

Bosonization, vertex operators, and maximal violation of the Bell-Clauser-Horne-Shimony-Holt inequality in wedge regions

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It is pointed out that the vertex operators of a chiral boson in $1 + 1$ dimensions provide an explicit realization of dichotomic, bounded, Hermitian operators that saturate the Tsirelson bound of the Bell-Clauser-Horne-Shimony-Holt inequality in the vacuum state.

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Introduction. Since the seminal work of Summers and Werner [1–3], the study of the Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality [4,5] in relativistic quantum field theory has attracted increasing interest. For recent overviews, see [6,7]. The modern program combines the Haag-Kastler framework, the Bisognano-Wichmann theorem, and Tomita-Takesaki modular theory [8–13]. In this setting, explicit Bell-CHSH constructions are already known for free fields, including scalar realizations in terms of bounded unitary or Hermitian operators [14,15] and maximal-violation constructions for massless spinors [16].

The Summers-Werner theorem states that the Bell-CHSH correlator in wedge regions can be made arbitrarily close to the Tsirelson bound in the vacuum of a free quantum field theory [1–3]. Thus, for suitable wedge-supported test functions, one can construct a set $\{A(f), A(f'), A(g), A(g')\}$ of dichotomic,¹ bounded, Hermitian operators such that the vacuum correlator²

$$\langle C \rangle = \langle 0 | (A(f) + A(f'))A(g) + (A(f) - A(f'))A(g') | 0 \rangle \quad (1)$$

can be made arbitrarily close to Tsirelson's bound [17], namely,

$$\langle C \rangle \approx 2\sqrt{2}. \quad (2)$$

Despite the generality of the theorem, the Fermi and Bose fields behave differently when one seeks explicit Bell operators. On the fermionic side, the canonical anticommutation relations allow one to write down the relevant dichotomic, bounded, Hermitian operators directly. For example, in the case of a massless Majorana spinor [16],

$$\psi(f) = \int_{-\infty}^{\infty} dx_+ f_R(x_+) \psi_R(x_+) + \int_{-\infty}^{\infty} dx_- f_L(x_-) \psi_L(x_-), \quad (3)$$

where $\psi_R(x_+) = \psi_R(t+x)$ and $\psi_L(x_-) = \psi_L(t-x)$ stand, respectively, for the real-valued right- and left-moving components, the operator

$$A_M(f) = \frac{\psi(f)}{\|\psi\|} \quad (4)$$

is bounded, Hermitian, and dichotomic for any two-component real-valued smooth test function, $f = (f_R(x_+), f_L(x_-))$. By suitably choosing a set $\{f, f', g, g'\}$ of four test functions, which play a role akin to the four Bell's angles of quantum mechanics, one can define a Bell-CHSH correlator that saturates Tsirelson's bound [1–3]. In the present

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¹A smeared operator $A(f)$ is said to be dichotomic when it satisfies $A^2(f) = 1$.

²The quantities (f, f') and (g, g') denote pairs of smooth spacelike test functions needed to properly define $\{A\}$ as operators acting on the one-particle Hilbert space of the theory.

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example, the inner product between two test functions is

$$\langle f|g \rangle = \int_0^\infty dk f_R(k)^* g_R(k) + \int_{-\infty}^0 dk f_L(k)^* g_L(k), \quad (5)$$

where $f_R(k)$ and $f_L(k)$ are, respectively, the Fourier transformations of $f_R(x_+)$ and $f_L(x_-)$,

$$\begin{aligned} f_R(k) &= \int dx_+ e^{ikx_+} f_R(x_+), \\ f_L(k) &= \int dx_- e^{ikx_-} f_L(x_-). \end{aligned} \quad (6)$$

It turns out that, for the operator (4),

$$\langle 0|A_M(f)A_M(g)|0 \rangle = -i \frac{\langle f|g \rangle}{\|f\| \cdot \|g\|}, \quad (7)$$

so that, for the Fermi field case, the Bell-CHSH correlator reads

$$\langle C \rangle = -i \left(\frac{\langle f|g \rangle}{\|f\| \cdot \|g\|} + \frac{\langle f'|g \rangle}{\|f'\| \cdot \|g\|} + \frac{\langle f|g' \rangle}{\|f\| \cdot \|g'\|} - \frac{\langle f'|g' \rangle}{\|f'\| \cdot \|g'\|} \right). \quad (8)$$

