

Casimir pressure with dissipation and quantum corrections from axion dark matter

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We study the effects of the oscillating axion field present in our environment on the Casimir pressure between two metallic plates. We take into account the finite conductivity of the boundary plates and model the interactions between matter and photons in the Schwinger-Keldysh formalism. This allows us to take into account dissipation in the quantum field description of this open quantum system and retrieve the Lifschitz results for the Casimir interaction between two metallic plates. We then compute the leading correction to the Lifschitz theory in inverse powers of the axion suppression scale and show that the Casimir pressure receives oscillating corrections depending on the product of the axion mass and the distance between the plates. This contribution is repulsive at large distance compared to the axion Compton wavelength as a consequence of the breaking of parity invariance by the axion dark matter background.

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I. INTRODUCTION

We consider the effects of an axionlike oscillating pseudoscalar field coupled to photons on the Casimir pressure [1–13]. The axion field is taken to be oscillating in our environment as expected from a field whose fast oscillatory behavior reproduces the effects of dark matter on large scales and inside the Milky Way. The quantum modes of the photon are affected by the coupling to the axion. This leads to corrections to the Casimir pressure between the plates.

In the ideal case of infinitely reflexive metallic plates, the electromagnetic field in the cavity shows resonances which are perfect and imply an infinite series of Dirac peaks for the induced electromagnetic field. This behavior is regularized physically by the finite permittivity of the metal, which in the simple Drude approximation depends on the plasma frequency ω_{pl} and the conductivity σ [14].¹ In this case the number of resonances becomes finite and their frequency values are all bounded by the plasma frequency ω_{pl} . The nonzero value of the resistivity moves the pole of the Green's functions away from the real axis. As a result,

the modes of the photon field between the boundary plates penetrate inside them over a finite length. The resonances are widened by the dissipation induced by the metallic plates. These effects are thoroughly discussed in [14] and constitute the classical treatment of this effect. In the present paper we present the quantum description of the Casimir effect in the presence of dissipation and axion dark matter.

The description of the quantum fluctuations of the photons when coupled to a dissipative metal goes beyond a usual quantum field theory treatment.² Indeed dissipation breaks unitarity and the Hamiltonian of the system becomes non-Hermitian.³ The core of this work corresponds to modifying the usual QFT methods to compute quantum averages in this dissipative case and then apply these results in the case of axions.⁴

At the classical level, dissipation can be treated in a Lagrangian framework by doubling the numbers of degrees of freedom, i.e., by introducing a “shadow” field corresponding to the time-reversed evolution. The shadow field

¹For a recent description of the issues of the Drude model when comparing with data, see [15].

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²By this we mean the *in-out* formalism of scattering amplitudes or the *in-in* description of cosmological expectation values for instance. In both cases unitarity plays a crucial role and is lost here. See Appendix B for details.

³Unitarity can be reintroduced by considering the complete system involving the photons and all the matter degrees of freedom. This approach to the Casimir effect with dissipation can be found in [16].

⁴For a recent description of electromagnetism in matter in the Schwinger-Keldysh formalism where gauge invariance is considered, see [17].

brings energy from late times backward so that the system comprised of the real field and its shadow becomes a conservative system [18]. In this paper we use this formal method, at the expense of a doubling of number of the degrees of freedom. In the end the shadow field is disregarded and the original field evolves in a dissipative system with increasing times.

The goal of this work is to use a path integral formulation to calculate vacuum expectation values of the energy momentum tensor of axion electrodynamics. This eventually leads to the computation of the Casimir pressure between two metallic plates. The usual way of doing this would be via the Gellman-Low formula in the *in-out* formalism.⁵ Here this connection is lost because in the zero-temperature treatment we pursue, the evolution operator is not unitary. The path integral formulation only stands for the vacuum expectation of operators $\mathcal{O}(t)$ evaluated at a single time $\langle 0|\mathcal{O}(t)|0\rangle$ where the vacuum is taken to be the one of the free theory when no dissipation is taken into account. In this case, a Schwinger-Keldysh representation of these expectation values is obtained. The action of each field is integrated along the Schwinger-Keldysh path going from $t = -\infty$ to $t = +\infty$ and back. One can introduce the Schwinger-Keldysh representation of the action as involving two copies of each field, one on the lower branch of the Schwinger-Keldysh path from $t = -\infty$ to $t = +\infty$ and another one on the upper branch going from $t = +\infty$ to $t = -\infty$. This back-and-forth path is unrelated to the existence of the shadow field and shall be used for both, leading to another doubling of number of the fields.⁶

In addition, in the Coulomb gauge, the two electromagnetic polarizations are described by two scalar fields. All in all, the two original polarizations of the photons are then described by eight fields, two of which are physical: the two polarizations going forward in time.

Using this formulation, we retrieve the usual Casimir pressure in the absence of dissipation and its Lifschitz generalization when dissipation in the plates is taken into account [20,21]. The Schwinger-Keldysh field theoretic treatment allows one to extend our results to the presence of an axion coupling to photons. The Casimir pressure is now simply corrected as the two-point correlation function of the two photon polarizations is modified by the axion interaction. This can be calculated using the usual rules of perturbative quantum field theory in the Schwinger-Keldysh formulation. The pressure due to the coupling to the axion field vanishes with the mass of the axion and grows with the distance until it reaches the Compton length of the axion

field. For larger distances, the pressure oscillates eventually and changes sign when $md = \mathcal{O}(1)$. This is similar to the results in [22] where the breaking of parity invariance, e.g. by the background value of the axion field, allows for a change of the sign of the Casimir effect. Here the amplitude of the oscillating part due to the axion is not large enough to induce a change of sign and the breaking of the Kenneth-Klich theorem on the attractiveness of the Casimir interaction [23]. It simply reduces the magnitude of the Casimir interaction on large distances $d \gg m^{-1}$.

The Casimir system with two metallic plates where dissipation and dispersion take place is a testbed for the dissipative open quantum system. Indeed, as soon as the plates are not perfect, the reflection coefficients of the walls differ from unity and therefore energy flows from the vacuum cavity to the plates implying that the cavity is not a closed quantum system anymore. There is an exchange of energy with the plates which is reinforced in the dissipative case where currents inside the plaques can be excited and the Joule effect prevents the existence of thermal equilibrium if the transfer of heat into radiation is not taken into account.

Difficulties in the description of the Casimir effect in the presence of dissipation from first principle have already been noticed [24,25]. In particular the analytic continuation from the sum over the resonant frequencies in the cavity to the continuous integral used in the Lifschitz theory can be problematic. In this paper, we use the *in-in* formalism for which the definition of the Feynman propagator with its T -product decomposition is crucial. As we will see, the retarded propagator is naturally defined with its resonances below the real axis in the complex plane of pulsations. The Feynman propagator requires one to shift the resonances from below the negative real axis to above the negative real axis. This can be achieved using a modification of the Schwarz reflection principle. In the limit of weak dissipation $\gamma \rightarrow 0$, where γ is a damping time related to the conductivity and the plasma frequency via $\sigma\gamma = \omega_p^2$, the Feynman propagator is well defined apart from the usual poles at the resonances and the branch cuts for the continuum modes with a pulsation above the plasma frequency. For a plasma with $\gamma = 0$, one can analytically continue the integral along the real axis defining the propagator to the imaginary axis, i.e. a Wick's rotation. For finite values of the conductivity when $\gamma \neq 0$, the definition of the Feynman propagator is obstructed by a branch cut along the imaginary axis in the complex plane of pulsations. Using contour integrals, this induces a new contribution to the Feynman propagator from the finite discontinuity across the imaginary axis. The analytic continuation to the imaginary axis of observables such as the Casimir pressure can still be performed and is well defined, i.e. one can Wick rotate and retrieve the Lifschitz theory [20,21]. Then one can perform a standard perturbative calculation to take into account the effects of the axion coupling to the photons.

⁵For the link between the *in-in* and the *in-out* formalisms in the case of a closed and nondissipative quantum system, see [19] for instance.

⁶A simplified formalism where the shadow fields and the Keldysh copies are identified may be obtained using the classical action presented in Appendix A. We do not pursue this approach here and leave it to future work.

The effects that axions can have on the Casimir pressure have been already discussed in the literature. First of all, the coupling to matter of axions leads to quantum effects, such as Casimir-Polder forces, induced by the presence of light axions [26]. We focus here on a different aspect, when the coupling of axions to photons leads to a modification of the mode eigenfrequencies and a direct effect on the Casimir pressure. This has been discussed in the context of linear time or spatial variations of the axion field [27–30]. Some results concerning Green's functions and energy densities are discussed in the presence of an oscillating axion are presented in [13]. In this paper, we consider the regime where the axion time variation is very rapid and cannot be linearized. We average over the fast axion variations and deduce the effects of the axion on very large time scales compared to the inverse axion mass. We find that the correction to the Casimir pressure oscillates in space with a period proportional to the Compton wavelength $1/m$. For large values of md , the pressure becomes repulsive and vanishes as $m^2 d^2$ at short distance [27], d being the size of the cavity.

The paper is arranged as follows. In Sec. II we describe the coupling between photons and the metallic plates, and the existence of narrow resonances. Then in Sec. III, we consider the path integral associated to correlators in the *in-in* formalism. In Sec. IV, we recover the classic Lifschitz results without axions when the plates have a dissipative behavior. Finally in Sec. V we calculate the leading correction to the Casimir pressure coming from the axion contribution to the photon vacuum energy in an idealized experimental setup. A few technical appendixes complete the paper.

II. PHOTONS IN MATTER COUPLED TO AXIONS

A. The action

As we are interested in the axion in the presence of metallic plates, we will calculate the quantum Casimir effect using the vacuum Lagrangian and its modified version taking into account the interaction of photons with matter. In vacuum we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\phi}{4M}F^{\mu\nu}\tilde{F}_{\mu\nu} \quad (1)$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is the dual field strength and M is an energy scale of the order of the Peccei-Quinn symmetry breaking scale. We define $\epsilon^{0123} = 1$. The pseudoscalar field ϕ will be assumed to be time dependent, as representing the oscillations of the dark matter field in our environment. Notice that when the axion field becomes constant, the coupling between the axion and photons becomes irrelevant as $F\tilde{F}$ is a derivative term and we do not consider nontrivial topological effects. We work in the Coulomb gauge where $A^0 = 0$ and $\partial_i A^i = 0$ which selects two

transverse polarizations. We are interested in situations where there is a background magnetic field parallel to the plates [14]. In terms of the fluctuation a^μ of the potential vector where $A^\mu = \bar{A}^\mu + a^\mu$ and $\bar{A}^\mu$ corresponds to a constant magnetic field parallel to the boundary plates, we have for the action governing the dynamics of the fluctuating field [14]

$$S = \int d^4x \left(\frac{1}{2}(\vec{e}^2 - \vec{b}^2) + \frac{\phi}{M}\vec{e} \cdot \vec{b} - \vec{\mathcal{J}} \cdot \vec{a} \right) \quad (2)$$

where they have introduced the current

$$\mathcal{J}^\mu = \frac{1}{2M}\partial_\nu \phi \epsilon^{\mu\nu\rho\sigma} \bar{F}_{\rho\sigma}. \quad (3)$$

In a time-dependent axionic background this corresponds to the current induced by the axion and the external background field

$$\vec{\mathcal{J}} = -\frac{\dot{\phi}}{M}\vec{B}. \quad (4)$$

This action reproduces the equations of motion for the photon field in the presence of the axion field.

When coupled to matter, in particular a metal where dissipation takes place, we would like to find an action that reproduces the phenomenological field equations of motion

$$\partial_0 \vec{a} - \vec{\nabla} \wedge \vec{b} = -\frac{\dot{\phi}}{M}\vec{b} + \vec{\mathcal{J}}, \quad (5)$$

which corresponds to

$$-\partial_0(\epsilon \star_t \partial_0 \vec{a}) + \Delta \vec{a} = -\frac{\dot{\phi}}{M}\vec{b} + \vec{\mathcal{J}}, \quad (6)$$

where we work in the Coulomb gauge with $a^0 = 0$ and $\vec{e} = -\partial_0 \vec{a}$. Here the displacement field is represented by the convolution

$$\vec{d} = \vec{e} + \vec{p} = \epsilon \star_t \vec{e} \quad (7)$$

where $\epsilon(t)$ is the permittivity of the medium, $\vec{p}$ its polarization and $\star_t$ represents the convolution in the time domain. This phenomenological equation is postulated and leads to the wave equation in the medium

$$(\Delta + \epsilon(\omega)\omega^2)\vec{a} = 0 \quad (8)$$

in the absence of axion source. The propagation speed is now $1/\sqrt{\epsilon(\omega)}$. In real space, this change of speed becomes the convolution in time with the permittivity function. The source term is not modified as this depends only on the external magnetic field and the axion background.

One way of finding the appropriate action governing the interaction of photons with matter and the axion field would be to go back to a microscopic model where atoms interact with the local electric and magnetic fields [25]. Here we will use a more effective description where we introduce a second gauge field $\tilde{a}$. This doubling of degrees of freedom can be seen as equivalent to introducing the time-reversed process in order to restore the Lagrangian structure of the theory despite dissipation [18]. In this approach we introduce the new action

$$S = \int d^4x \left(\vec{e} \cdot \vec{a} - \vec{b} \cdot \vec{b} + \frac{\phi}{M} \vec{e} \cdot \vec{b} + \frac{\tilde{\phi}}{M} \vec{e} \cdot \vec{b} - \vec{J} \cdot \vec{a} - \vec{J} \cdot \vec{a} \right), \quad (9)$$

where we have coupled the axion to the “shadow” gauge field $\tilde{a}$. We have introduced the shadow axion field and source

$$\tilde{\phi}(t) \equiv \phi(-t), \quad \vec{J}(t) \equiv \vec{J}(-t). \quad (10)$$

The shadow source involves a “shadow” background magnetic field which is identical to the original one in practice as the external magnetic field is time independent.

The shadow axion couples to $\vec{b}$ in the same manner as it appears together with the shadow background magnetic field in $\vec{J}$. This gives the correct sources in the Euler-Lagrange equations. The Euler-Lagrange equation for $\tilde{a}$ reduces to the phenomenological equation of motion

$$-\partial_0(\epsilon \star_r \partial_0 \vec{a}) + \Delta \vec{a} = -\frac{\dot{\phi}}{M} \vec{b} + \vec{J}, \quad (11)$$

whilst the equation for a satisfies the time reversed equation

$$-\partial_0(\hat{\epsilon} \star_r \partial_0 \vec{a}) + \Delta \vec{a} = -\frac{\dot{\tilde{\phi}}}{M} \vec{b} + \vec{J}. \quad (12)$$

Hence the Lagrangien (9) reproduces the two equations of motion including dissipation at the price of doubling the number of fields. This extension of the usual electromagnetic setting is minimal in the sense that this new Lagrangian is still quadratic in the fields as the original Lagrangian of electrodynamics. In the following, we will offer two further tests that the new Lagrangian reproduces the physics of electromagnetism. We will see in Sec. III E that the energy-momentum tensor following from (9) is related to the Minkowski tensor of classical electrodynamics in matter. Moreover, in the quantum case, the calculation of the Casimir interaction between two dissipative plates reproduces the Lifschitz theory as shown in Sec. IV.

A classical solution of the two equations is given by $\tilde{a}(t) = a(-t)$ where $a(t)$ satisfies (11) as $\phi(t) = \phi_0 \cos mt$ satisfies $\tilde{\phi}(t) = \phi(t)$ in the equations of motion. This

justifies the identification of $\tilde{a}$ as a shadow field whose behavior sees time in reverse.

Quantization is made easier by separating the classical effects from the quantum effects. We first separate $a^\mu = a_{\text{cl}}^\mu + a_{\text{qu}}^\mu$ where a_{cl}^μ is the classical field sourced by $\mathcal{J}^\mu$ and obeying the usual Maxwell equations. We apply the same decomposition to $\tilde{a}$. The resulting action can be written as

$$S = S_{\text{qu}} + S_{\text{cl}} \quad (13)$$

where the classical action is

$$S_{\text{cl}} = \int d^4x \left(\vec{e}_{\text{cl}} \cdot \vec{a}_{\text{cl}} - \vec{b}_{\text{cl}} \cdot \vec{b}_{\text{cl}} + \frac{\phi}{M} \vec{e}_{\text{cl}} \cdot \vec{b}_{\text{cl}} + \frac{\tilde{\phi}}{M} \vec{e}_{\text{cl}} \cdot \vec{b}_{\text{cl}} - \vec{J} \cdot \vec{a}_{\text{cl}} - \vec{J} \cdot \vec{a}_{\text{cl}} \right) \quad (14)$$

and the quantum action contains only quadratic terms in the quantum fields

$$S_{\text{qu}} = \int d^4x \left(\vec{e}_{\text{qu}} \cdot \vec{a}_{\text{qu}} - \vec{b}_{\text{qu}} \cdot \vec{b}_{\text{qu}} + \frac{\phi}{M} \vec{e}_{\text{qu}} \cdot \vec{b}_{\text{qu}} + \frac{\tilde{\phi}}{M} \vec{e}_{\text{qu}} \cdot \vec{b}_{\text{qu}} \right) \quad (15)$$

where the classical source term has decoupled. It is convenient to separate the quantum action into two parts: the free theory with the action

$$S_0 = \int d^4x (\vec{e}_{\text{qu}} \cdot \vec{a}_{\text{qu}} - \vec{b}_{\text{qu}} \cdot \vec{b}_{\text{qu}}), \quad (16)$$

and the interaction terms

$$S_I = \int d^4x \frac{\phi}{M} (\vec{e}_{\text{qu}} \cdot \vec{b}_{\text{qu}} + \vec{e}_{\text{qu}} \cdot \vec{b}_{\text{qu}}). \quad (17)$$

The free quantum modes satisfy

$$-\partial_0(\epsilon \star_r \partial_0 \vec{a}_{\text{qu}}) + \Delta \vec{a}_{\text{qu}} = 0 \quad (18)$$

and

$$-\partial_0(\hat{\epsilon} \star_r \partial_0 \vec{a}_{\text{qu}}) + \Delta \vec{a}_{\text{qu}} = 0, \quad (19)$$

which correspond in Fourier space to

$$\epsilon(\omega) \omega^2 \vec{a}_{\text{qu}} + \Delta \vec{a}_{\text{qu}} = 0 \quad (20)$$

and

$$\epsilon(-\omega) \omega^2 \vec{a}_{\text{qu}} + \Delta \vec{a}_{\text{qu}} = 0. \quad (21)$$

Notice that the modes for $\tilde{a}$ are obtained from the ones of a by the exchange $\omega \rightarrow -\omega$. The modes taking into account

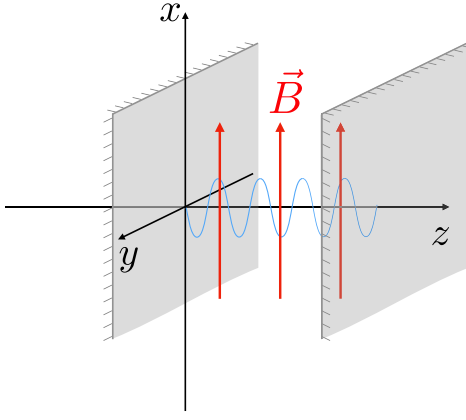


FIG. 1. Typical Casimir experimental setup. We add a constant magnetic field parallel to the plates. This leads to a classical Casimir pressure due to the presence of the axion, which is studied in [14]. The quantum case studied here involves the modes in the presence of the background axion field.

the interactions will be obtained later within perturbation theory.

B. Resonances

We are interested in the Casimir situation where boundary conditions are imposed in the z direction. We consider metallic plates along the (x, y) plane as sketched in Fig. 1.

The problem becomes $(1+1)$ -dimensional and the modes for $\vec{a}$ satisfy in momentum space

$$\vec{a}'' + (\omega^2 \epsilon(\omega) - p_{\parallel}^2) \vec{a} = 0, \quad (22)$$

where $\vec{p}_{\parallel}$ is the momentum parallel to the plates. Similarly we have

$$\vec{\bar{a}}'' + (\omega^2 \bar{\epsilon}(\omega) - p_{\parallel}^2) \vec{\bar{a}} = 0, \quad (23)$$

where we have used that $\bar{\epsilon}(\omega) = \epsilon(-\omega)$. This implies that the modes for $\vec{\bar{a}}$ are simply the complex conjugate of the modes for $\vec{a}$. We can therefore restrict ourselves to finding the modes of $\vec{a}$.

We will consider a simple case and describe the permittivity in the material of the plates using the Drude model.⁷ Inside the plates the permittivity is not equal to one and is given by

$$\epsilon = 1 - \frac{\omega_{\text{Pl}}^2}{\omega^2 + i\gamma\omega} \quad (24)$$

⁷The constant magnetic field parallel to the plates does not alter the Lorentz force term in the equation of motion for the charge carriers and therefore does not change the permittivity [14].

where ω_{Pl} is the plasma frequency and γ is the inverse of a damping time related to the conductivity as $\sigma = \frac{\omega_{\text{Pl}}^2}{\gamma}$. Although this model is controversial when comparing to experimental data of the Casimir effect [15], it is a good theoretical testbed to describe dissipation in a simple setting.⁸

As the permittivity jumps across boundaries, we must impose conditions across these interfaces. There are two types of boundary conditions corresponding to the two modes inside a cavity: the transverse electric (TE) and magnetic (TM) modes. We will give a full description of them below. In this section, we first consider the *TE* modes where both a and $\partial_z a$ are continuous across the boundaries. The *TM* case is such that a and $\epsilon^{-1}(\omega) \partial_z a$ are continuous. The spectrum in this case is described in Appendix F 3.

We define in matter

$$\xi = (p_{\parallel}^2 - \omega^2 \epsilon(\omega))^{1/2} \quad (25)$$

where $p_{\parallel}$ is the modulus of the momentum in the x - y plane, and the square root is such that the real part of ξ is always positive along the real axis in the ω -complex plane. In the vacuum we define

$$\Delta = (\omega^2 - p_{\parallel}^2)^{1/2}. \quad (26)$$

The modes are of the form $e^{-i\omega t + i\vec{p}_{\parallel} \cdot \vec{x}_{\parallel} \pm i\Delta z}$ in the vacuum, $e^{-i\omega t + i\vec{p}_{\parallel} \cdot \vec{x}_{\parallel} + \xi z}$ for $z < 0$ and $e^{-i\omega t + i\vec{p}_{\parallel} \cdot \vec{x}_{\parallel} - \xi z}$ for $z > d$. This choice guarantees the convergence of the modes at both $z = \pm\infty$. There are resonances in the cavity between the plates, and the resonance equation is given by

$$\Theta^{-1}(\omega, p_{\parallel}) = 0 \quad (27)$$

where, see Appendix E,

$$\Theta(\omega, p_{\parallel}) = \frac{1}{\Delta((1 - \frac{\xi^2}{\Delta^2}) \sin \Delta d - 2 \frac{\xi}{\Delta} \cos \Delta d)}. \quad (28)$$

This has solutions for

$$\tan \Delta d = \frac{2 \frac{\xi}{\Delta}}{1 - \frac{\xi^2}{\Delta^2}}. \quad (29)$$

In the case of a metal with dissipation, we will only treat the case of weak dissipation implying that γ is smaller than the frequencies of interest. This is justified because in real situations, the plates are good conductors. In particular as the resonances will be for $\omega_n \propto n/d$, this requires that

⁸For a modern perspective on the permittivity of matter, see [31] for instance.

$d \ll \gamma^{-1}$.⁹ In this regime where $\gamma \ll |\omega| \ll \omega_{\text{Pl}}$ implying that $\Delta \ll \omega_{\text{Pl}}$ we have

$$\frac{\xi}{\Delta} \simeq \frac{\omega_{\text{Pl}}}{\Delta} \left(1 - i \frac{\gamma}{2\omega}\right), \quad (30)$$

which is large and therefore

$$\tan \Delta d = -\frac{2\Delta}{\xi}. \quad (31)$$

The solutions are obtained by iteration and are given by

$$\Delta_n d = n\pi - 2 \frac{n\pi}{\omega_{\text{Pl}} d} \left(1 + i \frac{\gamma}{2\omega_n^{(0)}}\right), \quad n \neq 0 \quad (32)$$

where to leading order

$$\omega_n^{(0)} = \text{sign}(n) \sqrt{\frac{n^2 \pi^2}{d^2} + p_{\parallel}^2}. \quad (33)$$

The imaginary part of the resonance frequency is obtained as

$$\omega_n^2 = \frac{n^2 \pi^2}{d^2} + p_{\parallel}^2 - i \frac{2n^2 \pi^2}{\omega_{\text{Pl}} d^2} \frac{\gamma}{\omega_n^{(0)}}. \quad (34)$$

The solutions are

$$\omega_n = \omega_n^{(0)} - i \frac{n^2 \pi^2}{\omega_{\text{Pl}} d^2} \frac{\gamma}{\frac{n^2 \pi^2}{d^2} + p_{\parallel}^2}, \quad n \neq 0 \quad (35)$$

corresponding to the retarded choice of poles below the real axis. Here the fact that the poles are not on the real axis is due to the finite conductivity which induces the presence of finite-lifetime resonances. There are only a finite number of resonances of energy lower than the plasma frequency.

Above the plasma frequency, the plates become transparent and no resonances are present. Indeed the resonance equation becomes

$$\tan \Delta d = 2i \quad (36)$$

or equivalently $\tanh \Delta d = 2$, which has no solution. Hence, there are no resonances above the plasma frequency. Of course we have only studied the two limiting cases of frequencies well below and well above the plasma frequencies. This is enough to state that no resonances are with a large frequency compared to ω_{Pl} and that the spectrum contains only a finite number of resonances of finite width.

