

Exploring nonperturbative behavior of moments and cumulants in quantum theories

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The dynamics of quantum fields become nonperturbative when their interactions are probed by a large number of particles. To explore this regime, we study correlation functions that involve a large number of fields, focusing on massive scalar theories that feature arbitrary self-interactions, ϕ^{2p} . Treating quantum fields as operator-valued distributions, we investigate n -point correlation functions at ultrashort distances and compute moments and cumulants of fields, using a semiclassical saddle point approximation in the double scaling limit of weak coupling, $\lambda \rightarrow 0$, large quantum number, $n \rightarrow \infty$, while keeping λn constant. Addressing the nonperturbative regime, where $\lambda n \gtrsim 1$, requires a resummation of the effective saddle point to all orders in λn . We perform this resummation in zero and one dimensions and show that the moments, corresponding to correlation functions including disconnected contributions, grow exponentially with n . This growth is significantly reduced for higher-order self-interactions, i.e., for larger p . On the other hand, we argue that the cumulants, which represent connected correlation functions, grow even more rapidly and are mostly independent of p .

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I. INTRODUCTION

Modern quantum field theory (QFT) calculations achieve a remarkable precision that is consistently confirmed by experimental results. This high level of accuracy is facilitated by the powerful framework of perturbation theory, which employs formal series expansions to systematically organize small deviations from the vacuum state at weak coupling. However, a comprehensive assessment of QFT dynamics far away from the vacuum is still lacking. These dynamics typically include particle interactions at extremely high energies, for instance, through scattering of many quanta, thereby probing the ultraviolet (UV) properties of fields.

Despite its power, perturbation theory has been observed to break down at very high energies across various scenarios, including the electroweak instanton sector [1–3], generic scalar QFTs [4–16], or massless string scattering amplitudes [17,18]. This breakdown is typically attributed to the divergent, asymptotic nature of perturbative expansions [19] and associated with the rapidly growing number of Feynman diagrams at each

order of the series [20,21]. Intuitively, one would expect this growth, and hence, the corresponding energy scale at which nonperturbative effects emerge, to depend on the underlying QFT model. For instance, in multiparticle scattering processes, higher-order self-interactions, ϕ^{2p} for integer p , in principle generate an arbitrary number of particles at each interaction vertex, suggesting that such interactions could significantly enhance particle production rates. On the other hand, the staggering number of Feynman diagrams and their interference effectively can lead to low-point interactions, such as ϕ^3 or ϕ^4 , governing n -particle rates at the kinematic threshold [22]. This results in an intricate interplay between different operators in an effective field theory [23].

In this work, we shed some more light on this question by addressing a conceptually identical yet technically simpler aspect of the problem. Specifically, we investigate n -point correlation functions, which serve as a prototype for field dynamics where nonperturbative behavior is expected as n increases. These scenarios are typically analyzed through a semiclassical expansion around a nontrivial saddle point that accounts for the large number of field insertions, often enabling analytic [13–15,24,25] as well as numerical treatments [26–28]. For example, the scaling dimension of the operator ϕ^n in conformal field theories (CFTs) at the Wilson-Fisher fixed point has been determined in the double scaling limit of perturbatively small fixed-point coupling, $\lambda \rightarrow 0$, large quantum number,

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$n \rightarrow \infty$, while keeping λn fixed [29–31], and recently violations of superadditivity in large-charge sectors of CFTs have been explored [32].

Here, we take initial steps to go beyond CFTs by exploring the behavior of large- n correlation functions of massive fields in the short-distance limit, where all fields are inserted at the same spacetime point, $\langle \phi^n(x) \rangle$. While these expectation values of composite operators can be formally understood as the limit of correlation functions, where $|x_i - x_j| \rightarrow 0$ for all spacetime points, they diverge in this regime, revealing the UV behavior of the QFT. Consequently, composite operators require a careful renormalization, for instance, provided by the operator product expansion [33]. To regulate the divergent behavior, we adopt an axiomatic formulation of QFT, treating quantum fields as operator-valued distributions with test functions of localized support [34–36]. Physically, the local spacetime points where the fields are inserted cannot be distinguished beyond a finite energy scale. In this context, the correlation functions correspond to moments and cumulants of the probability density defined by the classical action. To compute these quantities, we first employ a semiclassical approach expanding around a nontrivial saddle point of an effective action that explicitly accounts for the large number of fields, in the double scaling limit $\lambda \rightarrow 0$ and $n \rightarrow \infty$ with λn fixed. However, this saddle point expansion is only accurate in the perturbative regime where $\lambda n \ll 1$. In zero and one dimensions, we perform a resummation of this expansion to all orders in λn , allowing us to investigate the non-perturbative regime where λn is no longer small, $\lambda n \gtrsim 1$. In this domain, the moments, which correspond to correlation functions that include disconnected contributions, grow exponentially. This growth is significantly reduced for higher-order self-interactions, i.e., for larger values of p . In contrast, the cumulants, representing fully connected correlation functions that encapsulate all physical information of the quantum theory, exhibit universal growth with n . That is, their growth is mostly independent of p . While this result provides insights into the non-perturbative behavior of large- n correlation functions in quantum theories, we should nevertheless be aware that realistic QFT settings introduce additional complications, such as a nontrivial phase space or the renormalization of operators.

This work is organized as follows. In Sec. II, we briefly review connected as well as nonconnected correlation functions in QFTs from a path integral perspective. In Sec. III, we build intuition for these correlation functions by examining ordinary integrals as an analog of a QFT in zero dimensions. We then scrutinize this intuition for functional path integrals in Sec. IV, where we employ a semiclassical approach to gain insights into the large- n behavior of correlation functions. Finally, we summarize our findings and conclude in Sec. V.

II. CORRELATION FUNCTIONS INVOLVING A LARGE NUMBER OF FIELDS

In a quantum theory, all physical information such as the spectra of excitations and their scattering amplitudes is encoded in fully connected correlation functions of local operators. If one were to know all correlation functions precisely, one would effectively have “solved” the quantum theory. In practice, these correlation functions are often determined by evaluating Feynman diagrams with fully connected topologies. This construction corresponds to a perturbative expansion in terms of a small parameter, typically the coupling times Planck’s constant, $\lambda \hbar$.

A. Generating functionals for correlation functions

Formally, connected correlation functions can be derived from their nonconnected counterparts, i.e., correlation functions that also include disconnected contributions such as vacuum bubbles. These are defined as vacuum expectation values of time-ordered operator products,

$$\gamma_n(x_1, \dots, x_n) = \langle 0 | \phi(x_1) \dots \phi(x_n) | 0 \rangle. \quad (2.1)$$

They are generated by the partition function $Z[J]$ of the quantum theory, such that

$$\gamma_n(x_1, \dots, x_n) = \frac{\delta^n}{\delta J(x_1) \dots \delta J(x_n)} Z[J] \Big|_{J=0}. \quad (2.2)$$

For a theory with Euclidean action S_E , the partition function is defined as

$$Z[J] = \int \mathcal{D}\phi e^{-S_E + \int d^d x J(x) \phi(x)}. \quad (2.3)$$

The partition function measures the response of the field dynamics to an external local source $J(x)$, which enters the equation of motion as an inhomogeneous source term. In other words, it quantifies the probability of the vacuum state in the asymptotic past remaining in the vacuum state in the asymptotic future, $Z[J] = \langle 0 | 0 \rangle_J$, under the influence of a driving force J .

