacuum ultraviolet divergences such that the energy density describes a local finite quantity. The second is the energy contribution from imposing the chromoelectric boundary condition (geometry of two parallel plates) on the QCD vacuum, i.e., the act of putting the plates modifies the zero-point energy.

The dimensionless physical Casimir potential per unit area of the plates is shown in Fig. 6(a) in SU(2) and Fig. 6(b) in SU(3) in units of the string tension. The different filled



(a) SU(2)



(b) SU(3)

FIG. 6. Casimir potential per unit area (filled symbols) between two parallel plates separated by a distance $R\sqrt{\sigma}$ in (top) SU(2) and (bottom) SU(3) non-Abelian gauge theory in $3 + 1$ dimensions. As the data line up along a common curve within two sigma of uncertainty, we also present the values of a global fit (dashed line) of the potential function Eq. (27).

symbols describe the values of the potential obtained from lattice ensembles with different coupling and thus different physical lattice spacing. The larger the value of beta, the smaller the associated lattice spacing. We find that for the statistics we were able to collect, within two sigma, the data points line up and can be described by a single potential function (dashed line). This potential function is negative, its functional form indicates that the Casimir force, described by the slope of the potential, experienced by the plates in the non-Abelian gauge theory is attractive.

Let us take a more quantitative look at the potential. In order to carry out the global fit to all data points, we employ a fitting function that is a straightforward generalization of the function used in the study of the Casimir potential between wires in $(2 + 1)\text{D}$ [23]. It reads

$$V_{\text{Cas}}^{\text{fit}}(R) = -2(N_c^2 - 1) \frac{\pi^2}{1440} \frac{1}{R^{(\nu+3)} \sigma^{(\nu+3)/2}} e^{-M_{\text{Cas}} R}, \quad (27)$$

where the power of σ is chosen to give a dimensionless potential, i.e., $V_{\text{Cas}}/\sqrt{\sigma^3}$. The dependence on the distance and numerical coefficients are obtained from the tree level behavior of the potential for a noninteracting theory, $M_{\text{Cas}} = \nu = 0$ [24],

$$V_{\text{Cas}}^{\text{tree}}(R) = -2(N_c^2 - 1) \frac{\pi^2}{1440} \frac{1}{R^3}, \quad (28)$$

describing $2(N_c^2 - 1)$ additive contributions from the non-interacting gluon fields.

The results for fitting our lattice data from Fig. 6 with the above fitting function are collected in Table I. They show that our results are compatible with a vanishing anomalous scaling dimension $\nu = 0$ in both SU(2) and SU(3). For the Casimir mass, in units of $\sqrt{\sigma}$ we obtain values smaller than unity. The statistical errors are obtained from the χ^2 fit used to determine the optimal fitting parameters, while the systematic errors summarize several sources of uncertainty. They include differences between the global fit of all data points, compared to carrying out the fits on each beta value separately and averaging in the end. They also include the differences observed when fitting the data directly or fitting their logarithm to avoid instabilities in the exponential function fit.

TABLE I. Results of the fit of the potential function Eq. (28) to the data presented in Fig. 6. Systematic errors quantify differences between fitting the potential for each β parameter separately, averaging *a posteriori* versus carrying out a global fit on all datapoints combined. In addition differences from fitting the data directly or fitting their logarithm are included.

	ν	$M_{\text{Cas}}/\sqrt{\sigma}$
SU(2)	0.00 ± 0.02	$0.42 \pm 0.09(\text{stat}) \pm 0.16(\text{sys})$
SU(3)	0.00 ± 0.02	$0.68 \pm 0.07(\text{stat}) \pm 0.17(\text{sys})$

The Casimir potential for the plates has also been recently computed in Ref. [30] where a comprehensive discussion is given for the presence of a boundary induced deconfined phase transition in the region between the plates. We will expand on this discussion when we look at the temperature dependence of the potential in the next Section. The SU(3) Casimir potential that we obtain has similar qualitative features to the potential obtained in Ref. [30], however the rate of decay of the potential varies particularly at short separation distances where our measured potential decays more rapidly. The value of the Casimir mass computed by Chernodub *et al.* $M_{\text{Cas}} = 1.0(1)\sqrt{\sigma}$ is a larger than what we obtained here, but the values differ by only 1.3σ . Note that no continuum extrapolation has been performed. As our study focuses on extending the study of the Casimir effect to different geometries we refrain from a detailed quantitative comparison to their results, which is left for future work.

2. Symmetrical tube

The next geometry that we consider in the non-Abelian theory is a hollow symmetrical tube. In the preceding Section, we showed that there exists an equivalence of the $\hat{x}$ and $\hat{y}$ field components according to the F3 symmetry in Eq. (9) such that the expression for the energy density of the system in Euclidean space reduces to only the $\hat{z}$ components,

$$\varepsilon_{\text{tube}}(x, y) = \frac{1}{2} [\langle B_z^2 \rangle - \langle E_z^2 \rangle], \quad (29)$$

which depends on both x and y because the gluon fields are now enclosed in a hollow cavity and the Casimir force acts in two directions.

On the lattice, the energy density is given by a sum over the plaquettes on the world volume of the tube,

$$\varepsilon(x, y)_{\text{tube}}^{\text{lat}} = \langle S_{P_{ij}} \rangle = \frac{1}{N_\tau} \sum_{N_z} \sum_{N_\tau} S_{P_{ij}}, \quad (30)$$

and these plaquettes contributing to the action are reduced to P_{xy} and P_{zt} according to Eq. (29). The expression for the Casimir potential of the tube reads,

$$V_{\text{Cas}}^{\text{lat}}(R) = \left[\int_{dA} \varepsilon(x, y)_{\text{tube}} \right]_{R-R_0} \quad (31)$$

$$= \left[\sum_{N_x, N_y} \varepsilon(x, y)_{\text{tube}}^{\text{lat}} \right]_{R-R_0} = \langle \langle S_{P_{ij}} \rangle \rangle \quad (32)$$

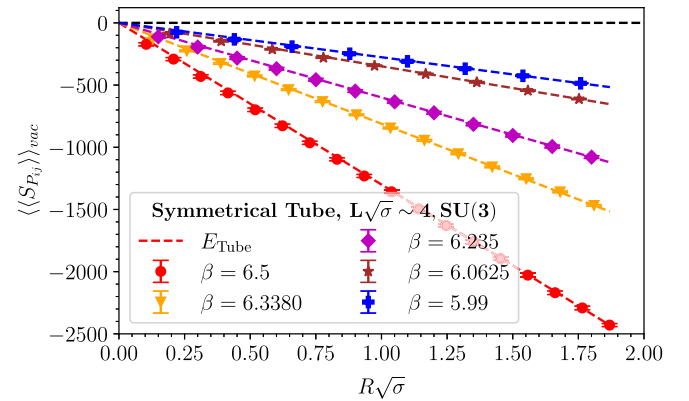
$$= \sum_{N_x, N_y} [\langle S_{P_{ij}} \rangle_R - \langle S_{P_{ij}} \rangle_{R_0}], \quad (33)$$

where $\langle \langle S_{P_{ij}} \rangle \rangle$ is given by Eq. (22).

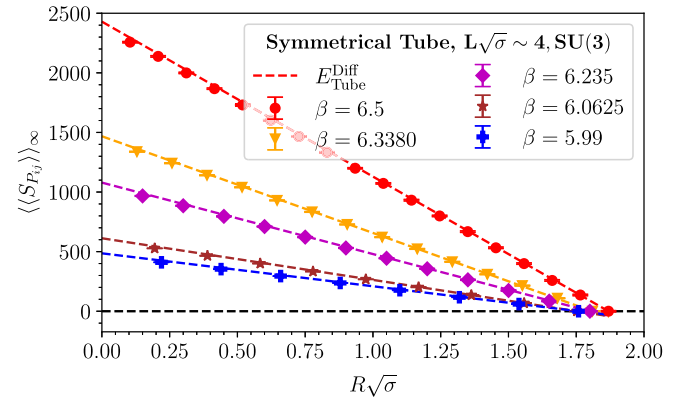
In the previous Section, i.e. the geometry of parallel plates, the normalization condition involved a simple constant since the area of the plates did not change with R . It is shown in Ref. [31] that the energy contribution from the plates was independent of the separation distance.