⁹Another family of resonances exists when $d \gg \gamma^{-1}$ corresponds to $\omega_n \ll \gamma$ [14]. We will not consider this case and consider only $d \ll \gamma^{-1}$.

Moreover, as we will see, the fact that the imaginary part of the eigenfrequencies along the positive real axis is negative follows from the fact that there is dissipation and the fields must decay at $t = +\infty$.

The case of the *TM* polarization is described in detail in Appendix F3. The spectrum is still composed of resonances below the real axis. We will denote by $+$ the *TE* polarization and $-$ the *TM* one.

C. The retarded and Feynman propagators

1. The retarded propagator

The Casimir pressure is most easily calculated using a quantum formalism where the Feynman propagator in the presence of the nontrivial permittivity and the boundary plaques are present. Typically in the Appendixes E and F, the Green's functions satisfying

$$-\partial_0(\epsilon \star_t \partial_0 G) + \Delta G = \delta^{(4)}(x^\mu - y^\mu) \quad (37)$$

are investigated. The electric and magnetic cases are simply distinguished by the type of boundary conditions that one must impose at the interface between the plates and the vacuum between the plaques. In the following we will use time-translation invariance and space-translation invariance in the (x, y) plane along the plaques. In terms of Fourier decomposition the Green's functions satisfy

$$(\partial_z^2 - p_{\parallel}^2 + \epsilon(\omega)\omega^2)G = \delta(z - z_0). \quad (38)$$

As is well known in the case of vacuum with no boundaries, i.e. quantum field theory in Minkowski space time, there are different Green's functions depending on prescriptions of contour integrals in the ω plane. The same cases can be obtained in the system with boundaries and a nontrivial permittivity. The solutions given in the Appendixes correspond to retarded propagators as we have shown in the previous section that the poles of the Green's functions are below the real axis. They are suited to the study of the radiation of the electric and magnetic fields when real currents and charge distributions are present inside the plates for instance. This follows from the explicit representation

$$G(z, \omega, p_{\parallel}; z_0) = \Theta(\omega, p_{\parallel}) H(z, \omega, p_{\parallel}; z_0) \quad (39)$$

where the functions $H(z, \omega, p_{\parallel}; z_0)$ can be found in the Appendixes. We have seen that the poles of Θ , giving rise to resonances, are in a finite number and located at $\omega = \omega_n$ in the right half plane $\Re \omega > 0$. The other resonances are situated at $-\bar{\omega}_n$ as we have the property $\bar{\Theta}(\omega, p_{\parallel}) = \Theta(-\bar{\omega}, p_{\parallel})$ when ω is extended to the complex ω plane. This implies that the resonances appear in pairs $(\omega_n, -\bar{\omega}_n)$. As all the resonances ω_n in the right half plane have a negative imaginary part, their images $-\bar{\omega}_n$ have also a

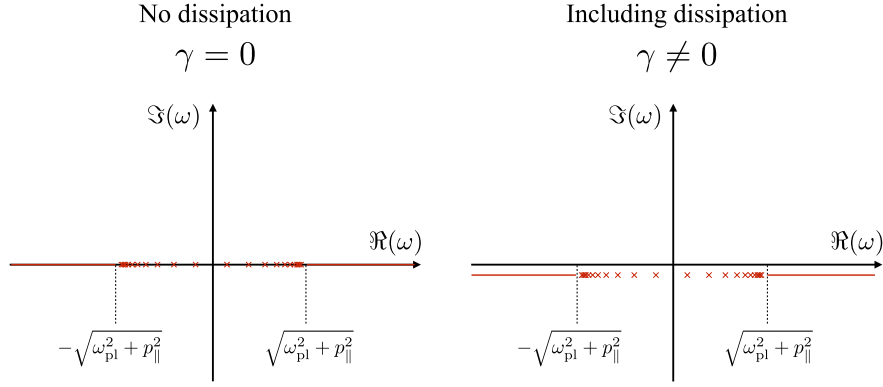


FIG. 2. Location of the poles in the case without dissipation (left) and with dissipation (right). The cuts above the plasma frequency correspond to the transparency of the metal at high energy.

negative imaginary part too and are situated in the left half plane. This is illustrated in Fig. 2.

We want to reconstruct

$$G(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_{0\parallel}, t_0) = \int d\omega d^2 p_{\parallel} e^{-i\omega(t-t_0)} e^{i\vec{p}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}_{0\parallel})} G(z, \omega, p_{\parallel}; z_0). \quad (40)$$

The integrand $G(z, \omega, p_{\parallel}; z_0)$ has poles in the complex ω plane, which are located below the real axis. When $t > t_0$, we can close the contour in the complex ω plane in the lower half-plane on a large circular contour. Inside the contour the poles are ω_n and $-\bar{\omega}_n$. The contributions from the ω_n 's with residues $G_n(\omega_n, p_{\parallel})$ given by

$$G_n(\omega_n, p_{\parallel}) = \frac{1}{\frac{d\Theta^{-1}}{d\omega} \big|_{\omega=\omega_n}} H(z, \omega_n, p_{\parallel}) \quad (41)$$

lead to

$$G_+(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_{0\parallel}, t_0) = -i \sum_n \int d^2 p_{\parallel} e^{-i\omega_n(t-t_0)} e^{i\vec{p}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}_{0\parallel})} G_n(\omega_n, p_{\parallel}) Y(t-t_0) \quad (42)$$

where the minus sign comes from the integral around each pole giving $-2\pi i G_n(p_{\parallel})$.¹⁰ The function $Y(t)$ is the Heaviside function. The function on the right-hand side is on-shell as the frequencies are the eigenfrequencies of the system.

The Green's function depends on two complex functions Δ and ξ , which have branch cuts in the complex ω plane. The Δ function has a branch cut between $-p_{\parallel}$ and $p_{\parallel}$ where it takes opposite values across the cut. Now the Green's function can be explicitly seen to be an even function of Δ ,

see the Appendixes E and F. As a result, there is no branch cut due to Δ for the Green's functions. This is not the same for ξ which has two branch cuts joining infinity from the two branch points

$$\omega_{\pm} = \pm(\omega_{\text{pl}}^2 + p_{\parallel}^2)^{1/2} - \frac{i\gamma\omega_{\text{pl}}^2}{2(p_{\parallel}^2 + \omega_{\text{pl}}^2)} \quad (43)$$

below the real axis. Above the branch cut starting at ω_+ , we have $\xi = -i\sqrt{|p_{\parallel}^2 - \epsilon(\omega)\omega^2|}$ and the opposite value below the cut. The situation is reversed along the branch cut starting at ω_- with a value $i\sqrt{|p_{\parallel}^2 - \epsilon(\omega)\omega^2|}$ above the cut and its opposite below the cut. As a result there is a contribution from the continuum along the branch cut of ξ given by

$$G_+(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_{0\parallel}, t_0) = \int_{\omega_+}^{\infty} d\omega \int d^2 p_{\parallel} e^{-i\omega(t-t_0)} e^{i\vec{p}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}_{0\parallel})} (G(-i|\xi|) - G(i|\xi|)) Y(t-t_0) \quad (44)$$

where we have denoted $|\xi| = \sqrt{|p_{\parallel}^2 - \epsilon(\omega)\omega^2|}$ and simplified the notation to emphasize the dependence of G only on ξ above and below the cut. The contributions from the resonances and the continuum must be added to give the full expression of G_+ .

Similarly the poles at $-\bar{\omega}_n$ have a residue

$$G_n(-\bar{\omega}_n, p_{\parallel}) = -\frac{1}{\frac{d\Theta^{-1}}{d\omega} \big|_{\omega=\omega_n}} \bar{H}(z, \omega_n, p_{\parallel}) \quad (45)$$

where $H(z, -\bar{\omega}_n, p_{\parallel}) = \bar{H}(z, \omega_n, p_{\parallel})$ leading to a contribution

¹⁰We define $d\mathbf{x} = \frac{d\mathbf{x}}{2\pi}$ and $\delta(x) = 2\pi\delta(x)$.

$$G_{-}(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_0, t_0) \supset -i \sum_n \int d^2 p_{\parallel} e^{i\bar{\omega}_n(t-t_0)} e^{i\vec{p}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}_0)} G_n(-\bar{\omega}_n, p_{\parallel}) Y(t-t_0) \quad (46)$$

which is also on-shell. The continuum gives rise to

$$G_{-}(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_0, t_0) \supset \int_{-\infty}^{\omega_{-}} d\omega \int d^2 p_{\parallel} e^{-i\omega(t-t_0)} e^{i\vec{p}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}_0)} (G(i|\xi|) - G(-i|\xi|)) Y(t-t_0). \quad (47)$$

The two contributions to $G_{\pm}$ are distinguished by separating the frequencies into the ones with positive real parts and negative ones. Now when $t < t_0$, one must close the contour in the upper-half plane where there are no poles and the function vanishes. As a result we have

$$G(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_0, t_0) = (G_{+}(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_0, t_0) + G_{-}(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_0, t_0)) Y(t-t_0). \quad (48)$$

This is a retarded Green's function where the on-shell resonances and the continuum enter in the evaluation. Notice that the on-shell term can be written as

$$G_{\text{onshell}}(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_0, t_0) = -i \int d\omega d^2 p_{\parallel} e^{-i\omega(t-t_0)} e^{i\vec{p}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}_0)} \delta(\Theta^{-1}(\omega, p_{\parallel})) H(z, \omega, p_{\parallel}) Y(t-t_0) \quad (49)$$

where the on-shell condition is apparent. The integration contours mentioned here are illustrated in Fig. 3.

2. The Feynman propagator

In the quantum field theoretic treatment of the Casimir effect, which will be expounded below, the Feynman propagator corresponding to the T product of the field is required. The Feynman propagator can be obtained using an adaptation of the Schwarz reflection principle where holomorphic functions are extended from the upper half plane to the lower one. In the right half plane with $\Re \omega > 0$, we take

$$\Re \omega \geq 0: G_F(\omega) = G(\omega) \quad (50)$$

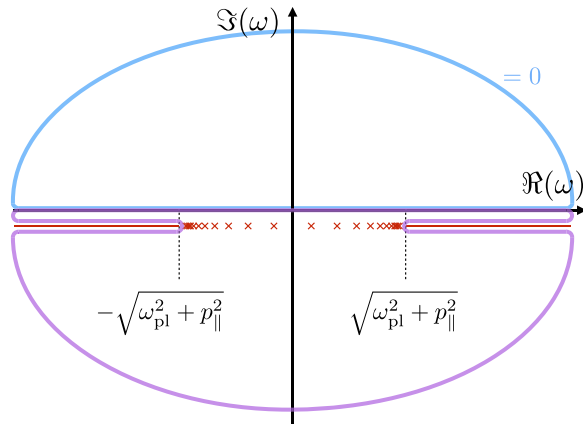


FIG. 3. Integration contours for the retarded Green's function. In the upper half plane, the retarded Green's function vanishes. This is not the case when closing the contour below the axis due to the poles and the transparency cuts.

where we have omitted the dependence on the other variables to simplify the notation. In the left half plane we define

$$\Re \omega \leq 0: G_F(\omega) = \bar{G}(\bar{\omega}) = G(-\omega). \quad (51)$$

The Feynman propagator is obtained through a reflection with respect to the origin, which sends the resonances in the right half plane to the left half plane above the negative real axis. This defines the Feynman propagator in the whole ω complex plane. In terms of integrals, the Feynman propagator reads explicitly

$$G_F(t, \vec{x}) = 2 \int_0^{\infty} d\omega \int d^2 p_{\parallel} \cos \omega t e^{i\vec{p}_{\parallel} \cdot \vec{x}_{\parallel}} G(\omega, \vec{p}_{\parallel}, z) \quad (52)$$

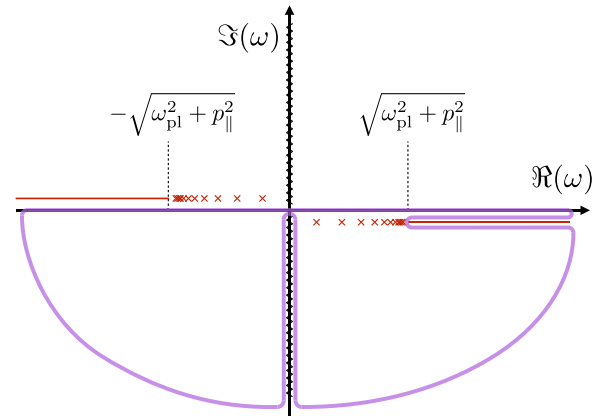


FIG. 4. Location of the poles and cuts for the definition of the Feynman propagator and relevant integration contours.

which satisfies the same partial derivative equation as the retarded propagator.¹¹ The locations of the poles for the definition of the Feynman propagator are shown in Fig. 4. There is a discontinuity on the imaginary axis if $\gamma \neq 0$, except at the origin.

Now the poles of the Feynman propagator are located at ω_n and $-\omega_n$ below and above the real axis. The residues of the poles at $-\omega_n$ are given by $-G_n(\omega_n, p_{\parallel})$. Closing the contour of the inverse Fourier integral in the lower half plane, we have

$$G_F(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_{0\parallel}, t_0) \supset -i \int d^2 p_{\parallel} e^{-i\omega_n(t-t_0)} e^{i\vec{p}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}_{0\parallel})} G_n(\omega_n, p_{\parallel}) Y(t - t_0) \quad (53)$$

whereas when closing in the upper half plane we have

$$G_F(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_{0\parallel}, t_0) \supset -i \int d^2 p_{\parallel} e^{i\omega_n(t-t_0)} e^{i\vec{p}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}_{0\parallel})} G_n(\omega_n, p_{\parallel}) Y(t_0 - t). \quad (54)$$

The contribution from the continuum along the branch cut of ξ is given by

$$G_F(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_{0\parallel}, t_0) \supset \int_{\omega_+}^{\infty} d\omega \int d^2 p_{\parallel} e^{-i\omega(t-t_0)} e^{i\vec{p}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}_{0\parallel})} (G(-i|\xi|) - G(i|\xi|)) Y(t - t_0) \quad (55)$$

where we have denoted $|\xi| = \sqrt{|p_{\parallel}^2 - \epsilon(\omega)\omega^2|}$ and simplified the notation to emphasize the dependence of G only on ξ above and below the cut. All in all we have

$$\begin{aligned} G_F^+(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_{0\parallel}, t_0) = & \left(-i \int d^2 p_{\parallel} e^{-i\omega_n(t-t_0)} e^{i\vec{p}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}_{0\parallel})} G_n(\omega_n, p_{\parallel}) \right. \\ & \left. + \int_{\omega_+}^{\infty} d\omega \int d^2 p_{\parallel} e^{-i\omega(t-t_0)} e^{i\vec{p}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}_{0\parallel})} (G(-i|\xi|) - G(i|\xi|)) \right) Y(t - t_0). \end{aligned} \quad (56)$$

Similarly, the closing of the contour in the upper half plane gives a contribution along the cut between $-\infty$ and $-\omega_+$ implying that the Feynman propagator is decomposed as

$$G_F(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_{0\parallel}, t_0) = G_F^+(z, \vec{x}_{\parallel}, t; z_0, \vec{x}_{0\parallel}, t_0) Y(t - t_0) + G_F^+(z_0, \vec{x}_{0\parallel}, t_0; z, \vec{x}_{\parallel}, t) Y(t_0 - t) \quad (57)$$

where G_F^+ is the positive frequency propagator comprising both the on-shell and the continuum contributions. This is a T product and coincides with what is expected for a Feynman propagator. Similar properties are valid for the magnetic Green's function upon substituting $G \rightarrow G_M$.

This definition of the Feynman propagator is such that the function $G_F(\omega)$ is not continuous across the imaginary axis. The Feynman propagator is only continuous on the imaginary axis provided

$$G(i\omega) = G(-i\omega) \quad (58)$$

which is only valid when $\gamma = 0$ and approximately in the weak limit of dissipation $\gamma \rightarrow 0$. In this case the resonances are displaced from the real axis by an infinitesimal amount.

¹¹We use $\int_0^{\infty} d\omega \cos \omega t = \pi \delta(t)$ implying that $-\partial_t(\epsilon(t) \star_t \partial_t) G_F + \Delta G_F = \delta^{(4)}(x^\mu)$.

When this is not the case, the Feynman propagator has a branch cut on the imaginary axis apart from the origin where it is continuous. This is discussed in the next section.

3. The Feynman propagator for finite dissipation

When $\gamma \neq 0$ and is not taken to be infinitesimal, the Feynman propagator defined in the complex ω plane by a reflection across the origin is not continuous along the imaginary axis apart from the origin. When calculating the Fourier transform as an integral along the real line $\omega \in \mathbb{R}$, this is not an issue. When going to the complex plane and closing the contour in the lower half plane for instance, this requires one to modify the integration path by including a contribution going from $-i\infty$ to 0 and back to $-i\infty$ along both sides of the branch cut, which are joined at the origin where the propagator is continuous. This gives a new contribution to G_F^+ , which reads

$$\delta G_F^+ = \int_{-i\infty}^0 d\omega \int d^2 p_{\parallel} e^{-i\omega(t-t_0)} e^{i\vec{p}_{\parallel} \cdot (\vec{x}_{\parallel} - \vec{x}_{0\parallel})} (G(\omega) - G(-\omega)) Y(t - t_0). \quad (59)$$

This is necessary to lift the obstruction to constructing the Feynman propagator as a continuous function across the imaginary axis. As expected, this new contribution preserves the structure of the Feynman propagator as a T product.

In practice, we will see that it is easier to calculate the Feynman propagator at equal times by first performing a Wick's rotation sending the real axis to the imaginary axis via a rotation of a $\pi/2$ angle. As the axis is rotated, it does not cross poles and branch cuts. The equality of the two integrals over ω is guaranteed as long the integrand converges to zero on a large circular quadrant joining the two axes. For very large $|\omega|$ the permittivity is close to unity and the Feynman propagator behaves like the one in empty space in $1/|\omega|$. This justifies the validity of Wick's rotation. Instead of using the contour drawn in Fig. 4, the integral can be performed very close to the positive part of the imaginary axis after Wick's rotation, as schematically shown in Fig. 5. Because of the parity on the imaginary axis, this gives half the desired result.

After the rotation the integral for the Feynman propagator lies to the right of the imaginary axis in the upper half plane and to its left in the lower half. Analytically we have $G_F(i\omega) = G(i\omega)$ along this path. As a result, we can use the expressions for the retarded Green's function to obtain the Casimir pressure for instance as we shall see below.

III. PATH INTEGRAL QUANTIZATION

A. The two polarizations

A direct approach to quantization bypasses the interpretation of the field a_i in terms of particles, which is

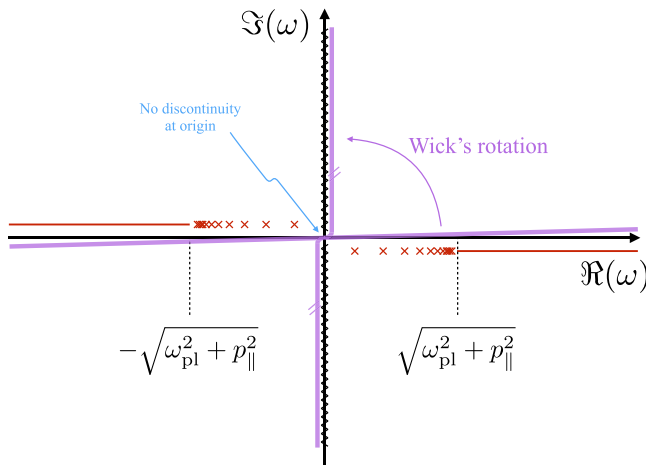


FIG. 5. Feynman propagator can be Wick-rotated to the vicinity of the imaginary axis. After this Wick's rotation the integral is computed on a line infinitely close to the positive part of the imaginary axis.

flawed as Poincaré invariance is broken and there is dissipation in the system. We first decompose the vectors $a_i, i = (x, y, z)$, as

$$a_i(\vec{x}, t) = \sum_{\alpha=\pm} \int d^3 k e_{\alpha i}(k) a_{\alpha}(k, t) e^{i\vec{k} \cdot \vec{x}} = \sum_{\alpha} e_{\alpha i} \star_x a_{\alpha} \quad (60)$$

where $\star_x$ is the convolution operator in space. The polarization vectors are complex such that $\bar{e}_{i\alpha}(k) = e_{i\alpha}(-k)$ and satisfy the familiar orthogonality relations

$$\begin{aligned} \sum_{\alpha} \bar{e}_{\alpha i}(k) e_{\alpha j}(k) &= \delta_{ij} - \frac{k_i k_j}{k^2}, & k^i e_{\alpha i}(k) &= 0, \\ \bar{e}_{\alpha i}(k) e_{\beta}^i(k) &= \delta_{\alpha\beta} \end{aligned} \quad (61)$$

where $k^2 = k_i k^i$. The momenta are real. The vector field is Hermitian as long as the two scalar polarization fields satisfy $a_{\alpha}^{\dagger}(k) = a_{\alpha}(-k)$. Similarly we introduce two scalar fields $\tilde{a}_{\alpha}$ for the shadow vector field $\vec{\tilde{a}}$. We define the Fourier transform for each polarization scalar as

$$a_{\alpha}(\vec{x}, t) = \int d^3 k a_{\alpha}(k, t) e^{i\vec{k} \cdot \vec{x}}. \quad (62)$$

They are real scalar fields. As a consequence we have for the displacement vector

$$d_i = - \sum_{\alpha=\pm} \int d^3 k e_{\alpha i}(k) (\epsilon \star_t \dot{a}_{\alpha})(k, t) e^{i\vec{k} \cdot \vec{x}} = \sum_{\alpha=\pm} e_{\alpha i} \star_t \dot{a}_{\alpha} \quad (63)$$

and we have used the property $\partial_0(\epsilon \star_t a_{\alpha})(k, t) = (\epsilon \star_t \dot{a}_{\alpha})(k, t)$.

B. The Hamiltonian and dissipation

Let us now come back to the quantization procedure. We can also calculate the associated Hamiltonian using the conjugate momenta

$$\vec{\pi}_a = \hat{\epsilon} \star_t \partial_0 \vec{a} - \frac{\phi}{M} \vec{b}, \quad \vec{\pi}_{\tilde{a}} = \epsilon \star_t \partial_0 \vec{a} - \frac{\tilde{\phi}}{M} \vec{b} \quad (64)$$

and then obtain

$$H = \int d^3 x (\partial_0 \vec{a} (\epsilon \star_t \partial_0 \vec{a}) + \partial_i \vec{a} \cdot \partial_i \vec{a} + \vec{J} \cdot \vec{a} + \vec{\tilde{J}} \cdot \vec{a}) \quad (65)$$

where the dependence on the axion field only appears in the source terms. From the quantum action S_{qu} we get the Hamiltonian

$$H_{\text{qu}} = \int d^3x (\partial_0 \tilde{a}^i (\epsilon \star_i \partial_0 a^i) + \partial_i \tilde{a}_j \partial_i a_j) \quad (66)$$

where we will suppress the qu index for convenience below as we are only dealing with the quantum properties from now on. Notice that the explicit axion interactions do not enter the Hamiltonian. They appear in the definition of the canonical momenta.

We will separate the Hamiltonian corresponding to the free theory in vacuum and the part containing the matter permittivity. We have

$$\begin{aligned} H_0 &= \int d^3x (\partial_0 \tilde{a}_i \partial_0 a_i + \partial_i \tilde{a}_j \partial_i a_j) \\ &= \sum_{\alpha=\pm} \int d^3x (\partial_0 \tilde{a}_\alpha \partial_0 a_\alpha + \partial_i \tilde{a}_\alpha \partial_i a_\alpha) \end{aligned} \quad (67)$$

corresponding to two systems of independent free scalar fields. The free Hamiltonian is corrected by interaction terms

$$H_I = \sum_{\alpha=\pm} \int d^3x ((\partial_0 \tilde{a}_\alpha) \chi \star_t \partial_0 a_\alpha) \quad (68)$$

where we have defined

$$\chi(\omega) = \epsilon(\omega) - 1. \quad (69)$$

Associated to the free Hamiltonian are the free conjugate momenta

$$\vec{\pi}_a^0 = \partial_0 \vec{a}, \quad \vec{\pi}_{\tilde{a}}^0 = \partial_0 \vec{\tilde{a}} \quad (70)$$

such that

$$H_0 = \int d^3x (\vec{\pi}_a^0 \cdot \vec{\pi}_{\tilde{a}}^0 + \partial_i \tilde{a}_j \partial_i a_j) \quad (71)$$

and the interaction Hamiltonian

$$H_I = \int d^3x (\vec{\pi}_{\tilde{a}}^0 \chi \star_t \vec{\pi}_a^0). \quad (72)$$

In the Schrödinger picture, the free conjugate momenta are associated to the functional derivatives $\vec{\pi}_a^0 = \frac{1}{i} \frac{\delta}{\delta \vec{a}}$ and similarly for $\vec{\pi}_{\tilde{a}}^0 = \frac{1}{i} \frac{\delta}{\delta \vec{\tilde{a}}}$. As operators the two momenta commute. Now let us take the Hermitian conjugate of the interaction Hamiltonian density

$$[(\partial_0 \tilde{a}_\alpha) (\chi \star_t \partial_0 a_\alpha)]^\dagger = (\hat{\chi} \star_t \partial_0 a_\alpha) (\partial_0 \tilde{a}_\alpha) \quad (73)$$

where $\hat{\chi}(t) = \chi(-t)$. We see that

$$H_I^\dagger \neq H_I \quad (74)$$

as soon as $\chi(-\omega) \neq \chi(\omega)$, i.e. the interaction Hamiltonian is not Hermitian. This is a consequence of dissipation.