The connected correlation functions correspond to the cumulants constructed from the nonconnected correlation functions. For instance, the one-point function in the presence of the external source satisfies the relation

$$G_1^J(x) = \frac{\gamma_1^J(x)}{Z[J]}. \quad (2.4)$$

Here, the superscript J indicates that the source has not yet been set to zero. Differentiating this expression with respect to the source and subsequently setting $J = 0$ yields the cumulants of arbitrary order, for instance,

$$G_1(x) = \frac{\gamma_1(x)}{Z_0}, \quad G_2(x, y) = \frac{\gamma_2(x, y)}{Z_0} - \frac{\gamma_1(x)\gamma_1(y)}{Z_0^2}, \quad (2.5)$$

where $Z_0 = Z[0]$. In general, following this prescription, the connected correlation functions are generated by the logarithm of the partition function,

$$G_n(x_1, \dots, x_n) = \frac{\delta^n}{\delta J(x_1) \dots \delta J(x_n)} \ln Z[J] \Big|_{J=0}. \quad (2.6)$$

In terms of probability theory, the correlation functions γ_n and G_n correspond to the moments and cumulants of the probability density defined by $\exp(-S_E)$, respectively.

For this work's purpose, we investigate the nonperturbative behavior of correlation functions at high energies, which is governed by both the number of field insertions n and their short distance limits. In this regime, the spacetime points where the field operators are inserted become indistinguishable below a certain energy scale. Consequently, the objects of interest include quantities such as $\langle \phi^2(x) \rangle$ or $G_2(x, x)$. When evaluating connected correlation functions at coincident spacetime points, we can calculate Eq. (2.6) explicitly using Faà di Bruno's formula (see, e.g., [37]), yielding

$$G_n(x, \dots, x) = \sum_{k=1}^n (-1)^{k-1} (k-1)! B_{n,k} \left(\frac{\gamma_1}{Z_0}, \dots, \frac{\gamma_{n-k+1}}{Z_0} \right). \quad (2.7)$$

Here, $B_{n,k}$ denote the partial exponential Bell polynomials, and the nonconnected correlation functions correspond to vacuum expectation values of the form $\gamma_n = \langle 0 | \phi^n(x) | 0 \rangle$. Due to Poincaré symmetry, the choice of x is arbitrary.

However, it is important to note that the formal expressions in Eqs. (2.2) and (2.6) are valid only for local field operators that are inserted at distinct spacetime points, $x_i \neq x_j$. As their separation approaches zero, $|x_i - x_j| \rightarrow 0$, we expect these expressions to diverge unless the UV behavior of the quantum theory is properly regulated. For instance, the propagator of a free real scalar field behaves as

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle \sim \frac{1}{|x - y|^2}, \quad (2.8)$$

which diverges at short distances as $|x - y| \rightarrow 0$. Therefore, the variance of a quantum field, $\langle \phi^2(x) \rangle$, is formally infinite. This highlights the fact that quantum fields are distributions that cannot be multiplied arbitrarily [38]. To illustrate this, let us follow [39] and consider the correlation function $\langle \phi(y_1) \dots \phi(y_r) \phi^n(x) \rangle$, which in terms of Feynman diagrams is given by all graphs featuring r external vertices and one internal vertex with n edges. Some of these diagrams include loops connecting the internal vertex to

itself, leading to divergent loop integrals. In the free theory, normal ordering effectively removes these divergent contributions, thereby defining the renormalized composite operator $\phi^n(x)$ in the quantum theory. In an interacting quantum theory, a rigorous regularization and renormalization involves the mixing of different operators. Nevertheless, this procedure is essential for providing a meaningful interpretation of composite operators in QFTs [33]. To address the divergent nature of these operators, we can formally treat quantum fields as operator-valued distributions that map test functions to operators acting on the Hilbert space of states. These test functions serve as regulators for the UV behavior of correlation functions in the short-distance limit, a point we will elaborate on in the following.

B. Quantum fields as operator-valued distributions

In a scenario where all spacetime points of local field operator insertions coincide, correlation functions are ill defined, and their divergent structure needs to be carefully examined. For instance, this can be done in axiomatic formulation of QFT, where quantum fields are not local operators; rather, they are operator-valued distributions [34,35]. These distributions are linear functionals that map test functions to operators that act on the Hilbert space of states,

$$\phi_f = \int d^d x \phi(x) f(x). \quad (2.9)$$

Here, $\phi(x)$ is the quantum field understood as an operator-valued distribution, f is a local test function, and ϕ_f is the corresponding operator acting on the Hilbert space. We require the test functions to be smooth and either have compact support in spacetime or are rapidly decreasing as $|x| \rightarrow \infty$, along with their derivatives. Both scenarios provide the field with localized support in spacetime, effectively introducing a UV cutoff for the quantum theory.¹ A simple prototype of this concept is realized by smearing the quantum field with a Gaussian test function (see also [47]),

$$f_\sigma(x, t) = \frac{\delta(t)}{(2\pi\sigma^2)^{\frac{d-1}{2}}} \exp\left(-\frac{|x|^2}{2\sigma^2}\right), \quad (2.10)$$

¹Quantum fields involving test functions with compact support are called *strictly localizable*, while those that are rapidly decreasing are referred to as *tempered distributions*. The spectral density of tempered quantum fields does not grow faster than a fixed-order polynomial, whereas that of strictly localizable operators can grow exponentially with energy [36], such as in Galileon theories (see, e.g., [40]). For further details and phenomenological consequences, we refer the reader to [22,41–46].

where σ defines its width. This choice of test function localizes the operator around the origin, approximately over a distance σ in each spatial direction.

A rigorous treatment of quantum fields as operator-valued distribution allows us to assign a meaningful interpretation to the variance of a quantum field, $\langle \phi^2(x) \rangle$. The localized support in spacetime, characterized by the width σ , effectively introduces a UV cutoff to the quantum theory, which regulates the short-distance behavior of correlation functions. Naively, the inverse width σ^{-1} determines the energy scale beyond which two spacetime points become indistinguishable. To illustrate this, let us consider a free real scalar field of mass m , which is decomposed into momentum modes,

$$\phi(x) = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{\sqrt{2E_p}} (a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}). \quad (2.11)$$

Here, $a_{\mathbf{p}}^\dagger$ and $a_{\mathbf{p}}$ are the standard creation and annihilation operators of momentum $\mathbf{p}$. By employing the canonical commutation relations for $a_{\mathbf{p}}^\dagger$ and $a_{\mathbf{p}}$, we can determine the associated operator after smearing it with the Gaussian test function (2.10),

$$\phi_{f_\sigma} = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{\sqrt{2E_p}} e^{-\frac{\sigma^2 \mathbf{p}^2}{2}} (a_{\mathbf{p}} + a_{\mathbf{p}}^\dagger). \quad (2.12)$$

This enables us to define the short-distance limit of the correlation functions by the asymptotic relation

$$\langle \phi(x) \phi(y) \rangle \sim \langle \phi_{f_\sigma}^2 \rangle, \quad (2.13)$$

as $|x - y| \rightarrow 0$. Similarly, we can express correlation functions at coinciding spacetime points involving $2n$ fields as

$$\langle \phi_{f_\sigma}^{2n} \rangle = (2n - 1)!! I^n, \quad (2.14)$$

where we have defined the momentum integral

$$I = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{2E_p} e^{-\sigma^2 \mathbf{p}^2}. \quad (2.15)$$

Evaluating I in spherical coordinates yields Tricomi's confluent hypergeometric function,

$$I \propto U\left(\frac{1}{2}, 2 - \frac{d}{2}, m^2 \sigma^2\right). \quad (2.16)$$

Notably, the localized support of the quantum field parametrized by the width σ , only shows up in combination with its mass, $m\sigma$. Therefore, since the case $\sigma = 0$ does not belong to the space of smooth test functions, any finite σ serves to regulate the theory in the UV.