However, the energy contribution from the tube depends on the separation distance R , because as R increases, the size of the four rectangular faces forming the tube also increases. The fact that introducing a boundary into the simulation domain is associated with a change of energy can be understood from the fact that at the interface we enforce a larger coupling according to Eq. (15). Hence the number of plaquettes set close to unity differs among the differently sized tubes, affecting the overall energy. In the following we carefully disentangle these contributions from the physics of the Casimir potential. We devise two approaches to account for this contribution.

In order to capture the tube's energy dependence on the separation distance, R and consequently the size of the plates forming the tube, we plot the total energy of a symmetrical tube in Fig. (7) using different normalization



(a) Vacuum Subtracted Energy



(b) R_∞ Subtracted Energy

FIG. 7. Total energy of the system in a symmetrical tube in (3 + 1)D SU(3) for different couplings and different normalization schemes in lattice units.

schemes. In Fig. 7(a) we plot the vacuum subtracted total energy of the tube,

$$E_{\text{Tot}}^{\text{Vac}} = E_{\text{Cas}} + E_{\text{Tube}}. \quad (34)$$

Note that this definition explicitly excludes the energy needed to create the faces of the tube from the Casimir energy.

An inspection of Fig. 7 reveals that the total energy, varies linearly with separation distance at distances $R\sqrt{\sigma} > 1$. However, we also know from our findings for the plates, that at large separation distances, the Casimir energy contribution should be negligible.

We therefore conclude that the energy contribution from creating the tube, E_{Tube} increases linearly in magnitude with increasing plate size, thus has the functional form,

$$E_{\text{Tube}} = -mR\sqrt{\sigma} + c, \quad (35)$$

with fit parameters, m capturing the plate size dependence and c providing the overall scaling. In the absence of the tube, i.e., at $R\sqrt{\sigma} = 0$, the energy contribution from the tube vanishes. This allows us to set the constant term, $c = 0$. At $R = L\sqrt{\sigma}/2$, which is the an create in the periodic lattice, the Casimir energy vanishes because there is an equal number of modes inside and outside the tube. We show the vacuum normalized total energy of the system at $R = L\sqrt{\sigma}/2$ in Fig. (8), which is equal to the energy of creating the large tube as per Eq. (34).

We use this result of the energy for the largest possible tube to fix the energy at the second endpoint of the linear function, and consequently calculate the slope of the resulting straight line which describes the energy from creating the tube. Subtracting this energy from the system's total energy in Eq. (34) isolates the Casimir potential. The resulting slope of the energy of the walls of a symmetrical tube shown in Fig. 7 is provided in Table II at various gauge

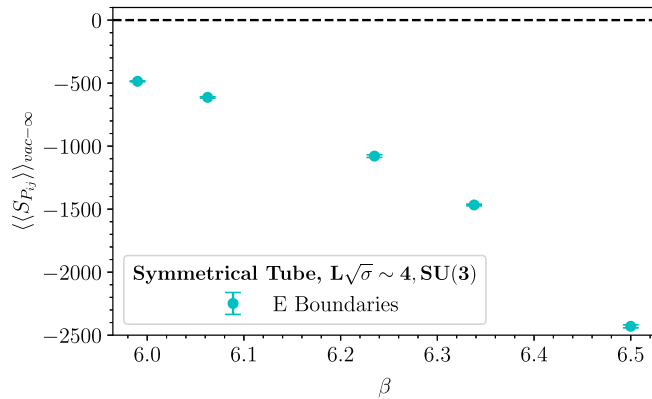


FIG. 8. Total energy required to create a symmetrical tube with side lengths $L\sqrt{\sigma}/2$ in $(3+1)$ D SU(3) at varying lattice couplings.

TABLE II. Linear fit parameters for the energy contributions from the boundaries of a symmetrical tube in SU(3) for lattice size, $L\sqrt{\sigma} \sim 4$.

Energy	β	m	c
$\langle\langle S_{P_{ij}} \rangle\rangle_{\infty}$	6.500	-1300	2430
	6.338	-811	1468
	6.235	-599	1079
	6.063	-349	612
	5.990	-276	486

coupling constants, where $\langle\langle S_{P_{ij}} \rangle\rangle_{\text{vac}}$ has the same gradient, m with $c = 0$ for corresponding β .

In summary, the Casimir energy is isolated from the energy contributions associated with the creation of the tube faces as follows;

- (1) Compute the total energy of the system and apply the vacuum subtraction to remove UV divergences, remaining with the total energy in Eq. (34).
- (2) Approximate the energy of the walls of the tube using a linear function. Take the constant term, $c = 0$ because the energy from creating the boundaries vanishes in the absence of boundaries. Take the energy at $R = L\sqrt{\sigma}/2$ to be equal to the energy of creating the largest possible tube with side-length R (where the Casimir energy vanishes). Draw a straight line between these two endpoints and calculate the slope.
- (3) Use the slope to perform a second subtraction on the total energy in Eq. (34) in order to remove the energy associated with creating the tube. This isolates the Casimir energy.

The resulting physical absolute Casimir potential for the symmetrical tube as defined in Eq. (23) is shown in Fig. 9(a) and is attractive, while the Casimir energy per unit surface area of the symmetrical tube defined in Eq. (24) is shown in Fig. 9(b). The corresponding total surface area of the symmetrical tube is, $A_{\text{lat}} = 4\sigma RL$, where R is the tube side-length and separation distance. Computing the potential per unit surface area introduces an additional dependence on the side-length of the tube, $R\sqrt{\sigma}$ which scales the diminishing potential with separation distance nonlinearly.

In Fig. 7(b), we show an alternative way of accounting for the energy contributions of the walls of the tube. Instead of performing a vacuum normalization, we use the R_{∞} normalization where we subtract from the total energy, the energy contribution with a large tube of side length, $R = R_{\infty} = L\sqrt{\sigma}/2$. This is the same normalization employed in the parallel plates geometry, except, the energy contribution from creating a larger tube of side length, $R = R_{\infty} = L\sqrt{\sigma}/2$ is greater than the energy required to create a smaller tube with $R < L\sqrt{\sigma}/2$. Hence by subtracting the energy contribution at R_{∞} , we eliminate the vacuum energy but remain with an energy

difference between creating a small tube and a large tube according to,

$$E_{\text{Tot}}^{\infty} = E_{\text{Cas}} + E_{\text{Tube}}^{\text{Diff}}, \quad E_{\text{Tube}}^{\text{Diff}} = E_{\text{Tube}}^{\text{R}} - E_{\text{Tube}}^{\text{R}_{\infty}}. \quad (36)$$

Again, the resulting energy difference is linear and vanishes at $R = L\sqrt{\sigma}/2$ because the Casimir energy is negligible and the “inner tube” of side length, R has equal dimensions to the R_{∞} tube. Similarly to the vacuum normalization, we employ a linear function to describe this energy difference and use it to renormalize the total energy in Eq. (36), thus isolating the Casimir energy. This approach is discussed comprehensively in Ref. [31], including a method of employing a linear fitting function for $R \gtrsim 1$ to obtain the energy from the walls of the tube.