At this stage, one can follow two equivalent routes to suitably choose the test functions $\{f, f', g, g'\}$. In the first, one employs the original Summers-Werner construction based on the Bisognano-Wichmann theorem and Tomita-Takesaki modular theory [1–3]. In the second, developed recently in [18], one relates the inner products in (8) to the spectral properties of Carleman and Hankel operators. In both cases, maximal violation is achieved. For Bose fields, explicit examples of Bell operators are already known by making use of the Weyl unitaries [14,15], although not leading to maximal violation. The narrower question addressed here is whether one can realize the same Bell maximal violating mechanism exhibited by Fermi fields directly within the framework of the bosonization of chiral fields. The aim of the present Letter is to show that the answer is affirmative: by relying on the bosonization of $(1+1)$ -dimensional models [19,20], we obtain a vertex-operator realization of the fermionic Bell construction. More precisely, for the bosonization value $\alpha^2 = 4\pi$, Hermitian smeared combinations of the chiral vertex operators reproduce the two-point function and normalization entering the Majorana Bell operators. There is no need to rederive the modular choice of wedge-supported test functions; once the bosonized operator family is identified, the near-Tsirelson result follows from the same Summers-Werner input as in the fermionic case [1–3].

Vertex operators and the saturation of the tsirelson bound. One starts by considering a free massless scalar field in $(1+1)$ -dimensional Minkowski spacetime. As one learns

from the bosonization setup [19,20], the scalar field splits into right and left movers $\varphi_R(x_+)$ and $\varphi_L(x_-)$, obeying

$$\partial_- \varphi_R(x_+) = 0, \quad \partial_+ \varphi_L(x_-) = 0. \quad (9)$$

Our strategy is simple. First, at the bosonization value $\alpha^2 = 4\pi$, introduce conjugate vertex operators satisfying a fermionic exchange relation. Second, construct Hermitian smeared combinations of these operators reproducing the two-point function entering the Majorana Bell construction. Finally, one uses the same wedge-supported test functions as in the fermionic case, so that the localization in causal complementary wedge algebras and the modular optimization of the CHSH correlator are inherited directly from the Summers-Werner framework [1–3]. The relevant operators of the theory as, for instance, the chiral currents (J_+, J_-) are obtained through the right and left movers. A particular class of operators, called vertex operators, is obtained by exponentiating the left and right movers and taking the normal ordering, namely,

$$V_R^\alpha(x_+) =: e^{i\alpha\varphi_R(x_+)}:, \quad (10)$$

where α is a free parameter. An analogous expression holds for the left vertex operator $V_L^\alpha(x_-)$. A useful feature of the vertex operators is the exchange relation

$$V_R^\alpha(x_+) V_R^\beta(x'_+) = e^{\frac{i\alpha\beta}{4} \text{sign}(x_+ - x'_+)} V_R^\beta(x'_+) V_R^\alpha(x_+). \quad (11)$$

For conjugate charges $\beta = -\alpha$ and $\alpha^2 = 4\pi$, the phase factor equals -1 for $x_+ \neq x'_+$, so the pair V_R^α and $V_R^{-\alpha}$ behaves as a pair of fermions. Needless to say, the vertex operators (10) are a fundamental tool in order to map Fermi theories into Bose ones and vice versa. Furthermore,

$$\langle 0|V_R^\alpha(x_+) V_R^{-\alpha}(x'_+)|0 \rangle = \frac{1}{x_+ - x'_+ - i\varepsilon}, \quad \alpha^2 = 4\pi, \quad (12)$$