Intuitively, the short-distance limit of correlation functions in Eq. (2.14) can be interpreted as the moments of the probability density defined by the classical action. These moments include disconnected contributions, which in the free theory factorize into products of propagators. To determine the connected correlation functions at coinciding points, which correspond to the cumulants of the probability density, we can apply Eq. (2.7). While the moments exhibit factorial growth, we find that the cumulants for a free theory vanish, except for the propagator at $n = 1$. This matches our expectation from Feynman diagrams, as there are no higher-point graphs for a free field with fully connected topologies. In fact, the moments and cumulants are represented by Feynman diagrams with a vertex with n edges, which corresponds to the local insertion of the composite operator. When we include self-interactions, these graphs are similar to petal diagrams where the edges of the vertex are fully connected for the cumulants, while for the moments, the edges can be connected to each other in separate subgraphs. Let us now investigate their non-perturbative behavior in quantum theories with nontrivial self-interactions. To build intuition, we first consider toy models that resemble quantum theories in zero dimensions.

III. ANALOGIES WITH ORDINARY INTEGRALS

Ordinary integrals serve as a valuable test bed for exploring path integrals in the context of interacting QFTs. They often permit exact solutions while effectively capturing the essential features of QFTs in higher dimensions. In this toy model, we aim to illustrate the behavior of correlation functions involving a large number of fields. We begin by considering the moments γ_n . These correspond to the nonconnected correlation functions in zero dimensions and can be expressed as an ordinary integral,

$$\gamma_n = \int_{-\infty}^{\infty} d\phi \phi^n \exp\left(-\frac{1}{2} \phi^2 - \frac{1}{2p} g^{2(p-1)} \phi^{2p}\right). \quad (3.1)$$

Here, both n and p are positive integers, and g^2 denotes the coupling. The power of the latter is chosen such that, in any perturbative expansion, it aligns with the loop-counting parameter through the field redefinition, $\varphi = g\phi$.

In zero dimensions, we can compute the moments γ_n in Eq. (3.1) exactly for any integer n . The result is illustrated in Fig. 1, where we show γ_n as a function of n . For simplicity, we have set the coupling to $g^2 = 1$. We find that the moments exhibit factorial growth with increasing n . However, this growth severely depends on the power p of the self-interaction term ϕ^{2p} . Clearly, for larger values of p , the factorial growth is significantly suppressed at non-perturbatively large n . This behavior can also be confirmed analytically by neglecting the quadratic terms of the potential, leading to the asymptotic form

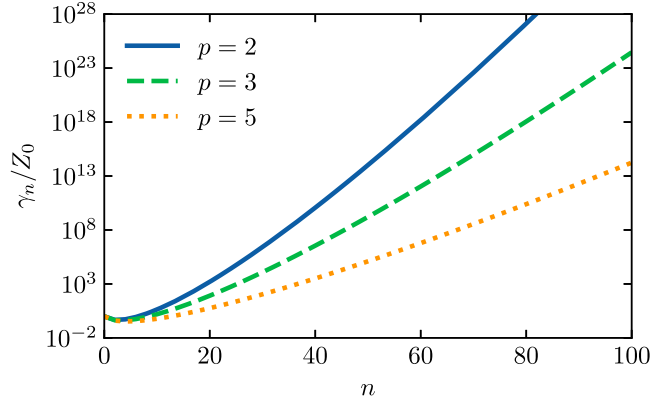


FIG. 1. Moments γ_n normalized to Z_0 , as a function of the number of field insertions n in zero dimensions. The different colors represent different self-interaction terms of the theory, ϕ^{2p} . For simplicity, the coupling is set to one, $g^2 = 1$.

$$\gamma_n \sim \Gamma\left(\frac{n}{2p}\right). \quad (3.2)$$

This demonstrates that the factorial growth is drastically reduced by the higher-order self-interactions of the field.

We now turn our attention to the cumulants G_n , which correspond to the connected correlation functions in zero dimensions. These can be obtained by substituting the exact expression for the moments γ_n into Eq. (2.7). An example of the cumulants is presented in the left panel of Fig. 2. We again observe a rapid factorial growth. However, unlike γ_n , this growth appears to be largely independent of the specific value of p , at least at the factorial level. Furthermore, the cumulants grow faster than the moments, even for the smallest possible values of p . This observation is both surprising and counterintuitive, as the connected correlation functions are naively derived from the non-connected ones by subtracting all factorizable diagrams. Similar phenomena have been noted in other contexts using Dyson-Schwinger equations [48,49].

To establish the universal growth of the cumulants at large n more precisely, it is useful to consider their ratio,

$$R_n = \frac{G_{n+2}}{G_n}. \quad (3.3)$$

If we assume that the cumulants grow factorially, $G_n \propto n!$, then the ratio will be a quadratic function of n , $R_n \propto n^2$. This ratio is illustrated in the right panel of Fig. 2 for various values of p . Indeed, we find that schematically

$$R_n \sim c_p n^2, \quad (3.4)$$

where the constant c_p exhibits only moderate dependence on p . Substituting this result into Eq. (3.3) allows us to obtain the large- n behavior of the cumulants,

$$G_n \sim 2^{n-1} c_p^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right)^2. \quad (3.5)$$

This illustrates that any dependence on the self-interaction term contributing to the cumulants G_n is at most a power law, which is markedly different from the reduced factorial growth observed for the moments γ_n . Consequently, we conclude that the factorial growth of the cumulants is universal with respect to n , i.e., mostly independent of the self-interaction term of the theory. This suggests that, in zero dimensions, the moments γ_n are primarily governed by the dynamics of the theory, while the corresponding cumulants G_n are predominantly governed by combinatorial factors.

We note that our results align with established theorems from complex analysis. In fact, in zero dimensions, we can formally prove that the cumulants grow factorially, without relying on saddle point approximations or the successive application of the chain rule as was done to derive Eq. (2.7). Closely following [50], let us first assume that the generating function for the cumulants (which in zero dimensions is a function rather than a functional) has a formal series expansion with coefficients a_n ,

$$\ln Z(J) = \sum_{n=0}^{\infty} a_n J^n, \quad (3.6)$$

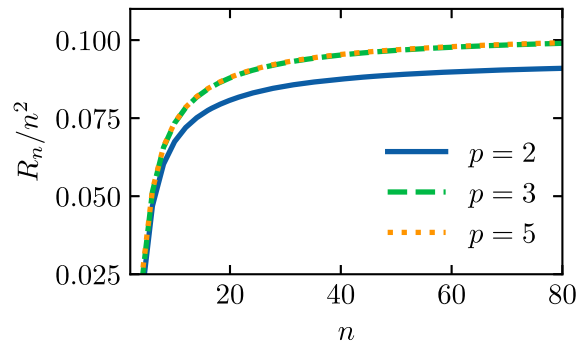
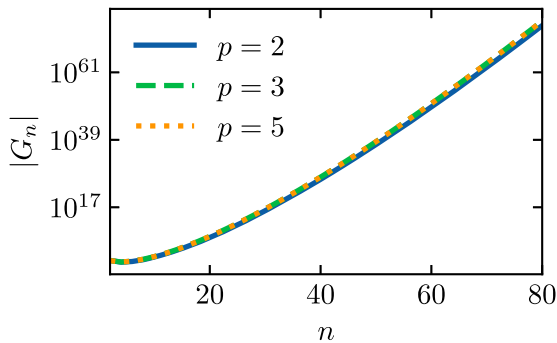


FIG. 2. Cumulants G_n (left) and their rescaled ratios (right), as a function of the number of fields n in zero dimensions. The different colors represent different self-interaction terms of the theory, ϕ^{2p} . For simplicity, the coupling is set to one, $g^2 = 1$.