C. Schwinger-Keldysh formalism

We are interested in vacuum expectation values in this theory with dissipation. In the Heisenberg representation of quantum mechanics where operators evolve and states are fixed, we shall quantize by taking as states (and the full Hilbert space of states) the one of the free theory. The operators evolve in time and the generator of the time evolution is the Hamiltonian. The free evolution operator satisfies

$$\partial_t U_0 = -i H_0 U_0 \quad (75)$$

whose solution is taken as the time ordered product

$$U_0(t) = T(e^{-i \int_{-\infty}^t H_0(t') dt'}). \quad (76)$$

It is convenient to define the operators in the interaction representation where this free evolution is taken into account

$$a_{\alpha I}(t, \vec{x}) = U_0^\dagger(t) a_{\alpha I}(-\infty, \vec{x}) U_0(t). \quad (77)$$

The free evolution operator is unitary as H_0 is Hermitian.

The time evolution of the operators involves the total Hamiltonian which is not Hermitian hence the evolution operator is not unitary. It satisfies

$$\partial_t U = -i H U \quad (78)$$

whose solution is again the time ordered product, i.e. operators are ordered from left to right from the greatest time to the lowest

$$U(t) = T(e^{-i \int_{-\infty}^t H(t') dt'}). \quad (79)$$

In the interaction picture we introduce

$$U_I(t) = U_0^\dagger(t) U(t) \quad (80)$$

which satisfies the equation

$$\partial_t U_I = -i U_0^\dagger H_I(a_\alpha, t) U_0 U_I. \quad (81)$$

We define

$$\mathcal{H}_I(a_{I\alpha}) = U_0^\dagger H_I(a_\alpha, t) U_0 \quad (82)$$

as the interaction the Hamiltonian in the interaction picture, i.e.

$$\mathcal{H}_I(a_{\alpha I}) = \sum_{\alpha=\pm} \int d^3x (\partial_0 \tilde{a}_{\alpha I}) (\chi \star_t \partial_0 a_{\alpha I}). \quad (83)$$

As result we have the T-product exponentiation

$$U_I = T(e^{-i \int_{-\infty}^t \mathcal{H}_I(t') dt'}) \quad (84)$$

which satisfies $U_I(-\infty) = \mathcal{I}$ where $\mathcal{I}$ is the identity operator. Similarly, we have the time evolution

$$a_{\alpha}(t) = U_I^{\dagger}(t) a_{\alpha I}(t) U_I(t) \quad (85)$$

between the operators in the interaction picture $a_{\alpha I}$ and the quantum field operators a_{α} . The same is true for any operator

$$\mathcal{O}(t) = U_I^{\dagger}(t) \mathcal{O}_I(t) U_I(t) \quad (86)$$

where $\mathcal{O}_I = U_0^{\dagger} \mathcal{O} U_0$. Notice that we have

$$\partial_t U_I^{\dagger} = i U_I^{\dagger} \mathcal{H}_I^{\dagger}(a_{\alpha}, t), \quad (87)$$

whose solution is

$$U_I^{\dagger} = \bar{T}(e^{i \int_{-\infty}^t \mathcal{H}_I^{\dagger}(t') dt'}) \quad (88)$$

where the $\bar{T}$ time ordering puts all the fields from left to right in increasing time order.

We will be interested in simple observables involving only one time and defined by the vacuum expectation value of operators $\mathcal{O}(t)$. For instance we will take $\mathcal{O}(t) = a_{\alpha}(\vec{x}, t) \tilde{a}_{\beta}(\vec{y}, t)$. We will consider the steady state situation reached after bringing the two plates together and waiting for a long time. In this case, we will have access to

$$\langle 0 | \mathcal{O}(+\infty) | 0 \rangle = \langle 0 | T_C[\mathcal{O}_I(+\infty) U_I(-\infty, -\infty)] | 0 \rangle \quad (89)$$

where we have identified the time-ordered product along the double line from $-\infty$ to $+\infty$ and back, see Appendix B for details.

$$\begin{aligned} & \langle 0 | T_C[\mathcal{O}_I(+\infty) U_I(-\infty, -\infty)] | 0 \rangle \\ & \equiv \langle 0 | U_I^{\dagger}(+\infty) \mathcal{O}_I(+\infty) U_I(+\infty) | 0 \rangle. \end{aligned} \quad (90)$$

Along the upper part of the path C the evolution operator involves $\mathcal{H}_I$ and on the lowest part $\mathcal{H}_I^{\dagger}$. This representation allows one to write this vacuum expectation value as the path integral

$$\langle 0 | \mathcal{O}(+\infty) | 0 \rangle = \int \mathcal{D}a_{\alpha}^1 \mathcal{D}a_{\alpha}^2 \mathcal{D}\tilde{a}_{\alpha}^1 \mathcal{D}\tilde{a}_{\alpha}^2 \mathcal{O}_I e^{iS(a_{\alpha}^1, \tilde{a}_{\alpha}^1) - iS^{\dagger}(a_{\alpha}^2, \tilde{a}_{\alpha}^2)} \quad (91)$$

involving two copies of the two polarization scalars $a_{\alpha}^a, \tilde{a}_{\alpha}^a$ with $a = 1, 2$ labelling the upper and lower parts of the Schwinger-Keldysh contour. The operator $\mathcal{O}_I$ is inserted on the upper path. Of course as a boundary condition we must impose that $a_{\alpha}^1(\infty) = a_{\alpha}^2(\infty)$ and similarly $\tilde{a}_{\alpha}^1(\infty) = \tilde{a}_{\alpha}^2(\infty)$, which imposes constraints on the Green's functions, see Appendix C.

The action in the Schwinger-Keldysh formalism is defined by

$$S(a_{\alpha}^a, \tilde{a}_{\alpha}^a) = S(a_{\alpha}^1, \tilde{a}_{\alpha}^1) - S^{\dagger}(a_{\alpha}^2, \tilde{a}_{\alpha}^2). \quad (92)$$

The quantum action can be written explicitly as

$$\begin{aligned} S(a_{\alpha}^1, \tilde{a}_{\alpha}^1) = & \int d^4x \left(\partial_0 \tilde{a}_{\alpha}^1 \epsilon \star_t \partial_0 a_{\alpha}^1 - \partial_i \tilde{a}_{\alpha}^1 \partial^i a_{\alpha}^1 \right. \\ & \left. + \frac{\phi}{M} \partial_0 \tilde{a}_{\alpha}^1 P_{i\alpha\beta} \star_x \partial^i a_{\beta}^1 + \frac{\tilde{\phi}}{M} \partial_0 a_{\alpha}^1 P_{i\alpha\beta} \star_x \partial^i \tilde{a}_{\beta}^1 \right) \end{aligned} \quad (93)$$

and similarly on the lower branch where we have defined the coupling tensor

$$P_{i\alpha\beta} = \int d\mathbf{k} P_{i\alpha\beta}(k) e^{i\vec{k} \cdot \vec{x}} \quad (94)$$

where

$$P_{i\alpha\beta}(k) = \epsilon_{ijk} \bar{e}_{j\alpha} e_{k\beta} \quad (95)$$

is the vector product of the polarization vectors. This is obviously zero if $\alpha = \beta$ by antisymmetry and orthogonal to the two polarization vectors. This coupling introduces a mixing between the two polarization scalars. We will separate the quantum action into the free part

$$S_{\text{free}}(a_{\alpha}^a, \tilde{a}_{\alpha}^a) = \int d^4x (\partial_0 \tilde{a}_{\alpha}^a \epsilon \star_t \partial_0 a_{\alpha}^a - \partial_i \tilde{a}_{\alpha}^a \partial^i a_{\alpha}^a) \quad (96)$$

with

$$S_{\text{free}} = S_{\text{free}}(a_{\alpha}^1, \tilde{a}_{\alpha}^1) - S_{\text{free}}^{\dagger}(a_{\alpha}^2, \tilde{a}_{\alpha}^2) \quad (97)$$

and the interaction part with the axion

$$\begin{aligned} S_{\text{int}}(a_{\alpha}^a, \tilde{a}_{\alpha}^a) = & \int d^4x \left(\frac{\phi}{M} \partial_0 \tilde{a}_{\alpha}^a P_{i\alpha\beta} \star_x \partial^i a_{\beta}^a \right. \\ & \left. + \frac{\tilde{\phi}}{M} \partial_0 a_{\alpha}^a P_{i\alpha\beta} \star_x \partial^i \tilde{a}_{\beta}^a \right) \end{aligned} \quad (98)$$

such that

$$S_{\text{int}} = S_{\text{int}}(a_{\alpha}^1, \tilde{a}_{\alpha}^1) - S_{\text{int}}^{\dagger}(a_{\alpha}^2, \tilde{a}_{\alpha}^2). \quad (99)$$

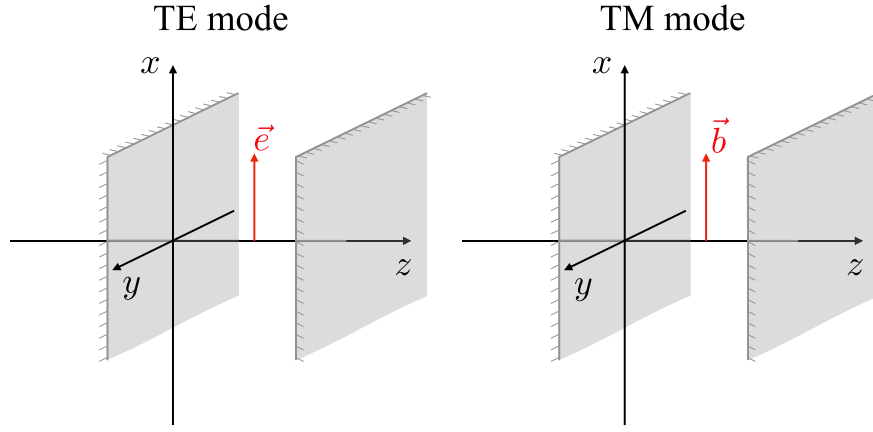


FIG. 6. Illustration of the field's vectors in the TE and TM modes.

This allows one to use perturbation theory techniques when calculating vacuum expectation values of operators depending on a single time.

D. The propagators and the two-point functions

1. Polarizations and boundary conditions

So far we have not distinguished the two polarizations and the associated scalar fields a_α . This is valid in empty space when the permittivity is uniformly equal to unity in space. When the permittivity is not a constant like in the case of the Casimir experiment setup, we must specify the boundary conditions at the interface between two media. To simplify the discussion, let us consider a planar interface at $z = 0$ between two media of permittivities $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon(\omega)$ in Fourier space. There are two types of electromagnetic waves which can be transmitted across the surface. The first one is the TE polarization when the electric field is parallel to the surface, say in the x direction. The other polarization is the TM polarization where the magnetic field is parallel to the interface, again say in the x direction. The two modes correspond to the two panels of Fig. 6.

Across the surface, both the transverse electric and magnetic fields must be continuous. Let us first consider the TE case. We can write

$$e_x = -\partial_0 a_x \quad (100)$$

implying that a_x must be continuous across the interface. Maxwell's equation reads

$$\partial_0 \vec{b} = -\vec{\nabla} \wedge \vec{e} \quad (101)$$

and implies that the parallel component of $\vec{b}$ is b_y , which is proportional to $\partial_z a_x$. The boundary conditions imply that for the TE polarization we must impose

$$[a_x] = 0, \quad [\partial_z a_x] = 0 \quad (102)$$

where $[f] = f_2 - f_1$ is the jump of f across the interface. Considering the case of an incident wave with momentum Δ along the z direction being transmitted to a wave of momentum ξ , we have the relations between the amplitudes of the reflected and transmitted waves

$$\begin{aligned} A_+ + A_- &= B_+ \\ \Delta(A_+ - A_-) &= \xi B_+ \end{aligned} \quad (103)$$

where $A_\pm$ are the amplitudes of the incident and reflected waves, and B_+ the transmitted one. We get that the reflection coefficient $r_{\text{TE}} = \frac{A_-}{A_+}$ is

$$r_{\text{TE}} = \frac{1 - \frac{\xi}{\Delta}}{1 + \frac{\xi}{\Delta}}. \quad (104)$$

Let us now turn to the case where the magnetic field is along the x direction. This implies that $b_x = -\partial_z a_y$ and the Maxwell equation

$$\partial_0 \vec{d} = \vec{\nabla} \wedge \vec{b} \quad (105)$$

implies that in Fourier modes

$$e_y = -i \frac{\partial_z^2 a_y}{\epsilon \omega}. \quad (106)$$

Now integrating this equation between $z = -\delta$ and $z = \delta$ and using the continuity of e_y across the interface implies that $\frac{1}{\epsilon} \partial_z a_y$ must be continuous. As a result we find that the TM polarization boundary condition is

$$[a_y] = 0, \quad \left[\frac{1}{\epsilon} \partial_z a_y \right] = 0. \quad (107)$$

For incident waves we find that amplitudes of the transmitted and reflected waves must satisfy

$$\begin{aligned} A_+ + A_- &= B_+ \\ \Delta(A_+ - A_-) &= \frac{\xi}{\epsilon} B_+ \end{aligned} \quad (108)$$

implying that the reflection coefficient for the TM modes is

$$r_{\text{TM}} = \frac{1 - \frac{\xi}{\epsilon\Delta}}{1 + \frac{\xi}{\epsilon\Delta}} \quad (109)$$

for a wave of energy ω . We will see that these boundary conditions follow from the action and influence the calculations of the propagators. In the following we will denote by $+$ the TE polarization and $-$ the TM polarization.

2. The TE propagators

The two polarizations with their specific boundary conditions can be related to the operators defining the action. This leads to the appropriate propagators with the TE and TM boundary conditions. In order to simplify the notation we define the electric doublet

$$A_+^a = \begin{pmatrix} a_+^a \\ \tilde{a}_+^a \end{pmatrix} \quad (110)$$

associated with the TE polarization. The free action for the TE polarization is obtained after integration by parts

$$S_{\text{free}} = \frac{1}{2} \int d^4x (A_+^a)^T \mathbf{D}_{ab}^E A_+^b. \quad (111)$$

The differential operator is of-diagonal

$$\mathbf{D}_{ab}^E = \begin{pmatrix} 0 & \hat{\mathcal{D}}_{ab}^E \\ \mathcal{D}_{ab}^E & 0 \end{pmatrix} \quad (112)$$

where $a = 1, 2$ labels the two Keldysh branches of the $\mathcal{C}$ path. We have introduced the differential operators

$$\begin{aligned} \mathcal{D}_{11}^E &= \mathcal{D}_E, & \mathcal{D}_{22}^E &= -\tilde{\mathcal{D}}_E \\ \hat{\mathcal{D}}_{11}^E &= \tilde{\mathcal{D}}_E, & \hat{\mathcal{D}}_{22}^E &= -\mathcal{D}_E \end{aligned} \quad (113)$$

and all the other entries of the matrix vanish, i.e. there is no coupling between the upper and lower branches of the Schwinger-Keldysh contour. The minus signs originate from the complex conjugation of e^{iS} on the lower contour. More precisely the pair $(\mathcal{D}, \tilde{\mathcal{D}})$ is such that

$$\mathcal{D}_E = -\partial_0(\epsilon \star_t \partial_0) + \Delta, \quad \tilde{\mathcal{D}}_E = -\partial_0(\hat{\epsilon} \star_t \partial_0) + \Delta, \quad (114)$$

or in Fourier space (for the time part only)

$$\mathcal{D}_E = \epsilon(\omega)\omega^2 + \Delta, \quad \tilde{\mathcal{D}}_E = \epsilon(-\omega)\omega^2 + \Delta, \quad (115)$$

where we can now see that $\hat{\mathcal{D}}_E = \mathcal{D}_E^\dagger$ upon using $\bar{\epsilon}(\omega) = \epsilon(-\omega)$. As a result the differential operator $\mathbf{D}^E$ is Hermitian. We also define the contraction $x^a y_a = c_{ab} x^a y^b$ where $c_{11} = 1, c_{22} = -1$ and 0 otherwise. This is the “metric” in the Schwinger-Keldysh formalism allowing one to raise and lower indices. This also allows one to treat the fields on the two parts of the Schwinger-Keldysh path simultaneously. We can invert the operator $\mathbf{D}^E$ by defining

$$\mathbf{D}_{ab}^E \Delta_E^{bc}(x; y) = i\delta^{(4)}(x^\mu - y^\mu) \delta_a^c \quad (116)$$

or more explicitly as

$$\Delta_E^{ab} = \begin{pmatrix} 0 & \Delta_E^{ab} \\ \hat{\Delta}_E^{ab} & 0 \end{pmatrix}. \quad (117)$$

Let us define the two propagators

$$\mathcal{D}_E \Delta_E(x; y) = i\delta^{(4)}(x^\mu - y^\mu) \quad (118)$$

and

$$\tilde{\mathcal{D}}_E \tilde{\Delta}_E(x; y) = -i\delta^{(4)}(x^\mu - y^\mu). \quad (119)$$

As a result we get the set of propagators

$$\begin{aligned} \Delta_E^{11} &\equiv \Delta_E, & \Delta_E^{22} &\equiv \tilde{\Delta}_E \\ \hat{\Delta}_E^{11} &= \tilde{\Delta}_E, & \hat{\Delta}_E^{22} &= \Delta_E \end{aligned} \quad (120)$$

corresponding to the fact that the propagators on the second branch of the Schwinger-Keldysh path are the time reversed of the propagators on the first branch. In the following we will use

$$\Delta_E = iG_{\text{EF}} \quad (121)$$

where G_{EF} satisfies

$$-\partial_0(\epsilon \star_t \partial_0 G_{\text{EF}}) + \Delta G_{\text{EF}} = \delta^{(4)}(x^\mu - y^\mu). \quad (122)$$

This is the Feynman propagator where we will retrieve its T-product decomposition, see below. Finally notice that across a boundary where ϵ jumps, like in the case of the transition between the vacuum and the plates in Casimir experiments, we can integrate both sides between $z = -\delta$ and $z = \delta$ where $z = 0$ is the interface and get that

$$\left[\frac{dG_{\text{EF}}}{dz} \right]_{z=0} = 0 \quad (123)$$

i.e. the first derivative of G_F does not jump across the interface. This is the TE boundary condition applied to the Green's function.

Let us connect these propagators with correlation functions. As the path integral of the free theory is Gaussian, we can immediately obtain

$$e^{W(J)} = \int \mathcal{D}A_+^1 \mathcal{D}A_+^2 e^{iS_{\text{free}} + \int d^4x A_+^{T a} \mathbf{J}_{a+}} \quad (124)$$

where $\mathcal{D}A_+^a = \mathcal{D}a_+^a \mathcal{D}\tilde{a}_+^a$ and

$$\mathbf{J}_{a+} = \begin{pmatrix} J_{a+} \\ \tilde{J}_{a+} \end{pmatrix}. \quad (125)$$

Here $W(J)$ is the generating functional of connected correlation functions. The Gaussian integral is obtained by solving

$$\mathbf{D}_{ab}^E A_+^b = i\mathbf{J}_{a+} \quad (126)$$

corresponding to

$$A_+^a = \Delta_E^{ab} \mathbf{J}_{b+} \quad (127)$$

and substituting back in the action

$$W(J) = \frac{1}{2} \mathbf{J}_{a+}^T (\Delta_E^{ab})^T \mathbf{J}_{b+}. \quad (128)$$

Here the transpose operators acts on the discrete matrix indices. As a result we get for the two-point functions obtained by Gaussian integration

$$\langle A_+^a(x) A_+^b(y) \rangle = \Delta_E^{ab}(x, y) \quad (129)$$

and we denote by

$$\langle A_+^a(x) A_+^b(y) \rangle = \int \mathcal{D}A_+^1 \mathcal{D}A_+^2 A_+^a(x) A_+^b(y) e^{iS_{\text{free}}}. \quad (130)$$

These two-point functions are the TE two-point functions.

In a more explicit fashion, only the two-point functions between a_α^a and $\tilde{a}_\alpha^b$ for any (a, b) are nonvanishing with

$$\langle a_+^1(x) \tilde{a}_+^1(y) \rangle = \Delta_E(x; y), \quad \langle \tilde{a}_+^1(x) a_+^1(y) \rangle = \tilde{\Delta}_E(x; y). \quad (131)$$

We can now decompose the two-point functions as T-products. More details can be found in Appendix C. For instance the boundary condition at infinity $a^1(\infty) = a^2(\infty)$ implies that

$$\begin{aligned} \langle a_+^1(x) \tilde{a}_+^1(y) \rangle &= Y(x^0 - y^0) \langle a_+^2(x) \tilde{a}_+^1(y) \rangle \\ &+ Y(y^0 - x^0) \langle a_+^1(x) \tilde{a}_+^2(y) \rangle \end{aligned} \quad (132)$$

where Y is the Heaviside distribution. For instance as $x^0 \rightarrow \infty$, the field located at x^μ on the first branch reaches infinity where it becomes the field on the second branch. This implies that the propagator for $x^0 > y^0$ should be identified with $\langle a_\alpha^2(x) a_\beta^1(y) \rangle = \Delta_E^{21} \delta_{\alpha\beta}$ by continuity. This defines Δ_E^{21} . We have also the identification between the path integration and the quantum vacuum expectation values

$$\langle 0 | T[a_+^a(x) \tilde{a}_+^b(x)] | 0 \rangle = \lim_{x^\mu \rightarrow y^\mu} \langle a_+^a(x) \tilde{a}_+^b(y) \rangle \quad (133)$$

where the limit has to be understood after the appropriate renormalization procedure.

3. The TM propagator

We now define the magnetic doublet as

$$A_-^a = \begin{pmatrix} a_-^a \\ \hat{\epsilon} \star_i \tilde{a}_-^a \end{pmatrix} \quad (134)$$

associated with the TM polarization. Notice that the second component is convolved with the time-reversed permittivity function. The free action for the TM polarization is obtained after integration by parts

$$S_{\text{free}} = \frac{1}{2} \int d^4x (A_-^a)^T \mathbf{D}_{ab}^M A_-^b \quad (135)$$

and involves the off-diagonal magnetic differential operator

$$\mathbf{D}_{ab}^M = \begin{pmatrix} 0 & \hat{\mathcal{D}}_{ab}^M \\ \mathcal{D}_{ab}^M & 0 \end{pmatrix} \quad (136)$$

where we have introduced the differential operators

$$\begin{aligned} \mathcal{D}_{11}^M &= \mathcal{D}_M, & \mathcal{D}_{22}^M &= -\tilde{\mathcal{D}}_M, \\ \hat{\mathcal{D}}_{11}^M &= \tilde{\mathcal{D}}_M, & \hat{\mathcal{D}}_{22}^M &= -\mathcal{D}_M. \end{aligned} \quad (137)$$

The pair of magnetic operators $(\mathcal{D}_M, \tilde{\mathcal{D}}_M)$ is such that

$$\mathcal{D}_M = -\partial_0^2 + \epsilon^{-1} \star_i \Delta, \quad \tilde{\mathcal{D}}_M = -\partial_0^2 + \hat{\epsilon}^{-1} \star_i \Delta, \quad (138)$$

or in Fourier space (for the time part only)

$$\mathcal{D}_M = \omega^2 + \frac{\Delta}{\epsilon(\omega)}, \quad \tilde{\mathcal{D}}_M = \omega^2 + \frac{\Delta}{\epsilon(-\omega)}, \quad (139)$$

i.e. the operator $\epsilon^{-1} \star_i$ is the convolution with the function whose Fourier transform is $1/\epsilon(\omega)$, respectively,

$\hat{\epsilon}^{-1} \star_t$ the convolution with the function of Fourier transform $1/\epsilon(-\omega)$. We see that $\hat{\mathcal{D}}_M = \mathcal{D}_M^\dagger$ upon using $\bar{\epsilon}(\omega) = \epsilon(-\omega)$. As a result the differential operator $\mathbf{D}_M$ is Hermitian.

We can invert the operator $\mathbf{D}^M$ by defining the propagators

$$\mathbf{D}_{ab}^M \Delta_M^{bc}(x; y) = i\delta^{(4)}(x^\mu - y^\mu) \delta_a^c \quad (140)$$

or more explicitly as

$$\Delta_M^{ab} = \begin{pmatrix} 0 & \Delta_M^{ab} \\ \hat{\Delta}_M^{ab} & 0 \end{pmatrix}. \quad (141)$$

This requires the two magnetic propagators

$$\mathcal{D}_M \Delta_M(x; y) = i\delta^{(4)}(x^\mu - y^\mu) \quad (142)$$

and

$$\tilde{\mathcal{D}}_M \tilde{\Delta}_M(x; y) = -i\delta^{(4)}(x^\mu - y^\mu). \quad (143)$$

Hence we get the set of propagators

$$\begin{aligned} \Delta_M^{11} &\equiv \Delta_M, & \Delta_M^{22} &\equiv \tilde{\Delta}_M, \\ \hat{\Delta}_M^{11} &\equiv \tilde{\Delta}_M, & \hat{\Delta}_M^{22} &\equiv \Delta_M. \end{aligned} \quad (144)$$

We will use in the following the magnetic Green's function

$$\Delta_M = iG_{MF} \quad (145)$$

where G_{MF} satisfies

$$-\partial_0^2 G_{MF} + \epsilon^{-1} \star_t \Delta G_{MF} = \delta^{(4)}(x^\mu - y^\mu) \quad (146)$$

or in Fourier space

$$\omega^2 G_{MF} + \frac{\Delta}{\epsilon(\omega)} G_{MF} = \delta^{(3)}(\vec{x} - \vec{y}). \quad (147)$$

Again across an interface where ϵ jumps, like at the interface between the vacuum and the plates in a Casimir experiment, we have

$$\left[\frac{1}{\epsilon(\omega)} \frac{dG_{MF}}{dz} \right]_{z=0} = 0, \quad (148)$$

which is the TM boundary condition that applies to the Green's function.