which coincides with the two-point function of a chiral massless Fermi field. Strictly speaking, the fermionic exchange relation is obtained in the standard bosonization convention in which the chiral vertex operators are completed by the appropriate cocycles, or Klein factors [1–3]. We shall use this convention below and absorb these factors into the notation. In particular, at $\alpha^2 = 4\pi$, the right- and left-moving vertex operators generate the same canonical anticommutation relation (CAR) algebra as the corresponding chiral Majorana fields. This point is important because the Bell-CHSH observables below are used as bounded self-adjoint CAR fields, not merely as formal normal-ordered exponentials. To approach the Bell-CHSH inequality, we introduce the Hermitian, smeared, vertex operators

$$Q_R^\alpha(f) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} dx_+ f(x_+) \left(V_R^\alpha(x_+) + V_R^\alpha(x_+)^{\dagger} \right), \quad (13)$$

where f is a real-valued smooth test function. It follows that

$$\langle 0 | Q_R^\alpha(f) Q_R^\alpha(g) | 0 \rangle = \frac{1}{2} \int dx_+ dx'_+ f(x_+) g(x'_+) \frac{1}{x_+ - x'_+ - i\epsilon} = -i \int_0^\infty dk f(k)^* g(k), \quad \alpha^2 = 4\pi. \quad (14)$$

Thus, at the level relevant for the Bell construction, $Q_R^\alpha(f)$ reproduces the chiral Majorana two-point function. Adding now the left component, one obtains a full Hermitian operator whose vacuum two-point function matches the massless Hermitian Majorana operator (3), namely,

$$\hat{W}^\alpha(f) = \frac{1}{\sqrt{2}} \int_{-\infty}^\infty dx_+ f_R(x_+) \left(V_R^\alpha(x_+) + V_R^\alpha(x_+)^{\dagger} \right) + \frac{1}{\sqrt{2}} \int_{-\infty}^\infty dx_- f_L(x_-) \left(V_L^\alpha(x_-) + V_L^\alpha(x_-)^{\dagger} \right), \quad (15)$$

with

$$\langle 0 | \hat{W}^\alpha(f) \hat{W}^\alpha(g) | 0 \rangle = -i \langle f | g \rangle = -i \left(\int_0^\infty dk f_R(k)^* g_R(k) + \int_{-\infty}^0 dk f_L(k)^* g_L(k) \right), \quad (16)$$

which is exactly the inner product of expression (5). In this sense, the bosonization dictionary identifies $\hat{W}^\alpha(f)$ with the bosonic realization of the Majorana Bell operator. It follows that the Bell-CHSH correlator built from vertex operators, namely,

$$W^\alpha(f) = \frac{\hat{W}^\alpha(f)}{\|f\|}, \quad (17)$$

$$\begin{aligned} \langle C \rangle_{\text{vertex}} &= \langle 0 | (W^\alpha(f) + W^\alpha(f')) W^\alpha(g) + (W^\alpha(f) - W^\alpha(f')) W^\alpha(g') | 0 \rangle \\ &= -i \left(\frac{\langle f | g \rangle}{\|f\| \cdot \|g\|} + \frac{\langle f' | g \rangle}{\|f'\| \cdot \|g\|} + \frac{\langle f | g' \rangle}{\|f\| \cdot \|g'\|} - \frac{\langle f' | g' \rangle}{\|f'\| \cdot \|g'\|} \right), \end{aligned} \quad (18)$$

can therefore be made arbitrarily close to Tsirelson's bound,

$$\langle C \rangle_{\text{vertex}} \approx 2\sqrt{2}. \quad (19)$$

Equation (19) deserves a few additional remarks. The first one concerns the eligibility of expression (17) as a Bell operator. At the bosonization point $\alpha^2 = 4\pi$, the vertex operators realize the same CAR algebra as the corresponding Majorana fields. Hence, for real test functions, the smeared operator $\hat{W}^\alpha(f)$ is self-adjoint and satisfies

$$\left(\hat{W}^\alpha(f) \right)^2 = \|f\|^2 \mathbf{1}.$$

Therefore, the normalized operator

$$W^\alpha(f) = \frac{\hat{W}^\alpha(f)}{\|f\|}$$

is bounded, Hermitian, and dichotomic,

$$\left(W^\alpha(f) \right)^2 = \mathbf{1}.$$

This is the property required for the Bell-CHSH observables. The second observation relies on the use of the powerful