where J is a complex source. Suppose that this series representation has a finite radius of convergence, $|J_c|$. According to Abel's theorem, the asymptotic growth of the coefficients is governed by inverse powers of this radius,

$$a_n \sim \left(\frac{1}{|J_c|}\right)^n, \quad (3.7)$$

as $n \rightarrow \infty$. Consequently, the cumulants necessarily exhibit factorial growth in this regime,

$$G_n \sim \left(\frac{1}{|J_c|}\right)^n n!. \quad (3.8)$$

Thus, if the series expansion of the generating function in Eq. (3.6) has a finite radius of convergence, the factorial growth of G_n is a direct consequence. In fact, $\ln Z(J)$ possesses a finite radius of convergence if the partition function $Z(J)$, as an analytic function, has a root somewhere in the complex plane. Recall that $Z(J)$ is the generating function of the moments,

$$Z(J) = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} J^n, \quad (3.9)$$

where, according to Eq. (3.2), the coefficients grow as $\gamma_n \sim \Gamma[n/(2p)]$ for large n . This leads to the asymptotic behavior of the partition function in the complex plane,

$$Z(J) \sim \exp(|J|^{\frac{2p}{2p-1}}), \quad (3.10)$$

as $|J| \rightarrow \infty$. Thus, through elementary complex analysis, we find that the analytic function $Z(J)$ is of order $2p/(2p-1)$. This order is rational for any value of $p \geq 2$, i.e., for any nontrivial self-interaction. Consequently, by Picard's theorem, $Z(J)$ must have infinitely many roots in the complex plane (see, e.g., [51]). This completes our proof that the cumulants, corresponding to the connected correlation functions, grow factorially with n . With this understanding derived from ordinary integrals, let us now proceed to examine higher-dimensional quantum theories.

IV. MOMENTS AND CUMULANTS FROM QUANTUM PATH INTEGRALS

While ordinary integrals provide a zero-dimensional test bed that captures the main features of a quantum theory, functional integrals used in QM or QFT settings are more involved to evaluate. Similarly, in zero dimensions, the moments of a probability distribution are well defined, while in a QFT, they are not and need to be regulated. According to our discussion in Sec. II, this can be done by treating quantum fields as operator-valued distributions that are smeared by smooth test functions. In this case, we can

formally write the correlation function at coinciding space-time points as an object localized around the origin,

$$\gamma_n = \langle \phi_{f_\sigma}^n \rangle. \quad (4.1)$$

Schematically, this means that the correlation function, as a distribution, is convoluted with the Gaussian test function (2.10) of width σ ,

$$\gamma_n \simeq \int \prod_{i=1}^n d^d x_i f_\sigma(x_i) \langle \phi(x_1) \dots \phi(x_n) \rangle. \quad (4.2)$$

In the regime of small width (compared to their relevant mass scale), each field is sharply localized around the origin. Therefore, in the limit $\sigma \rightarrow 0$, the moments asymptotically approach the path integral expression

$$\gamma_n \sim \int \mathcal{D}\phi \phi^n(0) e^{-S_E[\phi]}. \quad (4.3)$$

Here, we have defined the Euclidean action

$$S_E[\phi] = \int d^d x \left(\frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + \frac{1}{2p} g^{2(p-1)} \phi^{2p} \right). \quad (4.4)$$

This expression is the higher-dimensional generalization of the ordinary integral in Eq. (3.1). However, an exact solution for an interacting quantum theory is lacking, such that we have compute γ_n using approximation methods.

A. An effective saddle point approximation

To determine the nonconnected correlation functions, we need to find an appropriate saddle point approximation of Eq. (4.3). It is convenient to perform a field redefinition $\varphi = g\phi$, such that the moments read

$$\gamma_n = g^{-n-1} \int \mathcal{D}\varphi \varphi^n \exp\left(-\frac{S_E[\varphi]}{g^2}\right). \quad (4.5)$$

In these coordinates, the Euclidean action does not explicitly depend on the coupling of the theory,²

$$S_E[\varphi] = \int d^d x \left(\frac{1}{2} (\partial\varphi)^2 + \frac{1}{2} \varphi^2 + \frac{1}{2p} \varphi^{2p} \right). \quad (4.6)$$

In addition, we have normalized all dimensionful quantities to the scalar field mass, setting $m^2 = 1$.

In practice, modern QFT calculations evaluate the path integral perturbatively, i.e., as a series expansion around the vacuum, although they often involve only a limited number

²This is a consequence of the fact that the magnitude of a coupling does not have a physical meaning at the classical level (see, e.g., [52]).

of local field insertions. However, this approach may fail as n increases, since the saddle point approximation is typically performed around the vacuum $\phi = 0$. Similar to Eq. (3.1), as n grows, the support of the integrand shifts away from the perturbative vacuum at the origin and extends toward infinity along the real axis, significantly diminishing the accuracy of the naive approximation. This limitation of the perturbative approach can also be understood through Feynman diagrams. Typically, the number of Feynman diagrams contributing to the n th order of the perturbative expansion grows factorially, $c_n \sim n!$, highlighting the asymptotic nature of perturbation theory. Consequently, for large n , the combination of n and coupling, $g^2 n$, distinguishes the perturbative regime, where $g^2 n \ll 1$, from the nonperturbative regime of the theory, where $g^2 n \gtrsim 1$. This implies that a naive summation of Feynman diagrams in the latter regime does not yield an accurate and meaningful result for large $g^2 n$. To remedy this issue, we instead follow a semiclassical approach and express the moments as (see also [29,30])

$$\gamma_n = g^{-n-1} \int \mathcal{D}\phi \exp\left(-\frac{S_{\text{eff}}[\phi]}{g^2}\right), \quad (4.7)$$

with an effective action that accounts for the local operator insertion,³

$$S_{\text{eff}}[\phi] = S_E[\phi] - \epsilon \log \phi(0). \quad (4.8)$$

Here, for convenience, we have defined the combination

$$\epsilon = g^2 n, \quad (4.9)$$

which allows us to systematically construct a double scaling limit where $g^2 \rightarrow 0$ and $n \rightarrow \infty$ while keeping ϵ fixed. In this case, ϵ captures the dynamics relevant to the weakly coupled quantum theory. As previously mentioned, $\epsilon \ll 1$ characterizes the perturbative regime, while $\epsilon \gtrsim 1$ characterizes the nonperturbative domain of the theory. In other words, perturbation theory yields reliable and accurate results for small ϵ , but breaks down for large ϵ .

Minimizing the effective action S_{eff} accounts for the shifted support of the integrand away from the perturbative vacuum toward infinity as n increases. Thus, we can construct a suitable saddle point approximation around this nontrivial field configuration, satisfying $\delta S_{\text{eff}}/\delta\phi = 0$. The corresponding equations of motion for ϕ are given by

$$\partial^2 \phi - \phi - \phi^{2p-1} = -\epsilon \frac{\delta^{(d)}(x)}{\phi(0)}. \quad (4.10)$$

³Due to parity symmetry, we can disregard any imaginary part arising from the logarithmic term due to negative field values, without loss of generality.