We highlight the following important features of the symmetrical tube geometry based on our numerical results:

- (i) The Casimir potential is negative and has a positive slope, which implies that the resulting Casimir force is attractive. This result contrasts with the Casimir effect of a massless noninteracting scalar field computed in Ref. [14] using Dirichlet boundary conditions. We highlight that in the lattice formulation, if one does not correctly account for the energy contributions from the boundaries in the symmetrical tube geometry, then one would incorrectly find a repulsive Casimir potential as can be seen directly from the slope of the potential Fig. (7).
- (ii) The functional form of the total Casimir potential (not per unit surface area) shown in Fig. 9(a) appears linear as compared to the form in Eq. (27) for the case of parallel plates. Hence the Casimir force experienced by the surfaces of the tube diminishes slower with increasing separation distance.
- (iii) The potential of the tube at varying inverse coupling forms a smooth curve, but it does not collapse to a single curve as was the case for parallel plates. There are no additional energy contributions to the system that could introduce a different scaling, and we note this as a subject for further investigations.

3. Symmetrical box

The last geometry that we discuss for the non-Abelian gauge theory in SU(3) is a hollow box shown in Fig. 1(c). The box has finite extents on the three-dimensional spatial axis with side lengths $R_x = R_y = R_z = R\sqrt{\sigma}$ in a physical lattice volume $L\sqrt{\sigma} \sim 4$. We refer the reader to the preceding Section for a discussion on the electromagnetic field-strength tensor components and the numerical field configurations results for the symmetrical box.

The geometry of a box requires the fixing of all spatial coordinates, leaving only the Euclidean time axis free. Consequently, all rotational symmetries are broken in this geometry because rotations in four-dimensional space require a minimum of two free axes. Therefore, as opposed

to the geometries that we have discussed so far, where we explored various symmetry relations to reduce the full expression of the energy density of the system into simpler expressions, such an approach is not possible here.

The energy density of the box reads,

$$\varepsilon_{\text{box}}(x, y, z) = \frac{1}{2} \sum_i [\langle B_i^2 \rangle - \langle E_i^2 \rangle], \quad i = x, y, z \quad (37)$$

and is dependent on x, y and z because the gluon fields are enclosed in a finite volume (box) and the Casimir force acts on the walls of the box in all three spatial directions. The lattice expression of the energy density is a sum over the plaquettes on the world volume of the box,

$$\varepsilon(x, y, z)_{\text{box}}^{\text{lat}} = \langle S_{P_{ij}} \rangle = \frac{1}{N_{\tau}} \sum_{N_{\tau}} S_{P_{ij}}, \quad (38)$$

which comprise of all the plaquettes in the spatial and temporal directions according to Eq. (37). The resulting expression for the Casimir potential of the box is,

$$V_{\text{Cas}}^{\text{lat}}(R) = \left[\int_{dV} \varepsilon(x, y, z)_{\text{box}} \right]_{R-R_0} \quad (39)$$

$$= \left[\sum_{N_x, N_y, N_z} \varepsilon(x, y, z)_{\text{box}}^{\text{lat}} \right]_{R-R_0} \quad (40)$$

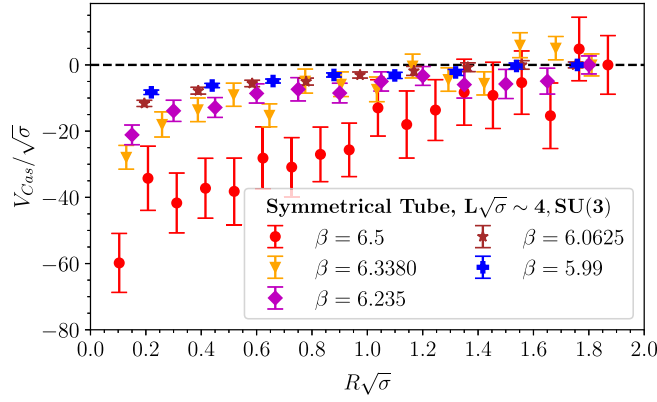
$$= \sum_{N_x, N_y, N_z} [\langle S_{P_{ij}} \rangle_R - \langle S_{P_{ij}} \rangle_{R_0}], \quad (41)$$

also represented by $\langle\langle S_{P_{ij}} \rangle\rangle$, and the expressions for the physical energies follow similarly to other cases already discussed in four-dimensional space.

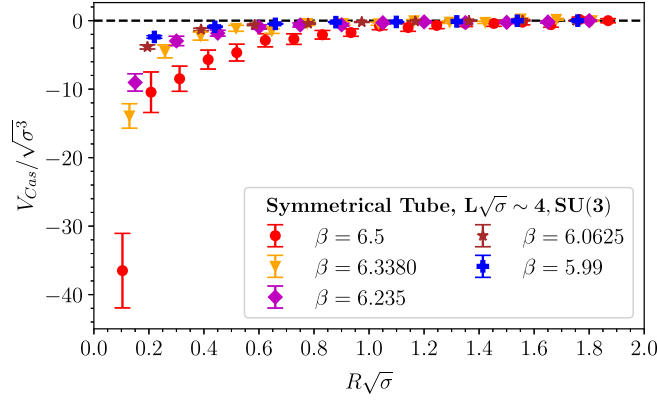
In the previous Subsection, we discussed the intricacies of accounting for the energy contributions from the boundaries of the tube. Such difficulties were overcome by exploiting the linear dependence of the energy of the walls of the tube with respect to tube size. We encounter the same problem in the case of the box geometry, and once again, have to redefine the normalization condition at R_0 .

We start by approaching this boundary problem for the symmetrical box in similar fashion to the symmetrical tube. We compute the lattice total energy of the system using Eq. (41) and compare the result obtained using the vacuum subtraction and its R_{∞} counterpart where we subtract the energy of a larger, symmetrical outer box of side length, $L\sqrt{\sigma}/2$. These results are shown in Fig. (10), and we observe that the energy dependence does not vary linearly with increasing box size as was the case for the tube, allowing for the simplification of the calculation.

The main difference in the observed total energy in the symmetric box is that it is nonmonotonous. We have already seen in the previous subsection that the energy



(a) Absolute Casimir Potential



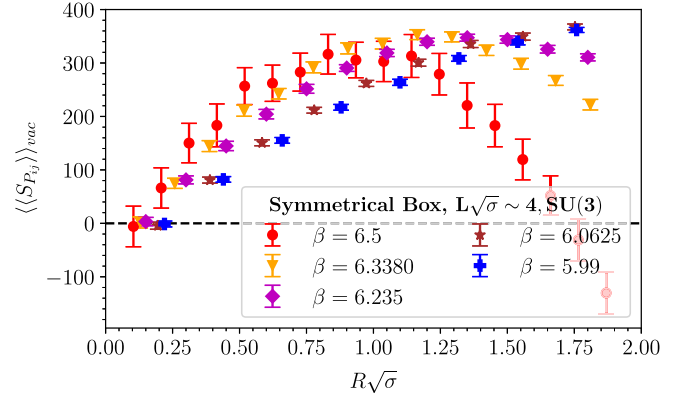
(b) Potential/Surface Area

FIG. 9. Total Casimir potential for a symmetrical tube with side lengths $R\sqrt{\sigma}$ in SU(3).

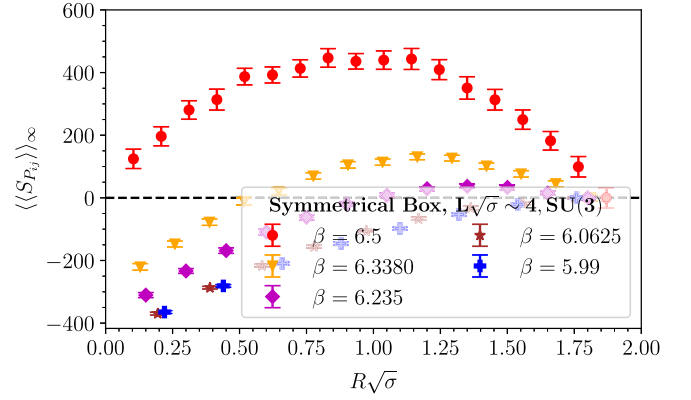
contribution from the boundaries increases with the size of the boundary, i.e., the larger the box, the larger the energy contribution from the introduction of its walls should be. Hence this is a puzzling result because intuitively, the Casimir effect should vanish with increasing separation distance or box size.