Just like in the TE case, we can connect these propagators with the correlation functions. The same reasoning leads to

$$\langle A_-^a(x) A_-^b(y) \rangle = \Delta_M^{ab}(x, y) \quad (149)$$

where we denote by

$$\langle A_-^a(x) A_-^b(y) \rangle = \int \mathcal{D}A_-^1 \mathcal{D}A_-^2 A_-^a(x) A_-^b(y) e^{iS_{\text{free}}}. \quad (150)$$

These two-point functions are the TM two-point functions. More explicitly, the only nonvanishing two-point functions between a_-^a and $\tilde{a}_-^a$ are

$$\begin{aligned} \langle a_-^1(x) \hat{\epsilon} \star_t \tilde{a}_-^1(y) \rangle &= \Delta_M(x; y), \\ \langle \hat{\epsilon} \star_t \tilde{a}_-^1(x) a_-^1(y) \rangle &= \tilde{\Delta}_M(x; y), \end{aligned} \quad (151)$$

with the T-product decomposition

$$\begin{aligned} \langle a_-^1(x) \hat{\epsilon} \star_t \tilde{a}_-^1(y) \rangle &= Y(x^0 - y^0) \langle a_-^2(x) \hat{\epsilon} \star_t \tilde{a}_-^1(y) \rangle \\ &\quad + Y(y^0 - x^0) \langle a_-^1(x) \hat{\epsilon} \star_t \tilde{a}_-^2(y) \rangle \end{aligned} \quad (152)$$

where $\langle a_-^2(x) \hat{\epsilon} \star_t \tilde{a}_-^1(y) \rangle = \Delta_M^{21}$. Finally we identify

$$\langle 0 | T[a_-^a(x) \hat{\epsilon} \star_t \tilde{a}_-^b(y)] | 0 \rangle = \lim_{x^\mu \rightarrow y^\mu} \langle a_-^a(x) \hat{\epsilon} \star_t \tilde{a}_-^b(y) \rangle \quad (153)$$

with the appropriate renormalization procedure.

E. The link with electrodynamics

We will elaborate further the link between the action involving the two fields a_α^1 and $\tilde{a}_\alpha^1$ and classical electrodynamics. We focus on the part of the action which does not contain the axion field as the axion coupling to photon does not contribute to the energy-momentum tensor. All in all we consider eight fields. The two physical fields a_+ and a_- are copied and time reversed for accounting for dissipation, and a second doubling of the field is necessary for integrating along the Schwinger-Keldysh contour, as

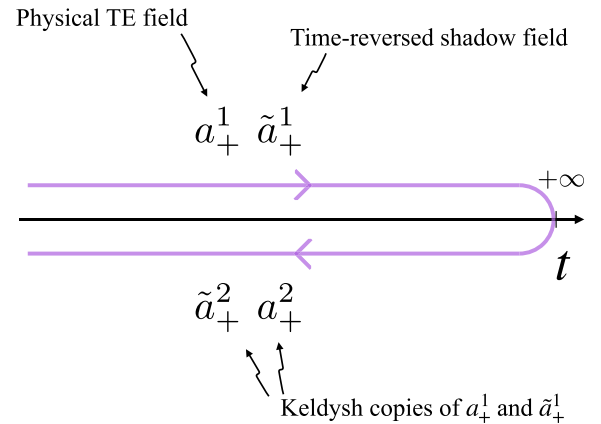


FIG. 7. Illustration of the Keldysh contour in the case of the TE field.

shown in Fig. 7 where the example of the TE field is considered.

We have seen that classically the shadow field is nothing but the time-reversed electromagnetic field. We first decompose the electromagnetic energy momentum tensor obtained from the action

$$S(a_\alpha^1, \tilde{a}_\alpha^1) = \int d^4x \mathcal{L} \quad (154)$$

where we only focus on the fields on the first branch on the Keldysh-Schwinger contour. This leads to the decomposition into two separate parts

$$T_{\mu\nu}^\alpha = T_{\mu\nu}^\alpha + \hat{T}_{\mu\nu}^\alpha \quad (155)$$

for each polarization where we have

$$T_{\mu\nu}^\alpha = \partial_\nu a_\alpha^1 \frac{\partial \mathcal{L}}{\partial \partial^\mu a_\alpha^1} - \frac{\eta_{\mu\nu}}{2} \mathcal{L}, \quad \hat{T}_{\mu\nu}^\alpha = \partial_\nu \tilde{a}_\alpha^1 \frac{\partial \mathcal{L}}{\partial \partial^\mu \tilde{a}_\alpha^1} - \frac{\eta_{\mu\nu}}{2} \mathcal{L}. \quad (156)$$

Notice that this energy-momentum tensor is not symmetric and is conserved. More explicitly we have

$$\begin{aligned} T_{00}^\alpha &= \frac{1}{2} (\partial_0 a_\alpha^1 (\hat{e} \star_t \partial_0 \tilde{a}_\alpha^1) + \partial_i a_\alpha^1 \partial^i \tilde{a}_\alpha^1), \\ T_{0i}^\alpha &= \partial_i a_\alpha^1 (\hat{e} \star_t \partial_0 \tilde{a}_\alpha^1), \quad T_{i0}^\alpha = \partial_0 a_\alpha^1 \partial_i \tilde{a}_\alpha^1, \\ T_{ij}^\alpha &= \partial_j a_\alpha^1 \partial_i \tilde{a}_\alpha^1 - \frac{\delta_{ij}}{2} (-\partial_0 a_\alpha^1 (\hat{e} \star_t \partial_0 \tilde{a}_\alpha^1) + \partial_i a_\alpha^1 \partial^i \tilde{a}_\alpha^1) \end{aligned} \quad (157)$$

and

$$\begin{aligned} \hat{T}_{00}^\alpha &= \frac{1}{2} (\partial_0 \tilde{a}_\alpha^1 (\epsilon \star_t \partial_0 a_\alpha^1) + \partial_i \tilde{a}_\alpha^1 \partial^i a_\alpha^1), \\ \hat{T}_{0i}^\alpha &= \partial_i \tilde{a}_\alpha^1 (\epsilon \star_t \partial_0 a_\alpha^1), \quad \hat{T}_{i0}^\alpha = \partial_0 \tilde{a}_\alpha^1 \partial_i a_\alpha^1, \\ \hat{T}_{ij}^\alpha &= \partial_j \tilde{a}_\alpha^1 \partial_i a_\alpha^1 - \frac{\delta_{ij}}{2} (-\partial_0 \tilde{a}_\alpha^1 (\epsilon \star_t \partial_0 a_\alpha^1) + \partial_i \tilde{a}_\alpha^1 \partial^i a_\alpha^1). \end{aligned} \quad (158)$$

The axion does not play a role in the energy momentum tensor as the original $F\tilde{F}$ term in the action of electrodynamics coupled to axions is topological, i.e. does not depend on the metric.

This energy-momentum tensor has interesting properties in the classical case and in the absence of dissipation $\gamma = 0$. There is no need for the doubling of the electromagnetic fields and in fact one can take $\tilde{a}(t) = a(t)$ as they both satisfy the same Maxwell equation. This implies that we have the relation

$$T_{\mu\nu} = T_{\mu\nu}^{\text{mink}} + \partial_\lambda \Theta_{\mu\nu}^\lambda \quad (159)$$

where the Minkowski energy-momentum tensor of electromagnetism in matter [32] is given by

$$\begin{aligned} T_{00}^{\text{mink}} &= \frac{1}{2} (\vec{d} \cdot \vec{e} + \vec{b} \cdot \vec{b}), \\ T_{0i}^{\text{mink}} &= (\vec{d} \wedge \vec{b})_i, \quad T_{i0}^{\text{mink}} = (\vec{e} \wedge \vec{b})_i, \\ T_{ij}^{\text{mink}} &= -e_i d_j - b_i b_j + \frac{\delta_{ij}}{2} (\vec{e} \cdot \vec{d} + \vec{b}^2). \end{aligned} \quad (160)$$

The correction tensor is given explicitly by

$$\begin{aligned} \Theta_{0i}^\lambda &= -\epsilon \star_t F_0^\lambda a_i, & \Theta_{i0}^\lambda &= -F_0^\lambda a_i, \\ \Theta_{ij}^0 &= -\epsilon \star_t F_i^0 a_j, & \Theta_{ij}^k &= -F_i^k a_j. \end{aligned} \quad (161)$$

When $\epsilon = 1$, i.e. in the case of electrodynamics in vacuum, this transformation is similar to the one between the Noether energy momentum tensor and the symmetric energy momentum tensor [33]. The two energy momentum tensors $T^{\mu\nu}$ and $T_{\text{mink}}^{\mu\nu}$ define the same conserved charge, i.e. the electromagnetic momentum

$$\mathcal{P}^\mu = \int d^3x T^{0\mu}. \quad (162)$$

Indeed using the explicit antisymmetry of $F^{\mu\nu}$ we have $\Theta^{00\mu} = 0$ and therefore

$$\int d^3x \partial_\lambda \Theta^{\lambda 0\mu} = \int d^3x \partial_i \Theta^{i 0\mu} = 0. \quad (163)$$

In this sense, the two energy momentum tensors are equivalent and we can identify $T^{\mu\nu}$ as the energy momentum tensor of electrodynamics in matter. Of course, the total derivatives in (159) would play a role when boundaries are present like in the Casimir setup. In the following section, we will show that the tensor $T^{\mu\nu}$ leads to the correct description of the Casimir effect in the presence of matter.

In the dissipative case when $\gamma \neq 0$ we have classically $\tilde{a}(t) = a(-t)$ and we find

$$\hat{T}_{\mu\nu}(t) = T_{\mu\nu}(-t), \quad (164)$$

i.e. the total energy-momentum tensor is the sum of the energy-momentum tensor $T_{\mu\nu}$ and its time-reversed $\hat{T}_{\mu\nu}$. In particular the energy density of the system is the sum of the energy density of the forward and backward energy densities

$$T_{00}^\alpha(t) = T_{00}^\alpha(t) + T_{00}^\alpha(-t). \quad (165)$$

This energy density is conserved which confirms that the dissipated energy by the forward process is brought back from infinity by the time-reversed dynamics. As a result, we identify $T_{\mu\nu}$ with the energy-momentum tensor of electrodynamics in the presence of dissipation. It involves the shadow field in a crucial way.

In the following we will calculate the quantum vacuum expectation values of $T_{\mu\nu}$ and identify this as the quantum vacuum contribution to the energy and pressure of electromagnetism in matter. We will confirm this identification by calculating the Casimir pressure when the plates are not ideal and retrieve the Lifschitz theory, i.e. the well-known results [20,21]. The vacuum energy is obtained as

$$E = \int d^3x \langle 0 | T_{00} | 0 \rangle \quad (166)$$

or equivalently

$$E = \frac{1}{2} \sum_{\alpha=\pm} \int d^3x \langle 0 | (\partial_0 a_\alpha^1 \hat{e}_t \star_t \partial_0 \tilde{a}_\alpha^1 + \partial_i a_\alpha^1 \partial_i \tilde{a}_\alpha^1) | 0 \rangle. \quad (167)$$

Similarly for the pressure we have

$$P_z = \frac{1}{2} \sum_{\alpha=\pm} \langle 0 | (\partial_z a_\alpha^1 \partial_z \tilde{a}_\alpha^1 + \partial_0 a_\alpha^1 \hat{e}_t \star_t \partial_0 \tilde{a}_\alpha^1 - \partial_x a_\alpha^1 \partial_x \tilde{a}_\alpha^1 - \partial_y a_\alpha^1 \partial_y \tilde{a}_\alpha^1) | 0 \rangle \quad (168)$$

coming from T_{zz} and involving the electric and magnetic two-point functions.

In the general case, we expect the quantum force on a given object of volume V_1 with boundary ∂V_1 exerted by another object of volume V_2 with boundary ∂V_2 to be given by the surface integral

$$F^i = \int_{\partial V_1} d^2 S n_1^j \langle 0 | T_j^i | 0 \rangle \quad (169)$$

where n_1^i is a unit normal vector at the surface of the object V_1 . In quantum field theory, the vacuum expectation values of the energy momentum tensor have divergences due to the coincident limit taken in the bilinear expressions in the fields, see (157) for instance. These divergences can be obtained by splitting the two arguments of the fields by a common distance ϵ and taking this regularization to zero eventually. Locally, in the UV limit, Lorentz invariance is restored and therefore the divergences only depend on the UV regulator ϵ . They do not depend on the long range infrared (IR) properties of the configuration such as the distance between the objects V_1 and V_2 . This implies that the divergent parts of the pressure should be renormalized by being absorbed in the coupling constants of the Lagrangian. This could involve renormalizing masses for instance. The end result should therefore be a finite force depending on the distance between the objects and the renormalized couplings. In the explicit calculations performed in this paper, the two objects are two finite plates. The calculation of the force acting on one of the plates should take into account the two sides of this plate. This requires calculating the Green's functions involving two finite-size plates and therefore five separate regions for the Green's functions. In order to reduce the algebraic

complexity of the calculation, we will follow a more direct route and calculate the pressure on the inner surface of a semi-infinite plate. Then we will remove the divergences by hand in a way mimicking the effect of the outer boundary at finite distance. As an infinite renormalization is always defined up to a finite constant, we will adjust the final result in such a way that for infinitely separated plates the pressure vanishes, i.e. retrieving the result of empty space. The full calculation with five regions and finite-size plate is more complex and left for future work.

In the following section, we will show that this way of calculating the pressure coincides with the result from Lifschitz's theory in the absence of axionic field. When the axion is taken into account, we will calculate the first correction to the quantum pressure due to the coupling to axions.

IV. THE LIFSCHITZ THEORY

Before calculating the effects of axions on the Casimir effect, let us revisit the case with nontrivial permittivities in the plaques and no axions. We will use the previous energy and pressure defined from the action involving the two polarization scalars and their shadow fields. We will retrieve the results of the Lifschitz theory which are commonly derived using different approaches [20,21].

A. The vacuum energy with no axions

We will calculate the vacuum expectation value of T^{00} as obtained in the previous section. Let us evaluate the potential energy first

$$E_P = \frac{1}{2} \sum_{\alpha=\pm} \langle 0 | \int d^3x \partial_i a_\alpha^1(x) \partial_i \tilde{a}_\alpha^1(x) | 0 \rangle \quad (170)$$

where the energy is the vacuum expectation value defined in the previous section. Here we consider the fields as living on the upper path of the Schwinger-Keldysh contour. The kinetic energy is given by

$$E_K = \frac{1}{2} \lim_{x^\mu \rightarrow x_0^\mu} \int d^3x \langle 0 | \partial_t a_\alpha^1(x) \hat{e}_t \star_{t_0} \partial_t \tilde{a}_\alpha^1(x_0) | 0 \rangle. \quad (171)$$

As the expectation values are taken at coincident points, the product of distributions is ill-defined and is understood in a limiting sense using the point-splitting method with $x^\mu \rightarrow x_0^\mu$. After an integration by parts the potential energy becomes

$$E_P = -\frac{1}{2} \sum_{\alpha=\pm} \langle 0 | \int d^3x \partial_x^2 a_\alpha^1(x) \tilde{a}_\alpha^1(x) | 0 \rangle \quad (172)$$

which is understood as the limit

$$E_P = -\frac{1}{2} \sum_{\alpha=\pm} \lim_{x^\mu \rightarrow x_0^\mu} \int d^3x \partial_x^2 \langle a_\alpha^1(x) \tilde{a}_\alpha^1(x_0) \rangle. \quad (173)$$

We first focus on the TE polarization. Now we use the identity

$$\langle a_+^1(x) \tilde{a}_+^1(x_0) \rangle = i G_{EF}(x; x_0). \quad (174)$$

As $x \neq x_0$ in the point-splitting limit we have also the equality

$$\lim_{x^\mu \rightarrow x_0^\mu} \int d^3x \partial_x^2 G_{EF}(x; x_0) = \lim_{x^\mu \rightarrow x_0^\mu} \int d^3x \epsilon \star_t \partial_t^2 G_{EF}(x; x_0). \quad (175)$$

Now as the Green's function is time-translation invariant, i.e. it only depends on $t - t_0$, we have $\partial_t^2 G_{EF}(x, x_0) = -\partial_t \partial_{t_0} G_{EF}(x; x_0)$. This implies that the potential energy and the kinetic energy for the TE polarization are equal

$$E_P^+ = E_K^+ = \frac{i}{2} \lim_{x^\mu \rightarrow x_0^\mu} \int d^3x \partial_t \partial_{t_0} (\epsilon \star_t G_{EF})(x; x_0). \quad (176)$$

The total energy is then

$$E_+ = i \lim_{x^\mu \rightarrow x_0^\mu} \int d^3x \partial_t \partial_{t_0} (\epsilon \star_t G_{EF})(x; x_0). \quad (177)$$

Translation invariance in the parallel direction to the plates and time translation invariance leads to

$$G_{EF}(x; x_0) = G_{EF}(t - t_0, \vec{x}_\parallel - \vec{x}_{\parallel 0}, z; 0, \vec{0}, z_0) \quad (178)$$

implying that the integral over the surface area of the plates factors out and we get finally in terms of Fourier modes

$$\frac{E_+}{A} = i \lim_{x \rightarrow x_0} \int dz d\omega d^2 p_\parallel e^{-i\omega(t-t_0)} \epsilon(\omega) \omega^2 G_{EF}(z, \omega, p_\parallel; z_0), \quad (179)$$

where we have defined the Fourier transform

$$G_{EF}(z, \omega, p_\parallel; z_0) = \int dt d^2 x_\parallel e^{-i\omega t} e^{i\vec{x}_\parallel \cdot \vec{p}_\parallel} G_{EF}(t, \vec{x}_\parallel, z; 0, \vec{0}, z_0), \quad (180)$$

which is given in Appendix E for the retarded Green's function and by the extension procedure in Sec. II C 2 for the Feynman propagator.

The TM polarization can be dealt with in the same fashion, see the Appendix F, using

$$\langle a_-^1(x) \tilde{a}_-^1(x_0) \rangle = i \epsilon^{-1} \star_t G_{MF}(x; x_0) \quad (181)$$

implying that the energy for the TM polarization is given by

$$E_+ = i \lim_{x^\mu \rightarrow x_0^\mu} \int d^3x \partial_t \partial_{t_0} G_{MF}(x; x_0) \quad (182)$$

or equivalently

$$\frac{E_-}{A} = i \lim_{x^\mu \rightarrow x_0^\mu} \int dz d\omega d^2 p_\parallel e^{-i\omega(t-t_0)} \omega^2 G_{MF}(z, \omega, p_\parallel; z_0). \quad (183)$$

Hence the total energy can be obtained from the following the integrals

$$\frac{E}{A} = i \lim_{x^\mu \rightarrow x_0^\mu} \int dz d\omega d^2 p_\parallel e^{-i\omega(t-t_0)} \omega^2 (\epsilon(\omega) G_{EF}(z, \omega, p_\parallel; z_0) + G_{MF}(z, \omega, p_\parallel; z_0)) \quad (184)$$

leading to the vacuum energy.

B. The ideal case

Before focusing on the case with metallic plates, let us retrieve the Casimir energy when the plates are ideal with a vanishing Green's function apart from inside the vacuum. This corresponds to the limit $\epsilon \rightarrow \infty$, see Appendix E for instance. In this case we have $G_F = 0$ for $z < 0$ and $z > d$ whilst in between the plates

$$G_{EF}(z; z_0, \omega, p_\parallel) = \frac{1}{\Delta} \frac{\sin \Delta(z_0 - d) \sin \Delta z}{\sin \Delta d},$$

$$G_{MF}(z; z_0, \omega, p_\parallel) = \frac{1}{\Delta} \frac{\cos \Delta(z_0 - d) \cos \Delta z}{\sin \Delta d}. \quad (185)$$

Explicitly we have

$$\int_0^d dz (G_F(z, \omega, p_\parallel; z) + G_{MF}(z, \omega, p_\parallel; z)) = \frac{d \cos \Delta d}{\Delta \sin \Delta d}. \quad (186)$$

We will change variable from ω to Δ which is a double cover of the real line into the positive real line. In order to take into account the double counting we extend Δ to the negative real line by taking $\Delta = \text{sign}(\omega) \sqrt{\omega^2 - p_\parallel^2}$. Then we get

$$\frac{E}{A} = i \frac{d}{2\pi} \lim_{t \rightarrow t_0} \int_{-\infty}^{\infty} d\omega \int d^2 p_\parallel e^{-i\omega(t-t_0)} \frac{\omega^2 \cos \Delta d}{\Delta \sin \Delta d}. \quad (187)$$

In the complex Δ plane, the integrand has poles for

$$\Delta_n = \frac{n\pi}{d} \Rightarrow \omega_n = \text{sign}(n) \left(\frac{\pi^2 n^2}{d^2} + p_\parallel^2 \right)^{1/2} \quad (188)$$

for $n \neq 0$. As $t > t_0$, the integral can be performed by closing the integration contour in the lower half-plane where ω has a negative imaginary parts guaranteeing that the exponential factor $e^{-i\omega(t-t_0)}$ decays at infinity. As a memory of the shift of the poles when dissipation is

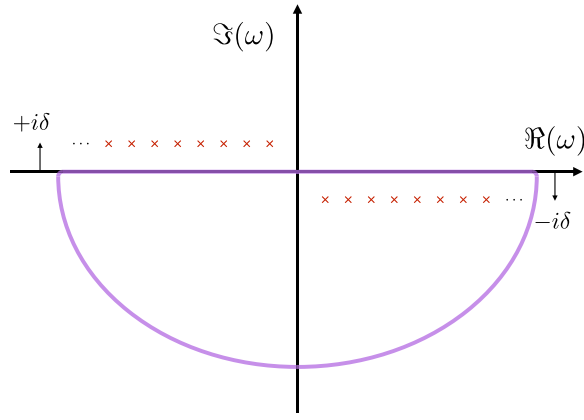


FIG. 8. The integration contour in the complex ω -plane in the ideal case. Notice that the poles have been shifted and are now off the real axis like in the Feynman propagator when dissipation is present. The limit $\delta \rightarrow 0$ should be taken once the integration has been performed.

present, we take the Feynman prescription that the poles are shifted from the real axis by

$$\omega_n = \text{sign}(n) \left(\left(\frac{\pi^2 n^2}{d^2} + p_{\parallel}^2 \right)^{1/2} - i\delta \right) \quad (189)$$

where δ is infinitesimal. As shown in Fig. 8 the poles are integrated over circles with residues $\text{Res}_n = \frac{\omega_n}{d}$ and we find that

$$\frac{E}{A} = i \frac{d}{2\pi} \times \int d^2 p_{\parallel} (-2\pi i) \sum_{n>0} \frac{\omega_n}{d} \quad (190)$$

leading to

$$G_{ER}(z, \omega, p_{\parallel}; z) = \frac{\Theta_E(\omega, p_{\parallel})}{2} \left(\frac{\xi}{\Delta} + \frac{\Delta}{\xi} \right) \left(\sin \Delta d + \frac{\xi}{\Delta} (\cos \Delta(2z - d) - \cos \Delta d) \right), \quad 0 \leq z \leq d \quad (193)$$

and

$$G_{MR}(z, \omega, p_{\parallel}; z) = \frac{\Theta_M(\omega, p_{\parallel})}{2} \left(\frac{\xi}{\epsilon \Delta} + \frac{\epsilon \Delta}{\xi} \right) \left(\sin \Delta d + \frac{\xi}{\epsilon \Delta} (\cos \Delta(2z - d) - \cos \Delta d) \right), \quad 0 \leq z \leq d. \quad (194)$$

In the plates we find

$$\begin{aligned} G_{ER}(z, \omega, p_{\parallel}; z) &= \frac{\Theta_E(\omega, p_{\parallel})}{2} \sin \Delta d \left(\frac{\xi}{\Delta} + \frac{\Delta}{\xi} \right) e^{-2\xi(z-d)} \quad z \geq d, \\ G_{ER}(z, \omega, p_{\parallel}; z) &= \frac{\Theta_E(\omega, p_{\parallel})}{2} \sin \Delta d \left(\frac{\xi}{\Delta} + \frac{\Delta}{\xi} \right) e^{2\xi z} \quad z \leq 0 \end{aligned} \quad (195)$$

and

$$\frac{E}{A} = \sum_{n>0} \int d^2 p_{\parallel} \omega_n. \quad (191)$$

This is the vacuum energy for the two polarizations, i.e. we retrieve that the vacuum energy is due to the presence of photon modes between the ideal plaques. Notice the complementary roles that the TE and TM polarizations play, i.e. they contribute equally to the vacuum energy. Moreover if we had shifted the poles below the real axis as for retarded Green's functions, the energy would have vanished. Using the Feynman prescription is crucial.