Here, the local field insertion acts as an inhomogeneous source term for the field's dynamics, which therefore explicitly depend on the effective coupling ϵ . Since there are no known analytic solutions to Eq. (4.10), we make a perturbative ansatz to find the classical field configuration corresponding to the effective saddle point,

$$\phi(x) = \sqrt{\epsilon}(\phi_0(x) + \epsilon^{p-1}\phi_1(x) + \epsilon^{2(p-1)}\phi_2(x) + \dots). \quad (4.11)$$

Using this ansatz, we can systematically solve the dynamics order by order in powers of ϵ by determining the coefficient functions $\phi_i(x)$. For instance, at leading order, the equations of motions take the form

$$\partial^2 \phi_0 - \phi_0 = -\frac{\delta^{(d)}(x)}{\phi_0(0)}. \quad (4.12)$$

Thus, ϕ_0 is proportional to the Green's function of the Helmholtz operator in d dimensions, $\phi_0 \propto r^{-d/2+1} K_{d/2-1}(r)$. Similarly, the effective action can be decomposed into a power series in ϵ through Eq. (4.11). At leading order, it involves only the field ϕ_0 and is given by

$$S_{\text{eff},0} = \epsilon \int d^d x \left(\frac{1}{2} (\partial \phi_0)^2 + \frac{1}{2} \phi_0^2 \right) - \epsilon \log [\sqrt{\epsilon} \phi_0(0)] = \frac{\epsilon}{2} - \frac{\epsilon}{2} \log \left(\frac{\epsilon}{2} \right). \quad (4.13)$$

Consequently, the moments at large n schematically read

$$\gamma_n \sim \exp \left[\frac{1}{g^2} \left(\frac{\epsilon}{2} \log \left(\frac{\epsilon}{2} \right) - \frac{\epsilon}{2} \right) \right]. \quad (4.14)$$

We remark that this result should be understood within the context of a double scaling limit where $g^2 \rightarrow 0$ and $n \rightarrow \infty$, while keeping ϵ fixed. In fact, Eq. (4.14) is a large- n representation of the familiar factorial growth of tree-level perturbation theory, through Stirling's approximation. For example, the exponential form of γ_n agrees with the overlap of an off-shell quantum state (created by an off-shell field acting on the vacuum) and an arbitrary n -particle state, $\langle n|\phi|0\rangle$. More precisely, it matches the tree-level computation of the latter in the double scaling limit [4–8]. Physically, this is expected, as in both scenarios, any tree-level approximation is dominated by the number of Feynman diagrams, which grow factorially, thus yielding the above behavior through Stirling's approximation (modulo appropriate powers of the coupling).

So far, Eq. (4.11) dictates a perturbative expansion around $\epsilon = 0$, which we expect to yield an accurate approximation in the perturbative regime of the quantum theory, where $\epsilon \ll 1$. In principle, it is viable to systematically compute higher-order terms of the classical field

configuration, φ_i , along with the corresponding corrections to the effective action. This would extend the range of validity of the saddle point approximation. However, going beyond perturbation theory into the nonperturbative domain where $\epsilon \gtrsim 1$ requires the resummation of a substantial number of terms in this expansion. While exploring this domain is feasible in CFTs (see [29–31]), scenarios involving a mass term present additional complications. For example, although formal solutions for φ_1 exist, determining the corresponding correction to the effective action in an arbitrary number of dimensions proves to be challenging. One of the simplest nontrivial examples where an explicit solution can be obtained is QM, which serves as a one-dimensional QFT. In this context, the first correction to the effective action is given by

$$S_{\text{eff},1} = \frac{\epsilon^p}{2^{p+1} p^2}. \quad (4.15)$$

A detailed derivation of this result is provided in Appendix. Accordingly, we will perform a complete resummation of all higher-order corrections to the saddle point approximation of the effective action in QM.

B. Resummation of the effective action in quantum mechanics

Addressing the nonperturbative behavior of correlation functions at short distances requires a resummation of the perturbative expansion of the effective action to all orders. We expect this approach to yield the exact results, as there are no instantons present on the real axis in the Borel plane that could spoil the resummation. In QM, this resummation of the effective action is feasible because instead of relying on a path integral formulation, we can compute the moments directly from the ground-state wave function,

$$\gamma_n = \int_{-\infty}^{\infty} dx x^n |\psi_0(x)|^2. \quad (4.16)$$

The wave functions, in turn, can be systematically derived from a perturbative expansion of Schrödinger's equation at weak coupling,

$$\left(-\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 + \frac{1}{2p} g^{2(p-1)} x^{2p} - E \right) \psi(x) = 0. \quad (4.17)$$

We make an ansatz for both the wave functions as well as the corresponding energy levels as a formal series expansion in powers of $g^{2(p-1)}$, allowing us to solve Eq. (4.17) order by order in this expansion. More generally, the perturbative construction of the spectrum is performed through a set of recursion relations introduced by Bender and Wu [53,54]. To implement the recursion relations in practice, we use the efficient code BenderWu [55].

For example, for the standard quartic anharmonic oscillator with $p = 2$, we obtain the ground-state wave function

$$\psi_0^{p=2}(x) = e^{-\frac{x^2}{2}} \left[1 - \frac{g^2}{16} (x^2 + 3) + \frac{g^4}{512} \left(x^8 + \frac{26}{3} x^6 + 31x^4 + 84x^2 \right) + \dots \right]. \quad (4.18)$$

Here, we have normalized the wave function such that $\int_{-\infty}^{\infty} dx |\psi_0|^2 = 1$.

Using Bender's and Wu's method, we can compute the spectrum of Eq. (4.17) to any desired order in the perturbative expansion. This, in turn, determines the moment in Eq. (4.16) up to the same perturbative order. Notably, we find that each series coefficient of γ_n is a polynomial in n . For instance, for $p = 2$, the first few terms take the form

$$\gamma_n^{p=2} = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}} \left[1 - \frac{g^2}{32} (n^2 + 10n) + \frac{g^4}{2048} \left(n^4 + \frac{92}{3} n^3 + 280n^2 + \frac{2968}{3} n \right) + \dots \right], \quad (4.19)$$

while for $p = 3$, they read

$$\gamma_n^{p=3} = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}} \left[1 - \frac{g^4}{144} \left(n^3 + \frac{33}{2} n^2 + 98n \right) + \frac{g^8}{41472} \left(n^6 + \frac{273}{5} n^5 + \frac{4807}{4} n^4 + 13104n^3 + 75079n^2 + \frac{1136382}{5} n \right) + \dots \right]. \quad (4.20)$$

In practice, we compute hundreds of orders of this formal series expansion and identify patterns in their structure. Regardless of the interaction's power p , we find a universal behavior of the leading-order (tree level) contribution,

$$\gamma_n^{\text{tree}} = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}}. \quad (4.21)$$

We have confirmed this result for self-interactions up to $p = 10$. It exemplifies the familiar factorial growth of observables associated with high multiplicities or large quantum numbers, highlighting the asymptotic nature of perturbation theory. Furthermore, we find that this leading contribution factorizes from the full expression, as indicated in Eqs. (4.19) and (4.20). This suggests that the remaining loop contributions can be effectively resummed in the large- n limit, similar to the case of the QM analog of high-multiplicity form factors, $\langle n|x|0 \rangle$ [56–58]. The

leading quantum corrections to the tree-level result indicate an exponential form,

$$\begin{aligned}\gamma_n^{p=2} &\sim 1 - \frac{g^2 n^2}{32} + \frac{g^4 n^4}{2048} - \frac{g^6 n^6}{196608} + \dots \\ &= \exp\left(-\frac{g^2 n^2}{32}\right),\end{aligned}\quad (4.22)$$

$$\begin{aligned}\gamma_n^{p=3} &\sim 1 - \frac{g^4 n^3}{144} + \frac{g^8 n^6}{41472} - \frac{g^{12} n^9}{17915904} + \dots \\ &= \exp\left(-\frac{g^4 n^3}{144}\right).\end{aligned}\quad (4.23)$$

The exponentiation is crucial for exploring the behavior of the correlation functions in the double scaling limit $g^2 \rightarrow 0$ and $n \rightarrow \infty$, while keeping their combination $\epsilon = g^2 n$ fixed. Specifically, if we adopt the general ansatz

$$\gamma_n = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}} \exp\left(\frac{F_c}{g^2}\right), \quad (4.24)$$

the large- n limit of γ_n is entirely encoded in both the tree-level contribution and, more importantly, in the exponent function F_c . If F_c is negative and sufficiently large to counteract the tree-level growth, one may conclude that γ_n is exponentially suppressed at high energies.