Nothing suggests that any other effects should materialize when the box is increased to a certain size. The same argument is true for thermal fluctuations, which should not fit inside the box. Therefore, we maintain that the observed effect of the total energy in the symmetric box is purely a consequence of the box's boundaries, which include non-trivial contributions from the edges and corners. This argument is motivated by the R_∞ subtracted energy in Fig. 10(b), which vanishes at $R = L\sqrt{\sigma}/2$.

In the case of the symmetrical tube we observed that the functional form of the system's energy was linear for $R \rightarrow \infty$ for both normalization schemes. However, in the case of the symmetrical box, the dependence of the total energy of the system on the separation distance (and consequently the expanding volume of the box) is more intricate. Perhaps a parabolic analytic form would work



(a) Vacuum Subtracted Fit

(b) R_∞ Subtracted FitFIG. 10. Total energy of the system in a symmetrical box in $(3+1)$ D SU(3) for different couplings and different normalization schemes in lattice units.

based on the observation of the energy at coupling, $\beta = 6.5$, but we would need to increase our physical lattice volume to validate whether this can be generalized to other couplings.

Despite the difficulty in assuming a simple analytical form for the system's energy, there are some common and consistent features. In the vacuum normalization, in the limit, $R\sqrt{\sigma} \rightarrow 0$, the total energy of the system,

$$E_{\text{Tot}}^{\text{vac}} = E_{\text{Cas}} + E_{\text{Box}}, \quad (42)$$

approaches zero because in this limit, the energy contribution from the walls of the box vanishes and so does the Casimir energy since it has to vanish in the absence of the box. On the other hand, in the R_∞ subtraction scheme, in the limit $R\sqrt{\sigma} \rightarrow L\sqrt{\sigma}/2$, the total energy of the system given in Eq. (36) with “Tube” replaced by “Box,” vanishes. This is consistent with our expectations because in this limit, the Casimir energy should be negligible and if the “inner box” is the same size as the “outer box,” then their energy contributions cancel.

We propose an alternative normalization scheme for the symmetrical box. Each box is formed by a combination of six sides/faces, therefore, we start by finding the energy contribution of two finitely extending faces of the same size placed a distance, $R = L\sqrt{\sigma}/2$ apart. As discussed in previous Sections, the energy contribution of the faces is independent of their separation distance. We merely place these faces far apart to ensure that there are no Casimir energy contributions to the total energy of the system.

The box system has total energy,

$$E_{\text{Box}}^{\text{Tot}} = E_{\text{Cas}}^{\text{Box}} + E_{\text{Box}} + E_{\text{Vac}}, \quad (43)$$

whereas the two finite parallel faces system has total energy,

$$E_{\text{Faces}}^{\text{Tot}} = E_{\text{Cas}}^{\text{Faces}} + E_{\text{Faces}} + E_{\text{Vac}}. \quad (44)$$

We numerically compute the energy contribution from two finite faces placed far apart. In the end we are interested in the six-faces configuration in order to compare the total energy of this configuration to the energy of the box system (whose geometry is formed by a total of six faces). Thus, we multiply $E_{\text{Faces}}^{\text{Tot}}$ by three to obtain,

$$E_{\text{Faces}}^{\text{Tot}_2} = 3E_{\text{Faces}}^{\text{Tot}} = 3(E_{\text{Faces}} + E_{\text{Vac}}) \quad (45)$$

$$= \tilde{E}_{\text{Faces}} + 3E_{\text{Vac}}. \quad (46)$$

Performing a vacuum subtraction on the energies of the two systems we obtain,

$$\tilde{E}_{\text{Box}}^{\text{Tot}} = E_{\text{Box}}^{\text{Tot}} - E_{\text{Vac}} = E_{\text{Cas}}^{\text{Box}} + E_{\text{Box}}, \quad (47)$$

$$\tilde{E}_{\text{Faces}}^{\text{Tot}} = E_{\text{Faces}}^{\text{Tot}_2} - 3E_{\text{Vac}} = \tilde{E}_{\text{Faces}}, \quad (48)$$

and we show the vacuum normalized energy for the box in Fig. (11) as colored points, along with the corresponding

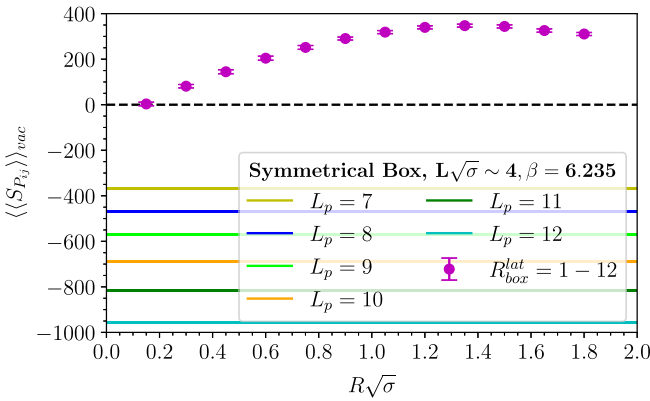


FIG. 11. Vacuum normalized energy of six faces compared to the vacuum normalized energy of a box with side lengths equal to the size of the faces.

energy of the six faces forming a box as colored lines, where $R = L_p$ is the lattice side-length of each face. It is reasonable that the energy contributions from the faces increases with the size of the faces.

Note that the energy contribution from six faces is not equal to the energy contribution of the box (composed of six faces). Hence, this is not a direct comparison of the energy contributions in the two systems because the geometrical setup of finite parallel faces is different from that of a single box with the same side lengths. The differences in energies can be attributed to the presence of corners in the box (see the field contributions from the corners in the preceding Section) and varying contributions from the edges in the two geometries.

The energy difference between the two systems (i.e. a box compared to a combination of six faces) is given by,

$$E_{\text{Box}}^{\text{Diff}}(R) = \tilde{E}_{\text{Box}}^{\text{Tot}} - \tilde{E}_{\text{Faces}}^{\text{Tot}} = E_{\text{Cas}}^{\text{Box}} + E_{\text{Box}} - \tilde{E}_{\text{Faces}}, \quad (49)$$

and shown in Fig. (12) for different gauge couplings. The Casimir energy contribution for the geometry with two finite faces is negligible irrespective of the size of the faces because the faces are placed far apart at a distance, $R = L\sqrt{\sigma}/2$. At large separation distances (large enough box size), the Casimir energy for the box becomes negligible, leaving only the energy difference between the boundaries of the two geometries. Therefore, for $R \gtrsim 1$, this energy difference should describe,

$$E_3 = E_{\text{Box}}^{\text{Diff}}|_{R \gtrsim 1} = E_{\text{Box}} - \tilde{E}_{\text{Faces}}. \quad (50)$$

We find that this energy difference varies linearly with the size of the box and faces. Therefore, instead of finding intricate functions to extract the energy of the box's boundaries from the observed behavior in Fig. 10, we have reduced the problem to finding the fitting parameters of a linear function. We find the constant of the fit by fixing the energy difference at the $R = 0$ end point to zero since the boundary contributions from both geometries vanish.

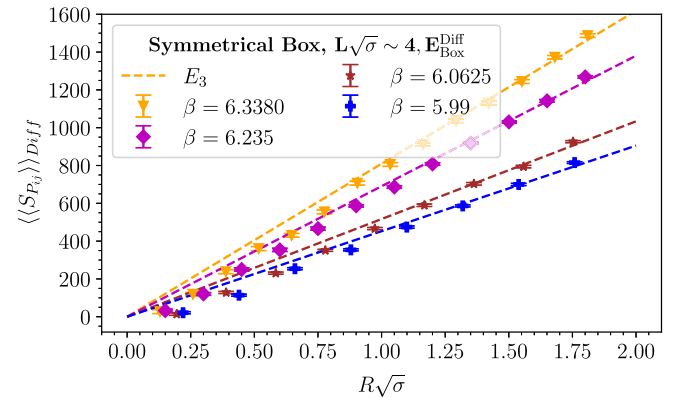


FIG. 12. Energy difference between the creation of six faces and a box with side lengths equal to the size of the faces.