C. Metallic plates

From now on and to simplify the notation, all the propagators are the Feynman propagators and we drop the F index for simplicity. In the case of metallic plates, the integrals of the Green's function inside the plates converges provided the two Green's functions $G_E(z, \omega, p_{\parallel}; z)$ and $G_M(z, \omega, p_{\parallel}; z)$ vanish when z is very large. It turns out that $\lim_{z \rightarrow \infty} G_E(z, \omega, p_{\parallel}; z) = -\frac{1}{2\xi}$ and $\lim_{z \rightarrow \infty} G_M(z, \omega, p_{\parallel}; z) = -\frac{\epsilon}{2\xi}$. This means that one must renormalize the Green's functions according to

$$\begin{aligned} G_{ER}(z, \omega, p_{\parallel}; z) &= G_E(z, \omega, p_{\parallel}; z) + \frac{1}{2\xi}, \\ G_{MR}(z, \omega, p_{\parallel}; z) &= G_M(z, \omega, p_{\parallel}; z) + \frac{\epsilon}{2\xi}. \end{aligned} \quad (192)$$

Physically this amounts to subtracting the pressure exerted on the plate by the vacuum when one regularizes the plate to be of finite and very large width L . The plate at $z = d + L$ compensates part of the pressure exerted at $z = d$. This is the origin of this renormalization. In practice this implies that

$$\begin{aligned}
G_{MR}(z, \omega, p_{\parallel}; z) &= \frac{\Theta_M(\omega, p_{\parallel})}{2} \sin \Delta d \left(\frac{\xi}{\epsilon \Delta} + \frac{\epsilon \Delta}{\xi} \right) e^{-2\xi(z-d)} \quad z \geq d, \\
G_{MR}(z, \omega, p_{\parallel}; z) &= \frac{\Theta_M(\omega, p_{\parallel})}{2} \sin \Delta d \left(\frac{\xi}{\epsilon \Delta} + \frac{\epsilon \Delta}{\xi} \right) e^{2\xi z} \quad z \leq 0.
\end{aligned} \tag{196}$$

As these expressions correspond to the renormalized Feynman propagators, they are defined in the right-hand plane where $\Re \omega > 0$. In the left-half plane the Feynman propagators are simply obtained by exchanging $\omega \rightarrow -\omega$. This implies that integrals involving the Feynman propagator along the real line are such that

$$\int_{-\infty}^{\infty} d\omega G_{EF}(\omega) = 2 \int_0^{\infty} d\omega G_{EF}(\omega) = 2 \int_0^{\infty} d\omega G_E(\omega) \tag{197}$$

as $G_{EF}(\omega) = G_E(\omega)$ for $\omega > 0$. We will use this below.

1. The electric energy

The contribution from the plaques to the energy is given by

$$\begin{aligned}
\frac{E_{\text{plaque}}^+}{A} &= i \lim_{t \rightarrow t_0} \int_0^{\infty} d\omega \int d^2 p_{\parallel} e^{-i\omega(t-t_0)} \epsilon(\omega) \omega^2 \frac{\Theta_E(\omega, p_{\parallel})}{2\xi} \left(\frac{\xi}{\Delta} + \frac{\Delta}{\xi} \right) \sin \Delta d \\
&\quad + i \lim_{t \rightarrow t_0} \int_{-\infty}^0 d\omega \int d^2 p_{\parallel} e^{-i\omega(t-t_0)} \epsilon(-\omega) \omega^2 \frac{\Theta_E(-\omega, p_{\parallel})}{2\xi(-\omega)} \left(\frac{\xi(-\omega)}{\Delta} + \frac{\Delta}{\xi(-\omega)} \right) \sin \Delta d
\end{aligned} \tag{198}$$

where on the second line all the contributions are evaluated at $-\omega$ and we do not specify $\Delta(-\omega)$ as Δ is an even function. The contribution from the vacuum between the plaques is

$$\begin{aligned}
\frac{E_V^+}{A} &= i \lim_{t \rightarrow t_0} \int_0^{\infty} d\omega \int d^2 p_{\parallel} e^{-i\omega(t-t_0)} \omega^2 \frac{\Theta_E(\omega, p_{\parallel})}{2\Delta} \xi d \left(\frac{\xi}{\Delta} + \frac{\Delta}{\xi} \right) \left(-\cos \Delta d + \left(\frac{\Delta}{\xi} - \frac{1}{\Delta d} \right) \sin \Delta d \right) \\
&\quad + i \lim_{t \rightarrow t_0} \int_{-\infty}^0 d\omega \int d^2 p_{\parallel} e^{-i\omega(t-t_0)} \omega^2 \frac{\Theta_E(-\omega, p_{\parallel})}{2\Delta} \xi(-\omega) d \left(\frac{\xi(-\omega)}{\Delta} + \frac{\Delta}{\xi(-\omega)} \right) \\
&\quad \times \left(-\cos \Delta d + \left(\frac{\Delta}{\xi(-\omega)} - \frac{1}{\Delta d} \right) \sin \Delta d \right).
\end{aligned} \tag{199}$$

Adding the two integrals we get

$$\begin{aligned}
\frac{E^+}{A} &= \frac{i}{2} \lim_{t \rightarrow t_0} \left[\int_0^{\infty} d\omega \int d^2 p_{\parallel} \Theta_E(\omega, p_{\parallel}) F(\omega, p_{\parallel}) e^{-i\omega(t-t_0)} \right. \\
&\quad \left. + \int_{-\infty}^0 d\omega \int d^2 p_{\parallel} \Theta_E(-\omega, p_{\parallel}) F(-\omega, p_{\parallel}) e^{-i\omega(t-t_0)} \right]
\end{aligned} \tag{200}$$

where we have introduced the even function of ω

$$F_E(\omega, p_{\parallel}) = \omega^2 \left(\frac{\xi}{\Delta} + \frac{\Delta}{\xi} \right) \left[\frac{\epsilon(\omega)}{\xi} \sin \Delta d + \frac{\xi d}{\Delta} \left(-\cos \Delta d + \left(\frac{\Delta}{\xi} - \frac{1}{\Delta d} \right) \sin \Delta d \right) \right]. \tag{201}$$

In these integrals, the limit $t \rightarrow t_0$ is taken to be with $t \leq t_0$ when $\omega \geq 0$ and $t \geq t_0$ when $\omega \leq 0$. This allows one to reduce the electric energy to a single integral with $t \leq t_0$

$$\frac{E^+}{A} = i \lim_{t \rightarrow t_0} \int_0^{\infty} d\omega \int d^2 p_{\parallel} \Theta_E(\omega, p_{\parallel}) F_E(\omega, p_{\parallel}) e^{-i\omega(t-t_0)} \tag{202}$$

which will be calculated after Wick's rotation.

2. The magnetic energy

The calculation runs similarly to the electric case with the magnetic energy in the plaques given by

$$\frac{E_{\text{plaque}}^-}{A} = i \lim_{t \rightarrow t_0} \int \not{d}\omega \not{d}^2 p_{\parallel} e^{-i\omega(t-t_0)} \omega^2 \frac{\Theta_M(\omega, p_{\parallel})}{2\xi} \left(\frac{\xi}{\epsilon\Delta} + \frac{\epsilon\Delta}{\xi} \right) \sin \Delta d \quad (203)$$

whereas the contribution from the vacuum between the plaques is

$$\frac{E_V^-}{A} = i \lim_{t \rightarrow t_0} \int \not{d}\omega \not{d}^2 p_{\parallel} e^{-i\omega(t-t_0)} \omega^2 \frac{\Theta_M(\omega, p_{\parallel})}{2\Delta\epsilon} \xi d \left(\frac{\xi}{\epsilon\Delta} + \frac{\epsilon\Delta}{\xi} \right) \left(-\cos \Delta d + \left(\frac{\epsilon\Delta}{\xi} - \frac{1}{\Delta d} \right) \sin \Delta d \right). \quad (204)$$

Adding the two integrals, where here $t \leq t_0$, we get

$$\frac{E^-}{A} = i \lim_{t \rightarrow t_0} \int_0^\infty \not{d}\omega \int \not{d}^2 p_{\parallel} \Theta_M(\omega, p_{\parallel}) F_M(\omega, p_{\parallel}) e^{-i\omega(t-t_0)} \quad (205)$$

where we have introduced

$$F_M(\omega, p_{\parallel}) = \omega^2 \left(\frac{\xi}{\epsilon\Delta} + \frac{\epsilon\Delta}{\xi} \right) \left[\frac{1}{\xi} \sin \Delta d + \frac{\xi d}{\epsilon\Delta} \left(-\cos \Delta d + \left(\frac{\epsilon\Delta}{\xi} - \frac{1}{\Delta d} \right) \sin \Delta d \right) \right]. \quad (206)$$

The integrals giving the energy and defined along the real axis such as (205) are oscillatory and difficult to manipulate. It is more convenient to perform a Wick's rotation in the complex plane and integrate along the imaginary axis.

D. Wick's rotation

1. The electric energy

The oscillating integrals we found in the previous section can be evaluated after performing Wick's rotation. This is only possible as the Feynman propagators do not have obstructions by poles and branch cuts when rotating the ω integral by an angle of $\pi/2$ in the complex plane. In taking the limit $t \rightarrow t_0$, we consider $t \leq t_0$ implying that the integral along a large circle in the first quadrant is exponentially convergent to zero. Then, after Wick's

rotation $\omega \rightarrow i\omega$ we have

$$\frac{E^+}{A} = -\frac{1}{2\pi} \int \not{d}^2 p_{\parallel} \int_0^\infty d\omega \Theta_E(i\omega, p_{\parallel}) F_E(i\omega, p_{\parallel}) \quad (207)$$

where now

$$\Delta = (\omega^2 + p_{\parallel}^2)^{1/2}, \quad \xi = (p_{\parallel}^2 + \omega^2 \epsilon(i\omega))^{1/2} \quad (208)$$

where in the metallic case

$$\epsilon(i\omega) = 1 + \frac{\omega_{\text{pl}}^2}{\omega^2 + \gamma\omega} \quad (209)$$

which has no singularity on the positive imaginary axis where $\omega > 0$. This transformation implies that

$$\Theta_E(i\omega, p_{\parallel}) = -\frac{1}{\Delta \left(\left(1 + \frac{\xi^2}{\Delta^2} \right) \sinh \Delta d + \frac{2\xi}{\Delta} \cosh \Delta d \right)} \quad (210)$$

and

$$F_E(i\omega, p_{\parallel}) = \omega^2 \left(-\frac{\xi}{\Delta} + \frac{\Delta}{\xi} \right) \left[\frac{\epsilon(i\omega)}{\xi} \sinh \Delta d + \frac{\xi d}{\Delta} \left(\cosh \Delta d + \left(\frac{\Delta}{\xi} - \frac{1}{\Delta d} \right) \sinh \Delta d \right) \right] \quad (211)$$

which is real. Finally we obtain

$$\begin{aligned} \frac{E^+}{A} = & -\frac{1}{2\pi} \int \not{d}^2 p_{\parallel} \int_0^\infty d\omega \frac{\omega^2}{\Delta \left(\left(1 + \frac{\xi^2}{\Delta^2} \right) \sinh \Delta d + \frac{2\xi}{\Delta} \cosh \Delta d \right)} \left(\frac{\xi}{\Delta} - \frac{\Delta}{\xi} \right) \\ & \times \left[\frac{\epsilon(i\omega)}{\xi} \sinh \Delta d + \frac{\xi d}{\Delta} \left(\cosh \Delta d + \left(\frac{\Delta}{\xi} - \frac{1}{\Delta d} \right) \sinh \Delta d \right) \right]. \end{aligned} \quad (212)$$

Let us take the large d limit corresponding to the infinite space case. We have

$$\frac{E_{\infty}^+}{A} = -\frac{1}{2\pi} \int d^2 p_{\parallel} \int_0^{\infty} d\omega \frac{\omega^2}{(1 + \frac{\xi}{\Delta})^2} \left(\frac{\xi}{\Delta} - \frac{\Delta}{\xi} \right) \left[\frac{\epsilon(i\omega)}{\xi} + \frac{\xi d}{\Delta} \left(1 + \left(\frac{\Delta}{\xi} - \frac{1}{\Delta d} \right) \right) \right]. \quad (213)$$

The energy of empty space time must be renormalized to zero, as we assume that locally space time is of the Minkowski type corresponding to a vanishing cosmological constant. As a result we define the renormalized energy $\frac{E_R^+}{A} = \frac{E^+}{A} - \frac{E_{\infty}^+}{A}$. This turns out to be

$$\frac{E_R^+}{A} = -\frac{d}{\pi} \int d^2 p_{\parallel} \int_0^{\infty} d\omega \frac{\omega^2}{\Delta \left(\frac{1 + \frac{\xi}{\Delta}}{1 - \frac{\xi}{\Delta}} \right)^2 e^{2\Delta d} - 1} \left(1 - \frac{2p_{\parallel}^2}{\omega^2} \frac{1}{\xi d} \right) \quad (214)$$

where the factor $(1 - \frac{2p_{\parallel}^2}{\omega^2} \frac{1}{\xi d})$ corresponds to the contribution from the boundary plates, i.e. it reduces to unity when concentrating on the cavity of size d only.

2. The magnetic energy

After Wick's rotation $\omega \rightarrow i\omega$, the magnetic energy becomes

$$\frac{E^-}{A} = -\frac{1}{2\pi} \int d^2 p_{\parallel} \int_0^{\infty} d\omega \Theta_M(i\omega, p_{\parallel}) F_M(i\omega, p_{\parallel}) \quad (215)$$

with

$$\Theta_M(i\omega, p_{\parallel}) = -\frac{1}{\Delta \left((1 + \frac{\xi^2}{\Delta^2}) \sinh \Delta d + \frac{2\xi}{\Delta} \cosh \Delta d \right)} \quad (216)$$

and

$$F_M(i\omega, p_{\parallel}) = \omega^2 \left(-\frac{\xi}{\epsilon(i\omega)\Delta} + \frac{\epsilon(i\omega)\Delta}{\xi} \right) \left[\frac{1}{\xi} \sinh \Delta d + \frac{\xi d}{\epsilon(i\omega)\Delta} \left(\cosh \Delta d + \left(\frac{\epsilon(i\omega)\Delta}{\xi} - \frac{1}{\Delta d} \right) \sinh \Delta d \right) \right] \quad (217)$$

which is real. Finally we obtain

$$\begin{aligned} \frac{E^-}{A} = & -\frac{1}{2\pi} \int d^2 p_{\parallel} \int_0^{\infty} d\omega \frac{\omega^2}{\Delta \left((1 + \frac{\xi^2}{\epsilon^2(i\omega)\Delta^2}) \sinh \Delta d + \frac{2\xi}{\epsilon(i\omega)\Delta} \cosh \Delta d \right)} \\ & \times \left(\frac{\xi}{\epsilon(i\omega)\Delta} - \frac{\epsilon(i\omega)\Delta}{\xi} \right) \left[\frac{1}{\xi} \sinh \Delta d + \frac{\xi d}{\epsilon(i\omega)\Delta} \left(\cosh \Delta d + \left(\frac{\epsilon(i\omega)\Delta}{\xi} - \frac{1}{\Delta d} \right) \sinh \Delta d \right) \right]. \end{aligned} \quad (218)$$

Let us take the large d limit corresponding to the infinite space case again. We have

$$\frac{E_{\infty}^-}{A} = -\frac{1}{2\pi} \int d^2 p_{\parallel} \int_0^{\infty} d\omega \frac{\omega^2}{(1 + \frac{\xi}{\epsilon(i\omega)\Delta})^2} \left(\frac{\xi}{\epsilon(i\omega)\Delta} - \frac{\epsilon(i\omega)\Delta}{\xi} \right) \left[\frac{1}{\xi} + \frac{\xi d}{\epsilon(i\omega)\Delta} \left(1 + \left(\frac{\Delta}{\xi} - \frac{1}{\Delta d} \right) \right) \right]. \quad (219)$$

We define the renormalized energy $\frac{E_R^-}{A} = \frac{E^-}{A} - \frac{E_{\infty}^-}{A}$. This turns out to be

$$\frac{E_R^-}{A} = -\frac{d}{\pi} \int d^2 p_{\parallel} \int_0^{\infty} d\omega \frac{\omega^2}{\Delta \left(\frac{1 + \frac{\xi}{\epsilon(i\omega)\Delta}}{1 - \frac{\xi}{\epsilon(i\omega)\Delta}} \right)^2 e^{2\Delta d} - 1} \left(1 - \frac{2p_{\parallel}^2}{\epsilon(i\omega)\omega^2} \frac{1}{\xi d} \right). \quad (220)$$

This can be explicitly evaluated in the ideal case.

3. The ideal case

The ideal case is easily retrieved by sending $\epsilon \rightarrow \infty$ and $\xi \rightarrow \infty$ implying that

$$\frac{E_R}{A} = -\frac{2d}{\pi} \int \not{d}^2 p_{\parallel} \int_0^{\infty} d\omega \frac{\omega^2}{\Delta} \frac{1}{e^{2\Delta d} - 1} \quad (221)$$

which can be written by parity as

$$\frac{E_R}{A} = -\frac{d}{\pi} \int \not{d}^2 p_{\parallel} \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{\Delta} \frac{1}{e^{2\Delta d} - 1}. \quad (222)$$

Notice that $\vec{\Delta} = (\omega, \vec{p}_{\parallel})$ is a 3-vector whose norm is Δ . Rotation invariance of the integral for the vector $\vec{\Delta}$ implies that

$$\begin{aligned} \frac{E_R}{A} &= -\frac{d}{3\pi(2\pi)^2} \int d^3 \Delta \frac{\Delta}{e^{2\Delta d} - 1} = -\frac{4d}{3(2\pi)^2} \int_0^{\infty} d\Delta \frac{\Delta^3}{e^{2\Delta d} - 1} \\ &= -\frac{1}{12(2\pi)^2 d^3} \Gamma(4) \zeta(4) \end{aligned} \quad (223)$$

or more concisely

$$\frac{E_R}{A} = -\frac{\pi^2}{720d^3} \quad (224)$$

as $\Gamma(4) = 6$ and $\zeta(4) = \frac{\pi^4}{90}$. The pressure is then given by taking the derivative with respect to d

$$P = -\frac{d(E_R/A)}{dd} = -\frac{\pi^2}{240d^4} \quad (225)$$

as expected.

In the nonideal case, the vacuum energy comprises the energy due to the fluctuations in the empty cavity and in matter. The fact that the reflection coefficients for waves coming from the cavity and leaving it into the plates are not

strictly equal to unity implies that the cavity as seen as a closed system of dimension d is in fact an open system where energy flows into the plate continually. As a result, the energy in the cavity and the pressure exerted by the quantum fluctuations on the interface between the cavity and matter are not simply related by one being the derivative of the other. We must calculate the pressure directly. This is done in the following section.

E. The vacuum pressure

We will evaluate directly the pressure acting on the plate at $z = d$. For this we calculate the vacuum expectation value of the T_{zz} part of the energy momentum tensor. This is given by

$$\begin{aligned} P_z &= \frac{1}{2} \sum_{\alpha=\pm} \langle 0 | (\partial_z a_{\alpha}^1 \partial_z \tilde{a}_{\alpha}^1 + \partial_0 a_{\alpha}^1 \hat{\epsilon} \star_t \partial_0 \tilde{a}_{\alpha}^1 \\ &\quad - \partial_x a_{\alpha}^1 \partial_x \tilde{a}_{\alpha}^1 - \partial_y a_{\alpha}^1 \partial_y \tilde{a}_{\alpha}^1) | 0 \rangle \end{aligned} \quad (226)$$

which can be conveniently written in terms of the electric and magnetic contributions

$$P_z^+ = \frac{i}{2} \lim_{z \rightarrow z_0} \int \not{d}\omega \not{d}^2 p_{\parallel} (\partial_z \partial_{z_0} - p_{\parallel}^2 + \omega^2) G_{EF}(z, \omega, p_{\parallel}; z_0) \quad (227)$$

and

$$P_z^- = \frac{i}{2} \lim_{z \rightarrow z_0} \int \not{d}\omega \not{d}^2 p_{\parallel} (\partial_z \partial_{z_0} - p_{\parallel}^2 + \omega^2) G_{MF}(z, \omega, p_{\parallel}; z_0) \quad (228)$$

evaluated at $z = z_0 = d$ after the appropriate renormalization. As we are interested in the pressure due to the vacuum on the plate, the electric contribution is given by

$$P_z(d)^+ = \frac{i}{2} \lim_{z \rightarrow d^-, z_0 \rightarrow d^-} \int \not{d}\omega \not{d}^2 p_{\parallel} (\partial_z \partial_{z_0} + \omega^2 - p_{\parallel}^2) G_{EF}(z, \omega, p_{\parallel}; z_0) \quad (229)$$

where the limit is taken with $z, z_0 \leq d$. A similar result holds for the magnetic pressure

$$P_z(d)^- = \frac{i}{2} \lim_{z \rightarrow d^-, z_0 \rightarrow d^-} \int \not{d}\omega \not{d}^2 p_{\parallel} (\partial_z \partial_{z_0} + \omega^2 - p_{\parallel}^2) G_{MF}(z, \omega, p_{\parallel}; z_0). \quad (230)$$

Let us evaluate these integrals.

1. The electric pressure

Let us start with the term involving the two derivatives in the electric case

$$\partial_z \partial_{z_0} G_E(z, \omega, p_{\parallel}; z_0)|_{z=z_0=d} = \Theta(\omega, p_{\parallel}) \xi \Delta \left(\sin \Delta d - \frac{\xi}{\Delta} \cos \Delta \right) \quad (231)$$

where G_E is the Fourier transform of the Green's function which can be then transformed into the Feynman propagator by the Schwarz reflection procedure. The same derivative expression evaluated as $z \rightarrow \infty$ gives

$$\partial_z \partial_{z_0} G_E(z, \omega, p_{\parallel}; z_0)|_{z=z_0 \rightarrow \infty} = \Theta(\omega, p_{\parallel}) \xi^2 \left(-\cos \Delta d + \frac{\sin \Delta d}{2} \left(\frac{\Delta}{\xi} - \frac{\xi}{\Delta} \right) \right). \quad (232)$$

Obviously this term corresponds to a contribution to the pressure exerted on the outer side of the plaque and we must subtract the two quantities to get

$$\partial_z \partial_{z_0} G_{ER}(z, \omega, p_{\parallel}; z_0) = \Theta(\omega, p_{\parallel}) \frac{\xi^2}{2} \left(\frac{\Delta}{\xi} + \frac{\xi}{\Delta} \right) \sin \Delta d, \quad (233)$$

where R stands for renormalized. Similarly the term in $(\omega^2 - p_{\parallel}^2) G_E(z, \omega, p_{\parallel}; z)$ must be understood as $(\omega^2 - p_{\parallel}^2) G_{ER}(z, \omega, p_{\parallel}; z)$ where G_R vanishes as $z \rightarrow \infty$, i.e. the pressure coming from the outer edge of the plaque has been removed. Here notice that we have taken the outer pressure as coming from vacuum, i.e. we have taken $\epsilon(\omega) = 1$ for the evaluation of the Green's functions. We have then

$$G_{ER}(d, \omega, p_{\parallel}; d) = \frac{\Theta_E(\omega, p_{\parallel})}{2} \left(\frac{\Delta}{\xi} + \frac{\xi}{\Delta} \right) \sin \Delta d. \quad (234)$$

As a result the contribution to the pressure can be written as

$$P_z^+ = \frac{i}{4} \int d\omega d^2 p_{\parallel} \Theta_E(\omega, p_{\parallel}) \left(\frac{\Delta}{\xi} + \frac{\xi}{\Delta} \right) (\xi^2 - p_{\parallel}^2 + \omega^2) \sin \Delta d. \quad (235)$$

This can be evaluated after Wick's rotation as

$$P_z^+ = \frac{1}{4} \int d\omega d^2 p_{\parallel} \frac{\xi^2 - \Delta^2}{\Delta} \left(\frac{\xi}{\Delta} - \frac{\Delta}{\xi} \right) \frac{\sinh \Delta d}{(1 + \frac{\xi^2}{\Delta^2}) \sinh \Delta d + \frac{2\xi}{\Delta} \cosh \Delta d}. \quad (236)$$

Again this result does not vanish when $d \rightarrow \infty$. This would correspond to the pressure in empty space time, which is taken to vanish as we assume that locally space time is the Minkowski one. After removing the pressure in empty space time when $d \rightarrow \infty$ we get

$$P_z^+ = - \int d\omega d^2 p_{\parallel} \frac{\Delta}{\left(\frac{1+\xi}{1-\xi} \right)^2 e^{2\Delta d} - 1}. \quad (237)$$

This is the Lifschitz's formula for the pressure coming from the TE modes with two plates of the same permittivity ϵ . Notice that this integral can be written as

$$P_z^+ = - \int d\omega d^2 p_{\parallel} \frac{\Delta}{r_{TE}^{-2} e^{2\Delta d} - 1}, \quad (238)$$

where r_{TE} is the reflection coefficient for the TE modes evaluated for imaginary frequencies $r_{TE}(i\omega) = \frac{1-\xi}{1+\xi}$ where $\xi = (p_{\parallel}^2 + \omega^2 \epsilon(i\omega))^{1/2}$ depends on the permittivity for imaginary frequencies.