To shed some light on this behavior, we note that F_c can be systematically constructed in an asymptotic $1/n$ -expansion (see also [56,57]),

$$F_c(\epsilon, n) = F_0(\epsilon) + \frac{1}{n} F_1(\epsilon) + \frac{1}{n^2} F_2(\epsilon) + \dots \quad (4.25)$$

This implies that in the limit where ϵ is kept fixed, the large- n behavior of $F_c(\epsilon)$ is dominated by the contribution $F_0(\epsilon)$. This holds true both in the perturbative, where $\epsilon \ll 1$, as well as in the nonperturbative regime, where $\epsilon \gtrsim 1$.

Let us now determine the exponent F_c by expressing its coefficient functions in Eq. (4.25) as

$$F_i(\epsilon) = \epsilon^p \sum_{k=0}^{\infty} c_{i,k} \epsilon^{k(p-1)}. \quad (4.26)$$

We can then fix all coefficients $c_{i,k}$ by substituting this ansatz into Eq. (4.24), performing the series expansion for the exponential, and matching the result to the analytic expressions for γ_n , such as those given in Eqs. (4.19) and (4.20). For the first few terms, we find

$$\begin{aligned}F_c^{p=2} &= -\frac{\epsilon^2}{32} + \frac{\epsilon^3}{192} - \frac{35}{24576} \epsilon^4 + \dots \\ &\quad + \frac{1}{n} \left(-\frac{5}{16} \epsilon^2 + \frac{45}{512} \epsilon^3 + \dots \right) \\ &\quad + \frac{1}{n^2} \left(\frac{371}{768} \epsilon^2 - \frac{3913}{12288} \epsilon^3 + \dots \right) + \mathcal{O}\left(\frac{1}{n^3}\right),\end{aligned}\quad (4.27)$$

and, for $p = 3$,

$$\begin{aligned}F_c^{p=3} &= -\frac{\epsilon^3}{144} + \frac{\epsilon^5}{1920} - \frac{5}{72576} \epsilon^7 + \dots \\ &\quad + \frac{1}{n} \left(-\frac{11}{96} \epsilon^3 + \frac{163}{9216} \epsilon^5 + \dots \right) \\ &\quad + \frac{1}{n^2} \left(-\frac{49}{72} \epsilon^3 + \frac{1645}{6912} \epsilon^5 + \dots \right) + \mathcal{O}\left(\frac{1}{n^3}\right).\end{aligned}\quad (4.28)$$

To determine the large- n behavior of the moments in the double scaling limit described above, we now write

$$\gamma_n = \frac{1}{g^{2n}} \exp\left(\frac{F}{g^2}\right), \quad (4.29)$$

such that we include the tree-level factor in the exponent using Stirling's formula,

$$F(\epsilon, n) = \frac{\epsilon}{2} \left(\log \frac{\epsilon}{2} - 1 \right) + F_c(\epsilon, n). \quad (4.30)$$

It is remarkable that this exponentiation is exact to all orders in perturbation theory, which is not guaranteed (for a detailed discussion of exponential series representations, see [56]).

By construction, this result is an expansion about the perturbative regime at $\epsilon = 0$. Therefore, exploring the nonperturbative domain of the theory, where $\epsilon \gtrsim 1$, requires a suitable resummation of F beyond the regime where ϵ is small. In principle, a resummation procedure based on a finite number of terms is provided by the Borel-Padé approximation (for an introduction to this method, see, e.g., [59]). However, in our case, a close inspection of a large number of terms reveals a closed-form expression for the corresponding coefficients $c_{0,k}$ in Eq. (4.26), which is valid for arbitrary p ,

$$c_{0,k} = \frac{(-1)^{k+1} (2^{p-1} p)^{-k-1}}{4} \frac{\Gamma\left[\frac{(p+1)(k+1)}{2}\right]}{(p-1)k + p \Gamma\left[\frac{(p-1)k + p + 1}{2}\right] \Gamma(k+2)}. \quad (4.31)$$

For instance, we find that the leading-order coefficient of the exponent beyond tree level is given by

$$c_{0,0} = -\frac{1}{2^{p+1} p^2}. \quad (4.32)$$

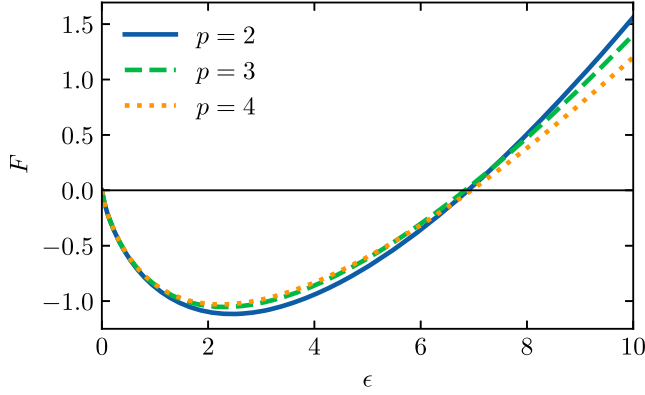


FIG. 3. Resummed exponent F of the moments γ_n , as a function of ϵ in the double scaling limit $g^2 \rightarrow 0$ and $n \rightarrow \infty$, where $\epsilon = g^2 n$ is kept fixed. The different colors represent different self-interaction terms of the theory, ϕ^{2p} .

This expression precisely matches the leading-order correction to the effective action S_{eff} in Eq. (4.15), which we have obtained from an effective saddle point approximation. This not only validates our approach to some extent but also facilitates an immediate identification of the exponent function F_0 with the effective saddle point. In this context, it would be interesting to investigate whether the $1/n$ -corrections of F correspond to the quantum fluctuations around the effective saddle point. At the same time, our result demonstrates that each coefficient of S_{eff} (or equivalently F_0) effectively resums an infinite number of loops at large n . This also means that the closed form in Eq. (4.31) enables us to resum the formal series expansion and obtain the analytic continuation of $F_0(\epsilon)$, which is valid everywhere in the complex ϵ plane. We omit its explicit form here, as it is lengthy and convoluted. Instead, we illustrate its shape in Fig. 3.

Clearly, F exhibits a turning point before becoming positive beyond a certain critical value ϵ_0 , where $F(\epsilon_0) = 0$. The turning point is reminiscent of the point of optimal truncation, ϵ_{opt} . It marks the order beyond which the individual terms of the asymptotic series expansion of F start to grow rapidly, and a careful resummation becomes necessary. We find that its location only mildly depends on the self-interaction power, and it appears to asymptotically approach $\epsilon_{\text{opt}} \simeq 2$ as p grows. We have verified this behavior for self-interactions up to $p = 8$. Similarly, the root of F marks the point ϵ_0 where exponential growth for the moments γ_n sets in. That is, if one chooses a weak coupling g^2 , ϵ_0 indicates the value of n for which the moments start to grow exponentially, $n \gtrsim \epsilon_0/g^2$. While, to the best of our knowledge, an analytic expression for ϵ_0 does not exist for any value of p , we observe that it is monotonically shifted toward larger values as p grows. Again, we have explicitly verified this for self-interactions up to $p = 8$. We expect that it asymptotically behaves as $\epsilon_0 \rightarrow \infty$ as $p \rightarrow \infty$, as this scenario naively corresponds to

the infinite square well potential. In this case, one can show explicitly that the moments are exponentially suppressed for large n , such that F is necessarily negative everywhere. We conclude that the growth rate depends on the self-interaction power, with larger values of p leading to a suppression of this growth. This behavior is consistent with our findings for ordinary integrals (see Fig. 1). Focusing on the nonperturbative domain, where $\epsilon \gtrsim 1$, this observation is further supported by an asymptotic expansion as $\epsilon \rightarrow \infty$,

$$F(\epsilon) \sim \frac{\epsilon}{p+1} \left[\log \left(\sqrt{\frac{p}{4}} \epsilon \right) - 1 \right], \quad (4.33)$$

which clearly shows a suppression factor for large p . In other words, at high energies, low-point interactions seem to dominate n -particle rates.