setup of the modular theory [1–3] in order to choose the test functions (f, f', g, g') . More precisely, according to [1–3], one sets

$$\begin{aligned} f &= (1 + s)\Phi, & f' &= (1 + s)i\Phi, \\ \tilde{g} &= i(1 + s^\dagger)\Phi, & \tilde{g}' &= -i(1 + s^\dagger)i\Phi, \end{aligned} \quad (20)$$

with

$$g = \frac{\tilde{g} - \tilde{g}'}{\sqrt{2}}, \quad g' = \frac{\tilde{g} + \tilde{g}'}{\sqrt{2}}, \quad (21)$$

where s stands for the Tomita-Takesaki operator, namely,

$$s = j\delta^{1/2}, \quad s^2 = 1. \quad (22)$$

The operator δ is the so-called modular operator, while j is the modular conjugation. [1–3]. In particular, in the case of wedge regions, the Bisognano-Wichmann results [9] state that

$$\delta = e^{-2\pi K}, \quad j = (CPT) - \text{operator}, \quad (23)$$

where $K = -i \frac{d}{d\theta}$ is the generator of the Lorentz boosts in the rapidity variable θ . As a consequence of the Bisognano-

Wichmann results, the spectrum of the operator δ is continuous, coinciding with the positive real line $\mathbb{R}_+$. Moreover, from [1–3], one learns that the quantity Φ in Eq. (20) stands for a normalized eigenstate of δ belonging to the spectral interval $[\lambda^2 - \epsilon, \lambda^2 + \epsilon]$, with $\lambda^2 \in [0, 1]$ being the spectral parameter of δ . All these properties ensure that the inner products of Eq. (18) fulfill the following conditions:

$$\begin{aligned} \frac{\langle f|g \rangle}{\|f\| \cdot \|g\|} &= \frac{\langle f'|g \rangle}{\|f'\| \cdot \|g\|} = \frac{\langle f|g' \rangle}{\|f\| \cdot \|g'\|} = -\frac{\langle f'|g' \rangle}{\|f'\| \cdot \|g'\|} \\ &= i\sqrt{2} \frac{\lambda}{1 + \lambda^2}, \end{aligned} \quad (24)$$

thus,

$$\langle C \rangle_{\text{vertex}} = 2\sqrt{2} \frac{2\lambda}{1 + \lambda^2} \xrightarrow{\lambda \rightarrow 1} 2\sqrt{2}. \quad (25)$$

It is worth mentioning here that, as pointed out in [1–3], the value $\lambda = 1$ has a deep meaning: it is the fixed point of the unitary operator $\delta^{it} = e^{it \log(\delta)}$ corresponding to the modular flow. Also, this value is directly linked to the fact that the von Neumann algebras of the local observables in wedge regions is of the type $III_{\lambda=1}$, a key feature of quantum field theories [1–3]. Let us close by emphasizing the scope and significance of the present construction. The result is not a new proof of the Summers-Werner theorem; rather, it gives an explicit vertex-operator realization of the same maximal Bell-CHSH mechanism in a bosonic chiral theory. The bosonic field

enters through its vertex operators at the bosonization point $\alpha^2 = 4\pi$, where the corresponding smeared fields realize the CAR normalization needed for dichotomic Bell observables. This provides a concrete bridge between the modular-theoretic construction of Bell correlations in wedge regions and the operator language of two-dimensional bosonization. It would be interesting to extend this analysis to compact-boson and conformal-field-theory settings and to understand which parts of the construction survive away from the free bosonization point or in thermal/Kubo-Martin-Schwinger states. Finally, we point out that the vertex operators exhibit a large variety of applications, from both theoretical as well as experimental sides. One might quote the fractional quantum Hall edge states, the Luttinger liquids in quantum wires, the anyonic interferometry, the cold atoms in one dimension, etc.; see, for instance, Refs. [21,22] for a recent account. As such, the present contributions could open interesting perspectives to devise Bell tests in such a systems.

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Data availability. The data that support the findings of this article are openly available [23].

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