2. The magnetic pressure

The two derivative term of the magnetic Green's function is

$$\partial_z \partial_{z_0} G_M(z, \omega, p_{\parallel}; z_0)|_{z=z_0=d} = \Theta_M(\omega, p_{\parallel}) \frac{\xi \Delta}{\epsilon} \left(\sin \Delta d - \frac{\xi}{\epsilon \Delta} \cos \Delta d \right). \quad (239)$$

The same derivative expression evaluated as $z \rightarrow \infty$ using the expressions in the plates gives

$$\partial_z \partial_{z_0} G_M(z, \omega, p_{\parallel}; z_0)|_{z=z_0 \rightarrow \infty} = \Theta_M(\omega, p_{\parallel}) \xi^2 \left(-\cos \Delta d + \frac{\sin \Delta d}{2} \left(\frac{\epsilon \Delta}{\xi} - \frac{\xi}{\epsilon \Delta} \right) \right). \quad (240)$$

Now we must remove the pressure exerted by the vacuum outside the plates as the size of the plates goes to infinity. This is obtained by subtracting $\epsilon^{-2} \partial_z \partial_{z_0} G_M(z, \omega, p_{\parallel}; z_0)|_{z=z_0 \rightarrow \infty}$. Indeed we know that for the TM polarization $\lim_{\delta \rightarrow 0} \partial_z G_M|_{L+\delta} = \lim_{\delta \rightarrow 0} \epsilon^{-1} \partial_z G_M|_{L-\delta}$ for a plate of size L and the derivatives are taken at the right boundary of the plates. The renormalized contribution to the pressure is then

$$\begin{aligned} \partial_z \partial_{z_0} G_{MR}(z, \omega, p_{\parallel}; z_0) \\ = \Theta_M(\omega, p_{\parallel}) \frac{\xi^2}{2\epsilon^2} \left(\frac{\epsilon\Delta}{\xi} + \frac{\xi}{\epsilon\Delta} \right) \sin \Delta d, \end{aligned} \quad (241)$$

where R stands for renormalized. Similarly the term in $(\omega^2 - p_{\parallel}^2)G_M(z, \omega, p_{\parallel}; z)$ must be understood as $(\omega^2 - p_{\parallel}^2)G_{MR}(z, \omega, p_{\parallel}; z)$ where G_{MR} vanishes as $z \rightarrow \infty$, i.e. the pressure coming from the outer edge of the plaque has been removed. Hence we have

$$G_{MR}(d, \omega, p_{\parallel}; d) = \frac{\Theta_M(\omega, p_{\parallel})}{2} \left(\frac{\epsilon\Delta}{\xi} + \frac{\xi}{\epsilon\Delta} \right) \sin \Delta d. \quad (242)$$

As a result the contribution to the pressure can be written as

$$\begin{aligned} P_z^- = \frac{i}{4} \int d\omega d^2 p_{\parallel} \Theta_M(\omega, p_{\parallel}) \\ \times \left(\frac{\epsilon\Delta}{\xi} + \frac{\xi}{\epsilon\Delta} \right) \left(\frac{\xi^2}{\epsilon^2} - p_{\parallel}^2 + \omega^2 \right) \sin \Delta d. \end{aligned} \quad (243)$$

This can be evaluated after Wick's rotation as

$$\begin{aligned} P_z^- = \frac{1}{4} \int d\omega d^2 p_{\parallel} \frac{\frac{\xi^2}{\epsilon^2(i\omega)} - \Delta^2}{\Delta} \left(\frac{\xi}{\epsilon(i\omega)\Delta} - \frac{\epsilon(i\omega)\Delta}{\xi} \right) \\ \times \frac{\sinh \Delta d}{(1 + \frac{\xi^2}{\epsilon(i\omega)\Delta^2}) \sinh \Delta d + \frac{2\xi}{\epsilon(i\omega)\Delta} \cosh \Delta d}. \end{aligned} \quad (244)$$

Again this result does not vanish when $d \rightarrow \infty$. This would correspond to the pressure in empty space time which is taken to vanish as we assume that locally space time is Minkowski AU: Please add a noun after "Minkowski." After removing the pressure in empty space time obtained when $d \rightarrow \infty$ we get

$$P_z^- = - \int d\omega d^2 p_{\parallel} \frac{\Delta}{\left(\frac{1 + \frac{\xi}{\epsilon(i\omega)\Delta}}{1 - \frac{\xi}{\epsilon(i\omega)\Delta}} \right)^2 e^{2\Delta d} - 1}. \quad (245)$$

This is the Lifschitz formula for the pressure coming from the TM modes with two plates with the same permittivity ϵ . Notice that the same integral can be written as

$$P_z^- = - \int d\omega d^2 p_{\parallel} \frac{\Delta}{r_{TM}^{-2} e^{2\Delta d} - 1}, \quad (246)$$

where r_{TM} is the reflection coefficient for the TM modes evaluated for imaginary frequencies.

3. The ideal case

In the ideal case where $r_{TE} = r_{TM} = 1$, the electric and magnetic contributions to the Casimir pressure are equal and the pressure reduces to

$$P_z = -2 \int d\omega d^2 p_{\parallel} \frac{\Delta}{e^{2\Delta d} - 1}, \quad (247)$$

which can be evaluated as

$$\begin{aligned} P_z &= - \frac{2 \times 4\pi}{16(2\pi)^3 d^4} \int_0^\infty dx \frac{x^3}{e^x - 1} \\ &= - \frac{2 \times 4\pi}{16(2\pi)^3 d^4} \Gamma(4) \zeta(4) \\ &= - \frac{\pi^2}{240 d^4}. \end{aligned} \quad (248)$$

This is the usual Casimir pressure in the ideal case.

Now that we have retrieved the classical Lifschitz theory in the Schwinger-Keldysh formalism, we can explore the effects of the axion coupling to photons on the Casimir pressure.

V. AXION CONTRIBUTION TO THE CASIMIR EFFECT

A. Perturbative expansion

The Casimir pressure is corrected by the axion interaction to photons as the two point correlation function $\langle a_\alpha(x) \tilde{a}_\alpha(y) \rangle$, where $\alpha = \pm$ corresponds to the two polarizations, is corrected. This gives rise to corrections to $G_E(z, \omega, p_{\parallel}; z_0)$ and $G_M(z, \omega, p_{\parallel}; z_0)$ which then induce a change in P_z . Let us start with the electric Green's function first. As we must take the average over the axion oscillations, the first correction to G_E reads at second order for the connected Green's function

$$\begin{aligned} i\delta G_E(x; x_0) \\ = - \frac{1}{2} \langle a_\gamma^1(x) \tilde{a}_\gamma^1(x_0) (S_{\text{int}}(a_\alpha^1, \tilde{a}_\alpha^1) - S_{\text{int}}^\dagger(a_\alpha^2, \tilde{a}_\alpha^2))^2 \rangle_c, \end{aligned} \quad (249)$$

as represented in Fig. 9. This can be calculated using Wick's theorem and contracting fields using the propagators. We are only interested in the corrections to the connected Green's function so we only keep contractions which do not contribute to disconnected pieces. This involves three contributions. Let us start with the first one

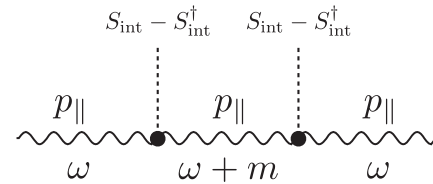


FIG. 9. Diagrammatic representation of the insertions in Eq. (249). Notice that eventually the transverse momentum is conserved at each interaction vertex whilst the axion insertion leads to a change of the energy by $\pm m$.

$$I = -\frac{1}{2} \langle a_\gamma^1(x) \tilde{a}_\gamma^1(x_0) S_{\text{int}}^2(a_\alpha^1, \tilde{a}_\alpha^1) \rangle_c, \quad (250)$$

or more explicitly

$$I = -\frac{1}{2} \int d^4x_1 d^4x_2 \frac{\phi(t_1)}{M} \frac{\phi(t_2)}{M} \langle a_\gamma^1(x) \tilde{a}_\gamma^1(x_0) (\partial_0 \tilde{a}_\alpha^1 P_{i\alpha\beta} \star_{x_1} \partial^i a_\beta^1 + \partial_0 a_\alpha^1 P_{i\alpha\beta} \star_{x_1} \partial^i \tilde{a}_\beta^1) \\ \times (\partial_0 \tilde{a}_\delta^1 P_{i\delta\kappa} \star_{x_2} \partial^i a_\kappa^1 + \partial_0 a_\delta^1 P_{i\delta\kappa} \star_{x_2} \partial^i \tilde{a}_\kappa^1) \rangle_c. \quad (251)$$

Notice that we have explicitly used that $\tilde{\phi} \equiv \phi$ as the cosine function is even. This can be evaluated using Wick's theorem and noticing that the coupling $P_{i\alpha\beta}$ mixes the polarizations, i.e. $\alpha \neq \beta$. In the following, we denote by $\bar{\alpha}$ the complementary polarization to α , i.e. $\bar{+} \equiv -, \bar{-} \equiv +$. We also denote

$$\langle a_+^1(x) \tilde{a}_+^1(y) \rangle = \Delta_+(x; y), \quad \langle a_-^1(x) \tilde{a}_-^1(y) \rangle = \Delta_-(x; y), \quad (252)$$

the latter being related to the magnetic Green's functions as

$$\Delta_-(x; y) = \epsilon^{-1} \star_{t_x} \Delta_{MF}(x; y) \quad (253)$$

where

$$\Delta_+(x; y) = iG_{MF}(x; y), \quad \Delta_M(x; y) = iG_{MF}(x; y). \quad (254)$$

Expanding the result we get

$$I = -\frac{1}{2} \int d^4x_1 d^4x_2 \frac{\phi(t_1)}{M} \frac{\phi(t_2)}{M} [\partial_{t_1} \Delta_\gamma(x; x_1) P_{i\gamma\bar{\gamma}}(x_1) \star_{x_1} P_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{x_1}^i \partial_{t_2} \Delta_{\bar{\gamma}}(x_1; x_2) \partial_{x_2}^j \tilde{\Delta}_\gamma(x_0; x_2) \\ + \partial_{t_1} \Delta_\gamma(x; x_1) P_{i\gamma\bar{\gamma}}(x_1) \star_{x_1} P_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{x_1}^i \partial_{x_2}^j \Delta_{\bar{\gamma}}(x_1; x_2) \partial_{t_2} \tilde{\Delta}_\gamma(x_0; x_2) \\ + P_{i\bar{\gamma}\gamma}(x_1) \star_{x_1} \partial_{x_1}^i \Delta_\gamma(x; x_1) P_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{t_1} \partial_{t_2} \Delta_{\bar{\gamma}}(x_1; x_2) \partial_{x_2}^j \tilde{\Delta}_\gamma(x_0; x_2) \\ + P_{i\bar{\gamma}\gamma}(x_1) \star_{x_1} \partial_{x_1}^i \Delta_\gamma(x; x_1) P_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{t_1} \partial_{x_2}^j \Delta_{\bar{\gamma}}(x_1; x_2) \partial_{t_2} \tilde{\Delta}_\gamma(x_0; x_2)]. \quad (255)$$

The other correlation functions involve the propagators such as Δ^{12} . The second term is

$$II = -\frac{1}{2} \int d^4x_1 d^4x_2 \frac{\phi(t_1)}{M} \frac{\phi(t_2)}{M} \langle a_\gamma^1(x) \tilde{a}_\gamma^1(x_0) (\partial_0 \tilde{a}_\alpha^2 \bar{P}_{i\alpha\beta} \star_{x_1} \partial^i a_\beta^2 + \partial_0 a_\alpha^2 \bar{P}_{i\alpha\beta} \star_{x_1} \partial^i \tilde{a}_\beta^2) \\ \times (\partial_0 \tilde{a}_\delta^2 \bar{P}_{i\delta\kappa} \star_{x_2} \partial^i a_\kappa^2 + \partial_0 a_\delta^2 \bar{P}_{i\delta\kappa} \star_{x_2} \partial^i \tilde{a}_\kappa^2) \rangle_c. \quad (256)$$

This is given explicitly by

$$II = -\frac{1}{2} \int d^4x_1 d^4x_2 \frac{\phi(t_1)}{M} \frac{\phi(t_2)}{M} [\partial_{t_1} \Delta_\gamma^{12}(x; x_1) \bar{P}_{i\gamma\bar{\gamma}}(x_1) \star_{x_1} \bar{P}_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{x_1}^i \partial_{t_2} \Delta_{\bar{\gamma}}^{22}(x_1; x_2) \partial_{x_2}^j \tilde{\Delta}_\gamma^{12}(x_0; x_2) \\ + \partial_{t_1} \Delta_\gamma^{12}(x; x_1) \bar{P}_{i\gamma\bar{\gamma}}(x_1) \star_{x_1} \bar{P}_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{x_1}^i \partial_{x_2}^j \Delta_{\bar{\gamma}}^{22}(x_1; x_2) \partial_{t_2} \tilde{\Delta}_\gamma^{12}(x_0; x_2) \\ + \bar{P}_{i\bar{\gamma}\gamma}(x_1) \star_{x_1} \partial_{x_1}^i \Delta_\gamma^{12}(x; x_1) \bar{P}_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{t_1} \partial_{t_2} \Delta_{\bar{\gamma}}^{22}(x_1; x_2) \partial_{x_2}^j \tilde{\Delta}_\gamma^{12}(x_0; x_2) \\ + \bar{P}_{i\bar{\gamma}\gamma}(x_1) \star_{x_1} \partial_{x_1}^i \Delta_\gamma^{12}(x; x_1) \bar{P}_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{t_1} \partial_{x_2}^j \Delta_{\bar{\gamma}}^{22}(x_1; x_2) \partial_{t_2} \tilde{\Delta}_\gamma^{12}(x_0; x_2)]. \quad (257)$$

The third term is obtained from

$$\begin{aligned}
 III_a = \frac{1}{2} \int d^4x_1 d^4x_2 \frac{\phi(t_1)}{M} \frac{\phi(t_2)}{M} \langle a_\gamma^1(x) \tilde{a}_\gamma^1(x_0) (\partial_0 \tilde{a}_\alpha^1 P_{i\alpha\beta} \star_{x_1} \partial^i a_\beta^1 + \partial_0 a_\alpha^1 P_{i\alpha\beta} \star_{x_1} \partial^i \tilde{a}_\beta^1) \\
 \times (\partial_0 \tilde{a}_\delta^2 \bar{P}_{i\delta\kappa} \star_{x_2} \partial^i a_\kappa^2 + \partial_0 a_\delta^2 \bar{P}_{i\delta\kappa} \star_{x_2} \partial^i \tilde{a}_\kappa^2) \rangle_c
 \end{aligned} \quad (258)$$

and the one where the two brackets are interverted. This gives rise to eight terms in two separate contributions

$$\begin{aligned}
 III_a = \frac{1}{2} \int d^4x_1 d^4x_2 \frac{\phi(t_1)}{M} \frac{\phi(t_2)}{M} [\partial_{t_1} \Delta_\gamma^{11}(x; x_1) P_{i\gamma\bar{\gamma}}(x_1) \star_{x_1} \bar{P}_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{x_1}^i \partial_{t_2} \Delta_\gamma^{12}(x_1; x_2) \partial_{x_2}^j \tilde{\Delta}_\gamma^{12}(x_0; x_2) \\
 + \partial_{t_1} \Delta_\gamma^{11}(x; x_1) P_{i\gamma\bar{\gamma}}(x_1) \star_{x_1} \bar{P}_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{x_1}^i \partial_{x_2}^j \Delta_\gamma^{12}(x_1; x_2) \partial_{t_2} \tilde{\Delta}_\gamma^{12}(x_0; x_2) \\
 + P_{i\gamma\bar{\gamma}}(x_1) \star_{x_1} \partial_{x_1}^i \Delta_\gamma^{11}(x; x_1) \bar{P}_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{t_1} \partial_{t_2} \Delta_\gamma^{12}(x_1; x_2) \partial_{x_2}^j \tilde{\Delta}_\gamma^{12}(x_0; x_2) \\
 + P_{i\gamma\bar{\gamma}}(x_1) \star_{x_1} \partial_{x_1}^i \Delta_\gamma^{11}(x; x_1) \bar{P}_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{t_1} \partial_{x_2}^j \Delta_\gamma^{12}(x_1; x_2) \partial_{t_2} \tilde{\Delta}_\gamma^{12}(x_0; x_2)]
 \end{aligned} \quad (259)$$

and

$$\begin{aligned}
 III_b = \frac{1}{2} \int d^4x_1 d^4x_2 \frac{\phi(t_1)}{M} \frac{\phi(t_2)}{M} [\partial_{t_1} \Delta_\gamma^{12}(x; x_1) \bar{P}_{i\gamma\bar{\gamma}}(x_1) \star_{x_1} P_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{x_1}^i \partial_{t_2} \Delta_\gamma^{21}(x_1; x_2) \partial_{x_2}^j \tilde{\Delta}_\gamma^{11}(x_0; x_2) \\
 + \partial_{t_1} \Delta_\gamma^{12}(x; x_1) \bar{P}_{i\gamma\bar{\gamma}}(x_1) \star_{x_1} P_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{x_1}^i \partial_{x_2}^j \Delta_\gamma^{21}(x_1; x_2) \partial_{t_2} \tilde{\Delta}_\gamma^{11}(x_0; x_2), \\
 + \bar{P}_{i\gamma\bar{\gamma}}(x_1) \star_{x_1} \partial_{x_1}^i \Delta_\gamma^{12}(x; x_1) P_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{t_1} \partial_{t_2} \Delta_\gamma^{21}(x_1; x_2) \partial_{x_2}^j \tilde{\Delta}_\gamma^{11}(x_0; x_2) \\
 + \bar{P}_{i\gamma\bar{\gamma}}(x_1) \star_{x_1} \partial_{x_1}^i \Delta_\gamma^{12}(x; x_1) P_{j\bar{\gamma}\gamma}(x_2) \star_{x_2} \partial_{t_1} \partial_{x_2}^j \Delta_\gamma^{21}(x_1; x_2) \partial_{t_2} \tilde{\Delta}_\gamma^{11}(x_0; x_2)].
 \end{aligned} \quad (260)$$

All these contributions give rise to Feynman integrals corresponding to tree-level Feynman diagrams where two vertices containing one axion insertion each are drawn and joined by lines corresponding to the propagators linked to the external points; see Fig. 9.

B. Feynman integrals

All the Feynman integrals are constructed in the same fashion. Let us give an explicit example using the first term of I that we denote by I_1 . First of all writing $\phi(t) = \frac{\phi_0(e^{imt} + e^{-imt})}{2}$, the source term contains terms in $\frac{\phi_0^2}{4} e^{i\epsilon_1 m t_1 + i\epsilon_2 m t_2}$ where $\epsilon_{1,2} = \pm$. As we take the average over $t = t_0$ to obtain the correction to the Casimir pressure, the end result is nonvanishing for contributions with $\epsilon_2 = -\epsilon_1$ only. If this is not the case then the diagram is associated to an oscillating integral whose average vanishes.

There are two contributions corresponding to the two possibilities for $\epsilon_1 = \pm$. The first one is given by

$$\begin{aligned}
 I_1 \supset -\frac{\phi_0^2}{8M^2} \int d^2p_\parallel d\omega dz_1 dz_2 e^{ip_\parallel \cdot (x-x_0)_\parallel - i\omega(t-t_0)} \omega(\omega + m) \\
 \times \Delta_\gamma(z; z_1, \omega, p_\parallel) \hat{\Delta}_\gamma(z_1; z_2, \omega + m, p_\parallel) \\
 \times \check{\Delta}_\gamma(z_0; z_2; \omega, p_\parallel).
 \end{aligned} \quad (261)$$

We have used the relation $k^i P_{i\gamma\bar{\gamma}}(\vec{k}) = |\vec{k}|$ as the triple $(\vec{k}, \vec{e}_\gamma, \vec{e}_{\bar{\gamma}})$ is of volume $|\vec{k}|$. Notice that reversing the order of the vectors changes the overall sign leading to the changes of signs in the second and third contributions I_2 and I_3 . We have introduced the modified propagator

$$\check{\Delta}_\gamma(z_1, z_2; \omega, p_\parallel) = \int d^2p_z e^{ip_z z_1} \sqrt{p_z^2 + p_\parallel^2} \Delta_\gamma(z_2; \omega, p_z, p_\parallel) \quad (262)$$

where we have Fourier-transformed the propagator with respect to the first argument only

$$\Delta_\gamma(z_2; \omega, p_z, p_\parallel) = \int dz_1 e^{-ip_z z_1} \Delta_\gamma(z_1, z_2; \omega, p_\gamma). \quad (263)$$

Similarly we define $\check{\check{\Delta}}_\gamma$ by first taking the Fourier transform of $\tilde{\Delta}$ with respect to the second argument and then multiplying by $\sqrt{p_z^2 + p_\parallel^2}$ before taking the inverse Fourier transform. The other contribution to I_1 is simply obtained by changing $m \rightarrow -m$. This corresponds to a contribution to the Fourier transform of the Green's function

$$i\delta G_{I_1}(z; z_0, \omega, p_{\parallel}) = \frac{\phi_0^2}{4M^2} \int dz_1 dz_2 \Delta_{\gamma}(z; z_1, \omega, p_{\parallel}) \check{\check{\Delta}}_{\gamma}(z_0; z_2; \omega, p_{\parallel}) \omega \\ \times [(\omega - m) \check{\check{\Delta}}_{\bar{\gamma}}(z_1; z_2, \omega - m, p_{\parallel}) + (\omega + m) \check{\check{\Delta}}_{\bar{\gamma}}(z_1; z_2, \omega + m, p_{\parallel})]. \quad (264)$$

The second contribution in the I list is given by

$$i\delta G_{I_2}(z; z_0, \omega, p_{\parallel}) = -\frac{\phi_0^2}{8M^2} \int dz_1 dz_2 \Delta_{\gamma}(z; z_1, \omega, p_{\parallel}) \check{\check{\Delta}}_{\gamma}(z_0; z_2; \omega, p_{\parallel}) \omega^2 \\ \times [\check{\check{\Delta}}_{\bar{\gamma}}(z_1; z_2, \omega - m, p_{\parallel}) + \check{\check{\Delta}}_{\bar{\gamma}}(z_1; z_2, \omega + m, p_{\parallel})]. \quad (265)$$

The third contribution in the I list is given by

$$i\delta G_{I_3}(z; z_0, \omega, p_{\parallel}) \supset -\frac{\phi_0^2}{8M^2} \int dz_1 dz_2 \Delta_{\gamma}(z; z_1, \omega, p_{\parallel}) \check{\check{\Delta}}_{\gamma}(z_0; z_2; \omega, p_{\parallel}) \\ \times [(\omega - m)^2 \check{\check{\Delta}}_{\bar{\gamma}}(z_1; z_2, \omega - m, p_{\parallel}) + (\omega + m)^2 \check{\check{\Delta}}_{\bar{\gamma}}(z_1; z_2, \omega + m, p_{\parallel})]. \quad (266)$$

Finally the last contribution reads

$$i\delta G_{I_4}(z; z_0, \omega, p_{\parallel}) = i\delta G_{I_1}(z; z_0, \omega, p_{\parallel}). \quad (267)$$

We can now collect the four terms and obtain the simplified result. This gives

$$i\delta G_I(z; z_0, \omega, p_{\parallel}) = -\frac{m^2 \phi_0^2}{8M^2} \int dz_1 dz_2 \Delta_{\gamma}(z; z_1, \omega, p_{\parallel}) \check{\check{\Delta}}_{\gamma}(z_0; z_2; \omega, p_{\parallel}) \\ \times [\check{\check{\Delta}}_{\bar{\gamma}}(z_1; z_2, \omega - m, p_{\parallel}) + \check{\check{\Delta}}_{\bar{\gamma}}(z_1; z_2, \omega + m, p_{\parallel})]. \quad (268)$$

We can comment on this result. First of all, this can be seen as the Feynman integral for the propagator of each polarization where two insertions of the interaction vertex are added. As this is a tree level Feynman diagram, momentum is conserved and the three propagators have the same transverse momentum $p_{\parallel}$. At each vertex the pulsation ω is shifted by either m or $-m$, see Fig. 9. The

time derivatives at the vertex result in factors of m corresponding to a vertex in $\check{\phi}$. In fact we know that the correction must vanish when m vanishes as the term $F\check{F}$ in the Lagrangian is a total derivative. In this correction to the propagator, only the propagator $\Delta \equiv \Delta^{11}$ is involved. We can now read off the other contributions

$$i\delta G_{II}(z; z_0, \omega, p_{\parallel}) = -\frac{m^2 \phi_0^2}{8M^2} \int dz_1 dz_2 \Delta_{\gamma}^{12}(z; z_1, \omega, p_{\parallel}) \check{\check{\Delta}}_{\gamma}^{12}(z_0; z_2; \omega, p_{\parallel}) \\ \times [\check{\check{\Delta}}_{\bar{\gamma}}^{22}(z_1; z_2, \omega - m, p_{\parallel}) + \check{\check{\Delta}}_{\bar{\gamma}}^{22}(z_1; z_2, \omega + m, p_{\parallel})]. \quad (269)$$

The third contribution gives

$$i\delta G_{III_a}(z; z_0, \omega, p_{\parallel}) = \frac{m^2 \phi_0^2}{8M^2} \int dz_1 dz_2 \Delta_{\gamma}^{11}(z; z_1, \omega, p_{\parallel}) \check{\check{\Delta}}_{\gamma}^{12}(z_0; z_2; \omega, p_{\parallel}) \\ \times [\check{\check{\Delta}}_{\bar{\gamma}}^{12}(z_1; z_2, \omega - m, p_{\parallel}) + \check{\check{\Delta}}_{\bar{\gamma}}^{12}(z_1; z_2, \omega + m, p_{\parallel})], \quad (270)$$

and the fourth one

$$i\delta G_{III_b}(z; z_0, \omega, p_{\parallel}) = \frac{m^2 \phi_0^2}{8M^2} \int dz_1 dz_2 \Delta_{\gamma}^{12}(z; z_1, \omega, p_{\parallel}) \check{\check{\Delta}}_{\gamma}^{11}(z_0; z_2; \omega, p_{\parallel}) \\ \times [\check{\check{\Delta}}_{\bar{\gamma}}^{21}(z_1; z_2, \omega - m, p_{\parallel}) + \check{\check{\Delta}}_{\bar{\gamma}}^{21}(z_1; z_2, \omega + m, p_{\parallel})]. \quad (271)$$

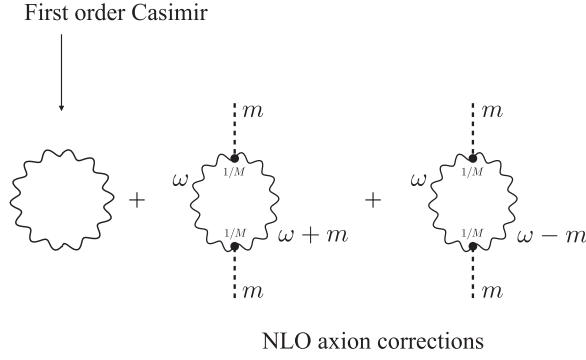


FIG. 10. Feynman diagrams corresponding to the quantum Casimir effect and the leading-order axion corrections. The first order Casimir effect corresponds to the Lifschitz theory.