In summary, we find that in the QM analog of self-interacting QFTs, the moments γ_n exponentiate and grow rapidly with a large number of field insertions at high energies, $n \rightarrow \infty$. However, this rapid growth is suppressed by the power p of the field's self-interaction. In the following, we will argue that this behavior is in strong contrast to that observed for the cumulants, which correspond to fully connected correlation functions.

C. Asymptotic behavior of cumulants

In general, having derived the resummed expression for the moments γ_n , the corresponding cumulants G_n , which represent connected correlation functions of fields inserted at a single spacetime point, can be read off from Eq. (2.7). Note that we have also normalized the wave functions to $\int_{-\infty}^{\infty} dx |\psi_0|^2 = 1$, which corresponds to the normalization $Z_0 = 1$.

The computation of the cumulants has to be treated with some care. Formally, we have determined the analytic continuation of γ_n in Eq. (4.29) in the double scaling limit $g^2 \rightarrow 0$ and $n \rightarrow \infty$, while keeping $\epsilon = g^2 n$ fixed. However, this limit is not inherently applicable to the expression for G_n given in Eq. (2.7). Therefore, in practice, we replace any power of the coupling constant by the corresponding expression involving ϵ as the effective coupling, $g^2 = \epsilon/n$. We then select an arbitrary but fixed value of ϵ , for instance, in the nonperturbative regime $\epsilon \gtrsim 1$, and subsequently take the large- n limit. In other words, to obtain a meaningful expression as $n \rightarrow \infty$, we express the cumulants as functions of both n and ϵ . Furthermore, we can organize any subleading corrections to G_n as a series expansion in inverse powers of n .

While we can study the cumulants' behavior for any value of ϵ in this manner, the evaluation of Eq. (2.7) becomes numerically unstable for large values of ϵ due to the combinatorial complexities arising from the addition and subtraction of large numbers. To address this instability, we numerically determine the ground-state wave function with high precision and then directly integrate the

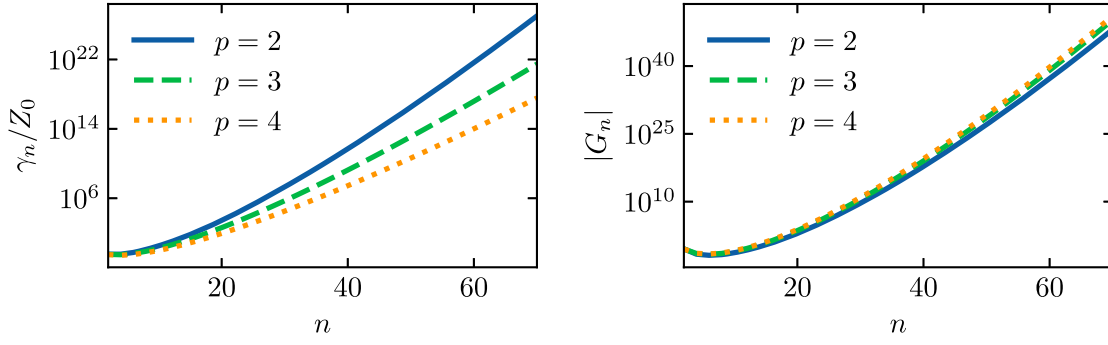


FIG. 4. normalized moments γ_n (left) and cumulants G_n (right), as a function of n in one dimension. The different colors represent the self-interaction terms of the theory, ϕ^{2p} . For simplicity, both γ_n and G_n are considered at $g^2 = 1$.

correlation functions γ_n , for arbitrary values of n and g^2 . These agree with our perturbative results and yield robust results for the cumulants G_n determined from Eq. (2.7). An example of both γ_n and G_n is illustrated in Fig. 4. Notably, the moments γ_n , shown in the left panel of Fig. 4, exhibit a strong dependence on the self-interaction term of the theory as $n \rightarrow \infty$. This behavior aligns with our expectations in the nonperturbative regime, where we have established a suppression of the exponent F through the self-interaction power p , given in Eq. (4.33). In contrast, the cumulants G_n are largely independent of the self-interaction term, meaning they do not significantly depend on p . More precisely, as $n \rightarrow \infty$, they grow asymptotically as

$$G_n \sim 2^{n-1} c_p^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right)^2. \quad (4.34)$$

This behavior is consistent with the analysis of ordinary integrals presented in Sec. III. We again find that the cumulants grow faster with the number of field insertions than their counterparts that include disconnected contributions. This observation is somewhat counterintuitive, as all disconnected contributions are subtracted from the correlation functions to obtain the fully connected ones.

Extending these results to higher-dimensional QFTs represents an intriguing avenue for future research. One potential approach is to systematically compute the series representation of the effective action in powers of ϵ , as outlined earlier. In the case of QM, we have shown that by comparing the effective action to the exponent function F , this method appears to yield reliable results and is indeed equivalent to a large-order perturbative analysis of Schrödinger's equation. However, the expansion of the effective action in higher-dimensional QFTs is inherently more complex. Unfortunately, given that an all-orders resummation is required, achieving this may not be feasible. An alternative approach could involve employing fully nonperturbative Dyson-Schwinger equations, as explored in [48,49]. We leave this for future work.

That said, it would be interesting to investigate the massless regime of our results, naively resembling the CFT

limit of the quantum theory. In practice, this may be achieved through the framework of “exact perturbation theory,” where the perturbative ansatz for a theory with massive degrees of freedom can be suitably reorganized to obtain nonperturbative results for its massless limit [60,61]. While this method, in principle, facilitates the precise computation of vacuum expectation values in the massless limit, their physical interpretation is somewhat more involved. On a technical level, the ultrashort distance limit of moments in a quantum theory featuring conformal symmetry may not be meaningful, as any test function providing a localized support for the field necessarily introduces a characteristic length scale, beyond which individual spacetime points become indistinguishable. This length scale breaks the conformal invariance of the quantum theory. More generally and strictly speaking, a CFT also does not feature the notion of a finite separation between quanta, such that there are no asymptotic in and out states to describe scattering processes. Consequently, connected correlation functions and their associated S -matrix elements cannot be properly defined.

On the other hand, the semiclassical expansion around the effective saddle point, as outlined above, is able to shed light on quantum theories featuring global symmetries with an associated Noether charge. Naively, this Noether charge corresponds to the number n of operator insertions at the same spacetime point. In this case, the method may facilitate an interpolation between Feynman-diagrammatic calculations, valid for $\epsilon \ll 1$, and the superfluid regime for CFTs at large charge, $\epsilon \gg 1$ [29,30]. Closely following this approach, it would be interesting to supplement these examples with a massive degree of freedom. In this case, even determining only the first few terms of the effective action may provide valuable insights into the dynamics of large-charge sectors of the quantum theory.