TABLE III. Linear fit parameters for the energy difference between the boundaries of six faces and those of a symmetrical box with side lengths equal to the faces.

Energy	β	m	c
$\langle\langle S_{P_{ij}} \rangle\rangle_{\text{Diff}}$	6.338	809	0
	6.235	690	0
	6.063	517	0
	5.990	453	0

The resulting fit parameters are provided in Table III at different couplings.

Once we have obtained this energy difference, we can normalize the energy of a symmetrical box using the energy from the boundaries of a finite parallel faces configuration, accounting for six faces that form a box. We then use the linear fit to remove the expected energy difference between the boundaries of a box and the parallel faces setup,

$$E_{\text{Cas}} = \tilde{E}_1 - \tilde{E}_2 - E_3. \quad (51)$$

This approach allows us to isolate the Casimir energy of the symmetrical box, as well as the approximate energy from the walls of the box,

$$E_{\text{Box}} = \tilde{E}_2 + E_3, \quad (52)$$

where E_1 is the total energy of the system with a box present, $E_{\text{Faces}}^{\text{Tot}}$ is the total energy of the system with two parallel faces placed far apart (times three) and E_3 is the energy difference of the respective boundaries.

Most importantly, we have not only isolated the Casimir energy for the box, but also, the energy contributions from the walls of the box given by Eq. (52). Despite not having an analytical formula to describe the energy from creating the box, we provide an approach to numerically extract it from lattice simulation data and we show this energy in Fig. (13). Although our result is sensitive to the fit parameters, it provides a reasonable estimate of the Casimir effect in this geometry.

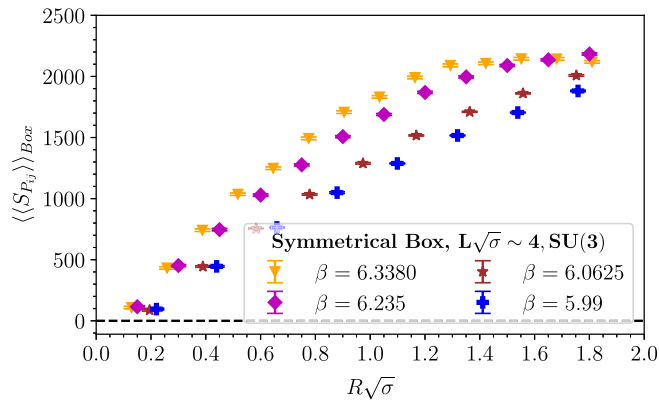
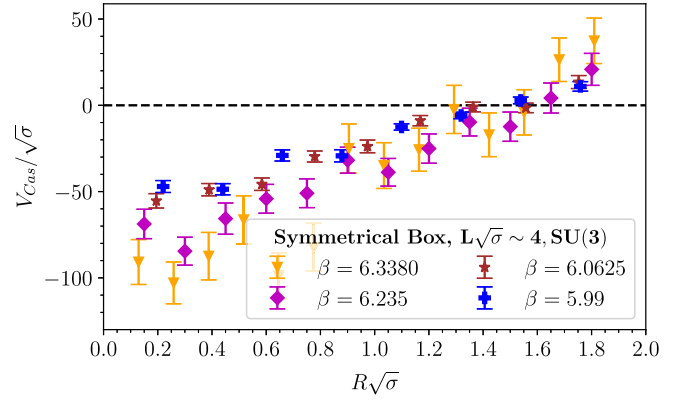
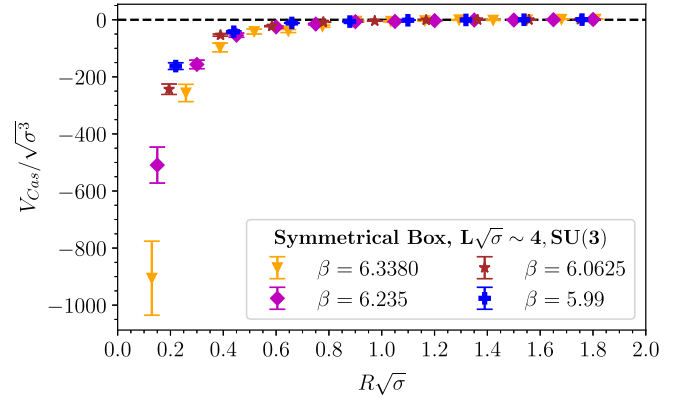


FIG. 13. Total energy contribution from the boundaries of a symmetrical box of different sizes.



(a) Total Casimir Potential



(b) Potential/Surface Area

FIG. 14. Total Casimir potential for a symmetrical box with side lengths $R\sqrt{\sigma}$ in SU(3).

The resulting absolute Casimir potential and the potential per unit surface area of a symmetrical box is shown in Fig. 14, where the area is given by, $A = 6R^2$. One key feature of this potential is that the Casimir force experienced by the symmetrical box is *attractive*. This result is consistent with the Casimir effect of a weakly coupled, massless noninteracting scalar field computed in Ref. [14]. As shown in Fig. 10(b), incorrectly accounting for the boundary energy contributions for the box would result in a repulsive Casimir effect for some couplings and an attractive potential for others, with a sign flip in between.

B. The Polyakov loop and deconfinement

The goal of this Section is to understand the temperature dependence of the Casimir potential. It is suggested in Ref. [23] that in the two-dimensional non-Abelian gauge theory, where the Casimir effect was studied for parallel wires in SU(2), the region between the wires is a Casimir-induced deconfined phase. As a result, the gluons in this finite region exhibit similar behavior to thermal glueballs in a heat bath at finite temperature. This is attributed to the absence of the periodicity of gluonic fields in the presence

of Casimir boundaries. The same behavior is observed for parallel plates in SU(3) [30] where these thermal glueballs are interpreted as colorless states of gluons bound to their negatively colored images on the chromoelectric boundary.

This suggestion is also supported by evidence of the measured mass of the relevant degrees of freedom in the Casimir interaction, M_{Cas} in Eq. (1). The Casimir mass was shown to assume a lower value than the lightest ground state glueball in SU(2) [23] and SU(3) [30], which should be the lowest mass in the pure gluonic system. The reduced glueball mass in the “heat-bath” Casimir cavity is inconsistent with finite temperature lattice measurements of thermal glueball properties, which show that the pole mass of the glueball ground state does not change below T_c , but increases with temperature above T_c [34]. This glueball mass dependence above T_c is opposite to the results of previous studies [35,36].

The discussion of the Casimir mass in the non-Abelian theory for parallel wires in (2 + 1)D and plates in (3 + 1)D for both SU(2) and SU(3) is expanded on in Ref. [31]. Similarly, a Casimir mass lower than the lightest glueball, M_{0++} , in the corresponding theories at temperatures corresponding to the confined phase is obtained. While we have studied the Casimir effect for the tube and box geometries, at the present stage, we have not performed Casimir mass calculations to test for any geometry dependence in the three-dimensional theory.

In Fig. (15), we show the temperature dependence of the measured Casimir effect of the parallel plate geometry as we move from the number of temporal lattice grid points N_τ associated with the confined to deconfined phase. Because the region between the plates is already a boundary induced deconfined phase where the relevant degrees of freedom have different properties compared to those measured for pure gluodynamics, decreasing N_τ (increasing the temperature) from the confined to deconfined phase does not alter the Casimir effect. The non-Abelian Casimir effect is a deconfined phase phenomenon due to the boundaries and is insensitive to temperature changes across T_c . However, it may be valuable in future studies to explore the limit $T \rightarrow 0$, which is resource intensive on the lattice.