The structure of these corrections are similar to the ones found in [13] here extended to the Schwinger-Keldysh case relevant to the dissipative case.

We can now concentrate on the corrections to the Green's function at coinciding points as illustrated in Fig. 10. In principle all the integrals are given above and could be evaluated numerically after an appropriate renormalization when necessary. In the following, we will focus on the case $\gamma \rightarrow 0$ as most of the calculations can be performed analytically and they illustrate the physics that one expects from the axion contribution to the Casimir pressure.

C. The vanishing dissipation limit $\gamma \rightarrow 0$

Let us recall some of the properties of the propagators before discussing the different contributions to the axionic Casimir pressure. We will restrict ourselves to the weak dissipation limit $\gamma \rightarrow 0$ as this simplifies the calculation and as for the ideal situation in the electromagnetic Casimir case, this provides a useful estimate of the effect without too many calculational complications.

For the diagonal propagators involving the first part of the Keldysh path we can use the decomposition defined in Sec. II C

$$\Delta_\gamma^{11}(z; z_0, \omega, p_\parallel) = i\Theta_\gamma(\omega, p_\parallel)H_\gamma(z; z_0, \omega, p_\parallel) \quad (272)$$

when $\Re(\omega) > 0$ and we replace $\omega \rightarrow -\omega$ when $\Re(\omega) < 0$. This corresponds to the definition of the Feynman propagator. Similarly we have

$$\Delta_\gamma^{21}(z; z_0, \omega, p_\parallel) \supset iY(\Re(\omega))\check{\theta}(\Theta_\gamma^{-1}(\omega, p_\parallel))H_\gamma(z; z_0, \omega, p_\parallel) \quad (273)$$

corresponding to the positive energy resonances, i.e. the resonances whose real parts are positive. This is complemented by a contribution coming from the continuum with pulsations above ω_{pl} . A similar result holds for Δ_γ^{12} with the negative frequency resonances. In the Feynman integrals, we also need the modified propagators $\check{\Delta}_\gamma$. They involve the modified Fourier transform $\check{H}$ which is defined like $\check{\Delta}_\gamma$. Finally, we notice that the Green's functions $\check{\Delta}$ exchange the positive and negative frequency parts with the ones of Δ . Using this, we see that δG_{II} vanishes as it is only nonvanishing when $\Re(\omega) = 0$ and this does not contribute to the pressure which involves an integral over ω . Similarly, the contribution δG_{III_b} is only nonvanishing when $\Re(\omega) \in [-m, 0]$ and δG_{III_b} when $\Re(\omega) \in [0, m]$.

In the following, we will only be interested in axion masses $m \lesssim \omega_{\text{pl}}$ corresponding to resonances for distances larger than the London length. In this regime, we see that the δG_{III} channels do not receive contributions from the continuum. Only the resonances can contribute. Collecting everything we have therefore

$$\begin{aligned} \delta G_I(z; z_0, \omega, p_\parallel) \supset & \frac{m^2 \phi_0^2}{8M^2} \Theta_\gamma(\omega, p_\parallel) \check{\Theta}_\gamma(\omega, p_\parallel) \int dz_1 dz_2 H_\gamma(z; z_1, \omega, p_\parallel) \check{H}_\gamma(z_0; z_2; \omega, p_\parallel) \\ & \times [\Theta_\gamma(\omega - m, p_\parallel) \check{H}_\gamma(z_1; z_2, \omega - m, p_\parallel) + \Theta_\gamma(\omega + m, p_\parallel) \check{H}_\gamma(z_1; z_2, \omega + m, p_\parallel)]. \end{aligned} \quad (274)$$

The third contributions give

$$\begin{aligned} \delta G_{III_a}(z; z_0, \omega, p_\parallel) \supset & -\frac{m^2 \phi_0^2}{8M^2} Y(-\Re(\omega)) Y(\Re(\omega) + m) \Theta_\gamma(\omega, p_\parallel) \check{\theta}(\Theta_\gamma^{-1}(\omega + m, p_\parallel)) \check{\theta}(\check{\Theta}_\gamma^{-1}(\omega, p_\parallel)) \\ & \times \int dz_1 dz_2 H_\gamma(z; z_1, \omega, p_\parallel) \check{H}_\gamma(z_2; z_0; \omega, p_\parallel) \check{H}_\gamma(z_1; z_2, \omega + m, p_\parallel), \end{aligned} \quad (275)$$

and similarly

$$\begin{aligned} \delta G_{III_b}(z; z_0, \omega, p_\parallel) \supset & -\frac{m^2 \phi_0^2}{8M^2} Y(\Re(\omega)) Y(m - \Re(\omega)) \Theta_\gamma(\omega, p_\parallel) \check{\theta}(\Theta_\gamma^{-1}(\omega - m, p_\parallel)) \check{\theta}(\check{\Theta}_\gamma^{-1}(\omega, p_\parallel)) \\ & \times \int dz_1 dz_2 H_\gamma(z; z_1, \omega, p_\parallel) \check{H}_\gamma(z_2; z_0; \omega, p_\parallel) \check{H}_\gamma(z_2; z_1, \omega - m, p_\parallel). \end{aligned} \quad (276)$$

As can be seen, the last two terms are only nonvanishing when two conditions are satisfied, i.e. either

$$\Theta_{\bar{\gamma}}^{-1}(\omega + m, p_{\parallel}) = 0, \quad \tilde{\Theta}_{\bar{\gamma}}^{-1}(\omega, p_{\parallel}) = 0 \quad (277)$$

or

$$\Theta_{\bar{\gamma}}^{-1}(\omega - m, p_{\parallel}) = 0, \quad \tilde{\Theta}_{\bar{\gamma}}^{-1}(\omega, p_{\parallel}) = 0. \quad (278)$$

These equations must have solutions for either $\Re(\omega) \in [-m, 0]$ or $\Re(\omega) \in [0, m]$. When solutions exist, they would be in a finite number. The resonances are given by (35) with an infinitesimal γ . The condition (277) must have solutions with the same imaginary parts. This implies that the integer numbers labeling the resonances must be equal. This is not compatible with the real parts being equal implying that

there is no resonance when the system is weakly dissipative. This implies that

$$\delta G(z; z_0, \omega, p_{\parallel}) = \delta G_I(z; z_0, \omega, p_{\parallel}) \quad (279)$$

which can be explicitly calculated analytically.

D. The axionic pressure in the ideal case

We will simplify the calculation of the correction to the Green's function by approximating the propagators by their ideal expressions. This is a good approximation when the penetration length is much smaller than the distance between the plates in such a way that one can neglect the penetration of the modes into the plaques. In this case the two Green's functions reduce to

$$G_E(z; z_0, \omega, p_{\parallel}) = \frac{1}{\Delta} \frac{\sin \Delta(z_0 - d) \sin \Delta z}{\sin \Delta d}, \quad G_M(z; z_0, \omega, p_{\parallel}) = \frac{1}{\Delta} \frac{\cos \Delta(z_0 - d) \cos \Delta z}{\sin \Delta d} \quad (280)$$

and vanish outside the interval $[0, d]$. As a result the quantum correction to the Green's functions becomes

$$\begin{aligned} \delta G_E(z; z_0, \omega, p_{\parallel}) &= \frac{m^2 \phi_0^2}{8M^2} \int dz_1 dz_2 G_E(z; z_1, \omega, p_{\parallel}) \check{G}_E(z_2; z_0; \omega, p_{\parallel}) \\ &\quad \times [\check{G}_E(z_1; z_2, \omega - m, p_{\parallel}) + \check{G}_E(z_1; z_2, \omega + m, p_{\parallel})] \end{aligned} \quad (281)$$

for the electric case. We have used $\tilde{G}(x; y) = G(y, x)$. The correction to the magnetic Green's function is given by

$$\begin{aligned} \delta G_M(z; z_0, \omega, p_{\parallel}) &= \frac{m^2 \phi_0^2}{8M^2} \int dz_1 dz_2 G_M(z; z_1, \omega, p_{\parallel}) \check{G}_M(z_2; z_0; \omega, p_{\parallel}) \\ &\quad \times [\check{G}_M(z_1; z_2, \omega - m, p_{\parallel}) + \check{G}_M(z_1; z_2, \omega + m, p_{\parallel})]. \end{aligned} \quad (282)$$

An easy calculation gives that for the modified propagators,

$$\check{G}_E(z; z_0, \omega, p_{\parallel}) = \omega G_E(z; z_0, \omega, p_{\parallel}); \quad \check{G}_M(z; z_0, \omega, p_{\parallel}) = \omega G_M(z; z_0, \omega, p_{\parallel}). \quad (283)$$

The explicit evaluation of the integrals gives for the corrections

$$\begin{aligned} \delta G_E(z; z_0, \omega, p_{\parallel}) &= -\frac{m^2 \omega \phi_0^2 \sin \Delta z \sin \Delta(z_0 - d)}{2M^2 \sin^2 \Delta d} \\ &\quad \times \left[\frac{(\cos \Delta d - \cos \Delta^- d)^2}{(\Delta^2 - (\Delta^-)^2)^2} \frac{\omega - m}{\Delta^- \sin \Delta^- d} + \frac{(\cos \Delta d - \cos \Delta^+ d)^2}{(\Delta^2 - (\Delta^+)^2)^2} \frac{\omega + m}{\Delta^+ \sin \Delta^+ d} \right] \end{aligned} \quad (284)$$

where $\Delta^{\pm}$ are associated to $\omega \pm m$. For the magnetic Green's function we have

$$\begin{aligned} \delta G_M(z; z_0, \omega, p_{\parallel}) &= -\frac{m^2 \omega \phi_0^2 \cos \Delta z \cos \Delta(z_0 - d)}{2M^2 \Delta^2 \sin^2 \Delta d} \\ &\quad \times \left[\frac{(\cos \Delta d - \cos \Delta^- d)^2}{(\Delta^2 - (\Delta^-)^2)^2} \frac{\Delta_- (\omega - m)}{\sin \Delta^- d} + \frac{(\cos \Delta d - \cos \Delta^+ d)^2}{(\Delta^2 - (\Delta^+)^2)^2} \frac{\Delta_+ (\omega + m)}{\sin \Delta^+ d} \right]. \end{aligned} \quad (285)$$

Notice that the Green's functions are simply corrected by momentum-dependent form factors. Using these expressions we can get the pressure after summing over the two polarizations. Let us first consider the correction to the electric pressure

$$\begin{aligned} \delta P_E(d) = & -i \frac{m^2 \phi_0^2}{4M^2} \int \not{d}\omega \not{d}^2 p_{\parallel} \frac{\omega \Delta^2 \cos \Delta d}{\sin^2 \Delta d} \\ & \times \left[\frac{(\cos \Delta d - \cos \Delta^- d)^2}{(\Delta^2 - (\Delta^-)^2)^2} \frac{\omega - m}{\Delta^- \sin \Delta^- d} + \frac{(\cos \Delta d - \cos \Delta^+ d)^2}{(\Delta^2 - (\Delta^+)^2)^2} \frac{\omega + m}{\Delta^+ \sin \Delta^+ d} \right]. \end{aligned} \quad (286)$$

This can be evaluated after Wick's rotation $\omega \rightarrow i\omega$ where we have $\Delta^{\pm} = \sqrt{(\omega \mp im)^2 + p_{\parallel}^2}$ leading to

$$\delta P_E(d) = \frac{m^2 \phi_0^2}{2M^2} \Re \left(\int \not{d}\omega \not{d}^2 p_{\parallel} \omega (\omega + im) \frac{\Delta^2}{\Delta^- (\Delta^2 - (\Delta^-)^2)^2} \frac{\cosh \Delta d (\cosh \Delta d - \cosh \Delta^- d)^2}{\sinh^2 \Delta d \sinh \Delta^- d} \right). \quad (287)$$

This expression can be simplified as

$$\delta P_E = \frac{\phi_0^2}{2M^2} \frac{1}{d^4} f_E(md), \quad (288)$$

where the form factor is given by the integral

$$f_E(md) = \Re \left(\int \not{d}x \not{d}^2 \vec{y} x (x + imd) \frac{\hat{\Delta}^2}{\hat{\Delta}^- (2x + imd)^2} \frac{\cosh \hat{\Delta} (\cosh \hat{\Delta} - \cosh \hat{\Delta}^-)^2}{\sinh^2 \hat{\Delta} \sinh \hat{\Delta}^-} \right), \quad (289)$$

where we have defined $x = \omega d$ and $\vec{y} = \vec{p}_{\parallel} d$ and obtain

$$\hat{\Delta} = (x^2 + \vec{y}^2)^{1/2}, \quad \hat{\Delta}^- = ((x + imd)^2 + \vec{y}^2)^{1/2}. \quad (290)$$

The same method yields for the magnetic contribution

$$\delta P_M(d) = \frac{m^2 \phi_0^2}{2M^2} \Re \left(\int \not{d}\omega \not{d}^2 \vec{p}_{\parallel} \omega (\omega + im) \frac{\Delta_-}{(\Delta^2 - (\Delta^-)^2)^2} \frac{\cosh \Delta d (\cosh \Delta d - \cosh \Delta^- d)^2}{\sinh^2 \Delta d \sinh \Delta^- d} \right), \quad (291)$$

corresponding to

$$\delta P_M = \frac{\phi_0^2}{2M^2} \frac{1}{d^4} f_M(md), \quad (292)$$

where the form factor is given by the integral

$$f_M(md) = \Re \left(\int \not{d}x \not{d}^2 \vec{y} x (x + imd) \frac{\hat{\Delta}_-}{(2x + imd)^2} \frac{\cosh \hat{\Delta} (\cosh \hat{\Delta} - \cosh \hat{\Delta}^-)^2}{\sinh^2 \hat{\Delta} \sinh \hat{\Delta}^-} \right). \quad (293)$$

Notice that the complete form factor $f_E + f_M$ vanishes when $m \rightarrow 0$ corresponding to the vanishing of the effect of the axion in the static limit. As expected the correction to the pressure has a dependence in $1/d^4$ and corrections in md .

E. The axionic pressure

The form factors $f_{E,M}$ are ultraviolet (UV) sensitive and dominated by the largest momenta and energies. In this case and in the limit $md \ll \omega_{\text{pl}} d$, we find that the electric form factor behaves like

$$f_E \simeq \frac{1}{4} \int \not{d}^3 \hat{\Delta} \frac{\hat{\Delta}}{\tanh^3 \hat{\Delta}} (1 - e^{imdx/\hat{\Delta}})^2, \quad (294)$$

where we have defined the vector $\vec{\hat{\Delta}} = (x, \vec{y})$. This integral is infrared convergent but diverges quartically in the UV where the cutoff scale is of the order of $\omega_{\text{pl}} d$ as no pressure is exerted at higher frequencies as the metal becomes transparent. Using this explicit cutoff, the form factor behaves like

$$f_E \simeq \frac{\pi}{8} (\omega_{\text{Pl}} d)^4 \left(2 - \frac{4}{md} \sin md + \frac{1}{md} \sin 2md \right). \quad (295)$$

This gives a contribution to the pressure in

$$\delta P_E \simeq \frac{\omega_{\text{Pl}}^4}{128\pi^2} \frac{\phi_0^2}{M^2} \left(2 - \frac{4}{md} \sin md + \frac{1}{md} \sin 2md \right). \quad (296)$$

This leads to a constant pressure in the limit $d \rightarrow \infty$ which must be subtracted as before. Doing so, the renormalized form factor becomes

$$f_E \simeq \frac{1}{4} \int d^3 \hat{\Delta} \hat{\Delta} \left(\frac{1}{\tanh^3 \hat{\Delta}} - 1 \right) (1 - e^{imdx/\hat{\Delta}})^2 \simeq \frac{3}{2} \int d^3 \hat{\Delta} \frac{\hat{\Delta}}{e^{2\hat{\Delta}} - 1} (1 - e^{imdx/\hat{\Delta}})^2. \quad (297)$$

This gives explicitly

$$f_E \simeq 3 \left(1 - \frac{2}{md} \sin md + \frac{1}{2md} \sin 2md \right) \int d^3 \hat{\Delta} \frac{\hat{\Delta}}{e^{2\hat{\Delta}} - 1}. \quad (298)$$

The last factor reconstructs the TE Casimir pressure in the ideal case. As a result we have

$$\delta P_E \simeq -\frac{3}{2} \left(1 - \frac{2}{md} \sin md + \frac{1}{2md} \sin 2md \right) \frac{\phi_0^2}{M^2} P_E. \quad (299)$$

There is also the magnetic contribution implying that the correction to the Casimir pressure due to the axion is given by

$$\frac{\delta P}{P} \simeq -\frac{3}{2} \left(1 - \frac{2}{md} \sin md + \frac{1}{2md} \sin 2md \right) \frac{\phi_0^2}{M^2}. \quad (300)$$

At short distance we have

$$\frac{\delta P}{P} \simeq \frac{\phi_0^2}{2M^2} (md)^2. \quad (301)$$

The dependence of the correction with md is plotted in Fig. 11. At short distance the correction is positive and evolves quadratically with d for a fixed mass, as shown in the inset of the figure. Interestingly we find that for $md = \mathcal{O}(1)$, the pressure changes sign and the plates repulse. This type of behavior is similar to the one obtained when the axion depends on time or space linearly [27,29]. For larger distances, the correction to the axion-induced Casimir pressure oscillates and becomes constant. The pressure itself vanishes as $1/d^4$ so that for large distances the usual CDM equation of state $p = 0$ is retrieved. The suppression of the quantum effect by ϕ_0^2/M^2 implies that for dark matter in the galactic halo this contribution is negligible. On the other hand, for axion clumps of very large

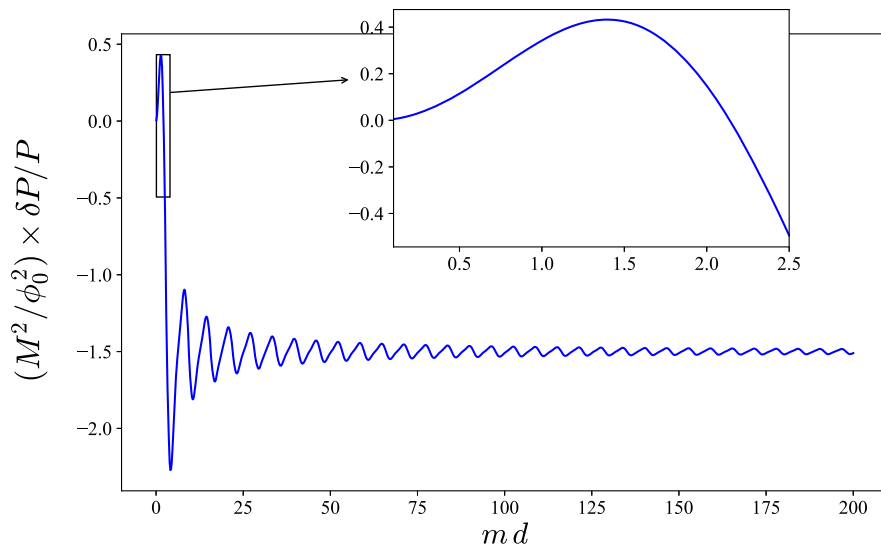


FIG. 11. Axionic quantum correction to the Casimir pressure as a function of the mass-distance product in the ideal case. Notice that at large distance compared to $1/m$ the axion correction is repulsive.

densities, we may hope that the quantum effects could be relevant. The analysis of this possibility is left for future work.

VI. CONCLUSION

We have described the effects of the axion oscillations in the environment of a Casimir experiment on the pressure exerted on the metallic plates. We have focused on metallic plates with dissipation and retrieved the Casimir pressure when axions are neglected, i.e. the Lifschitz theory, and then corrected this result at leading order in the axion coupling to the photons. We have developed these calculations in the Schwinger-Keldysh formulation of the *in-in* formalism as it is particularly suited to evaluate the vacuum expectation value of operators even in the case of dissipative systems. We have also used a field-doubling trick in order to describe dissipative systems in a Lagrangian way. For this we have introduced a “shadow” field whose role is to run history in reverse and bring back energy from infinity. This guarantees that a sound quantum field theory formulation can be defined. This doubling trick makes the connection with classical electrodynamics nontrivial. However we have shown how the conserved energy momentum tensor of the doubled theory can be written as the sum of a well-defined energy momentum tensor and its time reversed. Identifying the energy momentum tensor of electrodynamics in the presence of dissipation in this manner, we have shown how the pressure on the plates deduced from the spatial components of this tensor exactly reproduce the Lifschitz theory.

As expected, the calculation of the vacuum values of the energy-momentum tensor leading to the Casimir pressure shows divergences at coincident points. These UV divergences should be elegantly handled using the quantum extension of D’alembert’s principle of virtual work [34–36]. This is usually formulated in the *in-out* formalism where the variation of the free energy in the static case reduces to the quantum work of the Casimir interaction. Here we use the Schwinger-Keldysh *in-in* formalism where the number of fields is doubled and the action is the difference of two actions living on the two branches of the Keldysh contour. Moreover dissipation makes the interaction Hamiltonian non-Hermitian. In this context, the nontrivial extension of the principle of virtual work to this setting would be an important result both theoretically and for potential applications. This is left for future work.

We have calculated the corrections to the Green’s function induced by the axion field at second order in the axion coupling constant. As expected, the resulting pressure vanishes for massless axions. Indeed in this case, the axion term disappears from the action for trivial topologies and no axion effect is expected. On the other hand, we find that for small distances compared to the axion’s Compton wavelength, the induced pressure varies quadratically with the distance and is attractive. For larger

distances and a given mass, the axion contribution changes sign and oscillates with respect to the distance. This is a breaking of the Kenneth-Klich theorem [23] which happens when parity is violated [22]. Here the role of parity breaking medium is played by the dark matter background with an oscillating axion. We have not considered the phenomenology associated to this new effect. In particular, the axion-induced pressure is small due to the tiny coupling and the small background dark matter density in the galactic halo unless we take into account the possibility of enhanced effects in axion clumps. This is left for further investigation.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DISSIPATION

Let us analyze general properties of the permittivity function. We have given the example of the case of metals in the Drude model. More generally, $\epsilon(\omega) - 1$ is assumed to be an analytic function in the upper half plane with possible singularities in the lower half plane. As a result we have that

$$\epsilon(t) = \oint \omega e^{-i\omega t} (\epsilon(\omega) - 1). \quad (\text{A1})$$

By closing the contour in the upper half-plane, we see that the Cauchy theorem tells us that $\epsilon(t) = \oint(t)$, $t < 0$. When $\epsilon(\omega)$ has a pole at the origin as for metals, we define the integral by drawing a small circle around the origin in the

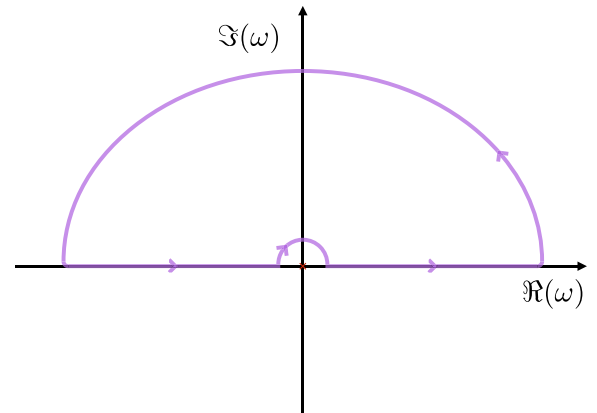


FIG. 12. Integration contour used for the evaluation of integral (A1).

complex plane as shown in Fig. 12. This guarantees that $\epsilon(t) - \delta(t)$ is only nonzero for $t \geq 0$.