Finally, our results are intriguing in the context of effective field theories (EFTs), which accommodate operators of arbitrary power beyond renormalizable QFT models. In this framework, our results suggest that the physical information of the quantum theory carried by the fully connected functions is only marginally dependent on

the precise operator structure in the UV. That is, in the large- n limit, all operators of the EFT contribute equally. In other words, in the nonperturbative regime of a quantum theory, we would not be able to detect the presence of higher-order interactions through the form factors discussed in this work. However, higher-dimensional QFTs introduce additional complications that must be carefully addressed. Notably, all coupling constants beyond a ϕ^4 term are dimensionful in four dimensions, which introduces an additional energy scale above which higher-dimensional operators become effective. This must be accounted for in the double scaling limit. Furthermore, it is important to recognize that the moments and cumulants discussed in this work do not represent physical observables in a strict sense. In a more realistic setting, physical correlation functions would be measured through scattering processes in particle collisions. This scenario would involve field insertions at different spacetime points, introducing nontrivial kinematics that can lead to suppression of interaction rates through phase space factors at large n , as the quantum theory would otherwise violate unitarity.

V. CONCLUSIONS

In this work, we have investigated the nonperturbative behavior of n -point correlation functions at short distances, with a particular focus on the role of self-interaction terms ϕ^{2p} . In the ultrashort distance limit, where all local field insertions coincide, the nonconnected and connected correlation functions correspond to the moments and cumulants of the probability density defined by the classical action, respectively. In the context of quantum theory, these objects can be given a well-defined meaning through an axiomatic formulation of QFT, where quantum fields are treated as operator-valued distributions.

To explore the dynamics of these moments and cumulants in the nonperturbative regime, we have adopted a double scaling limit of weak coupling, $g^2 \rightarrow 0$, and large quantum number, $n \rightarrow \infty$, while keeping the product $g^2 n$ fixed. We first follow a semiclassical approach, performing an expansion around the saddle points of an effective action that explicitly incorporates the field insertions inside the correlation function. This approximation is mainly accurate in the regime where the effective coupling is small, $g^2 n \ll 1$. Therefore, addressing the nonperturbative regime, where $g^2 n \gtrsim 1$, requires enhancing the approximation's range of validity, which can be achieved by a resummation of the perturbative saddle point expansion. While such computations may be feasible in scenarios featuring a lot of symmetry, they become increasingly challenging in more generic cases. We therefore perform the resummation of the saddle point expansion to all orders in $g^2 n$ in both zero and one dimensions, serving as a test bed for QFT. In this case, we derive exact solutions for the moments and cumulants, extending our analysis beyond the

perturbative regime. Our key finding is that the power p of the self-interaction term primarily influences the growth of moments, representing correlation functions that include disconnected contributions. In contrast, cumulants, representing the fully connected correlation functions that encapsulate the physical information of the quantum theory, exhibit a universal growth with respect to n , largely independent of the specific self-interaction term. Furthermore, we find that the cumulants grow more rapidly than the moments.

Our results challenge naive expectations, suggesting that the nonperturbative dynamics of quantum theories, encoded in fully connected correlation functions, exhibit a universal scaling behavior at large n . This indicates that their dynamics are predominantly governed by the combinatorial aspects of Feynman diagrams rather than the specific dynamics of the self-interactions. These insights may provide a crucial step toward understanding nonperturbative phenomena in QFTs, particularly in scenarios involving large quantum numbers.

Nevertheless, we note that our analysis is most robust in the context of QM, which can be interpreted as a one-dimensional QFT. While we anticipate that our main results may extend to higher-dimensional QFTs, a rigorous examination of these theories needs to address additional complications, including nontrivial phase space factors and the careful renormalization of composite operators. Future research endeavours should consider these aspects to fully elucidate the nonperturbative behavior of large- n correlation functions in realistic QFT models. In addition, investigating the interplay between different operators in effective field theories would provide a deeper understanding of the observed universality and its implications for low-energy phenomenology. These aspects certainly merit further investigation.

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DATA AVAILABILITY

The data supporting this study's findings are available within the article.

APPENDIX: PERTURBATIVE CORRECTIONS TO THE EFFECTIVE SADDLE POINT

In a d -dimensional Euclidean spacetime, we can define an effective action that accounts for the local operator insertions inside the path integral,

$$S_{\text{eff}} = \int d^d x \left(\frac{1}{2} (\partial \varphi)^2 + \frac{1}{2} \varphi^2 + \frac{1}{2p} \varphi^{2p} \right) - \epsilon \log \varphi(0). \quad (\text{A1})$$

The corresponding equations of motion for the field that minimize the action, $\delta S_{\text{eff}}/\delta \varphi = 0$, are given by

$$\partial^2 \varphi - \varphi - \varphi^{2p-1} = -\frac{\epsilon}{\varphi(0)} \delta^{(d)}(x), \quad (\text{A2})$$

such that the local operator insertions now act as an inhomogeneous source term. As discussed in the main text, we can systematically solve these equations of motion by employing the ansatz for the field

$$\varphi(x) = \sqrt{\epsilon}(\varphi_0(x) + \epsilon^{p-1} \varphi_1(x) + \epsilon^{2(p-1)} \varphi_2(x) + \dots). \quad (\text{A3})$$

The leading-order solution to the equations of motion, φ_0 , along with the corresponding effective action, $S_{\text{eff},0}$, are given in Eqs. (4.12) and (4.13).

Let us now outline how to obtain corrections to the effective action beyond leading order. Following our perturbative approach, we find the equations of motion for the first correction to the saddle point field configuration, φ_1 , to be

$$\partial^2 \varphi_1 - \varphi_1 = \varphi_0^{2p-1} + \frac{\varphi_1(0)}{\varphi_0^2(0)} \delta^{(d)}(x). \quad (\text{A4})$$

Here, all contributions on the right-hand side are known, in principle. Since the self-interactions of the field appear only as inhomogeneous terms in the equations of motion, we can express a formal solution for φ_1 using the free Green's functions D in position space,

$$\varphi_1(x) = \int d^d y D(x, y) \varphi_0^{2p-1}(y) + D(x, 0) \frac{\varphi_1(0)}{\varphi_0^2(0)}. \quad (\text{A5})$$

We note that this contribution to the classical field configuration at the saddle point is of order $\mathcal{O}(\epsilon^{p-1})$ with respect to the leading-order result φ_0 . Similarly, the first correction to the effective action is of order $\mathcal{O}(\epsilon^p)$ and reads

$$S_{\text{eff},1} = \epsilon^p \int d^d x \left(\partial \varphi_0 \partial \varphi_1 + \varphi_0 \varphi_1 + \frac{1}{2p} \varphi_0^{2p} \right) + \dots, \quad (\text{A6})$$

where the ellipsis denotes the logarithmic contributions from the local operator insertions. Although we have formal solutions for both φ_0 and φ_1 , Eq. (A6) is difficult to evaluate in general.

In the case of QM, we have $d = 1$, simplifying all multidimensional integrals to one-dimensional integrals. Here, the leading-order contribution to the saddle point configuration is given by

$$\varphi_0(t) = \frac{1}{\sqrt{2}} e^{-|t|}. \quad (\text{A7})$$

Furthermore, using the free Green's function

$$D(t, t') = \frac{1}{2} e^{-|t-t'|}, \quad (\text{A8})$$

we can explicitly evaluate the formal expression for φ_1 in Eq. (A5). This, in turn, allows us to determine $S_{\text{eff},1}$, by evaluating the spacetime integrals in Eq. (A6) and considering a suitable series expansion in ϵ of the logarithmic terms, such that we finally arrive at

$$S_{\text{eff},1} = \frac{\epsilon^p}{2^{p+1} p^2}, \quad (\text{A9})$$

presented in Eq. (4.15) in the main text.

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