In order to better understand gluodynamics along the temperature axis, we look at the finite temperature deconfinement order parameter described by the Polyakov loop (or *thermal Wilson line*) [37],

$$L_x = \frac{1}{N_c} \text{Tr} \prod_{\tau=0}^{N_\tau-1} U_{\mu=4}(\mathbf{x}, \tau), \quad (53)$$

which is a straight-line product of N_τ link variables in the Euclidean time direction, all situated at a single spatial point $\mathbf{x} := (x_1, x_2, x_3)$. The Polyakov loop is a trace over a closed loop (through periodic boundary conditions) and is therefore gauge invariant.

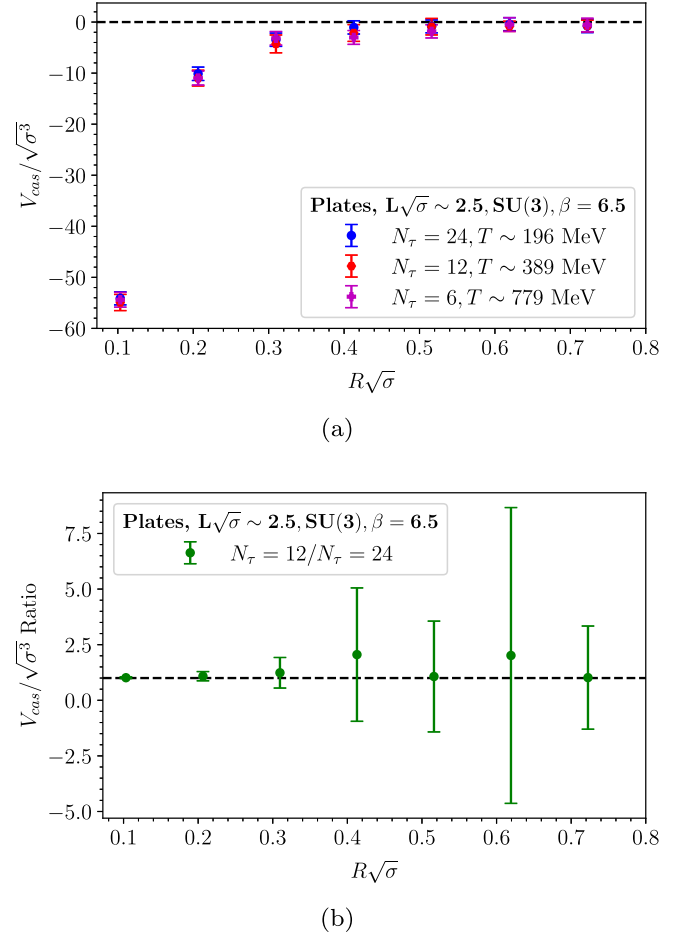


FIG. 15. Casimir potential per unit area between two parallel plates separated by a distance $R\sqrt{\sigma}$ in SU(3) at different temperatures.

The deconfinement order parameter is given by the spatial average

$$L \equiv \frac{1}{V_r} \left\langle \sum_{\mathbf{x} \in V_r} L_{\mathbf{x}} \right\rangle, \quad (54)$$

where $V_r = N_s^3$ is the physical volume. Given that we impose boundaries, note that we use the subscript r to specify between the regions in our Casimir effect geometries, where one region is inside (e.g., between the plates) and the other is outside. Thus we compute the Polyakov loop separately between these two regions.

The expectation value of a single Polyakov loop L_x in infinite volume is interpreted as the probability to observe a single static charge and is approximated by the exponential of the *free energy*, F_q [38],

$$\frac{1}{V} \left\langle \int d^3x L_x \right\rangle \sim e^{-F_q/T}, \quad (55)$$

of a single static charge.

We have shown in Fig. (15) that increasing the temperature from the confined to deconfined phase has no observable effect on the Casimir effect between parallel plates in SU(3). We now look at the corresponding Polyakov loop expectation values as N_τ is decreased from the confined to deconfined phase. In Fig. 16(a), we show the Polyakov loop expectation value in the confined phase as defined in Eq. (54). In the absence of the plates, $L \approx 0$ because the pure glueball system is in the confined phase and center symmetry is preserved.

In the region between the plates, the Polyakov loop expectation value is nonzero and increases with decreased separation distance between the plates, supporting the idea of a deconfined phase in this region. While there is a clear trend in the Polyakov loop expectation value, the error bars are large resulting in data points for $R > 1$ being consistent with zero, which would suggest the system remains confined at these distances. Improvement of the statistics is necessary to draw quantitative conclusions. Meanwhile the Polyakov loop, $L \sim 0$ in the large volume outside the plates where the gluonic system remains confined and increases only slightly with separation distance. As the separation distance between the plates, $R \rightarrow L\sqrt{\sigma}/2$ where the boundaries are mirrored, the expectation value of the Polyakov loop inside and outside the plates approaches the same value because the two regions become indistinguishable.

In Fig. (16), as the temperature is increased and the system moves into a deconfined phase, the vacuum (in the absence of the plates) Polyakov loop expectation value in the pure gluonic system becomes nonzero, thus signaling deconfinement. While in the deconfined phase, this vacuum expectation value continues to increase with temperature, however $L_{\text{vac}} < L_{\text{in}}$. In the confined phase, we observed that in the region outside the plates, $L \approx 0$ and equal to the vacuum expectation value. This is no longer the case in the deconfined phase, $L_{\text{vac}} < L_{\text{out}} < L_{\text{in}}$ and there is an observable linear increase in L_{out} with separation distance.

On the other hand, in the region between the plates, the magnitude of the deconfinement order parameter increases with temperature. However we have observed that this does not effect the Casimir effect. Given the observable change in the Polyakov loop with increasing temperature, it is surprising that the Casimir effect remains insensitive to the temperature change. Thermal glueballs studies [36] show that the glueball mass decreases with increasing temperature above T_c , hence the relevant degrees of freedom in the Casimir interaction should change with increased temperature. This temperature independence of the Casimir effect requires further investigation.

Lastly, we show the expectation value of the Polyakov loop in the confined phase for the symmetrical tube in Fig. (17), omitting the large error-bars for neatness. We emphasize that these results are not statistically significant but hint at an interesting trend. The Polyakov loop for both

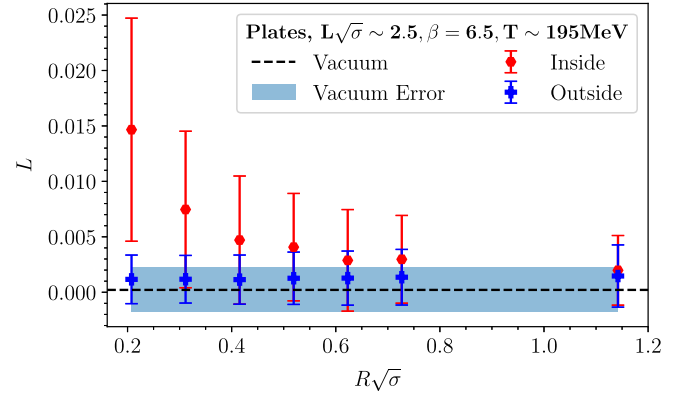
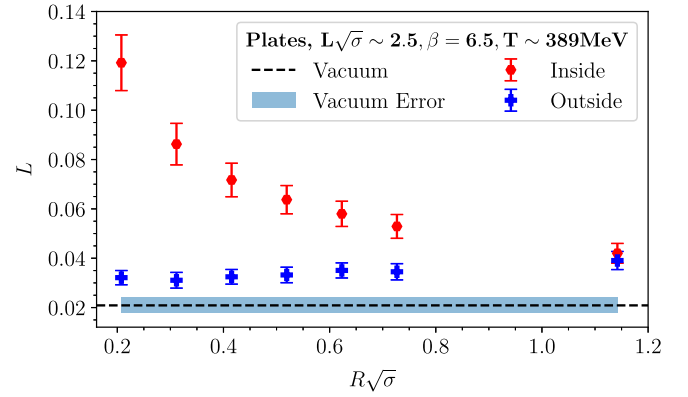
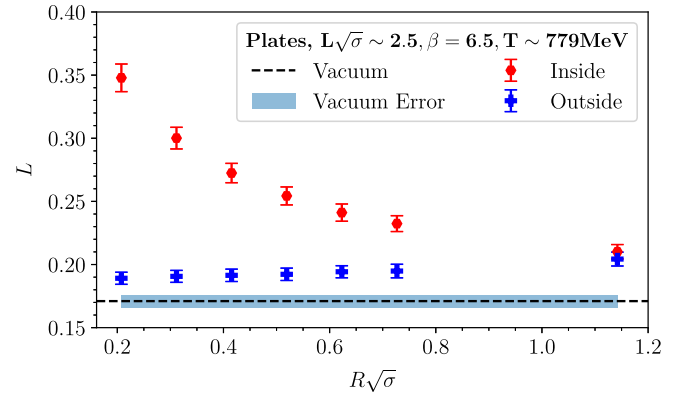
(a) $T \simeq 195$ MeV(b) $T \simeq 389$ MeV(c) $T \simeq 779$ MeV