The free Hamiltonian is expressed as

$$\begin{aligned} H_0 &= \int d^3x (\partial_0 \tilde{a}_i \partial_0 a_i + \partial_i \tilde{a}_j \partial_i a_j) \\ &= \sum_{\alpha=\pm} \int d^3x (\partial_0 \tilde{a}_\alpha \partial_0 a_\alpha + \partial_i \tilde{a}_\alpha \partial_i a_\alpha) \end{aligned} \quad (\text{A2})$$

corresponding to two systems of independent free scalar fields. In fact one can define

$$a_\alpha = \frac{a_\alpha^+ + a_\alpha^-}{2}, \quad \tilde{a}_\alpha = a_\alpha^+ - a_\alpha^- \quad (\text{A3})$$

or equivalently

$$a_\alpha^+ = a_\alpha + \frac{\tilde{a}_\alpha}{2}, \quad a_\alpha^- = a_\alpha - \frac{\tilde{a}_\alpha}{2} \quad (\text{A4})$$

to find that

$$H_0 = H_+ - H_- \quad (\text{A5})$$

where we have the free Hamiltonians

$$H_\pm = \sum_{\alpha=\pm} \frac{1}{2} \int d^3x (\partial_0 a_\alpha^\pm \partial_0 a_\alpha^\pm + \partial_i a_\alpha^\pm \partial_i a_\alpha^\pm). \quad (\text{A6})$$

The fact that the free Hamiltonian is the difference between two free Hamiltonians of scalar fields in the usual sense is reminiscent of the Keldysh formalism used in the main text. In fact, going back to the full action including the permittivity we find that we obtain

$$S(a_\alpha, \tilde{a}_\alpha) = S(a_\alpha^+) - S(a_\alpha^-) + K(a_\alpha^+, a_\alpha^-) \quad (\text{A7})$$

where

$$\begin{aligned} S(a_\alpha^\pm) &= \frac{1}{2} \int d^4x \left(\partial_0 a_\alpha^\pm \epsilon \star_t \partial_0 a_\alpha^\pm - \partial_i a_\alpha^\pm \partial^i a_\alpha^\pm \right. \\ &\quad \left. + \frac{\phi}{M} \partial_0 a_\alpha^\pm P_{i\alpha\beta} \star_x \partial^i a_\beta^\pm + \frac{\tilde{\phi}}{M} \partial_0 a_\alpha^\pm P_{i\alpha\beta} \star_x \partial^i a_\beta^\pm \right). \end{aligned} \quad (\text{A8})$$

Notice that here the $\pm$ fields with an upper index are not identified with the polarization basis of fields $\pm$ with a lower index. The $\pm$ fields with an upper index are simply combinations of the gauge fields and their shadows. This is the type of action of functional corresponding dissipative systems [18]. In this case, the influence functional is given by

$$K(a_\alpha^+, a_\alpha^-) = \frac{1}{2} \int d^4x \partial_0 a_\alpha^+ (\epsilon - \hat{\epsilon}) \star_t \partial_0 a_\alpha^- \quad (\text{A9})$$

and couples the two fields $a_\alpha^\pm$. Notice that it involves the permittivity combination $\epsilon - \hat{\epsilon}$ whose Fourier transform $\epsilon(\omega) - \epsilon(-\omega) = 2i\Im(\epsilon(\omega))$ only depends on the imaginary part of the permittivity function, i.e. the influence functional is only present when there is dissipation [18].

APPENDIX B: *in-in* VS *in-out*

In this section we will recall the link between the *in-in* and *in-out* formulations of quantum field theory when all the usual assumptions used in particle physics are respected. In particular, we assume that the Hamiltonian is Hermitian implying that the evolution operator is unitary. We also consider the case where Poincaré invariance is respected implying that the notion of particle state is well defined. The ground state of the theory is also assumed to be unique and corresponds to the absence of particles. We are interested in quantum averages in the Schrödinger picture

$$\langle \mathcal{O} \rangle_\psi(t) \equiv \langle \psi(t) | \mathcal{O} | \psi(t) \rangle \quad (\text{B1})$$

where the ket $|\psi\rangle$ evolves with time $|\psi\rangle(t) = U(t, -\infty)|\psi\rangle$. The system is prepared at time minus infinity and then evolves with the unitary evolution operator. We focus on vacuum averages where $|\psi\rangle = |0\rangle$ is the vacuum of the noninteracting theory at time $t = -\infty$ before the interactions are switched on. This vacuum is the *in* vacuum of the theory $|0\rangle = |\text{in}\rangle$. We can then use the fact that the free Hamiltonian $H_0|0\rangle = 0$ has vanishing energy in vacuum to write

$$\langle \mathcal{O} \rangle_{\text{vac}}(t) \equiv \langle \text{in} | U_I^\dagger(t, -\infty) \mathcal{O}_I(t) U_I(t, -\infty) | \text{in} \rangle \quad (\text{B2})$$

in the terms of the operator $\mathcal{O}_I(t)$ and the evolution operator in the interaction picture. We will be interested in the vacuum expectation value (vev) of $\mathcal{O}$ when all the transient effects have disappeared corresponding to $t = \infty$. Now we can identify the *out* vacuum as

$$|\text{out}\rangle = S|\text{in}\rangle \quad (\text{B3})$$

where the scattering S -matrix is defined as $S = U_I(\infty, -\infty)$. As a result we have

$$\langle \mathcal{O} \rangle_{\text{vac}}(\infty) = \langle \text{out} | \mathcal{O}_I(\infty) S | \text{in} \rangle. \quad (\text{B4})$$

This is the origin of the *in-out* evaluation of vevs using explicit S matrix elements. Stability of the vacuum and unitarity imply that

$$|\text{out}\rangle = e^{i\alpha} |\text{in}\rangle \quad (\text{B5})$$

where

$$e^{i\alpha} = \langle \text{in} | S | \text{in} \rangle \quad (\text{B6})$$

leading to

$$\langle \mathcal{O} \rangle_{\text{vac}}(\infty) = \frac{\langle \text{int} | \mathcal{O}_I(\infty) S | \text{in} \rangle}{\langle \text{in} | S | \text{in} \rangle}. \quad (\text{B7})$$

This is the Gellmann-Low formula. Now for Lagrangian field theories, we can use the Dyson formula

$$S = T(e^{i \int d^4x \mathcal{L}_I}) \quad (\text{B8})$$

where $\mathcal{L}_I$ is the interaction Lagrangian. This leads to the usual result in terms of path integrals

$$\langle \mathcal{O} \rangle_{\text{vac}}(\infty) = \frac{\int \mathcal{D}\phi \mathcal{O}(\infty) e^{i \int d^4x \mathcal{L}_I(\phi, \partial_\mu \phi)}}{\int \mathcal{D}\phi e^{i \int d^4x \mathcal{L}_I(\phi, \partial_\mu \phi)}} \quad (\text{B9})$$

where we have assumed that the field theory is defined by the field ϕ .

None of these results stand in the dissipative case as the evolution operator is not unitarity. Moreover, the notion of particle is ill defined in the Casimir setup as translation invariance is broken. As a result, the vevs must be evaluated in the Schwinger-Keldysh formalism where such assumptions are not made. All this implies that there are memory effects that cannot be cancelled, for instance

$$\langle 0 | a_\alpha(t_1) a_\beta(t_2) | 0 \rangle = \langle 0 | U_I^\dagger(t_1, -\infty) a_{\alpha I}(t_1) U_I(t_1, -\infty) U_I^\dagger(t_2, -\infty) a_{\beta I}(t_2) U_I(t_2, -\infty) | 0 \rangle. \quad (\text{B10})$$

In the absence of dissipation this reads

$$\langle 0 | a_\alpha(t_1) a_\beta(t_2) | 0 \rangle = \langle 0 | U_I(-\infty, t_1) a_{\alpha I}(t_1) U(t_1, t_2) a_{\beta I}(t_2) U_I(t_2, -\infty) | 0 \rangle, \quad (\text{B11})$$

which can be represented as

$$\langle 0 | a_\alpha(t_1) a_\beta(t_2) | 0 \rangle = \langle 0 | T_C(a_{\alpha I}(t_1) a_{\beta I}(t_2) U_I(-\infty, -\infty)) | 0 \rangle \quad (\text{B12})$$

where the time ordering is along the path C , which goes from $-\infty$ to ∞ and back to $-\infty$. When dissipation is present we obtain

$$U_I(t_1, -\infty) U_I^\dagger(t_2, -\infty) \neq U_I(t_1, t_2) \quad (\text{B13})$$

where $U(t_1, t_2) = T(e^{-i \int_{t_2}^{t_1} \mathcal{H}_I(t') dt'})$ due to $H_I^\dagger \neq H_I$. As a result the Schwinger-Keldysh representation of the correlators does not work because of dissipation. This is why in the main text, we only consider the time evolution of single operators.

APPENDIX C: T PRODUCT AND PROPAGATORS

The T-product decomposition of the propagator $\langle a_+^1(x) \tilde{a}_+^1(y) \rangle = \Delta_E(x; y)$ calculated using the Feynman integrals in the Schwinger-Keldysh formalism is obtained using the results of Sec. II C. In particular we have $\Delta_E = iG_F$ with the explicit decomposition (57). In this expression, the distribution iG_+ replaces, in the context of the Casimir effect with dissipation, the familiar positive frequency propagator of Feynman

$$iG_+(x_\mu) = -i \int d^4k Y(k^0) \delta(k^2 + m^2) e^{ik^\mu x_\mu} \quad (\text{C1})$$

for a scalar field of mass m . Notice that the contribution of the resonances to the positive frequency Green's function (49) involves the same structure where the on-shell condition $k^2 + m^2 = 0$ is replaced by $\Theta^{-1}(\omega, k_\parallel) = 0$. Of course in the case of Sec. II C, other contributions from the various branch cuts are also present.

The distribution G_+ is defined by a contour integral in the lower half-plane. One can also use the differential equation (118) to deduce that G_+ satisfies the on-shell equation

$$\mathcal{D}_E G_+ = 0 \quad (\text{C2})$$

together with the boundary condition

$$(\epsilon \star_t \partial_t G_+ - \epsilon \star_t \partial_t \tilde{G}_+)|_{t=0} = -\delta^{(3)}(\vec{x} - \vec{y}) \quad (\text{C3})$$

where $\tilde{G}_+(t, \vec{x}; t', \vec{y}) = G_+(t', \vec{y}; t, \vec{x})$. For the usual Feynman case (C1), the latter reads

$$(-i)^2 \int d^4k \frac{k^0}{2k^0} (\delta(k^0 - \sqrt{\vec{k}^2 + m^2}) + \delta(k^0 + \sqrt{\vec{k}^2 + m^2})) \equiv -1 \quad (\text{C4})$$

where the factor of k^0 in the numerator comes from the time derivative and in the denominator from the δ function.

The distribution iG_+ can be written as a linear combination of two independent solutions of the free equation. We know that one of these solutions can be identified with Δ_E^{21} from the fact that in (116) the right-hand side vanishes when the Keldysh indices $a \neq b$ do not coincide, i.e. the off-diagonal Schwinger-Keldysh propagators satisfy the free Klein-Gordon equation (C2). We can now use the boundary condition $a_+^1(\infty) = a_+^2(\infty)$ to identify the two distributions. Indeed taking $t \rightarrow \infty$ we have

$$iG_+(\infty, \vec{x}; t', \vec{y}) = \langle a_+^1(\infty) \tilde{a}_+^1(y) \rangle = \langle a_+^2(\infty) \tilde{a}_+^1(y) \rangle \quad (C5)$$

where the boundary condition at infinite time has been used. Similarly we have

$$\Delta_E^{21}(\infty, \vec{x}; t', \vec{y}) \equiv \langle a_+^2(\infty) \tilde{a}_+^1(y) \rangle \quad (C6)$$

by definition of the off-diagonal propagator. As this identity is valid for any $t' < t \rightarrow \infty$, we can identify the two distributions

$$\Delta_E^{21} = iG_+ \quad (C7)$$

in the T-product decomposition of the Feynman propagators. More details about the T-product decomposition and the Schwinger-Keldysh propagators can be found for instance in Chapter 6 of [37] and in the lecture notes [38].

APPENDIX D: QUANTUM DISSIPATION

We focus on the interior of the cavity of size d seen as an open system when the reflection coefficients are not unity. Its energy is given by the two contributions

$$\frac{E^+}{A} = -\frac{d}{\pi} \int \mathcal{P}^2 P_{\parallel} \int_0^\infty d\omega \frac{\omega^2}{\Delta} \frac{1}{\left(\frac{1+\epsilon}{1-\epsilon}\right)^2 e^{2\Delta d} - 1} \quad (D1)$$

for the TE polarization and

$$\frac{E^-}{A} = -\frac{d}{\pi} \int \mathcal{P}^2 P_{\parallel} \int_0^\infty d\omega \frac{\omega^2}{\Delta} \frac{1}{\left(\frac{1+\frac{\epsilon}{\epsilon(i\omega)\Delta}}{1-\frac{\epsilon}{\epsilon(i\omega)\Delta}}\right)^2 e^{2\Delta d} - 1} \quad (D2)$$

for the TM case. When dissipation is present, the first law of thermodynamics gives

$$dE = -PdV + \delta Q \quad (D3)$$

where δQ is the heat associated with the change of volume. This tells us that the pressure is not simply the derivative of E . For the cavity of size d we can define the quantum work $W_{\pm}$ for the two polarizations as

$$P_{\pm} = -\partial_d W_{\pm} \quad (D4)$$

which reads

$$W_{\pm} = \frac{A}{2} \int \mathcal{P}^2 P_{\parallel} \ln(1 - r_{TE/TM}^2 e^{-2\Delta d}). \quad (D5)$$

With this we can identify the loss of energy from the cavity as the energy penetrating to the plates

$$Q_{\pm} = E_R^{\pm} - W_{\pm} \quad (D6)$$

which is a function of the distance d . In the ideal case, the reflection coefficients are unity and the cavity behaves like a closed system. In this case no energy flows from the cavity to the plates, i.e. $Q_{\pm} = 0$. This is nonvanishing for real metals as the reflection coefficients are not equal to unity anymore and the cavity behaves like an open system. Some of the energy will disappear and $Q_{\pm}$ should account for some of the energy flowing away to infinity and some being dissipated in the metal when the imaginary part of ϵ is nonvanishing.

APPENDIX E: THE ELECTRIC GREEN'S FUNCTION

In this appendix we will give explicit formulas for the Green's function satisfying

$$-\partial_0(\epsilon \star_t \partial_0 G_E) + \Delta G_E = \delta^{(4)}(x^\mu - y^\mu). \quad (E1)$$

In the following we will use time-translation invariance and space-translation invariance in the (x, y) plane along the plaques to choose $y^\mu = (\vec{0}, z_0, 0)$. In terms of Fourier decomposition the Green's function satisfies

$$(\partial_z^2 - p_{\parallel}^2 + \epsilon(\omega)\omega^2)G_E = \delta(z - z_0). \quad (E2)$$

We will separate the z -axis into three intervals and give the solution in each case using continuity of the Green's function at the boundaries $z = 0$ and $z = d$. The first derivative is continuous there too whilst there is a jump $[\frac{dG_E}{dz}]_{z=z_0} = 1$ of the first derivative when G_E itself is continuous at $z = z_0$. Moreover we impose that $\lim_{|z| \rightarrow \infty} G_E = 0$. This determines a unique solution.

$$\mathbf{1. } z_0 \in [0, d]$$

When $0 \leq z_0 \leq d$ is between the plates the Green's function is then defined by

$$G_E = G_E^{E\epsilon z}, \quad z < 0 \quad (E3)$$

and

$$G_E = G_+^E e^{-\xi(z-d)}, \quad z > d \quad (\text{E4}) \quad \text{where}$$

where we have now

$$G_-^E = \Theta_E(\omega, p_{\parallel}) \left(\cos \Delta(z_0 - d) - \frac{\xi}{\Delta} \sin \Delta(z_0 - d) \right) \quad (\text{E5})$$

together with

$$G_+^E = \Theta_E(\omega, p_{\parallel}) \left(\cos \Delta z_0 + \frac{\xi}{\Delta} \sin \Delta z_0 \right) \quad (\text{E6})$$

2. $z_0 < 0$

This case is different from the previous case as

and the Green's function between the plates

$$G_E = G_-^E e^{\xi z}, \quad z \leq z_0 \quad (\text{E9})$$

$$0 \leq z \leq z_0, \quad G_E = G_-^E \left(\cos \Delta z + \frac{\xi}{\Delta} \sin \Delta z \right) \quad \text{and}$$

$$z_0 \leq z \leq d, \quad G_E = G_+^E \left(\cos \Delta(z - d) - \frac{\xi}{\Delta} \sin \Delta(z - d) \right). \quad G_E = A e^{\xi z} + B e^{-\xi z}, \quad z \in [z_0, 0] \quad (\text{E10})$$

(E7) whilst

$$\begin{aligned} 0 \leq z \leq d, \quad G_E &= G_+^E \left[\left(\sin \Delta d - \frac{\xi}{\Delta} \cos \Delta d \right) \sin \Delta z + \left(\cos \Delta d + \frac{\xi}{\Delta} \sin \Delta d \right) \cos \Delta z \right] \\ z \geq d, \quad G_E &= G_+^E e^{-\xi(z-d)}. \end{aligned} \quad (\text{E11})$$

We find that

$$\begin{aligned} A &= \frac{G_+^E}{2} \left[\cos \Delta d + \frac{\xi}{\Delta} \sin \Delta d + \frac{\Delta}{\xi} \left(\sin \Delta d - \frac{\xi}{\Delta} \cos \Delta d \right) \right] \\ B &= \frac{G_+^E}{2} \left[\cos \Delta d + \frac{\xi}{\Delta} \sin \Delta d - \frac{\Delta}{\xi} \left(\sin \Delta d - \frac{\xi}{\Delta} \cos \Delta d \right) \right] \end{aligned} \quad (\text{E12})$$

and we have

$$G_+^E = \Theta_E(\omega, p_{\parallel}) e^{\xi z_0}. \quad (\text{E13})$$

Finally we find that

$$G_-^E = \Theta_E(\omega, p_{\parallel}) \left[\cosh \xi z_0 \left(\cos \Delta d + \frac{\xi}{\Delta} \sin \Delta d \right) + \frac{\Delta}{\xi} \sinh \xi z_0 \left(\sin \Delta d - \frac{\xi}{\Delta} \cos \Delta d \right) \right]. \quad (\text{E14})$$

3. $z_0 > d$

We have now

$$z < 0, \quad G_E = G_-^E e^{\xi z} \quad (\text{E15})$$

and

$$z \in [0, d], \quad G_E = G_-^E \left(\cos \Delta z + \frac{\xi}{\Delta} \sin \Delta z \right) \quad (\text{E16})$$

whilst we have

$$z \in [d, z_0], \quad G_E = \frac{G_-^E}{2} \left[\sin \Delta d \left(\frac{\xi}{\Delta} + \frac{\Delta}{\xi} \right) e^{-\xi(z-d)} + \left(2 \cos \Delta d + \sin \Delta d \left(\frac{\xi}{\Delta} - \frac{\Delta}{\xi} \right) \right) e^{\xi(z-d)} \right] \quad (\text{E17})$$

and finally

$$z > z_0, \quad G_E = G_+^E e^{-\xi(z-d)} \quad (\text{E18})$$

where we find

$$G_-^E = \Theta_E(\omega, p_{\parallel}) e^{-\xi(z_0-d)} \quad (\text{E19})$$

and

$$G_+^E = \Theta_E(\omega, p_{\parallel}) \left[\cosh \xi(z_0 - d) \left(\cos \Delta d + \frac{\xi}{\Delta} \sin \Delta d \right) + \frac{\Delta}{\xi} \sinh \xi(z_0 - d) \left(-\sin \Delta d + \frac{\xi}{\Delta} \cos \Delta d \right) \right]. \quad (\text{E20})$$

APPENDIX F: THE MAGNETIC GREEN'S FUNCTIONS

1. The link to the electric Green's function

The magnetic Green's function satisfies the identity

$$-\partial_0^2 G_M + \epsilon^{-1} \star_t \Delta G_M = \delta^{(4)}(x^\mu - y^\mu). \quad (\text{F1})$$

Away from the point $x = x_0$, it is a solution of the homogeneous equation and therefore admits the same expansion in exponentials for $z < 0$ and $z > d$ whilst it is a linear combination of $\cos \Delta z$ and $\sin \Delta z$ between 0 and d . The only difference between the two Green's function comes from the boundary conditions involving the continuity of $\partial_z G_E$ for the TE case and $\epsilon^{-1}(\omega) \partial_z G_M$ for the TM polarization. This is enough to deduce the expression for the Green's function. Indeed we only have to substitute

$$\xi \rightarrow \frac{\xi}{\epsilon} \quad (\text{F2})$$

in the coefficients of the electric function to find the magnetic function. For instance when $0 \leq z_0 \leq d$ is between the plates the magnetic Green's is function obtained as

$$G_M = G_-^M e^{\xi z}, \quad z < 0 \quad (\text{F3})$$

and

$$G_M = G_+^M e^{-\xi(z-d)}, \quad z > d \quad (\text{F4})$$

where

$$G_-^M = \Theta_M(\omega, p_{\parallel}) \left(\cos \Delta(z_0 - d) - \frac{\xi}{\epsilon \Delta} \sin \Delta(z_0 - d) \right) \quad (\text{F5})$$

and

$$G_+^M = \Theta_M(\omega, p_{\parallel}) \left(\cos \Delta z_0 + \frac{\xi}{\epsilon \Delta} \sin \Delta z_0 \right). \quad (\text{F6})$$

The magnetic Green's function between the plates is

$$\begin{aligned} 0 \leq z \leq z_0, \quad G_M &= G_-^M \left(\cos \Delta z + \frac{\xi}{\epsilon \Delta} \sin \Delta z \right) \\ z_0 \leq z \leq d, \quad G_M &= G_+^M \left(\cos \Delta(z-d) - \frac{\xi}{\epsilon \Delta} \sin \Delta(z-d) \right). \end{aligned} \quad (\text{F7})$$

where

$$\Theta_M(\omega, p_{\parallel}) = \frac{1}{\Delta \left(\left(1 - \frac{\xi^2}{\epsilon^2 \Delta^2} \right) \sin \Delta d - 2 \frac{\xi}{\epsilon \Delta} \cos \Delta d \right)}. \quad (\text{F8})$$

Similar expressions are valid in the other two intervals. All this can be checked by an explicit calculation.

2. Convolution and the magnetic Green's function

Some properties of the TM two-point functions are useful. The magnetic two-point functions are given by

$$\Delta_M(x; y) = \langle a_-^\perp(x) (\hat{\epsilon} \star_{t_y} \tilde{a}_-^\perp(y) \rangle \quad (\text{F9})$$

where the permittivity function is convolved with the second field $\tilde{a}_\alpha^\perp$, and the convolution is expressed in time leading to

$$\Delta_M(x; y) = \int dt \epsilon(t - t_y) \langle a_-^1(\vec{x}, t_x) \tilde{a}_-^1(\vec{y}, t) \rangle. \quad (\text{F10})$$

As the two-point functions respect time translation invariance, we have $\Delta_M(x; y) = \Delta_M(\vec{x}, t_x - t_y; \vec{y}, 0)$ and therefore

$$\Delta_M(x; y) = \int dt \epsilon(t - t_y) \Delta_M(\vec{x}, t_x - t; \vec{y}, 0). \quad (\text{F11})$$

Changing the variable to $\tilde{t} = t_x + t_y - t$ we find

$$\begin{aligned} \Delta_M(x; y) &= \int d\tilde{t} \epsilon(t_x - \tilde{t}) \Delta_M(\vec{x}, \tilde{t} - t_y; \vec{y}, 0) \\ &= \int d\tilde{t} \epsilon(t_x - \tilde{t}) \langle a_-^1(\vec{x}, \tilde{t}) \tilde{a}_-^1(\vec{y}, t_y) \rangle \end{aligned} \quad (\text{F12})$$

and therefore

$$\begin{aligned} \Delta_M(x; y) &= (\epsilon \star_{t_x} \Delta_-^M)(x; y) \\ &= (\hat{\epsilon} \star_{t_y} \Delta_-^M)(x; y) = (\Delta_-^M \star_{t_y} \epsilon)(x; y), \end{aligned} \quad (\text{F13})$$

where

$$\Delta_-^M(x; y) = \langle a_-^1(\vec{x}, \tilde{t}) \tilde{a}_-^1(\vec{y}, t_y) \rangle. \quad (\text{F14})$$

In regions of space where $\epsilon(\omega) = 1$, we will identify Δ_M and Δ_-^M .

3. Magnetic resonances

The resonance equation for the *TM* polarization is

$$\Theta_M(\omega, p_{\parallel}) = 0 \quad (\text{F15})$$

where

$$\Theta(\omega, p_{\parallel}) = \frac{1}{\Delta((1 - \frac{\xi^2}{\epsilon^2 \Delta^2}) \sin \Delta d - 2 \frac{\xi}{\epsilon \Delta} \cos \Delta d)}. \quad (\text{F16})$$

This has solutions for

$$\tan \Delta d = \frac{2 \frac{\xi}{\epsilon \Delta}}{1 - \frac{\xi^2}{\epsilon^2 \Delta^2}}. \quad (\text{F17})$$

We consider the cases when $\gamma \ll |\omega| \ll \omega_{\text{Pl}}$ implying that $\Delta \ll \omega_{\text{Pl}}$ and

$$\frac{\xi}{\Delta} \simeq \frac{\omega_{\text{Pl}}}{\Delta} \left(1 - i \frac{\gamma}{2\omega} \left(1 + \frac{\Delta^2}{\omega_{\text{Pl}}^2} \right) \right) \quad (\text{F18})$$

which is large and therefore

$$\tan \Delta d = \frac{2\xi}{\epsilon \Delta} \simeq -2 \frac{\omega^2}{\omega_{\text{Pl}} \Delta} \left(1 + \frac{i\gamma}{2\omega} - \frac{\Delta^2}{2\omega_{\text{Pl}}^2} \right). \quad (\text{F19})$$

The solutions are obtained by iteration and are

$$\Delta_n d = n\pi - 2 \frac{(\omega_n^{(0)})^2 d}{n\pi \omega_{\text{Pl}}} \left(1 + i \frac{\gamma}{2\omega_n^{(0)}} \right), \quad n \neq 0 \quad (\text{F20})$$

where to leading order

$$\omega_n^{(0)} = \text{sign}(n) \sqrt{\frac{n^2 \pi^2}{d^2} + p_{\parallel}^2}. \quad (\text{F21})$$

The imaginary part of the resonance frequency is obtained by squaring (32) implying that

$$\omega_n^2 = \frac{n^2 \pi^2}{d^2} + p_{\parallel}^2 - 2i \frac{\gamma \omega_n^{(0)}}{\omega_{\text{Pl}} d}. \quad (\text{F22})$$

Similarly to the *TE* case, the resonances are below the real axis. Above the plasma frequency, the resonance condition is the same as the *TE* case and there are no solutions.

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