FIG. 16. Polyakov loop expectation value in the region between and outside the parallel plates in SU(3) in the confined and deconfined phases.

the symmetrical and asymmetrical tube is discussed in Ref. [31]. In both geometries, the Polyakov loop fluctuates around the vacuum expectation value outside the tube. Inside the symmetrical tube, the Polyakov loop behaves similarly to the parallel plate geometry and approaches zero with increased tube size.

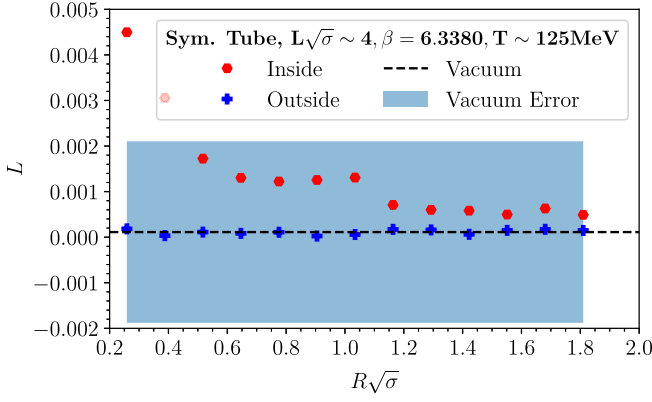


FIG. 17. Polyakov loop expectation value in the region inside and outside a symmetrical tube in SU(3) in the confined phase.

IV. CONCLUSIONS AND OUTLOOK

The Casimir effect has been successfully studied primarily in Abelian gauge theory using perturbative analytic methods. Complex geometries require simplifications such as fixed boundary conditions such as Dirichlet, Neumann etc. These simplifications in turn lead to inaccuracies in the predicted Casimir effect, for example thermal corrections [8–10]. These difficulties encountered in perturbative techniques provide a strong motivation for the exploration of the Casimir effect using nonperturbative methods such as lattice QED/QCD.

In order to better understand the implication of boundary conditions on the gauge fields, we have studied the effect of the chromoelectric boundary conditions on the individual field components in Sec. II. We find different degrees of change in the magnitude of the individual field components at the boundaries and the surrounding region.

We have presented Casimir potential results in $(3+1)$ D non-Abelian gauge theories for the geometries of parallel plates in SU(3) shown in Fig. (6), as well as SU(3) results for a symmetrical tube in Fig. (9) and a symmetrical box in Fig. (14). Our results show that the resulting Casimir potential is *attractive* in all these geometries.

A quantitative analysis of the Casimir potential between parallel plates in SU(3) non-Abelian gauge theory in $3+1$ D, based on the fitting function Eq. (27) shows preference for a vanishing value of the anomalous scaling dimension $\nu = 0$. Our value of the Casimir mass $M_{\text{Cas}}/\sqrt{\sigma} = 0.68 \pm 0.07(\text{stat}) \pm 0.17(\text{sys})$, is slightly smaller than previous estimates in the literature by Chernodub *et al.* $M_{\text{Cas}} = 1.0(1)\sqrt{\sigma}$, but differs only by 1.3σ . Note that neither of the values are continuum extrapolated.

Due to the changing area of the plates with separation distance in the tube and box geometry as opposed to the parallel plate geometry where the area was fixed, we show that a careful treatment of the energy contribution from the boundaries is necessary, else one could incorrectly measure a repulsive potential for the symmetrical tube, whereas, the

potential for the symmetrical box would appear to move from attractive to repulsive depending on the lattice coupling and box size. We have proposed two methods in which this energy of creating the boundaries could be accurately accounted in the geometry of a symmetrical tube employing linear fitting functions, whereas the energy contribution from the boundaries of a symmetrical box is nonlinear and unique in that it is nonmonotonous. The normalization is performed by subtracting the energy of creating the six faces that make up the box.

We then show that the Casimir potential is independent of a temperature increase from the confined to the deconfined phase. We provide results of the expectation value of the Polyakov loop for the plates and symmetrical tube. These results confirm that at temperatures below T_c , the Polyakov loop is zero outside the plates (consistent with a confined system) but it is nonzero in the region between the plates. This observation confirms that the region between each geometry is a boundary-induced deconfined phase [23], hence the measured Casimir mass of the relevant degrees of freedom is lower than the lightest glueball masses.

Our study has only been performed at microscopic separation distances between the boundaries, which allows for the interpretation of the matter inside the boundaries as a boundary induced deconfined phase. However, in the case of macroscopic separation distances between the boundaries, the matter in the region between the boundaries has no knowledge of the presence of the boundaries, as is the case for the region outside the boundaries even for microscopic distances considered in this study. Hence at low temperatures, a plausible scenario would be the Polyakov indicating confined matter in both the region between and outside the boundaries, while showing signs of deconfinement close to the boundaries (i.e., a deconfined “boundary layer”). This suggests a possible alternative to the interpretation of the region inside the boundaries and we highlight it as an idea for future investigations.

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DATA AVAILABILITY

The data containing the findings of this study are available in the Zenodo repository [39].

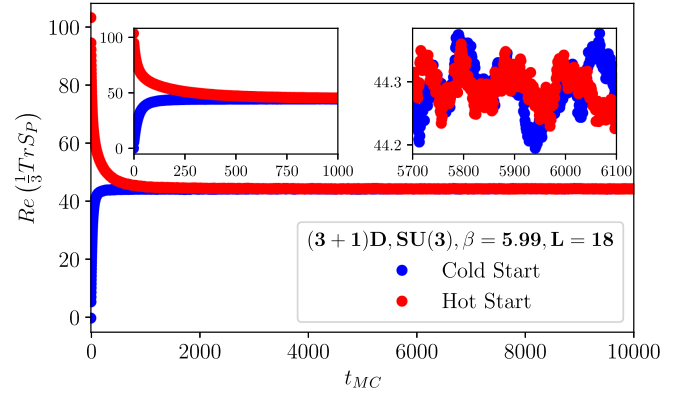
APPENDIX: THERMALIZATION

We have employed the Metropolis algorithm to generate the field configurations. One challenge with the Metropolis algorithm is picking a step size, which ultimately informs the convergence and efficiency of the algorithm. The consequence of a smaller step size is that the configuration space is explored much slower, resulting in large *thermalization times* depending on the starting configuration. In addition, a small step size leads to a high acceptance rate which results in long *autocorrelation times*, drastically increasing the statistical errors in the measurement. In order to overcome these large errors, one would need to increase the Monte Carlo sample size resulting in an inefficient algorithm. On the other hand, if the step size is too large, then the configuration space is explored more broadly, leading to a very small acceptance rate.

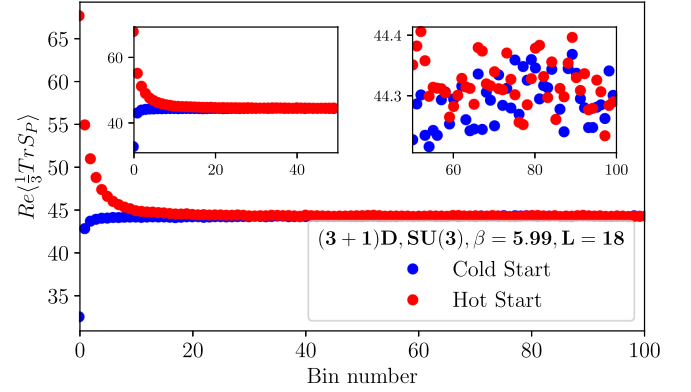
Irrespective of the step size that one picks for the Metropolis algorithm, it is expected that for large enough Monte Carlo time, the configurations converge (within uncertainty) on the stationary distribution. Although this is the case in scalar field theory, the statement does not in general hold for the gauge group $SU(N_c)$, possibly due to an asymmetry in the proposals leading to a modification of the acceptance probability. We refer the reader to Ref. [31] for an expansive discussion on the choice of step size in the Metropolis algorithm.

In Fig. 18, we show the thermalization steps of the vacuum normalized action defined in Eq. (13) in $(3+1)D$ $SU(3)$ for $\beta = 5.99$, $L = 18$. We generate proposals using a normal distribution for the generator algebra elements with $\sqrt{\sigma^2} = \Delta_{MC} \sim 0.8$. These are subjected to the standard Metropolis accept-reject step, starting from a cold or hot start. Figure 18(a) shows a comparison of these thermalization steps in Monte Carlo time. We observe a case where the cold start thermalizes faster than the hot start, as can be seen from the left inset figure. The right inset figure shows that over long Monte Carlo times, both starts converge to the same result.

For all our analysis, we test for thermalization as shown in Fig. 18(b) where the normalized action measurements along Monte Carlo time are binned with a bin size similar to the jackknife bin size. In this case, we have used a bin size of one hundred measurements. Then we compute the expectation value of the observable on each bin of these normalized action measurements and check where it plateaus. The data in bins at low Monte Carlo times with averages that deviate (because the system has not yet reached thermal equilibrium) from the average of thermalized measurements at large Monte Carlo times is then



(a) Vacuum Normalised Action



(b) Vacuum Normalised Action Averages

FIG. 18. Thermalization of the action for a cold and hot start in $(3+1)D$ $SU(3)$ for $\beta = 5.99$, $L = 18$.

discarded in this thermalization step. In the analysis, 5×10^3 thermalization steps are discarded.

We compute the *autocorrelation time* to quantify how strongly correlated any two measurements on the Markov chain are in order to measure the number Monte Carlo steps it takes to obtain statistically independent measurements. We use the autocorrelation function Γ_t and the normalized autocorrelation function $\rho_t = \Gamma_t/\Gamma_0$ [40],

$$\Gamma_t = \frac{1}{(N-t-1)} \sum_{i=1}^{N-t} [O_i - \langle O \rangle][O_{i+t} - \langle O \rangle], \quad (A1)$$

where t is the so called lag time (i.e., number of configurations between measurements along the Markov chain) allowing us to compare the running average to the first and last $(N-t)$ measurements and $\Gamma_0 = \sigma_O^2$. Note that in the absence of autocorrelations, the autocorrelation function vanishes as expected.

The normalized auto-correlation function exhibits exponential asymptotic behavior as the lag time becomes large and can be approximated by fitting a linear combination of exponentials [38],

$$\rho_t = \frac{\Gamma_t}{\Gamma_0} \sim \sum_{j=1}^n b_j e^{-t/\tau_j}, \quad (\text{A2})$$

where n is the number of exponential terms, b_j are coefficients determined from the fit and normalized to unity, and τ_j is also a fit parameter, where the asymptotically leading term, τ_n is called the *exponential autocorrelation time*. One then uses the value of τ_n to discard correlated measurements. An alternative to this approach which we employ in our analysis and which avoids dealing with the intricacies of fitting functions, is to instead compute the *integrated autocorrelation time*, τ_{int} defined as,

$$\tau_{\text{int}} = \frac{1}{2} + \sum_{t=1}^M \rho_t, \quad \text{with } M \geq 4\tau_{\text{int}} + 1. \quad (\text{A3})$$

See Ref. [41] for a detailed derivation of this result. The sum in Eq. (A3) should be taken along the entire length of the Markov chain, however, this becomes computationally costly and ρ_t becomes unreliable (e.g., negative values and some noise). Thus finding the right length to truncate can be nontrivial, and various techniques have been proposed. We have adopted the approach in Ref. [40], where one truncates at some chain length, M satisfying the above condition.

The corresponding error for the integrated autocorrelation time is given by [40],

$$\delta\tau_{\text{int}} = \tau_{\text{int}} \sqrt{\frac{4M+2}{N}}, \quad (\text{A4})$$

where M is our autocorrelation length. The statistical error in the observable, O accounting for the autocorrelations in the data is then given by,

$$\delta O = \frac{\sigma_O}{\sqrt{N}} \sqrt{2\tau_{\text{int}}}. \quad (\text{A5})$$

This is one way of accounting for the autocorrelations in the error analysis in order to ensure that the statistical errors in the measurements are not underestimated. In this work, we explicitly compute the autocorrelation time of our measurements and use this analysis to discard the correlated

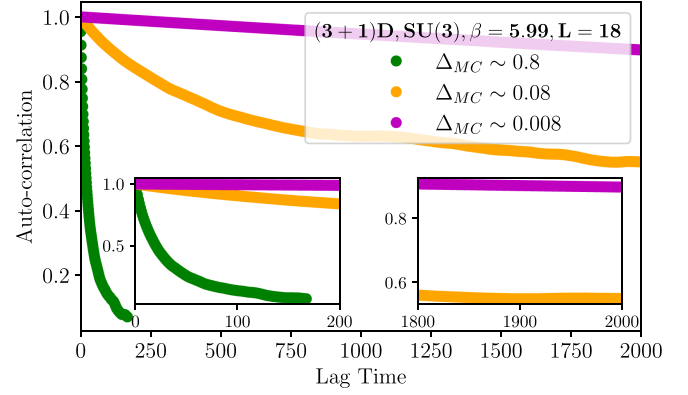


FIG. 19. Autocorrelations on the action measurements in $(3+1)\text{D}$ $\text{SU}(3)$ for $\beta = 5.99$, $L = 18$ using different step sizes sampling a normal distribution.

measurements, M . Then we apply the jackknife error technique. See Refs. [41–43] for more detailed discussions on the jackknife error method.

Lastly, we investigate the step-size (Δ_{MC}) dependence of the Metropolis algorithm and the effect that it has on the autocorrelations in the measurements. The reduction in the step size results in a higher acceptance rate as we sample around the same region in the phase space and the proposed state is very close to the current state. This, in turn leads to very strong autocorrelations in the measurements. We show, in Fig. 19, how rapidly the autocorrelation time of measured action grows with a reduction in step size. In our subsequent analysis, we have computed the autocorrelation length according to Eq. (A3).

In summary, we use the Wilson action with the Metropolis algorithm to sample the field configurations. We sample a normal distribution with standard deviation $\epsilon = 0.5$ to generate the candidate update matrices with a minimum of 5×10^3 thermalization steps and discard a minimum of 50 configurations between measurements. The autocorrelation time is computed for each dataset and the autocorrelation lengths are computed carefully according to Eq. (A3). We discard $n > M$ measurements such that we only use statistically independent measurements. Where possible, for example in the case of $(2+1)\text{D}$, more than 10^4 configurations have been used.

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